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A CLASS OF LONG-RUN AVERAGE CONTROL PROBLEMS OF LOTKA-VOLTERA SYSTEMS IN A STOCHASTIC ENVIRONMENT

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This paper is written for and dedicated to Professor Héléne Frankowska on the occasion of her 70th birthday

Abstract. This work is devoted to studying a class of biological control problems in a stochastic environment. Specifically, it focuses on stochastic Lotka-Volterra systems. Our effort is on treating average cost per unit time controlled diffusions. It is natural to use a vanishing discount argument. However, in contrast to the existing literature, neither the "near-monotone" nor the "stable" condition is satisfied in the current set up. In reference to one of our recent works, we divide the domain into two parts. In one sub-domain, the "near-monotone" condition is satisfied, whereas in the other sub-domain, the "stable" condition is satisfied. We then carefully work out the analysis in the two domains so as to obtain the desired optimal control.

1. **Controlled stochastic Predator-Prey model.** Stochastic Lotka-Volterra systems have been widely used in a wide variety of applications. Not only are they used in biological systems and ecological systems, but also they are applicable to the study of particle systems (see [5]). In addition, more recently, such systems have been used in social network modeling and related applications. This paper stems from mathematical models using Lotka-Volterra equations, but mainly concentrates

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on the study of optimal controls of such systems. In fact, we focus on a class of such systems and our primary concern is the optimal controls under a longrun average cost criterion.

In particular, our study is motivated by a class of biological control problems from the angle of biodiversity. Biodiversity or biological diversity is a term that refers to the variety of life on Earth, in all its forms and all its interactions, including genes, traits, species, and ecosystems. The popular abbreviation biodiversity came about in the mid-1980s by a symposium in 1986.

We value biodiversity both for what it provides to humans, and for the value it has in its own right. In short, biodiversity is the volume of life on Earth as well as how different species interact with each other and with the physical world around them. Specifically, in this paper, we are interested in the following aspects. It is well known that chemicals have been used widely for pest controls. However, they have many deleterious effects to the environment, human, and biodiversity; see [11, 14, 18, 19]. Resistance to pesticides also erodes their effectiveness and makes chemical control not cost-effective in the longrun; see [15, 7, 10]. Biological control is any alternative to chemical control where natural enemies are used to control insects, weed, and disease. Biological control has many benefits that chemical control does not offer. It is more environment-friendly, non-toxic, and it can even be more cost effective in a long run; see [3, 4, 9]. In this paper, we consider a control problem in a stochastic environment. Suppose we wish to control the population of a species X using its predator Y. The interaction of X and Y is assumed to follow a stochastic Lotka-Voltera system of the form

$$\begin{cases} dX(t) = & X(t)\left[(a_1 - b_{11}X(t) - b_{12}Y(t))dt + \sigma_1 dW_1(t)\right] \\ dY(t) = & Y(t)\left[(-a_2 + b_{21}X(t))dt + \sigma_2 dW_2(t)\right]. \end{cases} \tag{1.1}$$

However, we will control the density of $Y(t)$ by adding $u(t) \times \Delta$ to its density in a small interval $[t, t + \Delta)$. Then, we have the following controlled system

$$\begin{cases} dX(t) = & X(t)\left[(a_1 - b_{11}X(t) - b_{12}Y(t))dt + \sigma_1 dW_1(t)\right] \\ dY(t) = & u(t)dt + Y(t)\left[(-a_2 + b_{21}X(t))dt + \sigma_2 dW_2(t)\right]. \end{cases} \tag{1.2}$$

Denote by $\mathcal{F}_t$ the σ-algebra generated by $\{X(s), Y(s) : 0 \le s \le t\}$. We use the predator Y as a natural control of the prey population, and assume the control $u(t)$ takes values in a compact interval $[0, M]$ for some $M > 0$. Our objective is to minimize

$$\limsup_{T\to\infty} \frac{1}{T} \mathbb{E}^u_{x,y} \int_0^T X(t)dt,$$

where $\mathbb{E}^u_{x,y}$ denotes the expectation with initial data $X(0) = x$, $Y(0) = y$, and control u used over the class of admissible controls $u(t)$, where $u(t)$ is $\mathcal{F}_t$-adapted. That is, we aim to minimize the amount of substrate over the infinite horizon. The cost criterion is in the sense of an average cost per unit time (or longrun average cost). Note that we consider the case where the cost function does not depend on u. The main reason is for the bio-control problem that we are considering, our priority is on controlling $X(t)$. Moreover, our problem is constrained. The constraint on the economic aspect is imposed by limiting the input $u(t)$ so that $u(t) \le M$.

We will assume throughout the paper that

$$a_1 - \frac{\sigma_1^2}{2} > 0 \text{ and } a_1 - b_{12}\frac{a_2}{M} - \frac{\sigma_1^2}{2} > 0 \tag{1.3}$$

to exclude trivial cases. If $a_1 - \frac{\sigma_1^2}{2} \le 0$, it is easily seen that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^u_{x,y}\int_0^T X(t)dt = 0 \text{ w.p.1 for any control.}$$

Moreover, if $a_1 - b_{12}\frac{a_2}{M} - \frac{\sigma_1^2}{2} < 0$ with control $u(t) \equiv M$, $X(t)$ satisfies that $\lim_{t\to\infty} X(t) = 0$ then the optimal control problem is trivial.

We proceed to obtain the Hamilton-Jacobi-Bellman (HJB) equation for the optimal control and prove the existence and uniqueness of the solution to the HJB equation. We follow a "vanishing discount" argument using some ideas from the work [1], in which the HJB equation was obtained under either "near-monotone" or "stable" conditions. Nevertheless, we note that in our current setup, our system satisfies neither of these conditions. Thus, we need some new method to carry out the technical analysis of the limit of the value functions for discount control problems. In particular, for our models, we can divide the underlying region into two domains so that in one of the domains, the "near-monotone" condition is satisfied while the "stable" condition holds for the other domain. Difficulty arises when we examine how solutions switch back and forth between the two domains and how the movement causes changes in the value function. Although the idea is similar to that of [16], the model under consideration in this paper is more complicated. New techniques are needed to treat the problem and overcome the difficulty.

The rest of the paper is organized as follows. In Section 2, we recall the notion of stochastic ergodic control. It will be demonstrated that we can find an optimal control in the class of Markov controls. Then we state the main theorem on the existence and uniqueness of solution to the HJB equation, which characterizes the optimal control. Section 3 is devoted to the proof of the main theorem using some novel technical analysis.

2. **Main results.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, and $(W_1(t), W_2(t))$ is a standard $\mathcal{F}_t$-adapted two-dimensional Brownian motion such that $(W_1(t) - W_1(s), W_2(t) - W_2(s))$ and $\mathcal{F}_s$ are independent for all $t > s \ge 0$. Denote by $\mathbb{R}^{2,\circ}_+, \mathbb{R}^2_+$ the sets $(0,\infty)^2$ and $[0,\infty)^2$, respectively. To proceed, we recall some definitions introduced in [1, 12]. Denote by $M(\infty)$ the family of measures $\{m(\cdot)\}$ on the Borel subsets of $[0,\infty) \times [0, M]$ satisfying $m([0,t] \times [0,M]) = t$ for all $t \ge 0$. We say $m_n(\cdot)$ converges weakly to $m(\cdot)$ in $M(\infty)$, if

$$\lim_{n\to\infty} \int f(s,\alpha)m_n(ds \times d\alpha) = \int f(s,\alpha)m(ds\times d\alpha)$$

for any continuous function $f(\cdot) : [0,\infty) \times [0, M] \mapsto \mathbb{R}$ with compact support. An $M(\infty)$-valued random measure $m(\cdot)$ is an admissible relaxed control for (1.2) if $\int_0^M \int_0^t f(s,u)m(ds \times du)$ is independent of $\{W_i(t+s) - W_i(t), s > 0, i = 1, 2\}$ for each bounded and continuous function $f(\cdot)$. With a relaxed control $m(\cdot)$, (1.2) become

$$\begin{cases} dX(t) = & X(t)\left[(a_1 - b_{11}X(t) - b_{12}Y(t))dt + \sigma_1 dW_1(t)\right] \\ dY(t) = & \overline{m}_t + Y(t)\left[(-a_2 + b_{21}X(t))dt + \sigma_2 dW_2(t)\right], \end{cases} \tag{2.1}$$

where $\overline{m}_t = \int_0^M u m_t(du)$ and the "derivative" m_t is defined as the measure-valued function of (ω, t) such that for any smooth and bounded function f, we have

$$\iint f(s,u)m(ds \times du) = \int ds \int f(s,u)m_s(du).$$

The (t-dependent) operator associated with the controlled diffusion process (2.1), is given by

$$\begin{aligned}\mathcal{L}^m\phi(x,y) =& \frac{\partial\phi(x,y)}{\partial x}\left[x(a_1 - b_{11}x - b_{12}y)\right] + \frac{\partial\phi(x,y)}{\partial y}\left[\overline{m}_t + y(-a_2 + b_{21}x)\right] \\ &+ \frac{1}{2}\left(\sigma_1^2\frac{\partial^2\phi(x,y)}{\partial x^2}x^2 + \sigma_2^2\frac{\partial^2\phi(x,y)}{\partial y^2}y^2\right).\end{aligned}$$

Definition 2.1. We have the following definitions and notation.

