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Stability of coupled jump diffusions and applications

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Abstract

This paper develops stability and stabilization for systems of fully coupled jump diffusions. Such systems frequently arise in numerous applications where each subsystem (component) is operated under the influence of other subsystems (components). It derives sufficient conditions under which the underlying system of coupled jump diffusions is stable. The results are then applied to investigate the stability of linearizable jump diffusions, fast-slow coupled jump diffusions. Moreover, weak stabilization of interacting systems and consensus of leader-following systems are examined.

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1. Introduction

Networked systems have posed unprecedented opportunities as well as challenges. Such systems have numerous applications in control engineering, wireless communications, mathematical biology, financial engineering, and actuarial science. In many stochastic networked systems, subsystems and/or components are intertwined or highly coupled. This poses great challenges as one would like to study the system. Moreover, empirical studies reveal that there exist sudden rapid moments in the mid quotes of stock prices, i.e., jumps during trading periods [4,29]. Treating competitive Lotka-Volterra populations, as observed in [5], the population may suffer sudden environmental shocks such as earthquakes, hurricanes, epidemics, etc. The commonly used diffusion-type stochastic Lotka-Volterra models cannot explain such phenomena. To allow sudden changes, systems of stochastic differential equations involving Lévy process are often used to capture fluctuations as well as random jumps [43] (see also related problems due to random switching [24]).

When a system has been operated for a long time, its long-time behavior and stability become important features. As a result, they have been studied extensively. Given a system with coupled components/subsystems, can we derive the stability of one specific component based on the dynamics of the other components? Take for instance, a system involving fast-slow components, one uses different time scales to portray the fast-slow motions. A question is almost immediate. How can we determine the stability of the slow subsystem based on the information of the fast subsystem? In addition, for a coupled system of jump diffusions, can we design a feedback control so as to obtain the desired stability? This work addresses these questions.

Previous works on stability of jump-diffusion processes can be found in [41] for multi-dimensional jump-diffusion processes, [2] for constrained jump-diffusion processes, [18] for jump diffusions in a Hilbert space, [6,10,11,45,47] for regime-switching jump diffusions, [3] for jump diffusions with state-dependent densities. In contrast to the existing works in the literature, this work focuses on stability of fully coupled jump diffusions, where two jump-diffusion components interact with each other. Such systems have a wide range of applications to numerous physical, engineering, and biological problems such as chemical reactors [34], power transmission lines [13], flow regulation in deep mines [40], elastic beams linked to rigid bodies [31], blood flow model [12,17], mitochondrial swelling [16], to mention just a few among others. In such a situation, we are interested in the stability, averaging phenomena under the influence of the interacting processes in the environment. From a technical point of view, not only do the coupled systems possess many interesting properties, but also present many challenges; see coupled ordinary differential and partial differential equations (ODEs-PDEs) [35,25,28,38], coupled diffusion or stochastic differential equations [19,21,37], and coupled stochastic reaction-diffusion or stochastic partial differential equations [7,8,22]. The motivations and urgent need in both theory and applications motivate the current work.

Let $\mathbb{R}^{\ell_1}$ and $\mathbb{R}^{\ell_2}$ be two Euclidean spaces of dimensions $\ell_1 > 0$ and $\ell_2 > 0$, respectively. Denote by $\mathbf{0}$ a zero vector with appropriate dimension (which will be clear from the context). We assume $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are coupled jump-diffusion processes in $\mathbb{R}^{\ell_1}$, $\mathbb{R}^{\ell_2}$, respectively. More specifically, the pair $(\mathbf{X}_1(t), \mathbf{X}_2(t))$ is the solution of the system

$$\begin{cases} d\mathbf{X}_1(t) = b_1(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_1(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t-), \mathbf{X}_2(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ d\mathbf{X}_2(t) = b_2(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_2(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}_*^{n_2}} \gamma_2(\mathbf{X}_1(t-), \mathbf{X}_2(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}), \\ \mathbf{X}_1(0) = \mathbf{x}_1, \quad \mathbf{X}_2(0) = \mathbf{x}_2, \end{cases} \tag{1.1}$$

where, for $i = 1, 2$, $\mathbf{W}_i(t)$ are standard $\mathbb{R}^{d_i}$-valued Brownian motions that are mutually independent; $\mathbf{N}_i(dt, d\boldsymbol{\phi})$'s are Poisson random measures independent of $\mathbf{W}_1(t)$, $\mathbf{W}_2(t)$, and $\widetilde{\mathbf{N}}_i(dt, d\boldsymbol{\phi}) = \mathbf{N}_i(dt, d\boldsymbol{\phi}) - \boldsymbol{\nu}_i(d\boldsymbol{\phi})dt$ are the compensated Poisson random measures on $[0, \infty) \times \mathbb{R}_*^{n_i}$ with $\mathbb{R}_*^{n_i} := \mathbb{R}^{n_i} \setminus \{\mathbf{0}\}$; $b_1 : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \to \mathbb{R}^{\ell_1}$ and $b_2 : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \to \mathbb{R}^{\ell_2}$ are smooth functions; $\sigma_1 : \mathbb{R}_1^{\ell_1} \times \mathbb{R}_2^{\ell_2} \to \mathbb{R}^{\ell_1 \times d_1}$ and $\sigma_2 : \mathbb{R}_1^{\ell_1} \times \mathbb{R}_2^{\ell_2} \to \mathbb{R}^{\ell_2 \times d_2}$; $\gamma_1 : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \times \mathbb{R}_*^{n_1} \to \mathbb{R}^{\ell_1}$, $\gamma_2 : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \times \mathbb{R}_*^{n_2} \to \mathbb{R}^{\ell_2}$ are measurable functions. In this paper, it is assumed that

$$b_2(\mathbf{x}_1, \mathbf{0}) = \mathbf{0}, \ \sigma_2(\mathbf{x}_1, \mathbf{0}) = \mathbf{0}, \ \gamma_2(\mathbf{x}_1, \mathbf{0}, \boldsymbol{\phi}) = \mathbf{0}.$$

That is, $\mathbf{0}$ is an equilibrium point of $\mathbf{X}_2(t)$. We wish to derive mild conditions under which the equilibrium point $\mathbf{0}$ or the trivial solution of $\mathbf{X}_2(t)$ is stable.

In what follows, the assumptions for the stability are given in term of Lyapunov functions but the insight and intuition are derived from a dynamic system point of view combined with the averaging principle and ergodicity of coupled systems. We demonstrate that the conditions are easily applicable. The intuition and idea are that if the interacting process $\mathbf{X}_1(t)$ on the boundary (i.e., when $\mathbf{X}_2(t) = \mathbf{0}$) admits a unique invariant measure and the corresponding decoupled (averaging) equations (of the process $\mathbf{X}_2(t)$) at the stationary distribution of $\mathbf{X}_1(t)$ satisfy some appropriate stability conditions then $\mathbf{X}_2(t)$ is also stable. Taking the idea from a dynamic system point of view, the stability conditions are obtained by considering the Lyapunov exponent of the process $\mathbf{X}_2(t)$ corresponding to the invariant measure of the interacting process $\mathbf{X}_1(t)$ on the boundary. Such conditions coincide with the intuition that when the main process $\mathbf{X}_2(t)$ is close to the equilibrium $\mathbf{0}$, the interacting process is close to the solution on the boundary. Thus the stability conditions of the main process only need to be based on the information of the interacting process on the boundary, which however, poses great challenges. We need to reveal the behavior of the system around the boundary. Since two components are fully coupled, handling their interactions and analyzing their behavior require a careful analysis. We emphasize that one of the main challenges in this work is the coupled interaction of $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$. To overcome this difficulty, we modify the generalized coupling method in [20,26], in which the coupled processes are expected to approach each other rather than meet at a finite time in classical coupling methods, cf., [9]. The coupling will be done until a specified stopping time is reached. Further details can be found in Remark 2.3.

With the stability results at our hands, we study systems that commonly arise in applications. In particular, one of the main questions to answer is: What are the relationships between nonlinear systems and the associated linearized systems for jump diffusions? We address this question and provide sufficient conditions for the stability of linearizable jump diffusions. In various applications, the subsystems and/or components often display different time scales. It is often necessary to treat fast-slow coupled jump diffusions. We provide sufficient conditions for stability of the slow component based only on the limit system. Next, we design stabilizing strategies in a coupled system when only the interacting process is available to be controlled. Finally, leader-following systems are studied and conditions for the consensus of the systems are given.

The rest of paper is arranged as follows. Section 2 presents the main results on stability. Section 3 deals with linearizable systems and systems with fast and slow components. Section 4 focuses on stabilization and treats consensus problems. Finally, Section 5 concludes the paper and issues further remarks.

2. Stability of coupled jump diffusions

In this paper, we use $|\cdot|$ to denote the Euclidean norm for either vectors or matrices, and $A^\top$ the transpose of a vector or a matrix A. For two real numbers a and b, $a \vee b = \max(a, b)$, and $a \wedge b = \min(a, b)$. Denote $\mathbf{Z} = (\mathbf{X}_1^\top, \mathbf{X}_2^\top)^\top$, $\mathbf{z} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$, $b = (b_1^\top, b_2^\top)^\top$, $\sigma = (\sigma_1^\top, \sigma_2^\top)^\top$, $\gamma = (\gamma_1^\top, \gamma_2^\top)^\top$, $\mathbf{W} = (\mathbf{W}_1^\top, \mathbf{W}_2^\top)^\top$, and $\widetilde{\mathbf{N}} = (\widetilde{\mathbf{N}}_1^\top, \widetilde{\mathbf{N}}_2^\top)^\top$. We will use $\mathbf{Z}$ and $(\mathbf{X}_1, \mathbf{X}_2)$ exchangeably. Moreover, a function of $(\mathbf{x}_1, \mathbf{x}_2)$ can often be written as a function of $\mathbf{z}$ with $\mathbf{z} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$, which will be clear from the context. Note that the equation (1.1) in vector form becomes

$$d\mathbf{Z}(t) = b(\mathbf{Z}(t))dt + \sigma(\mathbf{Z}(t))d\mathbf{W}(t) + \int\limits_{\mathbb{R}_*^{n1}\times\mathbb{R}_*^{n2}} \gamma(\mathbf{Z}(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}(dt, d\boldsymbol{\phi}), \quad \mathbf{Z}(0) = \mathbf{z}.$$

We use $\mathbb{P}_\mathbf{z}$ and $\mathbb{E}_\mathbf{z}$ to denote the probability and expectation with initial data $\mathbf{z}$.

Next, define the operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}g(\mathbf{z}) :=&[\partial_\mathbf{z} g(\mathbf{z})]^\top b(\mathbf{z}) + \frac{1}{2}\mathrm{tr}[\sigma(\mathbf{z})\sigma^\top(\mathbf{z})\partial_\mathbf{z}^2 g(\mathbf{z})] \\ &+ \int\limits_{\mathbb{R}_*^{n1}\times\mathbb{R}_*^{n2}} [g(\mathbf{z} + \gamma(\mathbf{z}, \boldsymbol{\phi})) - g(\mathbf{z}) - (\partial_\mathbf{z} g)^\top\gamma(\mathbf{z}, \boldsymbol{\phi})]\boldsymbol{\nu}(d\boldsymbol{\phi}),\end{aligned}$$

for $g \in \mathcal{D}_\mathcal{L}$, where $\partial_\mathbf{z}$ and $\partial_\mathbf{z}^2$ denote the gradient and Hessian matrix with respect to $\mathbf{z}$, respectively, and

$$\begin{aligned}\mathcal{D}_\mathcal{L} := \Big\{& g : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \to \mathbb{R} : g(\mathbf{z}) \text{ is twice continuously differentiable and} \\ &\int\limits_{\mathbb{R}_*^{n1}\times\mathbb{R}_*^{n2}} |g(\mathbf{z} + \gamma(\mathbf{z}, \boldsymbol{\phi})) - g(\mathbf{z}) - \partial_\mathbf{z} g \cdot \gamma(\mathbf{z}, \boldsymbol{\phi})|\nu(d\boldsymbol{\phi}) < \infty\Big\}.\end{aligned}$$

It is noted that $\mathbf{z}$ in $\mathcal{L}g(\mathbf{z})$ represents the variable of $\mathcal{L}g$ rather than the variable of g. Indeed, later g can be plugged in by either functions of $\mathbf{x}_1$ or functions of $\mathbf{x}_2$. For example, if g is a function of $\mathbf{x}_1$ only, the gradient of g (with respect to $\mathbf{z}$) will be $([\partial_{\mathbf{x}_1} g(\mathbf{x}_1)]^\top, \mathbf{0}^\top)^\top$. However, because the coefficients are fully coupled, $\mathcal{L}g$ is a function of $\mathbf{z}$; we still write it as $\mathcal{L}g(\mathbf{z})$.

The following result is known as the generalized Itô formula (see e.g., [39,45])

$$\begin{aligned}g(\mathbf{Z}(t)) - g(\mathbf{Z}(0)) =& \int_0^t \mathcal{L}g(\mathbf{Z}(s-))ds + \int_0^t \partial_\mathbf{z} g(\mathbf{Z}(s-))\sigma(\mathbf{Z}(s-))d\mathbf{W}(s) \\ &+ \int_0^t \int\limits_{\mathbb{R}_*^{n1}\times\mathbb{R}_*^{n2}} \big[g(\mathbf{Z}(s-) + \gamma(\mathbf{Z}(s-), \boldsymbol{\phi})) - g(\mathbf{Z}(s-))\big]\widetilde{\mathbf{N}}(ds, d\boldsymbol{\phi}).\end{aligned}$$

To ensure the existence and uniqueness of the solution, we impose the following assumption.

Assumption 2.1. There are some constants $K_1, K_2 > 0$ such that $\forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$

$$|b(\mathbf{z}_1) - b(\mathbf{z}_2)|^2 + |\sigma(\mathbf{z}_1) - \sigma(\mathbf{z}_2)|^2 + \int\limits_{\mathbb{R}_*^{n_1} \times \mathbb{R}_*^{n_2}} |\gamma(\mathbf{z}_1, \boldsymbol{\phi}) - \gamma(\mathbf{z}_2, \boldsymbol{\phi})|^2 \boldsymbol{\nu}(d\boldsymbol{\phi}) \le K_1 |\mathbf{z}_1 - \mathbf{z}_2|^2, \tag{2.1}$$

and

$$\int\limits_{\mathbb{R}_*^{n_1} \times \mathbb{R}_*^{n_2}} |\gamma(\mathbf{z}, \boldsymbol{\phi})|^2 \boldsymbol{\nu}(d\boldsymbol{\phi}) \le K_2(1 + |\mathbf{z}|^2). \tag{2.2}$$

Remark 2.1. Assumption 2.1 can be replaced by local Lipschitz conditions together with a suitable condition imposed on a Lyapunov function.

To investigate the stability of the coupled jump diffusion system, we need the following assumptions.

Assumption 2.2. The following conditions hold.

