# A TROPICAL VIEW ON LANDAU-GINZBURG MODELS

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Abstract. This paper, largely written in 2009/2010, fits Landau-Ginzburg models into the mirror symmetry program pursued by the last author jointly with Mark Gross since 2001. This point of view transparently brings in tropical disks of Maslov index 2 via the notion of broken lines, previously introduced in two dimensions by Mark Gross in his study of mirror symmetry for $\mathbb{P}^2$.

A major insight is the equivalence of properness of the Landau-Ginzburg potential with smoothness of the anticanonical divisor on the mirror side. We obtain proper superpotentials which agree on an open part with those classically known for toric varieties. Examples include mirror LG models for non-singular and singular del Pezzo surfaces, Hirzebruch surfaces and some Fano threefolds.

## Contents



## Preface

This paper, largely written in 2009/2010, investigates the incorporation of Landau-Ginzburg models into the toric degeneration approach to mirror symmetry of the last author with Mark Gross [GS1, GS3]. At the time we could not answer a key question concerning the existence of the algorithm in [GS3] in the relevant unbounded case. We also felt that our construction poses many interesting questions and more should be said. With two of the authors leaving academia (M.C. 2010, M.P. 2011), the paper was eventually left in preliminary form on the

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last author's Hamburg webpage in the state of September 2010.[1] This preliminary version will remain available as an ancillary file in the arXiv-submission. A variation of the last three sections partly treating other cases appeared as part of the second author's doctoral thesis [Pu].

The key consistency proof of the superpotential in §3 and §4 has been repeatedly used in the construction of generalized theta functions in the surface and cluster variety cases, notably in [GHK, GHKK]. The question of existence of a consistent wall structure in the non-compact case eventually turned out to be best answered trivially by a compactification requirement (Definition 1.2, Proposition 1.5), or in the context of intrinsic mirror symmetry [GS6, GS7]. With a recent increase in studies of the smooth anticanonical divisor case, the case of central interest in this paper, it now seems the right time to finalize this paper.

To preserve the line of historical developments, we have mostly only done minor edits for accuracy and clarity. The exceptions are the mentioned compactification criterion in §1, Remark 4.11 on algebraizability of the superpotential, a new section on the fibers of the superpotential (§5), and a corrected treatment of the mirror of Hirzebruch surfaces taken from [Pu, §5.3]. We have also added some references to newer developments in the introduction and in some footnotes, but this did not seem the right place to give a comprehensive overview and due credit to all the wonderful developments that have happened around the topic in the last decade. We apologize with everybody whose newer contributions are not mentioned in this paper.

§5 existed in some form for many years and has been distributed occasionally, but the LG mirror map (Definition 5.11 and Theorem 5.12) has only rather recently been spelled out in discussions with Helge Ruddat in a joint project with Michel van Garrel on enumerative period integrals in Landau-Ginzburg models [GRS].

## Introduction

Mirror symmetry has been suggested both by mathematicians [Gi] and physicists [Wi, HV] to extend from Calabi-Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models. Mathematically a Landau-Ginzburg model is a non-compact Kähler manifold with a holomorphic function, the superpotential. Until recently, the majority of studies confined themselves to toric cases where the construction of the mirror is immediate. The one exception we are aware of is the work of Auroux, Katzarkov and Orlov on mirror symmetry for del Pezzo surfaces [AKO], where a symplectic mirror is constructed by a surgery construction. The general Floer-theoretic perspective for the mirror Landau-Ginzburg model of an SYZ fibered logarithmic Calabi-Yau manifold has been discussed by Auroux in [Au1, Au2]. In the following we use the phrase log Calabi-Yau to refer to a pair $(X, D)$ of a complete variety $X$ over a field with a non-zero effective anticanonical divisor $D \subset X$.[2]

The purpose of this paper is to fit the Fano/Landau-Ginzburg mirror correspondence into the mirror symmetry program via toric degenerations pursued by the last author jointly with

[1]The last named author presented our findings at the workshop "Derived Categories, Holomorphic Symplectic Geometry, Birational Geometry, Deformation Theory" at IHP/Paris in May 2010 and at VBAC 2010 in Lisbon in June 2010.

[2]In [GS3] log Calabi-Yau varieties were referred to as Calabi-Yau pairs to avoid confusion with the central fiber of toric degenerations of Calabi-Yau varieties.

Mark Gross [GS1, GS3]. The program as it stands suggests a non-compact variety as the mirror of a log Calabi-Yau variety, or rather toric degenerations of these varieties. So the main new ingredient is the construction of the superpotential.

The key technical idea of *broken lines* (Definition 4.2) for the construction of the superpotential has already appeared in a different context in the two-dimensional situation in Gross' mirror correspondence for $\mathbb{P}^2$ [Gr2]. We replace his case-by-case study of well-definedness with a scattering computation, making it work in all dimensions.

Our main findings can be summarized as follows.

(1) From our point of view, the natural data on the Fano side is a toric degeneration of log Calabi-Yau varieties as defined in [GS3], Definition 1.8. In particular, if arising from the localization of an algebraic family, the general fiber is a pair $(\check{X}, \check{D})$ of a complete variety $\check{X}$ and a reduced anticanonical divisor $\check{D}$. No positivity property is ever used in our construction apart from effectivity of the anticanonical bundle.

(2) The mirror is a toric degeneration of non-compact Calabi-Yau varieties, together with a canonically defined regular function on the total space of the degeneration (Proposition 1.5).

(3) The superpotential is proper if and only if the anticanonical divisor $\check{D}$ on the log Calabi-Yau side is locally irreducible (Theorem 2.5). These conditions also have clean descriptions on the underlying tropical models governing the mirror construction from [GS1, GS3]. §3 and §4 give the all order construction of the superpotential, summarized in Theorem 4.10.

(4) For smooth toric Fano varieties our construction provides a canonical (partial) compactification of the Hori-Vafa construction [HV] (Corollary 7.9 in the surface case, [Pu, Thm. 5.4] in all dimensions). But note Remark 4.11 concerning the general question of algebraizability of the superpotential.

(5) The terms in the superpotential can be interpreted in terms of virtual numbers of tropical disks, at least in dimension two (Proposition 6.15). On the Fano side these conjecturally count holomorphic disks with boundary on a Lagrangian torus.[3]

(6) The natural holomorphic parameters occurring in the construction on the Fano side lie in $H^1(\check{X}_0, \Theta_{(\check{X}_0,\check{D}_0)})$ where $\Theta_{(\check{X}_0,\check{D}_0)}$ is the logarithmic tangent bundle of the central fiber $(\check{X}_0, \check{D}_0)$ in (1). This group rules infinitesimal deformations of the pair $(\check{X}_0, \check{D}_0)$. We have not carefully analyzed the parameters on the Landau-Ginzburg side and their correspondence to the Kähler parameters on the log Calabi-Yau side.[4] Note however that all parameters come from deformations of the underlying space, our superpotential does not add extra parameters.

(7) Explicit computations include non-singular and singular del Pezzo surfaces, the Hirzebruch surfaces $\mathbb{F}_2$ and $\mathbb{F}_3$, $\mathbb{P}^3$ and a singular Fano threefold (§7–§8).

Throughout we work over an algebraically closed field $\Bbbk$ of characteristic 0. We use check-adorned symbols $\check{X}, \check{D}, \check{\mathfrak{X}}, \check{\mathfrak{D}}, \check{B}$ etc. for the log Calabi-Yau side, and unadorned symbols for the Landau-Ginzburg side. For an integral polyhedron $\tau \subset \mathbb{R}^n$ we denote by $\Lambda_\tau$ the free abelian group of integral tangent vector fields on $\tau$.

[3] This picture has recently been confirmed in [GS7] in terms of punctured Gromov-Witten invariants.

[4] The correspondence is transparent from the intrinsic mirror symmetry perspective [GS6, GS7].

We would like to thank Denis Auroux, Mark Gross and Helge Ruddat for valuable discussions.

## 1. Landau-Ginzburg tropical data and scattering diagram

Throughout the paper we assume familiarity with the basic notions of toric degenerations from [GS1], as reviewed in [GS3, §§1.1–1.2]. We quickly review the basic picture. Let $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ be the polarized intersection complex or *cone picture* [GS3, Expl. 1.13] associated to a (schematic or formal) proper polarized toric degeneration $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ of log Calabi-Yau varieties [GS3, Defs. 1.8/1.9]. Here $T$ is the (formal) spectrum of a discrete valuation $\Bbbk$-algebra, typically $\Bbbk[\![t]\!]$. Recall that $\check{B}$ is a topological manifold with a $\mathbb{Z}$-affine structure outside a codimension two cell complex $\check{\Delta} \subset \check{B}$, also called the discriminant locus; $\check{\mathscr{P}}$ is a decomposition of $\check{B}$ into integral, convex, but possibly unbounded polyhedra containing $\check{\Delta}$ as subcomplex of the first barycentric subdivision disjoint from vertices and the interiors of maximal cells; and $\check{\varphi}$ is a (generally multivalued) strictly convex piecewise linear function with integral slopes. The irreducible components of the central fiber $\check{X}_0 \subset \check{\mathfrak{X}}$ are the toric varieties with momentum polytopes the maximal cells in $\check{\mathscr{P}}$, and lower dimensional cells describe their intersections.

Equivalently, one has the discrete Legendre dual data $(B, \mathscr{P}, \varphi)$, referred to as the dual intersection complex or *fan picture* of the same degeneration [GS3, Expl. 1.11], or the cone picture of the mirror via discrete Legendre duality [GS3, Constr. 1.16].

While [GS1] only treated the case of trivial canonical bundle or closed $B$, [GS3] gave the straightforward generalization to the case of interest here of log Calabi-Yau varieties, that is, a variety $\check{X}$ and an anticanonical divisor $\check{D} \subseteq \check{X}$. These correspond to compact $\check{B}$ with locally convex boundary $\partial\check{B}$, with $\partial\check{B} \neq \emptyset$ iff $\check{D} \neq \emptyset$. It holds $\partial\check{B} \neq \emptyset$ iff the discrete Legendre-dual $B$ is non-compact. The proof of [GS1, Thm. 5.4] then still shows that under a primitivity assumption on the local affine geometry (“simple singularities, [GS1, Def. 1.60]), the corresponding central fibers $(\check{X}_0, \mathcal{M}_{\check{X}_0})$ of toric degenerations of log Calabi-Yau varieties $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$, as a log space, are classified by the cohomology groups

$$H^1(B, i_*\Lambda_B \otimes_{\mathbb{Z}} \mathbb{G}_m(\Bbbk)) = H^1(\check{B}, \check{\imath}_*\check{\Lambda}_{\check{B}} \otimes_{\mathbb{Z}} \mathbb{G}_m(\Bbbk))$$

computed from the affine geometry of $B$ or $\check{B}$. Here $\Lambda_B$ denotes the sheaf of integral tangent vectors on the complement of the real codimension two singular locus $\Delta \subseteq B$, $i : B \setminus \Delta \to B$ is the inclusion, and $\check{\Lambda}_B = \mathcal{H}om(\Lambda_B, \mathbb{Z}_B)$. Similar notations apply to $\check{B}$. The correspondence works via twisting toric patchings of standard affine toric models for the degeneration via so-called *open gluing data* $s$ [GS3, Def. 1.18] and showing that any choice of $s$ gives rise to a log structure on $\check{X}_0 = \check{X}_0(s)$ over the standard log point. This log structure is unique up to isomorphisms fixing $\check{X}_0$. Unlike in abstract deformation theory, the space of deformations is not just a torsor over the controlling cohomology group, but a group itself. In particular, trivial gluing data $s = 1$ lead to a distinguished choice of log Calabi-Yau central fiber $(\check{X}_0, \mathcal{M}_{\check{X}_0})$. The log structure also carries the information of the anticanonical divisor $\check{D}_0 \subseteq \check{X}_0$, which hence is suppressed in the notation.

Conversely, [GS3, Thm. 3.1] constructs a canonical formal polarized toric degeneration $\pi : (\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ with given logarithmic central fiber $(\check{X}_0, \mathcal{M}_{\check{X}_0})$, even under a weaker assumption (*local rigidity*, [GS3, Def. 1.26]) than simplicity. Thus we understand this side of the mirror correspondence rather well.

The objective of this paper is to give a similarly canonical picture on the mirror side. The mirrors of Fano varieties are suggested to be so-called Landau-Ginzburg models (LG-models). Mathematically these are non-compact algebraic varieties with a holomorphic function, referred to as *superpotential*, see e.g. [HV, CO, AKO, FOOO]. Following the general program laid out in [GS1] and [GS3], we construct LG-models via deformations of a non-compact union of toric varieties. The superpotential is constructed by canonical extension from the central fiber.

Our starting point in this paper is as follows.

**Assumption 1.1.** Let $(B, \mathscr{P}, \varphi)$ be a non-compact, polarized, (integral) tropical manifold without boundary [GS3, Def.1.2]. We further assume that $(B, \mathscr{P}, \varphi)$ comes with a compatible sequence of consistent (wall) structures $\mathscr{S}_k$ as defined in [GS3, Defs. 2.22, 2.28, 2.41], for some choice of open gluing data $s$.

For applications in mirror symmetry, $(B, \mathscr{P}, \varphi)$ is the Legendre-dual of a compact $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ with locally convex boundary. Unfortunately, the algorithmic construction of consistent structures in [GS3] is problematic at several places in the non-compact case.

The only practical general assumption we are aware of to make the algorithm work in the non-compact case adds the following convexity requirement at infinity.

**Definition 1.2.** We call a tropical manifold $(B, \mathscr{P})$ with or without boundary *compactifiable* if there exists a compact subset $K \subseteq B$ containing a neighborhood of the union of all bounded cells of $\mathscr{P}$ and a proper continuous map $\psi : B \setminus K \to \mathbb{R}_{\geq 0}$ with the following properties: (1) $\psi$ is locally on $B \setminus (K \cup \Delta)$ a convex function. (2) Each unbounded cell $\sigma \in \mathscr{P}$ has a finite integral polyhedral decomposition $\mathscr{P}_\sigma$ such that $\psi|_{\sigma \cap (B \setminus K)}$ is a convex, integral, piecewise affine function with respect to $\mathscr{P}_\sigma$.

The existence of $\psi$ in Definition 1.2 makes it possible to exhaust $B$ by tropical manifolds as follows.

**Lemma 1.3.** *Assume that $(B, \mathscr{P})$ is compactifiable (Definition 1.2). Then there exists a sequence of compact subsets $\overline{B}_1 \subseteq \overline{B}_2 \subseteq \ldots \subseteq B$ with (1) $B = \bigcup_{\nu \geq 1} \overline{B}_\nu$, and (2) $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$ with $\overline{\mathscr{P}}_\nu = \{\sigma \cap \overline{B}_\nu \,|\, \sigma \in \mathscr{P}\}$ is a tropical manifold in the sense of* [GS3, Def. 1.32].

*Proof.* Let $K \subseteq B$ and $\psi : B \setminus K \to \mathbb{R}_{\geq 0}$ be as in Definition 1.2. Define $\overline{B}_\nu = K \cup \psi^{-1}([0, \nu])$. Properness of $\psi$ implies $B = \bigcup_\nu \overline{B}_\nu$ as claimed in (1). Next observe that all bounded cells of $\mathscr{P}$ are contained in all $\overline{B}_\nu$ since they are contained in $K$. For an unbounded cell $\sigma \in \mathscr{P}$, Definition 1.2,(2) implies that the intersection $\sigma \cap \overline{B}_\nu$ is a compact convex polyhedron defined over $\mathbb{Q}$. The denominators appearing in the vertices of $\partial(\sigma \cap \overline{B}_\nu)$ disappear when going over to an appropriate multiple of $\nu$. Thus up to going over to a subsequence, $\mathscr{P}$ induces an integral polyhedral decomposition $\overline{\mathscr{P}}_\nu$ of $\overline{B}_\nu$.

Convexity at boundary points as required in [GS3, Def. 1.32,(2)] follows from convexity of $\psi$ posited in Definition 1.2,(1), provided $\partial \overline{B}_\nu \cap K \neq \emptyset$. The last condition clearly holds for $\nu \gg 0$, hence can be achieved for all $\nu$ by relabelling the $\overline{B}_\nu$. □

The proof of Lemma 1.3 motivates the following definition.

**Definition 1.4.** We call a tropical manifold $(\overline{B}, \overline{\mathscr{P}})$, with compact $\overline{B} \subset B$ containing all bounded cells of $\mathscr{P}$ and intersecting the interior of each unbounded cell, a *truncation* of $(B, \mathscr{P})$.

With an exhaustion as in Lemma 1.3 we can now apply the main theorem of [GS3] to produce a compatible sequence of consistent wall structures on $(B, \mathscr{P}, \varphi)$.

**Proposition 1.5.** *Let $(B, \mathscr{P}, \varphi)$ be a polarized, tropical manifold with $(B, \mathscr{P})$ compactifiable. Assume further that each $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$ from Lemma 1.3 describes the intersection complex of a locally rigid, positive, pre-polarized toric log Calabi-Yau variety $(\overline{X}_0, \overline{B}_0)$* [GS3, Defs. 1.4, 1.26, 1.23][5]*. Then there exists a compatible sequence of consistent (wall) structures $\mathscr{S}_k$ on $(B, \mathscr{P}, \varphi)$.*

*The formal toric degeneration* [GS3, Def. 1.9] $\pi : (\mathfrak{X}, \mathfrak{D}) \to \operatorname{Spf} \Bbbk[\![t]\!]$ *defined by the $\mathscr{S}_k$ according to* [GS3, Prop. 2.42] *has an open embedding into a proper formal family $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}}) \to$* $\operatorname{Spf} \Bbbk[\![t]\!]$*, with central fiber $\overline{X}_0$.*

*Moreover, assuming $H^1(\overline{X}_0, \mathcal{O}_{\overline{X}_0}) = H^2(\overline{X}_0, \mathcal{O}_{\overline{X}_0}) = 0$, this family is algebraizable: There exists a proper flat morphism $\overline{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{D}}) \to$* $\operatorname{Spec} \Bbbk[\![t]\!]$*, a toric degeneration in the sense of* [GS3, Def.1.8]*, and a divisor $\mathcal{Z} \subset \overline{\mathcal{X}}$ flat over* $\operatorname{Spec} \Bbbk[\![t]\!]$ *such that $\pi$ is isomorphic to the completion of $\overline{\pi}|_{\overline{\mathcal{X}} \setminus \mathcal{Z}}$ at the central fiber.*

*Proof.* For each $\nu$, [GS3, Thm. 3.1] produces a compatible sequence $\mathscr{S}_{k,\nu}$ of consistent wall structures on $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$. Comparing the inductive construction of $\mathscr{S}_{k,\nu}$ for fixed $k$, but taking $\nu \to \infty$ shows that the sets of walls stabilize on any compact subset of $B$. Note that by convexity no walls emanate from $\partial \overline{B}_\nu$. We can therefore take the limit over $\nu$ to define $\mathscr{S}_k$ on $B$. Mutual compatibility and consistency follows by consideration on compact subsets of $B$, using consistency and compatibility for the wall structure on $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$ for $\nu$ sufficiently large.

The compactification $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}})$ is constructed by observing that for each $k$ the proper schemes over $\Bbbk[t]/(t^{k+1})$ obtained from $\mathscr{S}_{k,\nu}$ stabilize for $\nu \to \infty$. Taking this limit first and then the limit over $k$ now defines the formal scheme $\overline{\mathfrak{X}}$ proper over $\operatorname{Spf} \Bbbk[\![t]\!]$ and containing $(\mathfrak{X}, \mathfrak{D})$ as an open dense formal subscheme.

With the additional assumptions, algebraizability follows from the Grothendieck algebraization theorem as in [GS3, Cor. 1.31]. □

*Remark* 1.6. The compactifying divisor $\mathfrak{Z} \subset \overline{\mathfrak{X}}$ with $\mathfrak{X} = \overline{\mathfrak{X}} \setminus \mathfrak{Z}$ can be described by inspection of the polyhedral decomposition $\mathscr{P}_\nu$ of $\overline{B}_\nu$ for any sufficiently large $\nu$ and the asymptotic behavior of $\mathscr{S}_k$. For each sequence $F = (F_\nu)_\nu$ of mutually parallel maximal flat affine subsets $F_\nu \subseteq \partial \overline{B}_\nu$ there exists a maximal closed reduced subscheme $\mathfrak{Z}_F \subseteq \mathfrak{Z}$, and $\mathfrak{Z}$ is the schematic union of the $\mathfrak{Z}_F$. The restriction of $\pi$ to $\mathfrak{Z}_F$ is itself a toric degeneration defined by the walls of $\mathscr{S}_k$ that intersect $F_\nu$ for all $\nu$, with wall functions obtained by disregarding all monomials not tangent to $F_\nu$.

These statements follow directly from the gluing construction [GS3, §2.6]. See §5 for a further discussion in the context of fibers of the superpotential.

[5]The assumptions are fulfilled for example if $(B, \mathscr{P})$ has simple singularities.

## 2. The superpotential at $t$-order zero

Recall from [GS3, Constr. 2.7] the definition of the rings $R^k_{g,\sigma}$ used to build $\mathfrak{X}$ over $\Bbbk[t]/(t^{k+1})$. These depend on an inclusion $g:\omega\to\tau$ of cells $\omega,\tau\in\mathscr{P}$ and a maximal reference cell $\sigma$ containing $\tau$ (hence $\omega$). The ring $R^k_{g,\sigma}$ provides the $k$-th order thickening inside $\mathfrak{X}$ of the stratum $X_\tau\subseteq X_0$ with momentum polytope $\tau$ in an affine open neighbourhood of the $\omega$-stratum $X_\omega\subseteq X_\tau$. The order is measured with respect to all irreducible components of $X_0$ containing the $\tau$-stratum. The reference cell $\sigma$ is necessary to fix the affine chart to work on. The rings obtained in this way from affine toric models are then localized at all functions carried by codimension one cells ("*slabs*") containing $\tau$.

The details of this construction are rather irrelevant for what follows except that the ring $R^k_{g,\sigma}$ is a localization of a monomial ring, with each monomial $z^m$ having an associated underlying tangent vector $\overline{m}\in\Lambda_\sigma$, and an order of vanishing $\operatorname{ord}_{\sigma'} z^m$ for each maximal cell $\sigma'\supseteq\tau$. If $\tau=\sigma$ and $\operatorname{ord}_\sigma z^m=l$ then $z^m$ restricts to $z^{\overline{m}}t^l$ in the Laurent polynomial ring $R^k_{\mathrm{id}_\sigma,\sigma}=A[\Lambda_\sigma]$ describing the trivial $k$-th order deformation of the big cell of the irreducible component $X_\sigma\subseteq X_0$ defined by $\sigma$ over $A=\Bbbk[t]/(t^{k+1})$.

We need the following definition.

**Definition 2.1.** We call unbounded edges $\omega,\omega'\in\mathscr{P}$ *parallel* if there exists a sequence of unbounded edges $\omega=\omega_0,\omega_1,\ldots,\omega_r=\omega'$ and maximal cells $\sigma_1,\ldots,\sigma_r\in\mathscr{P}$ with $\omega_{i-1},\omega_i\subseteq\sigma_i$ parallel ($\Lambda_\omega=\Lambda_{\omega'}$ as sublattices of $\Lambda_\sigma$) and unbounded in the same direction.

A tropical manifold $(B,\mathscr{P})$ with all unbounded edges parallel is called *asymptotically cylindrical.*

Let now $\sigma\in\mathscr{P}$ be an unbounded maximal cell. For each unbounded edge $\omega\subseteq\sigma$ there is a unique monomial $z^{m_\omega}\in R^0_{\mathrm{id}_\sigma,\sigma}$ with $\operatorname{ord}_\sigma(m_\omega)=0$ and $-\overline{m_\omega}$ a primitive generator of $\Lambda_\omega\subseteq\Lambda_\sigma$ pointing in the unbounded direction of $\omega$. Denote by $\mathscr{R}(\sigma)$ the set of such monomials $m_\omega$. We identify monomials for parallel unbounded edges $\omega,\omega'$. So these contribute only one exponent $m_\omega=m_{\omega'}$ to $\mathscr{R}(\sigma)$.

Now at any point of $\partial\sigma$, the tangent vector $-\overline{m_\omega}$ points into $\sigma$. Hence

$$W^0(\sigma):=\sum_{m\in\mathscr{R}(\sigma)} z^m \tag{2.1}$$

extends to a regular function on the component $X_\sigma\subseteq X_0$ corresponding to $\sigma$. For bounded $\sigma$ define $W^0(\sigma)=0$. To simplify the following discussion, from now on we only consider the following situation.[6]

**Assumption 2.2.** One of the following two conditions hold.

(1) If $\omega,\omega'\in\mathscr{P}$ are parallel edges there exists a maximal cell $\sigma\in\mathscr{P}$ with $\omega\cup\omega'\subseteq\sigma$.
(2) The open gluing data $s$ are trivial.

Since by Assumption 2.2 the restrictions of the $W^0(\sigma)$ to lower dimensional toric strata agree, they define a function $W^0\in\mathcal{O}(X_0)$. This is our *superpotential at order* 0. A motivation for this definition in terms of counts of tropical analogues of holomorphic disks will be given in Section 6.

[6] This assumption was missing in preliminary versions of this paper and fixes a subtlety arising with non-trivial gluing data. The correct treatment of gluing data is given in [GHS, §5.2].

One insight in this paper is that in studying LG-models tropically it is advisable to restrict to asymptotically cylindrical $B$.

**Proposition 2.3.** *The order zero superpotential $W^0 : X_0 \to \mathbb{A}^1$ is proper iff $(B, \mathscr{P})$ is asymptotically cylindrical.*

*Proof.* It suffices to show the claimed equivalence after restriction to a non-compact irreducible component $X_\sigma \subseteq X_0$, that is, for $W^0(\sigma)$ from (2.1). If all unbounded edges are parallel, $W^0(\sigma)$ is a multiple of a monomial with compact zero locus, and hence is proper.

For the converse we show that if $m_\omega \neq m_{\omega'}$ for some $\omega, \omega' \subseteq \sigma$ then $W^0(\sigma)$ is not proper. The idea is to look at the closure of the zero locus of $W^0(\sigma)$ in an appropriate toric compactification $X_{\tilde{\sigma}} \supset X_\sigma$.

Let $\omega_0, \ldots, \omega_r$ be the unbounded edges of $\sigma$ and write $m_i = m_{\omega_i}$ for their primitive generators. By assumption $\operatorname{conv}\{0, m_0, \ldots, m_r\}$ has a face not containing 0 of dimension at least one. Let $H \subseteq \Lambda_\sigma$ be a supporting affine hyperplane of such a face. After relabeling we may assume $m_0, \ldots, m_s$ are the vertices of this face. Note that all $m_i - m_0$ are contained in the affine half-space $H - \mathbb{R}_{\geq 0} m_0$. Now $\tilde{\sigma} = \sigma \cap (H - \mathbb{R}_{\geq 0} m_0)$ is a rational bounded polytope $\tilde{\sigma} \subseteq \sigma$ with a single facet $\tau \subset \tilde{\sigma}$ not contained in a facet of $\sigma$. Thus the toric variety $X_{\tilde{\sigma}}$ with momentum polytope $\tilde{\sigma}$ contains $X_\sigma$ as the complement of the toric prime divisor $X_\tau \subset X_{\tilde{\sigma}}$. Note that $\Lambda_\tau = H - m_0$ by construction.

To study the closure of the zero locus of $W^0(\sigma)$ in $X_{\tilde{\sigma}}$ consider the rational function $z^{-m_0} \cdot W^0(\sigma)$ on $X_{\tilde{\sigma}}$. This rational function does not contain $X_\tau$ in its polar locus, and its restriction to the big cell of $X_\tau$ is

$$1 + \sum_{i=1}^{s} z^{m_i - m_0} \in \Bbbk[\Lambda_\tau].$$

In fact, $z^{m_i - m_0}$ for $i > s$ vanishes along $X_\tau$. Since $s \geq 1$ this Laurent polynomial has a non-empty zero locus. This proves that unless $m_i = m_j$ for all $i, j$ the closure of the zero locus of $W^0(\sigma)$ in $X_{\tilde{\sigma}}$ has a non-empty intersection with $X_\tau$, and hence $W^0(\sigma)$ can not be proper. □

Thus if one is to study LG models via our degeneration approach, then to obtain the full picture one has to restrict to asymptotically cylindrical $(B, \mathscr{P})$.

