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Analyzing a class of stochastic SIRS models under imperfect vaccination

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ABSTRACT

This work is devoted to a class of stochastic infectious disease models, namely, stochastic SIRS models under imperfect vaccination. Because of the inclusion of the vaccination, the underlying models are more difficult to analyze than that of the previously considered in the literature. Our main effort is devoted to overcoming the difficulties and deriving a complete characterization of longtime behavior of systems. Numerical examples are provided to present computational evidence. These examples also provide insight that the discrete event process (the random switching) can reverse persistence to extinction, and vice versa.

1. Introduction

Motivations. Because of the importance, in the past decades, much attention has been devoted to modeling epidemic systems, analyzing their dynamic behaviors, and predicting the future. In the human history, infectious diseases have been making significant impacts on the health, economics, and social life of the population. At the forefront of scientific research, the compartment-type models introduced by Kermack and McKendrick [1,2] have received much attention. These models have been developed substantially and used widely for epidemic systems in recent years. The rational is that the population will be divided into several non-overlapping classes. For example, the so-called SIR epidemic model divides population into susceptible, infected, and recovered classes, which is suitable for diseases with permanent immunity such as rubella, whooping cough, measles, smallpox, etc. However, for some diseases such as common colds, influenza, etc., an infected individual can be reinfected again after recovery (that is, the individual returns to the susceptible class after recovery). This leads to the so-called SIRS model, in which the term SIRS refers for the cycle "susceptible (S) → infected (I) → recovered (R) → susceptible (S) again" in the dynamics. SIRS-type model has been recognized as one of the most important models in epidemiology and mathematical biology; see [3–8] and the references therein.

To control the spread of diseases, it is important to develop vaccines and investigate the efficiency of the vaccinations. In recent years, there is a resurgent effort in the study of epidemic models with vaccination; see [9–14] and the references therein. From

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a different angle, it has also been well-recognized that the so-called hybrid systems are more versatile and more appropriate for describing dynamic systems. Such systems feature the coexistence of continuous dynamics and discrete events and their interactions. They may be considered as a two component processes. One of the components describes the continuous dynamics and the other component depicts the discrete events. This added discrete component can be used to model factors such as the changes due to seasons, infection status with other diseases, health history, etc. A class of models that has become popular is the so-called regime-switching diffusion models, in which the continuous dynamics are given by diffusion processes whereas the discrete events are pure random jump processes. Suppose that the discrete environment (the switching process) has a finite number of configurations or regimes. Denote the totality of the discrete configurations by $\mathcal{M} = \{1, \dots, m_0\}$. At any given instance, the system resides in one of the states $\imath \in \mathcal{M}$. The corresponding continuous component is a diffusion with drift and diffusion coefficients depending on $\imath$. Then a random event trigged a jump from $\imath$ to $\ell \neq \imath$. The corresponding continuous component follows another diffusion with drift and diffusion coefficients depending on ℓ. A description of such a behavior can be found in [15, Chapter 1.3, pp. 4–5]; see also the applications [16,17] and references therein. Taking these factors into consideration, we study the following SIRS with vaccination, termed regime-switching SVIRS epidemic model:

$$
\begin{cases}
dS(t) = & \Big[\big(1-\widetilde{q}(\xi(t))\big) A(\xi(t)) - I(t)g(S(t)+\beta V(t), I(t), \xi(t)) - \big(\alpha_0(\xi(t)) + \widetilde{p}(\xi(t))\big) S(t) \\
& + \alpha_2(\xi(t))R(t) + \gamma(\xi(t))V(t) \Big] dt + \sigma_1(\xi(t))S(t)dW_1(t), \\
dV(t) = & \Big[\widetilde{q}(\xi(t))A(\xi(t)) + \widetilde{p}(\xi(t))S(t) - \big(\alpha_0(\xi(t)) + \gamma(\xi(t))\big)V(t)\Big]dt \\
& + \sigma_2(\xi(t))V(t)dW_2(t), \\
dI(t) = & \Big[I(t)g(S(t)+\beta V(t), I(t), \xi(t)) - \big(\alpha_0(\xi(t)) + \alpha_1(\xi(t)) + \alpha_3(\xi(t))\big)I(t)\Big]dt \\
& + \sigma_3(\xi(t))I(t)dW_3(t), \\
dR(t) = & \Big[\alpha_3(\xi(t))I(t) - \big(\alpha_0(\xi(t)) + \alpha_2(\xi(t))\big)R(t)\Big]dt + \sigma_4(\xi(t))R(t)dW_4(t), \\
S(0) = & s \ge 0, \quad I(0) = i \ge 0, \quad V(0) = v \ge 0, \quad R(t) = r \ge 0, \quad \xi(0) = k \in \mathcal{M},
\end{cases}
\tag{1.1}
$$

