Finance and Stochastics (2024) 28:497–551
https://doi.org/10.1007/s00780-023-00527-9

Optimal investment in a large population of competitive and heterogeneous agents

Ludovic Tangpi[1] · **Xuchen Zhou**[1]

Received: 22 February 2022 / Accepted: 7 April 2023 / Published online: 5 February 2024

Abstract

This paper studies a stochastic utility maximisation game under relative performance concerns in finite- and infinite-agent settings, where a continuum of agents interact through a graphon (see definition below). We consider an incomplete market model in which agents have CARA utilities, and we obtain characterisations of Nash equilibria in both the finite-agent and graphon paradigms. Under modest assumptions on the denseness of the interaction graph among the agents, we establish convergence results for the Nash equilibria and optimal utilities of the finite-player problem to the infinite-player problem. This result is achieved as an application of a general backward propagation of chaos type result for systems of interacting forward–backward stochastic differential equations, where the interaction is *heterogeneous* and through the control processes, and the generator is of quadratic growth. In addition, characterising the solution of the graphon game gives rise to a novel form of infinite-dimensional forward–backward stochastic differential equation of McKean–Vlasov type, for which we provide well-posedness results. An interesting consequence of our result is the computation of the competition indifference capital, i.e., the capital making an investor indifferent between whether or not to compete.

Keywords Stochastic graphon games · Propagation of chaos · FBSDE · McKean–Vlasov equations

Mathematics Subject Classification (2020) 91A06 · 91A13 · 91A15

JEL Classification C72 · C73

Both authors gratefully acknowledge partial support from the NSF grants DMS-2005832 and CAREER DMS-2143861. We thank the anonymous referees and the Editor for their constructive feedback as well as Stefanie Hesse for fruitful discussions.

✉ X. Zhou
xuchen.zhou@princeton.edu

L. Tangpi
ludovic.tangpi@princeton.edu

[1] ORFE, Princeton University, Princeton, USA

1 Introduction

We consider agents investing in a common riskless bond and a vector of stocks of their own choosing. Each agent aims to maximise their utility as a function of their terminal wealth, benchmarked by the industry average. In addition, agents' utility functions are exponential. This problem was first investigated by Espinosa and Touzi [15, 14] in the setting of a complete market with a finite number of agents. In these works, the benchmark for a particular agent was taken to be an empirical average of the other players' terminal wealth, multiplied by a constant factor between 0 and 1 representing the sensitivity of this particular agent to their peers' performance. Such a utility maximisation problem under *relative performance concerns* has since been explored extensively using various techniques; see for instance Frei [16], Frei and dos Reis [17], Lacker and Zariphopoulou [28], Fu et al. [18], Lacker and Soret [26], Hu and Zariphopoulou [21] and references therein for a small sample of works on the question. We also refer to Dos Reis and Platonov [12], Anthropelos et al. [1], dos Reis and Platonov [13] for more recent articles studying relative performance concerns through the lens of the forward criteria of Musiela and Zariphopoulou [36]. The papers [15, 16, 17, 18] approach the problem from a purely probabilistic perspective through characterising the solution of the game using systems of (forward–)backward stochastic differential equations ((F)BSDEs). In particular, Fu et al. [18] explore an extension of the game in an incomplete market framework in the following sense: all agents there invest in the same vector of stocks (with dimension d) and their strategies can take values in $\mathbb{R}^d$. In addition to having a common Brownian motion W^* representing the market uncertainty or "common noise" in the price dynamics of all stocks, they allow each stock to be driven by a separate Brownian motion representing the "idiosyncratic noise". These Brownian motions are i.i.d. and independent of W^*. In this setting, the characterising system of BSDEs has the particular feature that it is quadratic in the control variable. As first observed in the work of Frei and dos Reis [17], such systems are not always globally solvable, making the analysis of the problem in full generality particularly challenging. For instance, Frei and dos Reis [17, 16] provide specific counterexamples when equilibria do not exist even in the case of a complete market where stocks are driven by idiosyncratic noise only. Frei [16] showed that multidimensional quadratic BSDEs are in general only locally solvable (i.e., the solutions exist only on small time intervals), and he provided equilibria for the original game of Espinosa and Touzi [15] using the existence of local solutions to the characterising BSDEs. Fu et al. [18] established existence and uniqueness of the characterising BSDEs in an incomplete market framework where all investors invest in two stocks only and the strategies are *unconstrained*.

Recent developments in mean field games provide new avenues to approach the above described utility maximisation game by considering the infinite-population case. In fact, standard mean field games heuristics of Lasry and Lions [30], Huang et al. [23] and Carmona and Delarue [10] suggest that in a *homogeneous* game, that is, when agents are *symmetric* and *identical*, the infinite-population analogue of the game can be solved by considering a *single* representative player whose best response is obtained as solution of a (one-dimensional) McKean–Vlasov BSDE. More precisely, one thus expects to bypass the subtleties coming with studying multidimensional quadratic BSDEs by analysing a one-dimensional McKean–Vlasov quadratic BSDE. Despite also having quadratic growth, the latter equation seems much easier to analyse (both analytically and numerically) than the former, making the mean field game paradigm particularly attractive for this game. The mean field setting was first considered by Lacker and Zariphopoulou [28] and Lacker and Soret [26] in the Markovian setting with deterministic, constant coefficients, and equilibria were derived using Hamilton–Jacobi–Bellman SPDE methods. More recently, probabilistic techniques were proposed by Fu et al. [18] and dos Reis and Platonov [13].

In the present paper, we rigorously study the link between the finite- and infinite-population games. Our main modelling assumptions can be summarised as follows: we consider an incomplete market in which

– agents are allowed to invest in different vectors of stocks with *random coefficients*, driven by idiosyncratic noise and common noise;

– agents' strategies are *constrained* to take values in a closed, convex set;

– agents benchmark their performance by a *weighted average* of other agents' terminal wealth.

Let us elaborate on the last and probably less studied model feature mentioned above. For a particular agent i, instead of having a single factor λ_i representing their sensitivity to a plain average of the other agents' terminal wealth, we allow this agent to have different sensitivity factors $(\lambda_{ij})_{j\neq i}$ towards each agent. This assumption is a lot more realistic in the sense that funds usually aim to outperform a small, specific group of competitors and are usually completely indifferent to the performance of other funds that are for example on a much smaller or larger scale, utilise completely different strategies or operate in a widely different market sector. This leads to a *heterogeneous* game set on a (random) graph and deviates from the standard symmetric agent interaction assumption, which is arguably the main limitation of the mean field game formulation. Following the seminal works of Lovász [33, Chap. 11] and Lovász and Szegedy [34] on the convergence of graphs to so-called *graphons* (see precise definition and discussion below), the natural infinite-population analogue of the game we consider in the present heterogeneous setting is a utility maximisation graphon game. It is worth pointing out that in addition to the methodology, the main modelling difference between the present work and [18] resides in the heterogeneous interactions among agents and the consideration of constrained strategies here. These give rise to infinite-dimensional McKean–Vlasov type (F)BSDEs with quadratic generators, making the analysis more demanding and requiring new techniques.

Similarly to mean field games, graphon games provide an alternative to study large scale network games that in general suffers less from the curse of dimensionality. However, unlike mean field games, agents in a graphon game are no longer anonymous, as mentioned by Carmona et al. [9]. The benefit associated with a graphon interaction is that agents are now aware of who their neighbours are, and are allowed to possess different preference metrics towards different neighbours. As a result, when deriving the optimal strategy for a specific agent, one needs to take into account (an *aggregation* of) a continuum of infinitely many other agents, where the aggregation is established through the graphon. The analysis of graphon games has gained traction in recent years, mostly in the engineering community. Parise and Ozdaglar [38] were the first to analyse equilibria for static graphon games. Caines and Huang [7, 8] studied decentralised control for graphon mean field games and established an ϵ-Nash theory that relates the equilibria for an infinite-population game to that of a finite-population game. Gao et al. [19] explored linear–quadratic Gaussian mean field games. Outside of the engineering community, Carmona et al. [9] studied various static graphon games. Aurell et al. [2] studied stochastic graphon games in a linear–quadratic setting. We also refer to the recent works of Lacker and Soret [27] and Bayraktar et al. [6] for recent results on generic games. The main contributions of the present work can be summarised as follows:

– We derive explicit characterisation properties of the Nash equilibria in the finite and the graphon utility maximisation games.

– We show that if the sensitivity matrix in the finite-agent game stems from a graphon and follows a Bernoulli distribution, then the heterogeneous finite-agent game converges to the graphon game in the sense that every sequence of Nash equilibria converges (up to a subsequence) to a graphon equilibrium along with the associated value functions.

– We prove solvability of the graphon utility maximisation game.

For the characterisation properties, we adopt an extension of a well-known methodology proposed by Hu et al. [22]. The convergence and existence results are more involved. Convergence is obtained as a byproduct of a general *backward propagation of chaos* type result which appears to be of independent interest. Consider a general system of weakly interacting FBSDEs in which the interaction is given through a random graph (appropriately) stemming from a graphon. We prove strong convergence of the interacting particle system to a limit consisting of infinitely many coupled particles. Backward propagation of chaos type results and their link to the mean field game convergence problem were first developed in recent works by Laurière and Tangpi [31], Luo and Tangpi [35] and Possamaï and Tangpi [39]. Note that in these works, generators are Lipschitz-continuous. Our work contributes to the theory by extending it to systems in heterogeneous interactions through the control processes and where the generators are of quadratic growth. A case of FBSDEs with heterogeneous interactions was posted on arXiv a week before the present work by Bayraktar et al. [6]. See also Bayraktar et al. [4], Bayraktar and Wu [5] for results along the same lines for *forward* particle systems. The results and methods of the present work further allow us to introduce and compute the so-called competition indifference capital, which is the capital allowing to make the investor indifferent between being concerned by their peers' performance or not.

The paper is organised as follows. In Sect. 2, we first introduce the probabilistic setting and the market model, followed by the finite-agent and the graphon models and lastly the main result of this paper, namely the convergence of the finite-agent Nash equilibrium to the graphon Nash equilibrium. The BSDE characterisations of the finite-agent game and the graphon game are presented in Sects. 3.1 and 3.2, respectively. Section 4 is dedicated to the proofs of the existence results. In Sect. 5, we prove the main results which are propagation of chaos for heterogeneous particle systems. Section 6 establishes existence and uniqueness for solutions of general graphon FBSDEs of McKean–Vlasov type which then allows deriving existence of solutions to the graphon game.

2 Probabilistic setting and main results

Let us now present the probabilistic setting underpinning this work. In this section, we also describe the market model as well as the finite- and infinite-population games under consideration. At the end of the section, we present the main results of the article.

2.1 The market model

We fix a finite time horizon $T > 0$ and integers $n, d \in \mathbb{N}$. Let $(W^i)_{i\geq 1}$ be a sequence of independent d-dimensional Brownian motions on $[0, T]$ supported on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, this probability space supports another independent one-dimensional Brownian motion W^* on $[0, T]$ and independent $\mathbb{R}$-valued random variables $(\xi^i)_{i\geq 1}$. We denote by $\mathbb{F}^n := (\mathcal{F}^n_t)_{t\in[0,T]}$ the $\mathbb{P}$-completion of the natural filtration of $((W^i)_{i=1,\dots,n}, W^*, \xi^1, \dots, \xi^n)$. Let us define spaces and norms used throughout the paper. Fix a generic finite-dimensional normed vector space $(E, |\cdot|)$, let $\mathbb{G}$ be a filtration and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.

• For $p \in [1, \infty]$, $\mathbb{L}^p(E, \mathcal{G})$ is the space of E-valued, $\mathcal{G}$-measurable (equivalence classes of) random variables R such that

$$\|R\|_{\mathbb{L}^p(E,\mathcal{G})} := (\mathbb{E}[|R|^p])^{\frac{1}{p}} < \infty \qquad \text{for } p < \infty,$$
$$\|R\|_{\mathbb{L}^\infty(E,\mathcal{G})} := \inf\{\ell \geq 0 : |R| \leq \ell \ \mathbb{P}\text{-a.s.}\} < \infty.$$

• For $p \in [1, \infty)$, $\mathbb{H}^p(E, \mathbb{G})$ is the space of (equivalence classes of) E-valued, $\mathbb{G}$-predictable processes Z with

$$\|Z\|^p_{\mathbb{H}^p(E,\mathbb{G})} := \mathbb{E}\bigg[\bigg(\int_0^T |Z_s|^2 \,\mathrm{d}s\bigg)^{p/2}\bigg] < \infty.$$

• The space $\mathcal{L}^2(E, \mathbb{G})$ consists of all (equivalence classes of) E-valued, $\mathbb{G}$-predictable processes Z such that

$$\int_0^T |Z_s|^2 \,\mathrm{d}s < \infty \qquad \mathbb{P}\text{-a.s.}$$

• For $p \in [1, \infty]$, $\mathbb{S}^p(E, \mathbb{G})$ is the space of (equivalence classes of) E-valued, continuous, $\mathbb{G}$-adapted processes Y such that

$$\|Y\|_{\mathbb{S}^p(E,\mathbb{G})} := \Big\| \sup_{t\in[0,T]} |Y_t| \Big\|_{\mathbb{L}^p(E,\mathcal{G}_T)} < \infty.$$

When the probability measure in the definition of these norms is another $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we specify this by writing $\mathbb{L}^p(E, \mathcal{G}, \mathbb{Q})$, $\mathbb{H}^p(E, \mathbb{G}, \mathbb{Q})$ and $\mathbb{S}^p(E, \mathbb{G}, \mathbb{Q})$.

The financial market consists of n agents trading in a common riskless bond with interest rate $r = 0$ and $n \times d$ stocks. In particular, each agent trades in a d-dimensional vector of stocks S^i with price evolution following the dynamics

$$\mathrm{d}S^i_t = \mathrm{diag}(S^i_t)(\mu^i_t \mathrm{d}t + \sigma^i_t \mathrm{d}W^i_t + \sigma^{*i}_t dW^*_t), \qquad i = 1, \dots, n,$$

where we denote by $\operatorname{diag}(x)$ the $d \times d$ matrix with entries $x \in \mathbb{R}^d$ on the diagonal and 0 everywhere else. The coefficients μ^i, σ^i and σ^{*i} are predictable stochastic processes assumed to be bounded. Let $\Sigma_t^i := (\sigma_t^i, \sigma_t^{*i})$. **We assume throughout** that for all $i \in \{1, \dots, n\}$, the matrix $\Sigma^i \Sigma^{i\top}$ is uniformly elliptic, that is, $x^\top \Sigma^i \Sigma^{i\top} x \geq \varepsilon |x|^2$ $\mathbb{P}$-a.s. for some constant $\varepsilon > 0$ and all $x \in \mathbb{R}^d$. Let us introduce the process θ^i by

$$\theta_t^i := \Sigma_t^{i\top} (\Sigma_t^i \Sigma_t^{i\top})^{-1} \mu_t^i.$$

2.2 The n-agent game

A portfolio strategy is an $\mathbb{F}^n$-predictable, $\mathbb{R}^d$-valued process $(\pi_t)_{t\in[0,T]}$, with each component representing the amount invested in the corresponding stock at time t. Let $X_t^{i,\pi}$ denote the wealth of agent i at time t when starting with the initial position ξ^i and employing the trading strategy π, which we assume to be self-financing. Then $X_t^{i,\pi}$ satisfies

$$\mathrm{d}X_t^{i,\pi} = \pi_t \cdot (\Sigma_t^i \theta_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}W_t^i + \sigma_t^{*i} \mathrm{d}W_t^*), \qquad X_0^{i,\pi} = \xi^i.$$

Each agent aims at maximising their own utility from terminal wealth, and we assume the utility function to be exponential. In addition, each agent is concerned with the relative performance of their peers; see e.g. Espinosa and Touzi [15], Frei and dos Reis [17], Espinosa [14] for early works on such problems. Thus the terminal wealths are benchmarked by a *weighted average* of the other agents' terminal values, namely $\frac{1}{n-1}\sum_{j\neq i} \frac{\lambda_{ij}}{\beta_n} X_T^{j,\pi}$, where here and throughout this work, we use $\sum_{j\neq i} x^j$ as a shorthand notation for $\sum_{j\in\{1,\dots,n\}\setminus\{i\}} x^j$. The main modelling novelty considered in the present work is the addition of the term $\frac{\lambda_{ij}}{\beta_n}$ which measures agent i's sensitivity to agent j's wealth. The point is that each agent will try to perform better than the average of the other agents in the market, but they are not concerned with the performance of *all* agents. Think for instance of hedge funds which typically compete with "similar" hedge funds, for instance those raising capital from the same investors. Thus we always have $\lambda_{ij} = 1$ if agent i is concerned with agent j's performance and $\lambda_{ij} = 0$ if not. Denote for simplicity

$$\lambda_{ij}^n := \frac{1}{n-1} \frac{\lambda_{ij}}{\beta_n} \qquad \text{with } \lambda_{ii}^n := 0.$$

The utility of agent i takes the form

$$U_i\bigg(X_T^{i,\pi^i}, \sum_{j\neq i} \lambda_{ij}^n X_T^{j,\pi^j}\bigg) := -\exp\bigg(-\frac{1}{\eta^i}\Big(X_T^{i,\pi^i} - \rho \sum_{j\neq i} \lambda_{ij}^n X_T^{j,\pi^j}\Big)\bigg),$$

where $\eta^i > 0$ measures the risk aversion for agent i and ρ models the interaction weight. Since we are interested in competition, we fix $\rho \in (0, 1]$ **throughout the article**; see e.g. Hu and Zariphopoulou [21]. Let $\mathcal{A}^i$ denote the set of admissible strategies for agent i (which we define shortly). To avoid bulky notations, we use the abbreviated π^i for the rest of this section with the understanding that the strategy depends on the size n of the game. The optimisation problem for agent i thus takes the form

$$V_0^{i,n} := V_0^{i,n}\big((\pi^j)_{j\neq i}\big)$$
$$:= \sup_{\pi\in\mathcal{A}^i} \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^i}\Big(X_T^{i,\pi} - \rho\sum_{j\neq i}\lambda_{ij}^n X_T^{j,\pi^j}\Big)\bigg)\bigg]. \tag{2.1}$$

Definition 2.1 Let A^i be a closed convex subset of $\mathbb{R}^d$ that we call *constraint set*. A strategy π^i for player i is *admissible* if $\pi^i \in \mathbb{H}^2(A^i, \mathbb{F}^n)$ and for every $j \in \{1,\dots,n\}$, there is $p > 2$ such that the family

$$\big\{\mathrm{e}^{\frac{p}{\eta^i}\frac{\rho\lambda_{ij}}{\beta_n}X_\tau^{i,\pi^i}} : \tau \text{ is an } \mathbb{F}^n\text{-stopping time with values in } [0,T]\big\}$$

is uniformly integrable. In this case, we write $\pi^i \in \mathcal{A}^i$.

We are interested in Nash equilibria, whose definition we recall.

Definition 2.2 A vector $(\widetilde{\pi}^1, \widetilde{\pi}^2, \dots, \widetilde{\pi}^n) \in \mathcal{A}_1\times\mathcal{A}_2\times\cdots\times\mathcal{A}_n$ is a *Nash equilibrium* if for every $i = 1,\dots,n$, the strategy $\widetilde{\pi}^i$ is a solution to the portfolio optimisation problem given in (2.1). That is, for each i,

$$V_0^{i,n}\big((\tilde{\pi}^j)_{j\neq i}\big) = \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^i}\Big(X_T^{i,\tilde{\pi}^i} - \rho\sum_{j\neq i}\lambda_{ij}^n X_T^{j,\tilde{\pi}^j}\Big)\bigg)\bigg].$$

In this work, we assume that $(\lambda_{ij})_{1\leq i,j\leq n}$ are realisations of i.i.d. random variables which are independent of the randomness in $(W^i, W^*, \xi^i)_{i\in\{1,\dots,n\}}$. In particular, $(\lambda_{ij})_{1\leq i,j\leq n}$ is defined on a different probability space $(\mathfrak{O}, \mathfrak{F}, \mathfrak{P})$ and results are proved for almost every realisation of the graph. Therefore, we are actually working on the product space $(\Omega\times\mathfrak{O}, \mathcal{F}\otimes\mathfrak{F}, \mathbb{P}\otimes\mathfrak{P})$. We often use $\mathbb{P}$ to simplify the exposition. The interaction parameters $(\lambda_{ij})_{1\leq i,j\leq n}$ give rise to an *undirected random graph*. Notice at this point already that our setting will include Erdős–Renyi graphs and the traditional complete graph.

Let us conclude this subsection by introducing some more notation used in the paper. Given a vector $\boldsymbol{y} = (y^1,\dots,y^n)$, we define

$$\overline{y}^i := \sum_{j\neq i}\lambda_{ij}^n y^j,$$

the weighted average of the vector $\boldsymbol{y}$ (taking out y^i). Let $X_t^{\pi^i}$ be shorthand notation for X_t^{i,π^i}, and given a Nash equilibrium $(\widetilde{\pi}^1, \widetilde{\pi}^2, \dots, \widetilde{\pi}^n)$, denote by $\overline{X}_t^i := \sum_{j\neq i}\lambda_{ij}^n X_t^{\tilde{\pi}^j}$ the weighted average of the portfolio values for agents $j\neq i$ when they all use the Nash equilibrium strategy $\widetilde{\pi}^j$. These notations are used in the statement of the main results.

2.3 The graphon game

Let $I = [0, 1]$ denote the unit interval. Intuitively, in the context of an infinite-player graphon game, we label by $u \in I$ a given agent amid a continuum. The following probabilistic setup models the infinite-population game.

Let $\mathcal{B}_I$ be the Borel σ-field of I and μ_I Lebesgue measure on I. Given a probability space $(I, \mathcal{I}, \mu)$ which is an extension of the usual Lebesgue space $(I, \mathcal{B}_I, \mu_I)$, and the sample space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \mu \boxtimes \mathbb{P})$ of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \mu \boxtimes \mathbb{P})$. Unfamiliar readers can consult Sun [41] for a self-contained presentation of the theory of rich Fubini extensions. Let $C([0, T]; \mathbb{R}^d)$ denote the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$. By Sun [41], we can construct $\mathcal{I} \boxtimes \mathcal{F}$-measurable processes $(W, \xi) : I \times \Omega \to C([0, T]; \mathbb{R}^d) \times \mathbb{R}$ with *essentially pairwise independent* (e.p.i.) and identically distributed random variables $(W^u, \xi^u)_{u \in I}$ such that for each $u \in I$, the process $W^u = (W^u_t)_{0 \le t \le T}$ is a d-dimensional Brownian motion supported on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and ξ^u represents the starting wealth of agent u. Here, following [41, Definition 2.7], essentially pairwise independent means that for μ-almost all $u, v \in I$, the processes (W^u, ξ^u) and (W^v, ξ^v) are independent. Suppose that in addition to $(W^u)_{u \in I}$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports the independent one-dimensional Brownian motion W^*.

Throughout the paper, unless otherwise stated, we identify families of random variables on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \mu \boxtimes P)$ which are equal $(\mu \boxtimes P)$-almost surely.

Remark 2.3 By [41, Lemma 2.3], we have the usual Fubini property on the rich product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \mu \boxtimes \mathbb{P})$, i.e., we are free to exchange the order of integration. That is, given a measurable and integrable function f on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \mu \boxtimes \mathbb{P})$, we can write

$$\int_{I \times \Omega} f(u, \omega)(\mu \boxtimes \mathbb{P})(\mathrm{d}\omega, \mathrm{d}u) = \int_I \mathbb{E}[f(u)]\mu(\mathrm{d}u) = \mathbb{E}\bigg[\int_I f(u)\mu(\mathrm{d}u)\bigg].$$

This is used often in proofs without further mention. Moreover, we write

$$\mu(\mathrm{d}u) = \mathrm{d}u$$

to lighten the notation.