- Denote by $\mathcal{P}(M(\infty))$ the space of probability measures on $M(\infty)$. A relaxed control $m(\cdot)$ for (2.1) is said to be Markov if there exists a measurable function $v : \mathbb{R}^2_+ \mapsto \mathcal{P}(M(\infty))$ such that $m_t = v(X(t), Y(t)), t \geq 0$. Under a relaxed Markov control $v(X(t), Y(t))$, (2.1) generates a Markov process $(X(t), Y(t))$ with generator

$$\begin{aligned}\mathcal{L}^v\phi(x,y) \ =& \ \frac{\partial\phi(x,y)}{\partial x}\left[x(a_1 - b_{11}x - b_{12}y)\right] \\ &+\frac{\partial\phi(x,y)}{\partial y}\left[\overline{v}(x,y) + y(-a_2 + b_{21}x)\right] \\ &+\frac{1}{2}\left(\sigma_1^2\frac{\partial^2\phi(x,y)}{\partial x^2}x^2 + \sigma_2^2\frac{\partial^2\phi(x,y)}{\partial y^2}y^2\right).\end{aligned}$$

  where $\overline{v}(x,y) = \int_0^M u[v(x,y)](du)$, the expectation of a random variable with distribution $v(x,y)$.
- A Markov control v is a relaxed control satisfying that $v(z)$ is a Dirac measure on $[0, M]$ for each $z \in \mathbb{R}^2_+$.
- Denote the set of Markov controls and relaxed Markov controls by Π_M an Π_{RM}, respectively. With a relaxed Markov control, $(X(t), Y(t))$ is a Markov process that has the strong Feller property in $\mathbb{R}^{2,\circ}_+$; see [1, Theorem 2.2.12].
- Because of the nondegeneracy, any invariant probability measure in $\mathbb{R}^{2,\circ}_+$ of $(X(t), Y(t))$ is unique if it exits. In this case, the control v is said to be stable. Denote by Π_{SRM} the set of stable relaxed Markov controls.
- Let $\mathcal{P}(\mathcal{X})$ be the space of probability measures on a metric space $\mathcal{X}$. For any stable relaxed Markov control v, define

$$\pi_v(dz \times du) = [v(z)(du)] \times \eta_v(dz) \in \mathcal{P}(\mathbb{R}^{2,\circ}_+ \times [0, M]),$$

  and

$$\mathcal{G} = \{\pi_v : v \text{ is a stable relaxed Markov control } \} \subset \mathcal{P}(\mathbb{R}^{2,\circ}_+ \times [0, M]),$$

  where η_v is the invariant probability measure of $(X(t), Y(t))$ on $\mathbb{R}^{2,\circ}_+$.

Lemma 2.2. *For any $(x,y) \geq 0$, and any admissible control $m(t)$, there exists a unique solution to (2.1) satisfying $(X(t), Y(t)) \in \mathbb{R}^2_+, t \geq 0$. If $(x,y) \in \mathbb{R}^{2,\circ+}$ then $(X(t), Y(t)) \in \mathbb{R}^{2,\circ}_+, t \geq 0$ and if $x = 0, y \geq 0$ then $X(t) = 0, t \geq 0$.*

Pick out $p_0 = \frac{\sigma_1^2 \vee \sigma_2^2}{2a_2}$. There exists positive constants K_1, K_2, and K_3 such that

$$\mathcal{L}^m(b_{21}x + b_{12}y)^{1+p_0} \leq K_1 - K_2(2\kappa_0 s + x)^{1+p_0} - K_2(2\kappa_0 s + x)^{p_0} x f_1(x,y), \quad (2.2)$$

Consequently, it holds for any admissible relaxed control $m(\cdot)$ that

$$\mathbb{E}^m_{x,y}\big(b_{21}X(t)+b_{12}Y(t)\big)^{1+p_0} \le (b_{21}x+b_{12}y)^{1+p_0}e^{-K_2t}+\frac{K_2}{K_1} \text{ for } s>0, x\ge 0, \tag{2.3}$$

As a corollary, for any stable relaxed Markov control v, we have

$$\int_{\mathbb{R}^{2,\circ}_+}(b_{21}x+b_{12}y)^{1+p_0}\eta_v(ds,dx) \le \frac{K_2}{K_1}. \tag{2.4}$$

where η_v is the invariant probability measure of $(X(t),Y(t))$ on $\mathbb{R}^{2,\circ}_+$ under control v.

Proof. The existence and uniqueness of positive solutions are standard, which are thus omitted here. With

$$U_p(x,y) = [b_{21}x+b_{12}y]^{1+p_0},$$

we have

$$\begin{aligned}\mathcal{L}^m U_{p_0}(x,y) =&(1+p_0)[b_{21}x+b_{12}y]^{p_0}\left(b_{21}a_1x-b_{21}b_{11}x^2-\overline{m}_t-b_{12}a_2y\right)\\ &+p_0(1+p_0)[b_{21}x+b_{12}y]^{p_0-1}(b_{21}^2\sigma_1^2x^2+b_{12}^2\sigma_2^2y^2)\\ \le& k_{1p_0}-(1+p_0)(b_{21}x+b_{12}y)^{p_0}(b_{12}a_2y+b_{21}a_2x)\\ &+p_0(1+p_0)(\sigma_1^2\vee\sigma_2^2)(b_{21}x+b_{12}y)^{1+p_0}\\ \le& k_{1p_0}-[a_2-p_0(\sigma_1^2\vee\sigma_2^2)](1+p_0)U_{p_0}(x,y)\end{aligned}$$

where k_{1p_0} is a constant independent of (x,y). (2.2) therefore holds. Applying Dynkin's formula to the function $e^{K_2t}U_0(x,y)$ we can obtain easily (2.3) using (2.2). □

Lemma 2.3. *With the control $u(t)\equiv M$, there exist a unique invariant measure of $(X(t),Y(t))$ on $\mathbb{R}^{2,\circ +}$ and*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}^u_{x,y}X(t)dt < \frac{2a_1-\sigma_1^2}{b_{11}}.$$

Proof. Under condition (1.3), it was proved in [20] (see also [17] for a similar model) that the solution $(X(t),Y(t))$ to

$$\begin{cases} dX(t) = & X(t)\,[(a_1-b_{11}X(t)-b_{12}Y(t))dt+\sigma_1dW_1(t)]\\ dY(t) = & M+Y(t)\,[(-a_2+b_{21}X(t))dt+\sigma_2dW_2(t)]\end{cases} \tag{2.5}$$

has a unique invariant probability measure on $\mathbb{R}^{2,\circ}_+$ and the transition probability of $(X(t),Y(t))$ converges weakly to π. The weak convergence and the uniform boundedness of $\mathbb{E}(X(t)+Y(t))^{1+p_0}$ in (2.3) implies that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}^u_{x,y}X(t) &= \int_{\mathbb{R}^{2,\circ}_+} x\pi(dxdy)>0 \text{ and}\\ \lim_{t\to\infty}\mathbb{E}^u_{x,y}Y(t) &= \int_{\mathbb{R}^{2,\circ}_+} y\pi(dxdy)>0.\end{aligned}$$

On the other hand, we have from Itô's formula that

$$\mathbb{E}^u_{x,y}\ln X(t)-\ln x = (a_1-\frac{\sigma_1^2}{2})t+\int_0^t\mathbb{E}^u_{x,y}b_{11}X(s)ds+\int_0^t\mathbb{E}^u_{x,y}b_{12}Y(s)ds.$$

Since $\sup_{t\geq 0}\mathbb{E}^u_{x,y}\ln X(t) \leq \sup_{t\geq 0}\ln \mathbb{E}^u_{x,y}X(t) < \infty$, we have

$$\limsup_{T\to\infty}\frac{1}{T}\left(\mathbb{E}^u_{x,y}\ln X(t) - \ln x\right) \leq 0,$$

which leads to

$$\begin{aligned}&a_1 - b_{11}\int_{\mathbb{R}^{2,\circ}_+} x\pi(dxdy) - b_{12}\int_{\mathbb{R}^{2,\circ}_+} y\pi(dxdy)\\&\quad = a_1 - \frac{\sigma_1^2}{2} - \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}^u_{x,y}(b_{11}X(s) + b_{12}Y(s))dsdt \leq 0.\end{aligned}$$

As a result,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}^u_{x,y}X(t) &= \int_{\mathbb{R}^{2,\circ}_+} x\pi(dxdy)\\&\leq \frac{1}{b_{11}}\left(a_1 - \frac{\sigma_1^2}{2} - b_{12}\int_{\mathbb{R}^{2,\circ}_+} y\pi(dxdy)\right) < \frac{2a_1-\sigma_1^2}{b_{11}}.\end{aligned}$$

□

The following lemma enables us to find the optimal control in the class Π_{SRM}.