In the above, $S(t)$ denotes the number of susceptible individuals who are infection-prone, $I(t)$ denotes the number of the infected individuals who have already contracted the disease, $V(t)$ is the number of the individuals who are vaccinated, $R(t)$ denotes the number of individuals recovering from infection, $\{W_i(t)\}$, $i = 1, \dots, 4$, are independent standard Brownian motions representing the white noise, $\xi(t)$ is a finite-state continuous-time Markov chain that is independent of the Brownian motions $W_i(t)$ and that has state space $\mathcal{M} = \{1, \dots, m_0\}$. In the model, there are a number of parameters: $\widetilde{q}$ is a fraction of vaccinated newborns; A is an input of new members into the population; $1-\beta$ is the efficiency of vaccination (i.e., β is the proportion of vaccinated individuals for who the vaccination fails to protect); $\widetilde{p}$ is the proportional coefficient of vaccinated for the susceptible; γ is the rate for immunity loss of vaccines after some time; α_0 is the nature death rate of the population; α_1 is the disease-related death rate; α_2 is the rate of recovered individual becoming susceptible again; α_3 is the rate of recovery from infection; σ_i, $i = 1, 2, 3, 4$ are intensities of the noise. All these constants are assumed to be positive and depend on $\xi(t)$. The rate (in time) of how many individuals in susceptible group get infected is represented by a function named "incidence rate function". Numerous types of incidence rates have been modeled and considered in the literature such as the bilinear functional response k_0SI (see [18]), the nonlinear functional response $\frac{k_0SI^l}{1+k_1I^h}$ (see [19,20]), and the Beddington–DeAngelis functional response $\frac{k_0SI}{1+k_1S+k_2I}$ (see [21,22]), etc. It is clear that if there is no infection in the population, then there is no disease transmission and then, the incidence rate function will be 0. As a result, the incidence rate in general can also be factorized by $I(t)$. Because of this, and also for the convenience of stating results later, we will write the incidence rate function in the form Ig for some function $g(\cdot, \cdot, \cdot)$.

Our goal in this paper is to provide a complete characterization of longtime behavior of (1.1) so as to answer one of the most important questions: What is the sufficient and necessary condition that guarantee the infected class of individuals to be extinct eventually. To answer this question, we introduce a real-valued threshold λ, and prove that if λ is negative, the disease will die out together with the extinction rate of the disease. On the other hand, if λ is positive, the disease becomes endemic. We shall show that the system has an invariant probability measure, and the transition probability of the solution process converges to the invariant measure. Our method to determine the thresholds is from a dynamical system theory point of view, which is interesting in its own right. This methods can be generalized to deal with many other models. The algebraic representations of the thresholds are given that is easily computable. In addition, simulation studies and numerical examples are given to illustrate our results.

Our contributions. The contributions and novelties in this work can be summarized as follows.

- In the literature, much effort has been devoted to the study of SIR epidemic models with vaccination, namely, SVIR, which are suitable for disease with permanent immunity after recovery from infection. However, the study of SIRS models with vaccination, which are valid for disease without permanent immunity, is relatively scarce. It is worth noting that the model of SIRS-type models are more difficult and require more effort than that of SIR-type models. For many dynamical systems, to establish a complete picture of longtime behavior, the higher dimensional problems require much more efforts. Even in some cases, from two-dimensional problems to there-dimensional problems, there is a big gap; see [23] for the difference in competitive-type models in mathematical biology. Some of the reasons are that the comparisons of solutions are not generally extendable to higher dimensional systems, and that the behaviors of solution and ergodic measures are much richer. More importantly, to date, there is no complete characterization of SVIRS models, not to mention the associated regime-switching models. In this paper, we provide a satisfactory answer to the open question.

