ZDM – Mathematics Education
https://doi.org/10.1007/s11858-023-01536-w

ORIGINAL PAPER

On understanding mathematical problem-posing processes

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Accepted: 20 October 2023
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Abstract

Problem posing engages students in generating new problems based on given situations (including mathematical expressions or diagrams) or changing (i.e., reformulating) existing problems. Problem posing has been at the forefront of discussion over the past few decades. One of the important topics studied is the process of problem posing as experienced by students and teachers. This paper focuses on problem-posing processes and models thereof. We first provide an overview of previous research and then present the results of a scoping review regarding recent research on problem-posing processes. This review covers 75 papers published between 2017 and 2022 in top mathematics education research journals. We found that some of the prior research directly attempted to examine problem-posing processes, whereas others examined task variables related to problem-posing processes. We conclude this paper by proposing a model for problem-posing processes that encompasses four phases: orientation, connection, generation, and reflection. We also provide descriptions of the four phases of the model. The paper ends with suggestions for future research related to problem-posing processes in general and the problem-posing model proposed in particular.

Keywords Problem posing · Problem-posing processes · Problem-posing strategy · P-PBL

1 Introduction

Mathematical problem posing has been discussed for decades but only in recent years has it been at the forefront of research (Brown & Walter, 1983; Cai et al., 2015; Ellerton, 1986; English, 1998; Kilpatrick, 1987; Silver, 1994). The recent increased research activities in the domain of problem posing have been reflected in journal special issues (e.g., Cai & Leikin, 2020; Singer et al., 2013), books (e.g., Felmer et al., 2016), and conferences (e.g., TSG 17 of the ICME-14 and PME45 Research Forum by Cai, Koichu, Rott, Zazkis & Jiang, 2022). This increased research on problem posing has also been reflected in the wide range of problem-posing topics studied (see Cai et al., 2015, and Singer et al., 2013, for examples of such topics) and review papers (e.g., Baumanns & Rott, 2021; Cai & Leikin, 2020; Cai et al., 2015).

One of the important topics studied is the *process* of problem posing as experienced by students and teachers. Although we know that students and teachers are capable of posing mathematical problems, we have a considerably less fine-grained understanding of how they go about posing those mathematical problems in any given situation. Some researchers have identified general strategies students may use to pose problems (e.g., Brown & Walter, 1983; Christou et al., 2005; English, 1998; Koichu, 2020; Koichu & Kontorovich, 2013; Paolucci & Wessels, 2017; Pittalis et al., 2004). Others have explored some of the variables that may influence students' problem posing (e.g., Kontorovich et al., 2012; Leung & Silver, 1997; Zhang et al., 2022). Still others have explored the affective processes of mathematical problem posing (e.g., Schindler & Bakker, 2020). However, there is not yet a general problem-posing analogue to well-established frameworks for problem solving such as Pólya's (1945) four phases of problem solving, Garofalo and Lester's (1985) cognitive-metacognitive processes of problem solving, and Schoenfeld's (1985a, b) problem-solving attributes.

The purpose of this paper is to provide a synthesis of problem-posing research with a focus on understanding problem-posing processes, which involves models of such processes. Although the field is still trying to understand the cognitive and affective processes of problem posing, various researchers have attempted to understand these processes. One thing that is clear is that students need to understand the

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Published online: 16 November 2023

problem-posing situation and prompt before they can actually pose problems. There is also evidence that at least some (if not the majority of) students think about possible solutions to the problems they pose (Cai et al., 2015; Erkan & Kar, 2022). Understanding problem-posing processes could help with teaching through mathematical problem posing (Cai, 2022; English, 2020).

In this paper, we first present work about problem-posing processes before 2017 and then provide a detailed review of studies about problem-posing processes between 2017 and 2022. The purpose of this timeline is aligned with the guidelines of this special issue. We focus on understanding both the cognitive and affective processes of problem posing and, whenever possible, we discuss methodological issues related to understanding the processes of problem posing. We end by presenting a general problem-posing process model and a few directions for future studies.

2 Understanding problem-posing processes before 2017

Before 2017, there were two types of attempts made to understand problem-posing processes: through the products of problem posing and by direct investigation. Regarding the former, several earlier studies (e.g., Cai & Hwang, 2002; English, 1998; Silver & Cai, 1996) used students' posed problems as a basis for examining the problem-posing process. For example, Cai and Hwang (2002) used pattern situations to examine students' problem posing and problem solving. They observed that the sequence of pattern-based problems posed by students appeared to reflect a common sequence of thought when solving pattern problems (gathering data, analyzing the data for trends, making predictions). Silver and Cai (1996) found that students tended to pose related and parallel problems when they were asked to pose three problems based on a driving situation. They observed a clear tendency of students to pose later problems by varying a single element in earlier problems, which is known as the "what if not" strategy (Brown & Walter, 1983).

