Optimal Dynamic Mechanism Design With Stochastic Supply and Flexible Consumers

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***Abstract*—In this article, we consider the problem of designing an expected-revenue-maximizing mechanism for allocating multiple nonperishable goods of k varieties to flexible consumers over T time steps. In our model, a random number of goods of each variety may become available to the seller at each time, and a random number of consumers may enter the market at each time. Each consumer is present in the market for one time step and wants to consume one good of one of its desired varieties. Each consumer is associated with a flexibility level that indicates the varieties of goods it is equally interested in. A consumer's flexibility level and the utility it gets from consuming a good of its desired varieties are its private information. We characterize the allocation rule for a Bayesian-incentive-compatible, individually rational, and expected-revenue-maximizing mechanism in terms of the solution to a dynamic program. The corresponding payment function is also specified in terms of the optimal allocation function. We leverage the structure of the consumers' flexibility model to simplify the dynamic program. Our simplified dynamic program allows us to provide an explicit allocation procedure and a simple payment rule in terms of the solution of the dynamic program.**

***Index Terms*—Bayesian incentive compatibility, dynamic mechanism design, flexible demand, optimal mechanism, revenue maximization.**

I. Introduction

CONSIDER the scenario faced by a monopolist seller with multiple resources who wants to allocate them to consumers over time in order to maximize its expected total revenue. The seller offers goods of different varieties, and there may be new additions to its stock of each variety over time. Different consumers may interact with the market at various points in time, each for a limited duration. Such a scenario arises in many marketplaces where the available supply and the population of the consumers vary in an uncertain fashion over time. In cloud computing platforms [1], for example, the computational and data storage resources get freed up with the termination of previously submitted jobs and are dedicated to processing of the new tasks dynamically over time. In power distribution networks with partial reliance on the renewable energy resources [2], the energy supply varies over time depending on the availability of the intermittent source of energy; the amount of power demanded by the consumer base connected to the grid is also uncertain and constantly fluctuates over time. In wireless spectrum management platforms [3], the available spectrum bands are leased to secondary wireless service providers (WSPs) for temporary usage and are freed up as the interim lease contracts terminate dynamically over time. One particular feature that makes these resource allocation problems challenging is that in the face of uncertainty about the future supply and demand, the seller needs to decide whether to use its limited resources to serve a currently present consumer or keep them for potentially more profitable transactions in future. Moreover, in order to decide about the optimal way of allocating its resources, the seller needs the information about the consumers' preferences and restrictions, and their willingness to pay for their desired goods or services. Typically, however, this information is known privately to each consumer, and the seller needs to *elicit* these data from them. Since the consumers are self-interested and strategic, they may distort their privately held information when communicating it to the seller if they believe that they can benefit from such misrepresentations. The seller, thus, needs to *incentivize* the consumers to disclose their private information. The theory of mechanism design provides a systematic framework for designing the rules of interaction between multiple strategic agents in a way that the principal decision maker's desired outcome emerges at the equilibrium of the induced game.

Auctions as a special class of mechanisms have been extensively studied in the context of mechanism design, and they are being adopted in an increasing number of markets for pricing and selling various products and services. While the theory of auction design is well developed under static settings, its extension to dynamic markets that involve allocation and pricing of time-varying supply to accommodate time-varying demand under incomplete information is generally less mature and is still an active area of research [4]. Given the growing practical interest in auctions for allocating and pricing resources in dynamically operating markets, a deeper understanding of the design and implementation of such auctions is crucial.

In this article, we study the problem of designing expected-revenue-maximizing auctions for selling indivisible and durable goods of k varieties to consumers over a discrete finite-time

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horizon. Additional units of each variety may become available to the seller at each time step. In our setup, each consumer is associated with a flexibility level, which indicates the varieties of goods that the consumer finds equally desirable. Formally, the flexibility level of a consumer is a number in the range $1, \ldots, k$ such that a consumer with flexibility level j wants to get a good of any of the first j varieties. Each consumer is present in the market for one time period and wants to receive one good (of any of the desired varieties) prior to its departure. The flexibility level and the valuation a consumer has for a desirable good are both its private information.

There are several markets with a temporally fluctuating consumer base where the flexibility in demand described above arises. We describe two such scenarios as follows.

1) *Dynamic spectrum management in cognitive radio networks:* With the emergence of various wireless applications for mobile users, there has been a significant increase in the demand for radio frequency spectrum in recent years. While most of the available radio spectrum has already been licensed off to the existing WSPs, they are not being fully utilized by their primary owners. As pointed out in [3], dynamic spectrum access protocols can enable efficient use of these underutilized frequency bands—referred to as *spectrum holes* in [3]—by accommodating the demands of secondary users who can use these bands on a temporary basis. Cognitive radio systems can detect the presence of such spectrum holes in the frequency bands owned by a primary user. Consider the problem faced by one such spectrum owner who wants to allocate its underutilized frequency bands of various widths, as they become available over time, to secondary WSPs who have different minimum bandwidth requirements. Suppose that the primary owner has frequency bands of widths $w_1, \ldots, w_k$ such that $w_1 > w_2 > \cdots > w_k$. We say that a WSP is of flexibility level j if it requires a frequency band of width at least w_j. At the beginning of each time step t, a random number of WSPs arrive into the market to compete for the limited radio frequency bands available at that time. The resource allocation problem that a primary spectrum owner encounters when it aims to allocate its intermittently available frequency bands to secondary users' temporary usage can be modeled within the framework developed in this article.
2) *Allocation of computational resources in cloud computing platforms:* Consider Amazon's EC2 cloud computing platform that sells various types of computational resources, such as memory, CPU, storage capacity, computer applications, etc. In this market, clients can randomly enter and depart over time. Clients rent virtual machines or *instances* and are typically charged on an hourly basis per instance depending on the duration of their usage as well as the rented instance type. Each of the instance types are offered in different sizes to suit various purposes. As explained in [1], for example, Amazon's EC2 offers "standard" instances in three sizes: small, large, and extra large. A consumer can belong to one of the following three flexibility classes:
   a) *inflexible consumers* who need an extra-large instance;
   b) *somewhat flexible consumers* who need a large or extra-large instance;
   c) *flexible consumers* who are fine with receiving any of the three types of instances.

The allocation of computing instances to consumers of differing flexibilities can be modeled by our setup.

A. Prior Work

Much of the prior work in the area of dynamic auctions can be grouped in two categories [5]: 1) markets with a dynamic population of consumers whose private information remains unchanged over time and 2) markets with a fixed population of consumers whose private information evolves over time. Within each of these two categories, the important findings on efficiency (social-welfare maximization) and optimality (revenue maximization) as the two primary objectives are highlighted in [5]. Our work falls under the first category (dynamic population) above with the focus on revenue maximization as the main objective. Therefore, we will focus on comparing our model with prior works that have addressed revenue maximization under the first category above. We can broadly categorize the works in this strand of the literature based on certain features of the seller's supply and the consumers' demands as follows.

1) Dynamic Auctions With Multiple Identical and Durable Goods: The works in this area have studied dynamic revenue-maximizing auctions in settings where the seller has multiple identical goods and wants to sell them to unit-demand[1] consumers over a finite- or infinite-time horizon. The setups in [7]–[9] consider a seller that has multiple identical goods and wants to sell them to consumers over T time steps; each consumer may demand one [7], [8] or more units [9] of the good and may be present in the market for one [7] or more time steps [8], [9]. Gallien [10] studies a similar setup, where the seller offers K identical items for sale over an infinite-time horizon and consumers are assumed to be unit demand and time sensitive in the sense that they discount their future utility with a common time-discount factor. Gershkov *et al.* [11] design a revenue-maximizing mechanism for a setting where the seller has multiple identical goods for sale over an infinite continuous-time horizon. They assume in [11] that the consumers are unit demand and that each consumer's arrival time and its valuation are its private information.

The key feature that differentiates these setups from our model is that in all of them, the goods are assumed to be *identical*. In our model, each consumer, depending on its flexibility level, *subjectively differentiates* between the goods. In particular, a consumer with flexibility level j has the same positive valuation for any good of varieties $1, \ldots, j$ and zero valuation for a good of varieties $j+1, \ldots, k$. Furthermore, in our model, more units may be added to the seller's supply of different varieties over time. In the setups described above, the seller's supply is limited to the initial stock of goods available at the beginning of the time horizon.

[1] A unit-demand consumer wants to receive one unit of the good [6].

2) Dynamic Auctions With Multiple Identical and Perishable Goods: Said [12] considers a setup where a seller obtains an uncertain number of perishable[2] identical goods at each time step and wants to sell them to unit-demand consumers over an infinite discrete-time horizon. Each consumer may depart the market exogenously at any time period t after its arrival with a common probability $(1-\gamma_t) \in [0,1]$. Otherwise, a consumer continues to interact with the market until it gets an allocation. Unlike the setup in [12], in our model, we assume that goods are durable and could, thus, be stored for future allocations. Moreover, in contrast to [12], we assume that consumers are present in the market for one time step only. Finally, in our model, goods are valued differently by different consumers depending on their flexibility levels, whereas, in [12], the offered goods are all identical from the consumers' viewpoints.

3) Dynamic Auctions With Multiple Heterogeneous Goods: The papers in this line of work study the dynamic revenue-maximizing auction design problem in cases where the seller has multiple *heterogeneous* goods and wants to sell them to unit-demand consumers over a finite/infinite-time horizon. Gershkov and Moldovanu [13] study one such setup over a continuous- and finite-time horizon where the goods are *commonly ranked* by the consumers that are impatient, i.e., they want to get an allocation immediately upon arrival in the market. Unlike this setup, each consumer in our model has *subjective* preferences for different varieties of goods, and each consumer's desired varieties are its private information. Furthermore, in our model, additional goods of each variety may be added to the seller's supply over time, which is not the case in [13].

4) Dynamic Auctions With Private Departure Times: Mierendorff [14] considers a setup where a seller wants to sell a single indivisible good over T time steps to consumers who are privately informed about their valuations as well as their deadlines for buying the single item in the dynamic auction. The key differences between this work and our setup are the following: 1) in our model, the seller offers multiple goods that are differentiated by each consumer subjectively based on their privately known flexibility levels, while in [14], the seller has only one good for sale; 2) the consumers' exit times are known to the seller in our setup, while they are privately known to each consumer in [14]; and 3) in our model, each consumer is present in the market for one time step only, while in [14], a consumer may remain present in the market for multiple time periods.

In the model studied by Pai and Vohra [8], each consumer's departure time is assumed to be its private information. As mentioned in Section I-A1, however, unlike our model, in the setup in [8], goods are assumed to be identical.

Contributions: We first characterize the allocation rule for a Bayesian-incentive-compatible (BIC), individually rational, and expected-revenue-maximizing mechanism in terms of the solution to a dynamic program (see Lemma 3 and Section V-A). The corresponding payment function is also specified in terms of the optimal allocation function.

We then leverage the structure of the consumers' flexibility model to simplify the dynamic program. In particular, we simplify both the "information state" of the dynamic program (i.e., the argument of the value function; see Lemma 4) and the maximization problem in each stage of the dynamic program (see Lemmas 7–9).

Our simplified dynamic program allows us to provide an explicit allocation procedure and a simple payment rule (see Theorem 1) in terms of the solution of the dynamic program.

B. Notations

Random variables are denoted by uppercase letters (X, Y, N) or by Greek letters (θ), and their realizations by the corresponding lowercase letters (x, y, n) or by Greek letters with tilde $(\tilde{\theta})$. $\{0,1\}^{N\times M}$ denotes the space of $N \times M$ matrices with entries that are either 0 or 1. $\mathbf{0}_{1\times k}$ is the k-dimensional all-zero row vector. $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_+$ denote the sets of nonnegative and positive integers, respectively. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. x^+ is the positive part of the real number x, that is, $x^+ = \max(x, 0)$. $\mathbb{1}_{\{a\leq b\}}$ denotes 1 if the inequality in the subscript is true and 0 otherwise. $\mathbb{E}$ denotes the expectation operator. For a random variable/random vector θ, $\mathbb{E}_\theta$ denotes that the expectation is with respect to the probability distribution of θ.[3] $x_{1:n}$, $y^{1:m}$, and $z_{1:n}^{1:m}$ are shorthands for vectors $(x_1,\dots,x_n)$, $(y^1,\dots,y^m)$, and $(z_1^1,\dots,z_1^m,\dots,z_n^1,\dots,z_n^m)$, respectively. For the vector $y^{1:m}$, y^{-j} is the shorthand for $(y^1,\dots,y^{j-1},y^{j+1},\dots,y^m)$. The summation $\sum_{i=m}^n y_i$ equals zero when $n<m$ regardless of the indexed quantities y_i.

II. Problem Formulation

We consider a setup where a seller offers k varieties of goods for sale over T time steps. At each time step, a random number of consumers enter the market. Let the random variable N_t denote the number of consumers that arrive at time step t. N_t is an integer-valued random variable that takes values in the set $\{0,1,\dots,\bar{n}\}$ according to the probability mass function (PMF) $\lambda_t(\cdot)$. At each time step, a random number of goods of varieties $1,2,\dots,k$ become available to the seller. Let the random variable X_t^j denote the number of goods of variety j that become available at time step t. X_t^j is an integer-valued random variable that takes values in the set $\{0,1,\dots,\bar{x}^j\}$ according to the PMF $\gamma_t^j(\cdot)$. The random variables $N_{1:T}$ and $X_{1:T}^{1:k}$ are mutually independent. Let Y_t^j denote the number of unallocated goods of variety j at time t—this includes X_t^j as well as any unallocated variety j goods from the past. Let V_t^j denote the number of variety j goods allocated by the seller at time t. Y_t^j evolves according to the following dynamics:

$$Y_{t+1}^j = \sum_{s=1}^{t+1} X_s^j - \sum_{s=1}^{t} V_s^j = Y_t^j - V_t^j + X_{t+1}^j, \quad t \geq 1$$
$$Y_1^j = X_1^j, \quad j = 1,2,\dots,k. \tag{1}$$

[2]A good is perishable if it cannot be stored for future allocations.