- We also use novel and systematic approaches. Our methodology (using the Lyapunov exponents from dynamical systems theory to define the thresholds) is general and systematic that can be generalized for other models. Although a similar approach has been developed in [24] for stochastic systems with switching, those results are not applicable because we are considering higher dimensional problems. In addition, the returning to susceptible group from recovered individuals, and the effects of vaccination component $V(t)$ make the dynamics of the underlying SVIRS system is much different from the existing works. As a result, new methods and techniques are required. From a dynamic system point of view, we take advantage of examining the boundary behavior of systems to facilitate the study Lyapunov exponents. In [24], and many other works in the literature, the boundary behavior can be handled by considering an one-dimensional stochastic equation. However, for the SVIRS problem considered in this work, one needs to examine a two-dimensional problem. As a result, the study of stationary distribution and how the systems interact (be attracted or repelled) with these distributions are much more difficult. In addition, SVIRS-type equations are highly coupled introducing additional difficulties. The complexity leads to significant challenges in understanding when the extinction happens. To overcome the difficulty, we consider auxiliary perturbed systems to understand how small changes in infected groups affect to the whole systems.

Organization of the paper. The rest of the paper is organized as follows. Section 2 presents our main result. Specifically, Section 2.1 establishes preliminary results for system (1.1), Section 2.2 introduces the ideas and states main results, and Sections 2.3 and 2.4 are devoted to the proofs. Section 3 gives numerical examples. Finally, Section 4 provides additional discussion and interpretation.

2. Main results

2.1. Preliminary results for (1.1)

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. $W_1(t)$, $W_2(t)$, $W_3(t)$, and $W_4(t)$ are independent standard Brownian motions, and $\xi(t)$ is a finite-state continuous-time Markov chain that is independent of the Brownian motions $W_i(t), i = 1, \ldots, 4$ and that has a finite state space $\mathcal{M} = \{1, \ldots, m_0\}$. Assume that the generator $Q = (q_{kl})_{m_0\times m_0}$ of $\xi(t)$ is irreducible; see e.g., [15]. As a result, it has a unique invariant probability measure π. Note that by definition,

$$\begin{aligned}&\mathbb{P}\{\xi(t+\Delta) = j|\xi(t) = i, \xi(u), u \leq t\} = q_{ij}\Delta + o(\Delta) \text{ if } i \neq j \quad \text{and}\\&\mathbb{P}\{\xi(t+\Delta) = i|\xi(t) = i, \xi(u), u \leq t\} = 1 + q_{ii}\Delta + o(\Delta).\end{aligned} \tag{2.1}$$

Throughout the paper, we denote $\mathbb{R}^2_+ = \{(s, v) \in \mathbb{R}^2 : s \geq 0, v \geq 0\}$ with interior $\mathbb{R}^{2,\circ}_+ = \{(s, v) \in \mathbb{R}^2 : s > 0, i > 0\}$, $\mathbb{R}^4_+ = \{(s, v, i, r) \in \mathbb{R}^4 : s \geq 0, i \geq 0, v \geq 0, r \geq 0\}$ with interior $\mathbb{R}^{4,\circ}_+ = \{(s, v, i, r) \in \mathbb{R}^4 : s > 0, i > 0, v > 0, r > 0\}$, $\mathbb{R}^{4,*}_+ = \{(s, v, i, r) \in \mathbb{R}^4_+ : i > 0\}$. Moreover, to simplify notation, we denote $\mathbf{z} := (s, v, i, r)$ and $\mathbf{Z}(\cdot) = (S(\cdot), V(\cdot), I(\cdot), R(\cdot))$. We will denote $\mathbb{P}_{\mathbf{z},k}$ and $\mathbb{E}_{\mathbf{z},k}$ the probability and expectation corresponding to the initial condition $S(0) = s$, $I(0) = i$, $V(0) = v$, $R(0) = r$, $\xi(0) = k$, respectively.

Assumption 2.1. We impose the following assumption through out the paper.

- $g(u, i, k)$ is locally Lipschitz and bounded by $C(1 + u)$ for some constant $C > 0$ on $\mathbb{R}^2_+ \times \mathcal{M}$.
- $g(u, i, k)$ is continuous at $i = 0$ uniformly in (u, k), that is

$$\lim_{i\to 0^+} \sup_{u\geq 0, k\in\mathcal{M}} \{|g(u, i, k) - g(u, 0, k)|\} = 0.$$

- $g(u, 0, k)$ is increasing in $u \in [0, \infty)$ for each $k \in \mathcal{M}$.