Let $\mathbb{F}^u$ denote the completion of the filtration generated by (W^u, W^*, ξ^u) and $\mathbb{F}$ the completion of the filtration generated by $((W^u)_{u \in I}, W^*, (\xi^u)_{u \in I})$. As above, we assume to be given a continuum of stocks S^u with dynamics

$$\mathrm{d}S^u_t = \mathrm{diag}(S^u_t)(\mu^u_t \mathrm{d}t + \sigma^u_t \mathrm{d}W^u_t + \sigma^{*u}_t \mathrm{d}W^*_t), \qquad u \in I,$$

so that the wealth process for agent u when employing strategy π follows the dynamics

$$\mathrm{d}X^u_t = \pi_t \cdot (\Sigma^u_t \theta^u_t \mathrm{d}t + \sigma^u_t \mathrm{d}W^u_t + \sigma^{*u}_t \mathrm{d}W^*_t), \qquad X^u_0 = \xi^u,$$

where Σ, θ, σ and σ^* are $\mathcal{B}([0,T]) \otimes \mathcal{I} \boxtimes \mathcal{F}$-measurable stochastic processes on $[0,T] \times I \times \Omega$, bounded uniformly in $u \in I$, with

$$\Sigma_t^u := (\sigma_t^u, \sigma_t^{*u}) \qquad \text{and} \qquad \theta_t^u := \Sigma_t^{u\top}(\Sigma_t^u \Sigma_t^{u\top})^{-1}\mu_t^u,$$

with $\Sigma_t^u \Sigma_t^{u\top}$ assumed to be uniformly elliptic and where for almost every $u \in I$, σ^u, σ^{*u} and μ^u are $\mathbb{F}^u$-predictable.

We finally assume that $(\sigma^u)_{u \in I}$, $(\sigma^{*u})_{u \in I}$ and $(\mu^u)_{u \in I}$ are e.p.i. and identically distributed.

Definition 2.4 A *strategy profile* is a family $(\pi^u)_{u \in I}$ of $\mathbb{F}^u$-progressive processes taking values in $\mathbb{R}^d$ and such that $(u,t,\omega) \mapsto \pi_t^u(\omega)$ is $\mathcal{I} \otimes \mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable.

Let the mapping $\eta : I \to (0,\infty)$ be $\mathcal{I}$-measurable, bounded and bounded away from zero uniformly in u. Assume that the agent u is an exponential utility maximiser with risk aversion parameter η^u and is additionally concerned with the performance of their peers. The *interaction* among the continuum of agents is modelled by a graphon, which is a symmetric and measurable function

$$G : I \times I \to I.$$

Throughout the paper, we fix a graphon G. The utility function of an agent u is exponential with risk aversion parameter η^u. Let $\mathbb{F}^* := (\mathcal{F}_t^*)_{t \in [0,T]}$ denote the $\mathbb{P}$-completion of the filtration generated by W^*.

Given $u \in I$, consider the utility maximisation problem

$$\begin{aligned} &V_0^{u,G} \\ &:= V_0^{u,G}\big((\pi^v)_{v \neq u}\big) \\ &:= \sup_{\pi^u \in \mathcal{A}^G} \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^u}\Big(X_T^{u,\pi^u} - \mathbb{E}\Big[\rho \int_I X_T^{v,\pi^v} G(u,v)\mathrm{d}v \Big| \mathcal{F}_T^*\Big]\Big)\bigg)\bigg]. \end{aligned} \tag{2.2}$$

The set $\mathcal{A}^G$ of admissible strategies in the infinite-population game is defined as follows.

Definition 2.5 Let $u \in I$ and let A^u be a closed convex subset of $\mathbb{R}^d$. A strategy profile $(\pi^u)_{u \in I}$ is *admissible* if for μ-almost every $u \in I$, it holds $\pi^u \in \mathbb{H}^2(A^u, \mathbb{F}^u)$ and $\int_I \|\pi^u\|_{\mathbb{H}^2(A^u,\mathbb{F}^u)}\mathrm{d}u < \infty$.

Taking inspiration from the theory of mean field games, see e.g. Carmona and Delarue [10, Chaps. 1 and 3] or Lasry and Lions [30], we are interested in *graphon Nash equilibria* defined as follows.

Definition 2.6 A family of admissible strategy profiles $(\widetilde{\pi}^u)_{u \in I}$ is called a *graphon Nash equilibrium* if for μ-almost every u, the strategy $\widetilde{\pi}^u$ is optimal for (2.2) with $(\pi^v)_{v \neq u}$ replaced by $(\widetilde{\pi}^v)_{v \neq u}$, that is,

$$V_0^{u,G}\big((\widetilde{\pi}^v)_{v \neq u}\big) = \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^u}\Big(X_T^{u,\widetilde{\pi}^u} - \mathbb{E}\Big[\rho \int_I X_T^{v,\widetilde{\pi}^v} G(u,v)\mathrm{d}v \Big| \mathcal{F}_T^*\Big]\Big)\bigg)\bigg].$$

2.4 Main results

Let us now present the main results of this work. These are essentially the existence of solutions to graphon games, the convergence of the finite-population game to the graphon game and a new notion of competition indifference capital.

2.4.1 Existence of solutions to the graphon game

We begin with the solvability of the graphon utility maximisation problem. Existing results on well-posedness of solutions to graphon games largely focus on linear–quadratic or static games, see e.g. Aurell et al. [2] and Carmona et al. [9]; we also refer to the more recent works by Lacker and Soret [27] and Bayraktar et al. [6] for more general settings. Moreover, the case of games with common noise has remained untouched. The existence result given here relies on general solvability of graphon BSDEs and FBSDEs discussed in the final section of the paper.

Theorem 2.7 *Assume that* $\xi^u \in L^2(\mu \boxtimes \mathbb{P})$. *Then the following hold*:
(i) *If* $A^u = \mathbb{R}^d$ *for all* $u \in I$ *and* ρ *satisfies*

$$\rho < \mathrm{e}^{-(2\|\theta\|_\infty + \frac{1}{2})T}(2\|\Sigma\|_\infty \vee \|\theta\|_\infty)^{-1},$$

then the graphon game admits a graphon Nash equilibrium.
(ii) *If* $\sigma^{*u} = 0$ *for all* $u \in I$, *then the graphon game admits a graphon Nash equilibrium.*

In the existence result in Theorem 2.7, we consider two cases. The first is the common noise case. Here, we make the simplifying assumption that the strategies are unconstrained. This is a standard assumption in the literature. We additionally require the competition parameter ρ to be sufficiently small. The second case is the case without common noise. Here, existence of a solution is obtained in full generality.

2.4.2 Convergence

The second main result states that as $n \to \infty$, the n-agent problem *converges in the strong sense* to the graphon problem, of course given some link between the sensitivity parameters $(\lambda_{ij})_{1\le i,j\le n}$ of the n-agent problem and its counterpart $G(u, v)$ in the graphon problem. Essentially, we assume below that $(\lambda_{ij})_{1\le i,j\le n}$ forms the adjacency matrix of a (random) graph converging in an appropriate sense to the graph represented by the graphon G. See for instance Lovász [33, Chap. 11] and Lovász and Szegedy [34] for extensive accounts on convergence of graphs as well as the link between graphs and graphons. Here we recall for readers the definition of the cut metric which is used in the main results.

Definition 2.8 The *cut norm* for a graphon G is defined by

$$\|G\|_\square := \sup_{E,E'\in\mathcal{B}_I} \left| \int_{E\times E'} G(u, v)\mathrm{d}u\mathrm{d}v \right|,$$

and the corresponding *cut metric* for two graphons G_1 and G_2 is defined by

$$d_\square(G_1, G_2) := \|G_1 - G_2\|_\square.$$

Although $\|\cdot\|_\square$ is not exactly a norm, we can make it one by identifying graphons which agree $(\mathrm{d}u \times \mathrm{d}u)$-almost everywhere. Let us consider the usual $\mathbb{L}^2$-norm on graphons, which is defined as

$$\|G\|_2 := \left(\int_{I\times I} |G(u,v)|^2 \,\mathrm{d}u\,\mathrm{d}v\right)^{1/2}.$$

We make the following assumptions.

Condition 2.9 1) *There is a sequence* $(\beta_n)_{n\geq 1}$ *in* $\mathbb{R}_+$ *such that* $\lim_{n\to\infty} n\beta_n^2 = \infty$.
2) *There exists a sequence* $(G_n)_{n\geq 1}$ *of graphons such that*
2a) *the graphons* G_n *are step functions, i.e., they satisfy*

$$G_n(u,v) = G_n\left(\frac{\lceil nu\rceil}{n}, \frac{\lceil nv\rceil}{n}\right) \qquad \textit{for } (u,v) \in I \times I,$$

and for every $n \in \mathbb{N}$, *it holds*

$$n\|G_n - G\|_2 \longrightarrow 0 \qquad \textit{as } n \to \infty;$$

2b) $\lambda_{ij} = \lambda_{ji} \sim \mathrm{Bernoulli}(\beta_n G_n(\frac{i}{n}, \frac{j}{n}))$ *are independent for* $1 \leq i, j \leq n$ *and independent of* $(\xi^u, \sigma^u, \theta^u, \sigma^{u*}, \eta^u)_{u\in I}$, W^* *and* $(W^u)_{u\in I}$.

The graphons G_n introduced above are called *step graphons* because they are piecewise constant. The conditions 1) and 2) are the important modelling conditions. By Lovász [33, Theorem 11.22], 2b) says that the graph on which the finite-population game is written converges (in the cut metric) to an infinite-population graph. Condition 2b) implicitly implies that $\beta_n G_n(\frac{i}{n}, \frac{j}{n}) \in [0,1]$ and means that the finite-population graph is a simple graph with weights $\{0,1\}$ depending on the outcome of a "coin toss". The parameter β_n can be seen as a density parameter on the graph, and our condition 1) allows the graph to become more and more sparse as n becomes large. In fact, we have in mind the situation $\lim_{n\to\infty} \beta_n = 0$.

Before stating the results, we start by putting the n-agent problem and the graphon problem in the same probabilistic setting.

Remark 2.10 We re-brand the d-dimensional Brownian motions $(W^i)_{i\in\{1,\dots,n\}}$ from Sect. 2.2 by $(W^{u_i})_{i\in\{1,\dots,n\}}$ so that the completion of the natural filtration generated by $(W^{u_i})_{i\in\{1,\dots,n\}}$ and W^* is now a subfiltration of $\mathbb{F}$. Consequently, all indices $i \in \mathbb{N}$ that appeared in Sect. 2.2 should be interpreted as u_i. The coefficients for the price evolution in the n-agent game, namely $(\sigma^i, \sigma^{*i}, \theta^i)_{i\in\{1,\dots,n\}}$ which are now $(\sigma^{u_i}, \sigma^{*u_i}, \theta^{u_i})_{i\in\{1,\dots,n\}}$ after this re-branding, should obey the same conditions imposed on $(\sigma^u, \sigma^{u*}, \theta^u)_{u\in I}$ as stated in Sect. 2.3. To avoid unnecessarily complicated notations, we keep the original indexing in the following sections. This re-branding comes up again in the proofs of the main convergence theorem.

The following result is the main contribution of this work. It proves convergence of the heterogeneous n-player game to the graphon game.

Theorem 2.11 *Let Condition 2.9 be satisfied, assume that* $\mathbb{E}[\mathrm{e}^{\frac{2\rho}{\eta^i\beta_n}|\xi^i|}] < \infty$ *for all* (i,n) *and that* $\xi \in L^2(\mu \boxtimes P)$. *Further assume that one of the following two conditions is satisfied*:

(i) $A^u = \mathbb{R}^d$ *for all* $u \in I$ *and* δ *satisfies* $\delta < \mathrm{e}^{-(2\|\theta\|_\infty + \frac{1}{2})T}(2\|\Sigma\|_\infty \vee \|\theta\|_\infty)^{-1}$.

(ii) $\sigma^{*u} = 0$ *for all* $u \in I$.

If the n*-agent problem* (2.1) *admits a Nash equilibrium* $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$, *then for each* i, *the control* $\widetilde{\pi}^{i,n}$ *converges to* $\tilde{\pi}^u$ *for some* u *and a graphon Nash equilibrium* $(\tilde{\pi}^u)_{u\in I}$ *in the sense that up to a subsequence, as* $n \to \infty$,

$$|\widetilde{\pi}^{i,n}_t - \widetilde{\pi}^{u_i}_t|^2 \longrightarrow 0 \qquad (\mathrm{d}t \otimes \mathbb{P})\text{-}a.e.,$$
$$\left|V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) - V_0^{u_i,G}\big((\tilde{\pi}^v)_{v\neq u_i}\big)\right| \longrightarrow 0.$$

Theorem 2.11 will follow as a consequence of a general propagation of chaos result for (quadratic) FBSDEs with non-homogeneous interaction. These propagation of chaos results seem to be first of the kind; we devote Sect. 5 to them.

Before going any further, let us present an example where Theorem 2.11 becomes easy in that propagation of chaos is not needed, at least granted our characterisation results to come in Remark 3.3 and Corollary 3.5 below. This example deals with the case of a market with constant coefficients and further motivates the analysis of the random coefficients case done in this paper.

Proposition 2.12 *Assume for all* $u \in I$ *that* $A^u = \mathbb{R}^d$, $\sigma^{*u} = 0$ *and* σ^u, μ^u *are deterministic measurable functions of time. Consider a slight modification of the utility maximisation problem* (2.1) *where* $\lambda_{ii} \neq 0$, *i.e., agent* i *takes into account a weighted average of all agents' final wealths as their benchmark. Under this modification, the utility maximisation problem for agent* i *now reads*

$$\begin{aligned} V_0^{i,n} &:= V_0^{i,n}\big((\pi^j)_{j\neq i}\big) \\ &:= \sup_{\pi\in\mathbb{R}^d} \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^i}\Big(X_T^{i,\pi} - \frac{\rho}{n\beta_n}\sum_{j=1}^n \lambda_{ij} X_T^{j,\pi^j}\Big)\bigg)\bigg]. \end{aligned}$$

Then for all $n \in \mathbb{N}$, *there is a Nash equilibrium* $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$ *given by*

$$\sigma_t^i \widetilde{\pi}_t^{i,n} = \frac{n\beta_n}{n\beta_n - \rho\lambda_{ii}}\eta_t^i\theta_t^i \qquad \textit{for all } (n,i) \in \mathbb{N}\times\{1,\dots,n\} \textit{ and a.a. } t.$$

Furthermore, there is a graphon Nash equilibrium $(\widetilde{\pi}^u)_{u\in I}$ *given by*

$$\sigma_t^u \widetilde{\pi}_t^u = \eta^u \theta_t^u \qquad \textit{for a.e. } (u,t) \in I \times [0,T].$$

In particular, $\tilde{\pi}^{i,n}$ and $\tilde{\pi}^{u}$ are deterministic and we have

$$|\sigma_t^i \widetilde{\pi}_t^{i,n} - \sigma_t^{u^i} \widetilde{\pi}_t^{u_i}| \leq \frac{\rho\lambda_{ii}}{n\beta_n - \rho\lambda_{ii}} \|\eta^i\theta^i\|_\infty \qquad \textit{for all } (n,i) \in \mathbb{N} \times \{1,\dots,n\} \text{ and a.a. } t. \tag{2.3}$$

In addition to providing an easy way to prove convergence results, Proposition 2.12 is interesting in that it shows that in the present heterogeneous game, when the coefficients are constant, the Nash equilibria (both in the finite and the graphon games) are constant as well, at least up to the randomness of the graph. This is in line with the homogeneous case studied by Lacker and Zariphopoulou [28] using PDE techniques and by Espinosa and Touzi [15] using BSDE techniques.

2.4.3 Competition indifference capital

To conclude this section on the presentation of our main result, we use the rich literature on exponential utility maximisation to assess the effect of competition on an individual investor. As said repeatedly, our results build on characterisations of the Nash and graphon equilibria by systems of (F)BSDEs. However, in order to numerically simulate equilibria, one still needs to simulate the solutions (notably the control process) of a high-dimensional system of (F)BSDEs or of McKean–Vlasov type equations. As is well known in the numerical simulation literature, efficient simulation of the control process is much harder than that of the value process. One might then wonder whether appropriately choosing the initial capital could make the investor indifferent between being concerned with the relative performance of their peers or not. That is, denoting

$$J^{i,n}(\xi^i, F) := \sup_{\pi\in\mathcal{A}^i} \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^i}(X_T^{i,\pi} - F)\bigg)\bigg], \qquad \text{where } X_0^{i,\pi} = \xi^i,$$

we should like to compute $p^{i,n}$ such that

$$J^{i,n}(\xi^i - p^{i,n}, 0) = J^{i,n}\bigg(\xi^i, \rho\sum_{j\neq i}^{n} \lambda_{ij}^n X_T^{j,\widetilde{\pi}^{j,n}}\bigg), \tag{2.4}$$

where $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$ is a Nash equilibrium. (In the definition of J when $F = 0$, we take the elements of $\mathcal{A}^i$ to be $\mathbb{F}^i$-progressive instead of $\mathbb{F}^n$-progressive since in this case, the agent is not concerned with the performance (and thus investments) of other market participants.) This is precisely the (spirit of the) utility indifference pricing of Hodges and Neuberger [20]. In the infinite-population game, this indifference capital takes the form

$$J^u(\xi^u - p^u, 0) = J^u\bigg(\xi^u, \rho\mathbb{E}\bigg[\int_I X_T^{v,\widetilde{\pi}^v} G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\bigg)$$

with

$$J^u(\xi^u, F) := \sup_{\pi\in\mathcal{A}^u} \mathbb{E}\bigg[-\exp\bigg(-\frac{1}{\eta^u}(X_T^{u,\pi} - F)\bigg)\bigg], \qquad \text{where } X_0^{u,\pi} = \xi^u.$$

We thus have the following corollary. In the statement below and **throughout the paper**, we denote by $P^i_t(\zeta)$ the projection of a vector ζ onto the constraint set $\Sigma^i_t A^i$.

Corollary 2.13 *If* $\mathbb{E}[\mathrm{e}^{\frac{2\rho}{\eta^i\beta_n}|\xi^i|}] < \infty$, *the competition indifference capital is given by*

$$p^{i,n} = \eta^i \log \frac{\gamma_0^{i,n}}{\gamma_0},$$

where $\gamma^{i,n}$ *is the value process of the FBSDE system* (3.2) *below and* (γ, ζ, ζ^*) *solves the BSDE*

$$\gamma_t = \int_t^T \Big(- \binom{\zeta_s}{\zeta^*_s} \cdot \theta^i_s - \frac{\eta^i}{2}|\theta^i_s|^2 + \frac{1}{2\eta^i}\Big|(\mathrm{Id} - P^i_s)\Big(\binom{\zeta_s}{\zeta^*_s} + \eta^i\theta^i_s\Big)\Big|^2\Big)\mathrm{d}s \\ - \int_t^T \zeta_s \cdot \mathrm{d}W^i_s - \int_t^T \zeta^*_s \mathrm{d}W^*_s.$$

Moreover, if the conditions of Theorem 2.11 *are satisfied, then we have*

$$|p^{i,n} - p^{u_i}| \longrightarrow 0 \qquad \text{as } n \to \infty,$$

where p^u *is the competition indifference capital of player* u *in the graphon game.*

The gist here is that $p^{i,n}$ is given in terms of the value process of a system of BSDEs, so that an investor starting with capital $\xi^i - p^{i,n}$ (only) needs to simulate the control process of a one-dimensional BSDE in order to compute the optimal trading strategy.

Proof of Corollary 2.13 The proof starts with the general duality result of Delbaen et al. [11] which asserts that

$$\sup_{\pi\in\mathcal{A}^i} \mathbb{E}\Big[- \exp\Big(- \frac{1}{\eta^i}\Big(X^{i,\pi}_T - \rho\sum_{j\neq i}^n \lambda^n_{ij} X^{j,\tilde{\pi}^{j,n}}_T\Big)\Big)\Big] \\ = -\exp\Big(\frac{1}{\eta^i}\sup_{\mathbb{Q}\in\mathcal{Q}}\Big(\mathbb{E}_{\mathbb{Q}}\Big[\rho\sum_{j\neq i}^n \lambda^n_{ij}X^{j,\tilde{\pi}^{j,n}}_T\Big] - \xi^i - \eta^i H(\mathbb{Q}|\mathbb{P})\Big)\Big),$$

where $H(\mathbb{Q}|\mathbb{P})$ is the relative entropy given by

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{Q}}[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}] & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty & \text{else,} \end{cases}$$

and $\mathcal{Q}$ is the set of probability measures $\mathbb{Q}$ that are absolutely continuous with respect to $\mathbb{P}$ and such that the stock price processes are local $\mathbb{Q}$-martingales and

$H(\mathbb{Q}|\mathbb{P}) < \infty$. Applying this result to both sides of (2.4) yields

$$
\begin{aligned}
p^{i,n} &= \sup_{\mathbb{Q}\in\mathcal{Q}} \bigg(\mathbb{E}_{\mathbb{Q}}\Big[\rho \sum_{j\neq i}^{n} \lambda_{ij}^{n} X_T^{j,\widetilde{\pi}^{j,n}} \Big] - \xi^i - \eta^i H(\mathbb{Q}|\mathbb{P}) \bigg) - \sup_{\mathbb{Q}\in\mathcal{Q}} \big(-\xi^i - \eta^i H(\mathbb{Q}|\mathbb{P}) \big) \\
&= \eta^i \log \bigg(- \sup_{\pi\in\mathcal{A}^i} \mathbb{E}\Big[-\exp\Big(-\frac{1}{\eta^i}\Big(X_T^{i,\pi} - \rho \sum_{j\neq i}^{n} \lambda_{ij}^{n} X_T^{j,\widetilde{\pi}^{j,n}} \Big)\Big)\Big]\bigg) \\
&\quad - \eta^i \log \bigg(- \sup_{\pi\in\mathcal{A}^i} \mathbb{E}\Big[-\exp\Big(-\frac{1}{\eta^i} X_T^{i,\pi} \Big)\Big]\bigg) \\
&= \eta^i \log \frac{Y_0^{i,n}}{Y_0},
\end{aligned}
$$

where the last equality follows by Theorem 3.1 and Hu et al. [22, Theorem 7]. The above argument also shows that $p^u = \eta^u \log(Y_0^u / Y_0')$, where Y^u satisfies (3.6) and $(Y', \zeta', \zeta^{*\prime})$ solves

$$
\begin{aligned}
Y_t' = \int_t^T \bigg(-\begin{pmatrix} \zeta_s' \\ \zeta_s^{*\prime} \end{pmatrix} \cdot \theta_s^u - \frac{\eta^u}{2} |\theta_s^u|^2 + \frac{1}{2\eta^u} \bigg| (\mathrm{Id} - P_s^u)\bigg(\begin{pmatrix} \zeta_s' \\ \zeta_s^{*\prime} \end{pmatrix} + \eta^u \theta_s^u \bigg) \bigg|^2 \bigg) \mathrm{d}s \\
- \int_t^T \zeta_s' \cdot \mathrm{d}W_s^u - \int_t^T \zeta_s^{*\prime} \mathrm{d}W_s^*.
\end{aligned}
$$

The convergence statement therefore follows from Theorem 2.11. □

The rest of the paper is dedicated to the proofs of the convergence and existence results.

3 Characterisations of solutions to the utility maximisation games

This section provides characterisations of the Nash equilibria of the two games presented above in terms of solutions of backward SDEs. These characterisations will play a key role in the proofs of our main results.

3.1 FBSDE characterisation of the n-agent problem

The following result provides an FBSDE characterisation for the n-agent utility maximisation problem (2.1). In particular, it expresses the Nash equilibrium and the associated utilities as functions of solutions to a system of (quadratic) FBSDEs. This extends the main result of Espinosa and Touzi [15] to the case where both common noise and idiosyncratic noise are considered. Recall the notation $\overline{X}_t^i := \sum_{j\neq i} \lambda_{ij}^n X_t^{j,\tilde{\pi}^j}$.