Lemma 2.4. *For any admissible relaxed control m,*

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}^m_{x,y}\int_0^T X(t)dt \geq \rho^*\\&\quad := \inf\left\{\int_{\mathbb{R}^{2,\circ}_+\times[0,M]} x\pi_v(dx\times dy\times du), v\in\Pi_{SRM}\right\}.\end{aligned} \tag{2.6}$$

Proof of Lemma 2.4. We define $\mathcal{G}_1$ as the class of $\pi_{v,0} = [v(z)(du)]\times\eta_{v,1}(du)$ where v is a relaxed Markov control and $\eta_{v,1}$ is the invariant probability measure of the solution $(X(t), Y(t))$ to (2.1) with $X(0) = 0$, $Y(0)\geq 0$, and $m_t = v(X(t), Y(t))$. Similarly, we define $\mathcal{G}_2$ as the class of $\pi_{v,2} = [v(z)(du)]\times\eta_{v,0}(du)$, where v is a relaxed Markov control and $\eta_{v,2}$ is the invariant probability measure of the solution $(X(t), Y(t))$ to (2.1) with $X(0) > 0, Y(0)\geq 0$, $m_t = v(X(t), Y(t))$. Because of the tightness due to (2.3) and the invariance of the two sets $(0,\infty)\times[0,\infty)$ and $\{0\}\times[0,\infty)$, the existence of $\eta_{v,1}$ and $\eta_{v,2}$ is straightforward. We also have

$$\mathcal{G}_1\subset\mathcal{P}(\{0\}\times[0,\infty)\times[0,M]), \mathcal{G}_2\subset\mathcal{P}((0,\infty)\times[0,\infty)\times[0,M]).$$

Now, for any admissible relaxed control m, define an empirical measure ζ^m_T as a $\mathcal{P}(\mathbb{R}^{2,\circ}_+\times[0,M])$-valued process satisfying that

$$\int_{\mathbb{R}^{2,\circ}_+\times[0,M]} fd\zeta^m_T = \mathbb{E}^m_{x,y}\frac{1}{T}\int_0^T\left(\int_0^M f(X(t),Y(t),u)m_t(du)\right)dt.$$

In view of Lemma 2.2, the family $\{\zeta^m_T, T>0\}$ is tight on $\mathcal{P}(\mathbb{R}^2_+\times[0,M])$. As a result, we can decompose any limit point $\widehat{\zeta}\in\mathcal{P}(\mathbb{R}^2_+\times[0,M])$ as

$$\widehat{\zeta} = p_1\zeta_1 + p_2\zeta_2,$$

with $\zeta_1\in\mathcal{P}(\{0\}\times[0,\infty)\times[0,M])$, and $\zeta_2\in\mathcal{P}((0,\infty)\times[0,\infty)\times[0,M])$.

Because $(0,\infty)\times[0,\infty)$ and $\{0\}\times[0,\infty)$ are two invariant sets for $(X(t), Y(t))$ for any control $m(t)$, we can show that $\zeta_1\in\mathcal{G}_1$ and $\zeta_2\in\mathcal{G}_2$; see [1, Lemma 3.4.6]. For any Markov control v, we have

$$Y(t)\leq\widetilde{Y}(t) \text{ given } X(t) = 0, Y(0)\leq\widetilde{Y}(0)$$

where $\widetilde{Y}(t)$ be the solution to

$$d\widetilde{Y}(t) = (M - a_2\widetilde{Y}(t))dt + \sigma_2\widetilde{Y}(t)dW_2(t)$$

As a result, we have that

$$\begin{aligned}\int_{\mathbb{R}^2_+\times[0,M]} y\zeta_1(dx,dy,u) &\le \limsup_{t\to 0}\frac{1}{t}\int_0^t \mathbb{E}Y(s)ds\\ &\le \lim_{t\to 0}\frac{1}{t}\int_0^t \mathbb{E}\widetilde{Y}(s)ds = \frac{M}{a_2},\end{aligned}\tag{2.7}$$

where the last equality is proved in [16]. We have from (2.7) and (1.3) that

$$\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x - b_{12}y - \frac{\sigma_1^2}{2}\right)\zeta_1(dx,dy,du) \ge a_1 - \frac{Mb_{12}}{a_2} - \frac{\sigma_1^2}{2} > 0. \tag{2.8}$$

For a control v, let $\eta^e_{v,2}$ be an ergodic probability measure of $(X^v(t),Y^v(t))$, the stationary solution to (2.1) under control v with distribution $\eta^e_{v,2}$ on $(0,\infty)\times[0,\infty)$. We have from the strong law of large numbers that

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln X^v(t)}{t} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \left(a_1 - b_{11}X^v(s) - b_{12}Y^v(s)\right)ds + \frac{\sigma_2 W(t)}{t}\\ &= \int_{(0,\infty)\times[0,\infty)]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\eta^e_{v,2}(dx',dy')\\ &:= e_0\end{aligned}\tag{2.9}$$

If $e_0 \ne 0$, it would lead to either $\lim_{t\to\infty} X(t) = 0$ or $\lim_{t\to\infty} X^v(t) = \infty$, both contradicts the assumption that $(X^v(t),Y^v(t))$ is a stationary process with distribution $\eta^e_{v,2}$ on $(0,\infty)\times[0,\infty)$. As a result, for any control v and any invariant probability measure $\eta_{v,2}$ on $(0,\infty)\times[0,\infty)$ of the solution to (2.1) under control v, we have $e_0 = 0$, which yields

$$\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\xi_2(dx',dy',du) = 0. \tag{2.10}$$

Let $\{t_k\}$ be a sequence increasing to infinity such that $\zeta^m_{t_k}$ converges to $\widehat{\zeta}$ as $k\to\infty$. We have

$$\begin{aligned}\lim_{t_k\to\infty}\frac{\mathbb{E}^m_{x,y}\ln X(t)}{t_k} &= \lim_{t\to\infty}\frac{1}{t_k}\int_0^{t_k}\mathbb{E}^m_{x,y}\left(a_1 - b_{11}X^v(s) - b_{12}Y^v(s)\right)ds\\ &= \int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\widehat{\zeta}(dx',dy',du)\\ &= p_1\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\zeta_1(dx',dy',du).\end{aligned}\tag{2.11}$$

In view of (2.3),

$$\lim_{t_k\to\infty}\frac{\mathbb{E}^m_{x,y}\ln X(t)}{t_k} \le 0. \tag{2.12}$$

From (2.11), (2.12), and (2.8), we must have $p_1 = 0$ or $\widehat{\zeta} = \zeta_2$. Now, we can decompose ζ_2 into:

$$\widehat{\zeta} = \zeta_2 = p_3\zeta_3 + p_4\zeta_4$$

where $\zeta_3 \in \mathcal{P}(\mathbb{R}^{2,\circ}_+\times[0,M])$ and ζ_4 satisfying $\zeta_4((0,\infty)\times\{0\}\times[0,M]) = 1$.

Since ζ_3 is in $\mathcal{G}$, Similar to (2.10), we also have

$$\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\zeta_3(dx',dy',du) = 0, \tag{2.13}$$

which together with (2.10) implies that

$$\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - b_{12}y' - \frac{\sigma_1^2}{2}\right)\zeta_4(dx',dy',du) = 0. \tag{2.14}$$

Since $\int_{\mathbb{R}^2_+\times[0,M]} y'\zeta_4(dx',dy',du) = 0$, we deduce from (2.14) that

$$\int_{\mathbb{R}^2_+\times[0,M]}\left(a_1 - b_{11}x' - \frac{\sigma_1^2}{2}\right)\zeta_4(dx',dy',du) = 0$$

or

$$\int_{\mathbb{R}^2_+\times[0,M]} x'\zeta_4(dx',dy',du) = \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}}. \tag{2.15}$$

By the weak convergence, the uniform boundedness of

$$\mathbb{E}^m_{x,y}(b_{21}X(t) + b_{12}Y(t))^{1+p_0}$$

in (2.3), and using (2.15), we have that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{t_n}\mathbb{E}^m_{x,y}\int_0^{t_k} X(t)dt \\
&= p_3\int_{\mathbb{R}^2_+\times[0,M]} x'\zeta_3(dx',dy',du) + p_4\int_{\mathbb{R}^2_+\times[0,M]} x'\zeta_4(dx',dy',du) \\
&\geq p_3\rho^* + \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}}p_4 \geq \rho^*,
\end{aligned}$$

since

$$\rho^* < \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}},$$

because of Lemma 2.3. The proof is complete. □

Let $\mathcal{C}_{1+p}$ be the class of functions $V : \mathbb{R}^{2,\circ}_+ \mapsto \mathbb{R}$ such that

$$|V(x,y)| \leq c_V(1+x+y)^{1+p}, (x,y) \in \mathbb{R}^{2,\circ}_+ \text{ for some } c_V > 0 \text{ and } p \in (0,p_0).$$

To proceed, we state the main result of this paper. The proof of Theorem 2.5 is relegated to the next section.

Theorem 2.5. *There is a unique pair (V,ρ), where $V \in C^2(\mathbb{R}^{2,\circ}_+)\cap\mathcal{C}_{1+p}$ and $\rho \in \mathbb{R}$ satisfying the equation*

$$\min_{u\in[0,M]}\{\mathcal{L}^u V(x,y) + x\} = \rho.$$

Moreover, we have $\rho = \rho^*$ and $v^* \in \Pi_{RM}$ is an optimal control if and only if it is a measurable selector from the minimizer

$$\min_{u\in[0,M]}\{\mathcal{L}^u V(x,y) + x\}.$$

In fact, we can choose

$$v^*(x,y) = \begin{cases} 0 & \text{if } \dfrac{\partial V(x,y)}{\partial x} \geq 0, \\ M & \text{otherwise.}\end{cases}$$

3. **Technical lemmas and proofs of main results.** As mentioned in the introduction, we use the vanishing discount argument. Thus, we need to analyze $V_\gamma(x,y)$, the optimal γ-discounted cost, that is

$$V_\gamma(x,y)=\inf\left\{\mathbb{E}^v_{x,y}\int_0^\infty e^{-\gamma t}X(t)dt : v\in\Pi_{RM}\right\},\ (x,y)\in\mathbb{R}^{2,\circ}_+.$$

We deduce from [1, Theorem 3.5.6 & Remark 3.5.8] that $V_\gamma(x,y)\in C^2(\mathbb{R}^{2,\circ}_+)\cup C_b(\mathbb{R}^{2,\circ}_+)$ satisfies

$$\min_{u\in[0,M]}\{\mathcal{L}^uV_\gamma(x,y)+x\}=\gamma V_\gamma(x,y). \tag{3.1}$$

and the optimal Markov control v_γ is a selector of $\min_{u\in[0,M]}\{\mathcal{L}^uV_\gamma(x,y)+s\}$. The following lemma is from [1, Lemma 3.7.8].