Remark 2.1. We note that almost all of the incidence rate functions used in the literature as mentioned in the introduction (such as the bilinear incidence rate, the Beddington–DeAngelis incidence rate, the Holling type II functional response, etc.) satisfy these conditions. Recall that the incidence rate in our setting is Ig rather than g.

To proceed, we first establish the existence, uniqueness, and basic properties of the solution process of (1.1).

Theorem 2.1. *For any initial value $(\mathbf{z}, k) \in \mathbb{R}^4_+\times\mathcal{M}$, there exists a global solution $(\mathbf{Z}(t), \xi(t))$ to (1.1) such that $\mathbb{P}_{\mathbf{z},k}\{\mathbf{Z}(t) \in \mathbb{R}^4_+,\ \forall t \geq 0\} = 1$. Moreover, for all $\mathbf{z} = (s, v, i, r)$ with $i = 0$ and $s, v, r \geq 0$,*

$$\mathbb{P}_{\mathbf{z},k}\{I(t) = 0,\ \forall t \geq 0\} = \mathbb{P}_{\mathbf{z},k}\{S(t) > 0,\ V(t) > 0,\ R(t) > 0,\ \forall t \geq 0\} = 1,$$

and for all $\mathbf{z} = (s, v, i, r)$ with $i > 0, s \geq 0, v \geq 0, r \geq 0$,

$$\mathbb{P}_{\mathbf{z},k}\{\mathbf{Z}(t) \in \mathbb{R}^{4,\circ}_+,\ \forall t > 0\} = 1,$$

We also have that the joint-process $(\mathbf{Z}(t), \xi(t))$ is a Markov–Feller process on $\mathbb{R}^4_+ \times \mathcal{M}$.

Remark 2.2. Together with well-posedness property of the underlying equation, the results show the positivity property of the solution. That is, all components are always non-negative provided if they start from non-negative initial conditions. Moreover, $I(t) = 0$ for all t if started at $i = 0$, and always positive if started at $i > 0$; while all other components are always positive.

Proof. The proof is rather standard and therefore is omitted here. We can refer to [25, Theorem 2.1], or [26, Theorem 1] for similar proofs. □

Next, we establish the boundedness in probability and boundedness in moment of the solution process.

Lemma 2.1. *For any $q > 0$ sufficiently small, there exist $C_q > 0$ and $D_q > 0$ such that*

$$\mathbb{E}_{\mathbf{z},k}\left(S(t)+V(t)+I(t)+R(t)\right)^{1+q} \le \frac{(1+s+i+v+r)^{1+q}}{e^{D_q t}} + \frac{C_q}{D_q}, \quad \forall t \ge 0. \tag{2.2}$$

As a result, for any $H > 0$, $\varepsilon > 0$, $T > 0$, there is a constant $M_{H,\varepsilon,T} > 0$ such that

$$\mathbb{P}_{\mathbf{z},k}\left\{\sup_{t\in[0,T]}\{S(t)+V(t)+I(t)+R(t)\} \le M_{H,\varepsilon,T}\right\} \ge 1-\varepsilon, \quad \forall(\mathbf{z},k)\in[0,H]^4\times\mathcal{M}. \tag{2.3}$$

Proof. Denote by $\mathcal{L}$ the operator associated with the solution process of (1.1). It is well-known that for a function U of $(\mathbf{z},k)$ that is twice differentiable with respect to $\mathbf{z}=(s,v,i,r)$, we have

$$\begin{aligned}\mathcal{L}U(\mathbf{z},k) =& U_s(\mathbf{z},k)\big((1-\widetilde{q}(k))A - ig(s+\beta v,i,k) - (\alpha_0(k)+\widetilde{p}(k))s + \alpha_2(k)r + \gamma(k)v\big)\\ &+ U_i(\mathbf{z})\big(ig(s+\beta v,i,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k))i\big)\\ &+ U_v(\mathbf{z},k)\big(\widetilde{q}(k)A(k)+\widetilde{p}(k)s-(\alpha_0(k)+\gamma(k))v\big)\\ &+ U_r(\mathbf{z},k)\big(\alpha_3(k)i-\alpha_0(k)r-\alpha_2(k)r\big)\\ &+ \frac{1}{2}U_{ss}(\mathbf{z},k)\sigma_1^2(k)s^2 + \frac{1}{2}U_{vv}(\mathbf{z},k)\sigma_2^2(k)v^2 + \frac{1}{2}U_{ii}(\mathbf{z},k)\sigma_3^2(k)i^2 + \frac{1}{2}U_{rr}(\mathbf{z},k)\sigma_4^2(k)r^2\\ &+ \sum_{l\in\mathcal{M}} q_{kl}U(\mathbf{z},l).\end{aligned}$$