The interpretation on the mirror side of the condition that $(B, \mathscr{P})$ be asymptotically cylindrical brings us to one of the main insights of this paper. We first formulate the Legendre-dual of Definition 2.1.

**Definition 2.4.** A tropical manifold $(\check{B}, \check{\mathscr{P}})$ is said to have *flat boundary* if $\partial \check{B}$ is locally flat in the affine structure.

**Theorem 2.5.** *Let $\mathfrak{X} \to \operatorname{Spf} \Bbbk[\![t]\!]$ and $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ be polarized toric degenerations with Legendre dual intersection complexes $(B, \mathscr{P}, \varphi)$, $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. Then the following are equivalent.*

(1) *The order zero superpotential $W^0 : X_0 \to \mathbb{A}^1$ defined in* (2.1) *is proper.*
(2) $(B, \mathscr{P})$ *is asymptotically cylindrical.*
(3) $(\check{B}, \check{\mathscr{P}})$ *has flat boundary.*
(4) *The restriction $\pi|_{\check{\mathfrak{D}}} : \check{\mathfrak{D}} \to T$ of $\pi : \check{\mathfrak{X}} \to T$ is itself a toric degeneration.*

*Proof.* The equivalence of (1) and (2) is the content of Proposition 2.3.

The Legendre dual of the condition that $(B, \mathscr{P})$ is asymptotically cylindrical says that $\partial\check{B} \subseteq \check{B}$ is itself an affine manifold with singularities to which our program applies. If this is the case then from the definition of $\check{\mathfrak{D}} \subseteq \check{\mathfrak{X}}$ it follows that $\check{\mathfrak{D}} \to T$ is obtained by restricting the slab functions to $\check{D}_0 \subseteq \check{X}_0$ and run our gluing construction [GS3, §2.6,§2.7] on $\partial\check{B}$. The result is hence a toric degeneration of Calabi-Yau varieties.

Conversely, assume that $\rho, \rho' \subseteq \partial\check{B}$ are two neighboring $(n-1)$-faces with $\Lambda_\rho \neq \Lambda_{\rho'}$ as subspaces of $\Lambda_v$ for $v \in \rho \cap \rho'$. In the notation of [GS3, Constr. 2.7], the toric local model of $\check{\mathfrak{X}} \to T$ is given by $\Bbbk[P_{\rho\cap\rho',\sigma}]$ for $\sigma \in \mathscr{P}$ a maximal cell containing $\rho \cap \rho'$, with $\check{\mathfrak{D}}$ locally corresponding to $\rho \cup \rho'$. It is now easy to see that $\check{\mathfrak{D}} \to T$ is not formally a smoothing of $\check{D}_0$ at the generic point of $\check{X}_{\rho\cap\rho'}$ unless $\partial\check{B}$ is straight at $\rho \cap \rho'$. Hence $\partial\check{B}$ has to be smooth for $\check{\mathfrak{D}} \to T$ to be a toric degeneration. □

Theorem 2.3,(4) motivates the following definition.

**Definition 2.6.** A toric degeneration of log Calabi-Yau varieties $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ is called *of compact type* if $\check{\mathfrak{D}} \to T$ is as well a toric degeneration of Calabi-Yau varieties.

As a first example we consider the case of $\mathbb{P}^2$.

**Example 2.7.** The standard method to construct the LG-mirror for $\mathbb{P}^2$ is to start from the momentum polytope $\Xi = \mathrm{conv}\{(-1,-1),(2,-1),(-1,2)\}$ of $\mathbb{P}^2$ with its anticanonical polarization [HV] . The rays of the corresponding normal fan associated to this polytope (using inward pointing normal vectors as in [GS3]) are generated by $(1,0),(0,1),(-1,-1)$. Calling the monomials corresponding to the generators of the first two rays $x$ and $y$, respectively, we obtain the usual (non-proper) Landau-Ginzburg model on the big torus $(\mathbb{G}_m(\Bbbk))^2$, the function $x + y + \frac{1}{xy}$.

To obtain a proper superpotential we need to make the boundary of the momentum polytope flat in affine coordinates. To do this we trade the corners with singular points in the interior. The simplest choice is a decomposition $\check{\mathscr{P}}$ of $\check{B} = \Xi$ into three triangles with three singular points with simple monodromy, that is, conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, as depicted in Figure 2.1. A minimal choice of the PL-function $\check{\varphi}$ takes values 0 at the origin and 1 on $\partial\check{B}$.[7] For this choice of $\check{\varphi}$ the Legendre dual of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is shown in Figure 2.1 on the right. Note that the unbounded edges are indeed parallel, so each unbounded edge comes with copies of the other two unbounded edges parallel at integral distance 1.

Now let us compute $X_0$ and $W^0_{\mathbb{P}^2}$. The polyhedral decomposition has one bounded maximal cell $\sigma_0$ and three unbounded maximal cells $\sigma_1, \sigma_2, \sigma_3$. The bounded cell is the momentum polytope of the $X_{\sigma_0}$, a $\mathbb{Z}/3$-quotient of $\mathbb{P}^2$. Each unbounded cell is affine isomorphic to $[0,1] \times \mathbb{R}_{\geq 0}$, the momentum polytope of $\mathbb{P}^1 \times \mathbb{A}^1 =: X_{\sigma_i}$, $i = 1,2,3$. The $X_{\sigma_i}$ glue together by torically identifying pairs of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s as prescribed by the polyhedral decomposition to yield $X_0$. By definition, $W^0_{\mathbb{P}^2}$ vanishes identically on the compact component $X_{\sigma_0}$. Each of the unbounded components has two parallel unbounded edges, leading to the pull-back to $\mathbb{P}^1 \times \mathbb{A}^1$ of the toric coordinate function of $\mathbb{A}^1$, say $z_i$ for the $i$-th copy. Thus $W^0_{\mathbb{P}^2}|_{X_{\sigma_i}} = z_i$

[7] [GHS, Expl. 6.2] identifies the generic fiber of the algebraization $(\check{X} \to T, \check{D})$ of the resulting toric degeneration with the family of elliptic curves $g(t)(x_0^3 + x_1^3 + x_2^3) + x_0x_1x_2$ in the trivial deformation of $\mathbb{P}^2$. Here $g(t) = t + O(t^2)$ is an analytic change of parameter related to Jacobian theta functions.

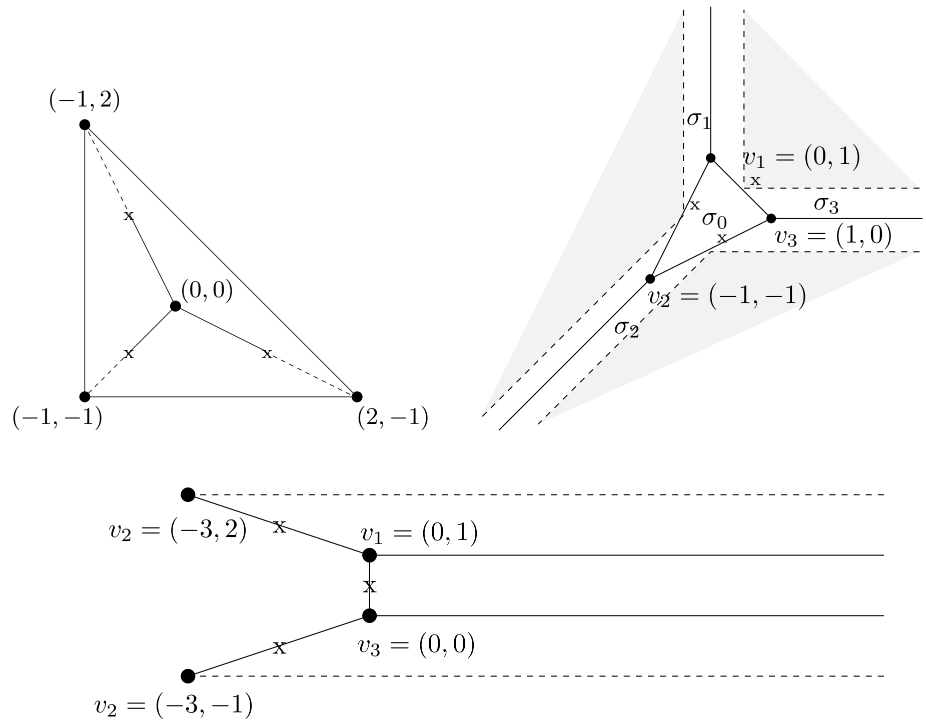


Figure 2.1. An intersection complex $(\check{B}, \check{\mathscr{P}})$ for $\mathbb{P}^2$ with straight boundary and its Legendre dual $(B, \varphi)$ for the minimal polarization, with a chart on the complement of the shaded region and a chart showing the three parallel unbounded edges.

for $i = 1, 2, 3$ and $W^0 : X_0 \to \mathbb{A}^1$ is proper. These functions are clearly compatible with the toric gluings.

*Remark* 2.8. An interesting feature of the degeneration point of view is that the mirror construction respects the finer data related to the degeneration such as the monodromy representation of the affine structure. In particular, this poses a question of uniqueness of the Landau-Ginzburg mirror. For the anticanonical polarization such as the chosen one in the case of $\mathbb{P}^2$, the tropical data $(\check{B}, \check{\mathscr{P}})$ is essentially unique, see Theorem 7.5 for a precise statement. For larger polarizations (thus enlarging $\check{B}$) there are certainly many more possibilities. For example, as an affine manifold with singularities one can perturb the location of the singular points transversally to the invariant directions over the rational numbers and choose an adapted integral polyhedral decomposition after appropriate rescaling. It is not clear to us if all $(\check{B}, \check{\mathscr{P}})$ with flat boundary leading to $\mathbb{P}^2$ can be obtained by this procedure.

## 3. Scattering of monomials

A central tool in [GS3] are scattering diagrams. The purpose of this section is to study the propagation of monomials through scattering diagrams. Assume $\mathscr{S}_k$ is a structure that is consistent to order $k$ and let $\mathfrak{j}$ be a *joint* of $\mathscr{S}_k$. Recall that a joint for a wall structure is a codimension two cell of the polyhedral decomposition of $B$ with codimension one skeleton the union of walls in $\mathscr{S}_k$. Thus each joint is the intersection of the walls that contain the joint. These walls containing $\mathfrak{j}$ define various *scattering diagrams* $\mathfrak{D} = (\mathfrak{r}_i, f_\mathfrak{c})$ that collect the data carried by $\mathscr{S}_k$ relevant around $\mathfrak{j}$. Each joint defines several, closely related scattering diagrams, one for each choice of vertex $v \in \sigma_\mathfrak{j}$ and $\omega \in \mathscr{P}$ with $\omega \subseteq \sigma_\mathfrak{j}$ see [GS3, Def. 3.3, Constr. 3.4]. Here $\sigma_\mathfrak{j} \in \mathscr{P}$ is the minimal cell containing $\mathfrak{j}$. A scattering diagram is a collection of half-lines $\mathbb{R}_{\ge 0}\overline{\overline{m}}$ in the two-dimensional quotient space $Q^v_{\mathfrak{j},\mathbb{R}} = \big(\Lambda_{B,v}/\Lambda_{\mathfrak{j},v}\big)_\mathbb{R}$ along with a function attached to each half-line. The double-bar notation denotes the image of an element or subset in $Q^v_{\mathfrak{j},\mathbb{R}}$, such as $\overline{\overline{m}}$, $\overline{\overline{\sigma}}$. The functions lie in the ring $\Bbbk[P_x]$ defining the local model of $\mathfrak{X}$ at some $x \in (\mathfrak{j} \cap \operatorname{Int}\omega) \setminus \Delta$. Any codimension one cell $\rho \in \mathscr{P}$ containing $\mathfrak{j}$ produces one or two half-lines, the latter in the case $\mathfrak{j} \cap \operatorname{Int}\rho \neq \emptyset$. Half-lines obtained from codimension one-cells are called *cuts*, all other half-lines *rays*.

For an exponent $m_0$ with $\overline{m}_0 \in \Lambda_v \setminus \Lambda_\mathfrak{j}$ we wish to define the scattering of the monomial $z^{m_0}$, which we think of traveling along the ray $-\mathbb{R}_{\ge 0}\overline{\overline{m}}_0$ into the origin of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} \simeq \mathbb{R}^2$. In a scattering diagram monomials travel along *trajectories*. These are defined in exactly the same way as rays [GS3, Def. 3.3], but will have an additive meaning.

**Definition 3.1.** A *trajectory* in $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ is a triple $(\mathfrak{t}, m_\mathfrak{t}, a_\mathfrak{t})$, where $m_\mathfrak{t}$ is a monomial on a maximal cell $\sigma \ni v$ with $\pm\overline{\overline{m}}_\mathfrak{t} \in \overline{\overline{\sigma}}$ and $m \in P_x$ for all $x \in \mathfrak{j} \setminus \Delta$, $\mathfrak{t} = \pm\mathbb{R}_{\ge 0}\overline{\overline{m}}$, and $a_\mathfrak{t} \in \Bbbk$. The trajectory is called *incoming* if $\mathfrak{t} = \mathbb{R}_{\ge 0}\overline{\overline{m}}$, and *outgoing* if $\mathfrak{t} = -\mathbb{R}_{\ge 0}\overline{\overline{m}}$. By abuse of notation we often suppress $m_\mathfrak{t}$ and $a_\mathfrak{t}$ when referring to trajectories.

Here is the generalization of the central existence and uniqueness result for scattering diagrams [GS3, Prop. 3.9] incorporating trajectories. This result is crucial for the existence proof of the superpotential in Lemmas 4.7 and 4.9.

**Proposition 3.2.** *Let $\mathfrak{D}$ be the scattering diagram defined by $\mathscr{S}_k$ for $\mathfrak{j} \in \operatorname{Joints}(\mathscr{S}_k)$, $g : \omega \to \sigma_\mathfrak{j}$, $v \in \omega$. Let $(\mathbb{R}_{\ge 0}\overline{\overline{m}}_0, m_0, 1)$ be an incoming trajectory and $\sigma \supset \mathfrak{j}$ a maximal cell with $\overline{\overline{m}}_0 \in \overline{\overline{\sigma}}$. For $\overline{\overline{m}} \in \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} \setminus \{0\}$ denote by*

$$\theta_{\overline{\overline{m}}} : R^k_{g,\sigma'} \to R^k_{g,\sigma}$$

*the ring isomorphism defined by $\mathfrak{D}$ for a path connecting $-\overline{\overline{m}}$ to $\overline{\overline{m}}_0$, where $\sigma'$ is a maximal cell with $-\overline{\overline{m}} \in \overline{\overline{\sigma'}}$.*

*Then there is a set $\mathfrak{T}$ of outgoing trajectories such that*

$$z^{m_0} = \sum_{\mathfrak{t} \in \mathfrak{T}} \theta_{\overline{\overline{m}}_\mathfrak{t}}(a_\mathfrak{t} z^{m_\mathfrak{t}}) \tag{3.1}$$

*holds in $R^k_{g,\sigma}$. Moreover, $\mathfrak{T}$ is unique if $a_\mathfrak{t} z^{m_\mathfrak{t}} \neq 0$ in $R^k_{g,\sigma'}$ for all $\mathfrak{t} \in \mathfrak{T}$, and if $m_\mathfrak{t} \neq m_{\mathfrak{t}'}$ whenever $\mathfrak{t} \neq \mathfrak{t}'$.*

*Proof.* The proof is by induction on $l \le k$. We first discuss the case that $\sigma_\mathfrak{j}$ is a maximal cell, that is, $\operatorname{codim}\sigma_\mathfrak{j} = 0$. Then $\mathfrak{D}$ has only rays, no cuts. In particular, any $\theta_{\overline{\overline{m}}}$ is an automorphism of $R^k_{g,\sigma}$ that is the identity modulo $I^{>0}_{g,\sigma}$. Thus for $l = 0$, (3.1) forces one outgoing trajectory $(-\mathbb{R}_{\ge 0}\overline{\overline{m}}_0, m_0, 1)$ if $\operatorname{ord}_{\sigma_\mathfrak{j}}(m_0) = 0$ or none otherwise. For the induction step assume (3.1) holds in $R^{l-1}_{g,\sigma}$. Then in $R^l_{g,\sigma}$ the difference of the two sides of (3.1) is a sum of monomials $az^m$ with $\operatorname{ord}_{\sigma_\mathfrak{j}}(m) = l$. Since $l > 0$ and since there are no cuts (these represent slabs containing $\mathfrak{j}$), it holds $\theta_{\overline{\overline{m}}}(az^m) = az^m$. Thus after adding appropriate trajectories

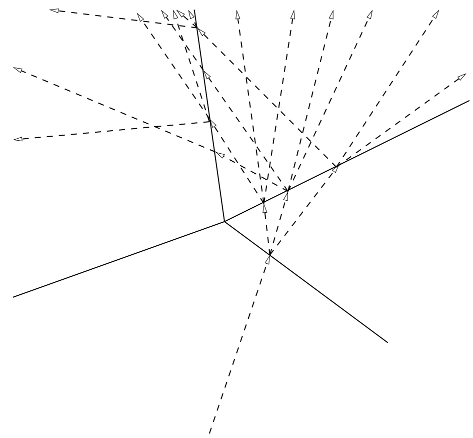

FIGURE 3.1. Scattering diagram with perturbed trajectories (cuts and rays solid, perturbed trajectories dashed).

$(-\mathbb{R}_{\geq 0}\overline{\overline{m}}, m, a)$ with $\mathrm{ord}_{\sigma_{\mathfrak{j}}}(m) = l$ to $\mathfrak{T}$, Equation (3.1) holds in $R^l_{g,\sigma}$. This is the unique minimal choice of $\mathfrak{T}$. This finishes the proof in the case $\operatorname{codim}\sigma_{\mathfrak{j}} = 0$.

Under the presence of cuts we have several rings $R^k_{g,\sigma'}$ for various maximal cells $\sigma' \supset \mathfrak{j}$. This possibly brings in denominators that are powers of $f_{\rho,v}$ for cells $\rho \supset \mathfrak{j}$ of codimension one. In this case we show existence by a perturbation argument. To this end consider first the simplest scattering diagram in codimension one consisting of only two cuts $\mathfrak{c}_\pm = (\pm\mathfrak{c}, f_{\rho,v})$ dividing $\mathcal{Q}$ into two halfplanes $\overline{\overline{\sigma}}_\pm$ and with the same attached function. The signs are chosen in such a way that $\overline{\overline{m}}_0 \in \overline{\overline{\sigma}}_-$. Let $\theta : R^k_{g,\sigma_-} \to R^k_{g,\sigma_+}$ be the isomorphism defined by a path from $\overline{\overline{\sigma}}_-$ to $\overline{\overline{\sigma}}_+$ and let $n \in \Lambda_\rho^\perp \subseteq \Lambda_v^*$ be the primitive integral vector that is positive on $\sigma_-$. Then $\langle m_0, n\rangle \geq 0$ and

$$\theta(z^{m_0}) = f_{\rho,v}^{\langle m_0,n\rangle} \cdot z^{m_0}.$$

Expanding yields the finite sum

$$\theta(z^{m_0}) = \sum_{\langle m,n\rangle \geq 0} a_m z^m = \theta\Big(\sum_{\langle m,n\rangle\geq 0} a_m \theta^{-1}(z^m)\Big) = \theta\Big(\sum_{\langle m,n\rangle\geq 0} \theta_{\overline{\overline{m}}}(a_m z^m)\Big), \tag{3.2}$$

for some $a_m \in \Bbbk$. Note that $\theta^{-1}(z^m) = \theta_{\overline{\overline{m}}}(z^m)$ by the definition of $\theta_{\overline{\overline{m}}}$. Now (3.2) equals $\theta$ applied to (3.1) for the set of trajectories

$$\mathfrak{T} := \big\{(-\mathbb{R}_{\geq 0}\overline{\overline{m}}, m, a_m) \,\big|\, \langle m,n\rangle \geq 0, a_m z^m \neq 0\big\}.$$

Hence existence is clear in this case.

In the general case we work with perturbed trajectories as suggested by Figure 3.1. More precisely, a perturbed trajectory is a trajectory with the origin shifted. There is one unbounded perturbed incoming trajectory, a translation of $(-\mathbb{R}_{\geq 0}\overline{\overline{m}}_0, m_0, 1)$, and a number of perturbed outgoing trajectories, each the result of scattering of other trajectories with rays or cuts. At each intersection point of a trajectory with a ray or cut, the incoming and outgoing trajectories at this point fulfill an equation analogous to (3.1). Similar to [GS3, Constr. 4.5] with our additive trajectories replacing the multiplicative $s$-rays[8], there is then an asymptotic scattering diagram with trajectories obtained by taking the limit $\lambda \to 0$ of rescaling

[8]For technical reasons $s$-rays were not asked to be piecewise affine. In the present situation we insist in piecewise affine objects.

the whole diagram by $\lambda \in \mathbb{R}_{>0}$. Any choice of perturbed incoming trajectory determines a unique minimal scattering diagram with perturbed trajectories. Moreover, for a generic choice of perturbed incoming trajectory the intersection points of trajectories with rays or cuts are pairwise disjoint, and they are in particular different from the origin. Hence the perturbed diagram can be constructed uniquely by induction on $l \le k$. Taking the associated asymptotic scattering diagram with trajectories now establishes the existence in the general case.

Next we show uniqueness for $\operatorname{codim} \sigma_{\mathrm{j}} > 0$. For $\operatorname{codim} \sigma_{\mathrm{j}} = 2$ any monomial $z^m$ in $\Bbbk[\Lambda_{\sigma_{\mathrm{j}}}]$ fulfills $\operatorname{ord}_{\sigma_{\mathrm{j}}}(m) = 0$, and all $\theta_{\overline{\overline{m}}}$ extend to $\Bbbk(\Lambda_{\sigma_{\mathrm{j}}})$-algebra automorphisms of $R^k_{g,\sigma} \otimes_{\Bbbk[\Lambda_{\sigma_{\mathrm{j}}}]} \Bbbk(\Lambda_{\sigma_{\mathrm{j}}})$. Hence we can deduce uniqueness by induction on $l \le k$ as in codimension 0 by taking the factors $a$ of trajectories to be polynomials with coefficients in $\Bbbk(\Lambda_{\sigma_{\mathrm{j}}})$. Thus we combine all trajectories $\mathfrak{t}$ with the same $\overline{\overline{m}}_{\mathfrak{t}}$ and the same $\operatorname{ord}_{\sigma_{\mathrm{j}}}(m_{\mathfrak{t}})$. It is clear that such generalized trajectories can be split uniquely into proper trajectories with all $m$ distinct, showing uniqueness in this case.

Finally, for uniqueness in codimension one we can not argue just with $\operatorname{ord}_{\sigma_{\mathrm{j}}}$ because there are monomials $z^m$ with $\operatorname{ord}_{\sigma_{\mathrm{j}}}(m) = 0$ but $\overline{\overline{m}} \neq 0$. Instead we look closer at the effect of adding trajectories. By induction it suffices to study the insertion of trajectories $(-\mathbb{R}_{\ge 0}\overline{\overline{m}}, m, a)$ with $\operatorname{ord}_{\sigma_{\mathrm{j}}}(m) = l$ for each $m$ and such that (3.1) continues to hold. By working with a perturbed scattering diagram as in Figure 4.1 of [GS3] and the asociated asymptotic scattering diagram as in [GS3, §4.3], it suffices to consider the case of only two cuts as already considered above. In this case we have

$$0 = \sum_m \theta_{\overline{\overline{m}}}(a_m z^m) \; = \sum_{i=0}^{l} \sum_{\langle m,n\rangle = i} \theta_{\overline{\overline{m}}}(a_m z^m) = \sum_{i=0}^{l} f_{\rho,v}^{-i} \sum_{\langle m,n\rangle = i} a_m z^m.$$

Since all monomials in $f_{\rho,v}$ have vanishing $\operatorname{ord}_{\sigma_{\mathrm{j}}}$ and only monomials $z^m$ with the same value of $\langle m,n\rangle$ can cancel, this equation implies

$$f_{\rho,v}^{-i} \sum_{\langle m,n\rangle = i} a_m z^m = 0$$

holds for each $i$. Multiplying by $f_{\rho,v}^{i}$ thus shows $\sum_{\langle m,n\rangle = i} a_m z^m = 0$ in $R^l_{g,\sigma}$, and hence $a_m = 0$ for all $m$. This proves uniqueness also in codimension one. □

## 4. The superpotential via broken lines

The easiest way to define the superpotential in full generality is by the method of broken lines. Broken lines have been introduced by Mark Gross for $\dim B = 2$ in his work on mirror symmetry for $\mathbb{P}^2$ [Gr2]. We assume we are given a locally finite scattering diagram $\mathscr{S}_k$ for the non-compact intersection complex $(B, \mathscr{P}, \varphi)$ of a polarized LG-model that is consistent to order $k$, as given by Assumption 1.1. The notion of broken lines is based on the transport of monomials by changing chambers of $\mathscr{S}_k$. Recall from [GS3, Def. 2.22] that a chamber is the closure of a connected component of $B \setminus |\mathscr{S}_k|$.

**Definition 4.1.** Let $\mathfrak{u}, \mathfrak{u}'$ be neighboring chambers of $\mathscr{S}_k$, that is, $\dim(\mathfrak{u} \cap \mathfrak{u}') = n - 1$. Let $az^m$ be a monomial defined at all points of $\mathfrak{u} \cap \mathfrak{u}'$ and assume that $\overline{m}$ points from $\mathfrak{u}'$ to $\mathfrak{u}$. Let $\tau := \sigma_{\mathfrak{u}} \cap \sigma_{\mathfrak{u}'}$ and

$$\theta : R^k_{\mathrm{id}_\tau, \sigma_{\mathfrak{u}}} \to R^k_{\mathrm{id}_\tau, \sigma_{\mathfrak{u}'}}$$

be the gluing isomorphism changing chambers. Then if

$$\theta(az^m) = \sum_i a_i z^{m_i} \tag{4.1}$$

we call any summand $a_i z^{m_i}$ with $\mathrm{ord}_{\sigma_{\mathfrak{u}'}}(m_i) \le k$ a *result of transport of* $az^m$ from $\mathfrak{u}$ to $\mathfrak{u}'$.

Note that since the change of chamber isomorphisms commute with changing strata, the monomials $a_i z^{m_i}$ in Definition 4.1 are defined at all points of $\mathfrak{u} \cap \mathfrak{u}'$.

**Definition 4.2.** (Cf. [Gr2, Def. 4.9]) A *broken line* for $\mathscr{S}_k$ is a proper continuous maps

$$\beta : (-\infty, 0] \to B$$

with image disjoint from any joint of $\mathscr{S}_k$, along with a sequence $-\infty = t_0 < t_1 < \ldots < t_{r-1} \le t_r = 0$ for some $r \ge 0$ with $\beta(t_i) \in |\mathscr{S}_k|$, and for $i = 1, \ldots, r$ monomials $a_i z^{m_i}$ defined at all points of $\beta([t_{i-1}, t_i])$ (for $i = 1$, $\beta((-\infty, t_1])$), subject to the following conditions.

(1) $\beta|_{(t_{i-1}, t_i)}$ is a non-constant affine map with image disjoint from $|\mathscr{S}_k|$, hence contained in the interior of a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}_k$, and $\beta'(t) = -\overline{m}_i$ for all $t \in (t_{i-1}, t_i)$. Moreover, if $t_r = t_{r-1}$ then $\mathfrak{u}_r \neq \mathfrak{u}_{r-1}$.

(2) $a_1 = 1$ and[9] there exists a (necessarily unbounded) $\omega \in \mathscr{P}^{[1]}$ with $\overline{m}_1 \in \Lambda_\omega$ primitive and $\mathrm{ord}_\omega(m_1) = 0$.

(3) For each $i = 1, \ldots, r-1$ the monomial $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$ (Definition 4.1).