[3]The subscript for $\mathbb{E}[\cdot]$ operator is added only when its absence is likely to cause ambiguity.

A. Consumer Flexibility and Consumer Type

Each consumer can consume at most one good. Each consumer has a flexibility level that indicates the varieties of goods the consumer is equally interested in. A consumer with flexibility level j wants to get one good of any of the first j varieties.

Each consumer is associated with a 4-tuple (θ, b, t^A, t^D).

1) θ is the consumer's utility if it receives one good of a desired variety. We refer to θ as the consumer's valuation.
2) b is the consumer's flexibility level.
3) t^A is the consumer's arrival time.
4) t^D is the consumer's departure time.

A consumer can receive a good at any time t, $t^A \leq t \leq t^D$.

Definition 1: We say that a consumer is impatient [6, Ch. 16] if its arrival and departure times are the same. Thus, an impatient consumer can only receive a good at its arrival time.

In this article we assume that all consumers are impatient.

The random variable b_t^i denotes the flexibility level of the ith consumer arriving at time t. b_t^i takes values in the set $\{1, \ldots, k\}$ according to the PMF $g_t(\cdot)$. The random variable θ_t^i denotes the valuation of the ith consumer arriving at time t. Given $b_t^i = j$, θ_t^i takes values in $\Theta := [\theta^{\min}, \theta^{\max}]$ with conditional probability density $\pi_t(\cdot|b_t^i = j)$.[4] We define the joint distribution function $f_t(\tilde{\theta}, j) := \pi_t(\tilde{\theta}|j).g_t(j)$, $j \in \{1, \ldots, k\}, \tilde{\theta} \in \Theta$. The probability distributions $\lambda_t(\cdot), f_t(\cdot), \gamma_t^j(\cdot), \forall t, \forall j$ are common knowledge.

For a consumer with valuation $\tilde{\theta}$ and flexibility level $\tilde{b}$, we refer to the pair $(\tilde{\theta}, \tilde{b})$ as its *type*. Each consumer's type is independent of the other consumers' types and of the random variables $N_{1:T}$ and $X_{1:T}^{1:k}$.

B. Direct Mechanisms

We consider a direct mechanism where each consumer arriving in the market reports a valuation from the set Θ and a flexibility level from the set $\{1, 2, \ldots, k\}$. Each consumer can misreport its valuation and flexibility level. Consider a consumer whose true type is $(\tilde{\theta}, \tilde{b})$, and let (r, c) denote the type it reports, where r is the reported valuation and c is the reported flexibility level. The consumers' arrivals are publicly observed. Hence, N_t is observed by the seller at time t and by the consumers who arrive at time t. We make the following assumptions about the consumers' reported types.

Assumption 1:

1) Each consumer reports its valuation and flexibility level simultaneously at its arrival time.
2) No consumer departs the market without reporting a type to the seller.
3) Consumers cannot over-report their flexibility levels, that is, c cannot exceed $\tilde{b}$.

C. Feasible Allocations

Suppose that n_t consumers arrive at time t, i.e., $N_t = n_t$. Let $h_t^R := \{(r_t^1, c_t^1), \ldots, (r_t^{n_t}, c_t^{n_t})\}$ be the collection of reports made by the consumers arriving at time t, where (r_t^i, c_t^i) denotes the type reported by the ith consumer arriving at time t. If $n_t = 0$, then $h_t^R = \emptyset$. Let $\mathcal{H}_t^R$ denote the set of all the possible values of h_t^R. At each time t if $h_t^R \neq \emptyset$, an allocation of the available goods among the currently present consumers can be described by a binary matrix $\boldsymbol{A}_t \in \{0,1\}^{n_t \times k}$. $\boldsymbol{A}_t(i,j) = 1$ if the ith consumer is allocated a good of the jth variety at time t and $\boldsymbol{A}_t(i,j) = 0$ otherwise. The matrix $\boldsymbol{A}_t$ is called an allocation matrix at time t. $\boldsymbol{A}_t$ must satisfy some feasibility constraints. In particular, $\sum_{i=1}^{n_t} \boldsymbol{A}_t(i,j) \leq y_t^j, \forall j$, where y_t^j is the number of variety j goods available for allocation at time t. Furthermore, we require that each consumer is allocated at most one good of its desired varieties and no goods of its undesired varieties, i.e., $\sum_{j \leq c_t^i} \boldsymbol{A}_t(i,j) \leq 1, \sum_{j > c_t^i} \boldsymbol{A}_t(i,j) = 0$ for $i = 1, \ldots, n_t$. A binary matrix that satisfies these constraints is called a *feasible* allocation matrix at time t. For $h_t^R \neq \emptyset$, let $\mathcal{S}(h_t^R, y_t^{1:k}) \subset \{0,1\}^{n_t \times k}$ denote the set of all feasible allocation matrices at time t.

D. Mechanism Setup

Let h_t denote all the information that the seller knows at time t. We call h_t the history at time t, which is given as

$$h_t := \left\{ h_{1:t}^R \ , \ y_{1:t}^{1:k} \ , \ x_{1:t}^{1:k} \right\}. \quad (2)$$

Let $\mathcal{H}_t$ denote the set of all possible values of h_t. We use H_t to denote a random history.

A mechanism needs to specify allocations and payments at each time t, for which the number of arriving consumers is nonzero, i.e., $h_t^R \neq \emptyset$. Such a mechanism consists of the following components.

1) A sequence of allocation functions $q_{1:T}$ such that for any h_t with $h_t^R \neq \emptyset$, $q_t(h_t) \in \mathcal{S}(h_t^R, y_t^{1:k})$. $q_t(h_t)$ describes the allocation matrix to be used at time t.
2) A sequence of payment functions $p_{1:T}$ such that for any h_t with $h_t^R \neq \emptyset$, $p_t(h_t) \in \mathbb{R}^{|h_t^R|}$. The ith component $p_t^i(h_t)$ of $p_t(h_t)$ describes the payment charged to the ith consumer at time t.

E. Consumer Utility Model

Suppose that h_t is the history at time t and $(\tilde{\theta}_t^i, \tilde{b}_t^i)$ is the true type of the ith consumer arriving at time t. Then, under the mechanism $(q_{1:T}, p_{1:T})$, this consumer's utility is given as

$$u(\tilde{\theta}_t^i, \tilde{b}_t^i, h_t) = \tilde{\theta}_t^i \left(\sum_{j \leq \tilde{b}_t^i} q_t^{i,j}(h_t) \right) - p_t^i(h_t) \quad (3)$$

where $q_t^{i,j}(h_t)$ is the entry in the ith row and the jth column of the allocation matrix $q_t(h_t)$ and $p_t^i(h_t)$ is the ith entry of the payments vector $p_t(h_t)$.

F. Incentive Compatibility and Individual Rationality

The seller needs to design a mechanism that satisfies incentive compatibility and individual rationality (IR) constraints, as described in the following.

In a BIC mechanism, truthful reporting of private information (valuations and flexibility levels in our setup) constitutes an

[4]Extension of the results under the case where different consumers may have different lower and upper bounds for their valuations (i.e., $\theta_i^{\min} \neq \theta_j^{\min}$ and/or $\theta_i^{\max} \neq \theta_j^{\max}$ for $i \neq j$) is straightforward.

equilibrium of the Bayesian game induced by the mechanism. In other words, each consumer would prefer to report its true type provided that all other consumers have adopted the truth-telling strategy. Moreover, according to the revelation principle [15], restriction to incentive compatible direct mechanisms is without loss of generality.

Suppose that n_t consumers arrive at time t and let $(\tilde{\theta}_t^i, \tilde{b}_t^i)$ be the true type of the ith consumer arriving at time t. Recall that N_t is observed by the consumers who arrive at time t (see Section II-B). Assuming that all other consumers report their types truthfully, consumer i's expected utility if it reports its type truthfully will be

$$\mathbb{E}_{H_t^{-i}}\left[\tilde{\theta}_t^i \sum_{j\leq \tilde{b}_t^i} q_t^{i,j}(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) - p_t^i(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) \mid N_t = n_t \right] \tag{4}$$

where the expectation is with respect to the collection of random variables H_t^{-i}, which includes all variables in the history at time t except the ith consumer's report.

Now, suppose that the consumer with type $(\tilde{\theta}_t^i, \tilde{b}_t^i)$ reports (r_t^i, c_t^i) as its type rather than $(\tilde{\theta}_t^i, \tilde{b}_t^i)$. That is, the consumer might *misreport* its valuation or its flexibility level or both. Assuming that all other consumers truthfully report their types, this consumer's expected utility if it reports (r_t^i, c_t^i) will be

$$\mathbb{E}_{H_t^{-i}}\left[\tilde{\theta}_t^i \sum_{j\leq \tilde{b}_t^i} q_t^{i,j}(H_t^{-i}, (r_t^i, c_t^i)) - p_t^i(H_t^{-i}, (r_t^i, c_t^i)) \mid N_t = n_t \right]. \tag{5}$$

The BIC constraint is satisfied if each consumer's expected utility is maximized when it reports its valuation and flexibility level truthfully, provided that all other consumers report their types truthfully. Therefore, from the viewpoint of the ith consumer at time t, the BIC constraint can be expressed as follows:

$$\begin{aligned}
&\mathbb{E}_{H_t^{-i}}\left[\tilde{\theta}_t^i \sum_{j\leq \tilde{b}_t^i} q_t^{i,j}(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) - p_t^i(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) \mid N_t = n_t \right] \\
&\geq \mathbb{E}_{H_t^{-i}}\left[\tilde{\theta}_t^i \sum_{j\leq \tilde{b}_t^i} q_t^{i,j}(H_t^{-i}, (r_t^i, c_t^i)) - p_t^i(H_t^{-i}, (r_t^i, c_t^i)) \mid N_t = n_t \right] \\
&\forall \tilde{\theta}_t^i, r_t^i \in \Theta\ , c_t^i \leq \tilde{b}_t^i,\ c_t^i, \tilde{b}_t^i \in \{1, 2, \ldots, k\}\ , \forall n_t, \forall t.
\end{aligned} \tag{6}$$

The IR constraint ensures that the consumer's expected utility at the truthful reporting equilibrium is nonnegative. Using (4), from the viewpoint of ith consumer at time t, the IR constraint can be described as follows:

$$\mathbb{E}_{H_t^{-i}}\left[\tilde{\theta}_t^i \sum_{j\leq \tilde{b}_t^i} q_t^{i,j}(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) - p_t^i(H_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) \mid N_t = n_t \right] \geq 0 \qquad \forall n_t,\ \forall t. \tag{7}$$

G. Expected-Revenue Maximization

Consider a BIC and IR mechanism $(q_{1:T}, p_{1:T})$. When all consumers adopt the truthful strategy, the history at time t is

$$H_t := \{\{(\theta_1^i, b_1^i)\}_{i=1}^{N_1}, \ldots, \{(\theta_t^i, b_t^i)\}_{i=1}^{N_t}, Y_{1:t}^{1:k}, X_{1:t}^{1:k}\} \tag{8}$$

and the expected total revenue is $\mathbb{E}\{\sum_{t=1}^T \sum_{i=1}^{N_t} p_t^i(H_t)\}$.

The mechanism design problem can now be formulated as

$$\max_{(q_{1:T}, p_{1:T})} \quad \mathbb{E}\left\{ \sum_{t=1}^{T} \sum_{i=1}^{N_t} p_t^i(H_t) \right\}\ , \text{ subject to } \quad (6) \text{ and } (7) \tag{9}$$

III. Characterization of BIC and IR Mechanisms

In this section, we provide a characterization of BIC and IR mechanisms that will be useful for solving the problem in (9).

A. Interim Allocation and Payment

Suppose that n_t consumers arrive at time t, and let $(\tilde{\theta}_t^i, \tilde{b}_t^i)$ be the true type of the ith consumer arriving at time t. Assuming that all other consumers report their types truthfully, this consumer's expected allocation and payment under the mechanism $(q_{1:T}, p_{1:T})$ if it reports the pair (r_t^i, c_t^i) are given as

$$Q_t^i(r_t^i, c_t^i, n_t) := \mathbb{E}_{H_t^{-i}}\left[\sum_{j\leq c_t^i} q_t^{i,j}(H_t^{-i}, (r_t^i, c_t^i)) \mid N_t = n_t \right] \tag{10}$$

$$P_t^i(r_t^i, c_t^i, n_t) := \mathbb{E}_{H_t^{-i}}\left[p_t^i(H_t^{-i}, (r_t^i, c_t^i)) \mid N_t = n_t \right]. \tag{11}$$

In the following lemmas, we provide an operational characterization of the BIC and IR mechanisms in terms of the interim quantities defined in (10) and (11).

Lemma 1: A mechanism $(q_{1:T}, p_{1:T})$ satisfies the BIC and IR constraints if given $N_t = n_t$, the following conditions hold true for all $i \in \{1, \ldots, n_t\}, \forall t$.

i) $Q_t^i(r, c, n_t)$ is nondecreasing in r for all $c \in \{1, \ldots, k\}$.
ii) $Q_t^i(r, c, n_t)$ is nondecreasing in c for all $r \in \Theta$.
iii) $P_t^i(\theta^{\min}, c, n_t) = 0$, for all $c \in \{1, 2, \ldots, k\}$.
iv) The interim payment in (11) takes the following form for all $r \in \Theta, c \in \{1, 2, \ldots, k\}$:

$$P_t^i(r, c, n_t) = r\ Q_t^i(r, c, n_t) - \int_{\theta^{\min}}^{r} Q_t^i(s, c, n_t)\ ds - \theta^{\min}\ Q_t^i(\theta^{\min}, c, n_t). \tag{12}$$

v) $\theta^{\min}\ Q_t^i(\theta^{\min}, c, n_t) \geq 0 \forall c$.