Theorem 3.1 *Assume that* $\mathbb{E}[\mathrm{e}^{\frac{2\rho}{\eta^i\beta_n}|\xi^i|}] < \infty$. *If the n-player game admits a Nash equilibrium* $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$, *then it holds*

$$\widetilde{\pi}_t^{i,n} = (\Sigma_t^i \Sigma_t^{i\top})^{-1}\Sigma_t^i P_t^i\left(\begin{pmatrix}\zeta_t^{ii}\\ \zeta_t^{*i}\end{pmatrix} + \eta^i\theta_t^i\right) \qquad (\mathrm{d}t\otimes\mathbb{P})\text{-a.e.},$$

$$V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) = -\mathrm{e}^{-\frac{1}{\eta^i}(\xi^i-\rho\overline{\xi}^i-\gamma_0^i)} \qquad \text{for } i\in\{1,\dots,n\}, \tag{3.1}$$

with $(X^i,\gamma^i,\zeta^{ij},\zeta^{*i}) \in \mathbb{S}^1(\mathbb{R},\mathbb{F}^n)\times\mathbb{S}^1(\mathbb{R},\mathbb{F}^n)\times\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{F}^n)\times\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{F}^n)$ *for* $(i,j)\in\{1,\dots,n\}^2$ *solving the FBSDE*

$$\begin{cases}\mathrm{d}\gamma_t^i = \Bigg(\begin{pmatrix}\zeta_t^{ii}\\ \zeta_t^{*i}\end{pmatrix}\cdot\theta_t^i + \dfrac{\eta^i}{2}|\theta_t^i|^2 - \dfrac{1}{2\eta^i}\displaystyle\sum_{j\neq i}^n|\zeta_t^{ij}|^2 \\ \qquad\qquad - \dfrac{1}{2\eta^i}\left|(\mathrm{Id}-P_t^i)\left(\begin{pmatrix}\zeta_t^{ii}\\ \zeta_t^{*i}\end{pmatrix}+\eta^i\theta_t^i\right)\right|^2\Bigg)\mathrm{d}t \\ \qquad\quad + \displaystyle\sum_{j=1}^n \zeta_t^{ij}\cdot\mathrm{d}W_t^j + \zeta_t^{*i}\mathrm{d}W_t^*, \qquad \gamma_T^i = \rho(\bar{X}_T^i - \bar{\xi}^i), \\ \mathrm{d}X_t^i = \widetilde{\pi}_t^{i,n}\cdot(\Sigma_t^i\theta_t^i\mathrm{d}t + \sigma_t^i\mathrm{d}W_t^i + \sigma_t^{*i}\mathrm{d}W_t^*), \qquad X_0^i = \xi^i.\end{cases} \tag{3.2}$$

The reader might wonder why our characterising equation is a multidimensional coupled FBSDE in contrast to the BSDEs usually derived in the literature; see for instance Espinosa and Touzi [15], Frei and dos Reis [17]. We can achieve a characterisation by a BSDE by "shifting" the value process γ^i and through introducing a function $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ allowing to decouple the FBSDEs (3.2) into the BSDE (3.4) given in the next corollary.

Corollary 3.2 *Assume that*

$$\sum_{j\neq i}\lambda_{ij}^n \in [0,1]. \tag{3.3}$$

If the n-player game admits a Nash equilibrium $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$, *then we have*

$$\Sigma_t^{i\top}\widetilde{\pi}_t^{i,n} = P_t^i\left(\begin{pmatrix}Z_t^{ii}\\ \psi_t^i(Z_t^*)\end{pmatrix}+\eta^i\theta_t^i\right) =: f_t^i\big(Z_t^{ii},\psi^i(Z_t^*)\big),$$

$$V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) = -e^{-\frac{1}{\eta^i}(\xi^i-\rho\overline{\xi}^i-Y_0^i)},$$

with $(Y^i, Z^{ij}, Z^{*i}) \in \mathbb{S}^1(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{F}^n) \times \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{F}^n)$ *for* $(i, j) \in \{1, \ldots, n\}^2$ *solving the* n*-dimensional BSDEs*

$$\begin{aligned} Y^i_t = \int_t^T \Bigg(&- \frac{\eta^i}{2}|\theta^i_s|^2 - \begin{pmatrix} Z^{ii}_s \\ \psi^i_s(Z^*_s) \end{pmatrix} \cdot \theta^i_s + \frac{1}{2\eta^i}\left|(\mathrm{Id} - P^i_s)\left(\begin{pmatrix} Z^{ii}_s \\ \psi^i_s(Z^*_s) \end{pmatrix} + \eta^i \theta^i_s\right)\right|^2 \\ &+ \frac{1}{2\eta^i}\sum_{j \neq i}^n \left| Z^{ij}_s + \rho\lambda^n_{ij}\sigma^j_s f^j_s\left(Z^{jj}_s, \psi^j_s(Z^*_s)\right)\right|^2 \\ &+ \sum_{j \neq i}^n \rho\lambda^n_{ij}\Big(f^j_s\left(Z^{jj}_s, \psi^j_s(Z^*_s)\right) \cdot \Sigma^j_s \theta^j_s\Big)\Bigg)\mathrm{d}s \\ - \sum_{j=1}^n \int_t^T & Z^{ij}_s \cdot \mathrm{d}W^j_s - \int_t^T Z^{*i}_s \mathrm{d}W^*_s, \qquad t \in [0, T], \end{aligned} \tag{3.4}$$

where for every fixed $t \in [0, T]$, $\psi_t(\zeta^*) = \psi_t(\zeta, \zeta^*)$, *where* $\psi_t(\zeta, \cdot)$ *is the inverse of the mapping* $\phi_t(\zeta, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$\phi^i_t(\zeta, \zeta^*) = \zeta^{i,*} - \sum_{j \neq i}^n \rho\lambda^n_{ij}\sigma^{j*}_t \cdot (\Sigma^j_t {\Sigma^j_t}^\top)^{-1}\Sigma^j_t P^j_t\left(\begin{pmatrix} \zeta^{jj} \\ \zeta^{j*} \end{pmatrix} + \eta^j \theta^j_t\right)$$

for all $(\zeta, \zeta^*) \in \mathbb{R}^{nd} \times \mathbb{R}^n$. *Here with abuse of notation,* $\phi^i_t(\zeta, \cdot)$ *maps from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *up to fixing a single trajectory of* $(\Sigma^j_t)_{j \in \{1,\ldots,n\}}$ *and* $(\theta^j_t)_{j \in \{1,\ldots,n\}}$. *Furthermore, for* $n \geq 3$, ψ_t *is Lipschitz-continuous in* ζ^* *with a constant depending on* n.

Observe that the dimension of the domain of the function ψ depends on n. Thus ψ_t will undoubtedly present a major obstacle when studying the limit of the game as $n \to \infty$. For instance, in the infinite-population game, this decoupling procedure does not seem to work. Furthermore, the condition (3.3) also presents an obstacle to the fact that we should like to consider the limit of the game on a relatively sparse graph. To avoid these difficulties while studying the limit, we rather work with the FBSDE (3.2).

Remark 3.3 In the absence of the common noise W^* (i.e., when $\sigma^{*u} = 0$ for all $u \in I$), the complications associated with ψ_t discussed above vanish. In fact, the system of BSDEs in Corollary 3.2 then takes the much simpler form

$$
\begin{aligned}
Y_t^i = \int_t^T \Big(&- \frac{\eta^i}{2}|\theta_s^i|^2 - Z_s^{ii}\cdot\theta_s^i + \frac{1}{2\eta^i}|(\mathrm{Id}-P_s^i)(Z_s^{ii}+\eta^i\theta_s^i)|^2 \\
&+ \frac{1}{2\eta^i}\sum_{j\neq i}^n |Z_s^{ij} + \rho\lambda_{ij}^n P_s^j(Z_s^{jj}+\eta^j\theta_s^j)|^2 \\
&+ \sum_{j\neq i}^n \rho\lambda_{ij}^n P_s^j(Z_s^{jj}+\eta^j\theta_s^j)\cdot\theta_s^j \Big)\mathrm{d}s \\
-\sum_{j=1}^n \int_t^T & Z_s^{ij}\cdot \mathrm{d}W_s^j, \qquad t\in[0,T],
\end{aligned} \tag{3.5}
$$

and the equilibrium strategy now takes the form

$$
\widetilde{\pi}_t^{i,n} = (\sigma_t^i)^{-1} P_t^i (Z_t^{ii} + \eta^i\theta_t^i) \qquad (\mathrm{d}t\otimes\mathbb{P})\text{-a.e.}
$$

3.2 FBSDE characterisation of solutions to the graphon problem

Similarly to the n-player game just discussed, we also derive (F)BSDE characterisations of solutions to the graphon game. This time, the characterisation obtained is with respect to a system of (infinitely many) McKean–Vlasov (F)BSDEs. We call these equations graphon (F)BSDEs to stress the fact that the dependence between the equations occurs through the graphon G. As above, we use the **notation** $P_t^u(\zeta)$ for the projection of a vector ζ onto the constrain set $\Sigma_t^u A^u$.

Proposition 3.4 *Assume that* $\xi\in L^2(\mu\boxtimes\mathbb{P})$ *and that a solution*

$$
(X^u, Y^u, Z^u, Z^{*u}) \in \mathbb{S}^2(\mathbb{R},\mathbb{F}^u)\times\mathbb{S}^2(\mathbb{R},\mathbb{F}^u)\times\mathbb{H}^2(\mathbb{R}^d,\mathbb{F}^u)\times\mathbb{H}^2(\mathbb{R},\mathbb{F}^u)
$$

such that $(u,t,\omega)\mapsto X_t^u(\omega)$ *is measurable exists for the graphon FBSDE*

$$
\begin{cases}
\mathrm{d}X_t^u = \tilde{\pi}_t^u\cdot(\Sigma_t^u\theta_t^u\mathrm{d}t + \sigma_t^u\mathrm{d}W_t^u + \sigma_t^{*u}\mathrm{d}W_t^*), \qquad X_0^u = \xi^u, \\
\widetilde{\pi}_t^u = (\Sigma_t^u\Sigma_t^{u\top})^{-1}\Sigma_t^u P_t^u\Big(\begin{pmatrix} Z_t^u \\ Z_t^{*u}\end{pmatrix} + \eta^u\theta_t^u\Big) \qquad (\mathrm{d}t\otimes\mu\boxtimes\mathbb{P})\text{-}a.e., \\
\mathrm{d}Y_t^u = \Big(\begin{pmatrix} Z_t^u \\ Z_t^{*u}\end{pmatrix}\cdot\theta_t^u + \frac{\eta^u}{2}|\theta_t^u|^2 - \frac{1}{2\eta^u}\Big|(\mathrm{Id}-P_t^u)\Big(\begin{pmatrix} Z_t^u \\ Z_t^{*u}\end{pmatrix} + \eta^u\theta_t^u\Big)\Big|^2\Big)\mathrm{d}t \\
\qquad + Z_t^u\cdot\mathrm{d}W_t^u + Z_t^{*u}\mathrm{d}W_t^*, \\
Y_T^u = \mathbb{E}\Big[\int_I \rho(X_T^v - \xi^v)G(u,v)\mathrm{d}v\Big|\mathcal{F}_T^*\Big].
\end{cases} \tag{3.6}
$$

Then the graphon game in (2.2) admits a graphon Nash equilibrium $(\tilde{\pi}^u)_{u\in I}$ *such that for almost every* $u\in I$, *we have*

$$
V_0^{u,G} = -\exp\Big(-\frac{1}{\eta^u}\Big(\xi^u - \int_I \rho\mathbb{E}[\xi^v]G(u,v)\mathrm{d}v - Y_0^u\Big)\Big) \tag{3.7}
$$

and

$$\widetilde{\pi}_t^u = (\Sigma_t^u \Sigma_t^{u\top})^{-1}\Sigma_t^u P_t^u \left(\begin{pmatrix} Z_t^u \\ Z_t^{*u} \end{pmatrix} + \eta^u \theta_t^u \right) \qquad (\mathrm{d}t \otimes \mu \boxtimes \mathbb{P})\text{-}a.e. \tag{3.8}$$

The above result characterises the solutions to the graphon game with common noise in the sense that solvability of the game reduces to solvability of the system (3.6). Moreover, the value function as well as the equilibrium strategies in the infinite-population game are given explicitly in terms of solutions to (3.6). In the case where there is no common noise, i.e., $\sigma^{*u} = 0$ for almost all $u \in I$, the above result simplifies as follows.

Corollary 3.5 *Assume that the graphon BSDE*

$$\begin{aligned} \mathrm{d}Y_t^u &= \bigg(\frac{\eta^u}{2}|\theta_t^u|^2 + Z_t^u \cdot \theta_t^u - \frac{1}{2\eta^u}|(\mathrm{Id} - P_t^u)(Z_t^u + \eta^u\theta_t^u)|^2 \\ &\qquad - \mathbb{E}\bigg[\int_I \rho P_t^v (Z_t^v + \eta^v \theta_t^v) \cdot \theta_t^v G(u,v)\mathrm{d}v \bigg] \bigg) \mathrm{d}t + Z_t^u \cdot \mathrm{d}W_t^u, \\ &\qquad t \in [0,T], \\ Y_T^u &= 0 \end{aligned} \tag{3.9}$$

admits a solution $(Y^u, Z^u)_{u\in I}$ such that $(u,t,\omega) \mapsto Z_t^u(\omega)$ is measurable and we have $(Y^u, Z^u) \in \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u)$ for almost every $u \in I$. Then the graphon game described in (2.2) admits a graphon Nash equilibrium $(\tilde{\pi}^u)_{u\in I}$ such that for almost every $u \in I$, we have

$$\widetilde{\pi}_t^u = (\sigma_t^u)^{-1} P_t^u (Z_t^u + \eta^u \theta_t^u) \qquad (\mathrm{d}t \otimes \mu \boxtimes \mathbb{P})\text{-}a.e., \tag{3.10}$$

$$V_0^{u,G} = -\exp\bigg(-\frac{1}{\eta^u}\Big(\xi^u - \int_I \mathbb{E}[\rho \xi^v] G(u,v)\mathrm{d}v - Y_0^u \Big) \bigg). \tag{3.11}$$

4 Proofs of existence and characterisation results

The proof of Theorem 2.11 is based on general propagation of chaos results that are given in Sect. 5. The existence result in Theorem 2.7 is a consequence of existence for solutions to graphon BSDEs discussed in Sect. 6, where we present existence results for graphon (F)BSDEs.

4.1 Proof of the existence result in Theorem 2.7

We distinguish two cases: the case with common noise and the case without.

(i) Case with common noise: In this case, when $A^u = \mathbb{R}^d$ for all u, the FBSDE (3.6) becomes

$$\begin{cases} \mathrm{d}X_t^u &= b^u(t, Z_t^u, Z_t^{*u})\mathrm{d}t + h_1^u(t, Z_t^u, Z_t^{*u})\mathrm{d}W_t^u + h_2^u(t, Z_t^u, Z_t^{*u})\mathrm{d}W_t^*, \\ \mathrm{d}Y_t^u &= -g^u(t, Z_t^u, Z_t^{*u})\mathrm{d}t + Z_t^u \cdot \mathrm{d}W_t^u + Z_t^{*u}\mathrm{d}W_t^*, \\ X_0^u &= \xi^u, \qquad Y_T^u = \mathbb{E}\left[\int_I \rho(X_T^v - \xi^v)G(u,v)\mathrm{d}v \middle| \mathcal{F}_T^*\right], \end{cases}$$

with

$$b^u(t,z,z^*) = (\Sigma_t^u \Sigma_t^{u\top})^{-1}\Sigma_t^u\left(\begin{pmatrix} z \\ z^* \end{pmatrix} + \eta^u\theta_t^u\right)\Sigma_t^u\theta_t^u,$$

$$h_1^u(t,z,z^*) = (\Sigma_t^u \Sigma_t^{u\top})^{-1}\Sigma_t^u\left(\begin{pmatrix} z \\ z^* \end{pmatrix} + \eta^u\theta_t^u\right)\cdot\sigma_t^u,$$

$$h_2^u(t,z,z^*) = (\Sigma_t^u \Sigma_t^{u\top})^{-1}\Sigma_t^u\left(\begin{pmatrix} z \\ z^* \end{pmatrix} + \eta^u\theta_t^u\right)\cdot\sigma_t^{*u},$$

$$g^u(t,z,z^*) = \begin{pmatrix} z \\ z^* \end{pmatrix}\cdot\theta_t^u + \frac{\eta^u}{2}|\theta_t^u|^2.$$

In particular, because the processes Σ^u, μ^u are bounded, the coefficients of this equation satisfy the conditions of Proposition 6.1. Thus it follows that (3.6) admits a unique square-integrable solution. Therefore the result follows from Proposition 3.4.

(ii) Case without common noise: When $\sigma^* = 0$, the proof is similar. In fact, it follows by Proposition 6.2 that the graphon BSDE (3.9) admits a unique solution such that $(Y^u, Z^u) \in \mathbb{S}^\infty(\mathbb{F}^u, \mathbb{R}^d) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}^u, \mathbb{R}^d)$ for almost every $u \in I$ with $(u,t,\omega) \mapsto Z_t^u(\omega)$ measurable and $\sup_u \|Z^u\|_{\mathbb{H}^2(\mathbb{R}^d,\mathbb{F}^u)} < \infty$. Then the result follows by Corollary 3.5. $\square$

4.2 Proof of Proposition 2.12

Under the given assumptions which include deterministic coefficients, the system of FBSDEs (3.2) characterising the n-agent optimisation problem simplifies to

$$\begin{cases} \mathrm{d}\gamma_t^i &= \left(\zeta_t^{ii}\cdot\theta_t^i + \frac{\eta^i}{2}|\theta_t^i|^2 - \frac{1}{2\eta^i}\sum_{j\neq i}^n |\zeta_t^{ij}|^2\right)\mathrm{d}t \\ &\quad + \sum_{j=1}^n \zeta_t^{ij}\cdot\mathrm{d}W_t^j, \qquad t\in[0,T], \\ \gamma_T^i &= \rho(\bar{X}_T^i - \bar{\xi}^i) = \rho\sum_{j=1}^n \lambda_{ij}^n \int_0^T \widetilde{\pi}_t^{j,n}\cdot\sigma_t^j(\theta_t^j\mathrm{d}t + \mathrm{d}W_t^j), \\ \mathrm{d}X_t^i &= \widetilde{\pi}_t^{i,n}\sigma_t^i(\theta_t^i\mathrm{d}t + \mathrm{d}W_t^i), \qquad X_0^i = \xi^i, \end{cases} \tag{4.1}$$

with the equilibrium strategies given by

$$\sigma_t^i\widetilde{\pi}_t^{i,n} = \zeta_t^{ii} + \eta^i\theta_t^i \qquad (\mathrm{d}t\otimes\mathbb{P})\text{-a.e.}$$

Let $Y_t^i = \gamma_t^i - \rho \sum_{j=1}^n \lambda_{ij}^n \int_0^t \widetilde{\pi}_s^{j,n} \sigma_s^j (\theta_s^j \mathrm{d}s + \mathrm{d}W_s^j)$. Recall that $\lambda_{ij}^n = \lambda_{ij}/n\beta_n$. Then we have $Y_T^i = 0$ and we can rewrite the FBSDE (4.1) as

$$
\begin{aligned}
Y_t^i = &\int_t^T \Big(- \zeta_s^{ii} \cdot \theta_s^i - \frac{\eta^i}{2} |\theta_s^i|^2 + \frac{1}{2\eta^i} \sum_{j \neq i}^n |\zeta_s^{ij}|^2 + \rho \sum_{j=1}^n \lambda_{ij}^n (\zeta_s^{jj} + \eta^j \theta_s^j) \theta_s^j \Big) \mathrm{d}s \\
&- \sum_{j=1}^n \int_t^T \big(\zeta_s^{ij} - \rho \lambda_{ij}^n (\zeta_s^{jj} + \eta^j \theta_s^j) \big) \mathrm{d}W_s^j .
\end{aligned}
$$

Observe that choosing

$$
\zeta_t^{ii} = \frac{\rho \lambda_{ii}^n}{1 - \rho \lambda_{ii}^n} \eta^i \theta_t^i, \qquad \zeta_t^{ij} = \frac{\rho \lambda_{ij}^n}{1 - \rho \lambda_{jj}^n} \eta^j \theta_t^j
$$

makes the stochastic integral in the above BSDE vanish, leaving Y^i a deterministic process. Thus

$$
Y_t^i = \int_t^T \Big(- \zeta_s^{ii} \cdot \theta_s^i - \frac{\eta^i}{2} |\theta_s^i|^2 + \frac{1}{2\eta^i} \sum_{j \neq i}^n |\zeta_s^{ij}|^2 + \sum_{j=1}^n \lambda_{ij}^n (\zeta_s^{jj} + \eta_j \theta_s^j) \theta_s^j \Big) \mathrm{d}s
$$

with

$$
\zeta_t^{ii} = \frac{\rho \lambda_{ii}^n}{1 - \rho \lambda_{ii}^n} \eta^i \theta_t^i, \qquad \zeta_t^{ij} = \frac{\rho \lambda_{ij}^n}{1 - \rho \lambda_{jj}^n} \eta^j \theta_t^j \quad \text{for } i \neq j
$$

is a (deterministic) solution to the above BSDE.

Similarly, the BSDE (3.9) characterising the solution to the graphon game simplifies to

$$
\begin{aligned}
Y_t^u = &\int_t^T \Big(- \frac{\eta^u}{2} |\theta_s^u|^2 - Z_s^u \cdot \theta_s^u + \mathbb{E}\Big[\rho \int_I (Z_s^v + \eta^v \theta_s^v) \cdot \theta_s^v G(u, v) \mathrm{d}v \Big] \Big) \mathrm{d}s \\
&- \int_t^T Z_s^u \cdot \mathrm{d}W_s^u ,
\end{aligned} \tag{4.2}
$$

with the equilibrium strategy given by

$$
\sigma_t^u \widetilde{\pi}_t^u = Z_t^u + \eta^u \theta_t^u .
$$

Using a change of measure argument, we can rewrite (4.2) as

$$
\begin{aligned}
Y_t^u &= \int_t^T \left(-\frac{\eta^u}{2}|\theta_s^u|^2 + \mathbb{E}\Big[\rho \int_I (Z_s^v + \eta_v \theta_s^v)\cdot \theta_s^v G(u,v)\mathrm{d}v\Big]\right)\mathrm{d}s \\
&\quad - \int_t^T Z_s^u \cdot (\mathrm{d}W_s^u + \theta_s^u \mathrm{d}s) \\
&= \int_t^T \left(-\frac{\eta^u}{2}|\theta_s^u|^2 + \mathbb{E}\Big[\rho \int_I (Z_s^v + \eta^v \theta_s^v)\cdot \theta_s^v G(u,v)\mathrm{d}v\Big]\right)\mathrm{d}s \\
&\quad - \int_t^T Z_s^u \cdot \mathrm{d}W_s^{u,\mathbb{Q}},
\end{aligned}
$$

where $W^{u,\mathbb{Q}}$ is a standard Brownian motion under a new probability measure $\mathbb{Q}$ such that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathrm{e}^{\int_t^T -\theta_s^u\cdot \mathrm{d}W_s^u - \frac{1}{2}\int_t^T |\theta_s^u|^2 \mathrm{d}u}$. Noting that $Z^u \in \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u, \mathbb{P})$, taking $\mathbb{Q}$-conditional expectations with respect to $\mathcal{F}_t^u$ on both sides, we can now conclude that

$$
\begin{aligned}
Y_t^u &= \int_t^T \left(-\frac{\eta^u}{2}|\theta_s^u|^2 + \mathbb{E}\Big[\rho \int_I \eta^v |\theta_s^v|^2 G(u,v)\mathrm{d}v\Big]\right)\mathrm{d}s, \\
Z_t^u &= 0, \qquad \sigma_t^u \widetilde{\pi}_t^u = \eta^u \theta_t^u,
\end{aligned}
$$

is a (deterministic) solution to the BSDE (4.2). The convergence results (2.3) thus follow from the boundedness of η and θ_t. $\square$

4.3 Proofs for Sect. 3.1

We now present the proof of the characterisation result for the n-player game. This section consists of the proof for Theorem 3.1 and two auxiliary results in Lemmas 4.1 and 4.2.