The *type* of $\beta$ is the tuple of all $\mathfrak{u}_i$ and $m_i$. By abuse of notation we suppress the data $t_i, a_i, m_i$ when talking about broken lines, but introduce the notation

$$a_\beta := a_r, \quad m_\beta := m_r.$$

For $p \in B$ the set of broken lines $\beta$ with $\beta(0) = p$ is denoted $\mathfrak{B}(p)$.

*Remark* 4.3. 1) If $(B, \mathscr{P})$ is asymptotically cylindrical (Definition 2.1) then in Definition 4.2 the existence of a one-cell $\omega \in \mathscr{P}$ with $\overline{m}_1 \in \Lambda_\omega$ in (2) follows from (1).

2) A broken line $\beta$ is determined uniquely by specifying its endpoint $\beta(0)$ and its type. In fact, the coefficients $a_i$ are determine inductively from $a_1 = 1$ by Equation (4.1).

According to Remark 4.3,(2) the map $\beta \mapsto \beta(0)$ identifies the space of broken lines of a fixed type with a subset of $\mathfrak{u}_r$. This subset is the interior of a polyhedron:

**Proposition 4.4.** *For each type* $(\mathfrak{u}_i, m_i)$ *of broken lines there is an integral, closed, convex polyhedron* $\Xi$, *of dimension* $n$ *if non-empty, and an affine immersion*

$$\Phi : \Xi \longrightarrow \mathfrak{u}_r,$$

*so that* $\Phi\big(\mathrm{Int}\,\Xi\big)$ *is the set of endpoints* $\beta(0)$ *of broken lines* $\beta$ *of the given type.*

*Proof.* This is an exercise in polyhedral geometry left to the reader. For the statement on dimensions it is important that broken lines are disjoint from joints. □

[9] The normalization condition $a_1 = 1$ has to be modified if there are parallel unbounded edges not contained in one cell and the gluing data are not trivial, see [GHS, §5.2].

*Remark* 4.5. A point $p \in \Phi(\partial\Xi)$ still has a meaning as an endpoint of a piecewise affine map $\beta : (-\infty, 0] \to B$ together with data $t_i$ and $a_i z^{m_i}$, defining a *degenerate broken line*. For $\beta$ not to be a broken line, $\mathrm{im}(\beta)$ has to intersect a joint.

Indeed, if $\beta$ violates Definition 4.2,(1) then $t_{i-1} = t_i$ or there exists $t \in (t_{i-1}, t_i)$ with $\beta(t) \in |\mathscr{S}_k|$. In the first case denote by $\mathfrak{p}_j \in \mathscr{S}$ the wall with $\beta'(t_j) \in \mathfrak{p}_j$ for all $\beta' \in \Phi(\mathrm{Int}\,\Xi)$. Then $\beta(t_{i-1}) = \beta(t_i)$ lies in $\mathfrak{p}_{i-1} \cap \mathfrak{p}_i$, hence in a joint. In the second case convexity of the chambers implies that $\beta([t_{i-1}, t]) \subset \partial\mathfrak{u}$ or $\beta([t, t_i]) \subset \partial\mathfrak{u}$. Thus $\beta$ maps a whole open interval to $|\mathscr{S}_k|$. The break point $t_{i-1}$ or $t_i$ contained in this interval again intersects two walls, hence is contained in a joint. The other conditions in the definition of broken lines are closed.

The set of endpoints $\beta(0)$ of degenerate broken lines of a given type is the $(n-1)$-dimensional polyhedral subset $\Phi(\partial\Xi) \subseteq \mathfrak{u}_r$. The set of degenerate broken lines *not transverse* to some joint of $\mathscr{S}_k$ is polyhedral of smaller dimension.

Any finite structure $\mathscr{S}_k$ involves only finitely many slabs and walls, and each polynomial coming with each slab or wall carries only finitely many monomials. Hence broken lines for $|\mathscr{S}_k|$ exist only for finitely many types. The following definition is therefore meaningful.

**Definition 4.6.** A point $p \in B$ is called *general* (for the given structure $\mathscr{S}_k$) if it is not contained in $\Phi(\partial\Xi)$, for any $\Phi$ as in Proposition 4.4.

Recall from [GS3, §2.6] that $\mathscr{S}_k$ defines a $k$-th order deformation of $X_0$ by gluing the sheaf of rings defined by $R^k_{g,\sigma_\mathfrak{u}}$, with $g : \omega \to \tau$ and $\mathfrak{u}$ a chamber of $\mathscr{S}_k$ with $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subseteq \sigma_\mathfrak{u}$. Given a general $p \in \mathfrak{u}$ we can now define the *superpotential* up to order $k$ locally as an element of $R^k_{g,\sigma_\mathfrak{u}}$ by

$$W^k_{g,\mathfrak{u}}(p) := \sum_{\beta \in \mathfrak{B}(p)} a_\beta z^{m_\beta}. \tag{4.2}$$

The existence of a canonical extension $W^k$ of $W^0$ to $X_k$ follows once we check that (i) $W^k_{g,\mathfrak{u}}(p)$ is independent of the choice of a general $p \in \mathfrak{u}$ and (ii) the $W^k_{g,\mathfrak{u}}(p)$ are compatible with changing strata or chambers [GS3, Constr. 2.24]. This is the content of the following two lemmas.

**Lemma 4.7.** *Let $\mathfrak{u}$ be a chamber of $\mathscr{S}_k$ and $g : \omega \to \tau$ with $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subseteq \sigma_\mathfrak{u}$. Then $W^k_{g,\mathfrak{u}}(p)$ is independent of the choice of $p \in \mathfrak{u}$.*

*Proof.* By Proposition 4.4 the set $A \subseteq \mathfrak{u}$ of non-general points is a finite union of nowhere dense polyhedra. Moreover, since all $\Phi$ in Proposition 4.4 are local affine isomorphisms, for each path $\gamma : [0,1] \to \mathfrak{u} \setminus A$ and broken line $\beta_0$ with $\beta_0(0) = \gamma(0)$ there exists a unique family $\beta_s$ of broken lines with endpoints $\beta_s(0) = \gamma(s)$ and with the same type as $\beta_0$. Hence $W^k_{g,\mathfrak{u}}(p)$ is locally constant on $\mathfrak{u} \setminus A$.

To pass between the different connected components of $\mathfrak{u} \setminus A$, consider the set $A' \subseteq A$ of endpoints of degenerate broken lines that are not transverse to the joints of $\mathscr{S}_k$. More precisely, for each type of broken line, the set of endpoints of broken lines intersecting a given joint defines a polyhedral subset of $\mathfrak{u}$ of dimension at most $n-1$. Then $A'$ is the union of the $(n-2)$-cells of these polyhedral subsets, for any joint and any type of broken line. Since $\dim A' = n-2$, we conclude that $\mathfrak{u} \setminus A'$ is path-connected. It thus suffices to study the following situation. Let $\gamma : [-1,1] \to \mathrm{Int}\,\mathfrak{u} \setminus A'$ be an affine map with $\gamma(0)$ the only point of intersection with $A$. Let $\overline{\beta}_0 : (-\infty, 0] \to B$ be the underlying map of a degenerate broken line with endpoint $\gamma(0)$. The point is that $\overline{\beta}_0$ may arise as a limit of several different types of broken lines with endpoints $\gamma(s)$ for $s \neq 0$. The lemma follows once we show that the contributions to $W^k_{g,\mathfrak{u}}\big(\gamma(s)\big)$ of such broken lines for $s < 0$ and for $s > 0$ coincide. Note we do not claim a bijection between the sets of broken lines for $s < 0$ and for $s > 0$, which in fact need not be true.

For later use let $V \subset \operatorname{Int}\mathfrak{u}$ be a local affine hyperplane intersecting $\overline{\beta}_0$ only in $\overline{\beta}_0(0) = \gamma(0)$ and containing $\operatorname{im}(\gamma)$. Note that $V$ is also transverse to $A$. Thus by the unique continuation statement at the beginning of the proof, each broken line $\beta$ with endpoint $\beta(0) \in V$ fits into a unique family of broken lines of the same type and with endpoint any other point of the connected component of $\beta(0)$ in $V \setminus A$.

In particular, since $\gamma^{-1}(A) = \{0\}$ any broken line $\beta$ with endpoint $\gamma(s_0)$ for $s_0 \neq 0$ extends uniquely to a family of broken lines $\beta_s$ for $s \in [-1, 0)$ or $s \in (0, 1]$. Thus $\beta$ has a unique limit $\lim \beta := \lim_{s\to 0} \beta_s$, a possibly degenerate broken line. For $s \neq 0$ denote by $\mathfrak{B}_s$ the space of broken lines $\beta$ with endpoint $\gamma(s)$ and such that the map underlying $\lim \beta$ equals $\overline{\beta}_0$. Since $\overline{\beta_0}$ is the underlying map of a degenerate broken line, $\mathfrak{B}_s \neq \emptyset$ for some sufficiently small $s$, hence also for all $s$ of the same sign, by unique continuation. Possibly by changing signs in the domain of $\gamma$ we may thus assume $\mathfrak{B}_s \neq \emptyset$ for $s < 0$. We have to show

$$\sum_{\beta\in\mathfrak{B}_{-1}} a_\beta z^{m_\beta} = \sum_{\beta\in\mathfrak{B}_1} a_\beta z^{m_\beta}. \tag{4.3}$$

Denote by $\mathfrak{T}_s$ the set of types of broken lines in $\mathfrak{B}_s$. Obviously, $\mathfrak{T}_s$ only depends on the sign of $s$.

The central observation is the following. Let $J \subset B$ be the union of the joints of $\mathscr{S}_k$ intersected by $\operatorname{im}\overline{\beta}_0$. Let $x := \overline{\beta}_0(t)$ for $t \ll 0$ be a point far off to $-\infty$. Thus $x$ lies in one of the unbounded cells of $\mathscr{P}$ and $\overline{\beta}_0$ is asymptotically parallel to an unbounded edge and does not cross a wall for $t' < t$. Let $U \subseteq B$ be a local affine hyperplane intersecting $\overline{\beta_0}$ transversally at $x$. By transversality with $J$ the images of the degenerate broken lines of types contained in any $\mathfrak{B}_s$ lie in a local affine hyperplane $H \subset U$ ($\dim H = n - 2$). Note that each joint that $\overline{\beta}_0$ meets defines such a local hyperplane $H \subset U$, and if $\overline{\beta}_0$ meets several joints, the hyperplanes agree locally near $x$ since $\gamma(0) \notin A'$. Moreover, the images of degenerate broken lines arising as the limit of broken lines of type in $\mathfrak{T}_{-1} \cup \mathfrak{T}_1$ separate a contractible neighbourhood of $\operatorname{im}\overline{\beta}_0$ in $B$ into two connected components. It follows that broken lines of type in $\mathfrak{T}_s$ for $s < 0$ intersect $U$ only on one side of $H$, and for $s > 0$ only on the other.

Now let $\breve{\beta}_s \in \mathfrak{B}_s$ be one family of broken lines, say for $s < 0$. Denote by $\mathfrak{t}$ the type of $\breve{\beta}_s$. Thus $\breve{\beta}_s$ is the unique broken line $\beta$ of type $\mathfrak{t}$ with endpoint $\beta(0) = \gamma(s)$. Since $\lim \breve{\beta}_s = \overline{\beta}_0$, the broken line $\breve{\beta}_s$ does not pass a wall before hitting $U$, thus is a straight line. Denote by $x_s \in U \setminus H$ the point of intersection of $\breve{\beta}_s$ with $U$. The $x_s$ vary affine linearly with $s$ with $\lim_{s\to 0} x_s = x$, hence define an affine line segment in $U$. This line segment can be viewed as a lift of $\gamma([-1, 0]) \subset \mathfrak{u}$ to a line segment in $U$ via broken lines of type $\mathfrak{t}$ and their limits.

If $\beta_s$ is another family of broken lines in $\mathfrak{B}_s$ for $s < 0$, of type $\mathfrak{t}'$, then by the same argument $\beta_s$ hits $U$ in another point $x'_s$ in the same connected component of $U \setminus H$. Moreover, there is a unique such $\beta'_s$ with endpoint $\beta'_s(0) \in V$, where $V \subset \operatorname{Int}\mathfrak{u}$ is the local affine hyperplane chosen above. In particular, for each $s$ there is a unique broken line $\beta'_s$ of type $\mathfrak{t}'$ passing through $x_s$. In other words, each broken line $\beta_s \in \mathfrak{B}_s$, $s < 0$, deforms uniquely to a broken line $\beta'_s$ with endpoint on $V$ and passing through $x_s$.

The set of $\beta'_s$ obtained in this way can alternatively be constructed as follows. For $s < 0$ all broken lines in $\mathfrak{B}_s$ have the same first chamber $\mathfrak{u}_1$ and monomial $z^{m_1}$. Now start with the straight broken line ending at $x_s$ and of type $(\mathfrak{u}_1, m_1)$. This broken line can be continued until it hits a wall or slab, where it splits into several broken lines, one for each summand in (4.1). Iterating this process leads to the infinite set of all broken lines with asymptotic given by $m_1$ and running through $x_s$. The $\beta'_s$ are the subset of the considered types, that is, with the unique deformation for $s \to 0$ having underlying map $\overline{\beta}_0$ and endpoint on $V$.

From this point of view it is clear that at each joint $\mathfrak{j}$ intersected by $\operatorname{im}\overline{\beta}_0$ the $\beta'_s$ compute a scattering of monomials as considered in §3. In fact, the union of the $\beta'_s$ with the same incoming part $az^m$ near $\mathfrak{j}$ induce a scattering diagram with perturbed trajectories as considered in the proof of Proposition 3.2. Thus the corresponding sum of monomials leaving a neighborhood of $\mathfrak{j}$ can be read off from the right-hand side of (3.1) in this proposition, applied to the incoming trajectory $(\mathbb{R}_{\ge 0}\overline{\overline{m}}, m, a)$.

We conclude that $\sum_{\beta\in\mathfrak{B}_s} a_\beta z^{m_\beta}$ for $s < 0$ computes the transport of $z^{m_1}$ along $\overline{\beta}_0$. This transport is defined by applying (3.1) instead of (4.1) at each joint intersected transversally by $\overline{\beta}_0$. The same argument holds for $s > 0$, thus proving (4.3). □

*Remark* 4.8. The proof of Lemma 4.7 really shows that the scattering of monomials introduced in §3 allows to replace the condition that broken lines have image disjoint from joints by transversality with joints. In the following we refer to these as *generalized broken lines.* The set of degenerate broken lines with endpoint $p$ is denoted $\overline{\mathfrak{B}}(p)$.

By Lemma 4.7 and Remark 4.8 we are now entitled to define

$$W^k_{g,\mathfrak{u}} := W^k_{g,\mathfrak{u}}(p) = \sum_{\beta\in\overline{\mathfrak{B}}(p)} a_\beta z^{m_\beta}, \tag{4.4}$$

for any choice of $p \in \mathfrak{u} \setminus A'$, $A'$ the set of endpoints in $\mathfrak{u}$ of degenerate broken lines *not* transverse to all joints of $\mathscr{S}_k$.

**Lemma 4.9.** *The $W^k_{g,\mathfrak{u}}$ are compatible with changing strata and changing chambers.*

*Proof.* Compatibility with changing strata follows trivially from the definitions. As for changing from a chamber $\mathfrak{u}$ to a neighboring chamber $\mathfrak{u}'$ ($\dim \mathfrak{u}\cap\mathfrak{u}' = n-1$) the argument is similar to the proof Lemma 4.7. Let $g : \omega \to \tau$ be such that $\omega \cap \mathfrak{u} \cap \mathfrak{u}' \neq \emptyset$, $\tau \subseteq \sigma_{\mathfrak{u}} \cap \sigma_{\mathfrak{u}'}$ and

$$\theta : R^k_{g,\mathfrak{u}} \longrightarrow R^k_{g,\mathfrak{u}'}$$

be the corresponding change of chamber isomorphism [GS3, §2.4]. We have to show $\theta\big(W^k_{g,\mathfrak{u}}\big) = W^k_{g,\mathfrak{u}'}$.

Let $A \subseteq \mathfrak{u} \cup \mathfrak{u}'$ be the set of endpoints of degenerate broken lines. Consider a path $\gamma : [-1, 1] \to \mathfrak{u}\cap\mathfrak{u}'$ connecting general points $\gamma(-1) \in \mathfrak{u}$, $\gamma(1) \in \mathfrak{u}'$ and with $\gamma^{-1}(\mathfrak{u}\cap\mathfrak{u}') = \{0\}$. We may also assume that $\gamma(s) \in A$ at most for $s = 0$, and that any degenerate broken line with endpoint $\gamma(0)$ is transverse to joints. For $s \neq 0$ we then consider the space $\mathfrak{B}_s$ of broken lines $\beta_s$ with endpoint $\gamma(s)$ and with deformation for $s \to 0$ a fixed underlying map of a degenerate broken line $\overline{\beta}_0$. By transversality of $\overline{\beta}_0$ with the set of joints the limits of families $\beta_s$, $s \to 0$, group into generalized broken lines (Remark 4.8). Each such generalized broken line $\beta$ has as endpoint $p_0 := \gamma(0)$, but viewed as an element either of $\mathfrak{u}$ or of $\mathfrak{u}'$. We call this chamber the *reference chamber* of $\beta$. Generalized broken lines with reference chambers $\mathfrak{u}$ and $\mathfrak{u}'$ contribute to $W^k_{g,\mathfrak{u}}$ and $W^k_{g,\mathfrak{u}'}$, respectively. Moreover, $m_\beta$ is either tangent to $\mathfrak{u}\cap\mathfrak{u}'$ or points properly into $\mathfrak{u}$ or into $\mathfrak{u}'$. We claim that $\theta$ maps each of the three types of contributions to $W^k_{g,\mathfrak{u}}$ to the three types of contributions to $W^k_{g,\mathfrak{u}'}$. Then $\theta\big(W^k_{g,\mathfrak{u}}\big) = W^k_{g,\mathfrak{u}'}$ and the proof is finished.

Let us first consider the set of degenerate broken lines $\beta$ with $m_\beta$ tangent to $\mathfrak{u}\cap\mathfrak{u}'$. Then changing the reference chamber from $\mathfrak{u}$ to $\mathfrak{u}'$ defines a bijection between the considered generalized broken lines with endpoint $p_0$ and reference cell $\mathfrak{u}$ and those with reference cell $\mathfrak{u}'$. Note that in this case $\beta$ has to intersect a joint, so this statement already involves the arguments from the proof of Lemma 4.7. Because $\theta(a_\beta z^{m_\beta}) = a_\beta z^{m_\beta}$ this proves the claim in this case.

Next assume $m_\beta$ points from $\mathfrak{u}\cap\mathfrak{u}'$ into the interior of $\mathfrak{u}$. This means that $\beta$ approaches $p_0$ from the interior of $\mathfrak{u}$. If we want to change the reference chamber to $\mathfrak{u}'$ we need to introduce one more point $t_{r+1} := t_r = 0$ and chamber $\mathfrak{u}_{r+1} := \mathfrak{u}'$. According to Equation (4.1) in Definition 4.1 the possible monomials $a_{r+1}z^{m_{r+1}}$ are given by the summands in $\theta(az^m) = \sum_i a_{r+1,i}z^{m_{r+1,i}}$. Thus for each summand we obtain one generalized broken line with reference cell $\mathfrak{u}'$. Clearly, this is exactly what is needed for compatibility with $\theta$ of the respective contributions to the local superpotentials.

By symmetry the same argument works for generalized broken lines $\beta$ with reference cell $\mathfrak{u}'$ and $m_\beta$ pointing from $\mathfrak{u}\cap\mathfrak{u}'$ into the interior of $\mathfrak{u}'$, and $\theta^{-1}$ replacing $\theta$. Inverting $\theta$ means that a number of generalized broken lines with reference cell $\mathfrak{u}_{r+1} = \mathfrak{u}$ and two points $t_{r+1} = t_r$ (and necessarily $\mathfrak{u}_r = \mathfrak{u}'$), one for each summand of $\theta^{-1}(a_\beta z^{m_\beta})$, combine into a single generalized broken line with reference cell $\mathfrak{u}_r = \mathfrak{u}'$. This process is again compatible with applying $\theta$ to the respective contributions to the local superpotentials. This finishes the proof of the claim, which was left to complete the proof of the lemma. □

Summarizing the construction and Lemmas 4.7 and 4.9, we now state the unique existence of the superpotential $W$ on canonical toric degenerations:

**Theorem 4.10.** *Let $\pi : \mathfrak{X} \to \operatorname{Spf} \Bbbk[\![t]\!]$ be the canonical toric degeneration given by the compatible system of wall structures in Assumption 1.1. Then there exists a unique formal function $W : \mathfrak{X} \to \mathbb{A}^1$ that modulo $t^{k+1}$ agrees with the expressions* (4.2) *and* (4.4) *at each point $p$.* □

Having defined the superpotential $W$ as a regular function on the formal scheme $\mathfrak{X}$, a natural question concerns algebraizablity of $W$ assuming $\mathfrak{X} \to \operatorname{Spf} \Bbbk[\![t]\!]$ is algebraizable. This generally appears to be a difficult question, but sometimes more can be said by methods going beyond the scope of this paper, as detailed in the following remark.

*Remark* 4.11. Given a toric degeneration, we have defined the superpotential $W^k$ locally at $p \in B$ in the interior of a chamber $\mathfrak{u}$ as an expression in the Laurent polynomial ring $A_k[\Lambda_\sigma]$, $A_k = \Bbbk[t]/(t^{k+1})$, with $\sigma = \sigma_\mathfrak{u}$ the maximal cell containing $\mathfrak{u}$. Increasing $k$ may make $\mathfrak{u}$ smaller, but if we choose $p$ sufficiently transcendental we can achieve that $p$ never hits a wall of any order. Then the all order potential $W := \varprojlim W^k$ still has an expression in the ring

$$\Bbbk[\Lambda_\sigma]\llbracket t\rrbracket = \varprojlim_k A_k[\Lambda_\sigma].$$

To understand the dependence on $p$ note that by the construction of $\mathfrak{X}$ as a colimit, a point $p \in B$ contained in the interior of a chamber for each $k$ defines an open embedding of the formal algebraic torus

$$\hat{\mathbb{G}}_m(\Lambda_\sigma) = \operatorname{Spf} \Bbbk[\Lambda_\sigma]\llbracket t\rrbracket \tag{4.5}$$

into $\mathfrak{X}$. Here $\sigma \in \mathscr{P}$ is the maximal cell containing $p$. The expression $W(p) = \varprojlim W^k(p)$ computes the pull-back of $W$ to this open formal subscheme. But the embedding $\hat{\mathbb{G}}_m(\Lambda_\sigma) \to \mathfrak{X}$ depends on $p$. Expressions for $W(p)$ for different choices of $p$ inside the same maximal cell $\sigma$ are related by a (typically infinite) series of wall crossing automorphisms. If $p$ moves to a $p'$ in a neighboring maximal cell then one needs to use the codimension one relations $XY = f_\rho t^e$ from the proof of [GS3, Lem. 2.34] to see that a similar wall crossing transformation involving quotients by slab functions relate $W(p)$ and $W(p')$.

In any case, for each general $p$ we obtain an expression for $W$ in $\Bbbk[\Lambda_\sigma]\llbracket t\rrbracket$ rather than in the algebraic subring $\Bbbk\llbracket t\rrbracket[\Lambda_\sigma]$.

However, as we will see in the examples in the last three section, sometimes $\mathfrak{X}$ is algebraizable and there exists a point $p \in B$ such that only finitely many broken lines have endpoint $p$. Then $W(p)$ lies in the subring of finite type

$$\Bbbk[\Lambda_\sigma][t] \subset \Bbbk\llbracket t\rrbracket[\Lambda_\sigma] \subset \Bbbk[\Lambda_\sigma]\llbracket t\rrbracket,$$

hence is even a Laurent polynomial with algebraic coefficients in $t$. It is then tempting to believe that this algebraic expression describes a lift of $W$ to an algebraization of $\mathfrak{X}$. But this may not be the case: Formal local representability by an algebraic expression neither means that $W$ lifts to an algebraization, nor that such a lift could locally be represented with polynomial coefficients in $t$ rather than with coefficients that are formal power series. In such a situation we therefore say that $W$ is *ostensibly algebraic*.

To conclude algebraizability of $W$, one rather has to write $W$ as a finite sum of quotients of sections of the polarizing line bundle $\mathcal{O}_{\mathfrak{X}}(1)$. This can indeed sometimes be achieved by using *generalized theta functions*, defined analogously to $W$ via sums of broken lines, see [GS5, GHS]. This method, which is beyond the scope of this paper, appears to work for example in all cases derived from reflexive polytopes via Construction 7.2.

## 5. Fibers of the superpotential over $0, 1, \infty$, and the LG mirror map

Given a finite scattering diagram $\mathscr{S}_k$ for a non-compact $(B, \mathscr{P}, \varphi)$ consistent to order $k$ as in Assumption 1.1, we have constructed in the last section the superpotential

$$W : \mathfrak{X} \longrightarrow \mathbb{A}^1. \tag{5.1}$$

as a morphism of formal schemes.

In this section we discuss some general features of $W$. Denote by $A \subset |\mathfrak{X}| = X_0$ the union of complete irreducible components of $X_0$, that is, the union of the irreducible components $X_\sigma \subset X_0$ defined by the bounded maximal cells $\sigma \in \mathscr{P}$.

We start with $W^{-1}(0)$, viewed as a Cartier divisor on the ringed space $\mathfrak{X}$. To compute the order of $W$ along $X_\sigma \subset \mathfrak{X}$ we define the following notion.

**Definition 5.1.** For a maximal cell $\sigma \in \mathscr{P}$, $\operatorname{depth}\sigma$ is the minimum of $\operatorname{ord}_\sigma m_\beta$ for all broken lines $\beta$ with endpoint in $\operatorname{Int}\sigma$.

**Proposition 5.2.** *The multiplicity of the Cartier divisor $W^{-1}(0) \subset \mathfrak{X}$ on the irreducible component $X_\sigma \subset X_0$ given by a maximal cell $\sigma \in \mathscr{P}$ equals* $\operatorname{depth}\sigma$.

*Proof.* This is obvious from the definition of $W_k$ in (4.2). □

If $(B, \mathscr{P})$ is asymptotically cylindrical, properness of $W|_{X_0}$ (Proposition 2.3) implies that if $\sigma \in \mathscr{P}$ is a maximal cell with $W^{-1}(0) \cap X_\sigma \neq \emptyset$ then $\sigma \in \mathscr{P}$ is bounded. In other words, $W^{-1}(0) \cap X_0 \subseteq A$. Otherwise not much can generally be said about $W^{-1}(0)$.

We next turn to the behavior of $W$ over $\infty$. This discussion makes only sense in the case that $\mathfrak{X} \to \operatorname{Spf}\Bbbk[\![t]\!]$ is compactifiable, that is, if it extends to a proper family $\overline{\mathfrak{X}} \to \operatorname{Spf}\Bbbk[\![t]\!]$. But even in this case, $W$ may not be a formal meromorphic function [SP, Tag 01X1] near $\infty$. In other words, $W$ may have essential singularities near a compactifiying formal closed subscheme. We therefore assume $(B, \mathscr{P})$ compactifiable and $W$ to extend to a meromorphic function on the corresponding partial completion $\overline{\mathfrak{X}}$ of $\mathfrak{X}$. A sufficient condition is if $W$ itself is algebraizable, as discussed in Remark 4.11.

Another problem is that the meromorphic lift may have a locus of indeterminacy containing components of the added divisor at $\infty$ on the central fiber. Here is a purely toric example.

