Queueing Systems (2022) 100:509–511
https://doi.org/10.1007/s11134-022-09835-x

Some open problems in exact simulation of stochastic differential equations

Jose H. Blanchet[1]

Received: 6 February 2022 / Accepted: 28 February 2022 / Published online: 20 May 2022

1 Introduction

A general d-dimensional SDE driven by Brownian motion takes the form (see [7]):

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t), \quad X(0) = x_0, \tag{1}$$

where $B(\cdot)$ is an m-dimensional standard Brownian motion, $\mu(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a drift vector field and $\sigma(\cdot) : \mathbb{R}^{d\times m} \to \mathbb{R}^d$ is a diffusion matrix. While these models are flexible and frequently used in engineering and science, they can also be analytically challenging to work with, so Monte Carlo simulation provides a natural approach to addressing the computational demands of modeling with SDEs. Diffusions are often motivated in queueing using a heavy-traffic approximation perspective. Regardless of how a model is motivated, once it is adopted, the question of efficient computation (for performance analysis or calibration) is essential to exploit the power of such a model.

Designing an efficient Monte Carlo strategy to sample *exactly* the solution of the SDE at a particular time, say $X(1)$, is an important topic in the theory of stochastic simulation, and it is precisely the topic of this note. Naturally, one can use a discretization such as the Euler scheme to approximate $X(1)$. These schemes generally produce a biased estimator. This bias can be reduced at a cost that explodes as the degree of precision required increases. Other path-dependent quantities are also of interest, but they can often be reduced to sampling a diffusion. For example, the running maximum follows a diffusion with a local-time-like constraint in the SDE. These constraints arise in queueing and are also a part of our later discussion.

An advantage of having an exact and efficient Monte Carlo strategy is that the estimator can be implemented easily in parallel. By doing this, we can reduce computational wall-clock time while guaranteeing any given error by just sampling over a

✉ Jose H. Blanchet
jose.blanchet@stanford.edu

[1] Department of Management Science and Engineering, Institute of Computational and Mathematical Engineering, Stanford University, Stanford, USA

large number of parallel computing cores. In contrast, it is not easy to directly reduce the wall-clock time by simply increasing the number of parallel cores using a biased Monte Carlo strategy because of the exploding cost mentioned earlier.

Our goal is to discuss the problem of designing reasonably general Monte Carlo algorithms for the exact generation of $X(1)$ *with finite expected termination time.*[1]

2 Discussion

The first algorithm for exact sampling of SDEs in $d = 1$ appeared in [2], under suitable boundedness assumptions. These assumptions were significantly weakened in [1] and further relaxed using a localization technique in [6], but using essentially the same sampling strategy as in [2]. The overall strategy in [2] relies on two essential properties which are non-restrictive in $d = 1$, but are very restrictive in the multidimensional setting. The first is the existence of an invertible transformation (i.e., the Lamperti transformation) which, when applied to the process $X(\cdot)$, results in a diffusion process with a constant diffusion matrix. Moreover, after this transformation is performed, the resulting drift coefficient of the transformed diffusion must be of gradient form.

So, the strategy introduced in [2], together with the extensions in [1] and [6], applies in the d-dimensional setting basically to diffusions of the form:

$$dX(t) = \bigtriangledown u(X(t))\,dt + dB(t)\,;\;\; X(0) = x_0, \tag{2}$$

where $\bigtriangledown u(\cdot)$ is the gradient of some function $u(\cdot)$.

In the queueing setting, there are diffusion processes of interest arising in the study of many server queues with phase-type service times which have a constant diffusion coefficient but with a drift function $\mu(\cdot)$ which is not of gradient form. This already leads to a problem of significant interest in queueing theory, namely *designing an exact sampling algorithm with finite expected termination time for* (1) *when* $\sigma(x) = \sigma$ *is a constant matrix and* $\mu(\cdot)$ *is a Lipschitz function.*

The only algorithm that exists to date for multidimensional diffusions under reasonably general assumptions was proposed in [5]. This algorithm terminates in finite time with probability one, but its expected termination time is infinite. However, it is worth contrasting the strategy developed in [5] with the one in [2].