Lemma 3.1. *Fix $(x_*,y_*)\in\mathbb{R}^{2,\circ}_+$. For any sequence $\gamma_n\downarrow 0$, there exists a subsequence, γ_{k_n}, a function $V\in C(\mathbb{R}^{2,\circ}_+)$, and a constant ρ such that as $k_n\to\infty$,*

$$\begin{aligned}&\gamma_{k_n}V_{\gamma_{k_n}}(x_*,y_*)\to\rho\ \text{ and}\\ &\overline{V}_{\gamma_{k_n}}(x,y):=V_{\gamma_{k_n}}(x,y)-V_{\gamma_{k_n}}(x_*,y_*)\to V(x,y)\end{aligned} \tag{3.2}$$

uniformly on each compact subset of $\mathbb{R}^{2,\circ}_+$. Moreover, we have

$$\min_{u\in[0,M]}\{\mathcal{L}^uV(x,y)+x\}=\rho\le\rho^*,\ (x,y)\in\mathbb{R}^{2,\circ}_+.$$

To prove Theorem 2.5, our goal is to show that a limit function V in Lemma 3.1 lies in the family $\mathcal{C}_{1+p}$. To enhance readability, we offer a road map of the proof outlined below. Using a Lyapunov function defined in (3.3), we adopt the idea in [6, 17] to show in Proposition 3.3 that the time $(X(t),Y(t))$ reaches a set $\mathcal{H}$ of the form $\{x+x^{-1}+y<H\}$ has bounded $1+p$-th moment for some $p>0$. The proof of Proposition 3.3 relies on an estimate for the long-run average of $\mathcal{L}^v\widetilde{V}(X(t),Y(t))$, provided in Lemma 3.2. Second, we show that $V(x,y)$, the limit in (3.2), is bounded in $\mathcal{H}$. Since $\mathcal{H}$ is noncompact in $\mathbb{R}^{2,\circ}_+$, we break $\mathcal{H}$ into two sets $\mathcal{H}^\gamma_1$ and $\mathcal{H}^\gamma_2$ in Proposition 3.5 determined by the average of $Y(t)$ on a time interval $[0,T^*]$ under a γ-discounted optimal control. Because $\mathcal{H}$ is noncompact, it is important to estimate the probability where $(X(t),Y(t))$ stays in a compact set of $\mathbb{R}^{2,\circ}_+$, which is given in Lemma 3.4. That estimate helps us to show that $V(x,y)$ is bounded in $\mathcal{H}$ by analyzing the dynamics of $(X(t),Y(t))$ in several cases.

Pick $b_0>0$ such that

$$\widetilde{V}(x,y)=\frac{2}{a_2}(b_{21}x+b_{12}y)-\ln x+b_0\ge 0 \text{ for all } x,y>0.$$

$$\begin{aligned}\mathcal{L}^v\widetilde{V}(x,y)=&\mathcal{L}^v\widetilde{V}_1(x,y)+\mathcal{L}^v\widetilde{V}_2(x,y)\\ =&\frac{2}{a_2}\left(b_{21}x(a_1-b_{11}x)+b_{12}(\overline{v}(x,y)-a_2y)\right)-(a_1-b_{11}x-b_{12}y)+\frac{\sigma_1^2}{2}\\ \le&\frac{2}{a_2}\left(b_{21}x(a_1-b_{11}x)+b_{12}M\right)+b_{11}x-b_{12}y+\frac{\sigma_1^2}{2}.\end{aligned} \tag{3.3}$$

It is easy to see that there exists $H>0$ such that

$$\mathcal{L}^v\widetilde{V}(x,y)\le -1 \text{ if } (x,y)\in\mathbb{R}^{2,\circ}_+, x+y\ge H. \tag{3.4}$$

Lemma 3.2. *Let*

$$\Delta := \frac{1}{5}\left(a_1 - b_{12}\frac{a_2}{M} - \frac{\sigma_1^2}{2}\right) > 0.$$

There exists $T_1 > 0$ such that for any $T_2 > 0$, we can find $\delta_2 = \delta_2(T_2, H) > 0$ satisfying that

$$\frac{1}{T_2}\int_0^{T_2} \mathbb{E}^v_{x,y}\mathcal{L}^v V(X(t),Y(t))dt \le -\Delta < 0 \text{ if } (x,y)\in\mathbb{R}^{2,\circ}_+, x\le\delta_2, x+y\le H.$$

Proof. Noting that

$$dX(t) \le a_1X(t)dt + \sigma_1X(t)dW_1(t),$$

we can easily obtain by using Dynkin's formula, Gronwall's inequality, and the Burkholder-Davis-Gundy inequality (see for e.g., [13, Theorem 2.4.4], that

$$\mathbb{E}\sup_{t\in[0,T]}[X(t)]^2 \le x_0^2e^{2a_3T} \text{ if } X(0)=x_0, \tag{3.5}$$

where $a_3 = a_1 + 2\sigma_1^2$. Let $\overline{Y}(t)$ be the solution to

$$d\overline{Y}(t) = M - (a_2-\delta_1)\overline{Y}(t))dt + \sigma_2\overline{Y}(t)dt.$$

We have the following properties of $\overline{Y}(t)$, which have been proved in [17] (see also [6]).

(C1)

$$\mathbb{E}_y\overline{Y}(t)^{1+p_0} \le (1+y)^{1+p_0}e^{-\overline{c}_0t} + c_1.$$

(C2) There exists a unique invariant probability measure $\overline{\boldsymbol{\mu}}$ of $\overline{Y}(t)$ on $(0,\infty)$ and

$$\mathbb{E}_\mu\overline{Y} = \frac{a_2-\delta_1}{M}.$$

(C3) For any $H>0, \Delta>0$, there exists a $T_1>0$ such that

$$\frac{1}{T}\int_0^T \mathbb{E}_y\overline{Y}(t)dt \le \frac{a_2-\delta_1}{M} + \frac{\Delta}{b_{12}},\ T\ge T_1.$$

Note that

$$\frac{1}{T}\left(\mathbb{E}_{x,y}(b_{21}X(T)+b_{12}Y(T)) - (b_{21}x+b_{12}y)\right) = \int_0^T \mathbb{E}^v_{x,y}\mathcal{L}^vV_1(X(t),Y(t))dt. \tag{3.6}$$

We have from (2.3) and (3.6) that

$$\frac{1}{T}\int_0^T \mathbb{E}^v_{x,y}\mathcal{L}^vV_1(X(t),Y(t))dt \le \frac{K_3(1+x+y)}{T}, T>1. \tag{3.7}$$

for some constant $K_3>0$ independent of T,x,y. Let $\delta_2>0$ to be determined. Define

$$\xi_{\delta_1} = \inf\{t\ge0 : X(t)\ge\delta_1\}.$$

By a comparison theorem, we have $Y(t)\le\overline{Y}(t)$ for $0\le t\le\xi_{\delta_1}$. As a result, for $T_2\ge T_1$, we have

$$\begin{aligned}\frac{1}{T_2}\mathbb{E}^v_{x,y}\int_0^{T_2}\mathbf{1}_{\{\xi_{\delta_1}>T_2\}}Y(t)dt &\le \mathbb{E}_y\frac{1}{T_2}\int_0^{T_2}\mathbf{1}_{\{\xi_{\delta_1}>T_2\}}\overline{Y}(t)dt\\ &\le \frac{1}{T_2}\mathbb{E}_y\int_0^{T_2}\overline{Y}(t)dt \le \frac{a_2-\delta_1}{M}+\frac{\Delta}{b_{12}}.\end{aligned} \tag{3.8}$$

On the other hand, we have from Holder's inequality that

$$\begin{aligned}\frac{1}{T_2}\int_0^{T_2}\mathbb{E}^v_{x,y}\mathbf{1}_{\{\xi_{\delta_1}\le T_2\}}Y(t)dt \le& \frac{1}{T_2}\int_0^{T_2}\left([\mathbb{P}_y\{\xi_{\delta_1}\le T_2\}]^{\frac{p_0}{1+p_0}}\mathbb{E}^v_{x,y}\left(Y(t)^{1+p_0}\right)^{\frac{1}{1+p_0}}\right)dt\\ \le& \left[\mathbb{P}^v_{x,y}\{\xi_{\delta_1}\le T_2\}\right]^{\frac{p_0}{1+p_0}}K_4(H+1)^{1+p_0}\\ \le& [\frac{xe^{a_3T_2}}{\delta_1}]^{\frac{p_0}{1+p_0}}K_4(H+1)^{1+p_0}\text{ if } x+y\le H,\end{aligned}\tag{3.9}$$

where K_4 is independent of x, y, T_1, T_2 and the last inequality is due to an application of Markov's inequality to (3.5). Let $\delta_2=\delta_2(\delta_1,T_2,H)>0$ such that

$$[\frac{\delta_2e^{a_3T_2}}{\delta_1}]^{\frac{p_0}{1+p_0}}K_4(H+1)^{1+p_0}\le\frac{\Delta}{b_{12}}\text{ and }\frac{b_{11}\delta_2}{a_3T_2}e^{a_3T_2}\le\Delta.\tag{3.10}$$