With a focus on problem-posing products, Christou et al., (2005) used confirmatory factor analysis to validate a theoretical model of four different processes that occur when individuals engage in problem posing: "editing quantitative information, their meanings or relationships, selecting quantitative information, comprehending and organizing quantitative information by giving it meaning or creating relations between provided information, and translating quantitative information from one form to another" (Christou et al., 2005, p. 149). The authors also showed that specific problem-posing tasks corresponded mostly to one of these four processes.

The second type of attempt that was made prior to 2017 to understand the problem-posing process was to directly investigate it. Three major descriptive models were proposed to describe the problem-posing process. First, Cruz (2006) described the process of problem posing in teaching–learning situations, including educational needs and goals (see Fig. 1). After setting a *goal*, a teacher *formulates a problem* and *tries to solve it*, which might fail or lead to regressions. After the problem has been solved, the problem is reflected upon, possibly *improved* to meet the goals, and then selected or rejected. This is a normative model of the problem-posing process intended to guide teachers; actually, it is based on a professional development program for teachers.

The second problem-posing process model—also a descriptive model—was developed by Pelczer and Gamboa (2009) and includes five phases, namely *setup*, *transformation*, *formulation*, *evaluation*, and *final assessment*. The setup phase is the starting point, including a reflection about the context of a given situation and the required knowledge. In the transformation phase, the given situation is analyzed, and possible modifications are reflected upon and then executed. During the formulation phase, problem formulations and possible alterations are explored. In the next phase, the posed problem is evaluated to see whether it satisfies the initial conditions. In the final phase, much like Pólya's looking-back phase in problem solving, the whole process is reflected upon.

Finally, Koichu and Kontorovich (2013) also developed a descriptive model. Based on two activities by prospective mathematics teachers called "success stories," they identified four phases of problem posing. The first phase is called *warming-up*, in which spontaneous ideas and typical problems regarding a given situation are posed. The next phase is called *searching for an interesting mathematical phenomenon*, in which the initially posed problems are critically considered and modified. Thereafter, problem posers are "hiding the problem-posing process in the problem formulation," which was a behavior that had not been observed before (Koichu & Kontorovich, 2013, p. 82). In the final *reviewing* phase, the posed problems are evaluated and possibly tested with peers.

In addition to the three descriptive models mentioned above, earlier studies have also tried to identify problem-posing strategies—as an analogue to heuristic strategies in

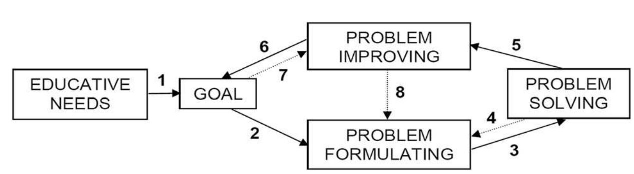

Fig. 1 Problem-posing phase model by Cruz (2006)

problem solving—as a way to understand problem-posing processes. There are consistent findings about the use of the "what if not" strategy in problem posing (Lavy & Bershadsky, 2003; Song, Yim, Shin, & Lee, 2007). For example, Lavy and Bershadsky (2003) identified two stages to posing problems. In the first stage, all the attributes included in the statement of the original problem are listed. In the second stage, each of the listed attributes is negated by asking "what if not attribute k?" and alternatives are proposed. Each of the alternatives could yield a new problem by varying the attributes.

3 Understanding problem-posing processes between 2017 – 2022

To understand recent developments in research on problem-posing processes, we performed a literature review. In the following subsections, we first describe the methodology for this review and then present the results.

3.1 Methodology of the review

Literature reviews have origins across a range of disciplines, which has resulted in inconsistent terminology used to identify review types (Horsley, 2019). However, initiatives like the PRISMA 2020 statement (Page et al., 2021) propose standards for review methodologies and terminologies. Using the terminology proposed by Horsley (2019), the review presented here is a *scoping review* in that it "addresses an exploratory (broad) research question aimed at mapping key concepts, types of evidence and gaps in research related to a defined area or field by systematically searching, selecting and synthesizing existing knowledge" (p. 55).