(i) There exist positive functions $V_0, V_1 : \mathbb{R}^{\ell_1} \mapsto \mathbb{R}_+$ satisfying

$$\mathcal{L}V_0(\mathbf{z}) \le K_3 - K_4 V_1(\mathbf{x}_1), \ \forall \mathbf{z} = (\mathbf{x}_1, \mathbf{0}), \tag{2.3}$$

for some constants $K_3, K_4 > 0$.
(ii) There exist a function $U : \mathbb{R}^{\ell_2} \mapsto \mathbb{R}_+$ and constants $m_0, \Delta_0 > 0$ such that

$$\lim_{\mathbf{x}_2 \to \mathbf{0}} U(\mathbf{x}_2) = \infty, \quad U(\mathbf{x}_2) - U(\mathbf{x}_2') \le m_0 \left(\ln \frac{|\mathbf{x}_2'|}{|\mathbf{x}_2|} + 1 \right), \ \forall \mathbf{x}_2, \mathbf{x}_2' \neq \mathbf{0}, |\mathbf{x}_1| < \Delta_0, |\mathbf{x}_2| < \Delta_0. \tag{2.4}$$

Moreover, assume that there are two Lipschitz functions $f_1 : \mathbb{R}^{\ell_1} \to \mathbb{R}$, and $f_2 : \mathbb{R}^{\ell_1} \to \mathbb{R}_+$, constants $\alpha_0 > 0$, and $K_5 > 0$ such that

$$\mathcal{L}U(\mathbf{z}) \ge f_1(\mathbf{x}_1), \forall \mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2), |\mathbf{x}_2| \le \Delta_0, \tag{2.5}$$

and

$$\begin{aligned} &|U_{\mathbf{x}_2}(\mathbf{x}_2)\sigma_2(\mathbf{z})|^2 + \int\limits_{\mathbb{R}_*^{n_2}} \left[\exp\left\{ \alpha_0 \big(U(\mathbf{x}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})) - U(\mathbf{x}_2) \big)_+ \right\} \right] \boldsymbol{\nu}_2(d\boldsymbol{\phi}) \\ &\le f_2(\mathbf{x}_1), \ \forall \mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2), |\mathbf{x}_2| \le \Delta_0, \end{aligned} \tag{2.6}$$

where $U_{\mathbf{x}_2}$ denotes the gradient of U; and

$$|f_1(\mathbf{x}_1)| + f_2(\mathbf{x}_1) < K_5 V_1(\mathbf{x}_1). \tag{2.7}$$

(iii) When $\mathbf{x}_2 = \mathbf{0}$ (yielding $\mathbf{X}_2(t) = \mathbf{0}$), the corresponding system for $\mathbf{X}_1$

$$d\mathbf{X}_1(t) = b_1(\mathbf{X}_1(t), \mathbf{0})dt + \sigma_1(\mathbf{X}_1(t), \mathbf{0})d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t-), \mathbf{0}, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}),$$

admits a unique invariant measure μ^* and

$$\Lambda_1 := \int f_1(\mathbf{x}_1)\mu^*(d\mathbf{x}_1) > 0.$$

Assumption 2.3. Suppose that $\sigma_1(\mathbf{x}_1, \mathbf{x}_2)$ admits a right inverse $\sigma_1^{-1}(\mathbf{x}_1, \mathbf{x}_2)$ (that is, $\sigma_1(\mathbf{x}_1, \mathbf{x}_2)\sigma_1^{-1}(\mathbf{x}_1, \mathbf{x}_2) = I_{\ell_1}$) such that

$$\|\sigma_1^{-1}(\mathbf{x}_1, \mathbf{x}_2)\| \le c_\sigma < \infty \text{ for all } |\mathbf{x}_2| \le \Delta_0,$$

where Δ_0 is as in Assumption 2.2. Moreover, assume that there is a constant K_6 such that $\forall \mathbf{z} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top, \mathbf{z}' = ((\mathbf{x}_1')^\top, (\mathbf{x}_2')^\top)^\top \in \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$

$$(b_1(\mathbf{z}) - b_1(\mathbf{z}'))^\top(\mathbf{x}_1 - \mathbf{x}_1') + |\sigma_1(\mathbf{z}) - \sigma_1(\mathbf{z}')|^2 \le K_6|\mathbf{z} - \mathbf{z}'|^2.$$

In addition, γ_1 and γ_2 are Lipschitz in $\mathbf{z}$ (uniformly in $\boldsymbol{\phi}$), i.e., there is some $K_7 > 0$ such that

$$|\gamma_1(\mathbf{z}, \boldsymbol{\phi}) - \gamma_1(\mathbf{z}', \boldsymbol{\phi})| + |\gamma_2(\mathbf{z}, \boldsymbol{\phi}) - \gamma_2(\mathbf{z}', \boldsymbol{\phi})| \le K_7|\mathbf{z} - \mathbf{z}'|, \ \forall \mathbf{z}, \mathbf{z}', \boldsymbol{\phi}.$$

Remark 2.2. Assumption 2.2 is the main assumption for stability. This assumption is rather mild and not restrictive. The function U in Assumption 2.2(ii) is used to bound the decay rate of $\mathbf{X}_2$ and f_1 is a bound for the growth rate of $U(\mathbf{X}_2(t))$. Condition (2.4) means that the function U does not tend to infinity faster than a negative logarithm rate, which is natural because in practice, we do not often expect the solution tends to 0 at a supergeometric rate. The functions U and f_1 are used to estimate the decay rate of $\mathbf{X}_2(t)$; Λ_1 in Assumption 2.2(iii) is a bound of the Lyapunov exponent. A simple but promising candidate of $U(\cdot)$ that satisfies the proposed conditions is $U(\mathbf{x}_2) = (-\ln|\mathbf{x}_2|) \vee 0$. Condition (2.6) gives a bound for the quadratic variation of the martingale component in the equation of $U(\mathbf{X}_2(t))$. Assumption 2.2(i) and (2.7) guarantee that the bound Λ_1 of the Lyapunov exponent is well-defined. Assumption 2.3 collects a form of strong non-degenerate condition of the diffusion and some technical conditions. Although the second condition in Assumption 2.3 is a consequence of (2.1), we impose this condition in case Assumption 2.1 is replaced by local Lipschitz conditions together with a suitable Lyapunov-type condition as we commented in Remark 2.1. It can be seen that the conditions are applicable to many systems in applications; see Example 2.1 below as well as Section 3.

Now, we state our main results.

Theorem 2.1. *Let $\gamma_0 \in (0, \frac{\Lambda_1}{m_0})$. For any $\mathbf{x}_1 \in \mathbb{R}^{\ell_1}$ and $\varepsilon > 0$, there exists $\theta_{\mathbf{x}_1,\varepsilon} > 0$ such that if $|\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| + |\widetilde{\mathbf{x}}_2| \le \theta_{\mathbf{x}_1,\varepsilon}$,*

$$\mathbb{P}_{\widetilde{\mathbf{z}}}\left\{\liminf_{t\to\infty} \frac{U(\mathbf{X}_2(t))}{t} \ge m_0\gamma_0\right\} \ge 1 - \varepsilon, \tag{2.8}$$

where $\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}_1^\top, \widetilde{\mathbf{x}}_2^\top)^\top$*; and thus,*

$$\mathbb{P}_{\widetilde{\mathbf{z}}}\Big\{ \limsup_{t\to\infty} \frac{\ln|\mathbf{X}_2(t)|}{t} \le -\gamma_0 \Big\} \ge 1-\varepsilon. \tag{2.9}$$

Example 2.1. To illustrate our results, let us provide a simple example. Consider a stochastic SIR epidemic model with Beddington-DeAngelis incidence rate given by the following SDEs with jumps,

$$\begin{cases} dS(t) = \Big[c_0 - c_1 S(t) - \frac{c_3 S(t) I(t)}{c_4 + c_5 S(t) + c_6 I(t)}\Big]dt + \sigma_1(S(t), I(t))dW_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(S(t), I(t), \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ dI(t) = \Big[-c_2 I(t) + \frac{c_3 S(t) I(t)}{c_4 + c_5 S(t) + c_6 I(t)}\Big]dt + c_7 I(t) dW_2(t) + I(t) \int\limits_{\mathbb{R}_*^{n_1}} \widehat{\gamma_2}(\boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}). \end{cases} \tag{2.10}$$

In the above, $\sigma_1, \gamma_1, \widehat{\gamma_2}$ are bounded functions such that $\sigma_1(s,i), \gamma_1(s,i,\boldsymbol{\phi}), \gamma_2(s,i,\boldsymbol{\phi}) = i\widehat{\gamma_2}(\boldsymbol{\phi})$ satisfy the technical conditions in Assumptions 2.1 and 2.3. Now, we check the stability conditions (Assumption 2.2). It is easily verified that Assumption 2.2(i) is satisfied with $V_0(s) = V_1(s) = s$; Assumption 2.2(ii) is satisfied with

$$U(i) = (-\ln i) \vee 0, \quad f_1(s) = c_2 + \frac{c_7^2}{2} + \int\limits_{\mathbb{R}_*^{n_2}} |\widehat{\gamma_2}(\phi)|^2 \boldsymbol{\nu}_2(d\boldsymbol{\phi}) - \frac{c_3 s}{c_4 + c_5 s},$$

and $f_2(s)$ is some large constant. Under certain conditions, when $I(t) = 0$, the corresponding system

$$d\widehat{S}(t) = (c_0 - c_1 \widehat{S}(t))dt + \sigma_1(\widehat{S}(t), 0)dW_1(t) + \int\limits_{\mathbb{R}_*^{n_1}} \gamma_1(\widehat{S}(t), 0, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi})$$

has a unique invariant measure μ^*. Therefore, if $\Lambda_1 := \int f_1(s)\mu^*(ds) > 0$, applying Theorem 2.1, $I(t)$ is stable at 0. Without jump, this result is consistent with the longtime characterization in stochastic SIR epidemic models in [14,15]; with jumps, it generalizes the results in the aforementioned references.

To prove Theorem 2.1, we begin with some auxiliary lemmas. Lemma 2.1 provides a local boundedness (uniform in finite intervals) in probability of the solution and Lemma 2.2 illustrates the continuity on initial value (at $\mathbf{0}$) in probability of $\sup_{t\in[0,T]} |\mathbf{X}_2(t)|$, for any finite time T.

Lemma 2.1. *For any* $T > 0$, $\varepsilon > 0$, $R > 0$, *there exists an* $H_{1,T,\varepsilon,R} > 0$ *such that*

$$\mathbb{P}_{\mathbf{z}}\Big\{ \sup_{t\in[0,T]} |\mathbf{Z}(t)| \le H_{1,T,\varepsilon,R} \Big\} > 1-\varepsilon, \text{ for all } |\mathbf{z}| \le R.$$

Proof. Under Assumption 2.1, using a standard argument (see e.g., [32, Lemma 6.9]), we obtain the following local boundedness

$$\mathbb{E}_{\mathbf{z}} \sup_{t\in[0,T]} |\mathbf{Z}(t)| < C_T(\mathbf{z}),$$

where $C_T(\mathbf{z})$ is some finite constant depending on T and $\mathbf{z}$ that is locally bounded in $\mathbf{z}$. As a result, the Markov inequality implies that for any $H > 0$,

$$\mathbb{P}_{\mathbf{z}}\Big\{ \sup_{t\in[0,T]} |\mathbf{Z}(t)| > H\Big\} \le \frac{\mathbb{E}_{\mathbf{z}} \sup_{t\in[0,T]} |\mathbf{Z}(t)|}{H} < \frac{C_T(\mathbf{z})}{H},$$

which yields that for any $T > 0$, $\varepsilon > 0$, and $R > 0$, there is an $H_{1,T,\varepsilon,R} > 0$ such that

$$\mathbb{P}_{\mathbf{z}}\Big\{ \sup_{t\in[0,T]} |\mathbf{Z}(t)| \le H_{1,T,\varepsilon,R}\Big\} > 1-\varepsilon, \text{ for all } |\mathbf{z}| \le R.$$

The proof is complete. □

Lemma 2.2. *For any $T > 0$, $\varepsilon > 0$, $R > 0$, and $\delta_1 > 0$, there exists $\delta_2 = \delta_2(T, \varepsilon, R, \delta_1) > 0$ such that*

$$\mathbb{P}_{\mathbf{z}}\Big\{ \sup_{t\in[0,T]} |\mathbf{X}_2(t)| \le \delta_1\Big\} > 1-\varepsilon, \textit{ for all } |\mathbf{x}_1| \le R, |\mathbf{x}_2| \le \delta_2.$$

Proof. Because of (2.4), there is an $L = L(\delta_1)$ such that whenever $U(\mathbf{x}_2) > L(\delta_1)$, $|\mathbf{x}_2| < \delta_1$. By the generalized Itô formula, we have

$$\begin{aligned} U(\mathbf{X}_2(t)) =& U(\mathbf{x}_2) + \int_0^t \mathcal{L}U(\mathbf{Z}(u))du + \int_0^t U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u-))d\mathbf{W}_2(u) \\ &+ \int_0^t \int_{\mathbb{R}^{n_2}_*} [U(\mathbf{X}_2(u-) + \gamma_2(\mathbf{Z}(u-), \boldsymbol{\phi})) - U(\mathbf{X}_2(u-))]\widetilde{\mathbf{N}}_2(du, d\boldsymbol{\phi}). \end{aligned} \tag{2.11}$$

Let $H(\mathbf{z}, \boldsymbol{\phi}) = U(\mathbf{x}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})) - U(\mathbf{x}_2)$, $\mathbf{z} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$, and α be such that $0 < \alpha < \frac{\alpha_0}{2}$, where α_0 is in Assumption 2.2. From the exponential martingale inequality (see e.g., [1, Theorem 5.2.9]), we have that with probability greater $1-\varepsilon$ the following inequality holds for all $t \in [0, T]$,

$$
\begin{aligned}
&-\int_0^t U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))d\mathbf{W}_2(u) - \int_0^t \int_{\mathbb{R}_*^{n_2}} H(\mathbf{Z}(u),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du,d\boldsymbol{\phi})\\
&\le \tfrac{\alpha}{2}\textstyle\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du + \tfrac{1}{\alpha}\int_0^t\int_{\mathbb{R}_*^{n_2}}\Big(e^{\alpha H(\mathbf{Z}(u),\boldsymbol{\phi})}-1-\alpha H(\mathbf{Z}(u),\boldsymbol{\phi})\Big)\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \tfrac{\ln\varepsilon}{\alpha}\\
&\le \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du + \frac{\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} H^2(\mathbf{Z}(u),\boldsymbol{\phi})e^{\frac{\alpha_0}{2}|H(\mathbf{Z}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha}\\
&\le \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\mathbf{Z}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha},
\end{aligned}
$$

where C_{α_0} is some finite constant such that $t^2e^{\frac{\alpha_0 t}{2}} \le C_{\alpha_0}e^{\alpha_0 t}, \forall t \ge 0$. As a result, we have

$$
\begin{aligned}
&\mathbb{P}\Bigg\{\int_0^t U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))d\mathbf{W}_2(u) + \int_0^t\int_{\mathbb{R}_*^{n_2}} H(\mathbf{Z}(u),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du,d\boldsymbol{\phi})\\
&\ge -\frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du - \frac{C_{\alpha_0}\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\mathbf{Z}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du + \frac{\ln\varepsilon}{\alpha},\ \forall t\in[0,T]\Bigg\}\\
&\ge 1-\varepsilon.
\end{aligned}
\tag{2.12}
$$

We obtain from Assumption 2.2 (ii) that

$$
\begin{aligned}
&-\int_0^t [\mathcal{L}U](\mathbf{Z}(u))du + \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\mathbf{Z}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du\\
&\qquad \le C_{\alpha,\alpha_0}\int_0^t V_1(\mathbf{X}_1(u),\mathbf{0})du.
\end{aligned}
$$

In addition, by Lemma 2.1, with probability greater than $1-\varepsilon$, $|\mathbf{Z}(t)| \le H_{1,T,\varepsilon,R}$, $\forall t \in [0,T]$. As a result, with probability greater $1-2\varepsilon$, we have

$$
\begin{aligned}
&-\int_0^t [\mathcal{L}U](\mathbf{Z}(u))du + \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\mathbf{X}_2(u))\sigma_2(\mathbf{Z}(u))|^2du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\mathbf{Z}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du\\
&\qquad \le C_{T,R,\varepsilon,\alpha,\alpha_0},
\end{aligned}
\tag{2.13}
$$

for some finite constant $C_{T,R,\varepsilon,\alpha,\alpha_0}$. The combination of (2.11), (2.12), and (2.13) yields that with probability greater $1-2\varepsilon$,

$$U(\mathbf{X}_2(t)) \geq U(\mathbf{x}_2) - C_{T,R,\varepsilon,\alpha,\alpha_0} + \frac{\ln\varepsilon}{\alpha}, \quad \forall t \in [0,T].$$