Proof of Theorem 3.1 Assume $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$ is a Nash equilibrium of problem (2.1). First note that our assumptions on σ_t^i, σ_t^{*i} and μ_t^i imply that $\overline{X}_t^i \in \mathbb{L}^2(\mathbb{R}, \mathcal{F}_t^n)$. Let $\mathcal{T}$ be the set of all $\mathbb{F}^n$-stopping times valued in $[0, T]$. Define the family of random variables

$$
\mathcal{J}^{i,\pi}(\tau) := \mathbb{E}\big[-\mathrm{e}^{-\frac{1}{\eta^i}(\int_\tau^T \pi_s\cdot(\Sigma_s^i\theta_s^i \mathrm{d}s + \sigma_s^i \mathrm{d}W_s^i + \sigma_s^{*i}\mathrm{d}W_s^*) - \rho(\overline{X}_T^i - \overline{\xi}^i))} \big| \mathcal{F}_\tau^n\big],
$$

and let

$$
\mathcal{V}^i(\tau) := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^i} \mathcal{J}^{i,\pi}(\tau) \qquad \text{for all } \tau \in \mathcal{T}
$$

so that

$$
\mathcal{V}^i(0) = \mathrm{e}^{\frac{1}{\eta^i}(\xi^i - \rho\overline{\xi}^i)} V_0^{i,n}\big((\widetilde{\pi}^j)_{j\neq i}\big).
$$

Now let

$$\beta_\tau^{i,\pi} := \mathrm{e}^{\frac{1}{\eta^i}\int_0^\tau \pi_u\cdot(\Sigma_u^i\theta_u^i\mathrm{d}u+\sigma_u^i\mathrm{d}W_u^i+\sigma_u^{*i}\mathrm{d}W_u^*)}.$$

Then it can be checked as in the proof of Espinosa and Touzi [15, Lemma 4.13] that for all $\pi \in \mathcal{A}^i$,

$$\beta_{\tau_1}^{i,\pi}\mathcal{V}^i(\tau_1) \geq \mathbb{E}[\beta_{\tau_2}^{i,\pi}\mathcal{V}^i(\tau_2)|\mathcal{F}_{\tau_2}^n] \qquad \text{for all stopping times } \tau_1 \leq \tau_2, \tag{4.3}$$

and by Karatzas and Shreve [24, Proposition I.3.14], the process $\mathcal{V}^i$ has a càdlàg modification again denoted by $(\mathcal{V}_t^i)_{t\in[0,T]}$. Moreover, this process also satisfies (4.3) so that for any $\pi \in \mathcal{A}^i$, the process $\beta^{i,\pi}\mathcal{V}^i$ is a $\mathbb{P}$-supermartingale. Now the definition of a Nash equilibrium implies that $\widetilde{\pi}^{i,n}$ is optimal for agent i. In other words,

$$\mathcal{V}_0^i = \sup_{\pi\in\mathcal{A}^i}\mathbb{E}\big[-\mathrm{e}^{-\frac{1}{\eta^i}(X_T^{i,\pi}-\xi^i-\rho(\overline{X}_T^i-\overline{\xi}^i))}\big] = \mathbb{E}\big[-\mathrm{e}^{-\frac{1}{\eta^i}(X_T^{i,\widetilde{\pi}^{i,n}}-\xi^i-\rho(\overline{X}_T^i-\overline{\xi}^i))}\big].$$

The above implies that $\beta^{i,\widetilde{\pi}}\mathcal{V}^i$ is a $\mathbb{P}$-martingale, where we write $\beta^{i,\widetilde{\pi}}$ as a shorthand notation for $\beta^{i,\widetilde{\pi}^{i,n}}$. Denote $\tilde{M}^i := \beta^{i,\tilde{\pi}}\mathcal{V}^i$. We now proceed to show that the adapted and continuous process

$$\gamma_t^i = X_t^{\widetilde{\pi}^{i,n}} - \xi^i + \eta^i\ln(-\tilde{M}_t^i), \qquad t \in [0,T], \tag{4.4}$$

solves a BSDE. Note already that by the definitions of $\mathcal{V}_t^i$ and $\tilde{M}_t^i$, we have

$$V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) = -\mathrm{e}^{-\frac{1}{\eta^i}(\xi^i-\rho\overline{\xi}^i-\gamma_0^i)}.$$

This proves the representation (3.1) of $V_0^{i,n}((\widetilde{\pi}^{j,n})_{j\neq i})$.

We first need to check that γ^i is in $\mathbb{S}^1(\mathbb{R},\mathbb{F}^n)$. On the one hand, using Jensen's inequality, we have

$$\frac{1}{\eta^i}\mathbb{E}[X_T^{i,\widetilde{\pi}^{i,n}} - \xi^i - \rho(\overline{X}_T^i - \overline{\xi}^i)|\mathcal{F}_t^n] \leq \ln(-\tilde{M}_t^i). \tag{4.5}$$

On the other hand, by the definition of $\mathcal{V}^i$, we have

$$-\tilde{M}_t^i = -\beta_t^{i,\widetilde{\pi}}\mathcal{V}_t^i \leq \beta_t^{i,\widetilde{\pi}}\mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big].$$

Thus using the inequality $\ln x \leq x$, we have

$$\begin{aligned}\ln(-\tilde{M}_t^i) &\leq \ln\beta_t^{i,\widetilde{\pi}} + \ln\mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big]\\ &\leq \frac{1}{\eta^i}\int_0^t \widetilde{\pi}_u^{i,n}\cdot(\Sigma_u^i\theta_u^i\mathrm{d}u + \sigma_u^i\mathrm{d}W_u^i + \sigma_u^{*i}\mathrm{d}W_u^*) + \mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big].\end{aligned}$$

Now, combining this with (4.5) and the definition of $\beta_t^{i,\tilde{\pi}}$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|\ln(-\tilde{M}_t^i)|\Big]\\
&\leq \mathbb{E}\Big[\sup_{t\in[0,T]}\frac{1}{\eta^i}\mathbb{E}\big[|X_t^{\tilde{\pi}^{i,n}}-\xi^i-\rho(\overline{X}_T^i-\overline{\xi}^i)|\big|\mathcal{F}_t^n\big]\Big]\\
&\quad+\mathbb{E}\Big[\sup_{t\in[0,T]}\frac{1}{\eta^i}|X_t^{\tilde{\pi}^{i,n}}-\xi^i|\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big]\Big]\\
&\leq\frac{1}{\eta^i}\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{E}\big[|X_T^{\tilde{\pi}^{i,n}}|\big|\mathcal{F}_t^n\big]\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^{\tilde{\pi}^{i,n}}|\Big]+2\mathbb{E}[\xi^i]\\
&\quad+\mathbb{E}\Big[\sup_{t\in[0,T]}\frac{1}{\eta^i}\mathbb{E}\big[|\rho(\overline{X}_T^i-\overline{\xi}^i)|\big|\mathcal{F}_t^n\big]\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big]\Big].
\end{aligned}$$

It is then sufficient to bound the last term. The Jensen and Hölder inequalities give

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{E}\big[\mathrm{e}^{\frac{\rho}{\eta^i}(\overline{X}_T^i-\overline{\xi}^i)}\big|\mathcal{F}_t^n\big]\Big]\\
&\leq\frac{1}{n-1}\sum_{j\neq i}^{n}\mathbb{E}\big[\mathrm{e}^{\frac{2\rho}{\eta^i}\frac{\lambda_{ij}}{\beta_n}\xi^j}\big]^{\frac{1}{2}}\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{E}\big[\mathrm{e}^{\frac{2\rho}{\eta^i}\frac{\lambda_{ij}}{\beta_n}X_T^{\tilde{\pi}^j}}\big|\mathcal{F}_t^n\big]^{\frac{1}{2}}\Big].
\end{aligned}$$

By the admissibility condition on π, it follows that $\ln(-\tilde{M}^i)\in\mathbb{S}^1(\mathbb{R},\mathbb{F}^n)$. It thus follows that $\gamma^i\in\mathbb{S}^1(\mathbb{R},\mathbb{F}^n)$ for every $i\in\{1,\ldots,n\}$. For an arbitrary $\pi\in\mathcal{A}^i$, define

$$M_t^{i,\pi}:=\mathrm{e}^{-\frac{1}{\eta^i}(X_t^\pi-\xi^i-\gamma_t^i)}=\tilde{M}_t^i\mathrm{e}^{-\frac{1}{\eta^i}(X_t^\pi-X_t^{\tilde{\pi}^{i,n}})}.$$

It follows from the same argument as in Espinosa and Touzi [15, Theorem 4.7, 2(b)] that $M^{i,\pi}$ is a supermartingale. Now by (4.4), the Doob–Meyer decomposition and Itô's formula, there is $(\zeta^i,\zeta^{*i})\in\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^{nd},\mathbb{F})\times\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{F})$ such that

$$\mathrm{d}\gamma_t^i=-b_t^i\mathrm{d}t+\sum_{j=1}^n\zeta_t^{ij}\cdot\mathrm{d}W_t^j+\zeta_t^{*i}\mathrm{d}W_t^*.$$

We first compute b^i, γ^i and $\widetilde{\pi}^i$ and then derive the BSDEs satisfied by $(\gamma^i,\zeta^i,\zeta^{*i})$.

By Itô's formula, we have

$$
\begin{aligned}
&-\mathrm{d}e^{-\frac{1}{\eta^i}(X_t^\pi-\xi^i-\gamma_t^i)}\\
&=e^{-\frac{1}{\eta^i}(X_t^\pi-\xi^i-\gamma_t^i)}\\
&\quad\times\bigg(\frac{1}{\eta^i}\Big((\sigma_t^i\pi_t^i)\cdot\mathrm{d}W_t^i+\sum_{j=1}^n\zeta_t^{ij}\cdot\mathrm{d}W_t^j+(\sigma_t^{*i}\cdot\pi_t^i+\zeta_t^{*i})\mathrm{d}W_t^*\Big)\\
&\qquad+\frac{1}{\eta^i}(b_t^i+\pi_t^i\cdot\Sigma_t^i\theta_t^i)\mathrm{d}t+\frac{1}{(\eta^i)^2}(\sigma_t^i\pi_t^i\cdot\zeta_t^{ii}+\sigma_t^{*i}\pi_t^i\zeta_t^{*i})\mathrm{d}t\\
&\qquad-\frac{1}{2(\eta^i)^2}\Big(|\sigma_t^i\pi_t|^2+|\sigma_t^{*i}\pi_t|^2+\sum_{j=1}^n|\zeta_t^{ij}|^2+|\zeta_t^{*i}|^2\Big)\mathrm{d}t\bigg).
\end{aligned}
\tag{4.6}
$$

Using the supermartingale property of $M^{i,\pi}$, the martingale property of $\tilde{M}^i$ together with (4.6), keeping in mind that $\Sigma_t^i := (\sigma_t^i, \sigma_t^{*i})$ and writing $\zeta_t^i := (\zeta_t^{ii}, \zeta_t^{*i})$, we get

$$
b_t^i \le \frac{1}{2\eta^i}|\Sigma_t^{i\top}\pi_t^i-(\zeta_t^i+\eta^i\theta_t^i)|^2+\frac{1}{2\eta^i}\sum_{j\neq i}^n|\zeta_t^{ij}|^2-\frac{\eta^i}{2}|\theta_t^i|^2-\zeta_t^i\cdot\theta_t^i, \tag{4.7}
$$

$$
b_t^i = \frac{1}{2\eta^i}|\Sigma_t^{i\top}\tilde{\pi}_t^i-(\zeta_t^i+\eta^i\theta_t^i)|^2+\frac{1}{2\eta^i}\sum_{j\neq i}^n|\zeta_t^{ij}|^2-\frac{\eta^i}{2}|\theta_t^i|^2-\zeta_t^i\cdot\theta_t^i.
$$

Thus $\widetilde{\pi}_t^{i,n}$ minimises the function (in π^i) on the right-hand side of (4.7). Therefore we can express $\widetilde{\pi}_t^{i,n}$ and b_t^i as

$$
\begin{aligned}
\widetilde{\pi}_t^i &= (\Sigma_t^i\Sigma_t^{i\top})^{-1}\Sigma_t^iP_t^i(\zeta_t^i+\eta^i\theta_t^i),\\
b_t^i &= \frac{1}{2\eta^i}\mathrm{dist}(\zeta_t^i+\eta^i\theta_t^i,\Sigma_t^iA^i)^2+\frac{1}{2\eta^i}\sum_{j\neq i}^n|\zeta_t^{ij}|^2-\frac{\eta^i}{2}|\theta_t^i|^2-\zeta_t^i\cdot\theta_t^i.
\end{aligned}
$$

Therefore, $(\gamma^i,\zeta^i,\zeta^{*i})\in\mathbb{S}^1(\mathbb{R},\mathbb{F})\times\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^{nd},\mathbb{F})\times\mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{F})$ solves the BSDE

$$
\begin{aligned}
\mathrm{d}\gamma_t^i &= \Big(\zeta_t^i\cdot\theta_t^i+\frac{\eta^i}{2}|\theta_t^i|^2-\frac{1}{2\eta^i}\sum_{j\neq i}^n|\zeta_t^{ij}|^2-\frac{1}{2\eta^i}|(\mathrm{Id}-P_t^i)(\zeta_t^i+\eta^i\theta_t^i)|^2\Big)\mathrm{d}t\\
&\quad+\sum_{j=1}^n\zeta_t^{ij}\cdot\mathrm{d}W_t^j+\zeta_t^{*i}\mathrm{d}W_t^*,\\
\gamma_T^i &= \rho(\overline{X}_T^i-\overline{\xi}^i)=\rho\sum_{j\neq i}\lambda_{ij}^n\int_0^T\widetilde{\pi}_s^j\cdot(\Sigma_s^j\theta_s^j\mathrm{d}s+\sigma_s^j\mathrm{d}W_s^j+\sigma_s^{j*}\mathrm{d}W_s^*).
\end{aligned}
$$

□

Proof of Corollary 3.2 The proof of this corollary builds upon that of Theorem 3.1, with exactly the same notation. Define the process

$$Y_t^i := \gamma_t^i - \sum_{j\neq i} \rho\lambda_{ij}^n \int_0^t \widetilde{\pi}_s^j \cdot (\Sigma_s^j \theta_s^j \mathrm{d}s + \sigma_s^j \mathrm{d}W_s^j + \sigma_s^{j*} \mathrm{d}W_s^*)$$

as well as

$$Z_t^{ij} := \zeta_t^{ij} - \rho\lambda_{ij}^n \sigma_t^j \widetilde{\pi}_t^j, \qquad Z_t^{*i} := \phi_t^i(\zeta_t^*) := \zeta_t^{*i} - \sum_{j\neq i}^n \rho\lambda_{ij}^n \sigma_t^{j*} \cdot \widetilde{\pi}_t^j. \tag{4.8}$$

Here, ϕ_t is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined componentwise as in (4.8). Moreover, notice that $Z_t^{ii} = \zeta_t^{ii}$ since $\lambda_{ii}^n = 0$, and that $\gamma_0^i = Y_0^i$. The processes (Y^i, Z^{ij}, Z^{*i}) thus satisfy

$$\begin{aligned} Y_t^i = \int_t^T \Big(&- \frac{\eta^i}{2}|\theta_s^i|^2 - \zeta_s^i \cdot \theta_s^i + \frac{1}{2\eta^i}|(\mathrm{Id} - P_s^i)(\zeta_s^i + \eta^i\theta_s^i)|^2 + \frac{1}{2\eta^i}\sum_{j\neq i}^n |\zeta_s^{ij}|^2 \\ &+ \sum_{j\neq i}^n \rho\lambda_{ij}^n (\widetilde{\pi}_s^j \cdot \Sigma_s^j \theta_s^j)\Big)\mathrm{d}s - \sum_{j=1}^n \int_t^T Z_s^{ij} \cdot \mathrm{d}W_s^j - \int_t^T Z_s^{*i} \mathrm{d}W_s^*. \end{aligned}$$

By Lemma 4.2 below, ϕ_t has an inverse ψ_t so that $\zeta_t^{*i} = \psi_t^i(Z_t^*)$. We can thus express the equilibrium strategy for player i as

$$\Sigma_t^{i\top} \widetilde{\pi}_t^i = P_t^i\left(\begin{pmatrix} Z_t^{ii} \\ \psi_t^i(Z_t^*)\end{pmatrix} + \eta^i\theta_t^i\right) =: f_t^i\big(Z_t^{ii}, \psi^i(Z_t^*)\big), \qquad t \in [0, T],$$

and

$$V_0^{i,n}\big((\widetilde{\pi}^j)_{j\neq i}\big) = -\mathrm{e}^{\frac{1}{\eta^i}(\xi^i - \rho\overline{\xi}^i - \gamma_0^i)}.$$

By construction, $(Y^i, Z^i, Z^{*i}) \in \mathbb{S}^1(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^{nd}, \mathbb{F}^n) \times \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{F}^n)$ solves the BSDE (3.4). $\square$

Lemma 4.1 *For any $t \in [0, T]$, fixed $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}^{d+1}$, the map*

$$H_{\alpha,\beta}(x) = x + \frac{1}{n-1}\sigma_t^{*i\top}(\Sigma_t^i\Sigma_t^{i\top})^{-1}\Sigma_t^i \cdot P_t^i\left(\begin{pmatrix}\alpha \\ x\end{pmatrix} + \beta\right)$$

is a bijection on $\mathbb{R}$ for every i. Furthermore, its inverse is a contraction.

Proof Fix $t \in [0, T]$ and i. The mapping $H_\alpha(\cdot)$ is a bijection if and only if the map

$$M^y(x) = y - \frac{1}{n-1}\sigma_t^{*i\top}(\Sigma_t^i\Sigma_t^{i\top})^{-1}\Sigma_t^i \cdot P_t^i\left(\begin{pmatrix}\alpha \\ x\end{pmatrix} + \beta\right)$$

has a unique fixed point. Notice that since the projection operator is 1-Lipschitz,

$$|M^y(x) - M^y(x')| \le \frac{1}{n-1}|\sigma_t^{*i\top}(\Sigma_t^i\Sigma_t^{i\top})^{-1}\Sigma_t^i||x - x'|.$$

It is thus sufficient to show that $|\sigma_t^{*i\top}(\Sigma_t^i\Sigma_t^{i\top})^{-1}\Sigma_t^i| < 1$. For notational convenience, let us omit all subscripts t. First notice that $\Sigma^i\Sigma^{i\top} = \sigma^i\sigma^i + \sigma^{*i}\sigma^{*i\top}$. Using the Sherman–Morrison formula in Bartlett [3], which gives conditions on a matrix A and vectors a, b under which the matrix $A + ab^\top$ is invertible, we have that

$$(\Sigma^i\Sigma^{i\top})^{-1} = \sigma^{-i}\sigma^{-i} - \frac{\sigma^{-i}\sigma^{-i}\sigma^{*i}\sigma^{*i\top}\sigma^{-i}\sigma^{-i}}{1 + \sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}$$

and

$$\begin{aligned}
&\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\Sigma^i \\
&= \big(\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\sigma^i\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\sigma^{*i}\big) \\
&= \bigg(\Big(1 - \frac{\sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}{1 + \sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}\Big)\sigma^{*i\top}\sigma^{-i}\,\frac{\sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}{1 + \sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}\bigg).
\end{aligned}$$

Thus

$$|\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\Sigma^i| = \frac{\sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}}{1 + \sigma^{*i\top}\sigma^{-i}\sigma^{-i}\sigma^{*i}} < 1,$$

where the last line follows from the fact that σ^i is uniformly elliptic for every i. We now proceed to show that the inverse H_α^{-1} of H_α is a contraction. For $x \neq x'$, we have

$$\begin{aligned}
&\left|x - x' + \frac{1}{n-1}\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\Sigma^i \cdot \left(P^i\left(\begin{pmatrix}\alpha\\x\end{pmatrix} + \beta\right) - P^i\left(\begin{pmatrix}\alpha\\x'\end{pmatrix} + \beta\right)\right)\right|^2 \\
&= |x - x'|^2 \\
&\quad + \frac{2}{n-1}(x - x')\cdot\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\Sigma^i \cdot \left(P^i\left(\begin{pmatrix}\alpha\\x\end{pmatrix} + \beta\right) - P^i\left(\begin{pmatrix}\alpha\\x'\end{pmatrix} + \beta\right)\right) \\
&\quad + \frac{1}{(n-1)^2}\left|\sigma^{*i\top}(\Sigma^i\Sigma^{i\top})^{-1}\Sigma^i \cdot \left(P^i\left(\begin{pmatrix}\alpha\\x\end{pmatrix} + \beta\right) - P^i\left(\begin{pmatrix}\alpha\\x'\end{pmatrix} + \beta\right)\right)\right|^2 \\
&\ge |x - x'|^2,
\end{aligned} \tag{4.9}$$

since the property of projection onto closed convex sets implies that the middle term is nonnegative. □

Lemma 4.2 *Define the map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ componentwise by*

$$\phi^i_t(\zeta^*_t) = \zeta^{i,*}_t - \sum_{j\neq i}^n \lambda^n_{ij}\sigma^{j*}_t \cdot (\Sigma^j_t {\Sigma^j_t}^\top)^{-1}\Sigma^j_t P^j_t\left(\begin{pmatrix} Z^{jj}_t \\ \zeta^{j*}_t \end{pmatrix} + \eta^j\theta^j_t\right). \tag{4.10}$$

If $\sum_{j\neq i}\lambda^n_{ij} \in [0,1]$, then for $t \in [0,T]$, the mapping ϕ_t is a bijection on $\mathbb{R}^n$ and has an inverse that we denote by ψ_t. Furthermore, ψ_t is measurable and Lipschitz-continuous with a constant that depends only on n when $n \geq 3$.

Proof Omit all subscripts t for notational convenience. Let Z^* and ζ^* denote the column vectors $(Z^{1,*},\dots,Z^{n,*})^\top$ and $(\zeta^{1,*},\dots,\zeta^{n,*})^\top$, respectively. Further, let

$$\mathcal{P}_j := \sigma^{j*}\cdot(\Sigma^j {\Sigma^j}^\top)^{-1}\Sigma^j P^j\left(\begin{pmatrix} Z^{jj} \\ \zeta^{j*} \end{pmatrix} + \eta^j\theta^j\right).$$

Let $\mathfrak{V}$ denote the column vector with the jth component equal to $\mathcal{P}_j$. Let Λ be the matrix $(\lambda_{ij})_{0\leq i,j,\leq n}$. By (4.8), we have

$$Z^* = \zeta^* - \frac{1}{n-1}\Lambda\mathfrak{V}$$

and

$$\frac{1}{n-1}\bar{1}^i(\Lambda + I_d)\mathfrak{V} + Z^{i,*} = H_{Z^{ii},\eta^i\theta^i}(\zeta^{i,*}),$$

where $\bar{1}^i$ denotes the n-dimensional vector with 1 at the ith position and 0 in all other positions. By Lemma 4.1, $H_{Z^{jj},\eta_j\theta^j}(\zeta^{i,*})$ is invertible. Using (4.8) again, we have

$$\begin{aligned}\zeta^{i,*} &= Z^{i,*} + \sum_{j\neq i}^n \lambda^n_{ij}\sigma^{j*}\cdot(\Sigma^j {\Sigma^j}^\top)^{-1}\Sigma^j \\ &\qquad\qquad \times P^j\left(\begin{pmatrix} Z^{jj} \\ H^{-1}_{Z^{jj},\eta_j\theta^j}(\frac{1}{n-1}\bar{1}^j(\Lambda+I)[\mathcal{P}_k] + Z^{i,*}) \end{pmatrix} + \eta_j\theta^j\right) \\ &=: N^{i,Z^*}(\zeta^*). \end{aligned} \tag{4.11}$$

We then proceed to show that $N^{i,Z^*}(\zeta^*)$ has a unique fixed point. Notice that for $x \neq y$, following the inequality in (4.9),

$$\begin{aligned}&|H_{Z^{jj},\eta^j\theta^j}(x) - H_{Z^{jj},\eta^j\theta^j}(y)|^2 \\ &\geq \left(1+\frac{1}{n-1}\right)^2\left|\left(P^j\left(\begin{pmatrix} Z^{jj} \\ x \end{pmatrix} + \eta^j\theta^j\right) - P^j\left(\begin{pmatrix} Z^{jj} \\ y \end{pmatrix} + \eta^j\theta^j\right)\right)\right|^2.\end{aligned}$$

Thus for fixed Z^{jj} and $\eta^j\theta^j$, the map $P^j((\overset{Z^{jj}}{H^{-1}_{Z^{jj},\eta^j\theta^j}(\cdot)}) + \eta^j\theta^j)$ is Lipschitz with constant $1/(1+\frac{1}{n-1})$. For $x, y \in \mathbb{R}^d$ and $x \neq y$,

$$
\begin{aligned}
|N^{i,Z^*}(x) - N^{i,Z^*}(y)| &\le \sum_{j\neq i}^{n} \lambda_{ij}^n \frac{|\tilde{\sigma}^{j*}| \sum_{k\neq j} \lambda_{jk}^n |(x-y)_k|}{n} + \sum_{j\neq i}^{n} \lambda_{ij}^n \frac{|\tilde{\sigma}^{j*}||x-y|}{n} \\
&\le \frac{1}{n-1}|x-y|,
\end{aligned}
$$

where the last inequality follows since $\sum_{j\neq i} \lambda_{ij}^n \in [0,1]$ for all i and $|\tilde{\sigma}^{j*}| < 1$ for all j (see the proof of Lemma 4.1). We can now conclude that for $n \ge 3$, N^{i,Z^*} admits a unique fixed point which we denote by $\psi^i(Z^*)$, and that $\boldsymbol{\zeta}^* = \psi(\boldsymbol{Z}^*)$ is the unique solution to (4.10).