Adding (3.8) and (3.9), we have

$$\begin{aligned}\frac{1}{T_2}\int_0^{T_2}\mathbb{E}^v_{x,y}Y(t)dt\le&\frac{a_2-\delta_1}{M}+\frac{\Delta}{b_{12}}+[\frac{xe^{a_3T_2}}{\delta_1}]^{\frac{p_0}{1+p_0}}K_4(H+1)^{1+p_0}\\ \le&\frac{a_2-\delta_1}{M}+\frac{2\Delta}{b_{12}}\text{ if } x+y\le H\text{ and } x\le\delta_2.\end{aligned}\tag{3.11}$$

If $x\le\delta_2$, we have from (3.5) that

$$\frac{1}{T_2}\mathbb{E}^v_{x,y}\int_0^{T_2}b_{11}X(t)dt=b_{11}\frac{1}{T_2}\int_0^{T_2}xe^{a_2t}dt\le\frac{b_{11}\delta_2}{a_3T_2}e^{a_3T_2}\le\Delta.\tag{3.12}$$

From (3.7), (3.11), and (3.12), if $T_1\ge\frac{K_3(1+H)}{\Delta}$ we have for any $T_2\ge T_1$ and δ_2 satisfying (3.10) that

$$\begin{aligned}\frac{1}{T_2}\int_0^{T_2}\mathbb{E}^v_{x,y}\mathcal{L}^v\widetilde{V}(X(t),Y(t))dt=&\frac{1}{T_2}\mathbb{E}^v_{x,y}\int_0^{T_2}\mathcal{L}^v\widetilde{V}_1(X(t),Y(t))dt\\ &-a_1+\frac{\sigma_1^2}{2}+b_{11}\frac{1}{T_2}\mathbb{E}^v_{x,y}X(t)dt+b_{12}\frac{1}{T_2}\mathbb{E}^v_{x,y}Y(t)dt\\ \le&-a_1+\frac{\sigma_1^2}{2}+b_{12}\frac{a_2-\delta_1}{M}+3\Delta\le-\Delta\end{aligned}\tag{3.13}$$

for any $(x,y)\in\mathbb{R}^{2,\circ}_+, x+y\le H, x\ge\delta_2$. □

Proposition 3.3. *Pick* $0<p<p_0$. *There exists* $A, c^U_1, T_2>0$ *such that*

$$U(x,y)=A(b_{21}x+b_{12}y)^{1+p}+\widetilde{V}^{1+p}(x,y)$$

satisfies

1. $$\mathbb{E}^v_{x,y}U(X(\tau),Y(\tau))\le U(x,y)+c^U_1\mathbb{E}^v_{x,y}\tau$$
   for any stopping time τ *with bounded expectation,*
2. $$\mathbb{E}^v_{x,y}U(X(t),Y(t))\le\max\{(C^*/\kappa^*)^{\frac{p+1}{p}},U(x,y)\}+c^U_1T_2, t\ge0,$$
3. *for any* B *sufficiently large, there exists* $c_B>0$ *such that*
   $$\mathbb{E}^v_{x,y}(\tau^*_B)^{1+p}\le c_BU(x,y),(x,y)\in\mathbb{R}^{2,\circ}_+,v\in\Pi_{RM}$$
   where $\tau^*_B=\inf\{t\ge0:U(X(t),Y(t))\le B\}$.

Proof. Because

$$\frac{1}{T_2}\int_0^{T_2}\mathbb{E}^v_{x,y}\mathcal{L}^v\widetilde{V}(X(t),Y(t))dt\le -\Delta<0 \text{ if } x\le\delta_2, x+y\le H,$$

and $\mathcal{L}^vV(x,y)\le -1$ if $x+y\ge H$, and because of (2.2), (2.2), we can verbatim follow the proof of [2, Proposition 4.9] to show that

$$\mathbb{E}^v_{x,y}U(X(T_2),Y(T_2))\le U(x,y)-\kappa^*U^{\frac{p}{p+1}}(x,y)+C^* \tag{3.14}$$

where

$$U(x,y)=A(b_{21}x+b_{12}y)^{1+p}+V^{1+p}(x,y)$$

for a sufficiently large $A>0$ and $0<p<p_0$, κ^*,C^*,T_2 are positive numbers independent of x,y. It is easy to check that when A is sufficiently large, we have

$$\mathcal{L}U(x,y)\le c_1^U \text{ for any } (x,y)\in\mathbb{R}^{2,\circ}_+. \tag{3.15}$$

As a consequence of (3.14), we have from (37) of [8, Theorem 3.6] that

$$\mathbb{E}^v_{x,y}(\tau^*_B)^{1+p}\le C\left(1+\mathbb{E}^v_{x,y}\sum_{k=0}^{\tau^*_B-1}(k+1)^p\right)\le C_BU(x,y) \tag{3.16}$$

where $\tau^*_B=\inf\{n\ge 0: U(X(nT_2),Y(nT_2))\le B\}$ for any $B>(C^*/\kappa^*)^{\frac{p+1}{p}}$. The proof of part (3) is complete.

We also have from (3.15) and Dynkin's formula that

$$\mathbb{E}^v_{x,y}U(X(\tau\wedge\xi_R),Y(\tau\wedge\xi_R))\le U(x,y)+c_2^U\mathbb{E}^v_{x,y}\tau\wedge\xi_R\le U(x,y)+c_2^U\mathbb{E}^v_{x,y}\tau$$

where $\xi_R=\inf\{t\ge 0: X(t)\vee Y(t)\ge R\}$. Letting $R\to\infty$ and applying Fatou's lemma we obtain part (1) of the proposition When $\tau=T_2$, we have

$$\mathbb{E}^v_{x,y}U(X(t),Y(t))\le U(x,y)+c_1^UT_2, t\in[0,T_2]. \tag{3.17}$$

On the other hand, it is easy to deduce from (3.14) that

$$\mathbb{E}^v_{x,y}U(X(kT_2),Y(kT_2))\le\max\{(C^*/\kappa^*)^{\frac{p+1}{p}},U(x,y)\}, k\in\mathbb{Z}_+. \tag{3.18}$$

In view of (3.18) and (3.17) and the Markov property of $(X(t),Y(t))$ we have that

$$\mathbb{E}^v_{x,y}U(X(t),Y(t))\le\max\{(C^*/\kappa^*)^{\frac{p+1}{p}},U(x,y)\}+c_1^UT_2 \text{ if } nT_2\le(n+1)T_2.$$

Thus, part (2) is proved. □

In what follows, we pick $B>(C^*/\kappa^*)^{p+1}p$. We let $\mathcal{H}=\{(x,y)\in\mathbb{R}^{2,\circ}_+: U(x,y)\le B\}$ and $\tau_{\mathcal{H}}:=\tau^*_B$.

Lemma 3.4. *For any $H>1$ and $\varepsilon>0$, there exists a $\delta=\delta(H,\varepsilon)>0$ such that*

$$\mathbb{P}^v_{x,\varepsilon}\{Y(\tau_{\mathcal{H}})\ge\delta\}\ge\frac{1}{2}, \text{ for any } H^{-1}\le x\le H, v\in\Pi_{RM}. \tag{3.19}$$

Proof. In view of Itô's formula, we have

$$\begin{aligned}\ln Y(\tau_{\mathcal{H}})&\ge\ln Y(0)+\left(-a_1-\frac{\sigma_2^2}{2}\right)\tau_{\mathcal{H}}+\int_0^{\tau_{\mathcal{H}}}X(s)ds+\sigma_2W_2(\tau_{\mathcal{H}})\\&\ge\ln Y(0)-\left(a_1+\frac{\sigma_2^2}{2}\right)\tau_{\mathcal{H}}+\sigma_2W_2(\tau_{\mathcal{H}}).\end{aligned}$$

Note that $\mathbb{E}_{x,\varepsilon}\tau_{\mathcal{H}} \leq C_{H,\varepsilon} < 0$ due to part (3) of Proposition 3.3. Furthermore, we have from the Burkholder-Davis-Gundy inequality that $\mathbb{E}^v_{x,\varepsilon}\sigma_2|W_2(\tau_{\mathcal{H}})| \leq 2\mathbb{E}^v_{x,\varepsilon}\tau_{\mathcal{H}} \leq 2C_{H,\varepsilon}$ As a result, we have from Markov's inequality that

$$\begin{aligned}
&\mathbb{P}^v_{x,\varepsilon}\left\{\ln Y(\tau_{\mathcal{H}}) \leq \ln\varepsilon - 2(a_1 + \frac{\sigma_2^2}{2} + 2)C_{H,\varepsilon}\right\}\\
&\leq \mathbb{P}^v_{x,\varepsilon}\left\{\left(a_1 + \frac{\sigma_2^2}{2}\right)\tau_{\mathcal{H}} + \sigma_2|W_2(\tau_h)| \geq 2(a_1 + \frac{\sigma_2^2}{2} + 2)C_{H,\varepsilon}\right\}\\
&\leq \frac{1}{2(a_1 + \frac{\sigma_2^2}{2} + 2)C_{H,\varepsilon}}\left(\left(a_1 + \frac{\sigma_2^2}{2}\right)\mathbb{E}^v_{x,\varepsilon}\tau_{\mathcal{H}} + \sigma_2\mathbb{E}^v_{x,\varepsilon}|W_2(\tau_h)|\right)\\
&\leq \frac{1}{2},
\end{aligned}$$

which leads to

$$\mathbb{P}^v_{x,\varepsilon}\left(Y(\tau_{\mathcal{H}}) \geq \varepsilon\exp\left(-2(a_1 + \frac{\sigma_2^2}{2} + 2)C_{H,\varepsilon}\right)\right) \geq \frac{1}{2}.$$