Proof: The proof closely follows the standard arguments in [16, Sec. III]. □

Lemma 2: Any BIC and IR mechanism $(q_{1:T}, p_{1:T})$ satisfies

$$P_t^i(r, c, n_t) \leq r\, Q_t^i(r, c, n_t) - \int_{\theta^{\min}}^{r} Q_t^i(s, c, n_t)\, ds$$
$$\text{for all} \quad t, n_t, i \in \{1, \dots, n_t\}, r \in \Theta, c \in \{1, 2, \dots, k\}. \quad (13)$$

Proof: The proof closely follows the standard arguments in [16, Sec. III]. □

Remark 1: The setup studied in [16] can be viewed as a special case of the framework considered in this article. Navabi and Nayyar [16] consider a model where the seller wants to sell a fixed number of goods of k varieties ($x_1^{1:k}$ are known) to a fixed number of consumers (n_1 is known) with different flexibility levels in one time step ($T = 1$).

IV. Revenue-Maximizing Mechanism

In this section, we characterize the expected-revenue-maximizing mechanism. Let us define

$$w_t(\tilde{\theta}, \tilde{b}) := \left(\tilde{\theta} - \frac{1 - \Pi_t(\tilde{\theta} \mid \tilde{b})}{\pi_t(\tilde{\theta} \mid \tilde{b})}\right) \quad (14)$$

where $\Pi_t(\cdot \mid \tilde{b})$ is the cumulative distribution function corresponding to the conditional probability density function (pdf) $\pi_t(\cdot \mid \tilde{b})$. In economics terminology, $w_t(\tilde{\theta}, \tilde{b})$ is referred to as the *virtual valuation* [15, Ch. 3] of a consumer with type $(\tilde{\theta}, \tilde{b})$ that arrives at time t.

We make the following assumptions to simplify the solution to the optimal mechanism design problem in (9).

Assumption 2:

i) The conditional pdfs $\pi_t(\cdot|c), t = 1, \dots, T, c = 1, \dots, k$ satisfy the generalized monotone hazard rate condition [8, Sec. 2], [16, Sec. IV]. That is, for all t, we assume that $\frac{\pi_t(x|c)}{1-\Pi_t(x|c)}$ is nondecreasing in x and c. Moreover, we assume that for all t if $x \geq x'$ and $c > c'$, then $\frac{\pi_t(x|c)}{1-\Pi_t(x|c)} > \frac{\pi_t(x'|c')}{1-\Pi_t(x'|c')}$.

ii) $w_t(\theta^{\min}, j) < 0$ for all j, t.

In the following lemma, we provide a characterization of the expected-revenue-maximizing mechanism.

Lemma 3: Suppose that $(q_{1:T}^*, p_{1:T}^*)$ is a BIC and IR mechanism for which the following conditions are true.

i) $(q_{1:T}^*)$ is the solution to the following functional optimization problem:

$$\max_{q_{1:T}} \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N_t} w_t(\theta_t^i, b_t^i)\left(\sum_{j \leq b_t^i} q_t^{i,j}(H_t)\right)\right] \quad (15)$$

where H_t is the history under truthful reporting.

ii) Given the history h_t and assuming that n_t consumers arrive at time t, the payment charged to the ith consumer arriving at time t with the true type $(\tilde{\theta}_t^i, \tilde{b}_t^i)$ is given as

$$p_t^{*i}(h_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) = \tilde{\theta}_t^i \sum_{j \leq \tilde{b}_t^i} q_t^{*i,j}(h_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) - \int_{\theta^{\min}}^{\tilde{\theta}_t^i}\left(\sum_{j \leq \tilde{b}_t^i} q_t^{*i,j}(h_t^{-i}, (\alpha, \tilde{b}_t^i))\right) d\alpha$$
$$\forall i \in \{1, \dots, n_t\}, \forall n_t, \forall t \quad (16)$$

where $h_t^{-i} = h_t \setminus \{(\tilde{\theta}_t^i, \tilde{b}_t^i)\}$.

Then, $(q_{1:T}^*, p_{1:T}^*)$ gives the highest expected revenue in the class of BIC and IR mechanisms.

Proof: See Appendix A. □

The results of Lemma 3 imply that in order for a BIC and IR mechanism to maximize the expected revenue, its allocation rules must solve the functional optimization problem in (15). This problem can be viewed as a stochastic control problem. In the following section, we describe this stochastic control problem and formulate a dynamic program to find the optimal allocation rules.

V. Solution to the Stochastic Control Problem

The optimization problem in (15) is a finite horizon stochastic control problem with the history at time t (with truthful reporting) as the state and the allocation matrix as the action at time t. The allocation function $q_{1:T}$ is the control strategy, and the optimization in (15) is to find the control strategy with the highest expected reward. This stochastic control perspective provides a dynamic program for the optimization in (15). We then leverage the structure of the consumers' flexibility model to simplify the dynamic program.

A. Dynamic Program

For a truthful history h_t at time t, let $R_t(h_t)$ denote the maximum expected reward from t to T for the stochastic control problem in (15). $R_t(h_t)$ is a value function and obeys the standard dynamic programming recursions given as

$$\text{If} \quad h_t^R = \emptyset: \quad R_t(h_t) := \mathbb{E}\left[R_{t+1}(H_{t+1}) \mid h_t\right] \quad (17)$$

$$\text{If} \quad h_t^R \neq \emptyset:$$

$$R_t(h_t) := \max_{\boldsymbol{A} \in \mathcal{S}(h_t^R, y_t^{1:k})} \left\{ \sum_{i=1}^{|h_t^R|} w_t(\tilde{\theta}_t^i, \tilde{b}_t^i) \sum_{j=1}^{k} \boldsymbol{A}(i,j) + \mathbb{E}\left[R_{t+1}(H_{t+1}) \mid h_t, \boldsymbol{A}_t = \boldsymbol{A}\right] \right\} \quad (18)$$

where $R_{T+1}(\cdot) = 0$.

In the above dynamic program, the information state at time t is h_t (since the value functions have h_t as the argument). It can be shown that the only relevant part of the history are the reports and the state of supply at current time. In the following lemma, we use this idea to simplify the information state and the dynamic program.

Lemma 4: Let $s_t = (h_t^R, y_t^{1:k})$. There exist functions $V_1(\cdot), \dots, V_T(\cdot)$ such that at each time t:

$$V_t(s_t) = R_t(s_t, x_t^{1:k}, h_{t-1})$$
$$\text{for all} \quad \{s_t, x_t^{1:k}, h_{t-1}\} \in \mathcal{H}_t. \quad (19)$$

Furthermore, these functions obey the following dynamic program:

$$\text{If} \quad h_t^R = \emptyset: \quad V_t(s_t) = \mathbb{E}\left[V_{t+1}(S_{t+1}) \mid s_t\right] \quad (20)$$

$$\text{If} \quad h_t^R \neq \emptyset:$$

$$V_t(s_t) = \max_{\boldsymbol{A}\in\mathcal{S}(h_t^R,y_t^{1:k})} \Bigg\{ \sum_{i=1}^{|h_t^R|} w_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\sum_{j=1}^{k}\boldsymbol{A}(i,j) + \mathbb{E}\left[V_{t+1}(S_{t+1}) \mid s_t, \boldsymbol{A}_t=\boldsymbol{A}\right]\Bigg\} \tag{21}$$

where $V_{T+1}(\cdot)=0$.

Proof: See Appendix B. □

Based on the results of Lemma 4, the optimal allocation functions can be described in terms of the solution to the dynamic program in (20) and (21) with the simplified information state $s_t=(h_t^R,y_t^{1:k})$ at time t.

In the dynamic program in (21) at each time t, the optimization variables comprise all the entries of the $|h_t^R|\times k$ allocation matrix $\boldsymbol{A}_t$. In the following, we simplify the dynamic program formulation in terms of alternate variables that need to be optimized at each time.

B. Alternative Optimization Variables

Consider the information state $(h_t^R,y_t^{1:k})$ with $h_t^R\neq\emptyset$ and an allocation matrix $\mathbf{A}\in\mathcal{S}(h_t^R,y_t^{1:k})$ at time t. Let u_t^j denote the number of consumers with flexibility level j that get a good at time t under $\mathbf{A}$. That is, $u_t^j := \sum_{\substack{i=1\\ i:\tilde{b}_t^i=j}}^{|h_t^R|}\sum_{l\le j}\mathbf{A}(i,l)$, where $\mathbf{A}(i,l)$ is the entry in the ith row and the lth column of the matrix $\mathbf{A}$. Let $\mathcal{U}(h_t^R,y_t^{1:k})$ denote the set of admissible values of $u_t^{1:k}$ given the information state $(h_t^R,y_t^{1:k})$. Thus, for every vector $u^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})$, there exists a matrix $\hat{\boldsymbol{A}}\in\mathcal{S}(h_t^R,y_t^{1:k})$ such that $\sum_{\substack{i=1\\ i:\tilde{b}_t^i=j}}^{|h_t^R|}\sum_{l\le j}\hat{\boldsymbol{A}}(i,l)=u^j\ \forall j$. If $h_t^R=\emptyset$, no consumer is present to be allocated, and thus, we define $\mathcal{U}(h_t^R,y_t^{1:k}):=\{\mathbf{0}_{1\times k}\}$. In the following lemma, we provide a more operational characterization of $\mathcal{U}(h_t^R,y_t^{1:k})$.

Lemma 5: Given the information state $(h_t^R,y_t^{1:k})$ at time t

$$\mathcal{U}(h_t^R,y_t^{1:k})=\left\{u^{1:k}\in\mathbb{Z}_{\ge 0}^k : \sum_{l=1}^{j}u^l\le\sum_{l=1}^{j}y_t^l\ ,\ u^j\le n_t^j,\forall j\right\} \tag{22}$$

where n_t^j denotes the number of consumers with flexibility level j that arrive at time t.

Proof: See Appendix C. □

Consider the information state $(h_t^R,y_t^{1:k})$ with $h_t^R\neq\emptyset$ and an allocation matrix $\mathbf{A}\in\mathcal{S}(h_t^R,y_t^{1:k})$ at time t. Let v_t^j denote the number of goods of variety j allocated at time t under $\mathbf{A}$, i.e., $v_t^j:=\sum_{i=1}^{|h_t^R|}\mathbf{A}(i,j)$. Given some vector $u_t^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})$, let $\mathcal{V}(u_t^{1:k},y_t^{1:k})$ denote the set of all values of $v_t^{1:k}$ that can fulfill the demand represented by $u_t^{1:k}$ under the supply $y_t^{1:k}$. More precisely, for every vector $v^{1:k}\in\mathcal{V}(u_t^{1:k},y_t^{1:k})$, there exists a matrix $\hat{\boldsymbol{A}}\in\mathcal{S}(h_t^R,y_t^{1:k})$ such that for each j, $\hat{\boldsymbol{A}}$ serves u_t^j consumers of flexibility level j (i.e., $\sum_{\substack{i=1\\ i:\tilde{b}_t^i=j}}^{|h_t^R|}\sum_{l\le j}\hat{\boldsymbol{A}}(i,l)=u_t^j$) and allocates v^j goods of variety j (i.e., $\sum_{i=1}^{|h_t^R|}\hat{\boldsymbol{A}}(i,j)=v^j$). If $h_t^R=\emptyset$, no consumer is present to be allocated, and thus, $\mathcal{V}(u_t^{1:k},y_t^{1:k})=\{\mathbf{0}_{1\times k}\}$. In the following lemma, we provide a more operational characterization of $\mathcal{V}(u_t^{1:k},y_t^{1:k})$.

Lemma 6: Given the information state $(h_t^R,y_t^{1:k})$ and the vector $u_t^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})$ at time t

$$\mathcal{V}(u_t^{1:k},y_t^{1:k})=\Bigg\{v^{1:k}\in\mathbb{Z}_{\ge 0}^k : v^j\le y_t^j\ ,j=1,\ldots,k,\ \sum_{l=1}^{j}u_t^l\le\sum_{l=1}^{j}v^l\ ,\ j=1,\ldots,k-1\ ,\ \sum_{l=1}^{k}u_t^l=\sum_{l=1}^{k}v^l\Bigg\}. \tag{23}$$

Proof: The proof is similar to the proof of Lemma 5 and is, therefore, omitted. □

In the following lemma, we show that the two k-dimensional vectors $u_t^{1:k}$ and $v_t^{1:k}$ constructed above can be treated as the optimization variables in the dynamic program in (21).

Lemma 7: The value function in (21) can be equivalently expressed as

$$V_t(h_t^R,y_t^{1:k})=\max_{u^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})}\Bigg\{\sum_{j=1}^{k}\sum_{i=1}^{u^j}w_t^{i,j} + \max_{v^{1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})}\mathbb{E}\left[V_{t+1}\left(H_{t+1}^R,\{y_t^j-v^j+X_{t+1}^j\}_{j=1}^k\right)\right]\Bigg\} \tag{24}$$

where $V_{T+1}(\cdot)=0$. In (24), $w_t^{i,j}$ denotes the ith largest element in $\mathcal{W}_t^j$ defined as

$$\mathcal{W}_t^j:=\{\ w_t(\tilde{\theta},\tilde{b}):(\tilde{\theta},\tilde{b})\in h_t^R\ ,\tilde{b}=j\ \} \tag{25}$$

that is, $\mathcal{W}_t^j$ denotes the set of virtual valuations [see (14)] of all the consumers with flexibility level j at time t.