In early September 2022, we conducted a thorough collection of all recent problem-posing papers by searching the websites of 11 journals for papers using a time filter ("2017–2022") and the following search terms: "problem posing," "problem-posing," and "pos* problem*." The website search options were not limited to the articles' titles, abstracts, and keywords but included full texts. We did not search for similar terms such as "problem finding," "problem generation," and so on because "problem posing" has become the widely used term for the type of studies of interest for this review. The journals we included in this procedure were the top-ranked journals in mathematics education research identified in the study by Williams and Leatham (2017, p. 390), which are identical to all A*-, A-, and B-ranked journals in mathematics education as classified by Törner and Azarello (2012, p. 53). Table 1 shows the names of the journals and the number of papers found, with a total of 75 problem-posing papers; full bibliographic details of all 75 papers are included in the appendix.

In the first round of review, the abstracts of all 75 papers were read to exclude papers that were clearly not concerned with problem posing, especially those that did not include empirical research on problem-posing processes. For example, the paper by Leavy and Hourigan (2022) was excluded from this review because it presents a framework that was "developed to focus prospective teacher noticing on desirable features of mathematics problems and inform decision-making processes around the selection of problems for use in elementary classrooms" (p. 147) rather than focusing on problem-posing processes. Similarly, the paper by Goos and Kaya (2019) was excluded because it presents a literature review but no empirical research regarding problem-posing

Table 1 Literature Review Journals and Papers (sorted alphabetically by journal name)

Journal name	# of initial problem-posing papers	# of problem-posing papers read	# of problem-posing papers reviewed
Educational Studies in Mathematics	17	6	5
For the Learning of Mathematics	3	2	2
International Journal of Mathematical Education in Science and Technology	9	6	3
International Journal of Science and Mathematics Education	3	1	1
Journal for Research in Mathematics Education	4	0	0
Journal of Mathematics Teacher Education	8	0	0
Mathematical Thinking and Learning	2	2	2
Mathematics Education Research Journal	3	0	0
Research in Mathematics Education	1	0	0
The Journal of Mathematical Behavior	6	2	1
ZDM Mathematics Education	19	5	4
Total	75	24	18

processes. This round of reductions led to 24 papers that were then fully read as the basis of the review (see Table 1, the third column).

At this stage, another two papers were excluded from the review results that are presented in the next section. In their abstract, Downton and Sullivan (2017) wrote that the "posing of appropriately complex tasks may actually prompt the use of more sophisticated strategies" (p. 303). Reading the paper, it became clear that the participating children of this study did not pose problems but were given problems to solve in task-based interviews. Similarly, Yao and Manouchehri (2019) presented a study that was not about problem posing but rather about a teaching experiment focused on experimenting, conjecturing, and generalizing related to geometric transformations in a dynamic geometry environment.

3.2 Results of the review

Although the final selection of 22 papers primarily dealt with better understanding problem-posing processes, not all of the studies described in those papers were directly aimed at the same aspects of such posing processes. It should be indicated that four of the reviewed studies used problem posing as an assessment tool to assess students' understandings of mathematical topics. These studies used different subjects and tasks to assess students' understanding of different mathematical topics. For example, Wessman-Enzinger and Mooney (2021) assessed eighth-grade students' thinking about integers and integer addition and subtraction using problem posing. Radmehr and Drake (2017) used a graphical problem-posing task to assess Year 13 and university students' understanding of the Fundamental Theorem of Calculus. It is not new to use problem posing to assess posers' mathematical thinking and understanding in problem-posing research. Reviewing the past 5 years of research shows that this tradition continues. Although problem posing used as an assessment tool to assess posers' understanding of different mathematical topics is not the same as understanding problem-posing processes, this aspect of problem-posing research shows the need to understand such processes. It is possible that with better understanding of problem-posing processes, we can better assess students' mathematical thinking and understanding.

We first divided the reviewed papers into those involving cognitive processes of problem posing and affective processes of problem posing. For the papers involving cognitive processes of problem posing, we further divided them into two categories: papers examining cognitive processes of problem posing through posed problems and papers directly examining problem-posing processes. In what follows, we present short summaries of the 18 papers left for the review (see Table 1, the fourth column), organized into three inductively formed categories based on the studies' foci: insights into cognitive processes of problem-posing through problem-posing products, insights into cognitive problem-posing processes through directly studying problem-posing processes, and insights into noncognitive aspects of problem-posing processes. None of the papers fell into more than one category.