Because of (2.4), there exists a $\delta_2 > 0$ such that for all $|\mathbf{x}_2| < \delta_2$,

$$U(\mathbf{x}_2) - C_{T,R,\varepsilon,\alpha,\alpha_0} + \frac{\ln\varepsilon}{\alpha} > L(\delta_1).$$

Therefore, with probability greater $1-2\varepsilon$, $\sup_{t\in[0,T]} U(\mathbf{X}_2(t)) \geq L(\delta_1)$, for all $|\mathbf{x}_1| < R$, $|\mathbf{x}_2| < \delta_2$, and thus, for all $|\mathbf{x}_1| < R$, $|\mathbf{x}_2| < \delta_2$, $\sup_{t\in[0,T]} |\mathbf{X}_2(t)| \leq \delta_1,\ \forall t \in [0,T]$. □

Let $\lambda > 0$ such that $\lambda > 20(1+K_1)$ with K_1 being the Lipschitz constant in Assumption 2.1. Consider the following coupled equations

$$\begin{cases} d\mathbf{X}_1(t) = b_1(\mathbf{X}_1(t),\mathbf{0})dt + \sigma_1(\mathbf{X}_1(t),\mathbf{0})d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t),\mathbf{0},\boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt,d\boldsymbol{\phi}), \\ d\widetilde{\mathbf{X}}_1(t) = b_1(\widetilde{\mathbf{Z}}(t))dt + \lambda(\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t))dt + \sigma_1(\widetilde{\mathbf{Z}}(t))d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\widetilde{\mathbf{Z}}(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt,d\boldsymbol{\phi}), \\ d\widetilde{\mathbf{X}}_2(t) = b_2(\widetilde{\mathbf{Z}}(t))dt + \sigma_2(\widetilde{\mathbf{Z}}(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}_*^{n_2}} \gamma_2(\widetilde{\mathbf{Z}}(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}), \\ \mathbf{X}_1(0) = \mathbf{x}_1, \widetilde{\mathbf{X}}_1(0) = \widetilde{\mathbf{x}}_1, \widetilde{\mathbf{X}}_2(0) = \widetilde{\mathbf{x}}_2, \end{cases} \tag{2.14}$$

where $\widetilde{\mathbf{Z}}(t) = ([\widetilde{\mathbf{X}}_1(t)]^\top,[\widetilde{\mathbf{X}}_2(t)]^\top)^\top$, $\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}_1^\top,\widetilde{\mathbf{x}}_2^\top)^\top$. We will use $\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}$ to indicate the initial conditions of the coupled equations (2.14).

Denote $\Lambda_2 := \int f_2(\mathbf{x}_1)\mu^*(d\mathbf{x}_1)$, where μ^* is the unique invariant measure of $\mathbf{X}_1(t)$ when $\mathbf{X}_2(t) = 0$ as in Assumption 2.2 (iii). Since $f_2(\cdot)$ satisfies condition (2.6), it is a positive function, and as a result, $\Lambda_2 > 0$. Let $C_{\alpha_0} > 1$ be a constant as in Lemma 2.2, i.e., a constant satisfying $t^2 e^{\frac{\alpha_0 t}{2}} \leq C_{\alpha_0} e^{\alpha_0 t}, \forall t \geq 0$. In the remainder of this section, let

$$\gamma_0 \in (0, \frac{\Lambda_1}{m_0}),\ \varsigma_0 = \frac{\Lambda_1 - m_0\gamma_0}{3} > 0,\ \alpha = \frac{\varsigma_0}{2C_{\alpha_0}\Lambda_2},\ \lambda_0 \in (0, \frac{\gamma_0}{4}),$$

and for each $\delta > 0$,

$$\widetilde{\tau}_\delta := \inf\{t > 0 : |\widetilde{\mathbf{X}}_2(t)| \geq \delta e^{-\gamma_0 t}\}.$$

To proceed, we present the following lemmas. Lemma 2.3 provides estimates of the coupling time of the coupled processes, while Lemma 2.4 handles the diffusion and jump parts.

Remark 2.3. As was mentioned, one of the main challenges in this work is the coupled interaction of $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$.

To overcome this difficulty, we modify the generalized coupling method in [20,26]. The coupling will be done until a stopping time $\widetilde{\tau}_\delta := \inf\{t > 0 : |\widetilde{\mathbf{X}}_2(t)| \geq \delta e^{-\gamma_0 t}\}$, which is defined as above. In contrast to existing literature, we do not couple until $|\widetilde{\mathbf{X}}_2(t)|$ exceeds a certain constant, which is not useful for our purpose. So we couple until a time $|\widetilde{\mathbf{X}}_2(t)|$ exceeds an exponential decay and then prove that time is infinite with a large probability. However, it is much harder to prove that $\widetilde{\tau}_\delta = \infty$ with a large probability that $\inf\{t > 0 : |\widetilde{\mathbf{X}}_2(t)| \geq \delta\} = \infty$ with a large probability. We will need to carefully look at the interaction of two components in the coupled system to show that $\widetilde{\tau}_\delta = \infty$ with a large probability.

Lemma 2.3. *There is a universal constant* $\widetilde{C} > 1$ *such that*

$$\mathbb{E}\Big(\sup_{t\le\widetilde{\tau}_\delta} e^{\lambda_0 t}|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)|^2\Big) \le \widetilde{C}(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2. \tag{2.15}$$

As a result, for any $\varepsilon > 0$ *one has*

$$\mathbb{P}\Bigg\{\int_0^{t\wedge\widetilde{\tau}_\delta}|v(s)|^2ds \ge \frac{(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\varepsilon} \text{ for some } t\ge 0\Bigg\} \le \frac{\widetilde{C}\lambda c_\sigma \varepsilon}{\lambda_0}, \tag{2.16}$$

where c_σ *is in Assumption 2.2 and*

$$v(t) := \lambda\sigma_1^{-1}(\widetilde{\mathbf{X}}_1(t),\widetilde{\mathbf{X}}_2(t))(\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)). \tag{2.17}$$

Proof. Use C to denote a finite constant, whose values may change at difference appearances. By the generalized Itô formula for jump diffusions, we have that

$$\begin{aligned}
&e^{\lambda_0 t}\big|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)\big|^2\\
&= |\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^2 + \int_0^t e^{\lambda_0 s}(\lambda_0-\lambda)\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^2 ds\\
&\quad + 2\int_0^t e^{\lambda_0 s}\big(\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big)^\top\big(b_1(\mathbf{X}_1(s),\mathbf{0})-b_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s))\big)ds\\
&\quad + \int_0^t e^{\lambda_0 s}\operatorname{tr}\Big[\big(\sigma_1(\mathbf{X}_1(s),\mathbf{0})-\sigma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s))\big)\big(\sigma_1(\mathbf{X}_1(s),\mathbf{0})-\sigma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s))\big)^\top\Big]ds\\
&\quad + \int_0^t e^{\lambda_0 s}\int_{\mathbb{R}_*^{n_1}}\Big[\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)+\gamma_1(\mathbf{X}_1(s-),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big|^2-\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^2\\
&\qquad\qquad -2\big(\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big)^\top\big(\gamma_1(\mathbf{X}_1(s),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big)\Big]\boldsymbol{\nu}_1(d\boldsymbol{\phi})ds\\
&\quad + 2\int_0^t e^{\lambda_0 s}\big(\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)\big)^\top\big(\sigma_1(\mathbf{X}_1(s-),\mathbf{0})-\sigma_1(\widetilde{\mathbf{X}}_1(s-),\widetilde{\mathbf{X}}_2(s-))\big)d\mathbf{W}_1(s)\\
&\quad + \int_0^t e^{\lambda_0 s}\int_{\mathbb{R}_*^{n_1}}\Big[\big|\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)+\gamma_1(\mathbf{X}_1(s-),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s-),\widetilde{\mathbf{X}}_2(s-),\boldsymbol{\phi})\big|^2\\
&\qquad\qquad -\big|\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)\big|^2\Big]\widetilde{\mathbf{N}}_1(ds,d\boldsymbol{\phi}).
\end{aligned} \tag{2.18}$$

By virtue of Kunita's first inequality [1, Theorem 4.4.23, p. 265], for all $T \geq 0$,

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\widetilde{\tau}_\delta} \int_{\mathbb{R}_*^{n_1}} e^{\lambda_0 s}\Big[\big|\mathbf{X}_1(s-) - \widetilde{\mathbf{X}}_1(s-) + \gamma_1(\mathbf{X}_1(s-), \mathbf{0}, \boldsymbol{\phi}) - \gamma_1(\widetilde{\mathbf{X}}_1(s-), \widetilde{\mathbf{X}}_2(s-), \boldsymbol{\phi})\big|^2 \\
&\qquad - \big|\mathbf{X}_1(s-) - \widetilde{\mathbf{X}}_1(s-)\big|^2\Big]\widetilde{\mathbf{N}}_1(ds, d\boldsymbol{\phi})\Big|^2 \\
&\leq C\mathbb{E} \int_0^{T\wedge\widetilde{\tau}_\delta} \int_{\mathbb{R}_*^{n_1}} e^{2\lambda_0 s}\Big[\big|\mathbf{X}_1(s-) - \widetilde{\mathbf{X}}_1(s-) + \gamma_1(\mathbf{X}_1(s-), \mathbf{0}, \boldsymbol{\phi}) - \gamma_1(\widetilde{\mathbf{X}}_1(s-), \widetilde{\mathbf{X}}_2(s-), \boldsymbol{\phi})\big|^2 \\
&\qquad - \big|\mathbf{X}_1(s-) - \widetilde{\mathbf{X}}_1(s-)\big|^2\Big]^2 \boldsymbol{\nu}_1(d\boldsymbol{\phi})ds \\
&\leq C\mathbb{E}\Big(\int_0^{T\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s) - \widetilde{\mathbf{X}}_1(s)\big|^4 ds + \delta^4 \int_0^{T\wedge\widetilde{\tau}_\delta} e^{(-4\gamma_0+2\lambda_0)s} ds\Big) \\
&\leq C\Big(\delta^4 + \mathbb{E} \int_0^{T\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s) - \widetilde{\mathbf{X}}_1(s)\big|^4 ds\Big).
\end{aligned} \tag{2.19}$$

On the other hand, the Burkholder-Davis-Gundy inequality [33, Theorem 2.13, p. 70] leads to

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\widetilde{\tau}_\delta} e^{\lambda_0 s}\big(\mathbf{X}_1(s-) - \widetilde{\mathbf{X}}_1(s-)\big)^\top \big(\sigma_1(\mathbf{X}_1(s-), \mathbf{0}) - \sigma_1(\widetilde{\mathbf{X}}_1(s-), \widetilde{\mathbf{X}}_2(s-))\big) d\mathbf{W}_1(s)\Big|^2 \\
&\leq C\mathbb{E}\Big(\int_0^{T\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s) - \widetilde{\mathbf{X}}_1(s)\big|^4 ds + \delta^4 \int_0^{T\wedge\widetilde{\tau}_\delta} e^{(-4\gamma_0+2\lambda_0)s} ds\Big) \\
&\leq C\Big(\delta^4 + \mathbb{E} \int_0^{T\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s) - \widetilde{\mathbf{X}}_1(s)\big|^4 ds\Big).
\end{aligned} \tag{2.20}$$

Now, applying (2.19) and (2.20) to (2.18), and using the Lipschitz continuity of $b_1(\cdot, \cdot)$, $\sigma_1(\cdot, \cdot)$, and $\gamma_1(\cdot, \cdot, \cdot)$, we obtain that

$$\begin{aligned}
&\mathbb{E} \sup_{t\leq T\wedge\widetilde{\tau}_\delta} e^{\lambda_0 t}\big|\mathbf{X}_1(t) - \widetilde{\mathbf{X}}_1(t)\big|^2 \\
&\qquad \leq |\mathbf{x}_1 - \widetilde{\mathbf{x}}_1|^2 + C\Big(\delta^2 + \int_0^T e^{(2\lambda_0-\gamma_0)s} ds + \Big[\mathbb{E} \int_0^{T\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s) - \widetilde{\mathbf{X}}_1(s)\big|^4 ds\Big]^{\frac{1}{2}}\Big).
\end{aligned} \tag{2.21}$$

To proceed, we estimate $\mathbb{E}\int_0^{t\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^4 ds$. Using the generalized Itô formula again, we have

$$
\begin{aligned}
&e^{2\lambda_0 t}\big|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)\big|^4\\
&=|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^4+\int_0^t e^{2\lambda_0 s}(2\lambda_0-\lambda)\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^4 ds\\
&+\int_0^t 4e^{2\lambda_0 s}\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^2\big(\mathbf{X}(s)-\widetilde{\mathbf{X}}_1(s)\big)^\top\big(b_1(\mathbf{X}_1(s),\mathbf{0})-b_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s))\big)ds\\
&+3\int_0^t e^{2\lambda_0 s}\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^2\\
&\quad\times \operatorname{tr}\Big[\big(\sigma_1(\mathbf{X}_1(s),\mathbf{0},\boldsymbol{\phi})-\sigma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big)\big(\sigma_1(\mathbf{X}_1(s),\mathbf{0},\boldsymbol{\phi})-\sigma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big)^\top\Big]ds\\
&+\int_0^t\int_{\mathbb{R}_*^{n_1}} e^{2\lambda_0 s}\Big[\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)+\gamma_1(\mathbf{X}_1(s-),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big|^4-\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^4\\
&\quad-4\big|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big|^2\big(\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\big)^\top\big(\gamma_1(\mathbf{X}_1(s),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s),\widetilde{\mathbf{X}}_2(s),\boldsymbol{\phi})\big)\Big]\boldsymbol{\nu}_1(d\boldsymbol{\phi})ds\\
&+4\int_0^t e^{2\lambda_0 s}\big|\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)\big|^2\\
&\quad\times\big(\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)\big)^\top\big(\sigma_1(\mathbf{X}_1(s-),\mathbf{0})-\sigma_1(\widetilde{\mathbf{X}}_1(s-),\widetilde{\mathbf{X}}_2(s-))\big)d\mathbf{W}_1(s)\\
&+\int_0^t\int_{\mathbb{R}_*^{n_1}} e^{2\lambda_0 s}\Big[\big|\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)+\gamma_1(\mathbf{X}_1(s-),\mathbf{0},\boldsymbol{\phi})-\gamma_1(\widetilde{\mathbf{X}}_1(s-),\widetilde{\mathbf{X}}_2(s-),\boldsymbol{\phi})\big|^4\\
&\quad-\big|\mathbf{X}_1(s-)-\widetilde{\mathbf{X}}_1(s-)\big|^4\Big]\widetilde{\mathbf{N}}_1(ds,d\boldsymbol{\phi}).
\end{aligned}
\tag{2.22}
$$

Taking expectation on both sides of (2.22), using the Lipschitz continuity of $b_1(\cdot,\cdot)$, $\sigma_1(\cdot,\cdot)$, $\gamma_1(\cdot,\cdot,\cdot)$, and noting λ being chosen to be sufficiently large, we obtain

$$
\begin{aligned}
&d\Big[\mathbb{E}e^{2\lambda_0(t\wedge\widetilde{\tau}_\delta)}\big|\mathbf{X}_1(t\wedge\widetilde{\tau}_\delta)-\widetilde{\mathbf{X}}_1(t\wedge\widetilde{\tau}_\delta)\big|^4\Big]\\
&\quad\le\mathbb{E}\Big[-D_1e^{2\lambda_0(t\wedge\widetilde{\tau}_\delta)}\big|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)\big|^4+D_2\delta^4e^{(-4\delta_0+2\lambda_0)(t\wedge\widetilde{\tau}_\delta)}\Big]dt,
\end{aligned}
\tag{2.23}
$$

for some finite positive constants D_1 and D_2. Integrating Eq. (2.23) implies that

$$\mathbb{E}e^{2\lambda_0(t\wedge\widetilde{\tau}_\delta)}\left|\mathbf{X}_1(t\wedge\widetilde{\tau}_\delta)-\widetilde{\mathbf{X}}_1(t\wedge\widetilde{\tau}_\delta)\right|^4-|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^4$$
$$\leq -D_1\mathbb{E}\int_0^{t\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\left|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\right|^4 ds + D_2\delta^4\int_0^t e^{-2\lambda_0 s}ds$$

or

$$\mathbb{E}\int_0^{t\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\left|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\right|^4 ds \leq C\left(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^4+\delta^4\right) \text{ for all } t\geq 0,$$

and thus,

$$\left[\mathbb{E}\int_0^{t\wedge\widetilde{\tau}_\delta} e^{2\lambda_0 s}\left|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)\right|^4 ds\right]^{\frac{1}{2}} \leq C\left(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^2+\delta^2\right) \text{ for all } t\geq 0. \tag{2.24}$$

Combining (2.24) and (2.21), we get that

$$\mathbb{E}\sup_{t\leq T\wedge\widetilde{\tau}_\delta} e^{\lambda_0 t}\left|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)\right|^2 \leq C\left(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|^2+\delta^2\right) \text{ for all } T\geq 0. \tag{2.25}$$

Therefore, (2.15) is proved.