**Example 5.3.** Let $(B, \mathscr{P})$ be given by the complete fan in $\mathbb{R}^2$ with rays generated by $(1, 0)$, $(0, 1)$, $(0, -1)$ and $(-1, k)$ for some $k \geq 2$, and $\varphi$ the convex PL function with value 1 at all the ray generators. The discriminant locus is empty. With trivial gluing data and trivial wall structure we obtain the completion at $t = 0$ of a toric threefold $\mathcal{X}$ over $\mathbb{A}^1 = \operatorname{Spec}\Bbbk[t]$. Letting $x, y$ be the monomial functions defined by $(1, 0)$ and $(0, 1)$, the superpotential equals

$$W = x + y + ty^{-1} + tx^{-1}y^k$$

on the big torus. Using $\varphi$ for the truncation, we obtain a compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ by intersecting $B$ with the 4-gon $\overline{B}$ with vertices $(k, 0)$, $(0, k)$, $(0, -k)$ and $(-1, k)$. Now express $W$ in the neighborhood

$$\operatorname{Spec}\Bbbk[u, v^{\pm 1}, t] \subset \overline{\mathcal{X}}, \quad y = u^{-1}, x = u^{-1}v,$$

of the divisor $\mathcal{D}_\rho \subset \partial\mathcal{X}$ defined by the face $\rho \subset \overline{B}$ with vertices $(k, 0), (0, k)$:

$$W = vu^{-1} + u^{-1} + tu + tu^{-k+1}v^{-1} = \frac{(1+v)u^{k-2} + tu^k + tv^{-1}}{u^{k-1}}.$$

This rational function on $\overline{\mathcal{X}}$ has locus of indeterminacy $u = t = 0$.

In the asymptotically cylindrical case a meromorphic extension of $W$ has empty locus of indeterminacy. We therefore now restrict ourselves to the following situation. Let $(B, \mathscr{P})$ be compactifiable and asymptotically cylindrical and let $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$ be an associated sequence of truncations obtained by Lemma 1.3. Denote by $\overline{\mathfrak{X}} \to \operatorname{Spf}\Bbbk[\![t]\!]$ the corresponding formal toric degeneration constructed in Proposition 1.5 from the associated sequence $(\overline{B}_\nu, \overline{\mathscr{P}}_\nu)$ of truncations from Lemma 1.3, and $\mathcal{Z} \subset \overline{\mathcal{X}}$ the compactifying reduced divisor. Assume further that $W$ lifts to a meromorphic function $\mathcal{W}$ on an algebraization $\overline{\mathcal{X}}$ of $\overline{\mathfrak{X}}$.

For a sequence $F = (F_\nu)$ of maximal flat affine subspaces of $\partial\overline{B}_\nu$ denote by $\mathcal{Z}_F \subset \overline{\mathcal{Z}}$ the irreducible component defined in Remark 1.6. For such an $F$ define a multiplicity $\mu_F \in \mathbb{N}\setminus\{0\}$ as follows. Let $\rho \subseteq F$ be an $(n-1)$-cell, $\sigma \in \mathscr{P}$ the unique maximal cell containing $\rho$ and $\xi \in \Lambda_\sigma$ the primitive generator of $\Lambda_\omega$ for an unbounded 1-cell $\omega$ pointing in the unbounded direction. Then

$$\mu_F = \big[\Lambda_\sigma : \Lambda_\rho + \mathbb{Z}\cdot\xi\big]. \tag{5.2}$$

Note that $\mu_F$ depends not on the choice of $\rho \subset F$.

**Proposition 5.4.** *Let $\mathcal{W}$ be a meromorphic lift of the superpotential $W$ to an algebraization $\overline{\mathcal{X}}$ of the partial compactification $\overline{\mathfrak{X}}$ of $\mathfrak{X}$ from Proposition 1.5 of the compactifiable, asymptotically cylindrical $(B,\mathscr{P},\varphi)$. Then $\mathcal{W}$ defines a morphism of schemes $\mathcal{W}:\overline{\mathcal{X}}\to\mathbb{P}^1$, and it holds*

$$\overline{\mathcal{W}}^{-1}(\infty) = \sum_F \mu_F\cdot\mathcal{Z}_F.$$

*The sum is over all sequences of parallel maximal flat affine subspaces of $\partial B_\nu$ and $\mu_F$ is defined in* (5.2).

*Proof.* Since $W$ defines a proper map to $\mathbb{A}^1$ the locus of indeterminacy of $\mathcal{W}$ is empty. Thus $\mathcal{W}$ defines a morphism to $\mathbb{P}^1$. Let $\sigma\in\mathscr{P}$ be an unbounded maximal cell and $X_\sigma \subseteq X_0$ the corresponding irreducible component of the central fiber. Then $\mathcal{W}|_{X_\sigma} = W|_{X_\sigma} \neq 0$, so the multiplicity of a prime divisor $\mathcal{Z}_F \subseteq \mathcal{W}^{-1}(\infty)$ can be computed after restriction to the central fiber $X_0$, that is, from $W^0$, and $X_\sigma$ is a component of the reduced divisor $\mathcal{Z}\cap X_0$. Moreover, in the coordinate ring $\Bbbk[\Lambda_\sigma]$ of the big torus of $X_\sigma$ we have $W^0 = z^\xi$. Thus the coefficient of $\mathcal{Z}_F \cap X_0$ in $(W^0)^{-1}(0)$ equals

$$\mathrm{ord}_{D_\rho} W^0 = \mathrm{ord}_{D_\rho} z^\xi = -[\Lambda_\sigma : \Lambda_\rho + \Lambda_{\omega_i}],$$

by a standard fact in toric geometry. □

The most interesting result in this section concerns a tropical description of the special fiber $W^{-1}(1)$ in the asymptotically cylindrical case. We will see that it is described by the asymptotic behaviors of $(B,\mathscr{P},\varphi)$ and $\mathscr{S}_k$. Moreover, $W^{-1}(1)$ is canonically the mirror toric degeneration of the anticanonical divisor $\check{\mathfrak{D}}\to T$ from Theorem 2.5, up to a tropically interesting mirror map reparametrizing the codomain $\mathbb{A}^1$ of $W$.

**Construction 5.5.** (*Asymptotic tropical manifold and asymptotic scattering diagram.*) Assume that $(B,\mathscr{P})$ is asymptotically cylindrical. Denote by $K\subseteq B$ the compact subset defined by the union of bounded cells of $\mathscr{P}$. Then there exists a non-zero integral vector field $\xi\in\Gamma(B\setminus K, i_*\Lambda)$ that is parallel to all unbounded 1-cells of $\mathscr{P}$. We fix $\xi$ uniquely by requiring it to be primitive (indivisible) and pointing in the unbounded direction. The integral curves of $\xi$ generate an equivalence relation $\sim$ on $B\setminus K$. Define the *asymptotic tropical manifold* $B_\infty$ associated to $B$ as the quotient $(B\setminus K)/\sim$.

An explicit description of $B_\infty$ runs as follows. Let $\sigma \in \mathscr{P}$ be an unbounded cell. Let $\bar\sigma$ be the convex hull of the vertices. Since $B$ is asymptotically cylindrical it holds

$$\sigma = \bar\sigma + \mathbb{R}_{\geq 0} \cdot \xi,$$

as an equation of subsets of $\Lambda_{\sigma,\mathbb{R}}$. Then define $\sigma_\infty$ as the image of $\sigma$ or $\bar\sigma$ under the canonical quotient

$$\Lambda_{\sigma,\mathbb{R}} \to \Lambda_{\sigma,\mathbb{R}}/\mathbb{R}_{\geq 0} \cdot \xi. \tag{5.3}$$

Clearly, this construction is compatible with the inclusion of faces. Taking the colimit of the $\sigma_\infty$ defines $B_\infty$ along with a polyhedral decomposition $\mathscr{P}_\infty$. The charts for the affine structure at vertices of $B_\infty$ are induced by the charts of $B$ along unbounded 1-cells of $\mathscr{P}$. Note that unbounded 1-cells of $\mathscr{P}$ are disjoint from $\Delta$. It is also clear that the strictly convex PL function $\varphi$ on $(B, \mathscr{P})$ induces a strictly convex PL function $\varphi_\infty$ on $(B_\infty, \mathscr{P}_\infty)$. We have thus constructed a polarized tropical manifold $(B_\infty, \mathscr{P}_\infty, \varphi_\infty)$ of dimension $\dim B - 1$, the *asymptotic tropical manifold* of the asymptotically cylindrical $(B, \mathscr{P}, \varphi)$.

A monomial at a point $x$ on an unbounded component of $\tau \setminus \Delta$ for $\tau \in \mathscr{P}$ induces a monomial at the image $x_\infty$ of the corresponding cell $\tau_\infty = \tau/\mathbb{R}_{\geq 0}\xi$. Since $\mathscr{S}_k$ is finite there exist a compact subset $K' \subseteq B$ such that only unbounded slabs or walls intersect $B \setminus K'$. We call these slabs or walls *asymptotic*. In the present asymptotically cylindrical case $\xi$ is then tangent to the suppport of any such asymptotic slab or wall. Thus for any asymptotic slab $\mathfrak{b}$ or wall $\mathfrak{p}$ in $\mathscr{S}_k$ the image under the quotient map $B \setminus K' \to B_\infty$ defines the support of a slab or wall in $(B_\infty, \mathscr{P}_\infty, \varphi_\infty)$. The associated slab function or exponent is defined via the projection of monomials (5.3). Define by $\mathscr{S}_k^\infty$ the structure on $B_\infty$ obtained in this way. □

To further relate the wall structures $\mathscr{S}_k$ on $B$ and $\mathscr{S}_k^\infty$ on $B_\infty$, only the rings $R^k_{g,\sigma}$ are relevant with $g : \omega \to \tau$ an inclusion of unbounded cells. Denote by $g_\infty : \omega_\infty \to \tau_\infty$ the induced inclusion of cells in $\mathscr{P}_\infty$. Taking a splitting of the inclusion $\mathbb{Z} \cdot \xi \subseteq \Lambda_\sigma$ provides a (non-canonical) isomorphism

$$\left(R^k_{g,\sigma}\right)_{z^\xi} \simeq R^k_{g_\infty,\sigma_\infty}[w, w^{-1}], \tag{5.4}$$

where by abuse of notation $\xi$ denotes the unique monomial $m$ of order 0 with $\overline{m} = \xi \in \Lambda_\sigma$. The isomorphism identifies $z^\xi$ with $w$.

Mapping $w$ to 1 now induces the canonical isomorphism of quotients

$$\left(R^k_{g,\sigma}\right)_{z^\xi}/(z^\xi - 1) \simeq R^k_{g_\infty,\sigma_\infty}. \tag{5.5}$$

Note this isomorphism is compatible with the map of monomials discussed in Construction 5.5 and does not depend on choices. In particular, there is a well-defined formal function $w$ on $\mathfrak{X} \setminus A$ locally given by $w$ in (5.4). From (5.4) it is also obvious that consistency of $\mathscr{S}_k$ implies consistency of $\mathscr{S}_k^\infty$.

**Proposition 5.6.** *Let $(\pi : \mathfrak{X} \to \operatorname{Spf} \Bbbk[\![t]\!], W)$ be the Landau-Ginzburg model with an asymptotically cylindrical intersection complex $(B, \mathscr{P})$. Then the composition $w^{-1}(1) \to \mathfrak{X} \xrightarrow{\pi} \operatorname{Spf} \Bbbk[\![t]\!]$ of the inclusion followed by $\pi$ is canonically isomorphic to the toric degeneration defined by the compatible system of wall structures $\mathscr{S}_k^\infty$ on $(B_\infty, \mathscr{P}_\infty, \varphi_\infty)$.*

*Proof.* It is enough to check the statement for a fixed finite order $k$. The key observation is that the asymptotic vector field $\xi$ is tangent to all unbounded walls on $B$. Now for fixed $k$ the complement $U \subset B$ of the compact subset $K' \subset B$ in Construction 5.5 only intersects unbounded walls. Moreover, gluing the rings associated to chambers in $U$ is enough to describe $X_k \setminus A$. The statement now readily follows from the construction of $\mathscr{S}_k^\infty$ and (5.5). □

*Remark* 5.7. Monomials in unbounded walls or slabs proportional to $z^\xi$ map to elements of the base ring $\Bbbk[\![t]\!]$ under the homomorphism 5.5. Such constant terms do not appear in the wall structures constructed either by the algorithm in [GS3] or in the canonical wall structure of [GS7]. Thus $\mathscr{S}_\infty$ is a more general wall structure than previously constructed that also involves *undirectional walls*, that is, walls $\mathfrak{p}$ with $f_\mathfrak{p} \in \Bbbk[\![t]\!]^\times$. This fact has been overlooked in earlier versions of this paper and affects the mirror statements for $w^{-1}(1)$.

The corrected statement is given in Proposition 5.9 below.

To understand the influence of undirectional walls, we observe a close relationship to gluing data.

*Remark* 5.8. Consider a wall structure $\mathscr{S}$ on an affine manifold with singularities $B$ with all walls and slabs undirectional. To emphasize the constant nature of the slab and wall functions, we now write $c_\mathfrak{p}, c_\mathfrak{b} \in \Bbbk[\![t]\!]^\times$ rather than $f_\mathfrak{p}, f_\mathfrak{b}$. Let $\mathfrak{j}$ be a zero-codimensional joint, that is, intersecting the interior of a maximal cell $\sigma$. Then the automorphism $\theta_\mathfrak{p}$ of $\Bbbk[\![t]\!][\Lambda_\sigma]$ associated to a wall $\mathfrak{p}$ containing $\mathfrak{j}$ equals

$$\theta_\mathfrak{p} : z^m \longmapsto \langle c_\mathfrak{p} \otimes n_\mathfrak{p}, m\rangle \cdot z^m, \tag{5.6}$$

with $n_\mathfrak{p} \in \check{\Lambda}_\sigma = \mathrm{Hom}(\Lambda_\sigma, \mathbb{Z})$ the primitive normal vector spanning $\Lambda_\mathfrak{p}^\perp$, with sign depending on the direction of wall crosssing. Now all such automorphisms $\theta_\mathfrak{p}$ with $\mathfrak{j} \subset \mathfrak{p}$ commute. Moreover, their product is trivial iff

$$\prod_\mathfrak{p} c_\mathfrak{p} \otimes n_\mathfrak{p} = 1 \otimes 0 \in \Bbbk[\![t]\!]^\times \otimes_\mathbb{Z} \check{\Lambda}_\sigma. \tag{5.7}$$

Note that in the tensor product the first factor is written multiplicatively, the second additively, so $1 \otimes 0$ is the unit in this abelian group.

We observe that the consistency condition (5.7) for $\mathscr{S}$ at $\mathfrak{j}$ is the cocycle condition at $\mathfrak{j}$ for the *tropical* 1*-cocycle* on $B$ supported on $|\mathscr{S}|$ that assigns $c_\mathfrak{p} \otimes n_\mathfrak{p}$, $c_\mathfrak{b} \otimes n_\mathfrak{p}$ to the elements of $\mathscr{S}$. This motivates to view a consistent, undirectional wall structure as a tropical 1-cocycle, with the cocycle condition reflected by consistency in all codimensions. Let us denote the group of tropical 1-cocyles by $C^{n-1}_{\mathrm{trop}}(B)$.

Next note that the ring homomorphisms defined by undirectional walls have the same form as in applying open gluing data, see e.g. [GHS, (5.2)]. In the simple singularity case, the group of equivalence classes of lifted open gluing data obeying a similar local consistency condition is given by the cohomology group $H^1(B, \iota_*\check{\Lambda} \otimes \Bbbk^\times)$ [GS1, Prop. 4.25, Def. 5.1, Lem. 5.5]. Now there is an obvious map

$$C^{n-1}_{\mathrm{trop}}(B) \longrightarrow H_{n-1}(B_0, \check{\Lambda} \otimes \Bbbk[\![t]\!]^\times),$$

which by consistency factors over $H_{n-1}(B, \iota_*\check{\Lambda} \otimes \Bbbk[\![t]\!]^\times)$. Moreover, by [Ru, Thm. 1] we have an isomorphism

$$H_{n-1}(B, \iota_*\check{\Lambda} \otimes \Bbbk[\![t]\!]^\times) \simeq H^1(B, \iota_*\check{\Lambda} \otimes \Bbbk[\![t]\!]^\times).$$

Thus we obtain a homomorphism

$$C^{n-1}_{\mathrm{trop}}(B) \longrightarrow H^1(B, \iota_*\check{\Lambda} \otimes \Bbbk[\![t]\!]^\times.$$

This map associates to the tropical 1-cocycle given by the undirectional wall structure $\mathscr{S}$ certain open gluing data that we denote $s_{\mathscr{S}}$.

Now one can run the algorithm in [GS3] in two ways. First as usual with a cocycle representative of the gluing data $s_{\mathscr{S}}$, leading to a wall structure $\mathscr{S}'$. Second, starting with an initial wall structure that takes into account the reduction modulo $t$ of $\mathscr{S}$, interpreted as gluing data; then run the algorithm with the undirectional walls inserted at each order to obtain a wall structure $\mathscr{S}''$. Consistency of the undirectional wall structure $\mathscr{S}$ should be a necessary and sufficient condition for this to work. We conjecture that the two families $\mathfrak{X}', \mathfrak{X}'' \to T$ obtained from $\mathscr{S}', \mathscr{S}''$ are isomorphic.

One can compare $\mathfrak{X}', \mathfrak{X}''$ by choosing a general point in each cell as a reference point and relate the diagrams of schemes in [GS3, § 2.6] by sequences of wall crossing automorphisms. To prove that this procedure induces an isomorphism of diagrams would require to carefully analyze the scattering algorithm on a neighborhood of the interior of each cell of $\mathscr{P}$, including the difference of the presence of undirectional walls versus the associated gluing data.

For the application to our asymptotic wall structure $\mathscr{S}_\infty$ on $B_\infty$, one takes for the walls of the undirectional wall structure all undirectional walls of the wall structure on the asymptotically cylindrical $B$. Applying [GS3, Prop. 3.10,(2)] one can prove consistency at codimension 0 joints order by order. For the slabs one observes that undirectional monomials in an unbounded slab $\mathfrak{b}$ are never of order 0. Hence as in [GS3, Thm. 5.2] we can factor

$$f_{\mathfrak{b}} = \bar{f}_{\mathfrak{b}} \cdot f^{\|}_{\mathfrak{b}}$$

with $f^{\|}_{\mathfrak{b}} \in \Bbbk[\![t]\!][z^{\pm\xi}]^\times$ and $\bar{f}_{\mathfrak{b}}$ having a product decomposition with no factors in $\Bbbk[\![t]\!][z^{\pm\xi}]$. We use $f^{\|}_{\mathfrak{b}}$ to define the slab of the undirectional wall structure on $B_\infty$ induced by $\mathfrak{b}$. A very careful analysis of the algorithm, which we have not carried out, should now show that this undirectional wall structure is consistent. □

We finally relate the mirror of $\check{\mathfrak{D}} \to T$ to the LG-fiber $w^{-1}(1)$. This makes precise and proves a conjecture of Auroux in our setup [Au1, Conj. 7.4].

**Proposition 5.9.** *If $\mathfrak{X} \to \mathrm{Spf}\, \Bbbk[\![t]\!]$ is mirror dual to $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ then $w^{-1}(1) \to \mathrm{Spf}\, \Bbbk[\![t]\!]$ is the mirror family to $\check{\mathfrak{D}} \to T$ twisted by an undirectional wall structure as discussed in Remarks 5.7 and 5.8.*[10]

Most interestingly, the function $w$ in Proposition 5.6 and (5.4) is closely related to the superpotential $W$. To explain this relation, note that for any fixed finite order $k$ we may restrict to the complement $U \subset B$ of a compact set such that each broken line $\beta$ ending in $U$ is parallel to the asymptotic vector field $\xi$. In other words, there exists $c \in \mathbb{Z} \setminus \{0\}$ with $m_\beta = c \cdot \xi$. There are two types of such broken lines, depending on the sign of $c$. If $c > 0$ then $\beta$ is (part of) a broken line not intersecting any wall, and hence $c = 1$ and the monomial carried by $\beta$ equals $z^\xi$. Otherwise $\beta$ is a broken line that returned from entering the compact

[10] In the interpretation of wall structures via punctured invariants [GS7], undirectional walls count punctured Gromov-Witten invariants with one positive contact order with $\check{\mathfrak{D}}$, hence relate to traditional log Gromov-Witten invariants of $(\check{\mathfrak{X}}, \check{\mathfrak{D}})$. Further details will appear in [GRS].

set due to some non-trivial interaction with walls outside of $U$. We call these broken lines $\beta$ and the corresponding monomial $m_\beta$ at the root vertex *outgoing*. In this case we can write $a_\beta z^{m_\beta} = a_\beta t^l z^{-d\xi}$ for some $l > 0$ and $d = -c > 0$. Summing over all such broken lines with general endpoint $p \in U$ leads to a polynomial with coefficients $N_{d,l} \in \Bbbk$:

$$h_k = \sum_{l=1}^{k} \sum_{d>0} N_{d,l} t^l z^{-d\xi} \in \Bbbk[t, z^{-\xi}]. \tag{5.8}$$

**Lemma 5.10.** *The coefficients $N_{d,l} \in \Bbbk$ in (5.8) do not depend on the choices of $U$, $p \in U$ or $k \geq l$.*

*Proof.* Observe first that $h_k + w$ equals $W^k_{g,\mathfrak{u}}(p)$ from (4.2), for $\mathfrak{u}$ any unbounded chamber of $\mathscr{S}_k$ containing the chosen endpoint $p \in \mathfrak{u}$ of broken lines in (5.8), and $g : \omega \to \tau$ any inclusion of cells relevant to $\mathfrak{u}$, that is, with $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subseteq \sigma_{\mathfrak{u}}$. Since $\xi$ is tangent to all walls intersecting $U$, this expression is unchanged under wall crossing automorphisms. The statement now follows from the independence of $W^k_{g,\mathfrak{u}}(p)$ of the choice of $p \in \mathfrak{u}$ (Lemma 4.7) and the compatibility of $W^k_{g,\mathfrak{u}}$ with changing strata and chambers (Lemma 4.9).

The statement on the independence of $k \geq l$ follows since a broken line $\beta$ interacting with a term in a wall of order $> k$ has an outgoing monomial $m_\beta$ of order $> k$. □

We are now in position to define the map relating $w$ and $W$ as the automorphism $\Phi$ of the formal algebraic torus

$$\hat{\mathbb{G}}_m = \mathbb{G}_m \times \operatorname{Spf} \Bbbk[\![t]\!] = \operatorname{Spf}\big(\Bbbk[u^{\pm 1}] \,\hat{\otimes}_{\Bbbk} \Bbbk[\![t]\!]\big) = \operatorname{Spf} \Bbbk[u^{\pm 1}][\![t]\!]$$

over $\Bbbk[\![t]\!]$ defined by

$$\Phi^\sharp(u) = u + \sum_{l>0} \Big( \sum_{d>0} N_{d,l} u^{-d} \Big) t^l = u \left( 1 + \sum_{l>0} \Big( \sum_{d>0} N_{d,l} u^{-d-1} \Big) t^l \right). \tag{5.9}$$

Note that for each fixed $l$ there are only finitely many broken lines contributing to the coefficient of $t^l$ in (5.8). Hence $\Phi^\sharp(u) \in \Bbbk[u^{\pm 1}][\![t]\!]$ as required to define $\Phi$. Note also that $\Phi$ induces the identity on $\mathbb{G}_m \subset \mathbb{G}_m \times \operatorname{Spf} \Bbbk[\![t]\!]$, the reduction modulo $t$.

**Definition 5.11.** We call the automorphism $\Phi$ of $\mathbb{G}_m \times \operatorname{Spf} \Bbbk[\![t]\!]$ the *Landau-Ginzburg (LG-) mirror map.*

We emphasize that $\Phi$ is not in general induced by an automorphism of the scheme

$$(\mathbb{G}_m)_{\operatorname{Spec} \Bbbk[\![t]\!]} = \operatorname{Spec} \Bbbk[\![t]\!][u^{\pm 1}].$$

For the following mirror map statement in the asymptotically cylindrical case we consider both $W|_{\mathfrak{X} \setminus A}$ and $w$ as maps to $\hat{\mathbb{G}}_m = \mathbb{G}_m \times \operatorname{Spf} \Bbbk[\![t]\!]$.

**Theorem 5.12.** *In the situation of Proposition 5.6 it holds $W|_{\mathfrak{X} \setminus A} = \Phi \circ w$.*

*Proof.* The statement follows by observing that modulo $t^{k+1}$,

$$(w^\sharp \circ \Phi^\sharp)(u) = w + \sum_{l>0} \Big( \sum_{d>0} N_{d,l} w^{-d} \Big) t^l$$

equals $W^k_{g,\mathfrak{u}} = h_k + w$ with $h_k$ as in (5.8), for all unbounded chambers $\mathfrak{u}$. □

Theorem 5.12 together with Proposition 5.9 yields the following.

**Corollary 5.13.** *Let $\mathfrak{X} \to \operatorname{Spf} \Bbbk[\![t]\!]$ be mirror dual to $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$. Then $(\Phi^{-1} \circ W)^{-1}(1) \to \operatorname{Spf} \Bbbk[\![t]\!]$ is the mirror family of $\check{\mathfrak{D}} \to T$ twisted by the undirectional wall structure discussed in Remarks 5.7 and 5.8.* □

## 6. Wall structures and broken lines via tropical disks

We now aim for an alternative construction of the potential $W$ in terms of tropical disks.[11]

6.1. **Tropical disks.** Our definition of tropical disks depends only on the integral affine geometry of $B$ and not on its polyhedral decomposition $\mathscr{P}$. As usual let $i : B \setminus \Delta \to B$ denote the inclusion for $\Delta$‘ the singular locus of the integral affine structure, and let $\Lambda_B$ be the sheaf of integral tangent vectors. We restrict to the asymptotically cylindrical case (Definition 2.1). Without reference to $\mathscr{P}$ we require that $B$ is non-compact and for some orientable compact subset $K \subset B$, $\Gamma(B \setminus K, i_*\Lambda)$ has rank one. Then there exists a unique primitive integral affine vector field $\xi$ on $B \setminus K$ pointing away from $K$. We assume the semiflow of $\xi$ is complete and call its integral curves the *asymptotic rays.*

**Definition 6.1.** Let $\Gamma$ be a tree with root vertex $V_{\mathrm{root}}$. Denote by $\Gamma^{[1]}, \Gamma^{[0]}, \Gamma^{[0]}_{\mathrm{leaf}}$ the set of edges, vertices, and leaf vertices (univalent vertices different from the root vertex), respectively. We allow unbounded edges, that is, edges adjacent to only one vertex, defining a subset $\Gamma^{[1]}_\infty \subseteq \Gamma^{[1]}$. Let $w : \Gamma^{[1]} \to \mathbb{N} \setminus \{0\}$ be a weight function.

Let $x \in B \setminus \Delta$. A *tropical disk bounded by $x$* is a proper, locally injective, continuous map

$$h : \big(|\Gamma|, \{V_{\mathrm{root}}\}\big) \longrightarrow \big(B, \{x\}\big)$$

with the following properties.

(1) $h^{-1}(\Delta) = \Gamma^{[0]}_{\mathrm{leaf}}$.
(2) For every edge $E \in \Gamma^{[1]}$ the image $h(E \setminus \partial E)$ is a locally closed integral affine submanifold of $B \setminus \Delta$ of dimension one.
(3) If $V \in \Gamma^{[0]}$ there is a primitive integral vector $m \in \Lambda_{B,h(V)}$ extending to a local vector field tangent to $h(E)$ and pointing away from $h(V)$. Define the *tangent vector of $h$ at $V$ along $E$* as $\overline{m}_{V,E} := w(E) \cdot m$.
(4) For every $V \in \Gamma^{[0]} \setminus \Gamma^{[0]}_{\mathrm{leaf}}$ the following *balancing condition* holds:

$$\sum_{\{E \in \Gamma^{[1]} \,|\, V \in E\}} \overline{m}_{V,E} = 0.$$

(5) The image of an unbounded edge is an asymptotic ray.

Two disks $h : |\Gamma| \to B$, $h' : |\Gamma'| \to B$ are identified if $h = h' \circ \phi$ for a homeomorphism $\phi : |\Gamma| \to |\Gamma'|$ respecting the weights.

The *Maslov index* of $h$ is defined as $\mu(h) := 2 \sum_{E \in \Gamma^{[1]}_\infty} w(E)$. □

Note that for a tropical disk $h^*(i_*\Lambda_B)$ is a trivial local system. In particular, there is a unique parallel transport of tangent vectors along $h$.