3.2.1 Insights into cognitive processes of problem-posing through problem-posing products

The authors of nine of the reviewed studies concentrated their analyses on problem-posing products—that is, the problems that were posed by the participants of their studies. Two of the nine studies used different designs to understand the problems that were posed and then directly or indirectly examined the problem-posing processes based on analyses of the posed problems. A problem-posing task has two components—a prompt and a situation (Cai et al., 2022). One study, Silber and Cai (2017), focused on variations of prompts, and the other study, Zhang et al. (2022), focused on variations of situations.

Silber and Cai (2017) examined the influence of two problem-posing prompts on preservice teachers' problem posing. The first prompt asked the teachers to pose problems (free) whereas the second prompt asked the teachers to pose problems with more specific conditions (structured). The results suggested that the posing prompt did influence the teachers' problem posing given that teachers in the structured posing condition more frequently showed evidence of addressing the mathematical concepts underlying the problem-posing tasks. In addition, findings from the interview component of the study suggested the need for problem posers to understand the task prior to posing problems, regardless of the posing conditions.

Zhang et al. (2022), meanwhile, focused on problem-posing situations. The goal of their study was to understand the cognitive process of problem posing by varying problem-posing situations, following Leung and Silver (1997). The authors conceptualized a framework with three problem-posing stages: (a) input, (b) processing, and (c) output. Examining the posed problems of 669 sixth-grade students, they especially focused on examining the role of the task format (i.e., with or without context and with or without specific numerical information). The students were given a problem-posing test with different problem-posing situations (randomly varied in eight sets of booklets) as well as a questionnaire eliciting their perceptions of understanding the problem-posing tasks. One month later, a problem-solving test was given using the same situations that had already been used on the problem-posing test. Analyses of the 2,376 responses (coded for being or not being mathematical problems, being clear or unclear, etc.) found that the participants were generally more successful with problem-posing situations that included specific numerical information than with those that did not. Regarding the task format, with or without context, students performed significantly better on tasks with versus without context. Also, students who were able to solve the problems on the problem-solving test posed more mathematical and more solvable problems than students who could not solve the respective problems. The findings from Zhang et al. (2022) not only suggest the impact of problem-posing situations on students' problem posing but also show the usefulness of examining problem-posing in different stages, such as during the input, processing, and output stages.

The remaining seven studies did not vary situations or prompts. One of these seven studies by Guo et al., (2021) confirmed findings about problem-posing processes based on analyses of the posed problems of 904 Chinese students in Grades 7, 8, and 9, aiming to identify grade-level differences in students' posed problems. They found that many students posed problems parallel to what was found in Silver and Cai (1996).

The remaining six studies examined knowledge involved in problem posing such as mathematical or real-life knowledge. For example, Silber and Cai (2021) gave 45 undergraduate students four problem-posing tasks, finding that many of the students were able to identify key mathematical ideas of the given situations and use them in their posed problems. This study suggests the role of knowledge in the problem-posing process. The authors concluded that problem-posing tasks could be used as a pedagogical tool to help students who have previously struggled with mathematics or have experienced mathematical anxiety.

Like Silber and Cai (2021), the study by Ergene (2021) also suggested the role of knowledge in problem posing. Ergene worked with 48 university students enrolled in an elementary mathematics education program. Students were asked to pose probability problems based on continuous and discrete sample spaces, to write reflection papers, and to participate in semi-structured interviews over a period of 4 weeks. Drawing on a framework by Christou et al. (2005), who differentiated between processes of editing, selecting, comprehending, and translating quantitative information when engaging in problem posing, Ergene used specially designed problem-posing tasks for each of those four processes. Similar to previously reported results, Ergene showed that his students mostly posed suitable and solvable problems and that most of those solvable problems were situated in real-life contexts, were applicable, and had clear language. Problems that were not suitable or solvable were influenced by the mathematical content (it was harder for students to pose problems for continuous than for discrete sample spaces) and students' unfamiliarity with the mathematical content.

The study by Nedaei et al. (2021) revealed that even though 80% of the posed problems were mathematically solvable, only very few of the problems were set in real-world contexts and those were often not realistic. The authors suggested using problem posing in teaching to show students real-world applications of the knowledge they have to acquire.