Therefore, by direct computations for $U=(1+s+v+i+r)^{1+q}$, we have

$$\begin{aligned}&\mathcal{L}(1+s+v+i+r)^{1+q}\\ =&(1+q)(1+s+v+i+r)^q\big((1-\widetilde{q}(k))A - ig(s+\beta v,i,k) - (\alpha_0(k)+\widetilde{p}(k))s + \alpha_2(k)r + \gamma(k)v\big)\\ &+ (1+q)(1+s+v+i+r)^q\big(ig(s+\beta v,i,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k))i\big)\\ &+ (1+q)(1+s+v+i+r)^q\big(\widetilde{q}(k)A(k)+\widetilde{p}(k)s-(\alpha_0(k)+\gamma(k))v\big)\\ &+ (1+q)(1+s+v+i+r)^q\big(\alpha_3(k)i-\alpha_0(k)r-\alpha_2(k)r\big)\\ &+ \frac{q(1+q)}{2}(1+s+v+i+r)^{q-1}\Big(\sigma_1^2(k)s^2+\sigma_2^2(k)v^2+\sigma_3^2(k)i^2+\sigma_4^2(k)r^2\Big)\\ =&(1+q)(1+s+v+i+r)^q\Big(A-\alpha_0(k)s-\alpha_0(k)v-(\alpha_0(k)+\alpha_1(k))i-\alpha_0(r)\Big)\\ &+ \frac{q(1+q)}{2}(1+s+v+i+r)^{q-1}\Big(\sigma_1^2(k)s^2+\sigma_2^2(k)v^2+\sigma_3^2(k)i^2+\sigma_4^2(k)r^2\Big)\\ \le&(1+q)(1+s+v+i+r)^q\Big(2A-\check{\alpha}(1+s+v+i+r)+q\widehat{\sigma}(1+s+v+i+r)\Big),\end{aligned}$$

where $\check{\alpha} := \min_{k\in\mathcal{M}}\{A,\alpha_0(k),\alpha_1(k)\} > 0$, and $\widehat{\sigma} := \frac{1}{2}\max_{k\in\mathcal{M},i=1,2,3,4}\{\sigma_i^2(k)\} < \infty$. Now, for any q, $0<q<\frac{\check{\alpha}}{2\widehat{\sigma}}$, we have

$$\mathcal{L}(1+s+v+i+r)^{1+q} \le C_q - D_q(1+s+v+i+r)^{1+q},$$

for $D_q = \frac{(1+q)\check{\alpha}}{4}$, and

$$C_q = \sup_{s,v,i,r\ge 0}\left\{(1+q)(1+s+v+i+r)^q\Big(2A-\frac{\check{\alpha}}{4}(1+s+v+i+r)\Big)\right\} < \infty.$$

Therefore, as an application of Itô's formula, one can obtain (2.2) and (2.3); see [27] or [28] for this well-known argument. □

2.2. A complete characterization of longtime behavior

In this section, a threshold λ will be introduced. This λ will completely characterize the longtime property of system (1.1). The threshold λ will be determined from the dynamics of the system on the boundary, that is, when $I(t)=R(t)=0$, in which, $(S(t),V(t))$ follows the system

$$\begin{cases} dS(t) = \Big[\big(1-\widetilde{q}(\xi(t))\big)A(\xi(t)) - \big(\alpha_0(\xi(t))+\widetilde{p}(\xi(t))\big)S(t)\\ \qquad +\gamma(\xi(t)V(t))\Big]dt + \sigma_1(\xi(t))S(t)dW_1(t),\\ dV(t) = \Big[\widetilde{q}(\xi(t))A(\xi(t))+\widetilde{p}(\xi(t))S(t) - \big(\alpha_0(\xi(t))+\gamma(\xi(t))\big)V(t)\Big]dt\\ \qquad +\sigma_2(\xi(t))V(t)dW_2(t).\end{cases} \tag{2.4}$$