The proof is complete. □

Proposition 3.5. *Let $V(x,y)$ be a limit in (3.2). We have*

$$\sup_{(x,y)\in\mathcal{H}} |V(x,y)| < \infty.$$

Proof. Let v_γ be the optimal Markov control of the γ-discounted control problem. Let $\varepsilon_* = \frac{1}{5}\left(\frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}} - \rho^*\right)$, and $\overline{\rho} = \rho^* + \varepsilon_* = \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}} - 4\varepsilon^*$. In view of (2.3), there is $C_{\mathcal{H}} > 0$ such that

$$\mathbb{E}^v_{x,y}X(t) \leq C_{\mathcal{H}} \text{ and } C_B\max\{(C^*/\kappa^*)^{\frac{p+1}{p}}, U(x,y)\} + C_Bc_1^UT_2 \leq C_{\mathcal{H}}$$

for any $(x,y)\in\mathcal{H}, t\geq 0$. Thus, we can pick $T^* > \dfrac{\overline{\rho}C_{\mathcal{H}} + 1}{\varepsilon_*}$ such that

$$\frac{\mathbb{E}^v_{x,y}\ln X(t) - \ln x}{t} \leq \frac{\mathbb{E}^v_{x,y}X(t) - \ln x}{t} \leq \varepsilon_* \text{ for any } t\geq T^*, (x,y)\in\mathcal{H}. \tag{3.20}$$

By virtue of (3.2), there is a γ_* sufficiently small such that

$$\gamma V_\gamma(x_*, y_*) \leq \overline{\rho}, \text{ for } \gamma < \gamma_*. \tag{3.21}$$

We can also assume γ_* satisfies

$$(1 - e^{-\gamma_*T^*})C_{\mathcal{H}} \leq \varepsilon_*. \tag{3.22}$$

Now for each $\gamma < \gamma_*$, we split $\mathcal{H}$ into two disjoint subsets (one of which is probably empty):

$$\mathcal{H}_1^\gamma = \left\{(x,y)\in\mathcal{H} : b_{12}\frac{1}{T^*}\int_0^{T^*}\mathbb{E}^{v_\gamma}_{x,y}Y(s)ds \leq b_{11}\varepsilon^*\right\} \text{ and } \mathcal{H}_2^\gamma = \mathcal{H}\setminus\mathcal{H}_1^\gamma.$$

In view of Itô's formula, we have

$$\frac{\mathbb{E}^v_{x,y}\ln X(t) - \ln x}{t} = a_1 - \frac{\sigma_1^2}{2} - b_{11}\frac{1}{t}\int_0^t\mathbb{E}^v_{x,y}X(s)ds - b_{12}\frac{1}{t}\int_0^t\mathbb{E}^v_{x,y}Y(s)ds$$

which together with (3.20) implies

$$\frac{1}{T^*}\int_0^{T^*}\mathbb{E}^{v_\gamma}_{x,y}X(s)ds \geq \frac{a_1 - \frac{\sigma_1^2}{2}}{b_{11}} - 2\varepsilon_* \text{ if } (x,y)\in\mathcal{H}_1^\gamma. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\begin{aligned}\mathbb{E}^{v_\gamma}_{x,y}\int_0^{T^*} e^{-\gamma t}X(t)dt =&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{T^*} X(t)dt - \mathbb{E}^{v_\gamma}_{x,y}\int_0^{T^*}(1-e^{-\gamma t})X(t)dt\\ \geq&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{T^*} X(t)dt - C_{\mathcal{H}}\int_0^{T^*}(1-e^{-\gamma t})dt\\ \geq&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{T^*} X(t)dt - \varepsilon_*T^*\\ \geq&(\frac{a_1-\frac{\sigma_1^2}{2}}{b_{11}}-2\varepsilon_*)T^*-\varepsilon_*T^*\geq(\rho^*+2\varepsilon_*)T^*,\\ &\text{for } (x,y)\in\mathcal{H}_1^\gamma, \gamma<\gamma_*.\end{aligned}\tag{3.24}$$

By (3.21), we deduce that

$$0\leq\inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)\leq\frac{\overline{\rho}}{\gamma}<\infty \text{ for } \gamma\leq\gamma_*.\tag{3.25}$$

Since $(X(t),Y(t))$ is a strong Markov process under a Markov control, we have from (3.25) that

$$\begin{aligned}V_\gamma(x,y) =&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{\tau_{\mathcal{H}}^{T^*}} e^{-\gamma t}X(t)dt + \mathbb{E}^{v_\gamma}_{x,y}\left[e^{-\gamma\tau_{\mathcal{H}}^{T^*}}V_\gamma\left(X(\tau_{\mathcal{H}}^{T^*}),Y(\tau_{\mathcal{H}}^{T^*})\right)\right]\\ \geq&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{\tau_{\mathcal{H}}^{T^*}} e^{-\gamma t}X(t)dt + \left(\inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)\right)\mathbb{E}^{v_\gamma}_{x,y}e^{-\gamma\tau_{\mathcal{H}}^{T^*}}\\ \geq&\mathbb{E}^{v_\gamma}_{x,y}\int_0^{\tau_{\mathcal{H}}^{T^*}} e^{-\gamma t}X(t)dt + \inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)-\frac{\overline{\rho}}{\gamma}\mathbb{E}^{v_\gamma}_{x,y}(1-e^{-\gamma\tau_{\mathcal{H}}^{T^*}})\\ &\text{for } (x,y)\in\mathbb{R}^{2,\circ}_+,\end{aligned}\tag{3.26}$$

where $\tau_{\mathcal{H}}^{T^*}=\inf\{t\geq T^*:(X(t),Y(t))\in\mathcal{H}\}$. By the Markov property of $(X(t),Y(t))$ and Proposition 3.3, we have

$$\begin{aligned}\frac{1}{\gamma}\mathbb{E}^{v_\gamma}_{x,y}(1-e^{-\gamma\tau_{\mathcal{H}}^{T^*}})\leq&\mathbb{E}^{v_\gamma}_{x,y}\tau_{\mathcal{H}}^{T^*}\leq T^*+\mathbb{E}^{v_\gamma}_{x,y}(\tau_{\mathcal{H}}^{T^*}-T^*)\\ =&T^*+\mathbb{E}^{v_\gamma}_{x,y}\mathbb{E}^{v_\gamma}_{X(T^*),Y(T^*)}\tau_{\mathcal{H}}\\ \leq&T^*+\mathbb{E}^{v_\gamma}_{x,y}C_BU(X(T^*),Y(T^*))\\ \leq&T^*+C_B\max\{(C^*/\kappa^*)^{\frac{p+1}{p}},U(x,y)\}+C_Bc_1^UT_2\leq T^*+C_{\mathcal{H}}.\end{aligned}\tag{3.27}$$

Combining (3.27), (3.24), and (3.26), we have

$$\begin{aligned}V_\gamma(x,y)\geq&\inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)+(\rho^*+2\varepsilon_*)T^*-\overline{\rho}(T^*+C_{\mathcal{H}})\\ \geq&\inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)+\varepsilon_*T^*-\overline{\rho}C_{\mathcal{H}} \text{ (since } \overline{\rho}=\rho^*+\varepsilon_*)\\ >&\inf_{(x,y)\in\mathcal{H}}V_\gamma(x,y)+1 \text{ (since } T^*>\frac{\overline{\rho}C_{\mathcal{H}}+1}{\varepsilon_*}),\end{aligned}$$

which is followed by

$$\overline{V}_\gamma(x,y)\geq\inf_{(x,y)\in\mathcal{H}}\overline{V}_\gamma(x,y)+1,(x,y)\in\mathcal{H}_1^\gamma,\gamma\leq\gamma_*.$$

As a consequence,

$$\inf_{(x,y)\in\mathcal{H}} \overline{V}_\gamma(x,y) = \inf_{(x,y)\in\mathcal{H}_2^\gamma} \overline{V}_\gamma(x,y) \text{ for } \gamma \le \gamma_*. \tag{3.28}$$

Now, for $(x,y)\in\mathcal{H}_2^\gamma$, there exists $t_{x,y}\in[0,T^*]$ such that

$$\mathbb{E}_{x,y}^{v_\gamma} Y(t_{x,y}) \ge \varepsilon_* \frac{b_{11}}{b_{12}}.$$