Jung and Magiera (2021) conducted a university course (a whole semester) on problem solving and modelling (with a background in social justice) with 36 preservice teachers, including two problem-posing cycles. Data from this study were the posed problems as well as written reflections from the participants. The results showed that the problem-posing activities could increase participants' awareness of sociocritical modeling, including realistic contexts.

There are very few problem-posing studies involving preschoolers; the study by Palmér and van Bommel (2020) was one such study. They conducted a teaching experiment with three preschool classes (children aged 6 years old) in Sweden. The pupils participated in a problem-solving lesson (with a problem on building blocks) and were then, in a second lesson, asked to pose a similar problem. Most of the 27 pupils were able to pose tasks, although not all of them included mathematical questions. The results also showed how the students interpreted the request to pose "similar" problems, with some students interpreting this to mean both constructing a very similar block building and a similar question whereas other students only constructed a similar building but did not pose a similar question.

We conclude this subsection with the study by Fosse and Meaney (2020) who presented a very different view of problem-posing products. They referred to the Norwegian tradition of *regnefortelling* wherein students pose number story problems that can be solved with arithmetical calculations. They showed that sometimes contexts and real-world references can make peers and teachers feel uneasy and uncertain because the stories can deal with unpleasant or even scary topics like stealing, murder, and so on. The authors discussed that such stories can indicate students' stress and other unfavorable emotions, why teachers should not tell students what are "acceptable" problems, and how to properly react in such unpleasant situations.

This category of studies clearly suggests (also confirmed by the review of studies in the next section) the need for problem posers to understand problem-posing tasks (including understanding the situations and prompts) before they actually pose problems. In particular, the studies that investigated task variables and problem posing found that task variables influenced posers' problem posing (Cai et al., 2022). Some of the studies reviewed here also point towards the role of problem posers' knowledge and prior familiarity with content in the problem-posing process. This finding may suggest problem posers' identification of knowledge in problem-posing tasks and using identified knowledge to pose problems.

3.2.2 Insights into cognitive processes of problem posing through directly studying problem-posing processes

It is quite encouraging that in four studies, problem-posing processes were addressed explicitly and explored via task-based interviews or close observations. Problem-solving activities were identified as an important part of these problem-posing processes. Xie and Masingila (2017) conducted two task-based interviews each with five pairs of preservice primary school teachers using a systematically varied selection of five tasks between the two interviews. They showed how problem posing contributes to problem solving (e.g., understanding a problem by posing an easier one) and vice versa (e.g., checking whether a posed problem has the anticipated solution or deepening understanding of the structure of mathematical problems).

Analyzing problem-posing task-based interviews with nine preservice mathematics teachers, Erkan and Kar (2022) showed that most of their participants tended to initiate the problem-posing process by finding the solution to the problem from the given problem-posing situation. This finding has confirmed what was found in other studies reviewed in Cai et al. (2015)—that is, problem posers tend to think about solutions for the problems they pose. Their findings suggest the importance of understanding the structure, finding a context, and determining the purpose of the initial situations in problem posing (Erkan & Kar, 2022). Additionally, they identified several metacognitive, cognitive, and instructional factors that played a role in the problem-posing process.

Similar to Xie and Masingila (2017), Baumanns and Rott (2022a) highlighted, among other things, the importance of the interplay between problem solving and problem posing. They developed a phase model to describe observed problem-posing processes based on an analysis of videotaped problem-posing processes of 64 preservice mathematics teachers. The theoretically and empirically grounded phases were *situation analysis* (understanding the problem-posing situation and the respective prompt), *variation* (altering an already given task, for example by using the "what if not" strategy, to pose a new task), *generation* (coming up with a new task that fits the situation but is not a variation of an already given task), *problem solving* (solving the self-posed tasks), and *evaluation* (reflection upon the solvability, quality, etc. of the self-posed tasks). Baumanns and Rott's situation analysis stage is similar to the input stage of Zhang et al. (2022) mentioned earlier.

Another perspective on problem-posing processes was offered by Armstrong (2017) who observed such processes of four groups of students with the goal of learning more about *bricolage*, which is best described as improvisation, working without a finished plan, or deciding on the spot. In this study, problem posing was used to facilitate processes that showed bricolage; Armstrong developed and presented a method to provide physical tracing of students' ideas and their ways of bridging gaps in understanding.

To summarize, previous attempts have been made in developing different phases of problem-posing processes. For example, like the studies that focused on problem-posing products (Sect. 3.2.1), the studies summarized in this category of the review highlight the importance of understanding the given situation. The problem-posing phases discussed in Baumanns and Rott (2022a) and Zhang et al. (2022) also confirm the importance of understanding the given situation.