Finally, we prove that ψ is Lipschitz with a constant that depends only on n when $n \ge 3$. From (4.11), we have that for all i and $n \ge 3$,

$$
\begin{aligned}
|\psi^i(Z_1^*) - \psi^i(Z_2^*)| \le |Z_1^{*i} - Z_2^{*i}| &+ \frac{2|\psi^i(Z_1^*) - \psi^i(Z_2^*)|}{n(n-1)} \\
&+ \frac{1}{n} \sup_{1\le j\le n} |Z_1^{*j} - Z_2^{*j}|.
\end{aligned}
$$

Then we have $\sup_{1\le j\le n} |\psi^j(Z_1^*) - \psi^j(Z_2^*)| \le \frac{n-1}{n-2} \sup_{1\le i\le n} |Z_1^{*j} - Z_2^{*j}|$. Therefore the function ψ is Lipschitz-continuous and hence Borel-measurable. □

4.4 Proofs for Sect. 3.2

We now prove results pertaining to the characterisation of solutions to the infinite-population game. These are direct consequences of the work of Hu et al. [22].

Proof of Proposition 3.4 Consider a solution

$$
(X^u, Y^u, Z^u, Z^{*u}) \in \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}, \mathbb{F}^u)
$$

to (3.6). Then for almost every $u \in I$, the process (Y^u, Z^u, Z^{*u}) solves the BSDE

$$
\begin{aligned}
\mathrm{d}Y_t^u &= \left(\frac{\eta^u}{2}|\theta_t^u|^2 + \begin{pmatrix} Z_t^u \\ Z_t^{*u} \end{pmatrix} \cdot \theta_t^u - \frac{1}{2\eta^u} \left| (\mathrm{Id} - P_t^u)\left(\begin{pmatrix} Z_t^u \\ Z_t^{*u} \end{pmatrix} + \eta^u \theta_t^u \right) \right|^2 \right) \mathrm{d}t \\
&\quad + Z_t^u \cdot \mathrm{d}W_t^u + Z_t^{*u} \mathrm{d}W_t^*, \\
Y_T^u &= F
\end{aligned}
$$

with $F := \mathbb{E}[\rho \int_I (X_T^{v,\tilde{\pi}^v} - \xi^v) G(u,v) \mathrm{d}v | \mathcal{F}_T^*]$. Thus it follows from [22, Theorem 7] that $\widetilde{\pi}^u$ given by (3.8) is an optimal strategy for the utility maximisation problem (2.2), while the value function satisfies (3.7). By linear growth of the projection operator, it follows that $\widetilde{\pi}^u \in \mathbb{H}^2(A^u, \mathbb{F}^u)$ for almost every $u \in I$, and by measurability of Z^u, we have that $\widetilde{\pi}^u$ is measurable. □

Remark 4.3 Hu et al. [22, Theorem 7] assumes a bounded terminal condition F. However, examining the proof reveals that boundedness is needed only to guarantee existence of a solution to the BSDE and the BMO property of $\int Z^u \cdot \mathrm{d}W^u + \int Z^{*u}\,\mathrm{d}W^*$.

Proof of Corollary 3.5 Let $(Y^u, Z^u)_{u\in I}$ solve (3.9) and introduce the processes

$$\gamma_t^u := Y_t^u + \int_0^t \mathbb{E}\Big[\rho \int_I P_s^v(Z_s^v + \eta^v\theta_s^v)\cdot\theta_s^v G(u,v)\mathrm{d}v\Big]\mathrm{d}s.$$

Then (γ^u, Z^u) satisfies

$$\mathrm{d}\gamma_t^u = \Big(-\frac{\eta^u}{2}|\theta_t^u|^2 - Z_t^u\cdot\theta_t^u + \frac{1}{2\eta^u}|(\mathrm{Id}-P_t^u)(Z_t^u+\eta^u\theta_t^u)|^2\Big)\mathrm{d}t - Z_t^u\cdot\mathrm{d}W_t^u$$

$(\mu\boxtimes\mathbb{P})$-a.s. for $t\in[0,T]$, and it follows by Fubini's theorem and the martingale property of $\int_0^t Z_s^u\cdot\mathrm{d}W_s^u$ that

$$\gamma_t^u = Y_t^u + \mathbb{E}\Big[\int_I \rho(X_t^v - \xi^v)G(u,v)\mathrm{d}v\Big].$$

In particular, $\gamma_T^u = \mathbb{E}[\int_I \rho(X_T^v-\xi^v)G(u,v)\mathrm{d}v]$ because $Y_T^u = 0$ in (3.9). Thus by [22, Theorem 7], the value function of the utility maximisation problem (2.2) (when $\sigma^*=0$) satisfies (3.11) and the process $\widetilde{\pi}^u$ given by (3.10) is an optimal strategy that is square-integrable. In the present case, we even have that

$$\Big\{\exp\Big(-\frac{1}{\eta^u}X_\tau^{\tilde{\pi}^u}\Big) : \tau \text{ is an } \mathbb{F}^u\text{-stopping time}\Big\}$$

is uniformly integrable. In particular, $(\tilde{\pi}^u)_{u\in I}$ is admissible. □

5 General backward propagation of chaos theorem: proof of Theorem 2.11

In this section, we present backward propagation of chaos results that are central in the proof of our main convergence result. We start by proving the case with common noise and then come back to the case without common noise. The two proofs are similar, but the case with common noise is slightly more involved because the representing backward particle system if fully coupled with a forward process.

5.1 Proof of Theorem 2.11: the common noise case

Consider an interacting particle system $(X^{i,n}, Y^{i,n}, Z^{ij,n}, Z^{*i,n})$. Within this system, the processes $(Y^{1,n}, Y^{2,n},\dots,Y^{n,n})$ evolve backward in time, and the processes $(X^{1,n}, X^{2,n},\dots,X^{n,n})$ evolve forward in time and characterise the Nash equilibrium, i.e., this particle system is such that

$$\widetilde{\pi}_t^{i,n} = (\Sigma_t^i {\Sigma_t^i}^\top)^{-1} \Sigma_t^i P_t^i \left(\begin{pmatrix} Z_t^{ii,n} \\ Z_t^{*i,n} \end{pmatrix} + \eta^i \theta_t^i \right) \qquad (\mathrm{d}t \otimes \mathbb{P})\text{-a.e.},$$

$$V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) = -\mathrm{e}^{-\frac{1}{\eta^i}(\xi^i - \rho\overline{\xi}^i - Y_0^{i,n})} \qquad \text{for } i \in \{1,\dots,n\};$$

see Theorem 3.1. We can find functions h and g such that the particle system satisfies the FBSDEs

$$\begin{cases} \mathrm{d}X_t^{i,n} = h^i(t, Z_t^{ii,n}, Z_t^{*i,n})(\theta_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}W_t^i + \sigma_t^{*i}\mathrm{d}W_t^*), \quad X_0^{i,n} = \xi^i, \\ \mathrm{d}Y_t^{i,n} = -\Big(g^i(t, Z_t^{ii,n}, Z_t^{*i,n}) + \dfrac{1}{2\eta^i} \displaystyle\sum_{j\neq i}^n |Z_t^{ij,n}|^2 \Big)\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{j=1}^n Z_t^{ij,n} \cdot \mathrm{d}W_t^j + Z_t^{*i,n}\mathrm{d}W_t^*, \\ Y_T^{i,n} = \rho \sum_{j\neq i}^n \lambda_{ij}^n (X_T^{j,n} - X_0^{j,n}). \end{cases} \tag{5.1}$$

Observe that due to the graph $(\lambda_{ij})_{1\leq i,j\leq n}$, the particles in the above system are not indistinguishable as in the homogeneous case considered by Laurière and Tangpi [31, 32] and Possamaï and Tangpi [39]. Our goal here is to show that as the number of particles in the system approaches infinity, the above particle system converges to the infinite particle system $(X^u, Y^u, Z^u, Z^{*u})_{0\leq u\leq 1}$ given by

$$\begin{cases} \mathrm{d}X_t^u = h^u(t, Z_t^u, Z_t^{*u})(\theta_t^u \mathrm{d}t + \sigma_t^u \mathrm{d}W_t^u + \sigma_t^{*u}\mathrm{d}W_t^*), \qquad X_0^u = \xi^u, \\ \mathrm{d}Y_t^u = -g^u(t, Z_t^u, Z_t^{*u})\mathrm{d}t + Z_t^u \mathrm{d}W_t^u + Z_t^{*u}\mathrm{d}W_t^*, \\ Y_T^u = \mathbb{E}\Big[\rho \displaystyle\int_I X_T^v G(u,v)\mathrm{d}v \Big| \mathcal{F}_T^*\Big] - \displaystyle\int_I \rho X_0^v G(u,v)\mathrm{d}v. \end{cases} \tag{5.2}$$

As above, this system is understood in the sense that the mapping

$$(u,t,\omega) \mapsto (X_t^u, Y_t^u, Z_t^u, Z_t^{*u})(\omega)$$

is measurable and for almost every $u \in I$, we have

$$(X^u, Y^u, Z^u, Z^{*u}) \in \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}, \mathbb{F}^u).$$

In particular, if we consider a specific particle $u = u_i$ in the continuum, we show that $(Y_t^{i,n}, Z_t^{ii,n}, Z_t^{*i,n})$ and $(Y_t^{u_i}, Z_t^{u_i}, Z_t^{u_i *})$ are "close" when $n \to \infty$. We consider the following assumption on the coefficients of the FBSDEs.

Condition 5.1 *The functions* $h^u : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ *and* $g^u : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ *are such that there exist nonnegative constants* ℓ_g

and ℓ_h such that for almost every $u \in I$, we have

$$\|h^u(t,x,z^*)\|_\infty \leq \ell_h(1+|z|+|z^*|),$$
$$|h^u(t,z,z^*) - h^u(t,z',z^{*\prime})| \leq \ell_h\big(|z-z'|+|z^*-z^{*\prime}|\big),$$
$$|g^u(t,z,z^*) - g^u(t,z',z^{*\prime})| \leq \ell_g(|z-z'|+|z^*-z^{*\prime}|)$$

for all $(t,z,z',z^*,z^{*\prime}) \in [0,T]\times(\mathbb{R}^d)^2\times\mathbb{R}^2$.

Remark 5.2 Recall that we use the same probability setting as in Remark 2.10. In other words, the indices in (5.1) should be considered as u_i. Further recall the link between λ_{ij} and $\beta_n > 0$ and the graphon G made in Condition 2.9.

Theorem 2.11 (i) is then a direct corollary of the following result.

Theorem 5.3 *Assume that Conditions 2.9 and 5.1 are satisfied. Further assume that the FBSDEs (5.1) and (5.2) admit respective solutions*

$$(X^{i,n},Y^{i,n},Z^{ij,n},Z^{*i,n}) \in \mathbb{S}^2(\mathbb{R},\mathbb{F}^n)\times\mathbb{S}^2(\mathbb{R},\mathbb{F}^n)\times\mathbb{H}^2(\mathbb{R}^d,\mathbb{F}^n)\times\mathbb{H}^2(\mathbb{R},\mathbb{F}^n),$$
$$(X^u,Y^u,Z^u,Z^{*u}) \in \mathbb{S}^2(\mathbb{R},\mathbb{F}^u)\times\mathbb{S}^2(\mathbb{R},\mathbb{F}^u)\times\mathbb{H}^2(\mathbb{R}^d,\mathbb{F}^u)\times\mathbb{H}^2(\mathbb{R},\mathbb{F}^u)$$

for every $(i,j)\in\{1,\dots,n\}^2$ and almost every $u\in I$. Then for every $i\in\mathbb{N}$, we have

$$|Y_0^{i,n} - Y_0^{u_i}| \longrightarrow 0 \qquad \text{as } n\to\infty. \tag{5.3}$$

Moreover, up to a subsequence, we have for almost every $t\in[0,T]$ that

$$\mathbb{E}\big[|Z_t^{ii,n} - Z_t^{u_i}| + |Z_t^{*i,n} - Z_t^{*u_i}|\big] \longrightarrow 0 \qquad \text{as } n\to\infty. \tag{5.4}$$

Proof Using Condition 5.1, Remark 2.3 and the definition of $\mathcal{F}_T^*$, we have that for almost all $(t,v)\in[0,T]\times I$,

$$\begin{aligned}
&\mathbb{E}\bigg[\int_I X_t^v G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\\
&= \int_I \mathbb{E}[X_0^v]G(u,v)\mathrm{d}v + \int_0^t \mathbb{E}\bigg[\int_I h^v(s,Z_s^v,Z_s^{*v})\theta_s^v G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\mathrm{d}s\\
&\quad + \mathbb{E}\bigg[\int_0^t\int_I h^v(s,Z_s^v,Z_s^{*v})\sigma_s^v G(u,v)\mathrm{d}v\mathrm{d}W_s^v\bigg|\mathcal{F}_T^*\bigg]\\
&\quad + \mathbb{E}\bigg[\int_0^t\int_I h^v(s,Z_s^v,Z_s^{*v})\sigma_s^{*v} G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\mathrm{d}W_s^*\\
&= \int_I \mathbb{E}[X_0^v]G(u,v)\mathrm{d}v + \int_0^t \mathbb{E}\bigg[\int_I h^v(s,Z_s^v,Z_s^{*v})\theta_s^v G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\mathrm{d}s\\
&\quad + \int_0^t \mathbb{E}\bigg[\int_I h^v(s,Z_s^v,Z_s^{*v})\sigma_s^{*v} G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\mathrm{d}W_s^*
\end{aligned}$$

where the first equality uses that X_0^u is independent of W^*, and the second follows from Lacker et al. [29, Lemma B.1]. Let us now introduce the "shifted" processes

$$
\begin{aligned}
\mathcal{Z}_t^{*u} &:= Z_t^{*u} - \mathbb{E}\bigg[\rho \int_I h^v(t, Z_t^v, Z_t^{*v})\sigma_t^{*v} G(u,v)\mathrm{d}v \bigg| \mathcal{F}_T^*\bigg],\\
\mathcal{Z}_t^u &:= Z_t^u,\\
\mathcal{Y}_t^u &:= Y_t^u - \rho\bigg(\mathbb{E}\bigg[\int_I X_t^v G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg] - \int_I \mathbb{E}[X_0^v]G(u,v)\mathrm{d}v\bigg),
\end{aligned}
$$

so that using (5.2), the processes $(\mathcal{Y}^u, \mathcal{Z}^u, \mathcal{Z}^{*u})$ satisfy

$$
\begin{aligned}
\mathcal{Y}_t^u = &\int_t^T g^u(s, Z_s^u, Z_s^{*u}) + \mathbb{E}\bigg[\rho \int_I h^v(Z_s^v, Z_s^{*v})\theta_s^v G(u,v)\mathrm{d}v\bigg|\mathcal{F}_T^*\bigg]\mathrm{d}s\\
&- \int_t^T \mathcal{Z}_s^u \cdot \mathrm{d}W_s^u - \int_t^T \mathcal{Z}_s^{*u}\mathrm{d}W_s^*.
\end{aligned}
$$

Observe that the drift term is not written with respect to the newly defined processes $(\mathcal{Y}^u, \mathcal{Z}^u, \mathcal{Z}^{*u})$, but rather with respect to the original (Y^u, Z^u, Z^{*u}). Similarly, for the prelimits, consider

$$
\begin{aligned}
\mathcal{Z}_t^{*i,n} &:= Z_t^{*i,n} - \rho\sum_{j\neq i}^n \lambda_{ij}^n \sigma_t^{*j} h^j(t, Z_t^{jj,n}, Z_t^{*j,n}),\\
\mathcal{Z}_t^{ij,n} &:= Z_t^{ij,n} - \rho\lambda_{ij}^n\sigma_t^j h^j(t, Z_t^{jj,n}, Z_t^{*j,n}),\\
\mathcal{Y}_t^{i,n} &:= Y_t^{i,n} - \rho\sum_{j\neq i}^n \lambda_{ij}^n(X_t^{j,n} - X_0^{j,n}),
\end{aligned}
$$

so that using (5.1), the processes $(\mathcal{Y}^{i,n}, \mathcal{Z}^{ij,n}, \mathcal{Z}^{*i,n})$ satisfy

$$
\begin{aligned}
\mathcal{Y}_t^{i,n} = &\int_t^T g^i(s, Z_s^{ii,n}, Z_s^{*i,n}) + \frac{1}{2\eta^i}\sum_{j\neq i}^n \|Z_s^{ij,n}\|^2\\
&+ \rho\sum_{j\neq i}^n \lambda_{ij}^n h^j(s, Z_s^{jj,n}, Z_s^{*j,n})\theta_s^j\mathrm{d}s - \sum_{j=1}^n\int_t^T \mathcal{Z}_s^{ij,n}\cdot\mathrm{d}W_s^j - \int_t^T \mathcal{Z}_s^{*i,n}\mathrm{d}W_s^*.
\end{aligned}
$$

To further simplify the notation, let us put

$$
\begin{aligned}
\Delta\mathcal{Y}_t^{i,n} &:= \mathcal{Y}_t^{i,n} - \mathcal{Y}_t^{u_i}, &\qquad \Delta Y_t^{i,n} &= Y_t^{i,n} - Y_t^{u_i},\\
\Delta\mathcal{Z}_t^{*i,n} &:= \mathcal{Z}_t^{*i,n} - \mathcal{Z}_t^{*u_i}, &\qquad \Delta Z_t^{*i,n} &:= Z_t^{*i,n} - Z_t^{*u_i},\\
\Delta\mathcal{Z}_t^{ij,n} &= \mathcal{Z}_t^{ij,n} - \delta_{ij}\mathcal{Z}_t^{u_i}, &\qquad \Delta Z_t^{ij,n} &= Z_t^{ij,n} - \delta_{ij}Z_t^{u_i}.
\end{aligned}
$$

Let $t \in [0, T]$ be fixed. We now define stopping times τ_k for $k \in \mathbb{N}$ by

$$\tau_k := T \wedge \inf\Big\{s \geq t : \int_t^s \sum_{j=1}^n \big(|\Delta Z_r^{ij,n}|^2 + |\Delta Z_r^{jj,n}| + |\Delta Z_r^{*j,n}|^2 + |h^j(r, Z_r^{jj,n}, Z_r^{*j,n})|^2\big)\mathrm{d}r + \sup_{r\in[t,s]} |\Delta \mathcal{Y}_r^{i,n}|^2 \geq k\Big\}.$$

Observe that τ_k depends on i and n, but this dependence is omitted to simplify notation. Since

$$(X^{i,n}, Y^{i,n}, Z^{ij,n}, Z^{*i,n}) \in \mathbb{S}^2(\mathbb{R}, \mathbb{F}^n) \times \mathbb{S}^2(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^n) \times \mathbb{H}^2(\mathbb{R}, \mathbb{F}^n)$$

and $(X^u, Y^u, Z^u, Z^{*u}) \in \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{S}^2(\mathbb{R}, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u) \times \mathbb{H}^2(\mathbb{R}, \mathbb{F}^u)$, it follows that for each n and i, τ_k converges to T $\mathbb{P}$-a.s as $k \to \infty$. Furthermore, put

$$\Gamma_s^{i,n} := \rho \sum_{j\neq i}^n \lambda_{ij}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\cdot\theta_s^{u_j} - \rho\mathbb{E}\Big[\int_I h^v(s, Z_s^v, Z_s^{*v})\cdot\theta_s^v G(u_i, v)\mathrm{d}v \Big| \mathcal{F}_T^*\Big]$$

and

$$\Gamma_s^{*i,n} := \rho \sum_{j\neq i}^n \lambda_{ij}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j}) \cdot \sigma_s^{*u_j} - \rho\mathbb{E}\Big[\int_I h^v(s, Z_s^v, Z_s^{*v}) \cdot \sigma_s^{*v} G(u_i, v)\mathrm{d}v \Big| \mathcal{F}_T^*\Big].$$

Now, applying Itô's formula to $|\Delta \mathcal{Y}_t^{i,n}|^2$, we get

$$\begin{aligned}
&|\Delta \mathcal{Y}_t^{i,n}|^2 + \int_t^{\tau_k} \Big(\sum_{j=1}^n |\Delta \mathcal{Z}_s^{ij,n}|^2 + |\Delta \mathcal{Z}_s^{*i,n}|^2\Big)\mathrm{d}s \\
&= |\Delta \mathcal{Y}_{\tau_k}^{i,n}|^2 \\
&\quad + \int_t^{\tau_k} 2\Delta \mathcal{Y}_s^{i,n}\Big(g^i(s, Z_s^{ii,n}, Z_s^{*i,n}) - g^{u_i}(s, Z_s^{u_i}, Z_s^{*u_i}) + \sum_{j\neq i}^n |Z_s^{ij,n}|^2\Big)\mathrm{d}s \\
&\quad + \int_t^{\tau_k} 2\Delta \mathcal{Y}_s^{i,n} \rho \sum_{j\neq i} \lambda_{ij}^n \theta_s^j \big(h^j(s, Z_s^{jj,n}, Z_s^{*j,n}) - h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\big)\mathrm{d}s \\
&\quad + \int_t^{\tau_k} 2\Delta \mathcal{Y}_s^{i,n} \Gamma_s^{i,n}\mathrm{d}s \\
&\quad - \sum_{j=1}^n \int_t^{\tau_k} 2\Delta \mathcal{Y}_s^{i,n} \Delta \mathcal{Z}_s^{ij,n} \cdot \mathrm{d}W_s^j - \int_t^{\tau_k} 2\Delta \mathcal{Y}_s^{i,n} \Delta \mathcal{Z}_s^{*i,n}\mathrm{d}W_s^*.
\end{aligned} \tag{5.5}$$

Recall that we have $\Delta\mathcal{Z}^{ij,n} = \Delta Z^{ij,n} - \frac{\rho}{n\beta_n}\lambda_{ij}\sigma^j h^j(Z^{jj,n}, Z^{*j,n})$ for $i \neq j$ and $\Delta\mathcal{Z}^{ii,n} = \Delta Z^{ii,n}$. Equation (5.5) now takes the form

$$
\begin{aligned}
&|\Delta\mathcal{Y}_t^{i,n}|^2 + \int_t^{\tau_k} \bigg(\sum_{j=1}^n |\Delta\mathcal{Z}_s^{ij,n}|^2 + |\Delta\mathcal{Z}_s^{*i,n}|^2\bigg)\,\mathrm{d}s \\
&= |\Delta\mathcal{Y}_{\tau_k}^{i,n}|^2 + \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\big(g^i(s, Z_s^{ii,n}, Z_s^{*i}) - g^{u_i}(s, Z_s^{u_i}, Z_s^{*u_i})\big)\mathrm{d}s \\
&\quad + \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\rho\sum_{j\neq i}^n \lambda_{ij}^n\theta_s^j\big(h^j(s, Z_s^{jj,n}, Z_s^{*j,n}) - h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\big)\,\mathrm{d}s \\
&\quad + \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\Gamma_s^{i,n}\mathrm{d}s + \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\Gamma_s^{*i,n}\mathrm{d}s \\
&\quad + \sum_{j\neq i}^n \rho\lambda_{ij}^n \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}h^j(s, Z_s^{jj,n}, Z_s^{*j,n})\sigma_s^j \cdot \Delta Z_s^{ij,n}\mathrm{d}s \\
&\quad - \sum_{j\neq i}^n \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\Delta\mathcal{Z}_s^{ij,n} \cdot (\mathrm{d}W_s^j - \Delta Z_s^{ij,n}\mathrm{d}s) \\
&\quad - \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\Delta Z_s^{ii,n} \cdot \mathrm{d}W_s^i - \int_t^{\tau_k} 2\Delta\mathcal{Y}_s^{i,n}\Delta\mathcal{Z}_s^{*i,n}\mathrm{d}W_s^*.
\end{aligned}
\tag{5.6}
$$