Both strategies ultimately rely on acceptance/rejection. However, in the context of (2), there is an obvious sampler that can be used as a proposal distribution, namely Brownian motion. The likelihood ratio of the solution to (2) up to $t = 1$ with respect to Brownian motion takes the form:

$$M(1) = \exp\left(\int_0^1 \bigtriangledown u(X(t))\,dX(t) - \frac{1}{2}\int_0^1 \|\bigtriangledown u(X(t))\|_2^2\,dt\right). \tag{3}$$

Applying Ito's lemma, one can rewrite the stochastic integral involving $\bigtriangledown u(X(t))$ in terms of a Riemann integral involving $\triangle u(X(t))$ (i.e., the Laplacian of $u(\cdot)$).

[1] We are vague with the model of computation, but generally a little common sense can be used to interpret the running time in terms of basic random variables needed to implement the algorithm.

Moreover, if $\triangle u(\cdot)$ and $\triangledown u(\cdot)$ are bounded, then the acceptance/rejection step can be executed by interpreting the Riemann integrand as a stochastic Poisson process intensity. Hence, as noted in [2], accepting the Brownian proposal, $X(1)$, boils down to checking if the first arrival of a suitable Poisson process with a stochastic intensity occurs before time 1. This can be done in finite expected time.

The key idea in the strategy of [5] can be explained in the context in which $\mu(\cdot)$ is not the gradient of any function but with a constant diffusion coefficient. The key idea, based on *strong simulation* (to be discussed), is extended in [5] to the non-constant diffusions and in [4] to diffusions with reflection often arising in queueing. In the constant diffusion case, as indicated earlier, we can use as a proposal Brownian motion. However, it is no longer possible to rewrite the stochastic integral in (3) as a Riemann integral. So, [5] propose a sequence of piecewise constant stochastic processes which approximate the stochastic integral within an error of size at most $1/n$ with probability 1. (Here, n is the n-th iteration in the sequence of approximations.) These approximations are called "$1/n$-strong simulation" estimators. The acceptance/rejection step then can be executed in finite time because this step is ultimately decided by comparing two random variables, namely the likelihood ratio, on the one hand, and an independent uniform random variable (suitably scaled), on the other. The chance of the algorithm running forever is zero because the uniform distribution is independent of the likelihood ratio, $M(1)$ defined in (3), so a tie occurs with zero probability.

Unfortunately, the computational cost of producing the $1/n$-strong approximation is of order $O\left(n^2\right)$ (ignoring logarithmic factors). For a finite expected termination time, a cost smaller than $o(n)$ is required. The approximation used is based on [3], which in turn uses the theory of rough paths. The general diffusion case is more involved, but a similar bottleneck in running time arises because of the $1/n$-strong approximation scheme developed in [3].

The acceptance/rejection strategy in [2] is based on a stochastic representation and additional randomization (via Poisson thinning). The strategy in [5] may benefit from including additional randomization, auxiliary processes or stronger couplings.

References

1. Beskos, A., Papaspiliopoulos, O., Roberts, G.O.: Retrospective exact simulation of diffusion sample paths with applications. Bernoulli **12**(6), 1077–1098 (2006)
2. Beskos, A., Roberts, G.O.: Exact simulation of diffusions. Ann. Appl. Probab. **15**(4), 2422–2444 (2005)
3. Blanchet, J., Chen, X., Dong, J.: ε-Strong simulation for multidimensional stochastic differential equations via rough path analysis. Ann. Appl. Probab. **27**(1), 275–336 (2017)
4. Blanchet, J., Murthy, K.: Exact simulation of multidimensional reflected Brownian motion. J. Appl. Probab. **55**(1), 137–156 (2018)
5. Blanchet, J., Zhang, F.: Exact simulation for multivariate Itô diffusions. Adv. Appl. Probab. **52**(4), 1003–1034 (2020)
6. Chen, N., Huang, Z.: Localization and exact simulation of Brownian motion-driven stochastic differential equations. Math. Oper. Res. **38**(3), 591–616 (2013)
7. Stroock, D., Varadhan, S.: Multidimensional Diffusion Processes. Grundlehren der mathematischen Wissenschaften. Springer, Berlin (1997)