[11]The interpretation of wall structures and the superpotential in terms of tropical disks of Maslov indices 0 and 2 in this section has a more speculative and inconclusive nature than the rest of the paper. Recent advances in our understanding of wall structures in the context of intrinsic mirror symmetry [GS7] makes it now feasible to develop the picture given in this section in full generality. This section is therefore included with only minor changes from the original version, but should be read with some caution.

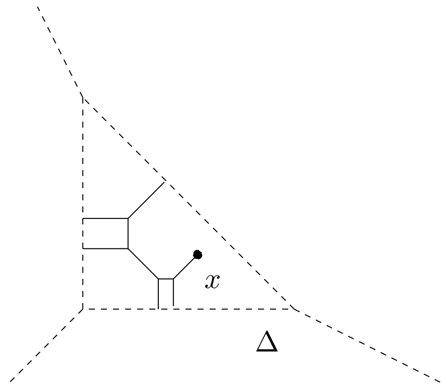


FIGURE 6.1. A tropical Maslov index zero disk bounding $x$ belonging to a moduli space of dimension 5. The dashed lines indicate a part of the discriminant locus.

**Example 6.2.** Suppose $\dim B = 3$ and $\Delta$ bounds an affine 2-simplex $\sigma$ with $T_x\sigma$ contained in the image of $i_*\Lambda_{B,x}$ for all $x \in \Delta$. Such a situation occurs in the mirror toric degenerations of local Calabi-Yau threefolds, for example in the mirror of $K_{\mathbb{P}^2}$ [GS4, Expl. 5.2]. Then any point $x \in \sigma \setminus \Delta$ bounds a family of tropical Maslov index zero disks of arbitrary dimension, as illustrated in Figure 6.1.

So far, our definition of tropical disks only depends on $|\Gamma|$ and not on its underlying graph $\Gamma$. A distinguished choice of $\Gamma$ is by assuming that there are no divalent vertices. At an interior vertex $V \in \Gamma^{[0]}$ (that is, neither the root vertex nor a leaf vertex) the rays $\mathbb{R}_{\geq 0} \cdot \overline{m}_{E,V}$ of adjacent edges $E$ define a fan $\Sigma_{h,V}$ in $i_*\Lambda_{B,h(V)} \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\Sigma^0_{h,V}$ the parallel transport along $h$ of $\Sigma_{h,V}$ to $V_{\text{root}}$. The *type* of $h$ consists of the weighted graph $(\Gamma, w)$ along with the $\Sigma^0_{h,V}$, $V \in \Gamma^{[0]} \setminus \Gamma^{[0]}_\infty$. For $x \in B \setminus \Delta$ and $\overline{m} \in \Lambda_{B,x}$ denote by $\mathcal{M}_\mu(\overline{m})$ the moduli space of tropical disks of Maslov index $\mu$ and root tangent vector $\overline{m}$. It comes with a natural stratification by type: A stratum consists of disks of fixed type, and the boundary of a stratum is reached when the image of an interior edge contracts to a vertex of higher valency.

From now on assume $B$ is equipped with a compatible polyhedral structure $\mathscr{P}$ as defined in [GS1, §1.3]. It is then natural to adapt $\Gamma$ to $\mathscr{P}$ by appending Definition 6.1 as follows:

(6) For any $E \in \Gamma^{[1]}$ there exists $\tau \in \mathscr{P}$ with $h(\operatorname{Int} E) \subseteq \operatorname{Int}(\tau)$, and if $V \in E$ is a divalent vertex then $h(V) \subseteq \partial\tau$.

In other words, we insert divalent vertices precisely at those points of $|\Gamma|$ where $h$ changes cells of $\mathscr{P}$ locally. Note however that we still consider the stratification on $\mathcal{M}_\mu(\overline{m})$ defined with all divalent vertices removed.

**Example 6.3.** As it stands, the type does not define a good stratification of the moduli space of tropical disks. For each vertex $V \in \Gamma$ mapping to a codimension one cell $\rho \in \mathscr{P}$ we also need to specify the connected component of $\rho \setminus \Delta$ containing $h(V)$ (that is, specify a reference vertex $v \in \rho$). This is illustrated in Figure 6.2. Here the dotted lines in the right picture correspond to generalized tropical disks, fulfilling all but (1) in Definition 6.1.

Tropical disks are closely related to broken lines as follows. We place ourselves in the context of §4. In particular, we assume given a structure $\mathscr{S}_k$ that is consistent to order $k$.

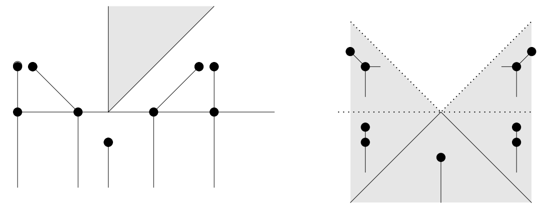

FIGURE 6.2. Disks near $\Delta$ (left) and their moduli cell complex (right).

**Lemma 6.4** (Disk completion)**.** [12] *As a map, any broken line is the restriction of a Maslov index two disk $h : |\Gamma| \to B$ to the smallest connected subset of $|\Gamma|$ containing the root vertex and the (unique) unbounded leaf. The restriction of $h$ to the closure of the complement of this subset consists of Maslov index zero disks.*

*Proof.* We continue to use the terminology of [GS3]. First we show that any projected exponent $\overline{m}$ at a point $p$ of a wall or slab in $\mathscr{S}_k$ is the root tangent vector of a Maslov index zero disk $h$ rooted in $h(V_{\mathrm{root}}) = p$. This is true for $\mathscr{S}_0$ since by simplicity the exponents of a slab function $f_{\rho,v}$ are root tangent vectors of Maslov index zero disks with only one edge. Assume inductively this holds as well for $\mathscr{S}_l$, $0 \le l \le k$, and show the claim for walls in $\mathscr{S}_{l+1} \setminus \mathscr{S}_l$ arising from scattering. We must show that the exponents of the outgoing rays are generated by those of the incoming rays or cuts. But if there existed an additional exponent, it would be preserved by any product with log automorphisms attached to the rays or cuts, as up to higher orders the latter are multiplications by polynomials with non trivial constant term. This contradicts consistency.

In particular, if $p = \beta(t_i)$ is a break point of a broken line $\beta$ then $t_i$ can be turned into a balanced trivalent vertex by attaching a Maslov index zero disk $h$ with root tangent vector $\overline{m}$ equal to the projected exponent taken from the unique wall or slab containing $p$. □

We call any tropical disk as in the lemma a *disk completion* of the broken line. The disk completion is in general not unique due to the following reasons:

(1) First, Example 6.2 shows that tropical Maslov index zero disks may come in families of arbitrarily high dimension.
(2) Even if the moduli space of tropical Maslov index zero disks is of expected finite dimension, there may be joints with different incoming root tangent vectors.
(3) There may exist several Maslov index zero disks with the same root tangent vector, for example a closed geodesic of different winding numbers.

We now take care of these issues.

6.2. **Virtual tropical disks.** Example 6.2 illustrates that for $\dim B \ge 3$, a tropical disk whose image is contained in a union of slabs leads to an unbounded dimension of the moduli space of tropical Maslov index $2\mu$ disks. In order to get enumerative invariants which recover broken lines we need a virtual count of tropical disks. Throughout we assume $B$ is oriented.

Suppose $\Delta$ is straightened as in [GS1, Rem. 1.49], that is, $\Delta$ defines a subcomplex $\Delta^\bullet$ of the barycentric refinement of the polyhedral decomposition $\mathscr{P}$ of $B$. Note that the simplicial structure of $\Delta^\bullet$ refines the natural stratification of $\Delta$ given by local monodromy type. Let

[12] Cf. [GS7, Prop.4.13] for a refined treatment in a slightly different setup.

$\Delta_{\max}$ denote the set of maximal cells of $\Delta^\bullet$ together with an orientation, chosen once and for all. Each $\tau \in \Delta_{\max}$ is contained in a unique $(n-1)$-cell $\rho \in \mathscr{P}$. Then monodromy along a small loop about $\tau$ defines a monodromy transvection vector $m_\tau \in \Lambda_\rho$, where the signs are fixed by the orientations via some sign convention. In view of the orientations of $\tau$ and $B$ we can then also choose a maximal cell $\sigma_\tau \supset \tau$ unambiguously.

For each $\tau \in \Delta_{\max}$ let $w_\tau$ be the choice of a partition of $|w_\tau| \in \mathbb{N}$ (with $w_\tau = \emptyset$ for $|w_\tau| = 0$). To separate leaves of tropical disks we will now locally replace $\Delta$ by a branched cover. We can then consider deformations of a disk $h$ whose leaves end on that cover instead of $\Delta$, with weights and directions prescribed by the partitions $\mathbf{w} := (w_\tau | \tau \in \Delta_{\max})$.

*Deformations of* $\Delta$. We first define a deformation of the barycentric refinement $\Delta_{\mathrm{bar}}$ of $\Delta$ as a polyhedral subset of $B$. For each $\tau \in \Delta_{\max}$, denote by $\mathfrak{s}_\tau \subseteq \sigma_\tau$ the 1-cell connecting the barycenter $b_\tau$ of $\tau$ to the barycenter of $\sigma_\tau$. Note that $\Lambda_{\mathfrak{s}_\tau} \otimes_{\mathbb{Z}} \mathbb{R}$ intersects $i_*\Lambda_{b_\tau} \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{span}(\Lambda_\tau, m_\tau)$ transversally. Moving the barycenter of the barycentric refinement $\tau_{\mathrm{bar}}$ of $\tau$ along $\mathfrak{s}_\tau$ while fixing $\partial\tau_{\mathrm{bar}}$ now defines a piecewise linear deformation $\tau_s$ of $\tau$ over $s \in \mathfrak{s}_\tau$ as a polyhedral subset of $\sigma_\tau$. Thus we obtain a deformation $\{\Delta_s | s \in S\}$ of $\Delta$ over the cone $S := \prod_\tau \mathfrak{s}_\tau$. It is trivial as deformation of cell complexes, as parallel transport in direction $\mathfrak{s}_\tau$ in each cell $\sigma_\tau$ induces an isomorphism of cell complexes $\Delta_{\mathrm{bar}} \cong \Delta_s$.

For an infinitesimal point of view let $i_\tau : \tau \to \sigma$ be the inclusion. Consider the preimage of the deformation of $\tau \subseteq \Delta$ under the natural inclusion $\sigma_\tau \hookrightarrow i_\tau^* T\sigma_\tau$. For $s = (s_1, \ldots, s_{\mathrm{length}\, w_\tau}) \in \mathfrak{s}_\tau^{\mathrm{length}\, w_\tau}$ with $s_i$ pairwise different,

$$\tau_s^{\mathbf{w}} := \bigcup_{k=1}^{\mathrm{length}\, w_\tau} \tau_{s_k} \subseteq i_\tau^* T\sigma_\tau$$

is then a length $w_\tau$-fold branched cover of $\tau$, via the natural projection

$$\pi : i_\tau^* T\sigma_\tau \to \tau.$$

Note that $\tau_s^{\mathbf{w}} = \emptyset$ if $|w_\tau| = 0$ and $\tau_s^{\mathbf{w}} \subseteq \tau_{s'}^{\mathbf{w}'}$ if $\mathrm{length}\, w_\tau \leq \mathrm{length}\, w'_\tau$ and if the entries of $s$ agree with the first $\mathrm{length}\, w_\tau$ entries of $s'$. We make $\tau_s^{\mathbf{w}}$ into a weighted cell complex by equipping each cell of $\tau_s^{\mathbf{w}}$ with the weight defined by the partition $w_\tau$. Finally, set $S^{\mathbf{w}} := \prod_\tau \mathfrak{s}_\tau^{\mathrm{length}\, w_\tau}$ and $\Delta_s^{\mathbf{w}} := \bigcup_\tau \tau_{s_\tau}^{\mathbf{w}}$, where $s \in S^{\mathbf{w}}$. We still call $\Delta_s^{\mathbf{w}}$ a deformation of $\Delta$, as for $\epsilon \to 0$, $\tilde{\Delta}_{\epsilon s}^{\mathbf{w}}$ converges to $\Delta$ as *weighted* complexes in an obvious sense.

*Deformations of tropical disks*. We now want to define a virtual tropical disk as an infinitesimal deformation $\tilde{h}$ of a tropical disk $h$ such that the leaves of $\tilde{h}$ end on $\Delta_s^{\mathbf{w}}$ as prescribed by $\mathbf{w}$.

The idea is that for small $\epsilon > 0$ and suitable environments $U_v \subseteq T_v B$ of 0, $v = h(V)$ the image of an internal vertex, the rescaled exponential map $\exp\big|_{\bigcup_v \left(\frac{1}{\epsilon} U_v\right)} \circ \epsilon\, \mathrm{id}_{T(B \setminus \Delta)}$ maps the union of the tropical curves $\tilde{h}_V$ in the following Definition 6.5 onto the image of a tropical disk $\tilde{h}_\epsilon : \tilde{\Gamma} \to B$ with leaves emanating from $\Delta_{\epsilon s}^{\mathbf{w}}$. By choosing $\epsilon > 0$ sufficiently small, the image of $\tilde{h}_\epsilon$ is contained in an arbitrary small neighborhood of the image of $h$. Thus $\tilde{h}_\epsilon$ indeed defines a deformation of $h$. Conversely, for $\epsilon$ sufficiently small, $\tilde{h}_\epsilon$ determines $\tilde{h}$ uniquely. Hence in order to simplify language and visualization, we may and will identify a virtual curve $\tilde{h}$ with its "images" $\tilde{h}_\epsilon$ in $B$ for small $\epsilon > 0$.

**Definition 6.5.** Let $h : |\Gamma| \to B$ be a tropical disk not intersecting $|\Delta^{[\dim B-3]}|$. A *virtual tropical disk* $\tilde{h}$ *of intersection type* $\mathbf{w}$ *deforming* $h$ consists of:

(1) For each interior vertex $V \in \Gamma^{[0]}$, a possibly disconnected genus zero ordinary tropical curve $\tilde{h}_V : \tilde{\Gamma}_V \to T_v B$ with respect to the fan $\Sigma_{h,V}$. This means that $\tilde{\Gamma}_v$ is a possibly disconnected graph with simply connected components and without di- and univalent vertices, the map $\tilde{h}_V$ satisfies conditions (2)–(4) of Definition 6.1, while instead of (5) the unbounded edges are parallel displacements of rays of $\Sigma_{h,V}$.
(2) A cover $\tilde{h}_E$ of each edge $E$ of $h$ by weighted parallel sections of the normal bundle $h|_E^* TB/Th(E)$. For each edge $E$ adjacent to an interior vertex $V$, we require that the inclusion defines a weight-respecting bijection between the cosets of $\tilde{h}_E^{-1}(V)$ over $V$ and rays of $\tilde{h}_V$ in direction $E$. Moreover, the intersection defines a weight-preserving bijection between the cosets of $\{\tilde{h}_E^{-1}(V) \,|\, h(V) \in \tau, E \ni V\}$ over the leaf vertices in $\tau$ and branches of $\tau_s^{\mathbf{w}}$.
(3) A virtual root position, that is, a point $\tilde{h}_{V_{\mathrm{root}}}(\widetilde{V_{\mathrm{root}}})$ in $T_{h(V_{\mathrm{root}})}B$ such that $\tilde{h}_{V_{\mathrm{root}}}(\widetilde{V_{\mathrm{root}}}) + Th(E) = \tilde{h}_E^{-1}(V_{\mathrm{root}})$, where $E$ is the root edge.

We denote by $\mathcal{M}_\mu(\Delta_s^{\mathbf{w}}, h)$ the moduli space of virtual Maslov index $\mu$ disks of intersection type $\mathbf{w}$ deforming $h$.

In order to exclude the phenomenons in Example 6.2, we now restrict to sufficiently general tropical disks. For such tropical disks a local deformation of the constraints on $\tilde{h}(\tilde{\Gamma}^{[0]})$ lifts to a local deformation of $\tilde{h}$ preserving the type. Formally, we define:

**Definition 6.6.** Let $s \in S^{\mathbf{w}}$ and $\mu \in \{0, 1\}$. A virtual tropical disk $\tilde{h} \in \mathcal{M}_{2\mu}(\Delta_s^{\mathbf{w}}, h)$ is *sufficiently general* if:

(1) $\tilde{h}$ has no internal vertices of valency higher than three,
(2) all intersections of $\tilde{h}$ with the codimension one cells of $\mathscr{P}$ are transverse intersections at divalent vertices outside $|\mathscr{P}^{[\dim B-2]}|$,
(3) there exists a subspace $L \subseteq T_{h(V_{\mathrm{root}})}B$ of dimension $\max\{1-\mu, 0\}$ and an open cone $C^{\mathbf{w}} \subseteq S^{\mathbf{w}}$ containing $s$ such that the natural map

$$\pi \times \mathrm{ev}_{\widetilde{V_{\mathrm{root}}}} : \bigcup_{s \in C^{\mathbf{w}}} \mathcal{M}_\mu(\Delta_s^{\mathbf{w}}, h) \longrightarrow C^{\mathbf{w}} \times \big(T_{h(V_{\mathrm{root}})}B\big)/L \tag{6.1}$$

is open.

$\Delta_s^{\mathbf{w}}$ is in *general position* if for all Maslov index zero disks $h$ the complement of the set $\mathcal{M}_0(\Delta_s^{\mathbf{w}}, h)^{gen}$ of sufficiently general disks in $\mathcal{M}_0(\Delta_s^{\mathbf{w}}, h)$ is nowhere dense.

**Lemma 6.7.** *Given* $\mathbf{w}$*, the space of non-general position deformations of* $\Delta$ *is nowhere dense in* $S^{\mathbf{w}}$*. For general position,* $\mathcal{M}_{2\mu}(\Delta_s^{\mathbf{w}}, h)$ *is of expected dimension* $\dim B + \mu - 1$.

*Proof.* (Sketch) Consider a generalized class of tropical disks by forgetting the leaf constraints, allowing edge contractions and replacing condition (6) by the assumption that the graph contains no divalent vertices. Fix a type with a trivalent graph $\Gamma$. Then any tropical disk of the given type is determined by the position of the root vertex $x$ and the length of the $N := |\Gamma^{[1]}| - \mu = 2|\Gamma^{[0]}_{\mathrm{leaf}}| - 1 - \mu$ bounded edges. This shows that the inverse map restricts to an open embedding $\bigcup_{s \in S^{\mathbf{w}}} \mathcal{M}_\mu(\Delta_s^{\mathbf{w}}) \to B \times \mathbb{R}^N_{\geq 0}$ in obvious identifications dictated by $\Gamma$. The statement now follows from the observation that the map (6.1) expressed in the induced affine structure on $\mathcal{M}_\mu(\Delta_s^{\mathbf{w}}, h)$ is piecewise linear, and any violation of stability defines a subset of a finite union of hyperplanes. In particular, the dimension statement follows by noting that the positions of the leaf vertices define constraints of codimension $2(|\Gamma^{[0]}_{\text{leaf}}| - \mu)$. □

*Remark* 6.8. A stratum of $\mathcal{M}_0(\overline{m})$ admits a natural affine structure. Hence a disk $h \in \mathcal{M}_0(\overline{m})^{[k]}$ belonging to a $k$-dimensional stratum naturally comes with the $k$-dimensional subspace of induced infinitesimal vertex deformations

$$\mathfrak{j}_V(h)^{[k]} := T_h \mathrm{ev}_V(T_h \mathcal{M}_0^{[k]}(\overline{m})) \subseteq T_{h(V)}B.$$

Likewise, infinitesimal deformations of a sufficiently general Maslov index zero disk $\tilde{h}$ give rise to *virtual joints*, that is, the codimension two subsets defined by restricting the deformation family to the vertices. Such virtual joints converge to some codimension two space $\mathfrak{j}_V(\tilde{h}) \in T_{h(V)}B$, as $s \to 0 \in C^{\mathbf{w}}$. Moreover, if the limiting disk $h$ of $\tilde{h}$ belongs to a $(\dim B - 2)$-stratum of $\mathcal{M}_0(\overline{m})$ such that (6.1) extends to an open map at $0 \in \partial C^{\mathbf{w}}$, then $\mathfrak{j}_V(h)^{[\dim B-2]} = \mathfrak{j}_V(\tilde{h})$. This may be used to define stability for tropical disks.

6.3. **Structures via virtual tropical disks.** We now relate the counting of virtual Maslov index zero disks with the structures of [GS3]. Let $\mathscr{B}$ be the set of closures of connected components of the codimension one cells of $\mathscr{P}$ when $\Delta$ is removed. For $\mathfrak{b} \in \mathscr{B}$ contained in $\rho \in \mathscr{P}^{[n-1]}$ and $v \in \rho$ a vertex contained in $\mathfrak{b}$ denote by $f_\mathfrak{b} := f_{\rho,v}$ the order zero slab function attached to $\mathfrak{b} \in \mathscr{B}$ via the log structure. Then $f_\mathfrak{b} \in \Bbbk[C_\mathfrak{b}]$ where $C_\mathfrak{b}$ is the monoid generated by one of the two primitive invariant $\tau$-transverse vectors $\pm m_\tau$ for each positively oriented $\tau \in \Delta^{[max]}$ with $\overline{\mathfrak{b}} \cap \tau \neq \emptyset$.

Let $k \in \mathbb{N}$. Define the order $k$ scattering parameter ring by

$$R^k := \Bbbk[t_\tau \,|\, \tau \in \Delta_{\max}]/\mathcal{I}_k, \quad \mathcal{I}_k := (t_\tau^{k+1} | \tau \in \Delta_{\max}),$$

and let $\widehat{R}$ be its completion as $k \to \infty$.

As $f_\mathfrak{b}$ has a non-trivial constant term, we can take its logarithm as in [GPS]

$$\log f_\mathfrak{b} = \sum_{\overline{m} \in C_\mathfrak{b}} \mathrm{length}(\overline{m}) a_{\mathfrak{b},\overline{m}} z^{\overline{m}} \in \Bbbk[\![C_\mathfrak{b}]\!], \tag{6.2}$$

defining virtual multiplicities $a_{\mathfrak{b},\overline{m}} \in \Bbbk$. We consider $\log f_\mathfrak{b}$ as an element of $\Bbbk[\![C_\mathfrak{b}]\!] \otimes_k \widehat{R}$ via the completion of the inclusions

$$\iota_k : \Bbbk[C_\mathfrak{b}] \to \Bbbk[C_\mathfrak{b}] \otimes_\Bbbk R^k, \quad z^{m_\tau} \mapsto z^{m_\tau} t_\tau.$$

**Definition 6.9.** Attach the following numbers to a sufficeintly general virtual tropical disk $\tilde{h} \in \mathcal{M}_\mu(\Delta_s^{\mathbf{w}}, h)^{gen}$:

(1) The virtual multiplicity of a vertex $V \in \widetilde{\Gamma}^{[0]}$ of $\tilde{h}$ is

$$\mathrm{vmult}_V(\tilde{h}) := \begin{cases} a_{\mathfrak{b},\overline{m}} & \text{if } V \text{ is univalent, } \pi(\tilde{h}(V)) \in \mathfrak{b} \\ s(\overline{m}) & \text{if } V \text{ is divalent} \\ \left|\overline{m} \wedge \overline{m}'\right|_{\mathfrak{j}_V(h)} & \text{if } V \text{ is trivalent} \end{cases}$$

where $\overline{m}$ denotes the tangent vector of $\tilde{h}$ at $V$ in the direction leading to the root, $\overline{m}'$ the tangent vector of a different edge of $\tilde{h}$ at $V$, $a_{\mathfrak{b},\overline{m}}$ the coefficients in (6.2), $s : \Lambda_{\tilde{h}(V)}B \to k^\times$ the change of stratum function at $\tilde{h}(V)$ coming from the gluing data, and the last expression the quotient density on $T_{\tilde{h}(V)}B/\mathfrak{j}_V(\tilde{h})$ induced from the natural density on $B \setminus \Delta$. Explicitly, letting $\overline{j}_1, \ldots, \overline{j}_{n-2}$ be generators of $\mathrm{j}_V(\tilde{h}) \cap \Lambda_{B,h(V)}$ (cf. Remark 6.8) then

$$\left|\overline{m} \wedge \overline{m}'\right|_{\mathrm{j}_V(h)} := |\overline{m} \wedge \overline{m}' \wedge \overline{j}_1 \wedge \ldots \wedge \overline{j}_{n-2}|.$$

(2) The virtual multiplicity $\mathrm{vmult}(\tilde{h})$ of $\tilde{h}$ is the total product

$$\mathrm{vmult}(\tilde{h}) := \frac{1}{|\operatorname{Aut}(\mathbf{w})|} \cdot \prod_{V \in \tilde{\Gamma}^{[0]}} \mathrm{vmult}_V(\tilde{h}).$$

Here $\mathbf{w}$ is the intersection type of $\tilde{h}$, and $\operatorname{Aut}(\mathbf{w})$ is the product of the automorphisms[13] of the partitions $w_\tau$ over all $\tau \in \Delta_{[\max]}$.

(3) The $t$-order of $\tilde{h}$ is the sum of the changes in the $t$-order of the tangent vectors $\overline{m}_V$ at divalent vertices $V$ under changing the adjacent maximal strata $\sigma_V^{\pm}$, that is

$$\operatorname{ord} \tilde{h} := \sum_{\substack{V \in \Gamma^{[1]}: \\ \tilde{h}(V) \in \sigma_V^+ \cap \sigma_V^- \in \mathscr{P}^{[n-1]}}} \left| \left\langle d\varphi|_{\sigma_V^-} - d\varphi|_{\sigma_V^+}, \overline{m}_V \right\rangle \right|.$$

*Remark* 6.10. The $t$-order may be considered as a combinatorial analogue of the symplectic area of a holomorphic disk.

Note that the virtual multiplicity of a sufficiently general tropical disk depends only on its type. Moreover, we have:

**Lemma 6.11.** *The virtual multiplicity of a (type of) sufficiently general tropical disk $\tilde{h}$ of intersection type $\mathbf{w}$ deforming a Maslov index zero disk $h$ is independent of the choice $s \in S^{\mathbf{w}}$ of the general position deformation $\Delta_s$.*

*Proof.* We only give a very rough sketch here: If the type only changes by the number of divalent vertices, the claim follows immediately from the definition of $\varphi$ as a continuous and piecewise linear function. In dimension two, the result then reduces to a standard one, cf. [GM]. In higher dimension, the only remaining instable *hyper*planes consist of disks with a four-valent vertex. Here the independence of their stable deformations reduces to the dimension two case, as the virtual multiplicity is invariant under splitting each edge of $\Gamma$, acting with the stabilizer $SL(n, \mathbb{Z})_{\mathrm{j}_V(h)}$ on each fan $\Sigma_{h,V}$, and regluing formally. □

We are now ready for our central definitions: Denote by $\#\mathcal{M}_0(\mathbf{w}, \overline{m}, \ell)^{gen}$ the number of types of sufficiently general virtual tropical disks with intersection type $\mathbf{w}$ and $t$-order $\ell$ which deform a tropical Maslov index zero disk with root tangent vector $\overline{m} \in \Lambda_{B \setminus \Delta, x}$, counted with virtual multiplicity. More generally, for $\mu \in \{0, 2\}$ we can define $\#\mathcal{M}_\mu(\mathbf{w}, \overline{m}, \ell)^{gen}$ by counting the corresponding disks themselves, but specifying the virtual root as follows: The virtual root is $0 \in T_x B$ if $\mu = 2$, and belongs to a line in $T_{h(V_{\mathrm{root}})} B$ transverse to $\mathrm{j}_{V_{\mathrm{root}}}(\tilde{h})$ if $\mu = 0$.

Let $\sigma \in \mathscr{P}_{\max}$, and $P_{v,\sigma}$ the associated monoid at $v \in \sigma$ which is determined by $\varphi$ as in [GS3, Constr. 2.7]. Define the *counting function* to order $k$ in $x \in \sigma$ by

$$\log f_{\sigma,x} := \sum_{\substack{\overline{m} \in \Lambda_{x,B}, \\ \ell \leq k}} \sum_{\mathbf{w}} \mathrm{length}(\overline{m}) \#\mathcal{M}_0(\mathbf{w}, \overline{m}, \ell)^{gen} z^{\overline{m}} t^{\ell} \prod_{\tau} t_\tau^{|w_\tau|}, \tag{6.3}$$

[13] that is, the number of permutations of the entries of $w_\tau$ that do not change the partition

which is an element of the ring $\Bbbk[P_{v,\sigma}] \otimes_{\Bbbk} R^k$.