Let $\mathbb{Q}$ be the probability measure with density

$$
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\bigg(\sum_{j\neq i}^n \int_t^{\tau_k} \Delta Z_s^{ij,n} \cdot \mathrm{d}W_s^j - \frac{1}{2}\sum_{j\neq i}^n \int_t^{\tau_k} |\Delta Z_s^{ij,n}|^2\mathrm{d}s\bigg).
$$

The probability measure $\mathbb{Q}$ depends on i and n, but its density has a second moment bounded by a constant C_k depending on k, but not on i and n. Taking conditional expectations under $\mathbb{Q}$ with respect to $\mathcal{F}_t^n$ in (5.6), we obtain

$$
\begin{aligned}
&|\Delta\mathcal{Y}^{i,n}_t|^2+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\bigg(\sum_{j=1}^n|\Delta\mathcal{Z}^{ij,n}_s|^2+|\Delta\mathcal{Z}^{*i,n}_s|^2\bigg)\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\le\mathbb{E}^{\mathbb{Q}}\big[|\Delta\mathcal{Y}^{i,n}_{\tau_k}|^2\big|\mathcal{F}^n_t\big]+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\frac{2\ell_g^2}{\varepsilon}|\Delta\mathcal{Y}^{i,n}_s|^2+\varepsilon|\Delta\mathcal{Z}^{ii,n}_s|^2+\varepsilon|\Delta\mathcal{Z}^{*i,n}_s|^2\,\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\quad+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}2\Delta\mathcal{Y}^{i,n}_s\Gamma^{i,n}_s\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}2\Delta\mathcal{Y}^{i,n}_s\Gamma^{*i,n}_s\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\quad+C\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\bigg(|\Delta\mathcal{Y}^{i,n}_s|^2\\
&\qquad\qquad+\frac{\rho}{(n-1)^2\beta_n^2}\bigg(\sum_{j=1}^n\lambda_{ij}^2\bigg)\\
&\qquad\qquad\times\sum_{j\neq i}^n|\theta^j_s|^2|h^j(s,Z^{jj,n}_s,Z^{*j,n}_s)-h^{u_j}(s,Z^{u_j}_s,Z^{*u_j}_s)|^2\bigg)\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\quad+C\mathbb{E}^{\mathbb{Q}}\bigg[\sum_{j\neq i}^n\int_t^{\tau_k}2\rho\lambda^n_{ij}|\Delta\mathcal{Y}^{i,n}_s||h^j(s,Z^{jj,n}_s,Z^{*j,n}_s)\sigma^j_s\cdot\Delta Z^{ij,n}_s|\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg].
\end{aligned}
$$

Recall $\mathfrak{P}$ from after Definition 2.2. Using $\mathbb{E}^{\mathfrak{P}}[\lambda_{ij}^2]\le\beta_n$ and by the definition of the stopping time τ_k, this estimate can be simplified to

$$
\begin{aligned}
&|\Delta\mathcal{Y}^{i,n}_t|^2+(1-\varepsilon)\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\bigg(\sum_{j=1}^n|\Delta\mathcal{Z}^{ij,n}_s|^2+|\Delta\mathcal{Z}^{*i,n}_s|^2\bigg)\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\le\mathbb{E}^{\mathbb{Q}}\big[|\Delta\mathcal{Y}^{i,n}_{\tau_k}|^2\big|\mathcal{F}^n_t\big]+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\bigg(1+\frac{2\ell_g^2}{\varepsilon}\bigg)|\Delta\mathcal{Y}^{i,n}_s|^2\,\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]\\
&\quad+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}2|\Delta\mathcal{Y}^{i,n}_s|(|\Gamma^{i,n}_s|+|\Gamma^{*i,n}_s|)\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg]+\frac{C_{\rho,\theta,h,k}}{(n-1)\beta_n}\\
&\quad+C\mathbb{E}^{\mathbb{Q}}\bigg[\sum_{j\neq i}^n\int_t^{\tau_k}2\rho\lambda^n_{ij}|\Delta\mathcal{Y}^{i,n}_s||h^j(Z^{jj,n}_s,Z^{*j,n}_s)\sigma^j_s\cdot\Delta Z^{ij,n}_s|\mathrm{d}s\bigg|\mathcal{F}^n_t\bigg].
\end{aligned}\tag{5.7}
$$

Applying Young's inequality and recalling the definition of τ_k, the last term above can be estimated as

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\sum_{j\neq i}^{n}\int_t^{\tau_k} 2\rho\lambda_{ij}^n|\Delta\mathcal{Y}_s^{i,n}||h^j(Z_s^{jj,n},Z_s^{*j,n})\sigma_s^j\cdot\Delta Z_s^{ij,n}|\mathrm{d}s\Big|\mathcal{F}_t^n\Big]\\
&\le \frac{\rho\|\sigma\|_\infty}{(n-1)\beta_n}\\
&\quad\times\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\le s\le\tau_k}|\Delta\mathcal{Y}_s^{i,n}|\int_t^{\tau_k}\sum_{j\neq i}^{n}\big(|h^j(s,Z_s^{jj,n},Z_s^{*j,n})|^2+|\Delta Z_s^{ij,n}|^2\big)\,\mathrm{d}s\Big|\mathcal{F}_t^n\Big]\\
&\le \frac{C_{\rho,\sigma,h,T,k}}{(n-1)\beta_n}.
\end{aligned}
$$

Thus choosing $\varepsilon<1$ and subsequently using in (5.7) Gronwall's inequality, taking expectations with respect to $\mathbb{P}$ and using the Cauchy–Schwarz and Doob inequalities, we are left with

$$
\begin{aligned}
\mathbb{E}[|\Delta\mathcal{Y}_t^{i,n}|^2]\le{}&\mathbb{E}\Big[\mathbb{E}^{\mathbb{Q}}\big[|\Delta\mathcal{Y}_{\tau_k}^{i,n}|^2\big|\mathcal{F}_t^n\big]\Big]\\
&+C_{k,T}\mathbb{E}\Big[\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)^4\Big]^{\frac14}\mathbb{E}\Big[\int_0^T(|\Gamma_s^{i,n}|^2+|\Gamma_s^{*i,n}|^2)\mathrm{d}s\Big]^{\frac12}\\
&+\frac{C_{\rho,\sigma,\theta,h,T,k}}{(n-1)\beta_n}.
\end{aligned}
\tag{5.8}
$$

Observe that using again Cauchy–Schwarz gives

$$
\mathbb{E}\Big[\mathbb{E}^{\mathbb{Q}}\big[|\Delta\mathcal{Y}_{\tau_k}^{i,n}|^2\big|\mathcal{F}_t^n\big]\Big]\le C\mathbb{E}\Big[\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)^2\Big]^{\frac12}\mathbb{E}[|\Delta\mathcal{Y}_{\tau_k}|^4]^{\frac12}.
$$

To proceed, first notice that for every n, $(|\Delta\mathcal{Y}_{\tau_k}^{i,n}|^4)_{k\ge1}$ converges to 0 in probability as $k\to\infty$, since τ_k converges to T $\mathbb{P}$-a.s. Thus there exists a subsequence $(\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n})_{m\ge1}$ such that

$$
\mathbb{P}[|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4>\varepsilon]\le\frac{\mathrm{e}^{-k^3}}{m}.
$$

Therefore, for every $\varepsilon>0$, we have

$$
\begin{aligned}
\mathbb{E}[|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4]&=\mathbb{E}\big[|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4 1_{\{|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4\le\varepsilon\}}\big]+\mathbb{E}\big[|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4 1_{\{|\Delta\mathcal{Y}_{\tau_{k,m}}^{i,n}|^4>\varepsilon\}}\big]\\
&\le\varepsilon+k^2\frac{\mathrm{e}^{-k^3}}{m}.
\end{aligned}
$$

Our definition of τ_k implies that all moments of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ are bounded by e^{Ck^2} for some constant $C>0$ independent of k. Thus coming back to (5.8), we continue the estimation as

$$\mathbb{E}[|\Delta\mathcal{Y}_t^{i,n}|^2] \le C\mathrm{e}^{Ck^2}\left(\varepsilon + k^2\frac{\mathrm{e}^{-k^3}}{m}\right)^{\frac{1}{2}} + C_{k,T}\mathbb{E}\left[\int_0^T |\Gamma_s^{i,n}|^2\mathrm{d}s\right]^{\frac{1}{2}}$$
$$+ C_{k,T}\mathbb{E}\left[\int_0^T |\Gamma_s^{*i,n}|^2\mathrm{d}s\right]^{\frac{1}{2}} + \frac{C_{\rho,\sigma,\theta,h,T,k}}{(n-1)\beta_n}. \tag{5.9}$$

Applying Lemma 5.4, first fix k and let $n \to \infty$, followed by letting $m \to \infty$ and $\varepsilon \to 0$. We then conclude that

$$\mathbb{E}[|\Delta\mathcal{Y}_t^{i,n}|^2] \longrightarrow 0 \qquad \text{as } n \to \infty.$$

In particular, starting with $t = 0$, it follows that the sequence $(\Delta\mathcal{Y}_0^{i,n})_{n\ge1}$ converges to zero, and we obtain (5.3) since $\Delta\mathcal{Y}_0^{i,n} = \Delta Y_0^{i,n}$.

Let us now turn to the convergence of the control processes. By (5.7), (5.9), the Cauchy–Schwarz inequality and the above estimates, we have

$$\mathbb{E}\left[\int_t^{\tau_{k,m}} (|\Delta\mathcal{Z}_s^{ii,n}| + |\Delta\mathcal{Z}_s^{*i,n}|)\,\mathrm{d}s\right]$$
$$\le T\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^2\right]^{\frac{1}{2}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau_{k,m}} (|\Delta\mathcal{Z}_s^{ii,n}|^2 + |\Delta\mathcal{Z}_s^{*i,n}|^2)\,\mathrm{d}s\right]^{\frac{1}{2}}$$
$$\le T\mathrm{e}^{Ck^2}\Bigg(\varepsilon + k^2\frac{\mathrm{e}^{-k^3}}{m} + C_{k,T}\mathbb{E}\left[\int_0^T |\Gamma_s^{i,n}|^2\mathrm{d}s\right]^{\frac{1}{2}}$$
$$+ C_{k,T}\mathbb{E}\left[\int_0^T |\Gamma_s^{*i,n}|^2\mathrm{d}s\right]^{\frac{1}{2}} + \frac{C_{h,\theta,T,k}}{(n-1)\beta_n}\Bigg)^{\frac{1}{2}}.$$

Using Lemma 5.4, first fix k and let $n \to \infty$, followed by letting $m \to \infty$ and $\varepsilon \to 0$. We then conclude that up to a subsequence, we have

$$\lim_{m\to\infty}\lim_{n\to\infty} |\Delta\mathcal{Z}_s^{ii,n}|1_{\{s\le\tau_{k,m}\}} + |\Delta\mathcal{Z}_s^{*i,n}|1_{\{s\le\tau_{k,m}\}} = 0 \qquad \mathbb{P}\text{-a.s. for a.e. } s \in [t, T].$$

Since $\mathbb{P}[\tau_{k,m} \ge t] = 1$, this shows that

$$|\Delta\mathcal{Z}_t^{ii,n}| + |\Delta\mathcal{Z}_t^{*i,n}| \longrightarrow 0 \qquad \mathbb{P}\text{-a.s. as } n \to \infty.$$

By the identity $\Delta \mathcal{Z}^{ii,n} = \Delta Z^{ii,n}$, we have thus obtained that $(\Delta Z_t^{ii,n})_{n\geq 1}$ converges to zero. For the convergence of $\Delta Z_t^{*i,n}$, observe that

$$\begin{aligned}
|\Delta Z_s^{*i,n}| &\leq |\Delta \mathcal{Z}_s^{*i,n}| + |\Gamma_s^{*i,n}| \\
&\quad + \frac{\rho}{n\beta_n} \sum_{j\neq i}^{n} \lambda_{ij} |\sigma_s^{*\frac{j}{n}}| |h^j(s, Z_s^{jj,n}, Z_s^{*j,n}) - h^j(s, Z_s^{u_j}, Z_s^{*u_j})| \\
&\leq |\Delta \mathcal{Z}_s^{*i,n}| + |\Gamma_s^{*i,n}| \\
&\quad + \rho \|\sigma^{*i}\|_\infty \ell_h \frac{1}{(n-1)\beta_n} \sum_{j\neq i}^{n} \lambda_{ij} (|\Delta Z_s^{jj,n}| + |\Delta Z_s^{*j,n}|).
\end{aligned}$$

Thus we get

$$\mathbb{E}\bigg[\int_0^{\tau_{k,m}} |\Delta Z_s^{*i,n}|\, \mathrm{d}s\bigg] \leq \mathbb{E}\bigg[\int_0^{\tau_{k,m}} (|\Delta \mathcal{Z}_s^{*i,n}|^2 + |\Gamma_s^{*i,n}|^2)\, \mathrm{d}s\bigg] + \frac{C_{\rho,\sigma^*,h,k}}{(n-1)\beta_n}.$$

Therefore, arguing as above and using again Lemma 5.4, we have that up to a subsequence,

$$|\Delta Z_t^{*i,n}| \longrightarrow 0 \qquad \mathbb{P}\text{-a.s. as } n \to \infty.$$

Therefore (5.4) follows by dominated convergence. This concludes the proof. $\square$

Lemma 5.4 *Under the conditions of Theorem 5.3, we have*

$$\mathbb{E}\bigg[\int_0^T |\Gamma_s^{*i,n}|^2 \mathrm{d}s\bigg] + \mathbb{E}\bigg[\int_0^T |\Gamma_s^{i,n}|^2 \mathrm{d}s\bigg] \longrightarrow 0 \qquad \textit{as } n \to \infty \textit{ for every } i \in \mathbb{N}.$$

Proof We consider only the term $\Gamma^{i,n}$; the term $\Gamma^{*i,n}$ is dealt with similarly. Using Condition 2.9, especially that λ_{ij} are i.i.d. and defined on a separate probability space from $(\Omega, \mathcal{F}, \mathbb{P})$ and $(W^1, \dots, W^n, W^*)$, we have

$$\begin{aligned}\mathbb{E}\big[|\Gamma_s^{i,n}|^2\big] &= \mathbb{E}\Big[\Big|\frac{1}{(n-1)\beta_n}\sum_{j\neq i}^n \lambda_{ij} h^j(s, Z_s^{u_j}, Z_s^{*u_j})\cdot\theta_s^{u_j} \\ &\qquad - \mathbb{E}\Big[\int_I h^v(s, Z_s^v, Z_s^{*v})\cdot\theta_s^v G(u_i,v)\mathrm{d}v\Big|\mathcal{F}_T^*\Big]\Big|^2\Big] \\ &\le 2\mathbb{E}\Big|\frac{1}{n-1}\sum_{j\neq i}^n\Big(\frac{\lambda_{ij}}{\beta_n}h^j(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j} \\ &\qquad - h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}G_n(u_i,u_j)\Big)\Big|^2 \\ &\quad + 2\mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j\neq i}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}G_n(u_i,u_j) \\ &\qquad - \mathbb{E}\Big[\int_I h^v(s, Z_s^v, Z_s^{*v})\cdot\theta_s^v G(u_i,v)\mathrm{d}v\Big|\mathcal{F}_T^*\Big]\Big|^2\Big].\end{aligned}$$

Continuing the estimation above,

$$\begin{aligned}\mathbb{E}\big[|\Gamma_s^{i,n}|^2\big] &\le \frac{C_\theta}{(n-1)^2\beta_n^2}\mathrm{Var}[\lambda_{ij}]\mathbb{E}\Big[\sum_{j\neq i}^n |h^j(s, Z_s^{u_j}, Z_s^{*u_j})|^2\Big] \\ &\quad + 4\mathbb{E}\Big[\Big|\frac{1}{n}\sum_{j\neq i}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}G_n(u_i,u_j) \\ &\qquad - \mathbb{E}\Big[\frac{1}{n}\sum_{j=1}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}G_n(u_i,u_j)\Big|\mathcal{F}_T^*\Big]\Big|^2\Big] \\ &\quad + \mathbb{E}\Big[\Big|\mathbb{E}\Big[\frac{1}{n}\sum_{j=1}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}G_n(u_i,u_j)\Big|\mathcal{F}_T^*\Big] \\ &\qquad - \mathbb{E}\Big[\int_I h^v(s, Z_s^v, Z_s^{*v})\cdot\theta_s^v G(u_i,v)\mathrm{d}v\Big|\mathcal{F}_T^*\Big]\Big|^2\Big].\end{aligned}$$

Using that the step function F_s^n given by

$$F_s^n(u) := \sum_{j=1}^n h^{u_j}(s, Z_s^{u_j}, Z_s^{*u_j})\theta_s^{u_j}1_{\{u\in(u_j,u_{j+1}]\}}$$

approximates the function $F_s : v \mapsto h^v(s, Z_s^v, Z_s^{*v})\cdot\theta_s^v$ in $L^2(I, \mathcal{B}(I), \mu)$, we have

$$
\begin{aligned}
&\mathbb{E}\big[|\Gamma_s^{i,n}|^2\big] \\
&\leq \frac{C_\theta}{(n-1)^2\beta_n^2}\mathrm{Var}[\lambda_{ij}]\mathbb{E}\bigg[\sum_{j\neq i}^{n}|h^j(s,Z_s^{u_j},Z_s^{*u_j})|^2\bigg] \\
&\quad+\frac{4}{(n-1)^2}\sum_{j\neq i}^{n}\mathbb{E}\big[\big|h^{u_j}(s,Z_s^{u_j},Z_s^{*u_j})\theta_s^{u_j}-\mathbb{E}[h^{u_j}(s,Z_s^{u_j},Z_s^{*u_j})\theta_s^{u_j}|\mathcal{F}_T^*]\big|^2\big] \\
&\quad+\mathbb{E}\bigg[\bigg|\int_I F_s^n(v)G_n(u_i,v)\,\mathrm{d}v-\int_I h^v(s,Z_s^v,Z_s^{*v})\cdot\theta_s^v G_n(u_i,v)\mathrm{d}v\bigg|^2\bigg] \\
&\quad+\mathbb{E}\bigg[\bigg|\int_I h^v(s,Z_s^v,Z_s^{*v})\cdot\theta_s^v G_n(u_i,v)\mathrm{d}v-\int_I h^v(s,Z_s^v,Z_s^{*v})\cdot\theta_s^v G(u_i,v)\mathrm{d}v\bigg|^2\bigg] \\
&\leq \frac{C_\theta}{(n-1)^2\beta_n^2}\mathbb{E}\bigg[\bigg|\sum_{j\neq i}^{n}(\lambda_{ij}-\mathbb{E}[\lambda_{ij}])h^j(s,Z_s^{u_j},Z_s^{*u_j})\bigg|^2\bigg] \\
&\quad+\frac{C_\theta}{(n-1)^2}\sum_{j\neq i}^{n}\mathbb{E}\big[|(h^{u_j}(s,Z_s^{u_j},Z_s^{*u_j})|^2\big]+\|F_s^n-F_s\|_{L^2(I,\mathcal{B}(I),\mu)} \\
&\quad+C_\theta\mathbb{E}\bigg[\int_I|h^v(s,Z_s^v,Z_s^{*v})|^2\Big(G_n(u_i,v)-G(u_i,v)\Big)^2\mathrm{d}v\bigg].
\end{aligned}
\tag{5.10}
$$

Because the Lipschitz constants of h^u and g^u do not depend on u, standard FBSDE estimates show that $\sup_{u\in I}\|(Z^u,Z^{*u})\|_{\mathbb{H}^2(\mathbb{R}^{d+1},\mathbb{F}^u)}<\infty$. Hence integrating on both sides above and using Condition 5.1, we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\int_0^T |\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big]\\
&\le \Big(\frac{C_{h,\theta,T}}{(n-1)\beta_n^2}+\frac{C_\theta}{(n-1)}\Big)\Big(\sup_{u\in I}\|(Z^u,Z^{*u})\|_{\mathbb{H}^2(\mathbb{R}^{d+1},\mathbb{F}^u)}+1\Big)\\
&\quad + C_T\|F_s^n-F_s\|_{L^2(I,\mathcal{B}(I),\mu)}\\
&\quad + \ell_h^2 C_\theta\Big(\sup_{u\in I}\|(Z^u,Z^{*u})\|_{\mathbb{H}^2(\mathbb{R}^{d+1},\mathbb{F}^u)}+1\Big)\\
&\qquad \times \frac{1}{n}\sum_{i=1}^{n}\int_I\Big(G_n(u_i,v)-G(u_i,v)\Big)^2\,\mathrm{d}v\\
&\le \frac{C_{h,\theta,T,Z}}{(n-1)\beta_n^2}+\frac{C_{\theta,Z}}{(n-1)}+C_T\|F_s^n-F_s\|_{L^2(I,\mathcal{B}(I),\mu)}\\
&\quad + \ell_h^2 C_{\theta,Z}\int_I\int_I\big(G_n(\omega,v)-G(\omega,v)\big)^2\,\mathrm{d}v\,\mathrm{d}u\\
&\le \frac{C_{h,\theta,T,Z}}{(n-1)\beta_n^2}+\frac{C_{\theta,Z}}{(n-1)}+C_T\|F_s^n-F_s\|_{L^2(I,\mathcal{B}(I),\mu)}\\
&\quad + \ell_h^2 C_{\theta,Z}\|G_n-G\|_2^2.
\end{aligned}
\tag{5.11}
$$

Therefore, since $n\|G_n-G\|_2^2\to 0$, it follows that for each i, we have

$$
\mathbb{E}\Big[\int_0^T |\Gamma_s^{i,n}|^2\mathrm{d}s\Big]\longrightarrow 0 \qquad \text{as } n\to\infty. \qquad \square
$$

Remark 5.5 If the function h is bounded (which is the case when the graphon equilibrium $(\widetilde{\pi}^u)_{u\in I}$ is bounded), it is enough the require that $n\|G_n-G\|_\square\to 0$, which is weaker that $\mathbb{L}^2$-convergence. This is due to the fact that the last term in (5.10) can be estimated as

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_I h^v(s,Z_s^v,Z_s^{*v})\cdot\theta_s^u\big(G_n(u_i,v)-G(u_i,v)\big)\,\mathrm{d}v\Big)^2\Big]\\
&\le C_{h,\theta}\Big|\int_I\big(G_n(u_i,v)-G(u_i,v)\big)\,\mathrm{d}v\Big|^2.
\end{aligned}
$$

Taking the average, we obtain the estimation

$$
\begin{aligned}
\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\int_I G_n(u_i,v)-G(u_i,v)\mathrm{d}v\Big|\Big)^2 &= \Big(\int_I\Big|\int_I\big(G_n(u,v)-G(u,v)\big)\mathrm{d}v\Big|\mathrm{d}u\Big)^2\\
&\le 4\|G_n-G\|^2,
\end{aligned}
$$

where $\|G\|$ is the so-called operator norm given by

$$\|G\| := \sup_{\|h\|_\infty \le 1} \int_I \left| \int_I h(u)G(u,v)\,\mathrm{d}v \right| \mathrm{d}u.$$

It follows from Lovász [33, Lemma 8.11] that $\|G\|_\square$ and $\|G\|$ are equivalent norms. Therefore the last term in (5.11) can be replaced by $\ell_h^2 C_{\theta,Z}\|G_n - G\|_\square^2$. $\square$