Now, we consider the second part.

By virtue of the definition of $v(t)$ in (2.17) and Assumption 2.2 (iv),

$$\begin{aligned}
&\mathbb{P}\left\{\int_0^{t\wedge\widetilde{\tau}_\delta}|v(s)|^2ds \geq \frac{(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\varepsilon} \text{ for some } t\geq 0\right\}\\
&\leq \mathbb{P}\left\{\int_0^{\widetilde{\tau}_\delta}\left|\lambda\sigma_1^{-1}(\mathbf{X}_1(s),\widetilde{\mathbf{X}}_2(s))(\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s))\right|^2ds \geq \frac{(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\varepsilon}\right\}\\
&\leq \mathbb{P}\left\{\int_0^{\widetilde{\tau}_\delta}|\mathbf{X}_1(s)-\widetilde{\mathbf{X}}_1(s)|^2ds \geq \frac{(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\lambda c_\sigma\varepsilon}\right\}.
\end{aligned} \tag{2.26}$$

A standard calculation shows that for the integrable function $h(s)$,

$$\int_0^t h(s)ds = \int_0^t \frac{e^{\lambda_0 s}h(s)}{e^{\lambda_0 s}}ds \leq \sup_{s\in[0,t]} e^{\lambda_0 s}h(s)\int_0^t e^{-\lambda_0 s}ds \leq \frac{1}{\lambda_0}\sup_{s\in[0,t]} e^{\lambda_0 s}h(s).$$

Therefore, it follows from (2.26) that

$$
\begin{aligned}
&\mathbb{P}\left\{ \int_0^{t\wedge\widetilde{\tau}_\delta} |v(s)|^2 ds \geq \frac{(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\varepsilon} \text{ for some } t\geq 0\right\} \\
&\leq \mathbb{P}\left\{ \sup_{t\leq\widetilde{\tau}_\delta} e^{\lambda_0 t}|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)|^2 \geq \frac{\lambda_0\,(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}{\lambda c_\sigma \varepsilon}\right\} \\
&\leq \frac{\lambda c_\sigma\varepsilon}{\lambda_0(|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta)^2}\mathbb{E}\sup_{t\leq\widetilde{\tau}_\delta} e^{\lambda_0 t}|\mathbf{X}_1(t)-\widetilde{\mathbf{X}}_1(t)|^2 \\
&\leq \frac{\widetilde{C}\lambda c_\sigma\varepsilon}{\lambda_0} \text{ (due to (2.15)).}
\end{aligned}
\tag{2.27}
$$

As a result, the proof is complete. □

Lemma 2.4. *Let $\widetilde{C}$ be as in Lemma 2.3 and L be a Lipschiz constant of $f_1(\cdot)$ and $f_2(\cdot)$. Suppose $|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+\delta<1$. Then we have*

$$
\begin{aligned}
&\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}\bigg\{ \int_0^t U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))d\mathbf{W}_2(u) + \int_0^t\int_{\mathbb{R}_*^{n_2}} H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du,d\boldsymbol{\phi}) \\
&\qquad \geq -C_{\alpha_0}\alpha\int_0^t f_2(\mathbf{X}_1(u))du + \frac{\ln\varepsilon}{\alpha} - \frac{2L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0},\ \forall 0\leq t\leq\widetilde{\tau}_\delta\bigg\} \geq 1-2\varepsilon.
\end{aligned}
$$

Proof. Let $H(\mathbf{z},\boldsymbol{\phi}) = U(\mathbf{x}_2+\gamma_2(\mathbf{z},\boldsymbol{\phi})) - U(\mathbf{x}_2)$. From the exponential martingale inequality (see e.g., [1, Theorem 5.2.9]), we have with probability greater than $1-\varepsilon$,

$$
\begin{aligned}
&-\int_0^t U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))d\mathbf{W}_2(u) - \int_0^t\int_{\mathbb{R}_*^{n_2}} H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du,d\boldsymbol{\phi}) \\
&\leq \tfrac{\alpha}{2}\textstyle\int_0^t |U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))|^2du + \frac{1}{\alpha}\int_0^t\int_{\mathbb{R}_*^{n_2}}\left(e^{\alpha H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})} - 1 - \alpha H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})\right)\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha} \\
&\leq \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))|^2du + \frac{\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} H^2(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})e^{\frac{\alpha_0}{2}|H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha} \\
&\leq \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))|^2du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t\int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha}.
\end{aligned}
$$

As a result, $\mathbb{P}(\Omega_1)\geq 1-\varepsilon$, where

$$\Omega_1 := \Bigg\{ -\int_0^t U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))d\mathbf{W}_2(u) - \int_0^t \int_{\mathbb{R}_*^{n_2}} H(\widetilde{\mathbf{Z}}(u), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du, d\boldsymbol{\phi})$$
$$\leq \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))|^2 du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t \int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha}\Bigg\}.$$

On the other hand, by (2.15), one has $\mathbb{P}(\Omega_2) \geq 1-\varepsilon$, where

$$\Omega_2 := \left\{ \sup_{t\leq\widetilde{\tau}_\delta} e^{\lambda_0 t}|\mathbf{X}_1(t) - \widetilde{\mathbf{X}}_1(t)|^2 \leq \frac{\widetilde{C}}{\varepsilon}\right\}.$$

For $t \leq \widetilde{\tau}_\delta$ and $\omega \in \Omega_1 \cap \Omega_2$, we have

$$\begin{aligned}
&-\int_0^t U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))d\mathbf{W}_2(u) - \int_0^t \int_{\mathbb{R}_*^{n_2}} H(\widetilde{\mathbf{Z}}(u), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(du, d\boldsymbol{\phi})\\
&\quad\leq \frac{\alpha}{2}\int_0^t |U_{\mathbf{x}_2}(\widetilde{\mathbf{X}}_2(u))\sigma_2(\widetilde{\mathbf{Z}}(u))|^2 du + \frac{C_{\alpha_0}\alpha}{2}\int_0^t \int_{\mathbb{R}_*^{n_2}} e^{\alpha_0|H(\widetilde{\mathbf{Z}}(u),\boldsymbol{\phi})|}\boldsymbol{\nu}_2(d\boldsymbol{\phi})du - \frac{\ln\varepsilon}{\alpha}\\
&\quad\leq C_{\alpha_0}\alpha\int_0^t f_2(\widetilde{\mathbf{X}}_1(u))du - \frac{\ln\varepsilon}{\alpha}\\
&\quad\leq C_{\alpha_0}\alpha\int_0^t f_2(\mathbf{X}_1(u))du - \frac{\ln\varepsilon}{\alpha} + L\int_0^t |\mathbf{X}_1(u) - \widetilde{\mathbf{X}}_1(u)|du\\
&\quad\leq C_{\alpha_0}\alpha\int_0^t f_2(\mathbf{X}_1(u))du - \frac{\ln\varepsilon}{\alpha} + \frac{L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}}\int_0^t e^{-\frac{\lambda_0 u}{2}}du\\
&\quad\leq C_{\alpha_0}\alpha\int_0^t f_2(\mathbf{X}_1(u))du - \frac{\ln\varepsilon}{\alpha} + \frac{2L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0}.
\end{aligned} \tag{2.28}$$

Therefore, the proof is complete. □

Proof of Theorem 2.1. We can assume that $\varepsilon \in (0, \frac{1}{2})$ and $e^{-3\varepsilon} \geq 1-4\varepsilon$. Let

$$\delta = \Delta_0 \wedge \left(\frac{1}{2e}\left(\frac{\varepsilon^3}{-\ln\varepsilon} \wedge \frac{\varepsilon^2}{2} \wedge \frac{\lambda_0}{2\widetilde{C}\lambda c_\sigma}\right)\right), \tag{2.29}$$

where $\widetilde{C}$ is in Lemma 2.3. We have from the definitions of Λ_1, Λ_2 that

$$\mathbb{P}_{\mathbf{x}_1,\mathbf{0}}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f_1(\mathbf{X}_1(u))du=\Lambda_1\right\}=\mathbb{P}_{\mathbf{x}_1,\mathbf{0}}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f_2(\mathbf{X}_1(u))du=\Lambda_2\right\}=1.$$

It is noted that $m_0\gamma_0+2\varsigma_0<\Lambda_1$ so that there exists $T_{\mathbf{x}_1,\varepsilon}\geq\frac{1}{\varsigma_0}\left(\frac{-\ln\varepsilon}{\alpha}+\frac{4L\sqrt{C}}{\sqrt{\varepsilon}\lambda_0}\right)$ such that $\mathbb{P}_{\mathbf{x}_1,\mathbf{0}}(\Omega_3)\geq 1-\varepsilon$, where

$$\Omega_3:=\left\{\frac{1}{t}\int_0^t f_1(\mathbf{X}_1(u))du\geq m_0\gamma_0+2\varsigma_0\text{ and }\frac{1}{t}\int_0^t f_2(\mathbf{X}_1(u))du\leq\Lambda_2,\ t\geq T_{\mathbf{x}_1,\varepsilon}\right\}.$$

Let $M_{\mathbf{x}_1,\varepsilon}>0$ be sufficient large such that $\mathbb{P}_{\mathbf{x}_1,\mathbf{0}}(\Omega_4)\geq 1-\varepsilon$, where

$$\Omega_4:=\left\{\frac{1}{t}\int_0^t f_1(\mathbf{X}_1(u))du-\frac{C_{\alpha_0}\alpha}{t}\int_0^t f_2(\mathbf{X}_1(u))du+\frac{\ln\varepsilon}{\alpha}-\frac{4L\sqrt{C}}{\sqrt{\varepsilon}\lambda_0}\geq -M_{\mathbf{x}_1,\varepsilon},\ t\leq T_{\mathbf{x}_1,\varepsilon}\right\},$$

and Ω_1, Ω_2 be as in Lemma 2.4. By the generalized Itô formula, we have

$$\begin{aligned}U(\widetilde{\mathbf{X}}_2(t))=&U(\widetilde{\mathbf{x}}_2)+\int_0^t\mathcal{L}U(\widetilde{\mathbf{Z}}(u))du+\int_0^t U_{\mathbf{x}_2}\sigma_2(\widetilde{\mathbf{Z}}(u-))d\mathbf{W}_2(u)\\&+\int_0^t\int_{\mathbb{R}^{n_2}_*}[U(\widetilde{\mathbf{X}}_2(u-)+\gamma_2(\widetilde{\mathbf{Z}}(u-),\boldsymbol{\phi}))-U(\widetilde{\mathbf{X}}_2(u-))]\widetilde{\mathbf{N}}_2(du,d\boldsymbol{\phi}).\end{aligned}\tag{2.30}$$

Definitions of Ω_1, Ω_2, and Ω_4 lead to

$$\begin{aligned}\int_0^t\mathcal{L}U(\widetilde{\mathbf{Z}}(u))du&\geq\int_0^t\mathcal{L}U(\widetilde{X}_1(u))du\geq\int_0^t\mathcal{L}U(X_1(u))du+L\int_0^t|X_1(u)-\widetilde{X}_1(u)|du\\&\geq\int_0^t\mathcal{L}U(X_1(u))du+\frac{L\sqrt{C}}{\sqrt{\varepsilon}}\int_0^t e^{-\frac{\lambda_0 u}{2}}du\\&\geq\int_0^t\mathcal{L}U(X_1(u))du+\frac{2L\sqrt{C}}{\sqrt{\varepsilon}\lambda_0},t\leq\widetilde{\tau}_\delta.\end{aligned}\tag{2.31}$$

We deduce from (2.28), (2.31), (2.30) and the definition of Ω_4 that

$$\begin{aligned}U(\widetilde{\mathbf{X}}_2(t))\geq&U(\widetilde{\mathbf{x}}_2)+\frac{1}{t}\int_0^t f_1(\mathbf{X}_1(u))du-\frac{C_{\alpha_0}\alpha}{t}\int_0^t f_2(\mathbf{X}_1(u))du+\frac{\ln\varepsilon}{\alpha}-\frac{4L\sqrt{C}}{\sqrt{\varepsilon}\lambda_0}\\\geq&U(\widetilde{\mathbf{x}}_2)-M_{\mathbf{x}_1,\varepsilon},\ \text{ for all }t\leq T_{\mathbf{x}_1,\varepsilon}\wedge\widetilde{\tau}_\delta,\ \omega\in\Omega_1\cap\Omega_2\cap\Omega_4.\end{aligned}\tag{2.32}$$

Let $\theta_{\mathbf{x}_1,\varepsilon} \in (0, \frac{\delta}{2})$ such that $U(\mathbf{x}_2) - U(\mathbf{x}_2') > M_{\mathbf{x}_1,\varepsilon}$ if $|\mathbf{x}_2| \le \theta_{\mathbf{x}_1,\varepsilon}, |\mathbf{x}_2'| \ge \delta e^{-T_{\mathbf{x}_1,\varepsilon}}$. Such a $\theta_{\mathbf{x}_1,\varepsilon}$ exists owing to (2.4). For $|\widetilde{\mathbf{x}}_2| < \theta_{\mathbf{x}_1,\varepsilon}$ and $\omega \in \Omega_2 \cap \Omega_3$, we must have $\widetilde{\tau}_\delta > T_{\mathbf{x}_1,\varepsilon}$. Otherwise, if $\widetilde{\tau}_\delta \le T_{\mathbf{x}_1,\varepsilon}$, we have $U(\widetilde{\mathbf{X}}_2(\widetilde{\tau}_\delta)) \ge U(\widetilde{\mathbf{x}}_2) - M_{\mathbf{x}_1,\varepsilon}$, which implies that $|\widetilde{\mathbf{X}}_2(\widetilde{\tau}_\delta)| < \delta e^{-T_{\mathbf{x}_1,\varepsilon}}$, and again contradicts to the definition of $\widetilde{\tau}_\delta$.