**Conjecture 6.12.** *For each $k \in \mathbb{N}$, the counting polynomial* (6.3) *modulo* $(t^{k+1})$ *stabilizes in* $\mathbf{w}$ *and then maps to the rings $R^k_{g,\sigma}$ via $t_\tau \mapsto 1$. The sets*

$$\mathfrak{p}^k[x] := \overline{\left\{ y \in \sigma \setminus \partial\sigma \mid \ \log f_{\sigma,y} = \log f_{\sigma,x} \neq 0 \in R^k_{g,\sigma} \right\}}, \tag{6.4}$$

*are either empty or define polyhedral subsets of codimension at least one. Up to refinement and adding trivial walls, the $\mathfrak{p}^k[x]$ together with their functions $\log f_{\sigma,y}$ reproduce the consistent wall structure $\mathscr{S}_k$ constructed in* [GS3].

*Remark* 6.13. Note that general position of $\Delta$ is not essential as long as we obtain the same virtual counts.

**Proposition 6.14.** *The conjecture is true for* $\dim B = 2$.

*Proof.* (Sketch) It is sufficient to show the following two claims:
Claim 1: Over $R^k$, the counting monomials arise via scattering. This can be proved by first decomposing $\exp f_{\sigma,x}$ into products of binomials and then proceeding inductively by applying [GPS, Thm. 2.7] to each joint. Alternatively, one can adapt their proof directly:

Consider the thickening

$$\iota_\tau : \ \Bbbk[t_\tau]/t^{k+1} \longrightarrow \frac{\Bbbk[u_{\tau i}|1 \leq i \leq k,]}{(u^2_{\tau i}|1 \leq i \leq k)}, \quad t_\tau \longmapsto \sum_{i=1}^{k} u_{\tau i}.$$

inducing a thickening $\bigotimes_\tau \iota_\tau : R^k \to \tilde{R}^k$ of the scattering parameter ring. Then consider virtual tropical disks with respect to the $2^k$-fold branched covering of $\Delta$ whose branches $\tau^J_s$ over $\tau$ are labeled by $J \subseteq \{1, ..., k\}$. Denote $u_{\tau J} := \prod_{i\in J} u_{\tau i}$. We say such a disk is special if it has the following additional properties: The weight of a leaf is $|J|$ if it emanates from $\tau^J_s$, and $u_{\tau J} u_{\tau J'} \neq 0$ whenever there are leaf vertices in $\tau^J_s$ and $\tau^{J'}_s$. We can now attach the following function to the root tangent vector $-\overline{m}_{\tilde{h}}$ of such disks:

$$f_{\tilde{h}} := 1 + \mathrm{length}(m_{\tilde{h}})\, \mathrm{vmult}'(\tilde{h}) \cdot z^{\overline{m}_{\tilde{h}}} t^{\mathrm{ord}\,\tilde{h}} \prod_{\tau^J_s \cap \tilde{h}(\tilde{\Gamma}^{[0]}_{\mathrm{leaf}}) \neq \emptyset} |J|! \cdot u_{\tau J}. \tag{6.5}$$

where vmult′ equals vmult without the combinatorial factor $|\operatorname{Aut}(\mathbf{w})|$.

Now the order 0 terms appearing in the thickening of the exponential of (6.3) are precisely the $f_{\tilde{h}}$ of those special disks $\tilde{h}$ that contain only one edge. The others indeed arise from scattering, meaning the following: Whenever the root vertices of two special disks $\tilde{h}, \tilde{h}'$ map to the same point $p$ with transverse root leaves, there at most two ways to extend both disks beyond $p$ locally: Either glue them to a single tropical disk, which is possible only if $f_{\tilde{h}} f_{\tilde{h}'} \neq 0$, or enlarge the root leaves such that $p$ stays a point of intersection, which is always possible.

The functions attached to the two old and the three new roots then define a consistent scattering diagram, that is the counterclockwise product of the automorphisms

$$z^{\overline{m}} \mapsto z^{\overline{m}} f_{\tilde{h}}^{|\overline{m} \wedge \overline{m}_{\tilde{h}}|}$$

equals one. This is the content of [GPS, Lem. 1.9], to which we refer for details. Now the proof of [GPS, Thm. 2.7] shows that the sum $\sum_{\tilde{h}} \log f_{\tilde{h}}$ over all special disks with $k$-intersection type $\mathbf{w}$, root tangent vector $-\overline{m}$ and $t$-order $\ell$ equals the thickening of the corresponding monomial in (6.3).

Claim 2: The counting functions (6.3) can be lifted. We must show that the scattering diagrams at each joint $\mathfrak{j}_V(\tilde{h})$ produce liftable monomials:

In case of a codimension zero joint this follows from the observation that each incoming non constant ray monomial has $t$-order greater than zero. Hence working modulo $(t^\ell)$ implies working up to a finite $k$-order. In case of codimension one joints, by assumption there is only one non constant monomial of zero $t$-order present in each scattering diagram, namely that given by the log structure. In this case, we can apply [Gr3]. Finally, there are no codimension two joints by assumption. From both claims it follows that the gluing functions of both constructions must indeed coincide, as by our assumption on $\Delta$ both rely on scattering only, and scattering is unique up to equivalence. □

6.4. **Virtual counts of tropical Maslov index two disks.** Assume now $(B, \mathscr{P}, \varphi)$ fulfills Conjecture 6.12, for example $\dim B = 2$.

**Proposition 6.15.** *The coefficient $a_\beta$ of the last monomial $z^{m_\beta}$ of a general broken line $\beta$ is the virtual number of tropical Maslov index two disks with root tangent vector $m_\beta$ which complete $\beta$ as in Lemma 6.4 and whose $t$-order equals the total change in the $t$-order of the exponents along $\beta$.*

*Proof.* Let $az^m, a'z^{m'}$ be the functions attached to the edges adjacent to a fixed break point $\beta(t)$ of $\beta$. Let $h$ be a virtual Maslov index zero disk bounded by $\beta(t)$ and with root tangent vector the required difference $\overline{m} - \overline{m}'$. Define the completed multiplicity of $h$ as the virtual multiplicity of $h$ times that of the break point. By definition, $a'/a$ is a coefficient in the exponential of $a_{\mathfrak{b}} := |\overline{m} \wedge \overline{m}'| \log f_{\mathfrak{b}}$ for the function $f_{\mathfrak{b}}$ belonging to the wall or slab with tangent space containing $\overline{m} - \overline{m}'$. By formula (6.3), $a_{\mathfrak{b}}$ equals the virtual number of completing virtual Maslov index zero disks with root tangent leaf $\overline{m} - \overline{m}'$ and $t$-order $\ell$, completed by the break point multiplicity. Hence the required coefficient in $\exp a_{\mathfrak{b}}$ is given by counting *disconnected* virtual tropical disks of total $t$-order $\ell$ and total root tangent leaf $\overline{m}$ with completed multiplicity. □

## 7. Toric degenerations of del Pezzo surfaces and their mirrors

In this section we will compare superpotentials for mirrors of toric degenerations of del Pezzo surfaces, using broken lines and tropical Maslov index two disks. Recall that apart from $\mathbb{P}^1 \times \mathbb{P}^1$ all non-singular del Pezzo surfaces $dP_k$ can be obtained by blowing up $\mathbb{P}^2$ in $0 \leq k \leq 8$ points. Note that $dP_k$ for $k \geq 5$ is not unique up to isomorphism but has a $2(k-4)$-dimensional moduli space. For the anticanonical bundle to be ample the blown-up points need to be in sufficiently general position. This means that no three points are collinear, no six points lie on a conic and no eight points lie on an irreducible cubic which has a double point at one of the points. However, rather than ampleness of $-K_X$ the existence of certain toric degenerations is central to our approach. For example, our point of view naturally includes the case $k = 9$.

7.1. **Toric del Pezzo surfaces.** Up to lattice isomorphism there are exactly five toric del Pezzo surfaces $X(\Sigma)$ whose fans $\Sigma$ are depicted in Figure 7.1, namely $\mathbb{P}^2$ blown up torically in at most three distinct points and $\mathbb{P}^1 \times \mathbb{P}^1$. To construct distinguished superpotentials for these surfaces we consider the following class of toric degenerations. For the definition note

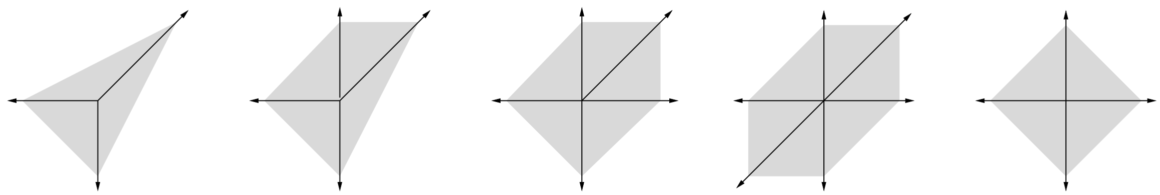

FIGURE 7.1. Fans of the five toric del Pezzo surfaces

that by the Grothendieck algebraization theorem a toric degeneration $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ with $\check{\mathfrak{D}}$ relatively ample can be algebraized. By abuse of notation we write $(\check{\mathfrak{X}}_\eta, \check{\mathfrak{D}}_\eta)$ for the generic fiber of an algebraization, and also simply speak of the *generic fiber of* $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$.

**Definition 7.1.** A *distinguished toric degeneration of Fano varieties* is a toric degeneration $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ of compact type (Definition 2.6) with $\check{\mathfrak{D}}$ relatively ample over $T$ and with generic fiber $\check{\mathfrak{D}}_\eta \subseteq \check{\mathfrak{X}}_\eta$ of an algebraization an anticanonical divisor in a Gorenstein surface.

The *associated intersection complex* $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is the intersection complex of $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ polarized by $\mathcal{O}_{\check{\mathfrak{X}}}(\check{\mathfrak{D}})$.

The point of this definition is both the irreducibility of the anticanonical divisor and the fact that this divisor extends to a polarization on the central fiber.

If the generic fiber $\check{\mathfrak{X}}_\eta$ is a surface then it is a $dP_k$ for some $k$, together with a smooth anticanonical divisor.

Starting from a reflexive polytope, there is a canonical construction of the intersection complex of a distinguished toric degeneration as follows.

**Construction 7.2.** Let $\Xi$ be a reflexive polytope and $v_0 \in \Xi$ the unique interior integral vertex. Define the polyhedral decomposition $\check{\mathscr{P}}$ of $\check{B} = \Xi$ with maximal cells the convex hulls of the facets of $\Xi$ and $v_0$. The affine chart at $v_0$ is the one defined by the affine structure of $\Xi$. At any other vertex define the affine structure by the unique chart compatible with the affine structure of the adjacent maximal cells and making $\partial\check{B}$ totally geodesic. This works because by reflexivity for any vertex $v$ the integral tangent vectors of any adjacent facet together with $v - v_0$ generate the full lattice. Moreover, $(\check{B}, \check{\mathscr{P}})$ has a natural polarization by defining $\check{\varphi}(v_0) = 0$ and $\check{\varphi}(v) = 1$ for each other vertex $v$.

The tropical manifold $(\check{B}, \check{\mathscr{P}})$ obtained in this way does not generally have simple singularities. Using a similar computation from [Gr1] on the Legendre dual side one can show, however, that $(\check{B}, \check{\mathscr{P}})$ has simple singularities iff the normal fan $\Sigma$ of $\Xi$ is elementary simplicial, meaning that each cone is the cone over an elementary simplex. In dimensions two and three this is equivalent to requiring the toric variety $\check{X}(\Xi)$ with momentum polytope $\Xi$ to be smooth. See [Pu, Constr. 5.2] for more details.

Assuming $(\check{B}, \check{\mathscr{P}})$ has locally rigid singularities (e.g. simple singularities or in dimension two) so we can run [GS3], or there is a consistent compatible sequence of wall structures on $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ by other means, we obtain a distinguished, anticanonically polarized toric degeneration $\check{\mathfrak{X}} \to \operatorname{Spf} \Bbbk[\![t]\!]$ together with an irreducible anticanonical divisor $\check{\mathfrak{D}} \subset \check{\mathfrak{X}}$.

The generic fiber $\check{\mathfrak{X}}_\eta$ of this toric degeneration may not be isomorphic to the toric variety $\check{X}(\Xi)$. But by introducing an additional parameter $s$ scaling the non-constant coefficients of

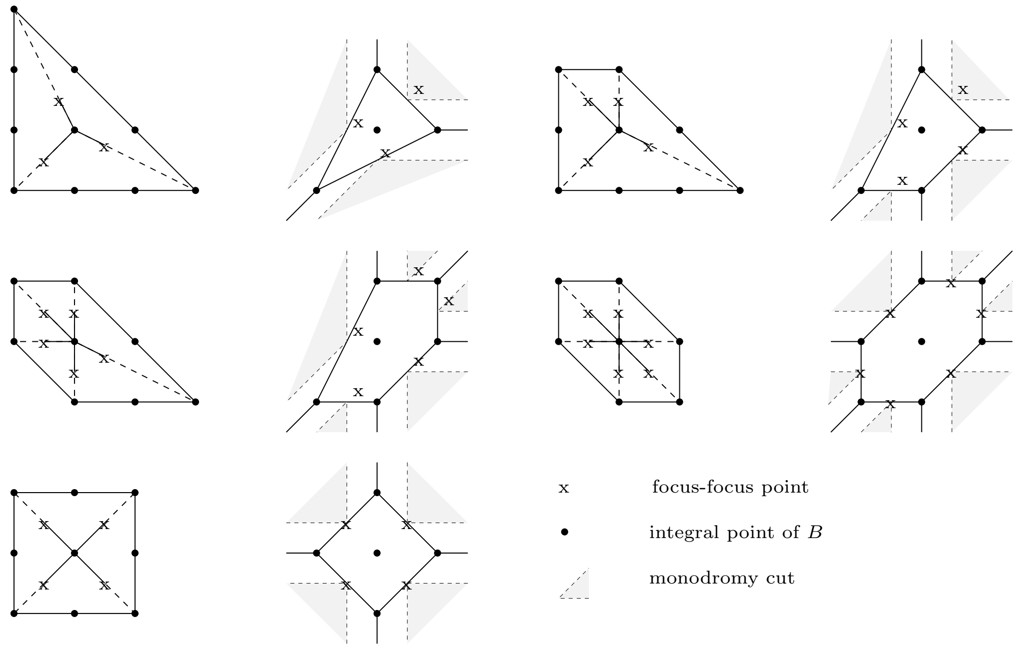


FIGURE 7.2. The intersection complexes $(\check{B}, \check{\mathscr{P}})$ of the five distinguished toric degenerations of del Pezzo surfaces with simple singularities and their Legendre duals $(B, \mathscr{P})$.

the slab functions one can produce a two-parameter family with $\check{\mathfrak{X}} \to \operatorname{Spf} \Bbbk[\![t]\!]$ the restriction to $s = 1$ and the constant family with fiber $\check{X}(\Xi)$ for $t \neq 0$ the restriction to $s = 0$.

The discrete Legendre transform $(B, \mathscr{P}, \varphi)$ has a unique bounded cell $\sigma_0$, isomorphic to the dual polytope of $\Xi$. Up to the addition of a global affine function, the dual polarizing function $\varphi$ is the unique piecewise affine function changing slope by one along the unbounded facets.[14]

*Remark* 7.3. Alternatively, one can use an MPCP resolution [Ba, Thm. 2.2.24] of $\check{X}(\Xi)$ defined by a simplicial subdivision of $\Sigma$ to split the discriminant locus of $(\check{B}, \check{\mathscr{P}}$ into simple singularities. On the Legendre-dual side the subdivision is given by writing the bounded maximal cell $\sigma_0$ as a union of elementary simplices. The resolution process leads to the introduction of more Kähler parameters on the log Calabi-Yau side, hence more complex parameters on the Landau-Ginzburg side, reflected in the choice of $\varphi$. See [Pu] for some discussions in this direction.

**Example 7.4.** Specializing to del Pezzo surfaces, we start from the momentum polytopes of the five non-singular toric Fano surfaces. The result of the construction is depicted in Figure 7.2, which shows a chart in the complement of the dotted segments. Note that the discrete Legendre transform $(B, \mathscr{P}, \varphi)$, also depicted in Figure 7.2, indeed has parallel outgoing rays.

Conversely, in dimension 2 we have the following uniqueness result.

**Theorem 7.5.** *Let* $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ *be a distinguished toric degeneration of del Pezzo surfaces. Then the associated intersection complex* $(\check{B}, \check{\mathscr{P}})$ *is a subdivision of the star subdivision of a*

[14]In the present case $\check{\varphi}$ is single-valued.

*reflexive polygon $\Xi$, with the edges containing a singular point precisely those connecting the interior integral point to a vertex of $\Xi$.*

*If furthermore the toric degeneration has simple singularities* [GS1, §1.5]*, then $(\check{B}, \check{\mathscr{P}})$ is isomorphic to an integral subdivision of one of the cases listed in Figure 7.2.*

*Proof.* Let $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ be the given toric degeneration. Thus the generic fiber $\check{\mathfrak{X}}_\eta$ is isomorphic to a del Pezzo surface $dP_k$ over $\eta$ for some $0 \leq k \leq 8$, or to $\mathbb{P}^1 \times \mathbb{P}^1$. By assumption $\partial\check{B}$ is locally straight in the affine structure.

First we determine the number of integral points of $\check{B}$. Let $\mathcal{L} = \mathcal{O}_{\check{\mathfrak{X}}}(\check{\mathfrak{D}})$ be the polarizing line bundle on $\check{\mathfrak{X}}$. By assumption

$$h^0(\check{\mathfrak{X}}_\eta, \mathcal{L}|_{\check{\mathfrak{X}}_\eta}) = h^0(dP_k, -K_{dP_k}) = \begin{cases} 10-k, & \check{\mathfrak{X}}_\eta \simeq dP_k \\ 9, & \check{\mathfrak{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases} \tag{7.1}$$

Let $t \in \mathcal{O}_{T,0}$ be a uniformizing parameter and $\check{X}_n := \mathrm{Spec}\left(\Bbbk[t]/(t^{n+1})\right) \times_T \check{\mathfrak{X}}$ the $n$-th order neighborhood of $\check{X}_0 := \pi^{-1}(0)$ in $\check{\mathfrak{X}}$. Denote by $\mathcal{L}_n = \mathcal{L}|_{\check{X}_n}$.

Then for any $n$ there is an exact sequence of sheaves on $\check{X}_0$,

$$0 \longrightarrow \mathcal{O}_{\check{X}_0} \longrightarrow \mathcal{L}_{n+1} \longrightarrow \mathcal{L}_n \longrightarrow 0.$$

By the analogue of [GS2, Thm. 4.2] for log Calabi-Yau varieties, we know

$$h^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = h^1(\check{\mathfrak{X}}_\eta, \mathcal{O}_{\check{\mathfrak{X}}_\eta}) = 0.$$

Thus the long exact cohomology sequence induces a surjection $H^0(\check{X}_0, \mathcal{L}_{n+1}) \twoheadrightarrow H^0(\check{X}_0, \mathcal{L}_n)$ for each $n$. By the theorem on formal functions and cohomology and base change [Ha, Thms. 11.1 & 12.11], we thus conclude that $\pi_*\mathcal{L}$ is locally free, with fiber over 0 isomorphic to $H^0(\check{X}_0, \mathcal{L}_0)$. In view of (7.1) we thus conclude

$$h^0(\check{X}_0, \mathcal{L}_0) = \begin{cases} 10-k, & \check{\mathfrak{X}}_\eta \simeq dP_k \\ 9, & \check{\mathfrak{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

Now on a toric variety the dimension of the space of sections of a polarizing line bundle equals the number of integral points of the momentum polytope. Since $\check{X}_0$ is a union of toric varieties, each integral point $x \in \check{B}$ provides a monomial section of $\mathcal{L}_0$ on any irreducible component $\check{X}_\sigma \subseteq \check{X}_0$ with $\sigma \in \check{\mathscr{P}}$ containing $x$. These provide a basis of sections of $H^0(\check{X}_0, \mathcal{L}_0)$.[15] Hence $\check{B}$ has $10-k$ integral points.

An analogous argument shows that the number of integral points of $\partial\check{B}$ equals

$$h^0(\check{D}_0, \mathcal{L}_0) = h^0(\check{\mathfrak{D}}_\eta, \mathcal{L}_\eta),$$

which by Riemann-Roch equals $K^2_{dP_k} = 9-k$ or $K^2_{\mathbb{P}^1\times\mathbb{P}^1} = 8$. In either case we thus have a unique integral interior point $v_0 \in \check{B}$. In particular, $\check{B}$ has the topology of a disk, and each interior edge connects $v_0$ to an integral point of $\partial\check{B}$.

Viewed in the chart at the interior integral point, $\check{B}$ is therefore a reflexive polygon $\Xi$. Moreover, since $\partial\check{B}$ is locally straight, each of the interior edges with endpoint a vertex of $\Xi$ has to contain a singular point. None of the other interior edges can contain a singular point for otherwise $\partial\check{B}$ would not be straight in the affine structure (and $\check{B}$ would not even be locally convex on the boundary).

[15] This also follows by the description of $(\check{X}_0, \mathcal{L}_0)$ by a homogeneous coordinate ring in [GS1, Def. 2.4].

If the singularities are simple, one finds that $\partial\check{B}$ is only locally straight in the five cases shown in Figure 7.2, up to adding some edges connecting $v_0$ to $\partial\check{B}$ without singular point. The remaining 11 cases are discussed in §8.1 below. □

*Remark* 7.6. 1) The proof shows that the five types can be distinguished by $\dim H^0(\check{\mathfrak{X}}_\eta, \mathcal{L}_\eta)$, except for $\mathbb{P}^1\times\mathbb{P}^1$ and $dP_1$. Alternatively, by Proposition 7.13 below one could use $H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta})$.

2) For each $(\check{B}, \check{\mathscr{P}})$ there is a discrete set of choices of $\check{\varphi}$, which determines the local toric models of $\check{\mathfrak{X}} \to \check{\mathfrak{D}}$. This reflects the fact that the base of (log smooth) deformations of the central fiber $\check{X}_0^\dagger$ as a log space over the standard log point $\Bbbk^\dagger$ is higher dimensional. In fact, let $r$ be the number of vertices on $\partial\check{B}$. Then taking a representative of $\check{\varphi}$ that vanishes on one maximal cell, $\check{\varphi}$ is defined by the value at $r-2$ vertices on $\partial\check{B}$. Convexity then defines a submonoid $Q \subseteq \mathbb{N}^{r-2}$ with the property that $\mathrm{Hom}(Q, \mathbb{N})$ is isomorphic to the space of (not necessarily strictly) convex, piecewise affine functions on $(\check{B}, \check{\mathscr{P}})$ modulo global affine functions. Running the construction of [GS3] with parameters then produces a log smooth deformation over the completion at the origin of $\operatorname{Spec} \Bbbk[Q]$ with central fiber $(\check{X}_0, \check{D}_0)$. For the minimal polyhedral decompositions of Figure 7.2 with $r = l$ the number of singular point we have $\operatorname{rk} Q = l - 2$, which by Remark 7.14,2 below agrees with the dimension of the space $H^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\Bbbk^\dagger})$ of infinitesimal log smooth deformations of $X_0^\dagger/\Bbbk^\dagger$. One can show that in this case the constructed deformation is in fact semi-universal.[16] □

The technical tool to compute superpotentials in the toric del Pezzo cases and in related examples in finitely many steps is the following lemma, suggested to us by Mark Gross. It greatly reduces the number of broken lines to be considered, especially in asymptotically cylindrical situations and with a finite structure on the bounded cells.

**Lemma 7.7.** *Let $\mathscr{S}$ be a structure for a non-compact, polarized tropical manifold $(B, \mathscr{P}, \varphi)$ that is consistent to all orders. We assume that there is a subdivision $\mathscr{P}'$ of a subcomplex of $\mathscr{P}$ with vertices disjoint from $\Delta$ and with the following properties.*

1. *Each $\sigma \in \mathscr{P}'$ is affine isomorphic to $\rho \times \mathbb{R}_{\ge 0}$ for some bounded face $\rho \subseteq \sigma$.*
2. *$B \setminus \mathrm{Int}(|\mathscr{P}'|)$ is compact and locally convex at the vertices (this makes sense in an affine chart).*
3. *If $m$ is an exponent of a monomial of a wall or unbounded slab intersecting some $\sigma \in \mathscr{P}'$, $\sigma = \rho + \mathbb{R}_{\ge 0}\overline{m}_\sigma$, then $-\overline{m} \in \Lambda_\rho + \mathbb{R}_{>0} \cdot \overline{m}_\sigma$.*

*Then the first break point $t_1$ of a broken line $\beta$ with $\mathrm{im}(\beta) \not\subseteq |\mathscr{P}'|$ can only happen after leaving $\mathrm{Int}\,|\mathscr{P}'|$, that is,*

$$t_1 \ge \inf\big\{t \in (-\infty, 0] \,\big|\, \beta(t) \notin |\mathscr{P}'|\big\}.$$

*Proof.* Assume $\beta(t_1) \in \sigma \setminus \rho$ for some $\sigma = \rho + \mathbb{R}_{\ge 0}\overline{m}_\sigma \in \mathscr{P}'$. Then $\beta|_{(-\infty,t_1]}$ is an affine map with derivative $-\overline{m}_\sigma$, and $\beta(t_1)$ lies on a wall. By the assumption on exponents of walls on $\sigma$, the result of nontrivial scattering at time $t_1$ only leads to exponents $m_2$ with $-\overline{m}_2 \in \Lambda_\rho + \mathbb{R}_{\ge 0}\overline{m}_\sigma$, the outward pointing half-space. In particular, the next break point can not lie on $\rho$. Going by induction one sees that any further break point in $\sigma$ preserves the condition that $\beta'$ does not point inward. Moreover, by the convexity assumption, this condition is also preserved when moving to a neighboring cell in $\mathscr{P}'$. Thus $\mathrm{im}(\beta) \subseteq |\mathscr{P}'|$. □

[16] [RS, Thm. 4.4] proves semi-universality for all simple toric degenerations of compact type.

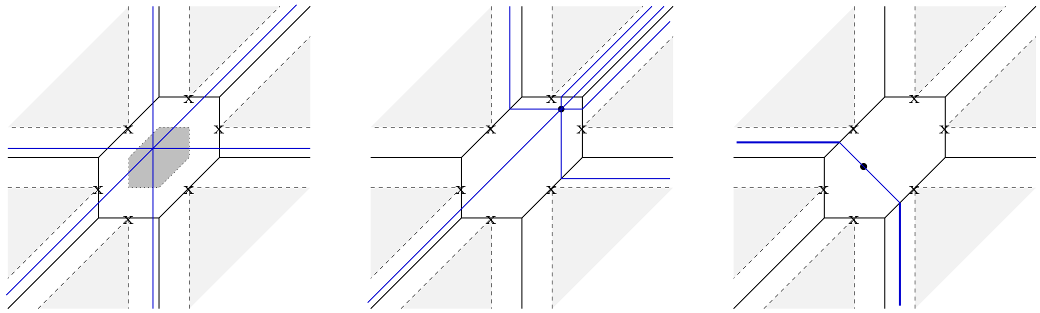

FIGURE 7.3. Tropical disks in the mirror of the distinguished base of $dP_3$ indicating the invariance under change of root vertex.