Proof: See Appendix D. □

In the following lemma, we establish a monotonicity property of the value functions in (24). In Lemma 9, we leverage this property to construct an optimal solution to the inner maximization over $v^{1:k}$ vector in (24).

Lemma 8: Consider two supply profiles $y_t^{1:k}$ and $z_t^{1:k}$ such that

$$\begin{aligned} y_t^i &= z_t^i+1\\ y_t^j &= z_t^j-1\\ y_t^l &= z_t^l \quad \text{for all}\quad l\neq i,j \end{aligned} \tag{26}$$

where $i<j$. Then, the value functions $V_t(\cdot)$ defined in Lemma 4 satisfy the following property:

$$V_t(h_t^R,y_t^{1:k})\ge V_t(h_t^R,z_t^{1:k})\quad \text{for all } h_t^R \text{ and t.} \tag{27}$$

Proof: See Appendix E. □

A more intuitive interpretation of the property established in Lemma 8 is that a good of variety j contributes more to the generation of revenue in comparison with a good of any of the varieties $j+1,j+2,\ldots,k$. Consequently, allocating a good of variety j is at least as costly as allocating a good of any of the varieties $j+1,j+2,\ldots,k$.

Remark 2: The value functions $V_t(\cdot)$ defined in Lemma 4 satisfy a more general version of the monotonicity property shown in Lemma 8. Consider any two supply profiles $y_t^{1:k}$ and $z_t^{1:k}$ such that $\sum_{l=1}^{j} y_t^l \geq \sum_{l=1}^{j} z_t^l, j = 1, \ldots, k$. Then, it can be shown that value functions $V_t(\cdot)$ still satisfy (27).

The property in (27) can be leveraged to provide an explicit solution for the inner maximization over $v^{1:k}$ in (24). This is shown in Lemma 9.

Lemma 9: At each time t, given $h_t^R, y_t^{1:k}$ and $u^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$, *recursively* define

$$v^{*k} := \min(u^k, y_t^k)$$
$$v^{*j} := \min\left(y_t^j, u^j + \left(\sum_{l=j+1}^{k} u^l - \sum_{l=j+1}^{k} v^{*l}\right)\right),$$
$$j = k-1, \ldots, 1. \tag{28}$$

Then

$$v^{*1:k} \in \underset{v^{1:k} \in \mathcal{V}(u^{1:k}, y_t^{1:k})}{\arg\max}\ \mathbb{E}\Big[V_{t+1}\Big(H_{t+1}^R, \{y_t^j - v^j + X_{t+1}^j\}_{j=1}^k\Big)\Big]. \tag{29}$$

Proof: See Appendix F. □

Based on the results of Lemma 9, the value functions in (24) can be simplified as follows:

$$V_t(h_t^R, y_t^{1:k}) = \max_{u^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})} \left\{ \sum_{j=1}^{k} \sum_{i=1}^{u^j} w_t^{i,j} + \mathbb{E}\Big[V_{t+1}\Big(H_{t+1}^R, \{y_t^j - v^{*j} + X_{t+1}^j\}_{j=1}^k\Big)\Big] \right\} \tag{30}$$

where $v^{*1:k}$ is obtained corresponding to each $u^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$, as described in (28). Note that since $\mathcal{U}(\cdot)$ and $\mathcal{V}(\cdot)$ are finite sets [see (22) and (23)], the dynamic program in (30) is guaranteed to have a solution.

VI. Optimal Mechanism

In Lemma 3, we established that the expected-revenue-maximizing allocation rules of a BIC and IR mechanism indeed coincide with the optimal control strategy for the stochastic control problem in (15). Based on this insight, Section V was devoted to the development of a characterization of the optimal control strategy for the problem in (15) in terms of the solution to a dynamic program [see (17) and (18)]. We leveraged the structure of the flexibility model to simplify the formulated dynamic program (see Lemmas 7–9). In the following theorem, we use the results of Lemmas 3–9 as well as the characterization of BIC and IR mechanisms provided in Lemma 1 to specify the allocation and payment rules of the optimal mechanism.

Theorem 1: Consider the information state $(h_t^{-i,R}, (r,j), y_t^{1:k})$, where consumer i reports (r,j) as its type and $h_t^{-i,R}$ denotes the set of reports from all consumers other than i. Let $u_t^{*1:k}$ and $v_t^{*1:k}$ denote the optimal vectors that result from solving the dynamic program in (30). Consider the mechanism $(q_{1:T}^*, p_{1:T}^*)$ described in the following.

1) *Allocations:* Let $q_t^*(h_t^{-i,R}, (r,j), y_t^{1:k}) \in \mathcal{S}(h_t^{-i,R}, (r,j), y_t^{1:k})$ denote the allocation matrix constructed according to the allocation procedure described as follows.
   i) Index the goods under the profile $v_t^{*1:k}$ in a nondecreasing flexibility order, i.e., the v_t^{*1} goods of variety 1 are indexed as $1, \ldots, v_t^{*1}$ and, the v_t^{*2} goods of variety 2 are indexed as $v_t^{*1}+1, \ldots, v_t^{*1}+v_t^{*2}$, and so on.
   ii) Sort consumers of flexibility level 1 in nonincreasing order of virtual valuations. Top u_t^{*1} consumers of flexibility level 1 get the first u_t^{*1} goods as arranged in (i). Ties are resolved randomly.
   iii) Sort consumers of flexibility level 2 in nonincreasing order of virtual valuations. Top u_t^{*2} consumers of flexibility level 2 get the next u_t^{*2} goods. Ties are resolved randomly.
   iv) Allocations to the top u_t^{*j} consumers with flexibility levels $j = 3, \ldots, k$ are carried out in the same fashion as above.
   v) The rest of the consumers do *not* get an allocation.
2) *Payments:* Suppose that n_t consumers arrive at time t. The payment function $p_t^*(h_t^{-i,R}, (r,j), y_t^{1:k}) \in \mathbb{R}^{n_t}$ is defined for $i = 1, \ldots, n_t$ as follows:

$$p_t^{*i}(h_t^{-i,R}, (r,j), y_t^{1:k}) = \begin{cases} \bar{\theta}_t^{i,j}, & \text{if consumer } i \text{ gets a good} \\ 0, & \text{otherwise} \end{cases} \tag{31}$$

where $\bar{\theta}_t^{i,j}$ is defined as

$$\bar{\theta}_t^{i,j} := \sup\Big\{x \in [\theta^{\min}, \theta^{\max}] : \sum_{l \leq j} q_t^{*i,l}(h_t^{-i,R}, (x,j), y_t^{1:k}) = 0\ ,\ w_t(x,j) \geq 0\Big\}. \tag{32}$$

Under Assumptions 1 and 2, $(q_{1:T}^*, p_{1:T}^*)$ is an expected-revenue-maximizing, BIC, and IR mechanism.

Proof: See Appendix G. □

Remark 3: The vectors $u_t^{*1:k}$ and $v_t^{*1:k}$ that characterize the optimal allocation matrix $q_t^*(\cdot)$ in Theorem 1 as well as the quantities $\bar{\theta}_t^{i,j}$ defined in (32) can be found by first discretizing the set $[\theta^{\min}, \theta^{\max}]$ with sufficient numerical precision and then applying the methods developed for solving Markov decision processes (MDPs) with discrete state and action spaces. Several methods for solving MDPs are available in the literature [17]–[19]. An extensive discussion of the computational complexity of different methods for solving MDPS can be found in [20] and [21].

A. Example

Consider a simple setup with $T = 2$ and $k = 2$, where the consumer arrival process $\lambda_t(\cdot)$ follows a Bernoulli distribution, that is, at each time step, a consumer may enter the market with probability p. For a consumer entering the market at time $t = 1, 2$, its flexibility level is equally likely to be 1 or 2, i.e.,

$g_t(j) = \frac{1}{2}$ for $j \in \{1, 2\}$ and, conditioned on its flexibility level being j, its valuation has truncated exponential distribution over the interval [0,1], i.e., $\pi_t(x|j) = \frac{\alpha_j \exp(-\alpha_j x)}{1-\exp(-\alpha_j)}, x \in [0,1]$, where $\alpha_2 > \alpha_1 > 0$. It is straightforward to verify that $\pi_t(\cdot|j)$ satisfies Assumption 2. The virtual valuation function [see (14)] associated with $\pi_t(\cdot|j)$ is of the following form:

$$w_t(x, j) = x - \frac{1}{\alpha_j}\left(1 - \exp(\alpha_j(x-1))\right), \quad t = 1, 2.$$

Suppose that the supply profile at time $t = 1$ is $(y_1^1, y_1^2) = (1, 1)$, and no more goods of either variety become available at time $t = 2$. Suppose that a consumer with type $(\tilde{\theta}, 2)$ arrives at time $t = 1$. Under the optimal mechanism characterized in Theorem 1, this consumer gets a good of variety 2 if $w_1(\tilde{\theta}, 2) > \mathbb{E}[V_2(H_2^R, (1,1)) - V_2(H_2^R, (1,0))] =: \rho_1^2$ [see (30)]. It is straightforward to verify that in this example

$$\begin{aligned}\mathbb{E}[V_2(H_2^R, (1,1))] &= \mathbb{E}[V_2(H_2^R, (1,0))] \\ &= \frac{p}{2}\Big(\mathbb{E}_{\theta|b=1}\left[\max\{w_2(\theta,1), 0\}\right] \\ &\qquad + \mathbb{E}_{\theta|b=2}\left[\max\{w_2(\theta,2),0\}\right]\Big)\end{aligned}$$

from which it follows that $\rho_1^2 = 0$.

If instead a consumer with type $(\tilde{\theta}, 1)$ arrives at time $t = 1$, it gets a good of variety 1 if $w_1(\tilde{\theta}, 1) > \mathbb{E}[V_2(H_2^R, (1,1)) - V_2(H_2^R, (0,1))] =: \rho_1^1$. We observe that in this example

$$\mathbb{E}[V_2(H_2^R, (0,1))] = \frac{p}{2}\mathbb{E}_{\theta|b=2}\left[\max\{w_2(\theta,2), 0\}\right]$$

and thus

$$\begin{aligned}\rho_1^1 &= \frac{p}{2}\mathbb{E}_{\theta|b=1}\left[\max\{w_2(\theta,1),0\}\right] \\ &= \frac{p}{2}\theta_{1,2}^{\text{res}}\frac{\exp(-\alpha_1\theta_{1,2}^{\text{res}}) - \exp(-\alpha_1)}{1-\exp(-\alpha_1)}\end{aligned} \tag{33}$$

where $\theta_{j,t}^{\text{res}}$ is defined as

$$\theta_{j,t}^{\text{res}} := \max\left\{x \in [0,1] : w_t(x, j) = 0\right\}. \tag{34}$$

For instance, if $\alpha_1 = 2$ and $p = 0.5$, we obtain $\theta_{1,2}^{\text{res}} \approx 0.36$ and, thus, $\rho_1^1 \approx 0.037$. In general, for the setup described above, it is easy to check that $\rho_1^1 \geq \rho_1^2$, which implies that a consumer with flexibility level 1 needs to have higher valuation to get an allocation at $t = 1$. If the consumer with type $(\tilde{\theta}, j)$ at time $t = 1$ gets a good, it pays $\bar{\theta}_1^j = w_1^{-1}(\rho_1^j; j)$ [see (32)], where $w_1^{-1}(\cdot; j)$ denotes the inverse of $w_1(\cdot, j)$. Notice that Assumption 2 combined with $\rho_1^1 \geq \rho_1^2$ implies that $\bar{\theta}_1^1 \geq \bar{\theta}_1^2$, i.e., a consumer with flexibility level 1 is charged a higher price upon allocation of a desired good. For instance, consider $\alpha_2 = 3, \alpha_1 = 2$, and $p = 0.5$. For this numerical setup, we obtain $\bar{\theta}_1^1 = w_1^{-1}(\rho_1^1; 1) \approx w_1^{-1}(0.037; 1) \approx 0.39$ and $\bar{\theta}_1^2 = w_1^{-1}(\rho_1^2; 2) = w_1^{-1}(0; 2) \approx 0.29$.

At time $t = 2$, which is the terminal time step, if a consumer with type $(\tilde{\theta}, j)$ arrives and a desired good is available, it gets an allocation if $w_2(\tilde{\theta}, j) > 0$ and is charged the reserve price $\theta_{j,t}^{\text{res}}$ at $t = 2$, as defined in (34). Notice that from Assumption 2, it follows that $\theta_{1,2}^{\text{res}} > \theta_{2,2}^{\text{res}}$, meaning that at time $t = 2$ also, a consumer with flexibility level 1 needs to have a higher valuation to get an allocation and is charged a higher price upon allocation of a desired good. For instance, for the case in which $\alpha_2 = 3, \alpha_1 = 2$, and $p = 0.5$, we see that $\theta_{1,2}^{\text{res}} \approx 0.36 > \theta_{2,2}^{\text{res}} \approx 0.29$.

Therefore, we observe that in the above setup under the optimal mechanism, at each time, the payment charged to the more flexible consumers is less than the payment charged to the less flexible consumers. Moreover, it is straightforward to verify that $\bar{\theta}_t^j$ is nonincreasing in t, that is, the payment charged to the consumers with flexibility level j decreases over time across all j (e.g., for the case in which $\alpha_1 = 2$ and $p = 0.5$, we see that $\bar{\theta}_1^1 \approx 0.39 > \bar{\theta}_2^1 = \theta_{1,2}^{\text{res}} \approx 0.36$). Intuitively, this is expected because unlike $t = 1$, the consumer that arrives at $t = 2$ faces no competition from future consumers. On the other hand, the virtual valuation of a consumer that arrives at $t = 1$ must outweigh the expected revenue that can be produced by saving the good for a consumer that may arrive at $t = 2$. As a result, a consumer that arrives at $t = 2$ is expected to be charged less (only the reserve price associated with its flexibility level) than a consumer of the same flexibility level that arrives at $t = 1$.