For $\omega \in \cap_{i=1}^4 \Omega_i$, we have $\widetilde{\tau}_\delta > T_{\mathbf{x}_1,\varepsilon}$. From (2.30), Assumption 2.2, and Lemma 2.4, one has

$$\begin{aligned}
U(\widetilde{\mathbf{X}}_2(t)) &\ge U(\widetilde{\mathbf{x}}_2) + \int_0^t f_1(\widetilde{\mathbf{X}}_1(u))du - C_{\alpha_0}\alpha \int_0^t f_2(\mathbf{X}_1(u))du + \frac{\ln\varepsilon}{\alpha} - \frac{2L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0} \\
&\ge U(\widetilde{\mathbf{x}}_2) + \int_0^t f_1(\mathbf{X}_1(u))du - L\int_0^t |\mathbf{X}_1(u) - \widetilde{\mathbf{X}}_1(u)|du \\
&\quad - C_{\alpha_0}\alpha \int_0^t f_2(\mathbf{X}_1(u))du + \frac{\ln\varepsilon}{\alpha} - \frac{2L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0} \qquad (2.33) \\
&\ge U(\widetilde{\mathbf{x}}_2) + (m_0\gamma_0 + 2\varsigma_0)t - \frac{L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}} \int_0^t e^{-\frac{\lambda_0 u}{2}}du - C_{\alpha_0}\alpha\Lambda_2 t + \frac{\ln\varepsilon}{\alpha} - \frac{2L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0} \\
&\ge U(\widetilde{\mathbf{x}}_2) + m_0\gamma_0 t + \varsigma_0 T_{\mathbf{x}_1,\varepsilon} + \frac{\ln\varepsilon}{\alpha} - \frac{4L\sqrt{\widetilde{C}}}{\sqrt{\varepsilon}\lambda_0} \\
&\ge U(\widetilde{\mathbf{x}}_2) + m_0\gamma_0 t, \text{ for all } \omega \in \cap_{i=1}^4 \Omega_i, t \in [T_{\mathbf{x}_1,\varepsilon}, \widetilde{\tau}_\delta).
\end{aligned}$$

The combination of (2.33) and (2.4) implies that $\ln|\widetilde{\mathbf{X}}_2(t)| \le \ln|\widetilde{\mathbf{x}}_2| - \gamma_0 t + 1, \forall t \in [T_{\mathbf{x}_1,\varepsilon}, \widetilde{\tau}_\delta)$, so that $|\widetilde{\mathbf{X}}_2(t)| < \frac{\delta}{2}e^{-\gamma_0 t}, t \in [T_{\mathbf{x}_1,\varepsilon}, \widetilde{\tau}_\delta)$. As a result, if $\widetilde{\mathbf{x}}_2 < \theta_{\mathbf{x}_1,\varepsilon}$, $\widetilde{\tau}_\delta = \infty$ for all $\omega \in \cap_{i=1}^4 \Omega_i$. It yields that for all $\omega \in \cap_{i=1}^4 \Omega_i$,

$$\liminf_{t\to\infty} \frac{U(\widetilde{\mathbf{X}}_2(t))}{t} \ge m_0\gamma_0 \text{ and } \limsup_{t\to\infty} \frac{\ln|\widetilde{\mathbf{X}}_2(t)|}{t} \le -\gamma_0.$$

An application of Lemma 2.3 leads to $\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}(\Omega_5) \ge 1 - \varepsilon$, where

$$\Omega_5 := \left\{ \int_0^{t\wedge\widetilde{\tau}_\delta} |v(s)|^2 ds \le \frac{\widetilde{C}\lambda c_\sigma (|\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| + \delta)^2}{\lambda_0\varepsilon} \ \forall t \ge 0 \right\}.$$

If $|\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| \le \delta$, we have from (2.29) that for all $\omega \in \Omega_5$

$$\int_0^{t\wedge\widetilde{\tau}_\delta} |v(s)|^2 ds \le \frac{4\widetilde{C}\lambda c_\sigma \delta^2}{\lambda_0\varepsilon} \le \frac{2\delta}{\varepsilon} < \varepsilon \ \forall t \ge 0.$$

By the exponential martingale inequality [32, Theorem 7.4, page 44], we have $\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}(\Omega_6) \ge 1 - e^{\frac{\varepsilon^3}{\delta}} \ge 1 - \varepsilon$ because $\delta \le \frac{\varepsilon^3}{-\ln\varepsilon}$, where

$$\Omega_6 := \left\{ \int_0^t v(s)d\mathbf{W}_1(s) \le \frac{\varepsilon^2}{2\delta}\int_0^t |v(s)|^2 ds + \varepsilon, t \le \widetilde{\tau}_\delta \right\}.$$

Define

$$\overline{\xi} := \inf\{t \ge 0 : \int_0^t |v(s)|^2 ds \ge \varepsilon\}$$

Let $\widehat{\mathbf{Z}}(t) = ([\widehat{\mathbf{X}}_1(t)]^\top, [\widehat{\mathbf{X}}_2(t)]^\top)^\top$ be solution of following coupled system

$$\begin{cases} d\mathbf{X}_1(t) = b_1(\mathbf{X}_1(t), \mathbf{0})dt + \sigma_1(\mathbf{X}_1(t), \mathbf{0})d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t), \mathbf{0}, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ d\widehat{\mathbf{X}}_1(t) = b_1(\widehat{\mathbf{Z}}(t))dt + \mathbf{1}_{\{t \le \overline{\xi}\wedge\widetilde{\tau}_\delta\}}\lambda(\mathbf{X}_1(t) - \widehat{\mathbf{X}}_1(t))dt + \sigma_1(\widehat{\mathbf{Z}}(t))d\mathbf{W}_1(t) + \int_{\mathbb{R}_*^{n_1}} \gamma_1(\widehat{\mathbf{Z}}(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ d\widehat{\mathbf{X}}_2(t) = b_2(\widehat{\mathbf{Z}}(t))dt + \sigma_2(\widehat{\mathbf{Z}}(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}_*^{n_2}} \gamma_2(\widehat{\mathbf{Z}}(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}), \\ \mathbf{X}_1(0) = \mathbf{x}_1, \widehat{\mathbf{X}}_1(0) = \widetilde{\mathbf{x}}_1, \widetilde{\mathbf{X}}_2(0) = \widetilde{\mathbf{x}}_2. \end{cases} \tag{2.34}$$

Since $\int_0^{\xi\wedge\widetilde{\tau}_\delta} v^2(s)ds \le \varepsilon$, we can apply the Cameron-Martin-Girsanov theorem to imply that under $\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}$, defining by $\frac{d\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}}{d\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}} = \exp\left\{-\int_0^{\overline{\xi}\wedge\widetilde{\tau}_\delta} v(s)d\mathbf{W}_1(s) - \frac{1}{2}\int_0^{\overline{\xi}\wedge\widetilde{\tau}_\delta} |v(s)|^2 ds\right\}$, we have $\mathbf{W}_1(t) + \int_0^{t\wedge\overline{\xi}\wedge\widetilde{\tau}_\delta} v(s)ds$ to be a Wiener process so that $\widehat{\mathbf{Z}}(t) = ([\widehat{\mathbf{X}}_1(t)]^\top, [\widehat{\mathbf{X}}_2(t)]^\top)^\top$ under $\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}$ is the solution to (1.1) with initial value $\widetilde{\mathbf{z}}$. Here, we use the fact that $\widehat{\mathbf{Z}}(t) = \widetilde{\mathbf{Z}}(t)$ for all $t \le \overline{\xi} \wedge \widetilde{\tau}_\delta$, which implies

$$\mathbf{W}_1(t) + \int_0^{t\wedge\overline{\xi}\wedge\widetilde{\tau}_\delta} \lambda\sigma_1^{-1}(\widehat{\mathbf{X}}_1(s), \widehat{\mathbf{X}}_2(s))(\mathbf{X}_1(s) - \widehat{\mathbf{X}}_1(s))ds = \mathbf{W}_1(t) + \int_0^{t\wedge\overline{\xi}\wedge\widetilde{\tau}_\delta} v(s)ds.$$

Moreover, since $\overline{\xi} \wedge \widetilde{\tau}_\delta = \infty$ in $\cap_{i=1}^5 \Omega_i$ if $|\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| + |\widetilde{\mathbf{x}}_2| \le \theta_{\mathbf{x}_1,\varepsilon}$, we have

$$\liminf_{t\to\infty} \frac{U(\widehat{\mathbf{X}}_2(t))}{t} = \liminf_{t\to\infty} \frac{U(\widetilde{\mathbf{X}}_2(t))}{t} \ge m_0\gamma_0 > 0, \text{ if } \omega \in \cap_{i=1}^5 \Omega_i, |\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| + |\widetilde{\mathbf{x}}_2| \le \theta_{\mathbf{x}_1,\varepsilon}.$$

On the other hand, if $|\mathbf{x}_1 - \widetilde{\mathbf{x}}_1| + |\widetilde{\mathbf{x}}_2| \le \theta_{\mathbf{x}_1,\varepsilon}$, for $\omega \in \Omega_5 \cap \Omega_6$, we have

$$\begin{aligned} \frac{d\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}}{d\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}} &= \exp\left\{-\int_0^{\overline{\xi}\wedge\widetilde{\tau}_\delta} v(s)d\mathbf{W}_1(s) - \frac{1}{2}\int_0^{\overline{\xi}\wedge\widetilde{\tau}_\delta} |v(s)|^2 ds\right\} \\ &= \exp\left\{-\int_0^{\widetilde{\tau}_\delta} v(s)d\mathbf{W}_1(s) - \frac{1}{2}\int_0^{\widetilde{\tau}_\delta} |v(s)|^2 ds\right\} \ge e^{-\varepsilon-\varepsilon-\varepsilon} \ge 1 - 4\varepsilon. \end{aligned}$$

Since

$$\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}(\cap_{i=1}^{6}\Omega_i) \geq 1-6\varepsilon,$$

we have

$$\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}(\cap_{i=1}^{6}\Omega_i) \geq (1-6\varepsilon)(1-4\varepsilon) \geq 1-10\varepsilon,$$

which implies

$$\mathbb{Q}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}\left\{\liminf_{t\to\infty}\frac{U(\widehat{\mathbf{X}}_2(t))}{t} \geq m_0\gamma_0\right\} \geq 1-10\varepsilon.$$

Therefore, we obtain that if $|\mathbf{x}_1-\widetilde{\mathbf{x}}_1|+|\widetilde{\mathbf{x}}_2| \leq \theta_{\mathbf{x}_1,\varepsilon}$ then

$$\mathbb{P}_{\mathbf{x}_1,\widetilde{\mathbf{z}}}\left\{\liminf_{t\to\infty}\frac{U(\mathbf{X}_2(t))}{t} \geq m_0\gamma_0\right\} \geq 1-10\varepsilon.$$

Hence, scaling ε by $\frac{\varepsilon}{10}$, we obtain (2.8), which together with (2.4) implies (2.9). The proof is complete. □

3. Stability of linearizable systems and fast-slow systems

In this section, we consider the system of equations with notations as in Section 1.

$$\begin{cases} d\mathbf{Y}_1(t)=b_1(\mathbf{Y}_1(t),\mathbf{Y}_2(t))dt+\sigma_1(\mathbf{Y}_1(t),\mathbf{Y}_2(t))d\mathbf{W}_1(t)+\int_{\mathbb{R}_*^{n_1}}\gamma_1(\mathbf{Y}_1(t-),\mathbf{Y}_2(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt,d\boldsymbol{\phi}), \\ d\mathbf{Y}_2(t)=b_2(\mathbf{Y}_1(t),\mathbf{Y}_2(t))dt+\sigma_2(\mathbf{Y}_1(t),\mathbf{Y}_2(t))d\mathbf{W}_2(t)+\int_{\mathbb{R}_*^{n_1}}\gamma_2(\mathbf{Y}_1(t-),\mathbf{Y}_2(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}). \end{cases} \tag{3.1}$$

Under the condition that the second equation can be linearized (see Assumption 3.1 below), we examine the stability of $\mathbf{Y}_2(\cdot)$. Then we consider the case when the two components have different time scales.

3.1. Stability of linearizable systems

Assumption 3.1. Assume that the assumptions in Theorem 2.1 for b_1,σ_1,γ_1 still hold and that b_2,σ_2,γ_2 are linearizable in $\mathbf{y}_2$. That is, there exist matrices $B_2(\mathbf{y}_1)$, $\Sigma_{21}(\mathbf{y}_1),\ldots,\Sigma_{2d_2}(\mathbf{y}_1)$, $\Gamma_2(\mathbf{y}_1)$ bounded in $\mathbf{y}_1$ such that $\Sigma_{2i}(\mathbf{y}_1)$ has bounded right inverse and

$$\begin{aligned} &\|b_2(\mathbf{z})-B_2(\mathbf{y}_1)\mathbf{y}_2\| \leq o(\mathbf{y}_2)V(\mathbf{y}_1), \\ &\|\sigma_2(\mathbf{z})-[\Sigma_{21}(\mathbf{y}_1)\mathbf{y}_2,\ldots,\Sigma_{2d_2}(\mathbf{y}_1)\mathbf{y}_2]\| \leq o(\mathbf{y}_2)\sqrt{V(\mathbf{y}_1)}, \\ &\gamma_2(\mathbf{z},\boldsymbol{\phi}) = \Gamma_2(\mathbf{y}_1,\boldsymbol{\phi})\mathbf{y}_2+o(\mathbf{y}_2)V(\mathbf{y}_1), \\ &\|b_2(\mathbf{z})\|+\|\sigma_2(\mathbf{z})\sigma_2^\top(\mathbf{z})\| \leq KV(\mathbf{y}_1), \end{aligned}$$

where $o(\mathbf{y}_2)$ is a matrix or vector depending on $\mathbf{z}$ satisfying $\lim_{\mathbf{y}_2\to 0}\frac{\sup_{\mathbf{y}_1\in\mathbb{R}^{\ell_1}}\{|o(\mathbf{y}_2)|\}}{|\mathbf{y}_2|}=0$.

Let $\Theta(t) = \frac{\mathbf{Y}_2(t)}{|\mathbf{Y}_2(t)|}$, $R(t) = |\mathbf{Y}_2(t)|^2$, by the generalized Itô formula for jump diffusions we have the following equations for $\mathbf{Y}_1(t), \Theta(t), R(t)$

$$\begin{cases} d\mathbf{Y}_1(t) = b_1(\mathbf{Z}(t))dt + \sigma_1(\mathbf{Z}(t))d\mathbf{W}_1(t) + \int_{\mathbb{R}^{n_1}_*} \gamma_1(\mathbf{Z}(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ d\Theta(t) = g_1(\mathbf{Y}_1(t), \Theta(t), R(t))dt + g_2(\mathbf{Y}_1(t), \Theta(t), R(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}^{n_2}_*} g_3(\mathbf{Y}_1(t), \Theta(t), R(t), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}), \\ dR(t) = h_1(\mathbf{Y}_1(t), \Theta(t), R(t))dt + h_2(\mathbf{Y}_1(t), \Theta(t), R(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}^{n_2}_*} h_3(\mathbf{Y}_1(t), \Theta(t), R(t), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}), \end{cases} \tag{3.2}$$

where $\mathbf{Z}(t) = ([\mathbf{Y}_1(t)]^\top, [\mathbf{Y}_2(t)]^\top)^\top = ([\mathbf{Y}_1(t)]^\top, \sqrt{R(t)}[\Theta(t)]^\top)^\top$. In (3.2), g_i and h_i with $i = 1, 2, 3$ are given as follows. If we denote $\mathbf{z} = (\mathbf{y}_1^\top, \mathbf{y}_2^\top)^\top = (\mathbf{y}_1^\top, \sqrt{r}\theta^\top)^\top$ then