5.2 Proof of Theorem 2.11: the case without common noise

Let us now present the proof of Theorem 2.11 (ii). **Throughout this subsection**, we assume $\sigma^{*u} = 0$ for all $u \in I$. By Theorem 3.1 and Remark 3.3, the Nash equilibrium $(\widetilde{\pi}^{i,n})_{i\in\{1,\dots,n\}}$ is characterised by the BSDE (3.5). That is, we have

$$\widetilde{\pi}_t^{i,n} = (\sigma_t^i)^{-1} P_t^i (Z_t^{ii} + \eta^i \theta_t^i)$$

and

$$V_0^{i,n}\big((\widetilde{\pi}^{j,n})_{j\neq i}\big) = -e^{-\frac{1}{\eta^i}(\xi^i - \rho\overline{\xi}^i - Y_0^i)} \qquad (\mathrm{d}t \otimes \mathbb{P})\text{-a.e.}$$

with $(Y^{i,n}, Z^{ij,n})_{(i,j)\in\{1,\dots,n\}^2}$ solving the BSDE (3.5). Moreover, by Corollary 3.5 and Proposition 6.2, there is a graphon equilibrium $(\widetilde{\pi}^u)_{u\in I}$ such that

$$\widetilde{\pi}_t^u = (\sigma_t^u)^{-1} P_t^u (Z_t^u + \eta^u \theta_t^u) \qquad (\mathrm{d}t \otimes \mu \boxtimes \mathbb{P})\text{-a.e.}$$

and

$$V_0^{u,G} = -\exp\bigg(-\frac{1}{\eta^u}\Big(\xi^u - \int_I \mathbb{E}[\rho\xi^v]G(u,v)\mathrm{d}v - Y_0^u\Big)\bigg)$$

with $(Y^u, Z^u)_{u\in I}$ solving (3.9). It thus suffices to show that

$$|Y_0^{i,n} - Y_0^{u_i}|^2 + |Z_t^{ii,n} - Z_t^{u_i}|^2 \longrightarrow 0 \qquad (\mathrm{d}t \otimes \mathbb{P})\text{-a.e. as } n \to \infty.$$

Let us put $\Delta Y^{i,n} := Y^{i,n} - Y^{u_i}$ and $\Delta Z^{ij,n} := Z^{ij,n} - Z^{u_i}\delta_{ij}$. Let $t \in [0,T]$ be fixed and consider the stopping time

$$\tau_k := \inf\bigg\{ s \ge t : \int_t^s \bigg(\sum_{j=1}^n |P_r^j (Z_r^{jj,n} + \eta^j \theta_r^j)|^2 + |\Delta Z_r^{jj,n}|^2 \bigg) \mathrm{d}r$$
$$+ \sup_{t\le r\le s} |\Delta Y_r^{i,n}|^2 \ge k \bigg\} \wedge T.$$

Observe that for each i, n, the sequence $(\tau_k)_{k\geq 1}$ converges to T $\mathbb{P}$-a.s. Applying Itô's formula to $\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2$ for some $\kappa > 0$ to be chosen, we have

$$\begin{aligned}
&\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2\\
&= \mathrm{e}^{\kappa\tau_k}(\Delta Y_{\tau_k}^{i,n})^2 + \int_t^{\tau_k} 2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\\
&\qquad\qquad \times\Big(\theta_s^i\cdot\Delta Z_s^{ii,n} + \frac{1}{2\eta^i}\big(|(\mathrm{Id}-P_s^i)(Z_s^{ii,n}+\eta^i\theta_s^i)|^2\\
&\qquad\qquad\qquad - |(\mathrm{Id}-P_s^{u_i})(Z_s^{u_i}+\eta^{u_i}\theta_s^{u_i})|^2\big)\Big)\,\mathrm{d}s\\
&\quad + \int_t^{\tau_k} 2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\sum_{j\neq i}^n |Z_s^{ij,n} + \sigma^j\lambda_{ij}^n\rho P_s^j(Z_s^{jj,n}+\eta^i\theta_s^i)|^2\,\mathrm{d}s\\
&\quad + \int_t^{\tau_k} 2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\rho\Big(\sum_{j\neq i}^n \lambda_{ij}^n P_s^j(Z_s^{jj,n}+\eta^j\theta_s^j)\theta_s^j\\
&\qquad\qquad - \mathbb{E}\Big[\int_I P_s^v(Z_s^v+\eta^v\theta_s^v)\theta_s^v G(u,v)\,\mathrm{d}v\Big]\Big)\,\mathrm{d}s\\
&\quad - \int_t^{\tau_k}\kappa\mathrm{e}^{\kappa s}(\Delta Y_s^{i,n})^2\,\mathrm{d}s - \sum_{j=1}^n\int_t^{\tau_k}\mathrm{e}^{\kappa s}|\Delta Z_s^{ij,n}|^2\,\mathrm{d}s\\
&\quad - \sum_{j=1}^n\int_t^{\tau_k} 2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\Delta Z_s^{ij,n}\,\mathrm{d}W_s^j.
\end{aligned}$$

Let us introduce the measure $\mathbb{Q}$ with density

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\Big(&\int_t^{\tau_k}\Big(\theta_s^i + \frac{1}{2\eta^i}\gamma_s(Z_s^{ii,n}, Z_s^{u_i})\Big)\cdot\mathrm{d}W_s^i\\
&+\sum_{j\neq i}^n\int_t^{\tau_k}\big(Z_s^{ij,n} + 2Z_s^{ij,n}\sigma_s^j\lambda_{ij}^n\rho P_s^j(Z_s^{jj,n}+\eta^j\theta_s^j)\big)\cdot\mathrm{d}W_s^j\Big),
\end{aligned}$$

where γ_s is the (linearly growing) function defined by

$$|(\mathrm{Id}-P_s^i)(Z_s^{ii,n}+\eta^i\theta_s^i)|^2 - |(\mathrm{Id}-P_s^{u_i})(Z_s^{u_i}+\eta^{u_i}\theta_s^{u_i})|^2 = \gamma_s(Z_s^{ii,n}, Z_s^{u_i})\Delta Z_s^{ii,n}.$$

The existence of such a function follows by the Lipschitz-continuity of the projection operator since A^j is convex (also recall the rebranding from i to u_i). Thus by Girsanov's theorem, the BMO martingale property of Z^{u_i} and square-integrability of

$Z^{ij,n}$, we have

$$
\begin{aligned}
&\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2\\
&=\mathbb{E}^{\mathbb{Q}}\bigg[\mathrm{e}^{\kappa\tau_k}(\Delta Y_{\tau_k}^{i,n})^2\\
&\qquad+\int_t^{\tau_k} 2e^{\kappa s}\Delta Y_s^{i,n}\sum_{j\neq i}^n \rho^2\frac{\lambda_{ij}^2}{n^2\beta_n^2}|\sigma_s^j|^2|P_s^j(Z_s^{jj,n}+\theta_s^j\eta^j)|^2\,\mathrm{d}s\bigg|\mathcal{F}_t^n\bigg]\\
&\quad+\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k} 2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\rho\bigg(\frac{1}{n\beta_n}\sum_{j\neq i}^n\lambda_{ij}P_s^j(Z_s^{jj,n}+\eta^j\theta_s^j)\theta_s^j\\
&\qquad\qquad-\frac{1}{n\beta_n}\sum_{j\neq i}^n\lambda_{ij}P_s^{u_j}(Z_s^{u_j}+\eta^{u_j}\theta_s^{u_j})\theta_s^{u_j}\bigg)\,\mathrm{d}s\\
&\qquad+\int_t^{\tau_k}\bigg(2\mathrm{e}^{\kappa s}\Delta Y_s^{i,n}\rho\Gamma_s^{i,n}-\kappa e^{\kappa s}(\Delta Y_s^{i,n})^2-\sum_{j=1}^n\mathrm{e}^{\kappa s}|\Delta Z_s^{ij,n}|^2\bigg)\,\mathrm{d}s\bigg|\mathcal{F}_t^n\bigg],
\end{aligned}
$$

where $\Gamma^{i,n}$ is the process given by

$$
\Gamma_s^{i,n}:=\frac{1}{n\beta_n}\sum_{j\neq i}^n\lambda_{ij}P_s^{u_j}(Z_s^{u_j}+\eta^{u_j}\theta_s^{u_j})\theta_s^{u_j}-\mathbb{E}\bigg[\int_I P_s^v(Z_s^v+\eta^v\theta_s^v)\theta_s^v G(u_i,v)\,\mathrm{d}v\bigg].
$$

Using Lipschitz-continuity of the projection operator and boundedness of Σ, we continue the estimation as

$$
\begin{aligned}
&\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2+\mathbb{E}^{\mathbb{Q}}\bigg[\sum_{j=1}^n\int_t^{\tau_k}\mathrm{e}^{\kappa s}|\Delta Z_s^{ij,n}|^2\,\mathrm{d}s\bigg|\mathcal{F}_t^n\bigg]\\
&=\mathbb{E}^{\mathbb{Q}}\bigg[\mathrm{e}^{\kappa\tau_k}(\Delta Y_{\tau_k}^{i,n})^2+\frac{C_{k,\rho,\sigma}}{n^2\beta_n^2}+\int_t^{\tau_k}e^{\kappa s}\bigg(\frac{C_{\theta,\sigma,\rho}}{\varepsilon}-\kappa\bigg)|\Delta Y_s^{i,n}|^2\,\mathrm{d}s\bigg|\mathcal{F}_t^n\bigg]\\
&\quad+\varepsilon\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^{\tau_k}\bigg(\mathrm{e}^{\kappa s}\Big(\frac{1}{n\beta_n}\sum_{j\neq i}^n\lambda_{ij}|\Delta Z_s^{jj,n}|\Big)^2+\mathrm{e}^{\kappa s}|\Gamma_s^{i,n}|^2\bigg)\,\mathrm{d}s\bigg|\mathcal{F}_t^n\bigg],
\end{aligned}
$$

where we also used Young's inequality with some $\varepsilon>0$. Choosing $\kappa>0$ large enough and using the Cauchy–Schwarz inequality, it follows that

$$\mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2+\sum_{j=1}^n\int_t^{\tau_k}\mathrm{e}^{\kappa s}|\Delta Z_s^{ij,n}|^2\,\mathrm{d}s\Big]$$

$$\leq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\kappa\tau_k}(\Delta Y_{\tau_k}^{i,n})^2]+\frac{C_{k,\rho}}{n^2\beta_n^2}$$

$$+\,\varepsilon\mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau_k}\Big(\mathrm{e}^{\kappa s}\frac{1}{n\beta_n^2}\Big(\sum_{j\neq i}\lambda_{ij}^2\Big)\frac{1}{n}\sum_{j\neq i}|\Delta Z_s^{jj,n}|^2+\mathrm{e}^{\kappa s}|\Gamma_s^{i,n}|^2\Big)\,\mathrm{d}s\Big]$$

$$\leq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\kappa\tau_k}(\Delta Y_{\tau_k}^{i,n})^2]+\frac{C_{k,\rho}}{n^2\beta_n^2}+\frac{C_{k,\kappa}}{n\beta_n}+\varepsilon\mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau_k}\mathrm{e}^{\kappa s}|\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big],$$

where we used that λ_{ij} is independent of $W^1,\dots,W^n$ with $\mathbb{E}^{\mathfrak{P}}[\lambda_{ij}^2]\leq\beta_n$ and the definition of the stopping time τ_k. As $(\Delta Y_{\tau_k}^{i,n})_{k\geq1}$ converges to 0 in $\mathbb{P}$-probability and thus in $\mathbb{Q}$-probability for each n, we can find a subsequence $(\Delta Y_{\tau_{k,m}}^{i,n})_{m\geq1}$ such that

$$\mathbb{Q}[|\Delta Y_{\tau_{k,m}}^{i,n}|\geq\varepsilon]\leq\frac{\mathrm{e}^{-k^2}}{m}.$$

Thus for every $\varepsilon>0$, we have

$$\mathbb{E}^{\mathbb{Q}}[|\Delta Y_{\tau_{k,m}}^{i,n}|^2]\leq\varepsilon+k\frac{\mathrm{e}^{-k^2}}{m}.$$

Hence, using again the definition of τ_k,

$$\mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2+\sum_{j=1}^n\int_t^{\tau_{k,m}}\mathrm{e}^{\kappa s}|\Delta Z_s^{ij,n}|^2\,\mathrm{d}s\Big]$$

$$\leq\varepsilon+k\frac{\mathrm{e}^{-k^2}}{m}+\frac{C_{k,\rho,\sigma}}{n\beta_n}+\varepsilon\mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau_k}\mathrm{e}^{\kappa s}|\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big].$$

Using the Cauchy–Schwarz inequality, we further have

$$\mathbb{E}\Big[\mathrm{e}^{\kappa t}|\Delta Y_t^{i,n}|+\int_t^{\tau_{k,m}}\mathrm{e}^{\kappa s}|\Delta Z_s^{ii,n}|\,\mathrm{d}s\Big]$$

$$\leq 2T\mathbb{E}\Big[\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)^2\Big]^{1/2}\mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\kappa t}(\Delta Y_t^{i,n})^2+\int_t^{\tau_{k,m}}\mathrm{e}^{\kappa s}|\Delta Z_s^{ii,n}|^2\,\mathrm{d}s\Big]^{\frac{1}{2}}$$

$$\leq C_k\Big(\varepsilon+k\frac{\mathrm{e}^{-k^2}}{m}+\frac{C_{k,\rho,\sigma}}{n\beta_n}+\varepsilon\mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau_{k,m}}\mathrm{e}^{\kappa s}|\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big]\Big)^{\frac{1}{2}}.$$

We show below that for each fixed k, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau_{k,m}}|\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big]\longrightarrow 0\qquad\text{as } n\to\infty. \tag{5.12}$$

Thus first taking the limit in n, then in m and then letting $\varepsilon \to 0$, it follows that

$$\mathbb{E}\bigg[\mathrm{e}^{\kappa t}|\Delta Y_t^{i,n}| + \int_t^{\tau_{k,m}} \mathrm{e}^{\kappa s}\|\Delta Z_s^{ii,n}\|\,\mathrm{d}s\bigg] \longrightarrow 0 \qquad \text{as } m,n \to \infty.$$

We thus obtain that $\Delta Y_0^{i,n} \to 0$ as $n \to \infty$ and, up to a subsequence,

$$|\Delta Z_s^{ii,n}|1_{\{s \le \tau_{k,m}\}} \longrightarrow 0 \qquad \text{for almost every } s \in [t,T], \mathbb{P}\text{-a.s., as } m,n \to \infty.$$

In particular, because $\mathbb{P}[\tau_{k,m} \ge 1] = 1$, we get $\Delta Z_t^{ii,n} \to 0$ $\mathbb{P}$-a.s. as $n \to \infty$.

Let us now come back to (5.12). Since the random variables $(Z^u)_{u\in I}$ are e.p.i., it follows by the exact law of large numbers, see Sun [41, Corollary 3.10], that

$$|\Gamma_s^{i,n}| \le \bigg|\frac{1}{n\beta_n}\sum_{j\ne i}^n \lambda_{ij} P_s^{u_j}(Z_s^{u_j} + \eta^{u_j}\theta_s^{u_j})\theta_s^{u_j} - \int_I P_s^v(Z_s^v + \eta^v\theta_s^v)\theta_s^v G(u_i,v)\,\mathrm{d}v\bigg|.$$

Therefore, using the triangle inequality and the fact that

$$\int_I F_s^n(v)G_n(u_i,v)(v)\,\mathrm{d}v = \frac{1}{n}\sum_{j=1}^n P_s^{u_j}(Z_s^{u_j} + \eta^{u_j}\theta_s^{u_j})\theta_s^{u_j} G_n(u_i,u_j)$$

with

$$F_s^n(u) := \sum_{j=1}^n P_s^{u_j}(Z_s^{u_j} + \eta^{u_j}\theta_s^{u_j})\theta_s^{u_j} 1_{\{u\in(u_j,u_{j+1}]\}},$$

it follows that

$$\begin{aligned}
|\Gamma_s^{i,n}| \le{}& \bigg|\frac{1}{n\beta_n}\sum_{j=1}^n \lambda_{ij} P_s^{u_j}(Z_s^{u_j} + \eta^{u_j}\theta_s^{u_j})\theta_s^{u_j} \\
&\quad - \frac{1}{n}\sum_{j=1}^n P_s^{u_j}(Z_s^{u_j} + \eta^{u_j}\theta_s^{u_j})\theta_s^{u_j} G_n(u_i,u_j)\bigg| \\
&+ \bigg|\int_I F_s^n(v)G_n(u_i,v)(v)\,\mathrm{d}v - \int_I F_s^n(v)G(u_i,v)(v)\,\mathrm{d}v\bigg| \\
&+ \bigg|\int_I F_s^n(v)G(u_i,v)\,\mathrm{d}v - \int_I P_s^v(Z_s^v + \eta^v\theta_s^v)\theta_s^v G(u_i,v)\,\mathrm{d}v\bigg|.
\end{aligned}$$

Proceeding as in the proof of Lemma 5.4, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_k} |\Gamma_s^{i,n}|^2 \,\mathrm{d}s\Big]\\
&\leq \frac{\mathrm{Var}[\lambda_{ij}]}{n\beta_n^2}\|\theta\|_\infty(\|Z^{u_i}\cdot W^{u_i}\|_{\mathrm{BMO}}+C_{\theta,\eta})\\
&\quad+\mathbb{E}^{\mathbb{Q}}\Big[\int_I\int_0^{\tau_k}|F_s^n(v)|^2\,\mathrm{d}s\big(G_n(u_i,v)-G(u_i,v)\big)^2\,\mathrm{d}v\Big]\\
&\quad+\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_k}\int_I|F_s^n(v)-P_s^v(Z_s^v+\eta^v\theta_s^v)\theta_s^v|^2\,\mathrm{d}v\,\mathrm{d}s\Big]\\
&\leq \frac{C_{\theta,\eta}}{n\beta_n^2}+\int_I\big(G_n(u_i,v)-G(u_i,v)\big)^2\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|F_s^n(v)|^2\,\mathrm{d}s\Big]\mathrm{d}v\\
&\quad+\mathbb{E}\Big[\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)^2\Big]^{1/2}\mathbb{E}\Big[\int_0^T\Big(\int_I|F_s^n(v)-P_s^v(Z_s^v+\eta^v\theta_s^v)\theta_s^v|^2\,\mathrm{d}v\Big)^2\mathrm{d}s\Big]^{1/2}.
\end{aligned}
$$

Since the intervals $(u_j, u_{j+1}]$ form a partition of I and by using the linear growth of the projection operator, it follows that

$$|F_s^n(u)|^2 \leq \sum_{j=1}^n |Z_s^{u_j}|^2 1_{\{u\in(u_j,u_{j+1}]\}} + C_{\theta,A}.$$

Thus using $\|\cdot\|_{\mathbb{H}^2(\mathbb{R}^d,\mathbb{F}^u)} \leq \|\cdot\|_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F}^u)}$, the fact that the BMO-norms under $\mathbb{P}$ and under $\mathbb{Q}$ are equivalent (due to the definition of the stopping time τ_k; see Kazamaki [25, Theorem 3.6]), and further $\sup_{u\in I}\|Z^u\|_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F}^u)} < \infty$, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T |F_s^n(u)|^2\,\mathrm{d}s\Big] \leq \sum_{j=1}^n \|Z^{u_i}\|_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F}^u)} 1_{\{u\in(u_j,u_{j+1}]\}} + C_{\theta,A} \leq C.$$

Hence we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_k}|\Gamma_s^{i,n}|^2\,\mathrm{d}s\Big]\\
&\leq \frac{C_{\theta,\eta}}{n\beta_n^2}+C\int_I\big(G_n(u_i,v)-G(u_i,v)\big)^2\,\mathrm{d}v+C_k\mathbb{E}\Big[\int_0^T\|F_s^n-F_s\|^2_{L^2(I,\mu)}\,\mathrm{d}s\Big]^{1/2},
\end{aligned}
$$

where $F_s(v) := P_s^v(Z_s^v+\eta^v\theta_s^v)\theta_s^v$. Since the sequence $(F^n)_{n\geq 1}$ of step functions converges to F in $L^2(I,\mu)$, it follows by dominated convergence that, fixing k, we have (5.12). The convergence to zero of the term $\int_I(G_n(u_i,v)-G(u_i,v))^2\,\mathrm{d}v$ is proved as at the end of the proof of Lemma 5.4. This concludes the proof. □

6 Well-posedness of graphon McKean–Vlasov BSDEs and FBSDEs

We conclude the article by the two existence results for graphon McKean–Vlasov (F)BSDEs used in the proof of existence of graphon equilibria. In the ensuing statements and proofs, we use the space $\mathbb{S}^p(\mathbb{F},\mathbb{R}^d,I)$ of families of (equivalence classes of) processes $(Y^u)_{u\in I}$ such that $(u,t,\omega)\mapsto Y^u_t(\omega)$ is $\mathcal{B}([0,T])\otimes\mathcal{I}\boxtimes\mathcal{F}$-measurable and for almost every u, we have $Y^u\in\mathbb{S}^p(\mathbb{F}^u,\mathbb{R}^d)$. This space is equipped with the norm

$$\|Y\|_{\mathbb{S}^p(\mathbb{F},\mathbb{R}^d,I)}:=\int_I\|Y^u\|_{\mathbb{S}^p(\mathbb{F}^u,\mathbb{R}^d)}\mathrm{d}u$$

which makes it a Banach space. We similarly define $\mathbb{H}^p(\mathbb{F},\mathbb{R}^d,I)$. We further denote by $\mathbb{H}_{\mathrm{BMO}}(\mathbb{F}^u,\mathbb{R}^d)$ the space of $\mathbb{F}^u$-predictable processes Z with values in $\mathbb{R}^d$ such that the process $\int Z\mathrm{d}W^u$ is a $(\mathbb{P},\mathbb{F}^u)$-BMO martingale. The space $\mathbb{H}_{\mathrm{BMO}}(\mathbb{F},\mathbb{R}^d,I)$ is defined analogously to $\mathbb{S}^p(\mathbb{F},\mathbb{R},I)$ with the norm

$$\|Z\|_{\mathbb{H}_{\mathrm{BMO}}(\mathbb{F},\mathbb{R}^d,I)}:=\int_I\|Z^u\|_{\mathbb{H}_{\mathrm{BMO}}(\mathbb{F}^u,\mathbb{R}^d)}\mathrm{d}u.$$

6.1 Graphon McKean–Vlasov FBSDEs

We start by proving existence of a solution to the graphon McKean–Vlasov FBSDEs with Lipschitz coefficients. Observe that this is a system involving a continuum of coupled equations, where the coupling is due to the graphon term.