B. Social Welfare Maximization

Consider a benevolent mechanism designer whose objective is to maximize the expected social welfare of all the consumers. In that case, the mechanism design problem can be formulated as follows:

$$\max_{(q_{1:T}, p_{1:T})} \quad \mathbb{E}\left\{\sum_{t=1}^{T}\sum_{i=1}^{N_t}\theta_t^i\left(\sum_{j \leq b_t^i} q_t^{i,j}(H_t)\right)\right\} \tag{35}$$

$$\text{subject to} \quad (6) \text{ and } (7).$$

Suppose that $(\hat{q}_{1:T}, \hat{p}_{1:T})$ is a BIC and IR mechanism, for which $\hat{q}_{1:T}$ is the solution to the following functional optimization problem:

$$\max_{q_{1:T}} \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N_t}\theta_t^i\left(\sum_{j \leq b_t^i} q_t^{i,j}(H_t)\right)\right] \tag{36}$$

where H_t is the history under truthful reporting.

It is straightforward to verify that $(\hat{q}_{1:T}, \hat{p}_{1:T})$ gives the highest expected social welfare in the class of BIC and IR mechanisms. That is, $(\hat{q}_{1:T}, \hat{p}_{1:T})$ is a solution to the optimization problem in (35).

One can easily show that the payment form in (16) can be used to further specify the above mechanism. Given the history h_t and assuming that n_t consumers arrive at time t, $\hat{p}_t$ can be specified as follows:

$$\begin{aligned}\hat{p}_t^i(h_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) &= \tilde{\theta}_t^i \sum_{j \leq \tilde{b}_t^i} \hat{q}_t^{i,j}(h_t^{-i}, (\tilde{\theta}_t^i, \tilde{b}_t^i)) \\ &- \int_{\theta^{\min}}^{\tilde{\theta}_t^i}\left(\sum_{j \leq \tilde{b}_t^i}\hat{q}_t^{i,j}(h_t^{-i}, (\alpha, \tilde{b}_t^i))\right) d\alpha \quad \forall i \in \{1, \ldots, n_t\}, \forall n_t, \forall t.\end{aligned} \tag{37}$$

By comparing (36) with (15), we observe that (36) is exactly the same optimization problem as (15) with virtual valuation $w_t(\theta_t^i, b_t^i)$ replaced by true valuation θ_t^i. Therefore, the analyses in Sections V– VI can be applied to specify the mechanism $(\hat{q}_{1:T}, \hat{p}_{1:T})$ as well. In particular, it can be verified that $\hat{q}_t$ has the same description as the one given for q_t^* in Theorem 1, where in steps (ii)–(iv), the consumers need to be sorted in nonincreasing order of *true valuations* rather than virtual valuations.

Moreover, using similar arguments as in Appendix G, we can simplify the payment form in (37) as follows:

$$\hat{p}_t^i(h_t^{-i,R},(r,j),y_t^{1:k}) = \begin{cases} \hat{\theta}_t^{i,j}, & \text{if consumer } i \text{ gets a good} \\ 0, & \text{otherwise} \end{cases} \tag{38}$$

where $\hat{\theta}_t^{i,j}$ is defined as

$$\hat{\theta}_t^{i,j} := \sup\left\{ x \in [\theta^{\min},\theta^{\max}] : \sum_{l\le j} \hat{q}_t^{i,l}(h_t^{-i,R},(x,j),y_t^{1:k}) = 0 \right\}. \tag{39}$$

VII. Conclusion

In this article, we studied the problem of designing a dynamic expected-revenue-maximizing, BIC, and IR mechanism for the allocation of multiple goods of k varieties to flexible consumers over T time steps. In our model, a random number of goods of each variety may become available to the seller at each time, and a random number of consumers may enter the market at each time. We considered impatient consumers that need to get one good of one of their desired varieties within the single time step of their arrival. Each consumer has a flexibility level, i.e., a number between 1 and k that indicates the varieties of goods the consumer finds equally desirable. A consumer's flexibility level and the utility it enjoys upon allocation of a desired good are its private information. We characterized the allocation and payment functions under the optimal mechanism in terms of the solution to a dynamic program. We leveraged the structure of the consumers' flexibility model to simplify the dynamic program and provided an alternative description of the optimal mechanism in terms of thresholds computed by the dynamic program.

Further exploration of the practical aspects of the developed mechanism through simulation experiments is an important task for future research. For instance, it would be interesting to deploy Monte Carlo simulation techniques to study the interplay between various parameters, such as revenue, payments, flexibility levels, etc., over time. Such studies would provide insights into the practical limitations of the developed mechanism and the scope of its applicability in different contexts.

An interesting extension to this article would be to study this setup with *patient* consumers, i.e., consumers may be present for more than one time step. In addition, studying the dynamic mechanism design problem under the settings where both the arrival and departure times of each consumer are privately known to them is an important direction for further exploration. In the present setup, we studied the case where each consumer wants to receive a *single* good of its desired varieties. Another interesting scenario would be the case where the consumers may need to get *multiple* goods of their desired varieties.

Appendix A
Proof of Lemma 3

Consider a BIC and IR mechanism $(q_{1:T}, p_{1:T})$. The expected revenue under this mechanism is

$$\mathbb{E}\left\{\sum_{t=1}^{T}\sum_{i=1}^{N_t} p_t^i(H_t)\right\} = \sum_{t=1}^{T}\sum_{n_t=1}^{\bar{n}} \lambda_t(n_t) \sum_{i=1}^{n_t} \mathbb{E}\left[p_t^i(H_t) \mid N_t = n_t\right]. \tag{40}$$

The conditional expectation in (40) can be written as

$$\sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} \mathbb{E}_{H_t^{-i}}[\, p_t^i(H_t^{-i},(\tilde{\theta}_t^i,\tilde{b}_t^i)) \,|N_t = n_t\,]\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i = \sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} P_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i \tag{41}$$

where $P_t^i(\cdot)$ is the interim payment defined in (11). Because of Lemma 2, we know that

$$P_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t) \le \tilde{\theta}_t^i\, Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t) - \int_{\theta^{\min}}^{\tilde{\theta}_t^i} Q_t^i(\alpha,\tilde{b}_t^i,n_t)\, d\alpha. \tag{42}$$

Using (42), (41) can be upper bounded as follows:

$$\begin{aligned}
&\sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} P_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i \\
&\le \sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} \Bigg(\tilde{\theta}_t^i\, Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t) \\
&\quad - \int_{\theta^{\min}}^{\tilde{\theta}_t^i} Q_t^i(\alpha,\tilde{b}_t^i,n_t)\, d\alpha\Bigg) f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i \\
&= \sum_{\tilde{b}_t^i=1}^{k}\Bigg[\int_{\theta^{\min}}^{\theta^{\max}} \tilde{\theta}_t^i\, Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i \\
&\quad - \int_{\theta^{\min}}^{\theta^{\max}} \left(\int_{\theta^{\min}}^{\tilde{\theta}_t^i} Q_t^i(\alpha,\tilde{b}_t^i,n_t)\, d\alpha\right) f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i\Bigg] \\
&= \sum_{\tilde{b}_t^i=1}^{k}\Bigg[\int_{\theta^{\min}}^{\theta^{\max}} \tilde{\theta}_t^i\, Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i \\
&\quad - \int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\alpha,\tilde{b}_t^i,n_t) \left(\int_{\alpha}^{\theta^{\max}} f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i\right) d\alpha\Bigg] \\
&= \sum_{\tilde{b}_t^i=1}^{k}\Bigg[\int_{\theta^{\min}}^{\theta^{\max}} \tilde{\theta}_t^i\, Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\, f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\, d\tilde{\theta}_t^i
\end{aligned}$$

$$-\int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\alpha,\tilde{b}_t^i,n_t)\left(g_t(\tilde{b}_t^i)\underbrace{\int_{\alpha}^{\theta^{\max}}\pi_t(\tilde{\theta}_t^i|\tilde{b}_t^i)\,d\tilde{\theta}_t^i}_{=1-\Pi_t(\alpha|\tilde{b}_t^i)}\right)d\alpha\Bigg]$$

$$=\sum_{\tilde{b}_t^i=1}^{k}\Bigg[\int_{\theta^{\min}}^{\theta^{\max}}\tilde{\theta}_t^i\,Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\,f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\,d\tilde{\theta}_t^i$$

$$-\int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\alpha,\tilde{b}_t^i,n_t)(1-\Pi_t(\alpha|\tilde{b}_t^i))\frac{f_t(\alpha,\tilde{b}_t^i)}{\pi_t(\alpha|\tilde{b}_t^i)}d\alpha\Bigg]$$

$$=\sum_{\tilde{b}_t^i=1}^{k}\Bigg[\int_{\theta^{\min}}^{\theta^{\max}}\tilde{\theta}_t^i\,Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\,f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\,d\tilde{\theta}_t^i$$

$$-\int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)(1-\Pi_t(\tilde{\theta}_t^i|\tilde{b}_t^i))\frac{f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)}{\pi_t(\tilde{\theta}_t^i|\tilde{b}_t^i)}d\tilde{\theta}_t^i\Bigg]$$

$$=\sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)$$

$$\times\left(\tilde{\theta}_t^i-\frac{1-\Pi_t(\tilde{\theta}_t^i|\tilde{b}_t^i)}{\pi_t(\tilde{\theta}_t^i|\tilde{b}_t^i)}\right)f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)d\tilde{\theta}_t^i$$

$$=\sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}} Q_t^i(\tilde{\theta}_t^i,\tilde{b}_t^i,n_t)\,w_t(\tilde{\theta}_t^i,\tilde{b}_t^i)f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\,d\tilde{\theta}_t^i. \quad (43)$$

The upper bound in (43) implies that the expected total revenue in (40) can be upper bounded by the following:

$$\sum_{t=1}^{T}\sum_{n_t=1}^{\bar{n}}\lambda_t(n_t)\sum_{i=1}^{n_t}\sum_{\tilde{b}_t^i=1}^{k}\int_{\theta^{\min}}^{\theta^{\max}}$$

$$\mathbb{E}\Bigg[w_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\sum_{j\le\tilde{b}_t^i}q_t^{i,j}(H_t^{-i},(\tilde{\theta}_t^i,\tilde{b}_t^i))\mid N_t=n_t\Bigg]f_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\,d\tilde{\theta}_t^i$$

$$=\sum_{t=1}^{T}\sum_{n_t=1}^{\bar{n}}\lambda_t(n_t)\sum_{i=1}^{n_t}\mathbb{E}\Bigg[w_t(\theta_t^i,b_t^i)\sum_{j\le b_t^i}q_t^{i,j}(H_t)\mid N_t=n_t\Bigg]$$

$$=\sum_{t=1}^{T}\mathbb{E}\Bigg[\sum_{i=1}^{N_t}w_t(\theta_t^i,b_t^i)\sum_{j\le b_t^i}q_t^{i,j}(H_t)\Bigg]$$

$$=\mathbb{E}\Bigg[\sum_{t=1}^{T}\sum_{i=1}^{N_t}w_t(\theta_t^i,b_t^i)\sum_{j\le b_t^i}q_t^{i,j}(H_t)\Bigg]$$

$$\le\max_{q_{1:T}}\mathbb{E}\Bigg[\sum_{t=1}^{T}\sum_{i=1}^{N_t}w_t(\theta_t^i,b_t^i)\left(\sum_{j\le b_t^i}q_t^{i,j}(H_t)\right)\Bigg].$$

Thus, the expected revenue of any BIC and IR mechanism is upper bounded by the maximum value in (15). Consequently, a BIC and IR mechanism $(q_{1:T}^*,p_{1:T}^*)$, for which $q_{1:T}^*$ achieves the maximum value in (15) and $p_{1:T}^*$ is of the form given in (16),[5] guarantees the largest expected revenue among all the BIC and IR mechanisms. This concludes the proof.

Appendix B
Proof of Lemma 4

We prove this by induction.

Base case: Clearly, the expression given for $R_T(h_T)$ in (17) and (18) solely depends on h_T^R and $y_T^{1:k}$ (recall that $R_{T+1}(\cdot)=0$). That is, the information in $h_T\setminus\{h_T^R,y_T^{1:k}\}$ is irrelevant for determining $R_T(h_T)$. Therefore, if we define the function $V_T(\cdot)$ as follows:

1) if $h_T^R=\emptyset$: $V_T(h_T^R,y_T^{1:k}):=0$
2) if $h_T^R\neq\emptyset$:

$$V_T(h_T^R,y_T^{1:k}):=\max_{\boldsymbol{A}\in\mathcal{S}(h_T^R,y_T^{1:k})}\left\{\sum_{i=1}^{|h_T^R|}w_T(\tilde{\theta}_T^i,\tilde{b}_T^i)\sum_{j=1}^{k}\boldsymbol{A}(i,j)\right\}$$

the equality in (19) holds true at time T.

Induction hypothesis: Suppose that there exists some function $V_{t+1}(\cdot)$ such that (19) holds true at time $t+1$.