$$\begin{aligned} g_1(\mathbf{y}_1, \theta, r) &= \frac{b_2(\mathbf{z})}{|\mathbf{y}_2|} - \frac{\sigma_2(\mathbf{z})\sigma_2^\top(\mathbf{z})\mathbf{y}_2}{|\mathbf{y}_2|^3} + \Big(- \mathbf{y}_2^\top b_2(\mathbf{z}) - \frac{1}{2}\operatorname{tr}(\sigma_2^\top(\mathbf{z})\sigma_2(\mathbf{z})) + \frac{3|\mathbf{y}_2^\top \sigma_2(\mathbf{z})|^2}{2|\mathbf{y}_2|^2}\Big]\Big)\frac{\mathbf{y}_2}{|\mathbf{y}_2|^3} \\ &\quad + \int_{\mathbb{R}^{n_2}_*} \left(\frac{\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})}{|\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})|} - \frac{\mathbf{y}_2}{|\mathbf{y}_2|} - \frac{|\mathbf{y}_2|^2\gamma_2(\mathbf{z}, \boldsymbol{\phi}) - (\mathbf{y}_2^\top \gamma_2(\mathbf{z}, \boldsymbol{\phi}))\mathbf{y}_2}{|\mathbf{y}_2|^3} \right) \boldsymbol{\nu}_2(d\boldsymbol{\phi}), \\ &= B_2(\mathbf{y}_1)\theta - \sum_{l=1}^{d_2} \big[\theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta\big]\big[\Sigma_{2l}(\mathbf{y}_1)\theta\big] \\ &\quad + \Big(- \theta^\top B_2(\mathbf{y}_1)\theta + \frac{1}{2}\sum_{l=1}^{d_2}\big[- |\Sigma_{2l}(\mathbf{y}_1)\theta|^2 + 3|\theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta|^2\big]\Big)\theta \\ &\quad + \int_{\mathbb{R}^{n_2}_*} \Big(\frac{\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta}{|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|} - \theta - |\theta|^2\Gamma_2(\mathbf{y}_1, \boldsymbol{\phi}) + (\theta^\top \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi}))\theta\Big)\boldsymbol{\nu}_2(d\boldsymbol{\phi}) \\ &\quad + o(1)\sqrt{V(\mathbf{y}_1)}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} g_2(\mathbf{y}_1, \theta, r) &= \frac{\sigma_2(\mathbf{z})}{|\mathbf{y}_2|} - \frac{\mathbf{y}_2\mathbf{y}_2^\top \sigma_2(\mathbf{z})}{|\mathbf{y}_2|^3} = (\Sigma_{21}(\mathbf{y}_1)\theta, \dots, \Sigma_{2d_2}(\mathbf{y}_1)\theta) - \theta\theta^\top(\Sigma_{21}(\mathbf{y}_1)\theta, \dots, \Sigma_{2d_2}(\mathbf{y}_1)\theta), \\ g_3(\mathbf{y}_1, \theta, r, \boldsymbol{\phi}) &= \frac{\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})}{|\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})|} - \frac{\mathbf{y}_2}{|\mathbf{y}_2|} = \frac{\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta}{|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|} - \theta, \end{aligned} \tag{3.4}$$

$$\begin{aligned} h_1(\mathbf{y}_1, \theta, r) &= 2\mathbf{y}_2 b_2(\mathbf{z}) + \operatorname{tr}(\sigma_2^\top(\mathbf{z})\sigma_2(\mathbf{z})) + \int_{\mathbb{R}^{n_2}_*} |\gamma_2(\mathbf{z}, \boldsymbol{\phi})|^2\boldsymbol{\nu}_2(d\boldsymbol{\phi}) \\ &= r\Big(2\theta^\top B_2(\mathbf{y}_1)\theta + \sum_{l=1}^{d_2}|\Sigma_{2l}(\mathbf{y}_1)\theta|^2 + \int_{\mathbb{R}^{n_2}_*} |\Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2\boldsymbol{\nu}_2(d\boldsymbol{\phi})\Big) + ro(r)V_1(\mathbf{y}_1), \\ h_2(\mathbf{y}_1, \theta, r) &= 2\mathbf{y}_2^\top \sigma_2(\mathbf{z}) = 2r\sum_{l=1}^{d_2}\theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta + ro(r)\sqrt{V_1(\mathbf{y}_1)}, \\ h_3(\mathbf{y}_1, \theta, r, \boldsymbol{\phi}) &= |\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})|^2 - |\mathbf{y}_2|^2 = r(|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 - 1). \end{aligned} \tag{3.5}$$

Let $\mathbf{X}_1(t) = ([\mathbf{Y}_1(t)]^\top, [\Theta(t)]^\top)^\top \in \mathbb{R}^{\ell_1} \times \mathcal{S}_d$ and $\mathbf{X}_2(t) = R(t) \in \mathbb{R}_+$. We have that

$$
\begin{aligned}
\ln R(t) =& h_4(\mathbf{Y}_1(t), \Theta(t), R(t))dt + h_5(\mathbf{Y}_1(t), \Theta(t), R(t))d\mathbf{W}_2(t) \\
&+ \int_{\mathbb{R}^{n_2}_*} h_6(\mathbf{Y}_1(t), \Theta(t), R(t), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}),
\end{aligned}
$$

where

$$
\begin{aligned}
h_4(\mathbf{y}_1, \theta, r) =& 2\theta^\top B_2(\mathbf{y}_1)\theta + \sum_{l=1}^{d_2} |\Sigma_{2l}(\mathbf{y}_1)\theta|^2 - 2|\theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta|^2 \\
&+ \int_{\mathbb{R}^{n_2}_*} \Big(\ln|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 - |\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 + 1\Big)\boldsymbol{\nu}_2(d\boldsymbol{\phi}) + o(1)V(\mathbf{y}_1), \\
h_5(\mathbf{y}_1, \theta, r) =& \frac{2\mathbf{y}_2^\top \sigma_2(\mathbf{z})}{|\mathbf{y}_2|^2} = 2\sum_{l=1}^{d_2} \theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta + o(1)\sqrt{V(\mathbf{y}_1)}, \\
h_6(\mathbf{y}_1, \theta, r, \boldsymbol{\phi}) =& \ln\frac{|\mathbf{y}_2 + \gamma_2(\mathbf{z}, \boldsymbol{\phi})|^2}{|\mathbf{y}_2|^2} = \ln|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 + o(1)\sqrt{V(\mathbf{y}_1)}.
\end{aligned}
$$

When $r = 0$, the equation for $\mathbf{X}_1(t) := ([\mathbf{Y}_1(t)]^\top, [\Theta(t)]^\top)^\top$ is

$$
\begin{cases}
d\mathbf{Y}_1(t) = b_1(\mathbf{Y}_1(t), \mathbf{0})dt + \sigma_1(\mathbf{Y}_1(t), \mathbf{0})d\mathbf{W}_1(t) + \displaystyle\int_{\mathbb{R}^{n_1}_*} \gamma_1(\mathbf{Y}_1(t-), \mathbf{0}, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\
d\Theta(t) = g_1(\mathbf{Y}_1(t), \Theta(t), 0)dt + g_2(\mathbf{Y}_1(t), \Theta(t), 0)d\mathbf{W}_2(t) + \int_{\mathbb{R}^{n_2}_*} g_3(\mathbf{Y}_1(t), \Theta(t), 0, \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}).
\end{cases}
\tag{3.6}
$$

Assume that $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are finite measures. Then the system has a unique invariant measure on $\mathbb{R}^{\ell_1} \times \mathcal{S}_d$ denoted by Π, due to the non-degeneracy of the diffusion, which follows from the explicit formula for g_i given above, and the boundedness of $\Theta(t)$ and the Assumption 2.2(iv). (Although $g_2 = 0$ if $\mathbf{Y}_2(t)$ is one-dimensional, it is noted that 0 is a stationary point for $\mathbf{Y}_2(t)$ so that it does not change sign. As a result, when we convert to polar coordinate, $\Theta(t)$ is a constant 1 or -1. Therefore, in this case, we are considering the invariant measure of $\mathbf{Y}_1(t)$ and it is a non-degenerate diffusion).

We have the following theorem.

Theorem 3.1. *Let* $(\mathbf{Y}_1, \mathbf{Y}_2)$ *be the solution to* (3.1). *Suppose Assumption 3.1 holds. If*

$$
\begin{aligned}
\lambda = \int_{\mathbb{R}^{\ell_1} \times \mathcal{S}_d} \Big(& 2\theta^\top B_2(\mathbf{y}_1)\theta + \sum_{l=1}^{d} \big(|\Sigma_{2l}(\mathbf{y}_1)\theta|^2 - 2|\theta^\top \Sigma_{2l}(\mathbf{y}_1)\theta|^2\big) \\
&+ \ln|\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 - |\theta + \Gamma_2(\mathbf{y}_1, \boldsymbol{\phi})\theta|^2 + 1\Big)\Pi(d\mathbf{y}_1, d\theta) < 0,
\end{aligned}
$$

then for any $\varepsilon > 0$, $\widetilde{\lambda} \in (\lambda, 0)$, $\mathbf{y}_1 \in \mathbb{R}^{\ell_1}_1$, *there exists* $\delta = \delta(\varepsilon, \mathbf{y}_1, \widetilde{\lambda})$ *such that*

$$\mathbb{P}_{\widetilde{\mathbf{y}}_1,\widetilde{\mathbf{y}}_2}\left\{\limsup_{t\to\infty}\frac{\ln|\mathbf{Y}_2(t)|^2}{t}<\widetilde{\lambda}\right\}\geq 1-\varepsilon.$$

Proof. Let $U(r)=(-\ln|r|)\vee 0$. Then we can straightforwardly apply Theorem 2.1 to show that $R(t)$ converges to 0 exponentially fast in probability under the hypothesis of the theorem. Note that the component Θ lives in a compact manifold $\mathcal{S}_d$ but Theorem 2.1 can be applied here because the coupling method in Lemma 2.3 and Theorem 2.1 can be done through the $(d-1)$-dimensional Euclidean coordinates of $\mathcal{S}_d$ and the right inverse of the diffusion coefficient for $(\mathbf{Y}_1(t),\Theta(t))$ is bounded. □

3.2. Stability of the slow component in a fast-slow system

We study the fast-slow coupled jump diffusions as follows

$$\begin{cases} d\mathbf{Y}_1^\varepsilon(t)=\frac{1}{\varepsilon}b_1(\mathbf{Y}_1^\varepsilon(t),\mathbf{Y}_2^\varepsilon(t))dt+\frac{1}{\sqrt{\varepsilon}}\sigma_1(\mathbf{Y}_1^\varepsilon(t),\mathbf{Y}_2^\varepsilon(t))d\mathbf{W}_1(t)+\int_{\mathbb{R}_*^{n_1}}\gamma_1(\mathbf{Y}_1^\varepsilon(t-),\mathbf{Y}_2^\varepsilon(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_1^\varepsilon(dt,d\boldsymbol{\phi}),\\ d\mathbf{Y}_2^\varepsilon(t)=b_2(\mathbf{Y}_1^\varepsilon(t),\mathbf{Y}_2^\varepsilon(t))dt+\sigma_2(\mathbf{Y}_2^\varepsilon(t))d\mathbf{W}_2(t)+\displaystyle\int_{\mathbb{R}_*^{n_2}}\gamma_2(\mathbf{Y}_2^\varepsilon(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}),\end{cases}\tag{3.7}$$

where $\widetilde{\mathbf{N}}_1^\varepsilon(dt,d\phi)=\mathbf{N}_1(dt,d\phi)-\frac{1}{\varepsilon}\boldsymbol{\nu}_1(d\boldsymbol{\phi})dt$, and σ_2 and γ_2 are assumed to be functions of $\mathbf{Y}_2^\varepsilon$ only. We will study the stability of the slow component for small ε based on the stability of the limit system. This problem is very important in applications. Typically, the limit system is often much easier to analyze and compute.

We assume that Assumption 3.1 holds with $\Sigma_{21},\dots,\Sigma_{2d_2}$ and $\Gamma_2(\boldsymbol{\phi})$ now being independent of $\mathbf{y}_1$. Let g_i,h_i be functions defined as in Section 3. Using the change of variable as in Section 3, by the generalized Itô formula for jump diffusions, we have the following equations for $\mathbf{Y}_1(t)$, $\Theta^\varepsilon(t)=\frac{\mathbf{Y}_2^\varepsilon(t)}{|\mathbf{Y}_2^\varepsilon(t)|}$, $R^\varepsilon(t)=|\mathbf{Y}_2^\varepsilon(t)|^2$

$$\begin{cases} d\mathbf{Y}_1^\varepsilon(t)= \dfrac{1}{\varepsilon}b_1(\mathbf{Z}^\varepsilon(t))dt+\dfrac{1}{\sqrt{\varepsilon}}\sigma_1(\mathbf{Z}^\varepsilon(t))d\mathbf{W}_1(t)+\displaystyle\int_{\mathbb{R}_*^{n_1}}\gamma_1(\mathbf{Z}^\varepsilon(t-),\boldsymbol{\phi})\widetilde{\mathbf{N}}_1^\varepsilon(dt,d\boldsymbol{\phi}),\\ d\Theta^\varepsilon(t)= g_1(\mathbf{Y}_1^\varepsilon(t),\Theta^\varepsilon(t),R^\varepsilon(t))dt+g_2(\Theta^\varepsilon(t),R^\varepsilon(t))d\mathbf{W}_2(t)+\displaystyle\int_{\mathbb{R}_*^{n_2}}g_3(\Theta^\varepsilon(t),R^\varepsilon(t),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}),\\ dR^\varepsilon(t)= h_1(\mathbf{Y}_1^\varepsilon(t),\Theta^\varepsilon(t),R^\varepsilon(t))dt+h_2(\Theta^\varepsilon(t),R^\varepsilon(t))d\mathbf{W}_2(t)+\displaystyle\int_{\mathbb{R}_*^{n_2}}h_3(\Theta^\varepsilon(t),R^\varepsilon(t),\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}),\end{cases}\tag{3.8}$$

where $\mathbf{Z}^\varepsilon(t)=([\mathbf{Y}_1^\varepsilon(t)]^\top,[\mathbf{Y}_2^\varepsilon(t)]^\top)^\top=([\mathbf{Y}_1^\varepsilon(t)]^\top,\sqrt{R^\varepsilon(t)}[\Theta^\varepsilon(t)]^\top)^\top$.

Let Π^ε be the family of invariant measures of the system

$$\begin{cases} d\mathbf{Y}_1^\varepsilon(t)=\dfrac{1}{\varepsilon}b_1(\mathbf{Y}_1^\varepsilon,\mathbf{0})dt+\dfrac{1}{\sqrt{\varepsilon}}\sigma_1(\mathbf{Y}_1^\varepsilon(t),\mathbf{0})d\mathbf{W}_1(t)+\displaystyle\int_{\mathbb{R}_*^{n_1}}\gamma_1(\mathbf{Y}_1^\varepsilon(t),\mathbf{0},\boldsymbol{\phi})\widetilde{\mathbf{N}}_1^\varepsilon(dt,d\boldsymbol{\phi}),\\ d\Theta^\varepsilon(t)=g_1(\mathbf{Y}_1^\varepsilon(t),\Theta^\varepsilon(t),0)dt+g_2(\Theta^\varepsilon(t),0)d\mathbf{W}_2(t)+\displaystyle\int_{\mathbb{R}_*^{n_2}}g_3(\Theta^\varepsilon(t),0,\boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt,d\boldsymbol{\phi}).\end{cases}\tag{3.9}$$

By a standard averaging principle (see e.g., [44]), we can show that Π^ε converges weakly to $\Pi_1 \times \Pi_2$, where Π_1 is the invariant measure of the system (due to the fast component of (3.9) is decoupled from the slow component),

$$d\mathbf{Y}_1(t) = b_1(\mathbf{Y}_1, \mathbf{0})dt + \sigma_1(\mathbf{Y}_1, \mathbf{0})d\mathbf{W}_1(t) + \int\limits_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{Y}_1(t), \mathbf{0}, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}),$$

and Π_2 is the invariant measure of the averaged system

$$d\Theta(t) = \overline{g}_1(\Theta(t))dt + g_2(\Theta(t), 0)dW_2(t) + \int\limits_{\mathbb{R}_*^{n_2}} g_3(\Theta(t), 0, \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}),$$

with

$$\begin{aligned}\overline{g}_1(\theta) = &\int\limits_{\mathbb{R}^{\ell_1}} g_1(\mathbf{y}_1, \theta, 0)\Pi_1(d\mathbf{y}_1)\\ =&\Big(-\theta^\top \overline{B}_2\theta + \frac{1}{2}\sum_{l=1}^d\big[-|\Sigma_{2l}\theta|^2 + 3|\theta^\top\Sigma_{2l}\theta|^2\big]\Big)\theta\\ &+\int\limits_{\mathbb{R}_*^{n_2}}\Big(\frac{\theta+\Gamma_2(\boldsymbol{\phi})\theta}{|\theta+\Gamma_2(\boldsymbol{\phi})\theta|} - \theta - |\theta|^2\Gamma_2(\boldsymbol{\phi}) + (\theta^\top\Gamma_2(\boldsymbol{\phi}))\theta\Big)\boldsymbol{\nu}_2(d\boldsymbol{\phi}),\end{aligned}$$

where

$$\overline{B}_2 = \int\limits_{\mathbb{R}^{\ell_1}} B_2(\mathbf{y}_1)\Pi_1(d\mathbf{y}_1).$$