On the other hand, since $\mathbb{E}_{x,y}^{v_\gamma} Y^{1+p_0}(t_{x,y}) \le C_{\mathcal{H}}$, we have

$$\begin{aligned}
&\mathbb{E}_{x,y}^{v_\gamma}\left[\mathbf{1}_{\{Y(t_{x,y})\le \varepsilon_*\frac{b_{11}}{2b_{12}}\}} Y(t_{x,y})\right]\\
&\le \left(\mathbb{P}_{x,y}^{v_\gamma}\{Y(t_{x,y}) \ge \varepsilon_* \frac{b_{11}}{2b_{12}}\}\right)^{\frac{p_0}{1+p_0}} \left(\mathbb{E}_{x,y}^{v_\gamma} Y^{1+p_0}(t)\right)^{\frac{p_0}{1+p_0}}\\
&\le \left(\mathbb{P}_{x,y}^{v_\gamma}\{Y(t_{x,y}) \ge \varepsilon_* \frac{b_{11}}{2b_{12}}\}\right)^{\frac{p_0}{1+p_0}} C_{\mathcal{H}}^{\frac{1}{1+p_0}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_{x,y}^{v_\gamma}\left[\mathbf{1}_{\{Y(t_{x,y})\ge \varepsilon_*\frac{b_{11}}{2b_{12}}\}} Y(t_{x,y})\right]\\
&= \mathbb{E}_{x,y}^{v_\gamma} Y(t_{x,y}) - \mathbb{E}_{x,y}^{v_\gamma}\left[\mathbf{1}_{\{Y(t_{x,y})\le \varepsilon_*\frac{b_{11}}{2b_{12}}\}} Y(t_{x,y})\right]\\
&\ge \varepsilon_* \frac{b_{11}}{b_{12}} - \varepsilon_* \frac{b_{11}}{2b_{12}} = \varepsilon_* \frac{b_{11}}{2b_{12}}.
\end{aligned}$$

Combining the two displayed equations above, we have

$$\mathbb{P}_{x,y}^{v_\gamma}\{Y(t_{x,y}) \ge \widehat{\varepsilon}_* := \varepsilon_* \frac{b_{11}}{2b_{12}}\} \ge \mathcal{C}_h^{-\frac{1}{p_0}} \left(\varepsilon_* \frac{b_{11}}{2b_{12}}\right)^{\frac{1+p_0}{p_0}} =: 4\overline{q}.$$

As a result,

$$\mathbb{P}_{x,y}^{v_\gamma}\{\eta \le T^*\} \ge 4\overline{q}, \text{ for } (x,y)\in\mathcal{H}_2^\gamma, \tag{3.29}$$

where

$$\eta = \inf\left\{t\ge 0 : Y(t) \ge \widehat{\varepsilon}_*\right\}.$$

Defining

$$\tau_{\mathcal{H}}^\eta = \inf\{t \ge \eta\wedge T^* : (X(t),Y(t))\in\mathcal{H}\},$$

we have

$$\begin{aligned}
\mathbb{E}_{x,y}^{v_\gamma}\tau_{\mathcal{H}}^\eta =& \mathbb{E}_{x,y}^{v_\gamma}(\eta\wedge T^*) + \mathbb{E}_{x,y}^{v_\gamma}(\tau_{\mathcal{H}}^\eta - (\eta\wedge T^*))\\
\le& T^* + \mathbb{E}_{x,y}^{v_\gamma}\left[\mathbb{E}_{X(\eta\wedge T^*),Y(\eta\wedge T^*)}^{v_\gamma}\tau_{\mathcal{H}}\right]\\
\le& T^* + \mathbb{E}_{x,y}^{v_\gamma}[c_B U(X(\eta\wedge T^*),Y(\eta\wedge T^*))]^{\frac{1}{1+p}}\\
\le& T^* + \left(\mathbb{E}_{x,y}^{v_\gamma}[c_B U(X(\eta\wedge T^*),Y(\eta\wedge T^*))]\right)^{\frac{1}{1+p}}\\
\le& T^* + c_B(U(x,y) + c_U\mathbb{E}_{x,y}^{v_\gamma}(\eta\wedge T^*))^{\frac{1}{1+p}}\\
\le& T^* + c_B(U(x,y)+c_U T^*)^{\frac{1}{1+p}}.
\end{aligned} \tag{3.30}$$

Since

$$\begin{aligned}
\mathbb{E}_{x,y}^{v_\gamma} U(X(\eta\wedge T^*),Y(\eta\wedge T^*)) &\le U(x,y) + c_1^U T^*\\
&\le H_1 := \sup_{(x,y)\in\mathcal{H}}\{U(x,y)\} + c_1^U T^*,
\end{aligned} \tag{3.31}$$

applying Markov's inequality and using (3.29), (3.31), we have

$$\mathbb{P}^{v_\gamma}_{x,y}\left\{\eta \le T^* \text{ and } U(X(\eta),Y(\eta)) \le \frac{H_1}{2\overline{q}}\right\} \ge 2\overline{q}, \text{ for } (x,y)\in\mathcal{H}_2^\gamma. \tag{3.32}$$

In view of Lemma 3.4, there exists a $\delta_*>0$ depending only on $\frac{H_1}{2\overline{q}}$ and $\widehat{\varepsilon}_*$ such that

$$\mathbb{P}^{v_\gamma}_{x,y}\{X(\tau_{\mathcal{H}})\ge\delta_*\} > \frac{1}{2}, \text{ if } U(x,y)\le\frac{H_1}{2\overline{q}}, y\ge\widehat{\varepsilon}_*, \gamma\in(0,1). \tag{3.33}$$

We can assume without loss of generality that $\delta_*<x_*$ and define $\mathcal{H}_3=\{(x,y)\in\mathcal{H}: y\ge\delta_*\}$. Observe that $\mathcal{H}_3$ is a compact subset of $\mathbb{R}^{2,\circ}_+$. Because of the strong Markov property of $(X(t),Y(t))$ under the Markov control v_γ and because of (3.32) and (3.33), we can estimate

$$\begin{aligned}
&\mathbb{P}^{v_\gamma}_{x,y}\left\{(X(\tau^\eta_{\mathcal{H}}),Y(\tau^\eta_{\mathcal{H}}))\in\mathcal{H}_3\right\}\\
&\ge \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{\left\{\eta\le T^* \text{ and } U(X(\eta),Y(\eta))\le\frac{H_1}{2\overline{q}}\right\}}\mathbf{1}_{\{(X(\tau^\eta_{\mathcal{H}}),Y(\tau^\eta_{\mathcal{H}}))\in\mathcal{H}_3\}}\right]\\
&= \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{\left\{\eta\le T^* \text{ and } U(X(\eta),Y(\eta))\le\frac{H_1}{2\overline{q}}\right\}}\mathbb{E}^{v_\gamma}_{X(\eta),Y(\eta)}\mathbf{1}_{\{X(\tau_{\mathcal{H}})\ge\delta_*\}}\right]\\
&\ge \frac{1}{2}\mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{\left\{\eta\le T^* \text{ and } U(X(\eta),Y(\eta))\le\frac{H_1}{2\overline{q}}\right\}}\right]\\
&\ge \overline{q}.
\end{aligned} \tag{3.34}$$

Define events $A=\{(X(\tau^\eta_{\mathcal{H}}),Y(\tau^\eta_{\mathcal{H}}))\in\mathcal{H}_3\}$ and $A^c=\Omega\setminus A$, we have the estimate

$$\begin{aligned}
\overline{V}_\gamma(x,y) =& \mathbb{E}^{v_\gamma}_{x,y}\int_0^{\tau^\eta_{\mathcal{H}}} e^{-\gamma t}X(t)dt + \mathbb{E}^{v_\gamma}_{x,y}\left[e^{-\gamma\tau^\eta_{\mathcal{H}}}V_\gamma\left(X(\tau^\eta_{\mathcal{H}}),Y(\tau^\eta_{\mathcal{H}})\right)\right] - V_\gamma(x_*,y_*)\\
\ge& \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_A e^{-\gamma\tau^\eta_{\mathcal{H}}}\inf_{(s',x')\in\mathcal{H}_3}V_\gamma(x',y')\right]\\
&\quad + \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{A^c} e^{-\gamma\tau^\eta_{\mathcal{H}}}\inf_{(x',y')\in\mathcal{H}}V_\gamma(x',y')\right] - V_\gamma(x_*,y_*)\\
=& \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_A\left(\inf_{(x',y')\in\mathcal{H}_3}V_\gamma(x',y') - V_\gamma(x_*,y_*)\right)\right]\\
&+ \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{A^c}\left(\inf_{(x',y')\in\mathcal{H}}V_\gamma(x',y') - V_\gamma(x_*,y_*)\right)\right]\\
&- \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_A(1-e^{-\gamma\tau^\eta_{\mathcal{H}}})\inf_{(x',y')\in\mathcal{H}_3}V_\gamma(x',y')\right]\\
&- \mathbb{E}^{v_\gamma}_{x,y}\left[\mathbf{1}_{A^c}(1-e^{-\gamma\tau^\eta_{\mathcal{H}}})\inf_{(x',y')\in\mathcal{H}}V_\gamma(x',y')\right].
\end{aligned}$$