Now, we want to show that there exists some function $V_t(\cdot)$ such that (19) holds true at time t. In other words, we want to show that given the history $h_t=\{h_t^R,y_t^{1:k},x_t^{1:k},h_{t-1}\}$, the expression given for $R_t(\cdot)$ in (17) and (18) is fully determined from $\{h_t^R,y_t^{1:k}\}$ and does not depend on $h_t\setminus\{h_t^R,y_t^{1:k}\}=\{x_t^{1:k},h_{t-1}\}$. For the case $h_t^R=\emptyset$, from (17), we see that $R_t(h_t)$ is expressed as

$$R_t(h_t)=\mathbb{E}\left[R_{t+1}(h_t,H_{t+1}^R,\{y_t^j+X_{t+1}^j\}_{j=1}^k,X_{t+1}^{1:k})\right].$$

Using the induction hypothesis, the above expression can be written as

$$R_t(h_t)=\mathbb{E}\left[V_{t+1}(H_{t+1}^R,\{y_t^j+X_{t+1}^j\}_{j=1}^k)\right].$$

Since H_{t+1}^R and X_{t+1}^j are independent of h_t, the expected value above depends only on $y_t^{1:k}$. Thus, when $h_t^R=\emptyset$, we can define

$$V_t(\emptyset,y_t^{1:k})=R_t(h_t)=\mathbb{E}_{H_{t+1}^R,Y_{t+1}^{1:k}}\left[V_{t+1}(H_{t+1}^R,Y_{t+1}^{1:k})\right].$$

Note that the above definition of $V_t(\cdot)$ satisfies (20).

For the case $h_t^R\neq\emptyset$, we see from (18) that $R_t(h_t)$ is expressed as

$$R_t(h_t)=\max_{\boldsymbol{A}\in\mathcal{S}(h_t^R,y_t^{1:k})}\Bigg\{\underbrace{\sum_{i=1}^{|h_t^R|}w_t(\tilde{\theta}_t^i,\tilde{b}_t^i)\sum_{j=1}^{k}\boldsymbol{A}(i,j)}_{\dagger}$$

$$+\mathbb{E}\left[R_{t+1}(H_{t+1})|h_t,\boldsymbol{A}_t=\boldsymbol{A}\right]\Bigg\}. \quad (44)$$

Clearly, the term † in the above expression does not depend on the information in $h_t\setminus\{h_t^R,y_t^{1:k}\}=\{x_t^{1:k},h_{t-1}\}$. Moreover, the set $\mathcal{S}(h_t^R,y_t^{1:k})$ over whose elements the $\max\{\cdot\}$ operation

[5] $p_{1:T}^*$ form in (16) makes the upper bound on the expected total revenue attainable, by ensuring that the inequality in (42) becomes an equality for $(q_{1:T}^*,p_{1:T}^*)$.

above is carried out is fully specified in terms of $h_t^R, y_t^{1:k}$ and does not depend on $\{x_t^{1:k}, h_{t-1}\}$. It, thus, remains to show that the second term in the $\max\{\cdot\}$ operation above does not depend on the information in $\{x_t^{1:k}, h_{t-1}\}$ either. Using the induction hypothesis, let us expand the second term above as follows:

$$\begin{aligned}
&\mathbb{E}\left[R_{t+1}(H_{t+1})|H_t = h_t, \boldsymbol{A}_t = \boldsymbol{A}\right] \\
&= \mathbb{E}\left[V_{t+1}(H_{t+1}^R, Y_{t+1}^{1:k})|H_t = h_t, \boldsymbol{A}_t = \boldsymbol{A}\right] \\
&= \mathbb{E}\Bigg[V_{t+1}\Bigg(H_{t+1}^R, \\
&\left\{y_t^j - \sum_{i=1}^{|h_t^R|}\boldsymbol{A}(i,j) + X_{t+1}^j\right\}_{j=1}^k\Bigg) \mid \{h_t^R, y_t^{1:k}, x_t^{1:k}, h_{t-1}\}, \boldsymbol{A}\Bigg] \\
&= \mathbb{E}\Bigg[V_{t+1}\Bigg(H_{t+1}^R, \Bigg\{y_t^j - \sum_{i=1}^{|h_t^R|}\boldsymbol{A}(i,j) \\
&\quad + X_{t+1}^j\Bigg\}_{j=1}^k\Bigg) \mid h_t^R, y_t^{1:k}, \boldsymbol{A}\Bigg] \\
&= \mathbb{E}\left[V_{t+1}(H_{t+1}^R, Y_{t+1}^{1:k}) \mid h_t^R, y_t^{1:k}, \boldsymbol{A}\right] \qquad (45)
\end{aligned}$$

where we used the fact that H_{t+1}^R and $X_{t+1}^{1:k}$ are independent of h_t. It is clear that the above conditional expectation is a function of $h_t^R, y_t^{1:k}$, and $\boldsymbol{A}$ and does not depend on the information in $\{x_t^{1:k}, h_{t-1}\}$. The above analysis allows us to conclude the following.

1) $R_t(h_t)$ is completely determined by h_t^R and $y_t^{1:k}$. Thus, we can define a function $V_t(h_t^R, y_t^{1:k}) = R_t(h_t)$.
2) Furthermore, using (44) and (45), it is clear that $V_t(\cdot)$ satisfies (21).

This completes the proof.

Appendix C
Proof of Lemma 5

Let $\mathcal{A} := \{u^{1:k} \in \mathbb{Z}_{\geq 0}^k : \sum_{l=1}^j u^l \leq \sum_{l=1}^j y_t^l \ ,\ u^j \leq n_t^j, \forall j\}$, i.e., $\mathcal{A}$ equals the set in the right-hand side of (22). Clearly, when $h_t^R = \emptyset, \mathcal{U}(h_t^R, y_t^{1:k}) := \{\mathbf{0}_{1\times k}\} = \mathcal{A}$.

Let us now consider the information state $s_t = (h_t^R, y_t^{1:k})$ with $h_t^R \neq \emptyset$. We start with showing that $\mathcal{U}(h_t^R, y_t^{1:k}) \subseteq \mathcal{A}$. Consider a vector $u_t^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$. Since n_t^j is the number of consumers with flexibility level j that arrive at time t and u_t^j is the number of consumers with flexibility level j that get a good at time t, we clearly have that $u_t^j \leq n_t^j$.

Now, consider $\sum_{l=1}^j u_t^l$. This is the total number of consumers with flexibility level less than or equal to j that get a good. Since consumers cannot get a good of variety higher than their flexibility level, it follows that $\sum_{l=1}^j u_t^l$ is less than or equal to the total number of available goods of variety less than or equal to j. In other words, $\sum_{l=1}^j u_t^l \leq \sum_{l=1}^j y_t^l$. Thus, $u_t^{1:k} \in \mathcal{A}$.

We now show that script $\mathcal{A} \subseteq \mathcal{U}(h_t^R, y_t^{1:k})$. Consider $u_t^{1:k} \in \mathcal{A}$. In order to prove that $u_t^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$, we need to show that there exists some matrix $\boldsymbol{D} \in \mathcal{S}(h_t^R, y_t^{1:k})$ such that $\sum_{i=1, i:\tilde{b}_t^i = j}^{|h_t^R|} \sum_{l\leq j} \boldsymbol{D}(i,l) = u_t^j, \forall j$. Let us construct such a matrix according to the following allocation procedure.

1) Select any u_t^1 consumers with flexibility level 1 and allocate each of them a good of variety 1. This is a feasible allocation since $u_t^1 \leq y_t^1$.
2) Select any u_t^2 consumers with flexibility level 2 and allocate each of them either an unallocated good of variety 1 (if $u_t^1 < y_t^1$) or a good of variety 2. These can be done since $u_t^1 + u_t^2 \leq y_t^1 + y_t^2$.
3) Proceed in a similar fashion for all flexibility levels: select any u_t^j consumers with flexibility level j and allocate each of them a good of any of the varieties $1, \ldots, j$ depending on their availability. Since $\sum_{l=1}^j u_t^l \leq \sum_{l=1}^j y_t^l$, the described allocation is feasible.
4) The other consumers that arrived at time t but were not selected for allocation in the above steps get zero allocation.

It is straightforward to verify that allocation matrix $\boldsymbol{D}$ constructed above belongs to $\mathcal{S}(h_t^R, y_t^{1:k})$ and that it serves u_t^j consumers of flexibility level j. Hence, every vector $u_t^{1:k} \in \mathcal{A}$ corresponds to a feasible allocation matrix $\boldsymbol{D} \in \mathcal{S}(h_t^R, y_t^{1:k})$. Hence, $u_t^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$. This establishes $\mathcal{A} \subseteq \mathcal{U}(h_t^R, y_t^{1:k})$ and completes the proof.

Appendix D
Proof of Lemma 7

Define

$$\mathcal{F}(h_t^R, y_t^{1:k}) := \left\{(u^{1:k}, v^{1:k}) \in \mathbb{Z}_{\geq 0}^{2k} : u^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})\ ,\ v^{1:k} \in \mathcal{V}(u^{1:k}, y_t^{1:k})\right\}. \qquad (46)$$

Furthermore, for any $(u^{1:k}, v^{1:k})$ in $\mathcal{F}(h_t^R, y_t^{1:k})$, define

$$\mathcal{S}_{\mathcal{F}}(h_t^R, y_t^{1:k}, u^{1:k}, v^{1:k}) := \Bigg\{\boldsymbol{A} \in \mathcal{S}(h_t^R, y_t^{1:k}) : \sum_{i=1}^{|h_t^R|}\boldsymbol{A}(i,j) = v^j,\ \sum_{\substack{i=1\\ i:\tilde{b}_t^i = j}}^{|h_t^R|}\sum_{l=1}^{j}\boldsymbol{A}(i,l) = u^j\ , j = 1, \ldots, k\Bigg\}. \qquad (47)$$

It is easy to check that the set of all feasible allocation matrices can be partitioned as

$$\mathcal{S}(h_t^R, y_t^{1:k}) = \bigcup_{(u^{1:k}, v^{1:k}) \in \mathcal{F}(h_t^R, y_t^{1:k})} \mathcal{S}_{\mathcal{F}}(h_t^R, y_t^{1:k}, u^{1:k}, v^{1:k}). \qquad (48)$$

Therefore, the value function in (21) can be written as

$$V_t(s_t) = \max_{(u^{1:k}, v^{1:k}) \in \mathcal{F}(s_t)} \Bigg\{ \max_{\boldsymbol{A} \in \mathcal{S}_{\mathcal{F}}(s_t, u^{1:k}, v^{1:k})} \Bigg\{ \sum_{i=1}^{|h_t^R|} w_t(\tilde{\theta}_t^i, \tilde{b}_t^i) \sum_{j=1}^{k}\boldsymbol{A}(i,j) + \underbrace{\mathbb{E}\left[V_{t+1}(S_{t+1})|s_t, \boldsymbol{A}_t = \boldsymbol{A}\right]}_{\dagger} \Bigg\}\Bigg\}. \qquad (49)$$

The † term in (49) can be written as

$$\dagger = \mathbb{E}\left[V_{t+1}\left(H^R_{t+1}, \{y^j_t - v^j + X^j_{t+1}\}^k_{j=1}\right)\right].$$

The above expectation depends only on $v^{1:k}$ and $y^{1:k}_t$ and not on the allocation matrix itself. Thus, (49) becomes

$$V_t(s_t) = \max_{(u^{1:k},v^{1:k})\in\mathcal{F}(s_t)} \left\{ \mathbb{E}\left[V_{t+1}\left(H^R_{t+1}, \{y^j_t - v^j + X^j_{t+1}\}^k_{j=1}\right)\right] + \underbrace{\max_{\boldsymbol{A}\in\mathcal{S}_\mathcal{F}(s_t,u^{1:k},v^{1:k})} \left\{ \sum_{i=1}^{|h^R_t|} w_t(\tilde{\theta}^i_t, \tilde{b}^i_t) \sum_{j=1}^{k} \boldsymbol{A}(i,j) \right\}}_{\ddagger} \right\}. \quad (50)$$

Considering the term ‡ in the above expression, it is straightforward to see that for all $\boldsymbol{A} \in \mathcal{S}_\mathcal{F}(s_t, u^{1:k}, v^{1:k})$

$$\sum_{i=1}^{|h^R_t|} w_t(\tilde{\theta}^i_t, \tilde{b}^i_t) \sum_{j=1}^{k} \boldsymbol{A}(i,j) \le \sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t$$

where $w^{i,j}_t$ denotes the ith largest element in $\mathcal{W}^j_t$ [see (25)]. Furthermore, an allocation matrix that, for each flexibility level j, gives goods to u^j consumers with highest virtual valuations satisfies the above inequality with equality. Hence

$$\max_{\boldsymbol{A}\in\mathcal{S}_\mathcal{F}(s_t,u^{1:k},v^{1:k})} \left\{ \sum_{i=1}^{|h^R_t|} w_t(\tilde{\theta}^i_t, \tilde{b}^i_t) \sum_{j=1}^{k} \boldsymbol{A}(i,j) \right\} = \sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t.$$

Plugging this result into (50), we obtain

$$V_t(h^R_t, y^{1:k}_t) = \max_{(u^{1:k},v^{1:k})\in\mathcal{F}(h^R_t,y^{1:k}_t)} \left\{ \sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t + \mathbb{E}\left[V_{t+1}\left(H^R_{t+1}, \{y^j_t - v^j + X^j_{t+1}\}^k_{j=1}\right)\right] \right\} \quad (51)$$

which can be rearranged in the form of the following nested maximization:

$$V_t(h^R_t, y^{1:k}_t) = \max_{u^{1:k}\in\mathcal{U}(h^R_t,y^{1:k}_t)} \left\{ \sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t + \max_{v^{1:k}\in\mathcal{V}(u^{1:k},y^{1:k}_t)} \left\{ \mathbb{E}\left[V_{t+1}\left(H^R_{t+1}, \{y^j_t - v^j + X^j_{t+1}\}^k_{j=1}\right)\right] \right\} \right\}. \quad (52)$$

This completes the proof.

Appendix E
Proof of Lemma 8

We provide an inductive proof of the lemma.