In view of Theorem 3.1, the condition for the stability of $\mathbf{Y}_2^\varepsilon$ is

$$\begin{aligned}\lambda_\varepsilon := \int\limits_{\mathbb{R}^{\ell_1}\times\mathcal{S}_d} &\Big(2\theta^\top B_2(\mathbf{y}_1)\theta + \sum_{l=1}^d\big(|\Sigma_{2l}\theta|^2 - 2|\theta^\top\Sigma_{2l}\theta|^2\big)\\ &+ \ln|\theta+\Gamma_2(\boldsymbol{\phi})\theta|^2 - |\theta+\Gamma_2(\boldsymbol{\phi})\theta|^2 + 1\Big)\Pi^\varepsilon(d\mathbf{y}_1, d\theta) < 0.\end{aligned}$$

Since $\Pi^\varepsilon(d\mathbf{y}_1, d\theta)$ converges weakly to $\Pi_1 \times \Pi_2$ as $\varepsilon \to \infty$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda_* := \int\limits_{\mathbb{R}^{\ell_1}\times\mathcal{S}_d} &\Big(2\theta^\top B_2(\mathbf{y}_1)\theta + \sum_{l=1}^d\big(|\Sigma_{2l}\theta|^2 - 2|\theta^\top\Sigma_{2l}\theta|^2\big)\\ &+ \ln|\theta+\Gamma_2(\boldsymbol{\phi})\theta|^2 - |\theta+\Gamma_2(\boldsymbol{\phi})\theta|^2 + 1\Big)(\Pi_1\times\Pi_2)(d\mathbf{y}_1, d\theta).\end{aligned} \tag{3.10}$$

Theorem 3.2. *Let Assumption 3.1 holds with $\Sigma_{21}, \ldots, \Sigma_{2d_2}$ and $\Gamma_2(\boldsymbol{\phi})$ now being independent of $\mathbf{y}_1$. Then, for sufficiently small $\varepsilon > 0$, the solution $(\mathbf{Y}_1^\varepsilon, \mathbf{Y}_2^\varepsilon)$ satisfies*

$$\lim_{\mathbf{y}_2 \to 0} \mathbb{P}_{\mathbf{y}_1, \mathbf{y}_2} \left\{ \lim_{t\to\infty} \frac{\ln |\mathbf{Y}_2^\varepsilon(t)|}{t} < \frac{\lambda_*}{2} < 0 \right\} = 1.$$

Proof. Because of (3.10), when ε is sufficiently small, we have $\lambda_\varepsilon < -\frac{\lambda_*}{2} < 0$. For that ε, applying Theorem 3.1 for (3.7) (λ_ε plays the same role for (3.7) as λ does for (3.1)), we have for any $\widetilde{\varepsilon} > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}_{\mathbf{y}_1, \mathbf{y}_2} \left\{ \limsup_{t\to\infty} \frac{\ln |\mathbf{Y}_2^\varepsilon(t)|}{t} < \frac{\lambda_*}{2} < 0 \right\} > 1 - \widetilde{\varepsilon}.$$

Letting $\widetilde{\varepsilon} \to \infty$, we have

$$\lim_{\mathbf{y}_2 \to 0} \mathbb{P}_{\mathbf{y}_1, \mathbf{y}_2} \left\{ \lim_{t\to\infty} \frac{\ln |\mathbf{Y}_2^\varepsilon(t)|}{t} < \frac{\lambda_*}{2} < 0 \right\} = 1. \quad \square$$

In view of [30], it is easy to check that $\lambda_* < 0$ is the necessary and sufficient condition for the following linear system to be exponentially stable

$$d\overline{\mathbf{Y}}_2(t) = \overline{B}_2 \overline{\mathbf{Y}}_2(t) dt + \sum_{l=1}^{d} \Sigma_{2d} \overline{\mathbf{Y}}_2(t) d\mathbf{W}_{2l}(t) + \int_{\mathbb{R}_*^{n_1}} \Gamma_2(\boldsymbol{\phi}) \overline{\mathbf{Y}}_2(t) \widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}). \tag{3.11}$$

Thus, we see from Theorem 3.2 that, when the averaged system is stable, so is the fast-slow system if ε is sufficiently small.

Remark 3.1. Treating two time-scale systems, one often uses the so-called freezing component argument; see [23] and [27, pp.88-90]. Here we use a somewhat different argument. Using polar decomposition (3.9), whether R^ε converges to 0 depends on the invariant probability measure Π^ε of $(Y_1^\varepsilon, \Theta^\varepsilon)$ for each ε. Then the Lyapunov exponent that determines stability and that is given by λ^ε, is computed based on Π^ε. Finally, we show λ^ε converges to λ^*, which is obtained based on the limit system.

4. Stabilization and consensus problems

4.1. Stabilization

In this section, we consider the controlled jump-diffusion system given by the following equations,

$$\begin{cases} d\mathbf{X}_1(t) = & b_1(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_1(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_1(t) + \int\limits_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t-), \mathbf{X}_2(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}), \\ & +u(t)dt \\ d\mathbf{X}_2(t) = & b_2(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_2(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_2(t) + \int\limits_{\mathbb{R}_*^{n_2}} \gamma_2(\mathbf{X}_1(t-), \mathbf{X}_2(t-), \boldsymbol{\phi})\widetilde{\mathbf{N}}_2(dt, d\boldsymbol{\phi}), \\ \mathbf{X}_1(0) = & \mathbf{x}_1, \quad \mathbf{X}_2(0) = \mathbf{x}_2, \end{cases} \tag{4.1}$$

where $u(t)$ is a control. We want to construct a control so as to stabilize the process $\mathbf{X}_2$. However, we cannot act directly to $\mathbf{X}_2$ but only the interacting process $\mathbf{X}_1$ can be controlled. We will apply our result to show that under certain conditions, we can control the interacting process $\mathbf{X}_1$ to have the stability of the process $\mathbf{X}_2$. A system may or may not have an invariant probability measure. The weak stabilization essentially means that we construct a control so that the resulting system is weakly stable. That is, the resulting system has an invariant measure. The term weak stability was originated from the work of Wonham [42].

Assumption 4.1. There exists a function $U : \mathbb{R}^{\ell_2} \mapsto [0, \infty)$ such that

$$\lim_{\mathbf{x}_2 \to \mathbf{0}} U(\mathbf{x}_2) = \infty, \qquad \sup_{|\mathbf{x}_2| \geq \theta_{\mathbf{x}_1, \varepsilon}} U(\mathbf{x}_2) < \infty, \qquad U(\mathbf{x}_2) - U(\mathbf{x}_1) \leq c_0 \ln \frac{|\mathbf{x}_1|}{|\mathbf{x}_2|};$$

and there are a constant Δ_0 and functions $f_1, f_2 : \mathbb{R}^{\ell_1} \to \mathbb{R}$ so that

$$[\mathcal{L}U](\mathbf{z}) \leq f_1(\mathbf{x}_1), \quad \left(U_{\mathbf{x}_2}\sigma_2(\mathbf{z})\right)^2 \leq f_2(\mathbf{x}_1), \quad \forall |\mathbf{x}_2| \leq \Delta_0.$$

Moreover, we suppose that f_1 and f_2 are bounded above by $K(1 + |\mathbf{x}_1|^2)$ and $f_1(\mathbf{0}) < 0$ and $\lim_{\mathbf{x}_1 \to \infty} f_1(\mathbf{x}_1) > 0$. Finally, suppose that there is a matrix Q such that

$$b_1^\top(\mathbf{z})Q + \operatorname{tr}\sigma_1^\top(\mathbf{z})Q\sigma_1(\mathbf{z}) + \int\limits_{\mathbb{R}_*^{n_1}} \gamma_1^\top(\mathbf{z}, \boldsymbol{\phi})Q\gamma_1(\mathbf{z}, \boldsymbol{\phi})\boldsymbol{\nu}_1(d\boldsymbol{\phi}) \leq c_1 + c_2|\mathbf{x}_1|^2, \text{ if } |\mathbf{x}_2| \leq \Delta_0.$$

From the assumption on f_1, we can write $f_1(\mathbf{x}_1)$ as $f_1(\mathbf{x}_1) \leq -K_1 + K_2|\mathbf{x}_1|^2$ for some constants K_1, K_2. Consider the control $u(t) = A\mathbf{X}_1(t)$, where A is a matrix satisfying

$$-\lambda_A := \max_{\mathbf{x}_1 \in \mathbb{R}^{\ell_1}} \frac{\mathbf{x}_1^\top QA\mathbf{x}_1}{|\mathbf{x}_1|^2} < -\frac{K_2c_1 + K_1c_2}{K_1}.$$

Now, when $\mathbf{X}_2 = \mathbf{0}$ the corresponding system for $\mathbf{X}_1$ is

$$d\mathbf{X}_1(t) = \Big(b_1(\mathbf{X}_1(t), \mathbf{0}) + u(t)\Big)dt + \sigma_1(\mathbf{X}_1(t), \mathbf{0})dW_1(t) + \int\limits_{\mathbb{R}_*^{n_1}} \gamma_1(\mathbf{X}_1(t), \mathbf{0}, \boldsymbol{\phi})\widetilde{\mathbf{N}}_1(dt, d\boldsymbol{\phi}).$$

We have for $V(\mathbf{x}_1) := \mathbf{x}_1^\top Q\mathbf{x}_1$ that

$$\mathcal{L}V(\mathbf{x}_1) = b_1^\top(\mathbf{x},\mathbf{0})Q + \operatorname{tr}\sigma_1^\top(\mathbf{x}_1,\mathbf{0})Q\sigma_1(\mathbf{x}_1,\mathbf{0}) + \int_{\mathbb{R}_*^{n_1}} \gamma_1^\top(\mathbf{x}_1,\mathbf{0},\boldsymbol{\phi})Q\gamma_1(\mathbf{x}_1,\mathbf{0},\boldsymbol{\phi})\boldsymbol{\nu}_1(d\boldsymbol{\phi}) + \mathbf{x}_1^\top QA\mathbf{x}_1 \le c_1 - (\lambda_A - c_2)|\mathbf{x}_1|^2.$$

As a result, when $\mathbf{x}_2 = \mathbf{0}$, there exists a unique invariant measure Π_1 for $\mathbf{X}_1(t)$ and

$$c_1 - (\lambda_A - c_2)\int_{\mathbb{R}^{\ell_1}} |\mathbf{x}_1|^2 \Pi_1(d\mathbf{x}_1) \ge 0.$$

That yields

$$\int_{\mathbb{R}^{\ell_1}} f_1(\mathbf{x}_1)\Pi_1(d\mathbf{x}_1) \le -K_1 + K_2\frac{c_1}{\lambda_A - c_2} < 0,$$

which implies the stability of the controlled system by an application of our main result (Theorem 2.1).

4.2. Leader-following consensus problems

In this section, we apply our results to the leader-following consensus problems. We consider a network with a leader and N identical followers. The dynamics of the leader is described by

$$d\mathbf{x}_0(t) = f(\mathbf{x}_0(t))dt + d\mathbf{W}(t), \tag{4.2}$$

and the dynamics of the ith follower are described by

$$d\mathbf{x}_i(t) = f(\mathbf{x}_i(t))dt + B\mathbf{u}_i(t)dt + d\mathbf{W}(t), \quad i = 1,\dots,N, \tag{4.3}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $i = 0,\dots,N$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{W}(t)$ is an n-dimensional Brownian motion, $\mathbf{u} = [\mathbf{u}_1^\top,\dots,\mathbf{u}_N^\top]^\top$ ($\mathbf{u}_i \in \mathbb{R}^n, i = 1,\dots,N$) is the control to be designed, $B \in \mathbb{R}^{n\times n}$.

Now, we model the information flow structure among different agents and the leader as follows. Let $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{A}\}$ be a connected graph, where:

- $\mathcal{V} = \{0,\dots,N\}$ denotes the set of nodes with 0 representing the leader, and $k \in \{1,\dots,N\}$ representing the k^{th} agent;
- $\mathcal{E}$ is the set of edges, $\mathcal{A} = [a_{kl}] \in \mathbb{R}^{(N+1)\times(N+1)}$ is the adjacency matrix of $\mathcal{G}$. To be more detailed, for $k,l \in \{1,\dots,N\}$, $a_{kl} = 1$ or 0 indicating whether or not there is a directed information flow between agents l and k if $k \ne l$, and $a_{kl} = 0$ if $k = l$. The connection between followers is undirected, i.e., $a_{kl} = a_{lk}$. The edge between vertex i (representing follower) and vertex 0 (representing the leader) is unidirectional, that is, follower i can receive information from the leader while the leader needs no information from any follower. Particularly, for $i = \{1,\dots,n\}$, $a_{0i} = 1$ if and only if i-th follower can receive information from the leader, and $a_{i0} = 0$.

Then we obtain the Laplacian matrix $\widetilde{\mathcal{H}}$ of $\mathcal{G}$ as follows

$$\mathcal{H}=\begin{bmatrix} 0 & \mathbf{a}_0 \\ \mathbf{0}_N^\top & \mathcal{H}\end{bmatrix},$$

where $\mathbf{a}_0=(-a_{01},\dots,-a_{0N})^\top$, $\mathbf{0}_N:=(0,\dots,0)^\top\in\mathbb{R}^N$ and

$$\mathcal{H}=\begin{bmatrix} \sum\limits_{j=0,j\neq 1}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum\limits_{j=0,j\neq 2}^{N} a_{2j} & \cdots & -a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \sum\limits_{j=0,j\neq N}^{N} a_{Nj}\end{bmatrix}.$$

For the i-th follower, we consider the following leader-following consensus protocol

$$\mathbf{u}_i(t)=K\sum_{j=0,j\in\mathcal{N}_i}^{N}\mathbf{z}_{ji}(t),\, i=1,\dots,N, \tag{4.4}$$

where the symmetric matrix $K\in\mathbb{R}^{n\times n}$ is the control gain to be designed, and

$$\mathbf{z}_{ji}(t)=\mathbf{x}_j(t)-\mathbf{x}_i(t)+(\mathbf{x}_i(t)-\mathbf{x}_j(t))\xi_{ji}(t),$$

is the measurement of the agent j from its neighbor agent i, and ξ_{ji}'s are some random noises. Denoted by

$$\mathcal{U}=\{\mathbf{u}(t)=([\mathbf{u}_1(t)]^\top,\dots,[\mathbf{u}_N(t)]^\top)^\top|\mathbf{u}_i(t) \text{ is given by (4.4)}, t\geq 0, \text{ and } i=1,\dots,N\},$$

the collection of all admissible distributed protocols. We refer the reader to [36,46] and references therein for motivation of the above system.

Assumption 4.2. We assume the following:

(i) $f(y)$ satisfies $|f(y)|\leq c|y|$, for all $y\in\mathbb{R}^n$ for some constant $c>0$.
(ii) The noise $\xi_{ji}(t)$ satisfies that

$$\int_0^t(\mathbf{x}_j(s)-\mathbf{x}_i(s))\xi_{ji}(s)ds=\int_0^t(\mathbf{x}_j(s)-\mathbf{x}_i(s))\left(\sigma_{ji}dw_{ij}(s)+\int_{\mathbb{R}_*^{n_1}}\gamma_{ji}(\boldsymbol{\phi})\widetilde{\mathbf{N}}_{ji}(ds,d\boldsymbol{\phi})\right),$$

where $w_{ji}(s)$ are independent standard Brownian motions, $\widetilde{\mathbf{N}}_{ji}(s,\phi)$ are jump processes. This formulation indicates that the systems can be perturbed by both noise and jump processes.