**Proposition 7.8.** *Let $(B, \mathscr{P})$ be the dual intersection complex of a distinguished toric degeneration of del Pezzo surfaces with simple singularities and let $\sigma_0 \subseteq B$ be the bounded cell. Then there is a neighborhood $U$ of the interior vertex $v_0 \in \sigma_0$ such that for any $p \in U$ there is a canonical bijection between broken lines with endpoint $p$ and rays of $\Sigma$, the fan over the proper faces of $\sigma_0$.*

*Proof.* We can embed $\Sigma$ in the tangent space at $v_0$ by extending the unbounded edges to $v_0$ in the chart shown in Figure 7.1. Each ray of $\Sigma$ can then be interpreted as the image of a unique degenerate broken line. Because each such degenerate broken line has a positive distance from the shaded regions in Figure 7.2 they can be moved with small perturbations of $p$ to deform to a proper broken line. Conversely, by inspection of the five cases, the result of non-trivial scattering at $\partial\sigma_0$ leads to a broken line not entering $\mathrm{Int}(\sigma_0)$. There are no walls entering $\mathrm{Int}\,\sigma_0$, so by Lemma 7.7 any such broken line can bend at most at the intersection with $\partial\sigma_0$. □

**Corollary 7.9.** *Let $(\mathfrak{X} \to \mathrm{Spf}\,\Bbbk[\![t]\!], W)$ be the Landau-Ginzburg model mirror to a distinguished toric degeneration of del Pezzo surfaces with simple singularities. Then there is an open subset $U \simeq \mathrm{Spf}\,\Bbbk[x^{\pm 1}, y^{\pm 1}][\![t]\!] \subseteq \mathfrak{X}$ such that $W|_U$ equals the usual Hori-Vafa monomial sum times $t$.* □

*Remark* 7.10. 1) For other than the anticanonical polarization of the del Pezzo surface, the terms in the superpotential receive different powers of $t$, just as in the Hori-Vafa proposal.

2) Analogous arguments work for Landau-Ginzburg mirrors of smooth toric Fano varieties of any dimension [Pu, Thm. 5.4].

**Example 7.11.** Let us study the mirror of the distinguished toric degeneration of $dP_3$ with the minimal polarization $\check{\varphi}$ explicitly. In Figure 7.3 the first two pictures show all Maslov index two tropical disks, respectively broken lines using disk completion (Lemma 6.4), for two choices of root vertex. Moving the root vertex within the shaded open hexagon yields the same result, that is, none of the six broken lines has a break point. In contrast, moving the root vertex inside $\sigma_0$ out of the shaded hexagon leads to some bent broken lines, but the set of root tangent vectors always remains $(1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1)$ and $(0, -1)$, all with coefficient 1. This illustrates the invariance under the change of root vertex proved in Lemma 4.7.

The potential in the chart for the bounded cell $\sigma_0$ is thus computed as

$$W_{dP_3}(\sigma_0) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t,$$

which for $t \neq 0$ has six critical points.

The picture on the right shows two tropical disks with weight two unbounded leaves. These do not contribute to the superpotential.

An analogous picture arises for the other four distinguished del Pezzo degenerations.

Morally speaking the last example shows that in toric situations ray generators of the fan are sufficient to compute the superpotential, but really they should be seen as special cases of tropical disks or broken lines.

7.2. **Non-toric del Pezzo surfaces.** In this section we consider del Pezzo surfaces $dP_k$ for $k \geq 4$, referred to as higher del Pezzo surfaces. Let us first determine the topology of $B$ and the number of singular points of the affine structure.

We need the following statement for Fano varieties with smooth anticanonical divisor

**Lemma 7.12.** *Let $X$ be a smooth Fano variety and $D \subset X$ a smooth anticanonical divisor. Then $H^0(X, \Omega_X(\log D)) = 0$.*

*Proof.* It is a folklore result in Hodge theory that the connecting homomorphism of the residue sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(\log D) \xrightarrow{\text{res}} \mathcal{O}_D \longrightarrow 0$$

maps $1 \in H^0(D, \mathcal{O}_D)$ to $c_1(\mathcal{O}_X(D)) \in H^{1,1}(X) = H^1(X, \Omega_X)$. By ampleness of $D$ this class is nonzero. Thus $H^0(X, \Omega_X(\log D)) \simeq H^0(X, \Omega_X)$.

The claim now follows from the Kodaira vanishing theorem:

$$H^0(X, \Omega_X) \simeq H^n(X, \mathcal{O}_X) = H^n(X, K_X \otimes K_X^{-1}) = 0. \qquad \square$$

**Proposition 7.13.** *Let $(B, \mathscr{P})$ be the dual intersection complex of a distinguished toric degeneration $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ of del Pezzo surfaces with simple singularities (Definition 7.1). In particular, we assume the generic fiber $\check{\mathfrak{X}}_\eta$ is a proper surface with $\check{\mathfrak{D}}_\eta$ a smooth anticanonical divisor.*

*Then $B$ is homeomorphic to $\mathbb{R}^2$, and the affine structure has $l = \dim H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta}) + 2$ singular points.*

*Proof.* Since the relative logarithmic dualizing sheaf $\omega_{\check{\mathfrak{X}}/\check{\mathfrak{D}}}(-\log \check{\mathfrak{D}})$ is trivial, the generalization of [GS1, Thm. 2.39] to the case of log Calabi-Yau varieties shows that $B$ is orientable. By the classification of surfaces with effective anticanonical divisor we know $H^i(\check{\mathfrak{X}}_\eta, \mathcal{O}_{\check{\mathfrak{X}}_\eta}) = 0$, $i = 1, 2$. As in the proof of Theorem 7.5 this implies $H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0$. Thus by the log Calabi-Yau analogue of [GS1, Prop. 2.37],

$$H^1(B, \Bbbk) = H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0.$$

In particular, $B$ has the topology of $\mathbb{R}^2$.

As for the number of singular points, the generalization of [GS2, Thms. 3.21 & 4.2] to the case of log Calabi-Yau varieties [Ts, Thm. 3.5 & Thm. 3.9] shows that $\dim H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta})$ is related to an affine Hodge group:[17]

$$\dim_{\Bbbk((t))} H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta}) = \dim_{\mathbb{R}} H^1(B, i_*\check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}). \tag{7.2}$$

To compute $H^1(B, i_*\check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R})$ we choose the following Čech cover of $B = \mathbb{R}^2$. Let $\ell_1, \ldots, \ell_l$ be disjoint real half-lines emanating from the singular points $p_1, \ldots, p_l$. Define $U_0 = \mathbb{R}^2 \setminus \bigcup_{i=1}^l \ell_i$, and $U_i = B_\varepsilon(p_i)$ for $\varepsilon$ sufficiently small to achieve $U_i \cap \ell_j = \emptyset$ unless $i = j$. Then $\mathfrak{U} := \{U_0, U_1, \ldots, U_l\}$ is a Leray covering of $B$ for $i_*\check{\Lambda}_{\mathbb{R}} := i_*\check{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}$, cf. [GS1, Lem. 5.5]. The terms in the Čech complex are

$$C^0(\mathfrak{U}, i_*\check{\Lambda}_{\mathbb{R}}) = \mathbb{R}^2 \times \prod_{i=1}^l \mathbb{R}, \quad C^1(\mathfrak{U}, i_*\check{\Lambda}_{\mathbb{R}}) = \prod_{i=1}^l \mathbb{R}^2, \quad C^k(\mathfrak{U}, i_*\check{\Lambda}_{\mathbb{R}}) = 0 \text{ for } k \geq 2.$$

The analogue of (7.2) for degree 0 cohomology groups shows that the kernel of the Čech differential $C^0(\mathfrak{U}, i_*\check{\Lambda}_{\mathbb{R}}) \to C^1(\mathfrak{U}, i_*\check{\Lambda}_{\mathbb{R}})$ computes $H^0(\mathfrak{X}_\eta, \Omega_{\mathfrak{X}_\eta}(\log \mathfrak{D}_\eta))$. This latter group vanishes by Lemma 7.12. Hence

$$\dim H^1(B, i_*\check{\Lambda}_{\mathbb{R}}) = 2l - (l+2) = l - 2$$

determines the number $l$ of focus-focus points as claimed. □

*Remark* 7.14. 1) From the analysis in Proposition 7.5 and Proposition 7.13 it is clear that for del Pezzo surfaces $dP_k$ with $k \geq 4$ the anticanonical polarization is too small to extend over a toric degeneration with simple singularities. The associated tropical manifold would simply not have enough integral points to admit the required number of singular points.

2) Essentially the same argument also computes the dimension of the space of infinitesimal deformations:

$$h^1(\check{\mathfrak{X}}_\eta, \Theta_{\check{\mathfrak{X}}_\eta}(\log \check{\mathfrak{D}}_\eta)) = h^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\Bbbk^\dagger}) = h^1(B, i_*\Lambda_{\mathbb{R}}) = l - 2.$$ □

It is easy to write down toric degenerations of non-toric del Pezzo surfaces, since they can be represented as hypersurfaces or complete intersections in weighted projective spaces, as for example done for $dP_6$ in [GS4, Expl. 4.4]. The most natural toric degenerations in this setup have central fiber part of the toric boundary divisor of the ambient space. But because this construction gives nodal $\mathfrak{D}_\eta$ such toric degenerations are never distinguished. To obtain proper superpotentials we therefore need a different approach.

**Construction 7.15.** Start from the intersection complex $(\check{B}, \check{\mathscr{P}})$ of the distinguished toric degeneration of $dP_3$ depicted as a hexagon in Figure 7.2. The six focus-focus points in the interior of the bounded two cell make the boundary $\rho$ straight. There is no space to introduce more singular points of the affine structure because all interior edges already contain a singular point. To get around this, polarize by $-2 \cdot K_{dP_3}$ and adapt $\mathscr{P}$ in the obvious way, see Figure 7.4. This scales the affine manifold $B$ by two, but keeps the singular points fixed. The new boundary now has 12 integral points and the union $\gamma$ of edges neither intersecting the central vertex nor $\partial B$ is a geodesic. We can then introduce new singular points on the boundary of the interior hexagon as visualized in Figure 7.4. Moreover, let $\check{\varphi}$

[17] The proof of [GS2, Thm. 4.2] has a gap fixed in [FFR, Thm. 1.10].

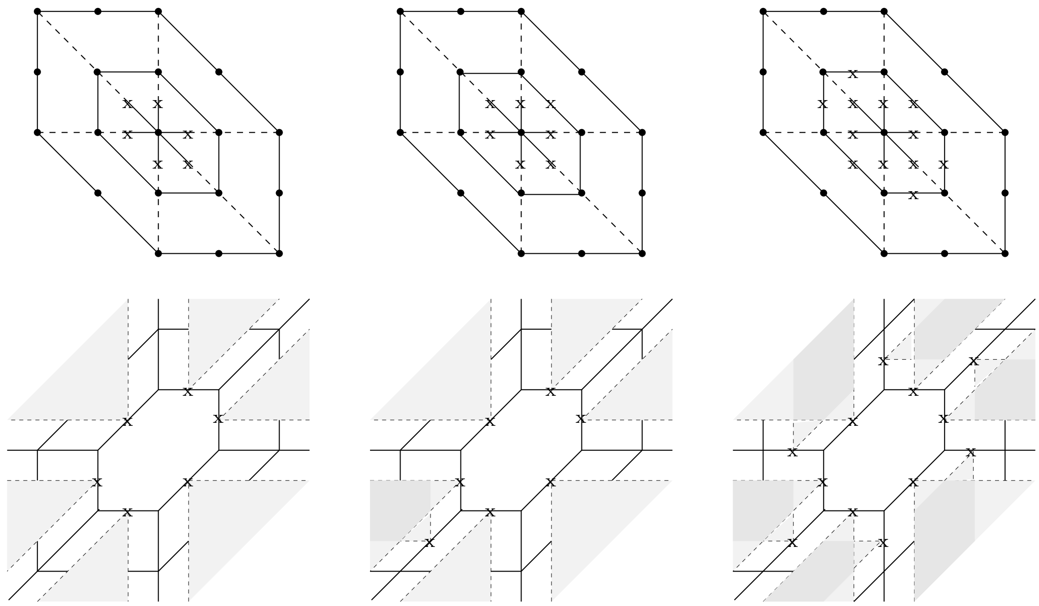

FIGURE 7.4. Straight boundary models for higher del Pezzo surfaces obtained by changing affine data for $dP_3$ and their Legendre duals.

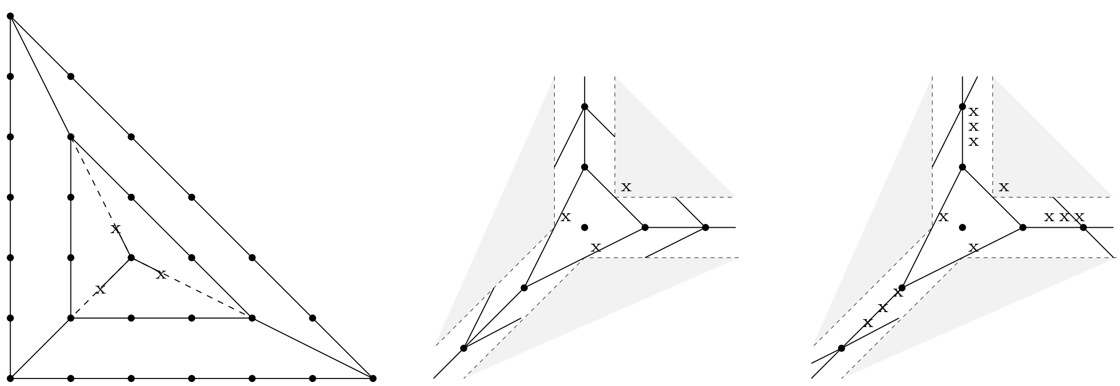

FIGURE 7.5. An alternative base for higher del Pezzo surfaces and their mirror.

be unchanged on the interior cells and change slope by one when passing to a cell intersecting $\partial B$. Plugging in up to five singular points, the Hodge numbers from Proposition 7.13 show that the toric degenerations obtained from the tropical data are in fact toric degenerations of $dP_k$, $4 \leq k \leq 8$. $\square$

Unlike in the anticanonically polarized case, the models constructed in this way are not unique. The geodesic $\gamma$ is divided into six segments by $\mathscr{P}$, and the choice on which of these segments we place the singular points, modulo the $\mathbb{Z}/6$-rotational symmetry, results in non-isomorphic models. We will see in Example 7.18 how this choice influences tropical curve counts.

Although there are other ways to define distinguished models for higher del Pezzo surfaces, for example by choosing another polarization, in this way we can extend the unique toric models most easily, since all tropical disks and broken lines we studied before arise in these models without any change.

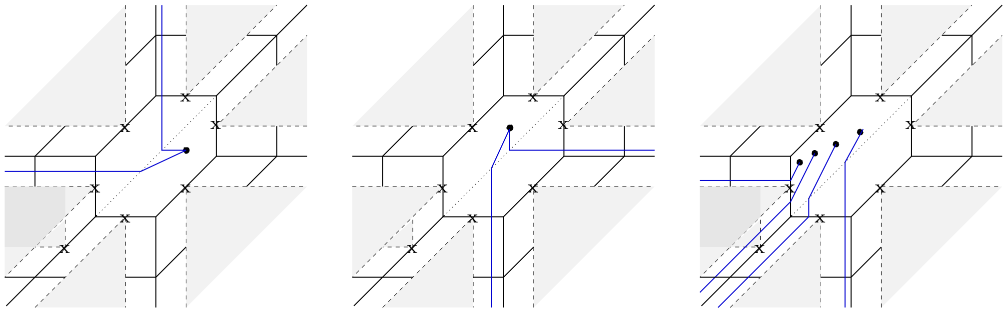

FIGURE 7.6. Mirror base to $dP_4$ showing new broken lines contributing to $W_{dP_4}$, indicating the wall crossing phenomenon and the invariance under change of endpoint within a chamber.

*Remark* 7.16. Note that introducing six new points, for instance as in the rightmost picture in Figure 7.4, corresponds to a blow up of $\mathbb{P}^2$ in nine points, which is not Fano anymore, but from our point of view still has a Landau-Ginzburg mirror.

From a different point of view this has already been noted in [AKO], where the authors construct a compactification of the Hori-Vafa mirror as a symplectic Lefschetz fibration as follows. Start with the standard potential $x + y + \frac{1}{xy}$ for $\mathbb{P}^2$ and compactify by a divisor at infinity consisting of nine rational curves. Then by a deformation argument it is possible to push $k$ of those rational curves to the finite part and decompactify to obtain a potential for $dP_k$, including $k = 9$.

We can reproduce this result from our point of view by starting with $\mathbb{P}^2$ rather than with $dP_3$, as illustrated in Figure 7.5. Moving rational curves from infinity to the finite part is analogous to introducing new focus-focus points. In the present case one may put three focus-focus points on each unbounded ray of $(B, \mathscr{P})$ until the respective Legendre dual vertex becomes straight. Figure 7.5 on the right shows nine such points (corresponding to the case $k = 9$ above), and any additional singular point would result in a concave boundary. This can be seen as an affine-geometrical explanation for why the compactification constructed by the authors in [AKO] has exactly nine irreducible components.

Note that it is possible to introduce more singular points when passing to larger polarizations, but in this way we will not end up with degenerations of del Pezzo surfaces.

In order to determine the superpotential, we depicted in Figure 7.4 an appropriate chart of the relevant $(B, \mathscr{P})$. When two regions to be removed overlap we shade them darker to indicate the non-trivial transformation there.

**Example 7.17.** Figure 7.6 shows the dual intersection complex $(B, \mathscr{P})$ of a toric degeneration of $dP_4$ from Construction 7.15. The additional focus-focus point changes the structure $\mathscr{S}$ and allows broken lines to scatter with the wall in direction $(1, 1)$ in the central cell $\sigma_0$ which subdivides $\sigma_0$ into two chambers $\mathfrak{u}, \mathfrak{u}'$ and yields new root tangent directions. A broken line coming from infinity in direction $(\pm 1, 0)$ produces root tangent vectors $(-1 \pm 1, -1)$, whereas one with direction $(0, \pm 1)$ takes directions $(-1, -1 \pm 1)$. By construction, every broken line reaching the interior cell $\sigma_0$ has $t$-order at least 2.

Note also that by [GS4, Expl. 4.3] only the wall indicated by a dotted line in Figure 7.6 enters $\sigma_0$. Thus $\sigma_0$ is subdivided into two chambers $\mathfrak{u}, \mathfrak{u}'$, in all orders. A broken line can

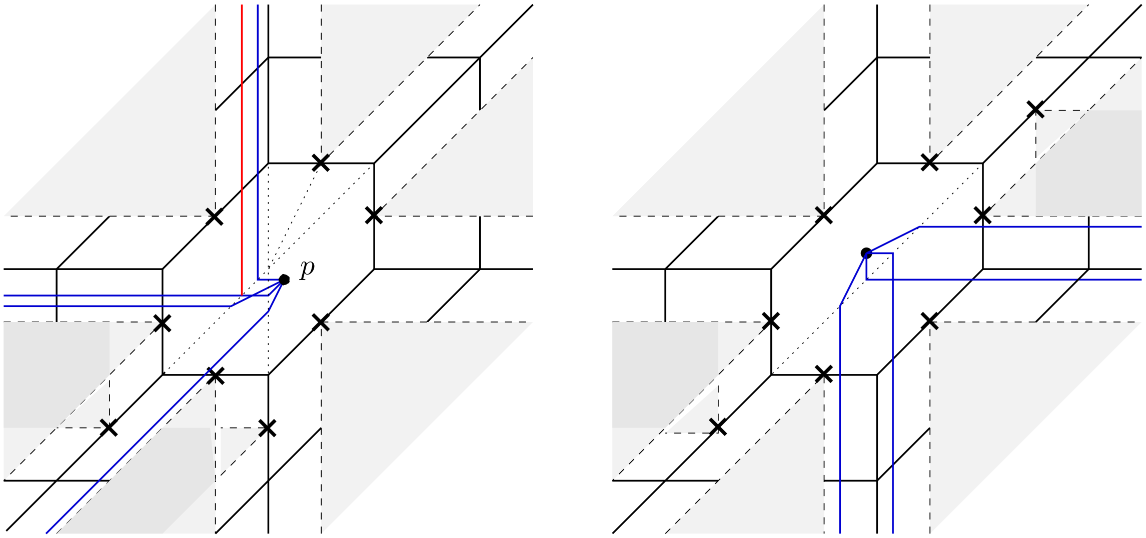


FIGURE 7.7. Mirror bases to $dP_5$ showing all new broken lines.

have at most one break point within $\sigma_0$, in which case the $t$-order increases by one. So let us compute $W^3_{dP_4}$. We get two new root tangent directions, namely $(-1,-2)$ and $(-2,-1)$, and possibly more contributions from directions $(0,-1)$ and $(-1,0)$. The two leftmost pictures in Figure 7.6 show all new broken lines for different choices of root vertex, apart from the six toric ones we have already encountered in Example 7.11. Depending on this choice, the superpotential to order three is therefore either given by

$$W_{dP_4}(\mathfrak{u}) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{x} + \frac{1}{x^2y}\Big) \cdot t^3 \quad \text{or}$$

$$W_{dP_4}(\mathfrak{u}') = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{y} + \frac{1}{xy^2}\Big) \cdot t^3,$$

both of which have seven critical points, as expected. These superpotentials are not only related by interchanging $x$ and $y$, for symmetry reasons, but also by wall crossing along the wall separating $\sigma_0$ into two chambers. This is the first non-trivial example of an ostensibly algebraic superpotential, as defined at the end of §4.

In the rightmost picture in Figure 7.6 we indicated the behaviour of a single broken line of root tangent direction $(-1,-2)$ under change of root vertex. If the root vertex changes a chamber by passing one of the dotted lines drawn, the broken lines change accordingly.

**Example 7.18.** Attaching another singular point on the unbounded ray in direction $(0,-1)$ as in Figure 7.7 on the left we arrive at a degeneration of $dP_5$. For a structure consistent to all orders there are three walls in the bounded maximal cell necessary, indicated by dotted lines in the figure. They are the extensions of the slabs with tangent directions $(1,1)$ and $(0,1)$ caused by additional singular points, and the result of scattering of these, the wall with tangent direction $(1,2)$. Because $(1,1)$ and $(0,1)$ form a lattice basis, the scattering procedure at the origin does not produce any additional walls. In any case, any broken line coming in from direction $(1,1)$ and with endpoint $p$ as indicated in Figure 7.7 can not interact with any of the scattering products. Tracing any possible broken lines starting from $t = -\infty$ one arrives at only five broken lines with endpoint $p$, with only one, drawn in red, having

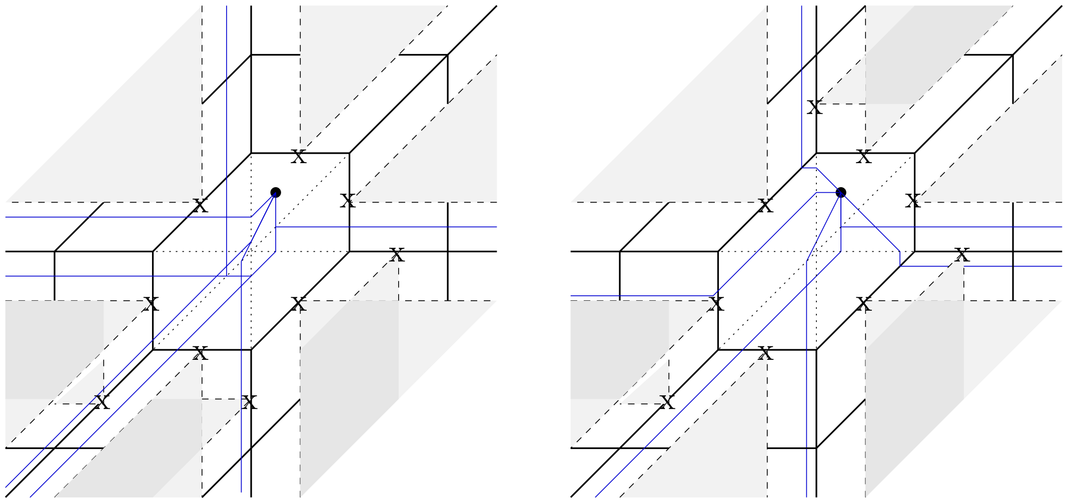

FIGURE 7.8. Two mirror bases to $dP_6$.

more than one breakpoint. We therefore obtain the following superpotential on the chamber $\mathfrak{u}$ containing $p$:

$$W_{dP_5}(\mathfrak{u}) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{y} + \frac{1}{xy} + \frac{1}{x^2y} + \frac{1}{xy^2}\Big) \cdot t^3 + \frac{1}{xy}t^4.$$

**Example 7.19.** We study another model of the mirror to $dP_5$, which differs from the last one in the position of the second focus-focus point. Instead of placing it on the ray with generator $(0,-1)$ we move it to the ray generated by $(1,1)$, as shown in Figure 7.7 on the right. This alternative choice yields the superpotential

$$W_{dP_5}(\mathfrak{u}) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{y} + x + x^2y + \frac{1}{xy^2}\Big) \cdot t^3$$

on the selected chamber $\mathfrak{u}$. It is an interesting question to understand in detail the effect of particular choices of singular points and the corresponding degenerations.

**Example 7.20.** As a last example, we study broken lines in the mirror of a distinguished model of $dP_6$, depicted on the left in Figure 7.8. This time we obtain the superpotential

$$W_{dP_6}(\mathfrak{u}) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(2 \cdot \frac{1}{xy^2} + \frac{1}{xy} + 2 \cdot \frac{1}{y}\Big) \cdot t^3 + \frac{1}{y} \cdot t^4 + \frac{1}{y} \cdot t^5,$$

with nine critical points. Again, this potential comes from a special choice of positions of critical points and root vertex among many others. This superpotential is ostensibly algebraic although the three walls meeting at the origin produce an infinite wall structure on the bounded cell $\sigma_0$.

In contrast, Figure 7.8 on the right shows the mirror base of an alternative $dP_6$-degeneration featuring a finite wall structure in the bounded cell $\sigma_0$ by [GS4, Expl. 4.3]. We again obtain an ostensibly algebraic superpotential

$$W_{dP_6}(\mathfrak{u}) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{y}{x} + \frac{1}{x} + \frac{1}{xy^2} + 2 \cdot \frac{1}{y} + \frac{x}{y}\Big) \cdot t^3.$$

These examples illustrate that if we leave the realm of toric geometry, Landau-Ginzburg potentials for del Pezzo surfaces can, at least locally, still be described by Laurent polynomials, as in the toric setting.

## 8. Singular Fano and smooth non-Fano surfaces

Having studied smooth Fano surfaces, we now show that our approach also works if we admit Gorenstein singularities or drop the Fano condition.

8.1. **Singular Fano surfaces.** We now classify the remaining distinguished toric degenerations of del Pezzo surfaces (Definition 7.1) from Theorem 7.5 with non-simple singularities. In this theorem we have already seen that the intersection complex $(\check{B}, \check{\mathscr{P}})$ is obtained from a star subdivision of a reflexive polygon $\Xi$ with a singular point on each of the interior edges, and then posssibly a further subdivision adding more edges without a singular point connecting the interior integral point to a non-vertex point on $\partial\check{B}$. There are 16 well-known isomorphism classes of reflexive polygons, among which five give rise to smooth varieties, studied in the last section. The remaining eleven polygons are characterized by the property that the dual polygon has at least one integral non-vertex boundary point. These 11 polygons are in one-to-one correspondence with isomorphism classes of singular toric del Pezzo surfaces.

To obtain a smooth boundary model $(\check{B} = \Xi, \check{\mathscr{P}})$, we now need to add a non-simple singular point on some of the added edges to straighten the boundary. Non-simple means that the affine monodromy along a counterclockwise loop about such a singular point is conjugate to $\left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix}\right)$ for some $k > 1$. We refer to $k$ as the *order* of the singular point. Legendre-duality then yields an affine singularity of the same order $k$ on the dual edge. As in the smooth case, we polarize $(\check{B}, \check{\mathscr{P}})$ by the minimal polarizing function $\check{\varphi}$ changing slope by one along each interior edge.

Figure 8.1 depicts the discrete Legendre duals $(B, \mathscr{P})$ thus obtained from star subdivision of the remaining 11 reflexive polygons. Note that the affine monodromy of the singular point is reflected in the shaded regions. The order $k$ of a singular point now equals the number of integral points on the edge of $\mathscr{P}$. We obtain the following addition to Theorem 7.5.