Proposition 6.1 *Let $g: I\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, $b,h_2: I\times[0,T]\times\Omega\times\mathbb{R}^{d+1}\to\mathbb{R}$ and $h_1: I\times[0,T]\times\Omega\times\mathbb{R}^{d+1}\to\mathbb{R}^d$ be measurable functions, where the sets I, $[0,T]$, $\mathbb{R}^d$ are equipped with their Borel-σ-algebras, and assume they are Lipschitz-continuous in the sense that for $(t,z,\bar z,z^*,\bar z^*)\in[0,T]\times(\mathbb{R}^d)^2\times\mathbb{R}^2$ and $f\in\{g^u,b^u,h^u_1,h^u_2\}$, we have*

$$|f(t,z,z^*)-f(t,\bar z,\bar z^*)|\le C(|z-\bar z|+|z^*-\bar z^*|)$$

for some $C>0$ which does not depend on (u,ω), where we write $C=:\ell_g$ for $f=g^u$ and $C=:\ell_h$ for $f\in\{b^u,h^u_1,h^u_2\}$. Moreover, we assume that

$$\int_0^1 E\left[\int_0^T|g^u(t,0,0)|^2\mathrm{d}t\right]\mathrm{d}u<\infty. \tag{6.1}$$

Further assume that we are given a family $(\xi^u)_{u\in I}$ such that $\xi\in L^2(\mu\boxtimes\mathbb{P})$. Then if $\rho<\frac{1}{2\ell_h}\mathrm{e}^{-(2\ell_g^2+\frac12)T}$, the graphon system

$$\begin{cases}\mathrm{d}X^u_t &= b^u(t,Z^u_t,Z^{*u}_t)\mathrm{d}t+h^u_1(t,Z^u_t,Z^{*u}_t)\mathrm{d}W^u_t+h^u_2(t,Z^u_t,Z^{*u}_t)\mathrm{d}W^*_t,\\ \mathrm{d}Y^u_t &= -g^u_t(Z^u_t,Z^{*u}_t)\mathrm{d}t+Z^u_t\mathrm{d}W^u_t+Z^{*u}_t\mathrm{d}W^*_t,\\ Y^u_T &= \mathbb{E}\left[\rho\int_I X^v_T G(u,v)\mathrm{d}v\,\middle|\,\mathcal{F}^*_T\right],\\ X^u_0 &= \xi^u\end{cases} \tag{6.2}$$

admits a unique solution $(X^u,Y^u,Z^u,Z^{*u})_{u\in I}$ such that

$$(X^u,Y^u,Z^u,Z^{*u})_{u\in I}\in\mathbb{S}^2(\mathbb{F},\mathbb{R},I)\times\mathbb{S}^2(\mathbb{F},\mathbb{R},I)\times\mathbb{H}^2(\mathbb{F},\mathbb{R}^d,I)\times\mathbb{H}^2(\mathbb{F},\mathbb{R},I).$$

Proof Let $(z^u, z^{*u})_{u\in I} \in \mathbb{H}^2(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}^2(\mathbb{F}, \mathbb{R}, I)$ be a given family of processes and consider $(X^u, Y^u, Z^u, Z^*_u)_{u\in I}$ which satisfy

$$\begin{cases} X^u_t := \xi^u + \int_0^t b(s, z^u_s, z^{*u}_s)\mathrm{d}s \\ \qquad\qquad + \int_0^t h_1(s, z^u_s, z^{*u}_s)\mathrm{d}W^u_s + \int_0^t h_2(s, z^u_s, z^{*u}_s)\mathrm{d}W^*_s, \\ Y^u_t = \mathbb{E}\Big[\rho \int_I X^v_T G(u,v)\mathrm{d}v \Big| \mathcal{F}^*_T\Big] + \int_t^T g^u(s, Z^u_s, Z^{*u}_s)\mathrm{d}s \\ \qquad - \int_t^T Z^u_s \mathrm{d}W^u_s - \int_t^T Z^{*u}_s \mathrm{d}W^*_s. \end{cases} \tag{6.3}$$

It follows by Stricker and Yor [40, Sect. 4] that $(u, t, \omega) \mapsto X^u_t(\omega)$ is measurable and thus that $\int_I X^v_T G(u, v)\mathrm{d}v$ is well defined. Arguing as in the proof of [40, Sect. 4] (in particular using Picard iteration), one establishes that $(u, t, \omega) \mapsto (Y^u_t, Z^u_t, Z^{*u}_t)(\omega)$ is measurable. Moreover, since X^u_T is square-integrable, it follows by the standard result of Pardoux and Peng [37] on Lipschitz BSDEs that (Y^u, Z^u) exists and is unique in $\mathbb{S}^2(\mathbb{F}^u, \mathbb{R}) \times \mathbb{H}^2(\mathbb{F}^u, \mathbb{R}^d) \times \mathbb{H}^2(\mathbb{F}^u, \mathbb{R})$ for almost every u. Thus the function

$$\Psi\big((z^u, z^{*u})_{u\in I}\big) := (Z^u, Z^{*u})_{u\in I}$$

maps the Banach space $\mathbb{H}^2(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}^2(\mathbb{F}, \mathbb{R}, I)$ into itself. This follows from the fact that the $\mathbb{H}^2(\mathbb{F}^u, \mathbb{R}^{d+1})$-norm of $(Z^u, Z^{*,u})$ depends only on the second moment of $\mathbb{E}[\rho \int_I X^v_T G(u, v)\mathrm{d}v|\mathcal{F}^*_T]$ as well as of $\mathbb{E}[\int_0^T |g^u(t, 0, 0)\mathrm{d}t|^2]$. Thus integrating over u and using (6.1) guarantees the claim. It remains to show that Ψ admits a unique fixed point.

Let $(z^u, z^{*u})_{u\in I}, (\bar z^u, \bar z^{*u})_{u\in I} \in \mathbb{H}^2(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}^2(\mathbb{F}, \mathbb{R}, I)$ be given. Put

$$\Psi\big((z^u, z^{*u})_{u\in I}\big) = (Z^u, Z^{*u})_{u\in I}, \qquad \Psi\big((\bar z^u, \bar z^{*u})_{u\in I}\big) = (\bar Z^u, \bar Z^{*u})_{u\in I},$$

where $(X^u, Y^u, Z^u, Z^*_u)_{u\in I}$ and $(\bar X^u, \bar Y^u, \bar Z^u, \bar Z^*_u)_{u\in I}$ both satisfy (6.3). Introduce the shorthand notations $\Delta X^u := X^u - \bar X^u$, $\Delta Y^u := Y^u - \bar Y^u$, $\Delta Z^u := Z^u - \bar Z^u$ and $\Delta Z^{*u} := Z^{*u} - \bar Z^{*u}$. Given some constant $\kappa > 0$, we apply Itô's formula to $\mathrm{e}^{\kappa t}|\Delta Y^u_t|^2$ to obtain

$$\begin{aligned} \mathrm{e}^{\kappa t}|\Delta Y^u_t|^2 \le{}& \mathrm{e}^{\kappa T}\rho^2 \mathbb{E}\Big[\int_I |\Delta X^v_T|^2 G(u,v)^2 \mathrm{d}v \Big| \mathcal{F}^*_T\Big] + \int_t^T \mathrm{e}^{\kappa s}\Big(2\frac{\ell_g^2}{\varepsilon} - \kappa\Big)|\Delta Y^u_s|^2 \\ & + (\varepsilon - 1)\int_t^T \mathrm{e}^{\kappa s}(|\Delta Z^u_s|^2 + |\Delta Z^{*u}_s|^2)\mathrm{d}s \\ & - \int_t^T \mathrm{e}^{\kappa s}\Delta Y^u_s \Delta Z^u_s \mathrm{d}W^u_s - \int_t^T \mathrm{e}^{\kappa s}\Delta Y^u_s \Delta Z^{*u}_s \mathrm{d}W^*_s. \end{aligned}$$

Taking expectations on both sides and choosing $\kappa = 2\ell_g^2/\varepsilon$, we have

$$\mathbb{E}\bigg[\mathrm{e}^{\kappa t}|\Delta Y_t^u|^2 + (1-\varepsilon)\int_t^T (\mathrm{e}^{\kappa s}|\Delta Z_s^u|^2 + \mathrm{e}^{\kappa s}|\Delta Z_s^{*u}|^2)\mathrm{d}s\bigg]$$
$$\le \mathrm{e}^{\kappa T}\rho^2 \int_I \mathbb{E}[|\Delta X_T^v|^2]\mathrm{d}v.$$

Applying Itô's formula to $\mathrm{e}^{\kappa t}|\Delta X_t^u|^2$ and using Lipschitz-continuity of b^u, h_1^u and h_2^u, we obtain via Young's inequality that

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{\kappa t}|\Delta X_t^u|^2]\\
&\le \mathbb{E}\bigg[\int_0^t \big(2\mathrm{e}^{\kappa s}\ell_h(|z_s^u-\bar z_s^u| + |z_s^{*u}-\bar z_s^{*u}|)|\Delta X_s^u|\\
&\qquad\qquad + \ell_h^2\mathrm{e}^{\kappa s}(|z_s^u-\bar z_s^u|^2 + |z_s^{*u}-\bar z_s^{*u}|^2)\big)\mathrm{d}s\bigg] + \kappa\mathbb{E}\bigg[\int_0^t \mathrm{e}^{\kappa s}|\Delta X_s^u|^2\,\mathrm{d}s\bigg]\\
&\le \mathbb{E}\bigg[(1+\kappa)\int_0^t \mathrm{e}^{\kappa s}|\Delta X_s^u|^2\mathrm{d}s\bigg] + 2\ell_h^2\mathbb{E}\bigg[\int_0^t \mathrm{e}^{\kappa s}(|z_s^u-\bar z_s^u|^2 + |z_s^{*u}-\bar z_s^{*u}|^2)\mathrm{d}s\bigg].
\end{aligned}$$

Thus by Gronwall's inequality, we have

$$\mathbb{E}[\mathrm{e}^{\kappa t}|\Delta X_t^u|^2] \le 2\ell_h^2\mathrm{e}^{(\kappa+1)T}\mathbb{E}\bigg[\int_0^t \mathrm{e}^{\kappa s}(|z_s^u-\bar z_s^u|^2 + |z_s^{*u}-\bar z_s^{*u}|^2)\mathrm{d}s\bigg].$$

So if $\varepsilon = 1/2$, we get

$$\int_I \mathbb{E}\bigg[\int_0^T \mathrm{e}^{\kappa s}|\Delta Z_s^u|^2 + \mathrm{e}^{\kappa s}|\Delta Z_s^{*u}|^2\mathrm{d}s\bigg]\mathrm{d}u$$
$$\le 4\ell_h^2\mathrm{e}^{(4\ell_g^2+1)T}\rho^2\int_I \mathbb{E}\bigg[\int_0^T \mathrm{e}^{\kappa s}(|z_s^u-\bar z_s^u|^2 + |z_s^{*u}-\bar z_s^{*u}|^2)\mathrm{d}s\bigg]\mathrm{d}u.$$

Thus by the choice of ρ and the Banach fixed point theorem, the mapping Ψ admits a unique fixed point, implying that the graphon FBSDE (6.2) admits a unique solution in $\mathbb{S}^2(\mathbb{F},\mathbb{R},I)\times\mathbb{S}^2(\mathbb{F},\mathbb{R},I)\times\mathbb{H}^2(\mathbb{F},\mathbb{R}^d,I)\times\mathbb{H}^2(\mathbb{F},\mathbb{R},I)$. □

6.2 Graphon McKean–Vlasov BSDE

Let us now turn to the well-posedness of graphon McKean–Vlasov BSDEs with Lipschitz-continuous coefficients.

Proposition 6.2 *Assume that $g: I\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ and $f: I\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ are Borel-measurable functions satisfying the local and global Lipschitz-continuity conditions, with some constants $\ell_g,\ell_f>0$ that do not depend on (u,ω),*

$$|g^u(t,z)-g^u(t,z')| \le \ell_g(|z|+|z'|)|z-z'|, \qquad |g^u(t,z)| \le \ell_g(1+|z|^2)$$

and

$$|f^u(t,z) - f^u(t,z')| \le \ell_f |z - z'|, \qquad |f^u(t,z)| \le \ell_f(1 + |z|)$$

for every $(t, z, z') \in [0, T] \times (\mathbb{R}^d)^2$ *and almost all* $u \in I$. *Further assume that we are given* $\mathcal{F}^u_T$*-measurable random variables* F^u *such that* $(u, \omega) \mapsto F^u(\omega)$ *is measurable and uniformly bounded. Then the graphon system*

$$Y^u_t = F^u + \int_t^T \left(g^u(s, Z^u_s) + \int_I \mathbb{E}[f^v(s, Z^v_s)] G(u, v) \mathrm{d}v \right) ds - \int_t^T Z^u_s \mathrm{d}W^u_s$$

admits a unique solution $(Y^u, Z^u)_{u \in I}$ *in* $\mathbb{S}^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}, \mathbb{R}^d, I)$ *and such that* $\sup_{u \in I} \|Z^u\|_{\mathbb{H}_{\mathrm{BMO}}(\mathbb{F}, \mathbb{R}^d)} < \infty$.

Proof Let $(y^u, z^u)_{u \in I} \in \mathbb{S}^\infty(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}, \mathbb{R}^d, I)$ be given and consider the (decoupled) quadratic BSDEs

$$Y^u_t = F^u + \int_t^T \left(g^u(s, Z^u_s) + \int_I \mathbb{E}[f^v(s, z^v_s)] G(u, v) \mathrm{d}v \right) ds - \int_t^T Z^u_s \mathrm{d}W^u_s.$$

It follows by Hu et al. [22] that for almost every $u \in I$, this equation admits a unique solution $(Y^u, Z^u) \in \mathbb{S}^\infty(\mathbb{F}^u, \mathbb{R}) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}^u, \mathbb{R}^d)$. Moreover, it follows by the arguments of Stricker and Yor [40, Sect. 4] that $(u, t, \omega) \mapsto (Y^u_t, Z^u_t)(\omega)$ is measurable. Thus the function

$$\Psi\big((y^u, z^u)_{u \in I}\big) := (Y^u, Z^u)_{u \in I}$$

is well defined and maps the Banach space $\mathbb{S}^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}, \mathbb{R}^d, I)$ into itself. This follows because by Huang et al. [23], the $\mathbb{S}^\infty(\mathbb{F}^u, \mathbb{R}) \times \mathbb{F}_{\mathrm{BMO}}(\mathbb{F}^u, \mathbb{R}^d)$-norm of (Y^u, Z^u) depends only on ℓ_f, ℓ_g and the bound of F^u. Since these constants are uniform in u, this yields the claim. It therefore remains to show that this mapping admits a unique fixed point.

Let $(y^u, z^u)_{u \in I}, (\bar{y}^u, \bar{z}^u)_{u \in I} \in \mathbb{S}^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}_{\mathrm{BMO}}(\mathbb{F}, \mathbb{R}^d, I)$ be given. Put

$$\Psi\big((y^u, z^u)_{u \in I}\big) = (Y^u, Z^u)_{u \in I}, \qquad \Psi\big((\bar{y}^u, \bar{z}^u)_{u \in I}\big) = (\bar{Y}^u, \bar{Z}^u)_{u \in I}.$$

Let $\kappa > 0$ be a constant to be determined and τ an $\mathbb{F}^u$-stopping time. Apply Itô's formula to $e^{\kappa t}|\Delta Y_t^u|^2 := e^{\kappa t}|Y_t^u - \bar{Y}_t^u|^2$ to obtain

$$\begin{aligned}
e^{\kappa\tau}|\Delta Y_\tau^u|^2 = \int_\tau^T 2e^{\kappa s}\Delta Y_s^u\Bigg(&g^u(s, Z_s^u) - g^u(s, \bar{Z}_s^u) \\
&+ \int_I \mathbb{E}[f^v(s, z_s^v) - f^v(s, \bar{z}_s^v)]G(u, v)\mathrm{d}v\Bigg)\mathrm{d}s \\
&- \kappa\int_\tau^T e^{\kappa s}|\Delta Y_s^u|^2\mathrm{d}s - \int_\tau^T e^{\kappa s}|\Delta Z_s^u|^2\mathrm{d}s - \int_t^T 2e^{\kappa s}\Delta Y_s^u\Delta Z_s^u\mathrm{d}W_s^u \\
\le \left(\frac{1}{\varepsilon} - \kappa\right)&\int_\tau^T e^{\kappa s}|\Delta Y_s^u|^2\mathrm{d}s + \varepsilon\ell_f^2\int_\tau^T e^{\kappa s}\int_I \mathbb{E}[|\Delta z_s^u|^2]G(u, v)^2\mathrm{d}s \\
&- \int_\tau^T e^{\kappa s}|\Delta Z_s^u|^2\mathrm{d}s - \int_\tau^T 2e^{\kappa s}\Delta Y_s^u\Delta Z_s^u\mathrm{d}W_s^{u,\mathbb{Q}},
\end{aligned}$$

where we used the shorthand notations $\Delta Z^u := Z^u - \bar{Z}^u$ and $\Delta z^u := z^u - \bar{z}^u$ and $W^{u,\mathbb{Q}}$ is a Brownian motion under the probability measure defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(\int_0^\cdot \beta^u(s, Z_s^u, \bar{Z}_s^u)\mathrm{d}W_s^u\right)$$

for a linearly growing function β such that $g^u(s, z) - g^u(s, \bar{z}) = \beta^u(s, z, \bar{z}) \cdot (z - \bar{z})$. Choose $\kappa > \frac{1}{\varepsilon}$. Taking conditional expectations on both sides yields

$$e^{\kappa\tau}|\Delta Y_\tau^u|^2 + \mathbb{E}^{\mathbb{Q}}\left[\int_\tau^T e^{\kappa s}|\Delta Z_s^u|^2\mathrm{d}s\middle|\mathcal{F}_\tau^u\right] \le \varepsilon\ell_f^2\int_I \mathbb{E}\left[\int_\tau^T e^{\kappa s}|\Delta z_s^v|^2\mathrm{d}s\middle|\mathcal{F}_\tau^u\right]\mathrm{d}v.$$

Taking the supremum over τ and integrating on both sides in u therefore gives

$$\|\Delta Y\|^2_{\mathbb{S}^\infty(\mathbb{R},\mathbb{F},I)} + \|\Delta Z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F},I)} \le \varepsilon C\ell_f^2\|\delta z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F},I)},$$

where we used as before that the BMO-norms under $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and that $\|\cdot\|_{\mathbb{H}^2(\mathbb{R}^d,\mathbb{F})} \le \|\cdot\|^2_{\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d,\mathbb{F})}$. Choosing $\varepsilon > 0$ small enough allows to conclude that Ψ is a contraction, and thus it follows by the Banach fixed point theorem that Ψ admits a unique fixed point in $\mathbb{S}^\infty(\mathbb{R}, \mathbb{F}, I) \times \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^d, \mathbb{F}, I)$. □

Declarations

Competing Interests The authors declare no competing interests.

References

1. Anthropelos, M., Geng, T., Zariphopoulou, T.: Competition in fund management and forward relative performance criteria. SIAM J. Financ. Math. **13**, 1271–1301 (2022)
2. Aurell, A., Carmona, R., Laurière, M.: Stochastic graphon games: II. The linear-quadratic case. Appl. Math. Optim. **85**, 39 (2022)

3. Bartlett, M.: An inverse matrix adjustment arising in discriminant analysis. J. Math. Stat. **22**, 107–111 (1951)
4. Bayraktar, E., Chakraborty, S., Wu, R.: Graphon mean field systems. Ann. Appl. Probab. **33**, 3587–3619 (2023)
5. Bayraktar, E., Wu, R.: Graphon particle system: uniform-in-time concentration bounds. In: Stochastic Processes and Their Applications, vol. 156, pp. 196–225 (2023)
6. Bayraktar, E., Wu, R., Zhang, X.: Propagation of chaos of forward-backward stochastic differential equations with graphon interactions. Appl. Math. Optim. **88**, 25 (2023)
7. Caines, P.E., Huang, M.: Graphon mean field games and the GMFG equations. In: 2018 IEEE 57th Conference on Decision and Control (CDC), pp. 4129–4134 (2018)
8. Caines, P.E., Huang, M.: Graphon mean field games and the GMFG equations: ε-Nash equilibria. In: 2019 IEEE 58th Conference on Decision and Control (CDC), pp. 286–292. IEEE Press, New York (2019)
9. Carmona, R., Cooney, D.B., Graves, C.V., Laurière, M.: Stochastic graphon games: I. The static case. Math. Oper. Res. **47**, 750–778 (2022)
10. Carmona, R., Delarue, F.: Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs. Control, and Games. Springer, Cham (2018)
11. Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M., Stricker, C.: Exponential hedging and entropic penalties. Math. Finance **12**, 99–123 (2002)
12. Dos Reis, G., Platonov, V.: Forward utility and market adjustments in relative investment–consumption games of many players. SIAM J. Financ. Math. **13**, 844–876 (2022)
13. Dos Reis, G., Platonov, V.: Forward utilities and mean-field games under relative performance concerns. In: Bernardin, C., et al. (eds.) From Particle Systems to Partial Differential Equations (International Conference, Particle Systems and PDEs VI, VII and VIII, vol. 2017–2019, pp. 227–251. Springer, Cham (2021)
14. Espinosa, G.E.: Stochastic Control Methods for Optimal Portfolio Investment. PhD Thesis, École Polytechnique, Palaiseau (2010). French version available online at https://pastel.hal.science/pastel-00512703/document
15. Espinosa, G.E., Touzi, N.: Optimal investment under relative performance concerns. Math. Finance **25**, 221–257 (2013)
16. Frei, C.: Splitting multidimensional BSDEs and finding local equilibria. Stoch. Process. Appl. **124**, 2654–2671 (2014)
17. Frei, C., dos Reis, G.: A financial market with interacting investors: does an equilibrium exist? Math. Financ. Econ. **4**, 161–182 (2011)
18. Fu, G., Su, X., Zhou, C.: Mean field exponential utility game: a probabilistic approach (2020). Preprint. Available online at https://arxiv.org/abs/2006.07684
19. Gao, S., Caines, P.E., Huang, M.: LQG graphon mean field games: graphon invariant subspaces. In: 2021 60th IEEE Conference on Decision and Control (CDC), pp. 5253–5260. IEEE Press, New York (2021)
20. Hodges, S., Neuberger, A.: Optimal replication of contingent claims under transaction costs. J. Futures Mark. **8**, 222–239 (1989)
21. Hu, R., Zariphopoulou, T.: N-player and mean-field games in Itô-diffusion markets with competitive or homophilous interaction. In: Yin, G., Zariphopoulou, T. (eds.) Stochastic Analysis, Filtering, and Stochastic Optimization, pp. 209–237. Springer, Berlin (2022)
22. Hu, Y., Imkeller, P., Müller, M.: Utility maximization in incomplete markets. Ann. Appl. Probab. **15**, 1691–1712 (2005)
23. Huang, M., Caines, P., Malhamé, R.: An invariance principle in large population stochastic dynamic games. J. Syst. Sci. Complex. **20**, 162–172 (2007)
24. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, Berlin (1991)
25. Kazamaki, N.: Continuous Exponential Martingales and BMO. Lecture Notes in Mathematics, vol. 1579. Springer, Berlin (1994)
26. Lacker, D., Soret, A.: Many-player games of optimal consumption and investment under relative performance criteria. Math. Financ. Econ. **14**, 263–281 (2020)
27. Lacker, D., Soret, A.: A label-state formulation of stochastic graphon games and approximate equilibria on large networks. Math. Oper. Res. **48**, 1987–2018 (2023)
28. Lacker, D., Zariphopoulou, T.: Mean field and n-agent games for optimal investment under relative performance criteria. Math. Finance **29**, 1003–1038 (2019)
29. Lacker, D., Shkolnikov, M., Zhang, J.: Superposition and mimicking theorems for conditional McKean–Vlasov equations. J. Eur. Math. Soc. **25**, 3229–3288 (2023)

30. Lasry, J.M., Lions, P.L.: Mean field games. Jpn. J. Math. **2**, 229–260 (2007)
31. Laurière, M., Tangpi, L.: Backward propagation of chaos. Electron. J. Probab. **27**, 1–30 (2022)
32. Laurière, M., Tangpi, L.: Convergence of large population games to mean field games with interaction through the controls. SIAM J. Math. Anal. **54**, 3535–3574 (2022)
33. Lovász, L.: Large Networks and Graph Limits. American Mathematical Soc., Providence (2012)
34. Lovász, L., Szegedy, B.: Limits of dense graph sequences. J. Comb. Theory, Ser. B **96**, 933–957 (2006)
35. Luo, P., Tangpi, L.: Laplace principle for large population games with control interaction (2021). Preprint. Available online at https://arxiv.org/abs/2102.04489
36. Musiela, M., Zariphopoulou, T.: Optimal asset allocation under forward exponential performance criteria. In: Stewart, N.E., et al. (eds.) Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz, pp. 285–300. IMS (2008)
37. Pardoux, E., Peng, S.: Adapted solution of a backward stochastic differential equation. Syst. Control Lett. **14**, 55–61 (1990)
38. Parise, F., Ozdaglar, A.: Graphon games. In: Proceedings of the 2019 ACM Conference on Economics and Computation, pp. 457–458 (2019)
39. Possamaï, D., Tangpi, L.: Non-asymptotic convergence rates for mean-field games: weak formulation and McKean–Vlasov BSDEs (2021). Preprint. Available online at https://arxiv.org/abs/2105.00484
40. Stricker, C., Yor, M.: Calcul stochastique dépendant d'un paramètre. Z. Wahrscheinlichkeitstheor. Verw. Geb. **45**, 109–133 (1978)
41. Sun, Y.: The exact law of large numbers via Fubini extension and characterization of insurable risks. J. Econ. Theory **126**, 31–69 (2006)