Definition 4.1. System (4.2) and (4.3) is said to be exponentially consentable in probability with respect to $\mathcal{U}$ if there exists a protocol $\mathbf{u} \in \mathcal{U}$ so that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $i = 1, \ldots, N$

$$\mathbb{P}\left\{|\mathbf{x}_i(t) - \mathbf{x}_0(t)| \text{ converges exponentially fast to } 0\right\} \geq 1 - \varepsilon,$$

whenever the initial values $(\mathbf{y}_0, \ldots, \mathbf{y}_n) \in \mathbb{R}^{nN}$ of (4.2) and (4.3) satisfying that

$$\sum_{i=1}^{N} |\mathbf{y}_i - \mathbf{y}_0|^2 < \delta.$$

To proceed, let $\mathbf{X}_i(t) = \mathbf{x}_i(t) - \mathbf{x}_0(t), i = 1, \ldots, N$, $\mathbf{x}(t) := \left[\mathbf{x}_1^\top(t), \ldots, \mathbf{x}_N^\top(t)\right]^\top$, $F_i(\mathbf{x}_0(t), \mathbf{x}_i(t)) := f(\mathbf{x}_i(t)) - f(\mathbf{x}_0(t))$, and $F(\mathbf{x}_0(t), \ldots, \mathbf{x}_N(t)) := \left[F_1^\top(\mathbf{x}_0(t), \mathbf{x}_1(t)), \ldots, F_N^\top(\mathbf{x}_0(t), \mathbf{x}_N(t))\right]^\top$, $\mathbf{X}(t) := \left[\mathbf{X}_1^\top(t), \ldots, \mathbf{X}_N^\top(t)\right]^\top$. For simplicity of notation, we will write $F(\mathbf{x}_0(t), \ldots, \mathbf{x}_N(t))$ as $F(\mathbf{x}_0(t), \mathbf{X}(t))$ by the identity $F(\mathbf{x}_0(t), \mathbf{X}(t)) = F(\mathbf{x}_0(t), \mathbf{x}_0(t) + \mathbf{X}_1(t), \ldots, \mathbf{x}_0(t) + \mathbf{X}_N(t))$. Then, one can obtain

$$\begin{cases} d\mathbf{x}_0(t) = f(\mathbf{x}_0(t))dt + d\mathbf{W}(t), \\ d\mathbf{X}(t) = \big(F(\mathbf{x}_0(t), \mathbf{X}(t)) - [\widetilde{\mathcal{H}} \otimes BK]\mathbf{X}(t)\big)\, dt + dM_1(t) + dM_0(t) + dM_0^N(t) + dM_1^N(t), \end{cases} \tag{4.5}$$

where $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, and

$$M_1(t) = \sum_{i,j=1}^{N} \int_0^t \sigma_{ji} \left[S_{ij} \otimes BK\right] \mathbf{X}(s) dw_{ji}(s),$$

$$M_0(t) = -\sum_{i=1}^{N} \int_0^t \sigma_{0i} \left[\overline{S}_i \otimes BK\right] \mathbf{X}(s) dw_{0i}(s),$$

$$M_1^N(t) = \sum_{i,j=1}^{N} \int_0^t \int_{\mathbb{R}_*^{n_1}} \left[S_{ij} \otimes BK\right] \mathbf{X}(s) \gamma_{ji}(\boldsymbol{\phi}) \widetilde{\mathbf{N}}_{ji}(ds, d\boldsymbol{\phi}),$$

$$M_0^N(t) = -\sum_{i=1}^{N} \int_0^t \int_{\mathbb{R}_*^{n_1}} \left[\overline{S}_i \otimes BK\right] \mathbf{X}(s) \gamma_{ji}(\boldsymbol{\phi}) \widetilde{\mathbf{N}}_{ji}(ds, d\boldsymbol{\phi}),$$

where $S_{ij} = [s_{kl}]_{N\times N}$ is an $N \times N$ matrix with $s_{ii} = -a_{ij}$ and $s_{ij} = a_{ij}$ and all other elements being 0, for $i, j = 1, \ldots, N$, and $\overline{S}_i = [\overline{s}_{kl}]_{N\times N}$ is an $N \times N$ matrix with $\overline{s}_{ii} = a_{0i}$ and all other entries being 0. It is easily seen that the consensus problem of (4.2) and (4.3) is equivalent to the stability of (4.5).

Assumption 4.3. There exists a matrix $K \in \mathbb{R}^{n\times n}$ such that there exists a function $U : \mathbb{R}^{nN} \mapsto [0, \infty)$ satisfying the following conditions,

$$\lim_{\mathbf{X}\to\mathbf{0}} U(\mathbf{X}) = \infty, \quad U(\mathbf{X}) - U(\mathbf{X}') \le c_0 \ln \frac{|\mathbf{X}'|}{|\mathbf{X}|},$$

and there is Δ_0 such that

$$[\mathcal{L}U](\mathbf{X}) \le c_1(\mathbf{x}_0), |\mathbf{X}| \le \Delta_0,$$

$$\sum_{i,j=1}^{N} \Big(U_{\mathbf{X}}^\top(\sigma_{ji}\left[S_{ij} \otimes BK\right]\mathbf{X})\Big)^2 + \sum_{i=1}^{N} \Big(U_{\mathbf{X}}^\top(\sigma_{0i}\left[\overline{S}_i \otimes BK\right]\mathbf{X})\Big)^2 \le c_2(\mathbf{x}_0), |\mathbf{X}| \le \Delta_0,$$

$$\int_{\mathbb{R}^{n_2}_*} \Big[\exp\Big\{-\alpha_0\big(U(\mathbf{X}_2 + \gamma_2(\mathbf{X}, \boldsymbol{\phi})) - U(\mathbf{X})\big)_+\Big\}\Big]\boldsymbol{\nu}_2(\boldsymbol{\phi}) \le c_3(\mathbf{x}_0), |\mathbf{X}| \le \Delta_0,$$

and

$$\int (c_1(\mathbf{x}_0) + c_2(\mathbf{x}_0) + c_3(\mathbf{x}_0))\, \mu^*(dx_0) > 0.$$

In the above, $\mu^*(\cdot)$ is the invariant measure of (4.2). Such a $\mu^*(\cdot)$ always exists because of Assumption 4.2.

Theorem 4.1. *Under Assumptions 4.2 and 4.3, system (4.5) is exponentially stable in probability. As a consequence, the leader-following system (4.2) and (4.3) is exponentially consentable in probability.*

In fact, our results above are verifiable. To illustrate that, we provide the following explicitly computational example.

Example 4.1. In this example, assume that $f(x) = Ax$ where $A \in \mathbb{R}^{n\times n}$. Assume that B is invertible. Let $U(\mathbf{X}) = -\ln|\mathbf{X}|$, by directed calculations, we have

$$\mathcal{L}U(\mathbf{X}) = -\frac{\mathbf{X}\left(A \otimes I_N - \widetilde{\mathcal{H}} \otimes BK\right)\mathbf{X}^\top}{|\mathbf{X}|^2}.$$

Then, it is easy to check the remaining conditions.

5. Conclusion

We studied stability and stabilization of a fully coupled system of jump diffusions. Sufficient conditions for stability are derived. We then investigate the stability of linearizable jump diffusions and fast-slow coupled jump diffusions. Next, we develop strategies for weak stabilization of a coupled system in which only one component can be controlled. Also considered are consensus problem of leader-following systems. This paper can be readily extended to systems with more than two components. There are many interesting important problems remain to be investigated. Future research could be extended for regime-switching with state-dependent diffusions or hidden Markov systems. Efforts can also be directed to studying systems with mean-fields interactions. These and other topics deserve to be carefully examined.

Data availability

No data was used for the research described in the article.

References

[1] D. Applebaum, Lévy Processes and Stochastic Calculus, 2nd ed., Cambridge Univ. Press, Cambridge, 2009.
[2] R. Atar, A. Budhiraja, Stability properties of constrained jump-diffusion processes, Electron. J. Probab. 7 (2002), 31 pp.
[3] V. Bally, D. Goreac, V. Rabiet, Regularity and stability for the semigroup of jump diffusions with state-dependent intensity, Ann. Appl. Probab. 28 (2018) 3028–3074.
[4] D.S. Bates, Jumps and stochastic volatility: exchange rate processes implicit in Deutsche mark options, Rev. Financ. Stud. 9 (1996) 69–107.
[5] J. Bao, X. Mao, G. Yin, C. Yuan, Competitive Lotka–Volterra population dynamics with jumps, Nonlinear Anal. 74 (2011) 6601–6616.
[6] Z. Chao, K. Wang, C. Zhu, Y. Zhu, Almost sure and moment exponential stability of regime-switching jump diffusion, SIAM J. Control Optim. 55 (2017) 3458–3488.
[7] S. Cerrai, Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative noise, SIAM J. Math. Anal. 43 (2011) 2482–2518.
[8] S. Cerrai, M. Freidlin, Averaging principle for stochastic reaction-diffusion equations, Probab. Theory Relat. Fields 144 (2009) 137–177.
[9] M.F. Chen, S.F. Li, Coupling methods for multidimensional diffusion processes, Ann. Probab. 17 (1989) 151–177.
[10] X. Chen, Z.-Q. Chen, K. Tran, G. Yin, Properties of switching jump diffusions: maximum principles and Harnack inequalities, Bernoulli 25 (2019) 1045–1075.
[11] X. Chen, Z-Q. Chen, K. Tran, G. Yin, Recurrence and ergodicity for a class of regime-switching jump diffusions, Appl. Math. Optim. 80 (2019) 415–445.
[12] F.T. Charbel, M. Zhao, S. Amin-Hanjani, W. Hoffman, X. Du, M.E. Clark, A patient-specific computer model to predict outcomes of the balloon occlusion test, J. Neurosurg. 101 (2004) 977–988.
[13] J. Daafouz, M. Tucsnak, J. Valein, Nonlinear control of a coupled PDE/ODE system modeling a switched power converter with a transmission line, Syst. Control Lett. 70 (2014) 92–99.
[14] N.T. Dieu, N.H. Du, N.N. Nguyen, Conditions for permanence and ergodicity of certain SIR epidemic models, Acta Appl. Math. 160 (2019) 81–99.
[15] N.H. Du, N.N. Nguyen, Permanence and extinction for the stochastic SIR epidemic model, J. Differ. Equ. 269 (2020) 9619–9652.
[16] A. Efendiev, M. Otani, J.H. Eberl, Mathematical analysis of a PDE-ODE coupled model of mitochondrial swelling with degenerate calcium ion diffusion, SIAM J. Math. Anal. 52 (2020) 543–569.
[17] M. Fernández, V. Milišic, A. Quarteroni, Analysis of a geometrical multiscale blood flow model based on the coupling of odes and hyperbolic pdes, SIAM J. Multiscale Model. Simul. 4 (2005) 215–236.
[18] D. Filipović, S. Tappe, J. Teichmann, Jump-diffusions in Hilbert spaces: existence, stability and numerics, Stochastics 82 (2010) 475–520.
[19] M. Hairer, X.M. Li, Averaging dynamics driven by fractional Brownian motion, Ann. Probab. 48 (2020) 1826–1860.
[20] M. Hairer, J.C. Mattingly, M. Scheutzow, Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations, Probab. Theory Relat. Fields 149 (1–2) (2011) 223–259.
[21] T. Hao, J. Li, Fully coupled forward-backward SDEs involving the value function and associated nonlocal Hamilton-Jacobi-Bellman equations, ESAIM Control Optim. Calc. Var. 22 (2016) 519–538.
[22] Y. Kao, C. Wang, Global stability analysis for stochastic coupled reaction-diffusion systems on networks, Nonlinear Anal., Real World Appl. 14 (2013) 1457–1465.
[23] R.Z. Khasminskii, On an averaging principle for Ito stochastic differential equations, Kybernetika 4 (1968) 260–279.
[24] R.Z. Khasminskii, C. Zhu, G. Yin, Stability of regime-switching diffusions, Stoch. Process. Appl. 117 (2007) 1037–1051.

[25] P. Krejĉi, A. Timoshin, Coupled ODEs control system with unbounded hysteresis region, SIAM J. Control Optim. 54 (2016) 1934–1949.
[26] A. Kulik, M. Scheutzow Michael, Generalized couplings and convergence of transition probabilities, Probab. Theory Relat. Fields 171 (1–2) (2018) 333–376.
[27] H.J. Kushner, Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems, Birkhäuser, Boston, 1990.
[28] C. Lattanzio, A. Maurizi, B. Piccoli, Moving bottlenecks in car traffic flow: a PDE-ODE coupled model, SIAM J. Math. Anal. 43 (2011) 50–67.
[29] S.S. Lee, J. Hannig, Detecting jumps from Lévy jump diffusion processes, J. Financ. Econom. 96 (2010) 271–290.
[30] C.W. Li, Z. Dong, R. Situ, Almost sure stability of linear stochastic differential equations with jumps, Probab. Theory Relat. Fields 123 (2002) 121–155.
[31] W. Littman, L. Markus, Exact boundary controllability of a hybrid system of elasticity, Arch. Ration. Mech. Anal. 103 (1988) 193–236.
[32] X. Mao, Stochastic Differential Equations and Their Applications, Horwood Publishing Chichester, 1997.
[33] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, London, 2006.
[34] A.A. Moghadam, I. Aksikas, S. Dubljevic, J.F. Forbes, LQ control of coupled hyperbolic PDEs and ODEs: application to a CSTR-PFR system, in: International Symposium on Dynamics and Control of Process Systems, Leuven, Belgium, 2010, pp. 721–726.
[35] A. Moghadam, I. Aksikas, S. Dubljevic, J. Forbes, Fraser boundary optimal (LQ) control of coupled hyperbolic PDEs and ODEs, Automatica 49 (2013) 526–533.
[36] W. Ni, X. Wang, C. Xiong, Consensus controllability, observability and robust design for leader-following linear multi-agent systems, Automatica 49 (2013) 2199–2205.
[37] A. Puhalskii, On large deviations of coupled diffusions with time scale separation, Ann. Probab. 44 (2016) 3111–3186.
[38] W. Ruan, A coupled system of ODEs and quasilinear hyperbolic PDEs arising in a multiscale blood flow model, J. Math. Anal. Appl. 343 (2008) 778–798.
[39] A.V. Skorokhod, Asymptotic Methods in the Theory of Stochastic Differential Equations, Translations of Math. Monographs, vol. 78, Amer. Math. Soc., Providence, RI, 1989.
[40] E. Witrant, A. D'Innocenzo, G. Sandou, F. Santucci, M.D. Di Benedetto, A.J. Isaksson, K.H. Johansson, S.I. Niculescu, S. laru, E. Serra, S. Tennina, U. Tiberi, Wireless ventilation control for large-scale systems: the mining industrial case, Int. J. Robust Nonlinear Control 20 (2010) 226–251.
[41] I-S. Wee, Stability for multidimensional jump-diffusion processes, Stoch. Process. Appl. 80 (1999) 193–209.
[42] W.M. Wonham, Liapunov criteria for weak stochastic stability, J. Differ. Equ. 2 (1966) 195–207.
[43] F. Xi, C. Zhu, On the martingale problem and Feller and strong Feller properties for weakly coupled Lévy type operators, Stoch. Process. Appl. 128 (2018) 4277–4308.
[44] J. Xu, J. Liu, Stochastic averaging principle for two time-scale jump-diffusion SDEs under the non-Lipschitz coefficients, Stochastics (2020), https://doi.org/10.1080/17442508.2020.1784897.
[45] G. Yin, F. Xi, Stability of regime-switching jump diffusions, SIAM J. Control Optim. 48 (2010) 4525–4549.
[46] X. You, C-C. Hua, H-N. Yu, X-P. Guan, Leader-following consensus for high-order stochastic multi-agent systems via dynamic output feedback control, Automatica 107 (2019) 418–424.
[47] X. Zong, F. Wu, G. Yin, Z. Jin, Almost sure and pth-moment stability and stabilization of regime-switching jump diffusion systems, SIAM J. Control Optim. 52 (2014) 2595–2622.