**Theorem 8.1.** *In the situation of Theorem 7.5 assume that $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ does not have simple singularities. Then $(\check{B}, \check{\mathscr{P}})$ is Legendre-dual to one of the cases listed in Figure 8.1.* □

With non-simple singularities on some edges in $(B, \mathscr{P})$, the slab functions are not uniquely determined by the gluing data. If $v$ is a vertex on an edge $\rho$ containing an order $k$ singularity, the slab function $f_{\rho,v}$ has the form $1+a_1x+\ldots a_{k-1}x^{k-1}+a_kx^k$ for $x$ the generating monomial for the $\rho$-stratum of $X_0$. The coefficient $a_k$ is determined by the gluing data, with $a_k = 1$ for trivial gluing data. Thus in any case, there are $k-1$ free coefficients for each singular point of order $k$. These coefficients reflect classes of exceptional curves on the resolution of the Fano side. Ignoring these classes, as suggested by working with the unresolved del Pezzo surface, leads to the slab function $(1+x)^k$.

**Theorem 8.2.** [18] *Let $(\mathfrak{X} \to \operatorname{Spec} \Bbbk[\![t]\!], W)$ be mirror to a distinguished toric degeneration $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ of del Pezzo surfaces (Definition 7.1), and $(B, \mathscr{P}, \varphi)$ the associated intersection complex for the anticanonical polarization on $\check{\mathfrak{X}}$.*

[18] The closely related superpotential of the corresponding 11 semi-Fano surfaces obtained by MPCP resolution has independently been computed in [CL] by other methods. See [Pu] for the reproduction and comparison of their results with our method. Add reference to MPCP

*Denote by $\sigma_0$ the bounded cell of the associated intersection complex $(B, \mathscr{P}, \varphi)$. For an integral point $m$ on an edge $\omega \subset \partial\sigma_0$ of integral length $k$ define $N_m = \binom{l}{k}$, where $l$ is the integral distance between $m$ and one of the vertices of $\omega$. Then it holds*

$$W(\sigma_0) = t \cdot \sum_{m \in \partial\sigma_0 \cap \Lambda_{\sigma_0}} N_m z^m.$$

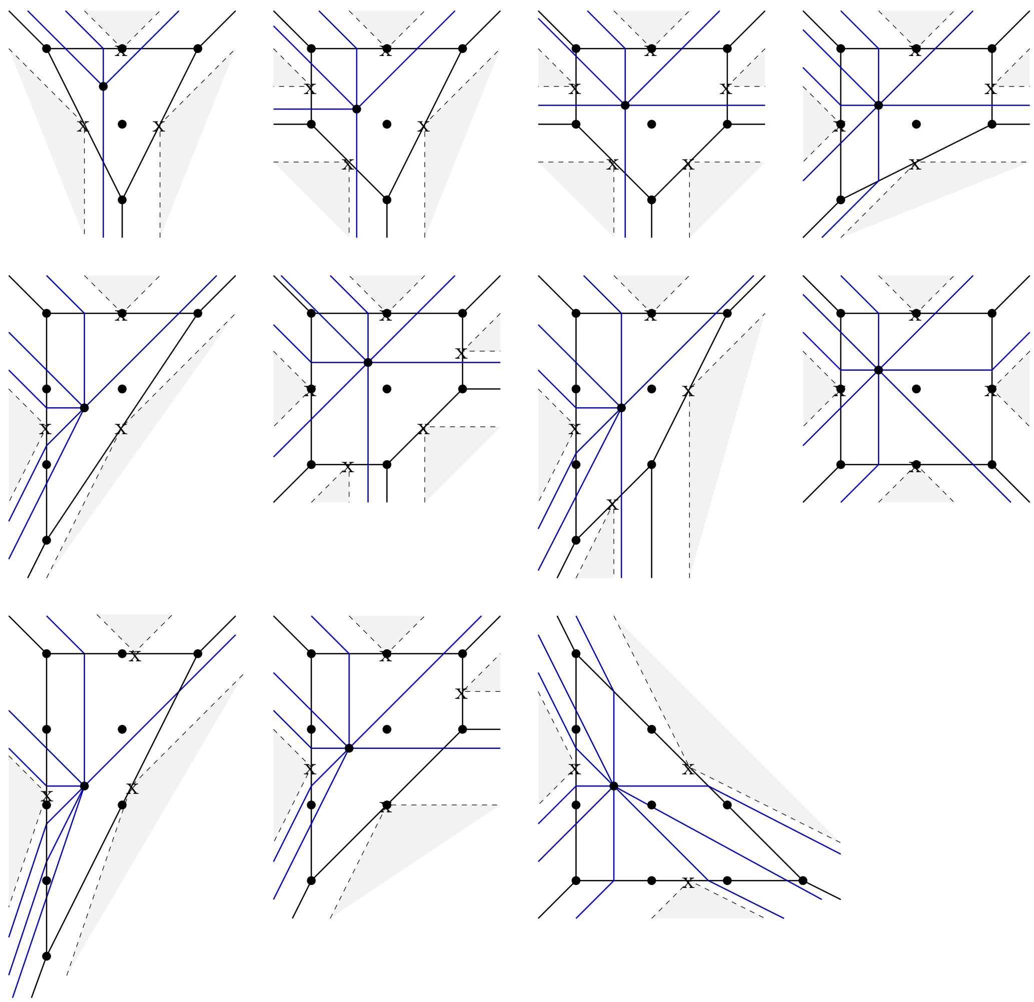

FIGURE 8.1. The broken lines contributing to the proper superpotential of the eleven singular toric del Pezzo surfaces.

*Proof.* Corollary 7.9 already treats the case with simple singularities. The remaining 11 cases are most easily done by inspection of the set of broken lines ending at a specified point $p$ as given in Figure 8.1. Note that there are no walls in $\operatorname{Int}\sigma_0$, so the result of this computation is independent of the choice of $p$. It is instructive to check this continuity property explicitly in some cases.

Alternatively one can argue more generally as follows. If a broken line ends at $p \in \operatorname{Int}(\sigma_0)$ then the next to last vertex of $\beta$ maps to the intersection of the ray $p + \mathbb{R}_{\geq 0} m_\beta$ with $\partial\sigma_0$. Denote by $\omega \subset \partial\sigma_0$ the edge containing this point of intersection. Then by Lemma 7.7 there are no more bending points of $\beta$, and hence the remaining part of $\beta$ has to be parallel to the unbounded edges of the unbounded cell containing $\omega$. This determines the kink of $\beta$ when crossing $\omega$. Note that this argument also limits the exponents $m$ appearing in $W(\sigma_0)$ to be contained in $\partial\sigma_0 \cap \Lambda_{\sigma_0}$. Moreover, each such exponent belongs to at most one broken line ending at $p$.

Now choose $p$ very close to the unique interior integral point of $\sigma_0$ and let $m \in \partial\sigma_0 \cap \Lambda_{\sigma_0}$. Then $p+\mathbb{R}_{\geq 0}$ intersects $\partial\sigma_0$ very close to $m$. A local computation now shows that depending on the choice of a vertex $v \in \omega$ the coefficient of $z^m$ comes from either the $l$-th or the $(k-l)$-th coefficient of the associated slab function $(1+x)^k$. In either case we obtain the stated binomial coefficient $\binom{l}{k}$. □

8.2. **Hirzebruch surfaces.** As a last application featuring some non-Fano cases, we will study proper superpotentials for Hirzebruch surfaces $\mathbb{F}_m$. We fix the fan $\Sigma$ in $N \cong \mathbb{Z}^2$ of $\mathbb{F}_m$ to be the fan with rays $\rho_0, \ldots, \rho_3$ generated by the four primitive vectors

$$v_0 = (0,1),\ v_1 = (-1,0),\ v_2 = (0,-1),\ v_3 = (1,m).$$

Since $\mathbb{F}_m$ is only Fano in the cases $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 = dP_1$, for $m \geq 2$ the normal fan of the anticanonical polytope is not the fan of a Hirzeburch surface.

We fix $m$ in the following and denote by $D_{\rho_i}$ the torus-invariant divisor associated to $\rho_i$. Instead of the anticanonical divisor, which is not ample for $m \geq 2$, we now consider a smooth divisor $D$ on $\mathbb{F}_m$ in the ample class

$$D_{\rho_0} + D_{\rho_1} + D_{\rho_2} + m \cdot D_{\rho_3}.$$

Define the tropical manifold $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ with straight boundary as follows. $\check{B}$ is obtained from the Newton polytope

$$\Xi_D = \operatorname{conv}\big\{(-1,-1),(2m,-1),(0,1),(-1,1)\big\}$$

of $D$ by joining each vertex with one of the endpoints of the line segment $[0, m-1] \times \{0\} \subset \operatorname{Int}\Xi_D$ as shown in Figure 8.2 on the top, and then introducing a single focus-focus singularity on each of the four joining one-cells. It is again elementary to check that this makes $\partial\check{B}$ totally geodesic. Moreover, setting $\check{\varphi}(v) = 1$ for all vertices $v$ of $\Xi_D$ and

$$\check{\varphi}(0,0) = \check{\varphi}(m-1,0) = 0$$

defines a strictly convex, integral PL-function $\check{\varphi}$ on $(\check{B}, \check{\mathscr{P}})$.

The Legendre dual $(B, \mathscr{P}, \varphi)$ has the four vertices $v_0, \ldots, v_3$ we started with, two bounded cells

$$\sigma_0 = \operatorname{conv}(v_0, v_1, v_2), \quad \sigma_1 = \operatorname{conv}(v_0, v_2, v_3),$$

and four unbounded one-cells contained in $\mathbb{R}_{\geq 0} v_i$, $i = 0, \ldots, 3$. These one-cells are indeed parallel in any chart with domain an open set in the complement of the union of bounded cells. Moreover, $\partial(\sigma_0 \cup \sigma_1)$ has precisely four integral points, namely $v_0$, $v_1$, $v_2$ and $v_3$. Note also that by the definition of the Legendre-dual, $\varphi$ is uniquely determined by

$$\varphi(v_0) = \varphi(v_1) = \varphi(v_2) = 1,\ \varphi(v_3) = m,$$

and by the requirement to change slope by one along $\partial(\sigma_0 \cup \sigma_1)$.

We see that for $m \geq 3$ the union $\sigma_0 \cup \sigma_1$ is non-convex. In this case the scattering of the two walls emanating from the singular points on the two edges with vertices $v_0, v_1, v_3$ produce walls entering $\operatorname{Int}(\sigma_0 \cup \sigma_1)$.

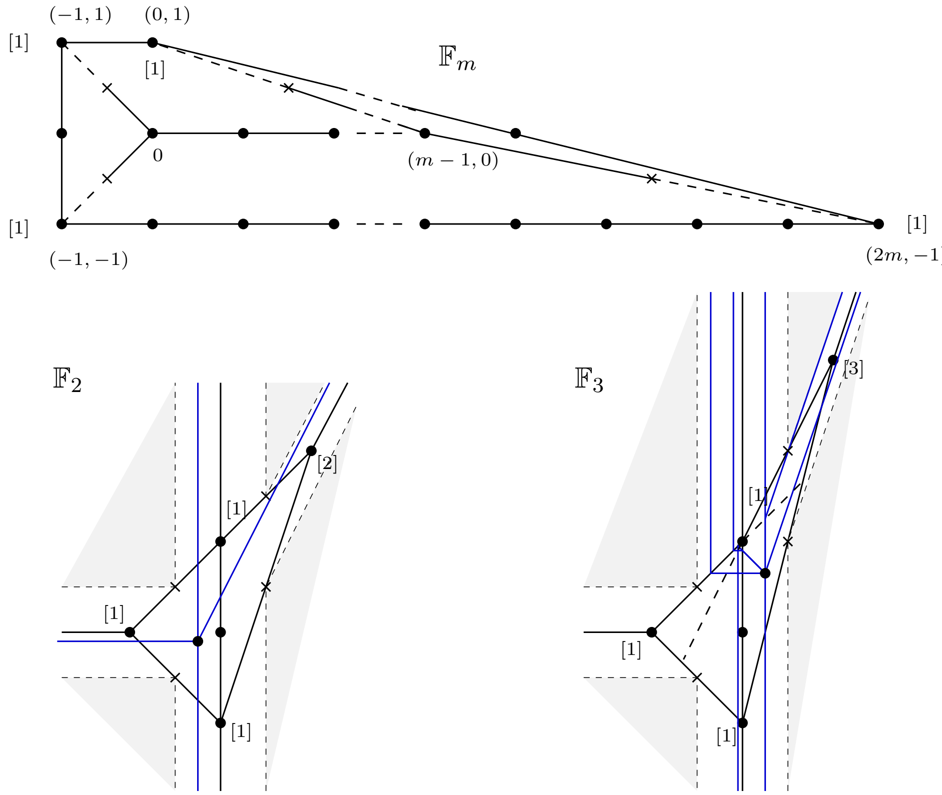


FIGURE 8.2. A straight boundary model for the Hirzebuch surfaces $\mathbb{F}_m$ with mirrors for $m = 2, 3$.

For $m > 3$ there is even an infinite number of walls to be added to make the wall structure consistent to all orders. While all these walls are added in the half-plane below the line $\mathbb{R} \cdot (1, m)$ and hence there is an open set contained in $\sigma_0$ not containing any wall, we have not carried out the necessary analysis to decide if $W(\sigma_0)$ is given to all orders by an algebraic expression.

We now restrict to the cases $m = 2, 3$ and explicitly compute the full superpotentials.

First, for the case $m = 2$ there are no walls in $\mathrm{Int}(\sigma_0 \cup \sigma_2)$. The broken lines ending at a specified point $p \in \mathrm{Int}(\sigma_0)$ are depicted on the lower left in Figure 8.2. The Landau-Ginzburg superpotential can then be read off as

$$W(\sigma_0) = \Big(\frac{1}{x} + \frac{1}{y} + y\Big) \cdot t + xy^2 \cdot t^2.$$

This is indeed the full potential, as there is no scattering in $\sigma_0 \cup \sigma_1$, so we can apply Lemma 7.7.

For $m = 3$ let us first compute the walls entering $\mathrm{Int}(\sigma_0 \cup \sigma_1)$. These come from scattering at the point $(0, 1)$ of the adjacent edges with focus-focus singular points in directions $(1, 1)$ and $(-1, -2)$. Locally this scattering situation is equivalent to the scattering of incoming walls from directions $(-1, 0)$ and $(0, -1)$ meeting at the origin. An explicit computation carried out in [GS4, §4.1] shows that this scattering diagram can be made consistent to all orders by introducing outgoing walls in directions $(1, 0)$, $(0, 1)$ and $(1, 1)$. These translate to walls in directions $(1, 1)$, $(-1, -2)$ and $(0, -1)$ in our situation, as indicated by the dashed lines in the lower right of Figure 8.2. Of course it will be necessary to insert more walls *outside* of the bounded part, but this is unessential for the computation of the superpotential. Hence scattering on the bounded part $\sigma_0 \cup \sigma_1$ is finite and we can once more apply Lemma 7.7. For the choice of root vertex $p' \in \sigma_1$ as indicated in the lower right of Figure 8.2 we get the following full superpotential for $(B_3, \mathscr{P}_3)$

$$W(\mathfrak{u}) = \Big(\frac{1}{x} + \frac{1}{y} + y + \frac{y}{x}\Big) \cdot t + \frac{y}{x} \cdot t^2 + \Big(xy^3 + y^2\Big) \cdot t^3.$$

Thus, neglecting $t$-orders for a moment, we have three new contributions that differ from the Hori-Vafa potential $\frac{1}{x} + \frac{1}{y} + y + xy^3$, namely the monomial $y^2$ and twice the term $\frac{y}{x}$. These come from broken lines that have a break point at the new walls emanating from $(0,1)$ in direction $(1,1)$, $(0,-1)$ and $(-1,-2)$, respectively. Note that these are precisely the terms Auroux found in [Au2, Prop. 3.2], when we make the coordinate change $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{1}{y}$.

The computation in [Au2, Prop. 3.2] is very explicit and rather long when compared with our derivation. Of course all the hard work is hidden in the scattering process of [GS3] and the propagation of monomials via broken lines, but still it is remarkably easy to compute Landau-Ginzburg models with this approach, once everything is set up.

## 9. Three-dimensional examples

So far we restricted ourselves to $\dim B = 2$. We now turn to a few simple examples illustrating some features of higher dimensional cases.

**Example 9.1.** Starting from the momentum polytope $\Xi$ for $\mathbb{P}^3$ with its anticanonical polarization, Construction 7.2 provides a model with a distinguished polarized tropical manifold $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ with $\check{B} = \Xi$ and with flat boundary. In dimension three this is done by trading corners *and edges* with a one-dimensional singular locus of the affine structure. Explicitly, $\check{\mathscr{P}}$ is the star-subdivision of

$$\Xi = \operatorname{conv}\big\{(2,-1,-1), (-1,2,-1), (-1,-1,2), (-1,-1,-1)\big\}$$

that introduces six two-faces spanned by the origin and two distinct vertices of $\Xi$. The discriminant locus $\check{\Delta}$ is the subcomplex of the first barycentric subdivision of these six affine triangles shown in Figure 9.1. The affine structure is fixed by the embedding of $\Xi$ into $\mathbb{R}^3$ at the origin, and by the affine charts at the vertices of $\check{B}$ making $\partial\Xi$ flat and inducing the given affine chart on the maximal cells of $\check{\mathscr{P}}$. Finally, $\check{\varphi}$ is determined by $\check{\varphi}(v) = 1$ for every vertex $v$ of $\Xi$ and $\check{\varphi}(0) = 0$.

The discrete Legendre dual $(B, \mathscr{P}, \varphi))$, also drawn in Figure 9.1, has four parallel unbounded rays and a discriminant locus $\Delta$ with six unbounded rays. Note that unlike in the case of closed tropical manifolds or in dimension two, $\check{\Delta}$ and $\Delta$ are not homeomorphic, but $\check{\Delta}$ is homeomorphic to the compactification of $\Delta$ that adds a point at infinity to each unbounded one-cell of $\Delta$. Every bounded two-face is subdivided into three 4-gons by $\Delta$ and at every vertex of $B$ three of these 4-gons meet. Denote the bounded three-cell by $\sigma_0$. As in the proof of Theorem 8.2 it now follows that any broken line ending at $p \in \operatorname{Int}(\sigma_0)$ is straight. This shows

$$W_{\mathbb{P}^3}(\sigma_0) = \Big(x + y + z + \frac{1}{xyz}\Big) \cdot t.$$

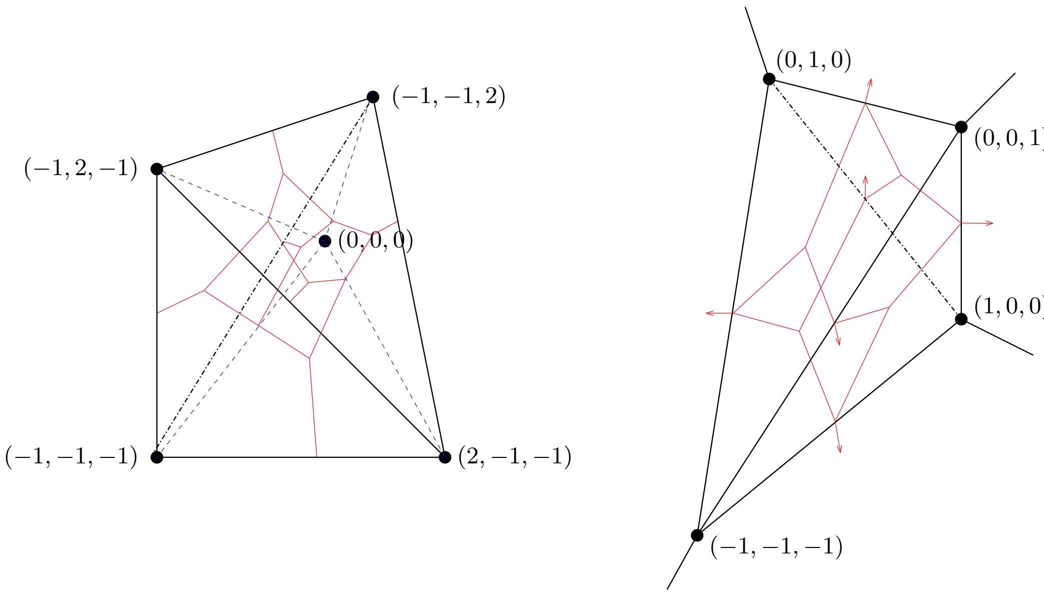


FIGURE 9.1. The distinguished model of $\mathbb{P}^3$ and its affine Legendre dual. The trivalent graphs indicate the discriminant loci, the dashed lines on the left the one-cells added in the star-subdivision. The little arrows indicate parts of unbounded cells of the discriminant locus.

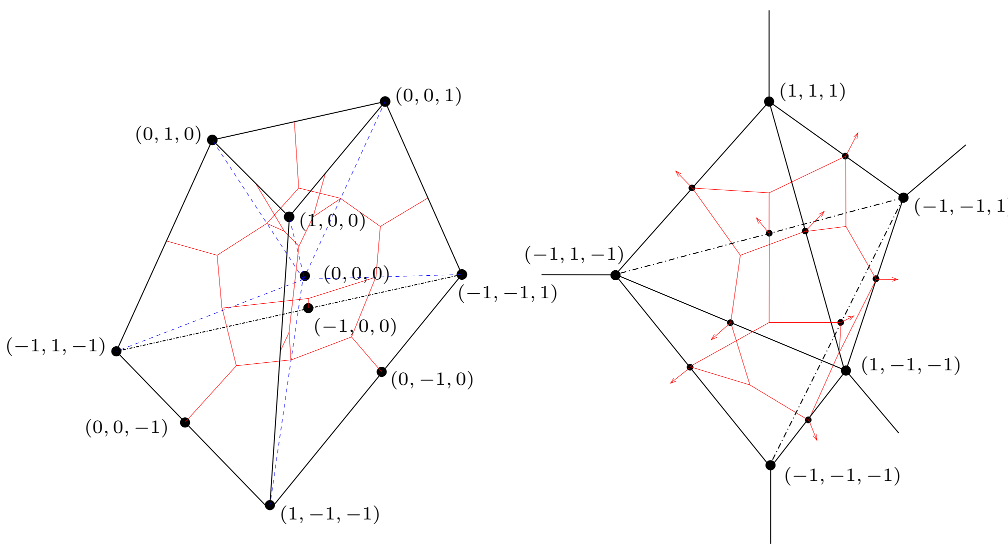


FIGURE 9.2. A reflexive polytope with edges and corners pushed in and its Legendre-dual, following the same conventions as in Figure 9.1.

**Example 9.2.** Consider the reflexive polytope $\Xi$ depicted on the left in Figure 9.2, a truncated tetrahedron with parallel top and bottom facets that is symmetric under cyclic permutation of the coordinates. The polar dual is the bounded polyhedron $\sigma_0$ on the right of the same figure.

Note that each edge of $\sigma_0$ has integral length two. This means that the toric Fano variety $\check{X}$ with anticanonical Newton polyhedron $\Xi$ is singular along each one-dimensional toric stratum; each such stratum has a neighborhood isomorphic to a product of a two-dimensonal $A_1$-singularity with $\mathbb{G}_m$.

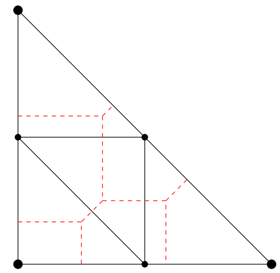

FIGURE 9.3. Refinement of the discriminant locus on a bounded two-cell of $\mathscr{P}$.

In exactly the same way as in Example 9.1, we arrive at a tropical manifold $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ with flat boundary and $\check{\mathscr{P}}$ given by the star subdivision of $\Xi$. The Legendre-dual $(B, \mathscr{P}, \varphi)$ has a double tetrahedron as the unique bounded maximal cell. Both tropical manifolds are depicted in a chart at the origin in Figure 9.2. The discriminant locus $\check{\Delta}$ of $(\check{B}, \check{\mathscr{P}})$ now is contained in the union of triangles added in the star-subdivision, with the intersection of $\check{\Delta}$ with one such triangle $\tau$ three edges meeting in the barycenter of $\tau$. The discriminant locus $\Delta$ of $(B, \mathscr{P})$ has nine rays emanating from the barycenters of the edges of $\sigma_0$, and then for each facet $\tau \subset \sigma_0$ again a union of three edges intersecting in the barycenter of $\tau$.

Now neither side has simple singularities. For example, the affine structure of $(B, \mathscr{P})$ at an interior point of an edge of $\Delta$ on $\partial\sigma_0$ is conjugate to $\left(\begin{smallmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, the square of the monodromy of a focus-focus singularity times an interval. Thus although it is not hard to see that $(B, \mathscr{P})$ is compactifiable (Definition 1.2), the assumptions of [GS3, Def. 1.26] can not be fulfilled for this compactification, and hence the existence of a consistent wall structure is not immediately clear.

In the present case we can proceed as follows. Each of the bounded two-cells of $(B, \mathscr{P})$ is integral affine isomorphic to the planar triangle with vertices $(0,0), (2,0), (0,2)$. Subdivide each such two-cell as in Figure 9.3 and refine $\mathscr{P}$ by intersection with the fan over the faces of this subdivision. The discriminant locus can then be refined as indicated by the dashed graph in Figure 9.3, along with replacing each unbounded cell of $\Delta$ by two copies joined with the rest of the graph at disjoint trivalent vertices lying on the edges of $\sigma_0$. Now we can indeed run the algorithm[19] to construct a compatible sequence $\tilde{\mathscr{S}}_k$ of consistent wall structures. In a second step undo the refinement process to show that the algorithm indeed works starting with the non-rigid data $(B, \mathscr{P}, \varphi)$.

As in Theorem 8.2 we also have a choice for the initial slab function, with a distinguished choice a square $f_{\rho,v} = (1 + x + y)^2$ for each bounded 2-cell $\rho$ and vertex $v \in \rho$.

Now the details of the construction of the wall structure are completely irrelevant for the computation of the superpotential in the bounded cell $\sigma_0 \in \mathscr{P}$: By the same arguments as in Theorem 8.2, it is again just the sum over broken lines with at most one bend when crossing $\partial\sigma_0$. Moreover, the set of such broken lines with endpoint at any $p \in \operatorname{Int}\sigma_0$ are in bijection with the integral points of $\partial\sigma_0$. For the distinguished slab functions the coefficient carried by the broken line equals 1 for the ones without bend and 2 for the others. The superpotential

[19] There is a technical problem to assure that the process is finite at each step. This can be done by working with $d\varphi + \psi$ for $d > 0$ and an appropriate $\psi$ or by inspection of the local scattering situations in the case at hand.

therefore equals

$$W(\sigma_0) = \Big(xyz + \frac{1}{xyz} + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + 2(x^2+y^2+z^2) + \frac{2}{x^2} + \frac{2}{y^2} + \frac{2}{z^2} + \frac{2}{xy} + \frac{2}{yz} + \frac{2}{xz}\Big)\cdot t.$$

A similar challenge concerns the existence of a consistent wall structure on $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$, but analogous arguments apply. The resulting toric degeneration $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ then fits into an algebraizable two-parameter family with the singular toric Fano manifold $\check{X}$ with its toric anticanonical divisor $\check{D}$ as another fiber. We have not performed a more detailed analysis to identify this family. Possibly it is just isomorphic to a deformation of $\check{D}$ inside $\check{X}$, as in Example 2.7 for $\mathbb{P}^2$ and $\check{D}$ a family of elliptic curves.

## Concluding remarks.

It would be interesting to more systematically analyze Landau-Ginzburg models for non-toric Fano threefolds with our method. In [Pr1, Pr2] so called *very weak Landau-Ginzburg potentials* are found. The terms and coefficients of these Laurent polynomials have to be chosen very carefully. As the potentials presented there do not come from a specific algorithm, but rather are written down in an ad hoc way, one would like to have an interpretation of the terms occurring. One might ask whether there are toric degenerations reproducing the potentials in [Pr1, Pr2] via tropical disk counting, as in the examples here.

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