Base case: At time T, consider a nonempty history h^R_T and supply profiles $y^{1:k}_T$ and $z^{1:k}_T$ such that $y^i_T = z^i_T + 1$, $y^j_T = z^j_T - 1$, and $y^l_T = z^l_T$ for $l \ne i, j$, where $i < j$. From the definition of $\mathcal{U}(\cdot)$ in (22), it follows that $\mathcal{U}(h^R_T, z^{1:k}_T) \subseteq \mathcal{U}(h^R_T, y^{1:k}_T)$. This fact combined with the definition of $V_T(\cdot)$ implies that (27) holds for $t = T$ and $h^R_T \ne \emptyset$. If $h^R_T = \emptyset$, then $V_T(h^R_T, z^{1:k}_T) = 0 = V_T(h^R_T, y^{1:k}_T)$. Hence, (27) holds true at time T for all h^R_T.

Induction hypothesis: Suppose that the statement of the lemma is true for $V_{t+1}(\cdot)$. Consider two supply profiles $y^{1:k}_t$ and $z^{1:k}_t$ such that $y^i_t = z^i_t + 1$, $y^j_t = z^j_t - 1$ and $y^l_t = z^l_t$ for $l \ne i, j$, where $i < j$. We now show that given such $y^{1:k}_t$ and $z^{1:k}_t$, the property in (27) holds true at time t, i.e.,

$$V_t(h^R_t, y^{1:k}_t) \ge V_t(h^R_t, z^{1:k}_t) \quad \forall h^R_t. \quad (53)$$

Let us first consider $h^R_t = \emptyset$. In this case, we have

$$\begin{aligned} V_t(\emptyset, y^{1:k}_t) &= \mathbb{E}\left[\, V_{t+1}(H^R_{t+1}, Y^{1:k}_{t+1}) \mid y^{1:k}_t \,\right] \\ &= \mathbb{E}\left[\, V_{t+1}(H^R_{t+1}, \{y^l_t + X^l_{t+1}\}^k_{l=1}) \,\right] \\ &\ge \mathbb{E}\left[\, V_{t+1}(H^R_{t+1}, \{z^l_t + X^l_{t+1}\}^k_{l=1}) \,\right] \\ &= \mathbb{E}\left[\, V_{t+1}(H^R_{t+1}, Y^{1:k}_{t+1}) \mid z^{1:k}_t \,\right] \\ &= V_t(\emptyset, z^{1:k}_t) \end{aligned} \quad (54)$$

where the inequality in (54) follows from the induction hypothesis by noting that for any realization $x^{1:k}_{t+1}$, $y^i_{t+1} = y^i_t + x^i_{t+1} = z^i_{t+1} + 1$, $y^j_{t+1} = y^j_t + x^j_{t+1} = z^j_{t+1} - 1$, and $y^l_{t+1} = y^l_t + x^l_{t+1} = z^l_{t+1}$ for $l \ne i, j$. This establishes the property in (27) for $h^R_t = \emptyset$.

Now, consider $h^R_t \ne \emptyset$. To prove (27), it suffices to show that for every $(u^{1:k}, v^{1:k}) \in \mathcal{F}(h^R_t, z^{1:k}_t)$, there exists $(u^{1:k}, a^{1:k}) \in \mathcal{F}(h^R_t, y^{1:k}_t)$ such that

$$\sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t + \mathbb{E}\left[V_{t+1}(H^R_{t+1}, \{y^l_t - a^l + X^l_{t+1}\}^k_{l=1})\right] \ge \sum_{j=1}^{k}\sum_{i=1}^{u^j} w^{i,j}_t + \mathbb{E}\left[V_{t+1}(H^R_{t+1}, \{z^l_t - v^l + X^l_{t+1}\}^k_{l=1})\right] \quad (55)$$

Recall that $(u^{1:k}, v^{1:k}) \in \mathcal{F}(h^R_t, z^{1:k}_t)$ implies that $v^l \le z^l_t$ for all l.

For $(u^{1:k}, v^{1:k}) \in \mathcal{F}(h^R_t, z^{1:k}_t)$ two cases arise.

1) Case 1: $v^j < z^j_t$. In this case, we define $a^{1:k} = v^{1:k}$. It is clear that $(u^{1:k}, a^{1:k}) \in \mathcal{F}(h^R_t, y^{1:k}_t)$ and (55) holds.
2) Case 2: $v^j = z^j_t$. In this case, we cannot set $a^{1:k} = v^{1:k}$ since $v^j > y^j_t$. Therefore, we define $a^{1:k}$ as follows: $a^i = v^i + 1, a^j = v^j - 1$, and $a^l = v^l$ for $l \ne i, j$. It is straightforward to verify that $(u^{1:k}, a^{1:k}) \in \mathcal{F}(h^R_t, y^{1:k}_t)$. Furthermore, using the induction hypothesis

$$\begin{aligned} &\mathbb{E}\left[V_{t+1}(H^R_{t+1}, \{y^l_t - a^l + X^l_{t+1}\}^k_{l=1})\right] \\ &= \mathbb{E}\big[V_{t+1}(H^R_{t+1}, y^1_t - a^1 + X^1_{t+1}, \ldots, y^i_t - a^i + X^i_{t+1}, \\ &\ldots, y^j_t - a^j + X^j_{t+1}, \ldots, y^k_t - a^k + X^k_{t+1})\big] \\ &= \mathbb{E}\big[V_{t+1}(H^R_{t+1}, y^1_t - v^1 + X^1_{t+1}, \ldots \\ &\ldots, y^i_t - (v^i + 1) + X^i_{t+1}, \ldots, y^j_t - (v^j - 1) + X^j_{t+1}, \ldots \\ &\ldots, y^k_t - v^k + X^k_{t+1})\big] \\ &= \mathbb{E}\big[V_{t+1}(H^R_{t+1}, y^1_t - v^1 + X^1_{t+1}, \ldots \end{aligned}$$

$$\ldots,(y_t^i-1)-v^i+X_{t+1}^i,\ldots,(y_t^j+1)-v^j+X_{t+1}^j,\ldots$$
$$\ldots,y_t^k-v^k+X_{t+1}^k)]$$
$$=\mathbb{E}\left[V_{t+1}(H_{t+1}^R,\{z_t^l-v^l+X_{t+1}^l\}_{l=1}^k)\right].$$

This proves (55) and, thus, establishes (27) for $h_t^R\neq\emptyset$ Therefore, (27) holds true for all h_t^R at time t. This completes the proof.

Appendix F
Proof of Lemma 9

We prove the lemma in the following steps.

Step 1: We first show that $v^{*1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})$. Clearly, $v^{*j}\leq y_t^j$ and $\sum_{l=j}^k v^{*l}\leq\sum_{l=j}^k u^l$ for all j. To show that $v^{*1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})$, it suffices to show that $\sum_{l=1}^k v^{*l}=\sum_{l=1}^k u^l$. We will utilize the following claim.

Claim: Suppose that for all $m=1,\ldots,j$: (i) $\sum_{l=m}^k v^{*l}<\sum_{l=m}^k u^l$ and (ii) $v^{*i}=y_t^i$, for all $i<j$. Then, (a) $v^{*j}=y_t^j$ and (b) $\sum_{l=j+1}^k v^{*l}<\sum_{l=j+1}^k u^l$.

Assume for now that the claim is true. We have already seen that $\sum_{l=1}^k v^{*l}\leq\sum_{l=1}^k u^l$. Suppose

$$\sum_{l=1}^k v^{*l}<\sum_{l=1}^k u^l. \tag{56}$$

Then, using the claim above with $j=1$ implies that $v^{*1}=y_t^1$ and $\sum_{l=2}^k v^{*l}<\sum_{l=2}^k u^l$. We can now use the above claim for $j=2$ to conclude that $v^{*2}=y_t^2$ and $\sum_{l=3}^k v^{*l}<\sum_{l=3}^k u^l$. Proceeding this way until $j=k-1$, we get that 1) $v^{*m}=y_t^m$ for all $m\leq k-1$ and 2) $v^{*k}<u^k$. Furthermore, 2) and the definition of v^{*k} imply that $v^{*k}=y_t^k$. Thus, the entire $v^{*1:k}$ vector is equal to the $y_t^{1:k}$ vector. But we started with the statement that (56) is true. Thus, $\sum_{l=1}^k y_t^l<\sum_{l=1}^k u^l$, which contradicts the fact that $u^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})$. Thus, (56) is false, and hence, $\sum_{l=1}^k v^{*l}=\sum_{l=1}^k u^l$.

The only thing left now is the proof of the claim.

Proof of claim: The inequality $\sum_{l=j}^k v^{*l}<\sum_{l=j}^k u^l$ implies that $v^{*j}<\sum_{l=j}^k u^l-\sum_{l=j+1}^k v^{*l}$. This, along with the definition of v^{*j}, implies that $v^{*j}=y_t^j$. Therefore, (a) holds true.

We already know that $\sum_{l=j+1}^k v^{*l}\leq\sum_{l=j+1}^k u^l$. If $\sum_{l=j+1}^k v^{*l}=\sum_{l=j+1}^k u^l$, then from (i) in the claim statement with $m=1$, it follows that $\sum_{l=1}^j v^{*l}<\sum_{l=1}^j u^l$. This combined with $v^{*i}=y_t^i$ for all $i<j$ in (ii) and $v^{*j}=y_t^j$ in part (a) implies that $\sum_{l=1}^j y_t^l<\sum_{l=1}^j u^l$, which contradicts the fact that $u^{1:k}\in\mathcal{U}(h_t^R,y_t^{1:k})$. Hence, $\sum_{l=j+1}^k v^{*l}=\sum_{l=j+1}^k u^l$ cannot be true. This establishes (b).

Step 2: For any $v^{1:k}\neq v^{*1:k}$ in $\mathcal{V}(u^{1:k},y_t^{1:k})$, consider the highest j such that $v^j\neq v^{*j}$. We argue that $v^j>v^{*j}$ cannot be true. Given the definition of v^{*j}, either $v^{*j}=y_t^j$ or $v^{*j}=u^j+(\sum_{l=j+1}^k u^l-\sum_{l=j+1}^k v^{*l})$. If $v^{*j}=y_t^j$, then $v^j>v^{*j}$ contradicts $v^{1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})$. Now, suppose $v^{*j}=u^j+(\sum_{l=j+1}^k u^l-\sum_{l=j+1}^k v^{*l})$. Given that $v^l=v^{*l},l=j+1,\ldots,k$, $v^j>v^{*j}$ would then imply that $v^j>u^j+(\sum_{l=j+1}^k u^l-\sum_{l=j+1}^k v^l)$ or arranged differently $\sum_{l=j}^k v^l>\sum_{l=j}^k u^l$. This combined with the fact that $\sum_{l=1}^k v^l=\sum_{l=1}^k u^l$ would then imply that $\sum_{l=1}^{j-1} v^l<\sum_{l=1}^{j-1} u^l$, which contradicts $v^{1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})$. Thus, it can only be the case that $v^j<v^{*j}$.

Step 3: For any $v^{1:k}\neq v^{*1:k}$ in $\mathcal{V}(u^{1:k},y_t^{1:k})$, we define a new vector $\mathcal{T}(v^{1:k})$ as follows: Pick the highest j such that $v^j<v^{*j}$.[6] Then, pick the highest $i<j$ with $v^i>0$. It can easily be shown that such i and j exist. Then, $\mathcal{T}^j(v^{1:k})=v^j+1,\mathcal{T}^i(v^{1:k})=v^i-1$, and $\mathcal{T}^l(v^{1:k})=v^l$ for $l\neq i,j$, where $\mathcal{T}^l(v^{1:k})$ denotes the lth entry in $\mathcal{T}(v^{1:k})$. We now argue that $\mathcal{T}(v^{1:k})\in\mathcal{V}(u^{1:k},y_t^{1:k})$. For $m<i$ or $m\geq j$, it is clear that

$$\sum_{l=1}^m u^l\leq\sum_{l=1}^m\mathcal{T}^l(v^{1:k}).$$

Since j is the highest index with $v^j\neq v^{*j}$, it follows that

$$\sum_{l=1}^{j-1}v^l>\sum_{l=1}^{j-1}v^{*l}\geq\sum_{l=1}^{j-1}u^l.$$

Now, for any m such that $i\leq m<j$,

$$\sum_{l=1}^m\mathcal{T}^l(v^{1:k})=(v^i-1)+\sum_{l=1}^{i-1}v^l=\left(\sum_{l=1}^{j-1}v^l\right)-1\geq\sum_{l=1}^m u^l.$$

Therefore, $\mathcal{T}(v^{1:k})$ satisfies all the inequalities $\sum_{l=1}^m\mathcal{T}^l(v^{1:k})\geq\sum_{l=1}^m u^l, m=1,\ldots,k$. Furthermore, it is easy to verify that $\mathcal{T}^j(v^{1:k})\leq y_t^j$. Thus, $\mathcal{T}(v^{1:k})\in\mathcal{V}(u^{1:k},y_t^{1:k})$. We now show that the objective value in (29) is (weakly) larger under $\mathcal{T}(v^{1:k})$ compared to that under $v^{1:k}$. Let $a^{1:k}:=\mathcal{T}(v^{1:k})$. Using Lemma 8

$$\begin{aligned}
&\mathbb{E}[V_{t+1}(H_{t+1}^R,\{y_t^l-v^l+X_{t+1}^l\}_{l=1}^k)]\\
&=\mathbb{E}[V_{t+1}(H_{t+1}^R,y_t^1-v^1+X_{t+1}^1,\ldots\\
&\ldots,y_t^i-v^i+X_{t+1}^i,\ldots,\\
&\ldots,y_t^j-v^j+X_{t+1}^j,\ldots,y_t^k-v^k+X_{t+1}^k)]\\
&\leq\mathbb{E}[V_{t+1}(H_{t+1}^R,y_t^1-v^1+X_{t+1}^1,\ldots\\
&\ldots,y_t^i-(v^i-1)+X_{t+1}^i,\ldots,\\
&\ldots,y_t^j-(v^j+1)+X_{t+1}^j,\ldots,y_t^k-v^k+X_{t+1}^k)]\\
&=\mathbb{E}[V_{t+1}(H_{t+1}^R,\{y_t^l-a^l+X_{t+1}^l\}_{l=1}^k)].
\end{aligned}$$

Therefore, the objective value in (29) can only improve after applying the transformation $\mathcal{T}(\cdot)$.