3.2.3 Insights into noncognitive aspects of problem-posing processes

Five papers addressed noncognitive aspects of problem posing like affect, self-efficacy, and metacognition. These studies used a similar design in the sense that relationships between problem posing and affect were examined but with different samples and sample sizes, from a single-case study to a sample of over 1600 students. Schindler and Bakker (2020) presented a case study of 18-year-old Anna who participated in an extracurricular enrichment program that lasted 1 year and focused heavily on collaborative problem-posing and problem-solving activities. They described how Anna's affective field (i.e., the interplay of a person's various affective factors) developed against the background of the activities and how her interest towards problem solving and problem posing grew considerably with her experiences.

On a larger scale, Segal et al. (2018) described a course over two semesters on mathematics didactics in which 61 preservice teachers learned to formulate hypotheses and generated interest and curiosity towards problem solving and posing. In the course, the preservice mathematics teachers worked in pairs on problem-posing assignments with dynamic geometry software using the "what if not" strategy. The analyzed data were the preservice teachers' presentations and discussions. One of the major findings was that the vast majority of the preservice teachers reported that they perceived themselves as participants rather than spectators. That is, they enjoyed the participation facilitated through problem posing.

Baumanns and Rott (2022b) conducted task-based interviews with 64 preservice teachers working in pairs on problem-posing tasks. The focus of the analyses of the videotaped processes was metacognitive behavior for which a framework was developed, encompassing activities of planning, monitoring, and evaluating. The authors found that the chosen perspective could reveal significant differences in the preservice teachers' processes which yielded identical products (i.e., posed problems). They concluded that considering metacognitive behavior could help in analyzing and understanding problem-posing activities.

Guo et al. (2020) conducted a quantitative study with 302 ninth graders from China. They used a test with one free and two semi-structured problem-posing tasks; coded for complexity, quantity, and accuracy; and correlated results of this test with self-report measures of self-concept, intrinsic value, and test anxiety. They found that self-concept was positively correlated with complexity and accuracy of posed problems and intrinsic value was positively correlated with complexity and quantity of posed problems.

Liu et al. (2020) used the largest sample size among the studies reviewed, measuring 1634 eighth-grade students' problem-posing performance and correlating it with their domain- and task-specific self-efficacy. The problem-posing tests contained three situations with three different prompts each (posing problems of differing difficulty levels), resulting in nine problem-posing items. Self-efficacy was measured with a self-report Likert-scale instrument. Students were more confident in their ability to pose problems when they were familiar with or knew more details about a real-life or mathematical scenario. They also found that the relationship between students' domain- and task-specific self-efficacy and their problem-posing performance was not always linear. In fact, the relationship between the students' task-specific self-efficacy and posing performance was different for the posing of easy versus the posing of difficult problems.

These studies indicate the beginning of explorations of affect in mathematical problem posing, with a focus on the relationship between problem posing and affect. The findings across these studies demonstrate that problem posing and affect are significantly correlated. Although these studies did not directly examine affective processes of problem posing, they clearly indicate the need to examine the role affect plays in problem-posing processes.

To summarize, several attempts have been made to better understand problem-posing processes by analyzing products and processes. For example, several studies pointed toward the importance of prior knowledge and familiarity with the context of the problem-posing situations, the importance of understanding given situations, and the influence of self-concept and self-efficacy. Although there is no commonly accepted general problem-posing process model, these studies suggest the critical need for developing such a model. These earlier studies assisted us in developing our general problem-posing process model in three ways. First, they suggested the need to understand problem-posing tasks. Actual posing is based on the comprehension of information embedded in problem-posing tasks. Second, our proposed model is consistent with earlier attempts at understanding problem-posing processes (cf. the three models in Sect. 2). Third, we extended these earlier attempts to obtain a full circle of problem-posing processes. We discuss our problem-posing process model in the next section.

4 A general problem-posing process model

Both the earlier attempts at understanding problem-posing processes (Sect. 2) and the review of papers spanning the period from 2017 to 2022 (Sect. 3) demonstrate that relatively few studies have focused on modeling problem-posing processes. However, we believe that a widely accepted model analogous to Pólya's (1945) problem-solving process model would advance the field of problem-posing research by serving as a foundation for future studies to refer to and extend on. Therefore, in this section, we propose and discuss a general problem-posing process model. Like Pólya's problem-solving model, this problem-posing model is heuristic in nature. This proposed model also parallels Pólya's four phases of problem solving. Figure 2 shows a graphic representation of our proposed model. It should be noted that in many cases, students are asked to pose multiple problems based on a single situation.