Since $\overline{V}_\gamma(x,y)=V_\gamma(x,y)-V_\gamma(x_*,y_*)\to V(x,y)$ as $\gamma\to0$ uniformly on each compact set, there exists an $H_2>0$ such that $|\overline{V}_\gamma(x,y)|<H_2$ for $(x,y)\in\mathcal{H}_3$ when γ is sufficiently small. We also have

$$\begin{aligned}
0\le \inf_{(x',y')\in\mathcal{H}}V_\gamma(x',y') &\le \inf_{(x',y')\in\mathcal{H}_3}V_\gamma(x',y')\\
&\le V_\gamma(x_*,y_*)\le\frac{\overline{\rho}}{\gamma}
\end{aligned}$$

when γ is sufficiently small. This together with (3.30), (3.34), and $\inf_{(x,y)\in\mathcal{H}}\overline{V}_\gamma(x,y)\le 0$ yields that

$$\begin{aligned}\overline{V}_\gamma(x,y)\ge&-H_2\mathbb{P}^{v_\gamma}_{x,y}(A)+\inf_{(x',y')\in\mathcal{H}}\overline{V}_\gamma(x',y')\mathbb{P}^{v_\gamma}_{x,y}(A^c)-\overline{\rho}\mathbb{E}^{v_\gamma}_{x,y}\left[\frac{1-e^{-\gamma\tau^\eta_{\mathcal{H}}}}{\gamma}\right]\\ \ge&-H_2\mathbb{P}^{v_\gamma}_{x,y}(A)+\inf_{(x',y')\in\mathcal{H}}\overline{V}_\gamma(x',y')\mathbb{P}^{v_\gamma}_{x,y}(A^c)-\overline{\rho}\mathbb{E}^{v_\gamma}_{x,y}\tau^\eta_{\mathcal{H}}\\ \ge&-H_2-\overline{\rho}\left(T^*+c_B(U(x,y)+c_UT^*)^{\frac{1}{1+p}}\right)+\inf_{(x',y')\in\mathcal{H}}\overline{V}_\gamma(x',y')\mathbb{P}^{v_\gamma}_{x,y}(A^c),\\ \ge&-H_2-\overline{\rho}\left(T^*+c_B(U(x,y)+c_UT^*)^{\frac{1}{1+p}}\right)+\inf_{(x',y')\in\mathcal{H}}\overline{V}_\gamma(x',y')(1-\overline{q}),\\ &\text{for any }(x,y)\in\mathcal{H}^\gamma_2,\end{aligned}$$

which combined with (3.28) leads to

$$\begin{aligned}\inf_{(x,y)\in\mathcal{H}}\overline{V}_\gamma(x,y)\ge-H_2-\overline{\rho}\sup_{(x,y)\in\mathcal{H}}&\left(T^*+c_B(U(x,y)+c_UT^*)^{\frac{1}{1+p}}\right)\\ &+\inf_{(x,y)\in\mathcal{H}}\overline{V}_\gamma(x,y)(1-\overline{q}),\end{aligned}$$

or

$$\inf_{(x,y)\in\mathcal{H}}\overline{V}_\gamma(x,y)\ge-\frac{1}{\overline{q}}\left(H_2+\overline{\rho}\sup_{(x,y)\in\mathcal{H}}\left(T^*+c_B(U(x,y)+c_UT^*)^{\frac{1}{1+p}}\right)\right)\ge-H_3 \tag{3.35}$$

for some H_3 independent of (x,y).

Let $v_c\equiv M$ be the constant control. Similar to the proof of [16, Lemma 4.2], we can show that

$$H_4:=\sup_{(x,y)\in\mathcal{H}}\mathbb{E}^{v_c}_{x,y}\tau_{\mathcal{H}_3}<\infty.$$

Noting that $\mathbb{E}^v_{x,y}|W(\tau_{\mathcal{H}_3})|\le 2\mathbb{E}^v_{x,y}\tau_{\mathcal{H}_3}<\infty$, and $\ln X(\tau_{\mathcal{H}_3})$ is bounded, we must have from

$$\ln X(\tau_{\mathcal{H}_3})=\ln X(0)+(a_1-\frac{\sigma_1^2}{2})\tau_{\mathcal{H}_3}-\int_0^{\tau_{\mathcal{H}_3}}(b_{11}X(s)+b_{12}Y(s))ds+\sigma_1W(\tau_{\mathcal{H}_3}).$$

that

$$\begin{aligned}b_{11}\mathbb{E}^v_{x,y}\int_0^{\tau_{\mathcal{H}_3}}X(s)\le&\int_0^{\tau_{\mathcal{H}_3}}(b_{11}X(s)+b_{12}Y(s))ds\\ &\le\ln x-\mathbb{E}^v_{x,y}\ln X(\tau_{\mathcal{H}_3})+(a_1-\frac{\sigma_1^2}{2})\mathbb{E}^v_{x,y}\tau_{\mathcal{H}_3}+\sigma_1\mathbb{E}^v_{x,y}W(\tau_{\mathcal{H}_3})\\ &\le\ln x-\sup_{\{(x',y')\in\mathcal{H}_3\}}\ln(x')+(a_1-\frac{\sigma_1^2}{2})H_4\le H_5\end{aligned} \tag{3.36}$$

where H_5 is a constant independent of $(x,y)\in\mathcal{H}$.

Using [1, Eq. (3.7.47)], we have

$$\begin{aligned}V(x,y)\le&\mathbb{E}^{v_c}_{x,y}\left[\int_0^{\tau_{\mathcal{H}_3}}X(t)dt+V(X(\tau_{\mathcal{H}_3}),Y(\tau_{\mathcal{H}_3}))\right]\\ \le&\frac{H_5}{b_{11}}+\sup_{(x',y')\in\mathcal{H}_3}V(x',y'),(x,y)\in\mathcal{H}.\end{aligned} \tag{3.37}$$

From (3.35) and (3.37), we obtain that

$$\sup_{(x,y)\in\mathcal{H}} |V(x,y)| < \infty.$$

The proof of Proposition 3.5 is complete. □

Now, we are in a position to prove Theorem 2.5.

Proof of Theorem 2.5. We first show that $V \in \mathcal{C}_1$. For any $(x,y) \in \mathbb{R}^{2,\circ}_+$, we have

$$\begin{aligned}\overline{V}_\gamma(x,y) =& \mathbb{E}^{v_\gamma}_{x,y}\int_0^{\tau_\mathcal{H}} e^{-\gamma t}X(t)dt + \mathbb{E}^{v_\gamma}_{x,y}\left[e^{-\gamma\tau_\mathcal{H}}V_\gamma\left(X(\tau_\mathcal{H}),Y(\tau^T_\mathcal{H})\right)\right] - V_\gamma(x_*,y_*)\\ \geq& \mathbb{E}^{v_\gamma}_{x,y}V_\gamma\left(X(\tau_\mathcal{H}),Y(\tau^T_\mathcal{H})\right) - \mathbb{E}^{v_\gamma}_{x,y}\left[(1-e^{-\gamma\tau_\mathcal{H}})V_\gamma\left(X(\tau_\mathcal{H}),Y(\tau^T_\mathcal{H})\right)\right]\\ &\qquad - V_\gamma(x_*,y_*)\\ \geq& \inf_{(x',y')\in\mathcal{H}} V_\gamma(x',y') - V_\gamma(x_*,y_*) - \mathbb{E}^{v_\gamma}_{x,y}\frac{\rho}{\gamma}(1-e^{-\gamma\tau_\mathcal{H}})\\ \geq& \inf_{(s',x')\in\mathcal{H}} \overline{V}_\gamma(s',x') - \mathbb{E}^{v_\gamma}_{x,y}\tau_\mathcal{H}\\ \geq& -\left(H_3 + (c_B U(x,y))^{\frac{1}{1+p}}\right) \quad \text{due to (3.35) and Proposition 3.3.}\end{aligned} \tag{3.38}$$

Similar to (3.37), we have

$$\begin{aligned}V(x,y) \leq& \mathbb{E}^{v_c}_{x,y}\left[\int_0^{\tau_\mathcal{H}} X(t)dt + V(X(\tau_\mathcal{H}),Y(\tau_\mathcal{H}))\right]\\ \leq& \sup_{(x',y')\in\mathcal{H}}|V(x',y')| + \frac{1}{b_{11}}\left(-\ln x - \sup_{(x',y')\in\mathcal{H}}\ln(x') + (a_1 - \frac{\sigma_1^2}{2})\mathbb{E}^{v_c}_{x,y}\tau_\mathcal{H}\right)\\ \leq& \sup_{(x',y')\in\mathcal{H}}|V(x',y')| + \frac{1}{b_{11}}\left(-\ln x - \sup_{(x',y')\in\mathcal{H}}\ln(x') + a_1\left(c_B U(x,y)\right)^{\frac{1}{1+p}}\right).\end{aligned} \tag{3.39}$$

Combining (3.38), (3.39), Proposition 3.5, and noting that

$$\begin{aligned}&\sup_{(x',y')\in\mathcal{H}} \ln(x') < \infty \quad \text{and}\\ &U(x,y)^{\frac{1}{1+p}} \leq C(x+y-\ln x+1),\end{aligned}$$

we have $V(x,y) \in \mathcal{C}_{1+p}$. This property together with Proposition 3.3 gives conditions needed to verbatim mimic [1, Theorem 3.7.11 and Theorem 3.7.12] to obtain the desired result. □

4. **Final remarks.** We have developed optimal strategies for a class of longrun average costs of Lotka-Volterra systems. We established the existence and uniqueness of optimal controls characterized by the solution to the HJB equation. Because the predator-prey model is a nonlinear non-monotone system, our conjecture is that the optimal strategy to control the population of the prey is not always adding maximum amount of its predator to the system. To prove this conjecture, we need to run numerical solutions to the optimal control problem. Note that numerical solutions to ergodic optimal control problems have not been very well understood. Moreover, we do not have an exponential bound for the returning time. That is, we are not able to show that $\mathbb{E}^v_{x,y}\exp\{\theta\tau^*_B\}$ is bounded, where τ^*_B is defined in Proposition 3.3, thus to show numerical solutions are good approximations are nontrivial and this will be the next step of our research.

Moreover, although our primary motivation is from biological controls, as alluded to in the introduction, Lotka-Volterra systems have also been used in particle systems, and social networks, among others. Thus the optimal controls obtained in this work have a wider range of applications.

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