As alluded to in the introduction, to examine the boundary behavior, we need to look at 2-dimensional problems, which presents an added difficult. The returning to susceptible group from recovered and vaccinated individuals is also another challenge. To ease the difficulty, we consider a perturbed system of (2.4). For each $\theta \in [-1,1]$, let $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t))$ be solution to

$$\begin{cases} d\widetilde{S}^\theta(t) = \Big[\big(1-\widetilde{q}(\xi(t))\big) A(\xi(t)) - \big(\alpha_0(\xi(t)) + \widetilde{p}(\xi(t))\big) \widetilde{S}^\theta(t) \\ \qquad + \gamma(\xi(t))\widetilde{V}^\theta(t) \Big] dt + \theta\alpha_2(\xi(t))dt + \sigma_1(\xi(t))\widetilde{S}^\theta(t) dW_1(t), \\ d\widetilde{V}^\theta(t) = \Big[\widetilde{q}(\xi(t))A(\xi(t)) + \widetilde{p}(\xi(t))\widetilde{S}^\theta(t) - \big(\alpha_0(\xi(t)) + \gamma(\xi(t))\big)\widetilde{V}^\theta(t)\Big]dt \\ \qquad + \sigma_2(\xi(t))\widetilde{V}^\theta(t) dW_2(t). \end{cases} \tag{2.5}$$

Denote by $\mathcal{L}_\theta$ the operator associated with the solution process of (2.5), i.e., for twice differentiable function $U((s,v),k)$ (with respect to (s,v))

$$\begin{aligned} \mathcal{L}_\theta U((s,v),k) =& U_s((s,v),k)\big((1-\widetilde{q}(k))A - (\alpha_0(k)+\widetilde{p}(k))s + \alpha_2(k)r + \gamma(k)v + \theta\alpha_2(k)\big) \\ &+ U_v((s,v),k)\big(\widetilde{q}(k)A(k) + \widetilde{p}(k)s - (\alpha_0(k)+\gamma(k))v\big) \\ &+ \frac{1}{2}U_{ss}((s,v),k)\sigma_1^2(k)s^2 + \frac{1}{2}U_{vv}((s,v),k)\sigma_2^2(k)v^2 + \sum_{l\in\mathcal{M}} q_{kl}U((s,v),k). \end{aligned}$$

Analogous to Theorem 2.1 and Lemma 2.1, we can easily show that for each initial value $((s,v),k) \in \mathbb{R}^2_+ \times \mathcal{M}$, there exists a global solution $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t))$ to (2.5) and that $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t)) \in \mathbb{R}^{2,\circ}_+$ for all $t>0$ almost surely and that

$$\mathbb{E}_{(s,v),k}\left(\widetilde{S}^\theta(t) + \widetilde{V}^\theta(t)\right)^{1+q} \le \frac{(1+s+v)^{1+q}}{e^{\widetilde{D}_q t}} + \frac{\widetilde{C}_q}{\widetilde{D}_q}, \quad \forall t \ge 0, \tag{2.6}$$

for some small $q>0$, and constants $\widetilde{C}_q, \widetilde{D}_q > 0$, which can be taken uniformly for any $\theta \in [-1,1]$.

Since the diffusion in (2.5) is nondegenerate on $\mathbb{R}^{2,\circ}_+ \times \mathcal{M}$, there exists uniquely an invariant probability distribution $\boldsymbol{v}_\theta$ of $((\widetilde{S}^\theta(t), \widetilde{V}^\theta(t)), \xi(t))$ on $\mathbb{R}^{2,\circ}_+ \times \mathcal{M}$. Due to (2.6) and [29, Lemma 3.4], we have

$$\sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} (s+v)^{1+q} \boldsymbol{v}_\theta(ds,dv,k) \le \frac{\widetilde{C}_q}{\widetilde{D}_q} < \infty.$$

Thus, we can define the following λ_θ which is a "perturbed" growth rate of $I(t)$ as its density is small:

$$\begin{aligned} \lambda_\theta :=& \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \left(g(s,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_3^2(k)}{2}\right) \boldsymbol{v}_\theta(ds,dv,k) \\ =& \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} g(s+\beta v,0,k)\boldsymbol{v}_\theta(ds,dv,k) - \sum_{k\in\mathcal{M}} \left(\alpha_0(k)+\alpha_3(k)+\alpha_1(k) + \frac{\sigma_3^2(k)}{2}\right)\pi_k. \end{aligned}$$