In fact, problem posing can be viewed as a special type of problem solving in the sense that problem posers are given information and the goal is to pose problems. Therefore, it is reasonable to view the problem-posing process as parallel to the problem-solving process, and Pólya's (1945) four phases of problem solving, which are understanding, planning, execution, and looking back, could correspond to four phases in problem posing. We propose the four phases of problem posing to be orientation, connection, generation, and reflection. Orientation is straightforward and corresponds to understanding. Reflection is also straightforward, corresponding to looking back. The parallels between the middle two phases, connection and generation, are less straightforward. We describe the four phases below.

It should be indicated that, in some respects, problem posing differs significantly from problem solving. For example, the "solutions" of a problem-posing task—that is, the posed problems—cannot be checked or derived by other methods to ensure correctness. Instead, totally different criteria are necessary to interpret the results of problem-posing activities. Also, totally different strategies are used in problem-posing processes compared to the heuristics in problem-solving processes. This warrants specific models for problem-posing processes.

4.1 Orientation

Orientation refers to understanding a problem-posing task and corresponds to Pólya's first phase, understanding of a given problem. As discussed elsewhere, a problem-posing

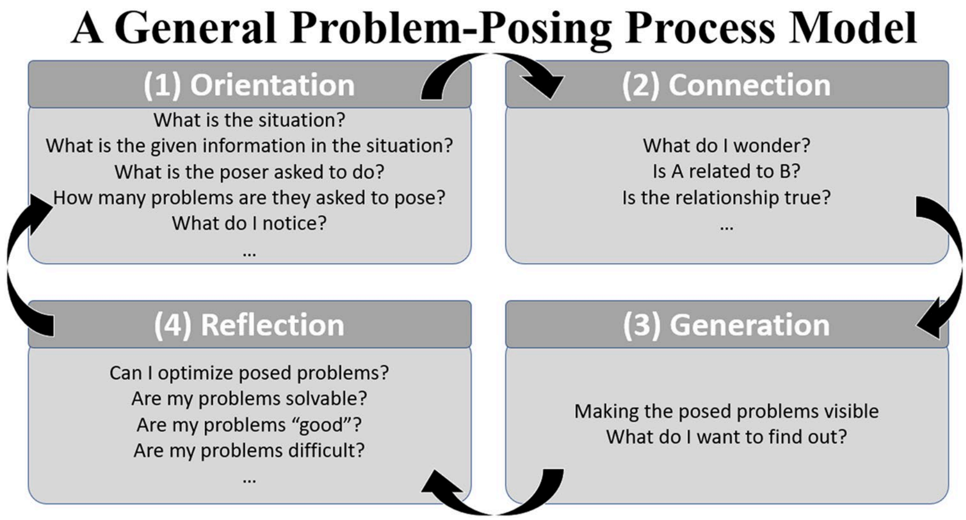

Fig. 2 Graphical Representation of a General Problem-Posing Process Model

task includes a situation (either mathematical or real life) and a prompt (Cai et al., 2022). The goal of orientation is to understand both the situation and the prompt. This orientation phase has been mentioned in several prior studies such as Zhang et al.'s (2022) input or Baumanns and Rott's (2022a) situation analyses stages and involves trying to understand the data involved and the requirements being asked of the problem poser. Again, several studies in this review suggest that such an understanding is important in problem posing, for example by Erkan and Kar (2022; see Sects. 3.2.2 and 3.2.3 for details). In this phase, problem posers might think of questions like the following:

What is the situation?

What is the given information in the situation?

What is the poser asked to do (prompt)?

How many problems are they asked to pose?

Are there any specifications for problems to be posed (such as the number, difficulty level, or any other requirements)?

What do I notice?

What do I want to find out?

What relationships are involved in the task?

4.2 Connection

The second phase is connection. Once the problem poser gradually comes to understand the problem-posing task, they might wonder about possible connections and make conjectures. In this phase, the problem poser wants to find out things involved in the problem-posing situation. They develop ideas for new problems by varying the given situation (often with the help of the famous "what if not" strategy or similar techniques that are analogous to problem-solving heuristics) or by generating new problems that are only vaguely tied to the given situation (cf. Baumanns & Rott, 2022a; Ergene, 2021). Developing ideas might be—more or less—straightforward or at times a difficult endeavor comparable to Schoenfeld's (1985a, 1985b) exploration phase in problem-solving processes. They might think of questions like the following:

What do I wonder?