Step 4: Starting with any $v^{1:k}\neq v^{*1:k}$ in $\mathcal{V}(u^{1:k},y_t^{1:k})$, we can keep applying transformation $\mathcal{T}(\cdot)$ to construct new vectors in $\mathcal{V}(u^{1:k},y_t^{1:k})$ that result in an objective value at least as large as that under $v^{1:k}$. This is conducted in the following while-loop:

```
1: while v^{1:k} ≠ v^{*1:k} do
2:     v^{1:k} ⟵ T(v^{1:k})
3: end while
4: return v^{1:k}
```

The above while-loop will terminate in a finite number of steps with $v^{1:k}=v^{*1:k}$ at termination. Thus, the objective value under $v^{*1:k}$ is at least as large as that under any $v^{1:k}\in\mathcal{V}(u^{1:k},y_t^{1:k})$. Thus, $v^{*1:k}$ is optimal.

[6]Note that the case $v^j>v^{*j}$ got ruled out in Step 2.

Appendix G
Proof of Theorem 1

Suppose that n_t consumers arrive at time t, and let (r, j) denote the type reported by the ith consumer arriving at time t. Assuming that all other consumers report their types truthfully, let $Q_t^{*i}(r, j, n_t)$ and $P_t^{*i}(r, j, n_t)$ denote the expected allocation and payment [see (10) and (11)], respectively, for this consumer under the mechanism $(q^*_{1:T}, p^*_{1:T})$, when it reports the pair (r, j).

Bayesian incentive compatibility and individual rationality: Because of Lemma 1, we can establish that $(q^*_{1:T}, p^*_{1:T})$ is BIC and individually rational by showing that the following conditions hold true.

i) $Q_t^{*i}(r, j, n_t)$ is nondecreasing in r for all i, t.
ii) $Q_t^{*i}(r, j, n_t)$ is nondecreasing in j for all i, t.
iii) $P_t^{*i}(\theta^{\min}, j, n_t) = 0$ for all j, n_t, t, i.
iv) $\theta^{\min}\, Q_t^{*i}(\theta^{\min}, j, n_t) = 0$ for all j, n_t, t, i.
v) $P_t^{*i}(r, j, n_t)$ is of the form given in (12) for all i, t.

We establish these conditions as follows.

(i) In order to establish that $Q_t^{*i}(r, j, \cdot)$ is nondecreasing in r, it suffices to show that $\sum_{l\le j} q_t^{*i,l}(h_t^{-i,R}, (r, j), y_t^{1:k})$ is nondecreasing in r, where $h_t^{-i,R}$ denotes the set of reports from all consumers other than i. Given $h_t^{-i,R}$ and $y_t^{1:k}$, consider two information states $\underline{s}_t := (h_t^{-i,R}, (\underline{r}, j), y_t^{1:k})$ and $\bar{s}_t := (h_t^{-i,R}, (\bar{r}, j), y_t^{1:k})$, where $\bar{r} > \underline{r}$. That is, consumer i has types $(\underline{r}, j)$ and $(\bar{r}, j)$ under $\underline{s}_t$ and $\bar{s}_t$, respectively. We now want to show that

$$\sum_{l\le j} q_t^{*i,l}(\bar{s}_t) \ge \sum_{l\le j} q_t^{*i,l}(\underline{s}_t). \tag{57}$$

Clearly, if $\sum_{l\le j} q_t^{*i,l}(\bar{s}_t) = 1$, (57) holds true. Let us consider the case where $\sum_{l\le j} q_t^{*i,l}(\bar{s}_t) = 0$. We need to argue that in this case, $\sum_{l\le j} q_t^{*i,l}(\underline{s}_t) = 0$. Let $u^{*1:k}$ denote the optimal $u^{1:k}$ vector obtained from solving the dynamic program in (30) under the information state $\bar{s}_t$. Since consumer i does not get served under $u^{*1:k}$ (recall that $\sum_{l\le j} q_t^{*i,l}(\bar{s}_t) = 0$), it can be shown that $u^{*1:k}$ is optimal under $\underline{s}_t$ as well and that consumer i will not get served under the information state $\underline{s}_t$. In other words, $\sum_{l\le j} q_t^{*i,l}(\underline{s}_t) = 0$ and (57) is true.

(ii) In order to establish that $Q_t^{*i}(r, j, \cdot)$ is nondecreasing in j, it suffices to prove that $\sum_{l\le j} q_t^{*i,l}(h_t^{-i,R}, (r, j), y_t^{1:k})$ is nondecreasing in j. We will use the following proposition in our proof.

Proposition 1: Let $u^{*1:k}$ denote the optimal vector that results from solving the dynamic program in (30) under the information state $s_t = (h_t^R, y_t^{1:k})$. Consider two flexibility levels j and j' with $j < j'$. Then, every consumer with flexibility level j' and virtual valuation greater than $w_t^{u^{*j}, j}$ gets served under $(q^*_{1:T}, p^*_{1:T})$.[7]

Proof: Suppose that the proposition is not true. Define the vector $\hat{u}^{1:k}$ as follows: $\hat{u}^j = u^{*j} - 1$, $\hat{u}^{j'} = u^{*j'} + 1$, and $\hat{u}^l = u^{*l}$ for all $l \ne j, j'$. Clearly, $\hat{u}^{1:k} \in \mathcal{U}(h_t^R, y_t^{1:k})$. Consider the expression of the value function given in (24) [which is equivalent to the definition in (30)]. It is straightforward to verify that the first term in (24), i.e., $\sum_{j=1}^{k}\sum_{i=1}^{u^j} w_t^{i,j}$, would be strictly larger under the vector $\hat{u}^{1:k}$ compared to that under $u^{*1:k}$. Moreover, since $\mathcal{V}(u^{*1:k}, y_t^{1:k}) \subseteq \mathcal{V}(\hat{u}^{1:k}, y_t^{1:k})$ [see (23)], the second term in (24) (i.e., the inner maximization over $v^{1:k}$ vector) cannot decrease under the vector $\hat{u}^{1:k}$ compared to that under $u^{*1:k}$. Therefore, the objective in (24) [equivalently, (30)] strictly improves under the vector $\hat{u}^{1:k}$ compared to that evaluated at $u^{*1:k}$, which contradicts the optimality of $u^{*1:k}$. This completes the proof. $\square$

Given $h_t^{-i,R}$ and $y_t^{1:k}$, consider two information states $\underline{s}_t := (h_t^{-i,R}, (r, \underline{c}), y_t^{1:k})$ and $\bar{s}_t := (h_t^{-i,R}, (r, \bar{c}), y_t^{1:k})$, where $\bar{c}, \underline{c} \in \{1, \ldots, k\}$, $\bar{c} > \underline{c}$. Thus, we need to show that

$$\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) \ge \sum_{l\le \underline{c}} q_t^{*i,l}(\underline{s}_t). \tag{58}$$

Clearly, if $\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) = 1$, (58) holds true. Let us consider the case where $\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) = 0$. We need to argue that in this case, $\sum_{l\le \underline{c}} q_t^{*i,l}(\underline{s}_t) = 0$. Let $u^{*1:k}$ denote the optimal $u^{1:k}$ vector obtained from solving the dynamic program in (30) under the information state $\bar{s}_t$. Let $n_t^j(s_t)$ denote the number of consumers with flexibility level j that arrive at time t under the information state s_t. Because consumer i with flexibility level $\bar{c}$ does not get a good under $\bar{s}_t$ (recall that $\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) = 0$), it clearly means that $u^{*\bar{c}} \le n_t^{\bar{c}}(\bar{s}_t) - 1 = n_t^{\bar{c}}(\underline{s}_t)$. Therefore, indeed, $u^{*1:k} \in \mathcal{U}(\underline{s}_t)$ [see (22)]. We now want to show that $u_t^{*1:k}$ is also optimal under $\underline{s}_t$.

Consider the following sequence of implications.

a) Since consumer i does not get served under the information state $\bar{s}_t$ (recall that $\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) = 0$), it follows from Proposition 1 that its virtual valuation must be no greater than the virtual valuations of the consumers that are served from flexibility levels *lower* than $\bar{c}$; in particular, $w_t(r, \bar{c}) \le w_t^{u^{*\underline{c}}, \underline{c}}$.
b) From Assumption 2, we know that $w_t(r, \underline{c}) < w_t(r, \bar{c})$. Therefore, (a) implies that $w_t(r, \underline{c}) < w_t^{u^{*\underline{c}}, \underline{c}}$.
c) $\sum_{l\le \bar{c}} q_t^{*i,l}(\bar{s}_t) = 0$ combined with (b) implies that the vector $u^{*1:k}$ results in the exact same objective value in (30) under both information states $\bar{s}_t$ and $\underline{s}_t$.
d) Since $\underline{c} < \bar{c}$ and $w_t(r, \underline{c}) < w_t(r, \bar{c})$, it is straightforward to show that under the information state $\underline{s}_t$, the value function in (30) is upper bounded by that under $\bar{s}_t$, i.e., $V_t(\underline{s}_t) \le V_t(\bar{s}_t)$.
e) Items (c) and (d) combined imply that $u^{*1:k}$ is optimal under $\underline{s}_t$ as well.

Items (e) and (b) above imply that consumer i with type $(r, \underline{c})$ does not get served under the information state $\underline{s}_t$, i.e., $\sum_{l\le \underline{c}} q_t^{*i,l}(\underline{s}_t) = 0$. Thus, (58) is true.

(iii)–(v): To establish conditions (iii)–(v), consider the payment form given as follows:

$$\rho_t^{*i}(h_t^{-i,R}, (r, j), y_t^{1:k}) = r \sum_{j'\le j} q_t^{*i,j'}(h_t^{-i,R}, (r, j), y_t^{1:k})$$

$$- \int_{\theta^{\min}}^{r} \left(\sum_{j'\le j} q_t^{*i,j'}(h_t^{-i,R}, (\alpha, j), y_t^{1:k}) \right) d\alpha. \tag{59}$$

[7] If $u^{*j} = 0$, then $w_t^{u^{*j}, j} := \infty$.

We first argue that monotonicity of $q_t^*(\cdot)$, as established above [condition (i)], implies that the payment form in (59) is equivalent to the one given in (31). Then, we show that this equivalent payment form in (59) indeed satisfies conditions (iii)–(v).

If consumer i with the report (r, j) does not get a good (i.e., $\sum_{j' \leq j} q_t^{*i,j'}(h_t^{-i,R}, (r, j), y_t^{1:k}) = 0$), then the monotonicity of q_t^* implies that the integral in (59) is also 0. Hence, in this case, consumer i pays nothing. On the other hand, if consumer i gets a good (i.e., $\sum_{j' \leq j} q_t^{*i,j'}(h_t^{-i,R}, (r, j), y_t^{1:k}) = 1$), then the definition of $\bar{\theta}_t^{i,j}$ [see (32)] implies that the integral in (59) is $(r - \bar{\theta}_t^{i,j})$. Hence, in this case, consumer i pays $\bar{\theta}_t^{i,j}$. Thus, the payment in (59) is identical to the payment in (31).

We now argue that the equivalent expression for $p_t^*(\cdot)$ in (59) satisfies the conditions in (iii)–(v).

To see that condition (iii) above holds true, recall that from Assumption 2, we have that $w_t(\theta^{\min}, j) < 0$ for all j, t. Hence, it must be that $\sum_{j' \leq j} q_t^{*i,j'}(h_t^{-i,R}, (\theta^{\min}, j), y_t^{1:k}) = 0$ for all $h_t^{-i,R}, y_t^{1:k}, j, i, t$. Otherwise, $q_{1:T}^*$ will not maximize the objective in (15) whose solution is given by the dynamic program in (20) and (21). Therefore, from (59), it follows that $\rho_t^{*i}(h_t^{-i,R}, (\theta^{\min}, j), y_t^{1:k}) = 0$ for all $h_t^{-i,R}, y_t^{1:k}, j, i, t$. This implies that $P_t^{*i}(\theta^{\min}, j, n_t) = 0$ for all j, n_t, t, i, which establishes condition (iii) above. Based on the same argument, condition (iv) above holds true as well.

By taking the expectation of $\rho_t^{*i}(H_t^{-i,R}, (r, j), Y_t^{1:k})$ in (59) over $(H_t^{-i,R}, Y_t^{1:k})$, where $H_t^{-i,R} = H_t^R \setminus \{(\theta_t^i, b_t^i)\}$, it is easily established that the expected payment $P_t^{*i}(\cdot)$ satisfies (12) with $\theta^{\min}\ Q_t^{*i}(\theta^{\min}, j, n_t) = 0$. Hence, condition (v) above holds true.

The above arguments establish that the mechanism $(q_{1:T}^*, p_{1:T}^*)$ is BIC and individually rational.

Expected-revenue maximization: The allocation functions $q_{1:T}^*$ constructed in Theorem 1 are the optimal control strategy for the stochastic control problem in (15). This is because Lemmas 4–9 established that the dynamic program in (30) is equivalent to the one in (17) and (18), which was formulated to address the control strategy optimization in (15). Moreover, the payment functions $p_{1:T}^*$ defined in (31) [which is equivalent to (59)] satisfy (16). Therefore, based on the results of Lemma 3, the mechanism $(q_{1:T}^*, p_{1:T}^*)$ is an expected-revenue-maximizing, BIC, and IR mechanism.

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