Is A related to B?

Is the relationship true?

The connections could be different kinds, such as mathematical representations and real-life contexts or different pieces of information in the situation (Zhang et al., 2022). It should be indicated that the poser might supply additional information to make possible connections (Silver & Cai, 1996).

4.3 Generation

The third phase is generation, which involves external representation of the potential connections made in the previous phase. Whereas orientation and connection may occur mentally, the generation phase involves making the posed problems visible so that others can see them. As in the problem-solving analogue—execution or carry out—generation consists of the things the problem poser wants to find out. Sometimes problem posers can present problems they are wondering about as conjectures and they simply need to prove that the conjectures are true or not. The goal is to present the relationship or certain facts in question format so that others will be able to understand the questions the problem poser has (Baumanns & Rott, 2022a). Often, problem posers solve their own problems or at least start to do so to see if they fulfill given (e.g., being solvable by mathematical means, being of the anticipated level of difficulty, etc. like in several of the studies discussed in Sect. 3.2.1) or self-imposed (e.g., being of pedagogical value, being fun to work on, etc.) criteria (Sect. 3.2.3).

4.4 Reflection

The last phase, reflection, can involve a variety of things such as reflecting after posing one problem (Baumanns & Rott, 2022a), optimizing a problem by reformulating it (e.g., Hartmann, in press), and reflecting during the move from one problem to the next problem or after posing all the required problems (Erkan & Kar, 2022). As in problem solving (see, for example, Schoenfeld, 1985a, 1985b), reflection is more like a meta-level that involves monitoring what was done during the problem-posing process. Reflection can help the problem poser evaluate what they did (Erkan & Kar, 2022; Jung & Magiera, 2021).

5 Conclusion

The field of mathematics education is still in the early stages of understanding problem-posing processes. In fact, there is a dilemma. On the one hand, there is growing interest in problem-posing research. On the other hand, we are nowhere near close to understanding problem-posing processes. In this paper, we have provided a brief review of research regarding these processes. As in problem solving, posers need to understand problem-posing tasks (including situations and prompts). Prior research shows that problem posers tend to pose related problems from one problem to another using the "what if not" strategy. In this paper, as a conclusion to our review, we proposed a general problem-posing process model (orientation, connection, generation, and reflection) based not only on the reviewed papers but on problem-posing research in general. In future studies, we will support this model with prior research as well as empirical studies, and we encourage fellow researchers to empirically examine the usefulness of this and other problem-posing models to help the field better understand such processes and help students become better problem posers.

We hope that this paper is helpful to researchers who engage in further studies to understand problem-posing processes. To foster our understanding of problem-posing processes, Cai, Koichu, Rott, Zazkis, and Jiang (2022) presented the state of the art on efforts to understand the impact of task variables on problem posing. They examined the impact of task variables at the individual, group, and classroom levels and found that although there are some studies investigating the impact of task variables on the processes and products of problem posing at the individual level, even fewer attempts have been made to examine the impact of task variables on the processes and products of problem posing at both the group and classroom levels. Research is thus needed to use the proposed general problem-posing model to examine the impact of task variables on the processes and products of problem posing at the individual level and especially at the group and classroom levels. In fact, given the potential of fostering students' learning and conceptual understanding through engaging in problem posing (Cai, 2022), understanding the impact of task variables at both the group and classroom levels is particularly important for us to understand the mechanisms of teaching mathematics through problem posing.

In addition, there is a need to expand the proposed general problem-posing model to include aspects related to affect (cf. Cai & Leikin, 2020), including motivation and beliefs. As Cai and Leikin (2020) indicated, the field of mathematics education has already paid close attention to affect in mathematical problem posing and found that students' affect is closely related to problem posing. However, more research is needed to examine problem posers' affective characteristics that promote or impede problem posing. We not only need to expand the problem-posing process model to understand how cognitive and affective processes are interweaved in the problem-posing process but also how different kinds of problem-posing tasks evoke posers' cognitive and affective mechanisms that promote effective problem-posing processes. In fact, Baumanns and Rott (2022b) showed not only the importance of examining the affective process of problem posing but also suggested the feasibility of doing so.

Funding During the preparation of this paper, Jinfa Cai is supported by a grant from the National Science Foundation (DRL- 2101552). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

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