The equality above follows from the fact that π is the marginal distribution of the third component in $\boldsymbol{v}_\theta$, which is because π is the unique invariant measure of $\xi(t)$. In particular, when $\theta=0$, we denote

$$\lambda := \lambda_0 = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \left(g(s+\beta v,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_3^2(k)}{2}\right) \boldsymbol{v}_0(ds,dv,k). \tag{2.7}$$

This λ will be shown to be the threshold that characterizes the longtime behavior of the disease. Note that, when $\theta=0$, (2.5) is the solution to (2.4). Thus, we can consider $\boldsymbol{v}_0$ as the unique invariant measure of $((S(t),V(t),I(t),R(t)),\xi(t))$ on the boundary $\mathbb{R}^{4,*}_+ \times \mathcal{M}$ (by embedding $\mathbb{R}^2_+ \times \mathcal{M}$ to $\mathbb{R}^{4,*}_+ \times \mathcal{M}$).

Lemma 2.2.

$$\lim_{\theta\to 0} \lambda_\theta = \lambda_0 = \lambda.$$

Proof. Owing to (2.6), the family $\{\boldsymbol{v}_\theta, \theta \in [-1,1]\}$ is tight. As a result, for any sequence θ_n that converges to 0 as n goes to ∞, we can find a subsequence, still denoted by $\{\theta_n\}_{n=1}^\infty$ for simplicity such that $\boldsymbol{v}_{\theta_n}$ converges weakly to a probability measure, denoted by $\overline{\boldsymbol{v}}$, as n goes to ∞.

On the other hand, because $\boldsymbol{v}_\theta$ is the unique invariant measure of (2.5), for any (s,v)-twice differentiable function $U((s,v),k)$ with compact support, we have

$$\sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \mathcal{L}_\theta U((s,v),k)\boldsymbol{v}_\theta(ds,dv,k) = 0.$$

Because of the weak convergence of $\{\boldsymbol{v}_{\theta_n}\}_{n=1}^\infty$ to $\overline{\boldsymbol{v}}$, we have

$$\lim_{n\to\infty} \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \mathcal{L}_{\theta_n} U((s,v),k)\boldsymbol{v}_{\theta_n}(ds,dv,k) = 0, \tag{2.8}$$

for any (s,v)-twice differentiable function $U((s,v),k)$ with compact support.

Note that $\mathcal{L}_{\theta_n}U((s,v),k) = \mathcal{L}U((s,v),k) + \theta_n\alpha_2(k)U_s((s,v),k)$ which implies

$$\begin{aligned}
&\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\mathcal{L}U((s,v),k)\overline{\boldsymbol{v}}(ds,dv,k)\\
&\quad=\lim_{n\to\infty}\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\mathcal{L}U((s,v),k)\boldsymbol{v}_{\theta_n}(ds,dv,k)\\
&\quad=\lim_{n\to\infty}\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\mathcal{L}_{\theta_n}U((s,v),k)\boldsymbol{v}_{\theta_n}(ds,dv,k)\\
&\qquad-\lim_{n\to\infty}\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\theta_n\alpha_2(k)U_s((s,v),k)\boldsymbol{v}_{\theta_n}(ds,dv,k)\\
&\quad=0.
\end{aligned}\tag{2.9}$$

The last equality is due to (2.8), the boundedness of $U_s(s,v,k)$ and the fact that $\lim_{n\to\infty}\theta_n = 0$.

Because

$$\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\mathcal{L}U((s,v),k)\overline{\boldsymbol{v}}(ds,dv,k) = 0$$

for any (s,v)-twice differentiable function $U((s,v),k)$ with compact support, $\overline{\boldsymbol{v}}$ must be an invariant measure of (2.4). Due to the uniqueness, $\overline{\boldsymbol{v}}$ is identical to $\boldsymbol{v}$. Therefore, we have showed that any weak limit of $\boldsymbol{v}_\theta$ as $\theta\to 0$ must be $\boldsymbol{v}$. This and the tightness of $\{\boldsymbol{v}_\theta, \theta\in[-1,1]\}$ implies that $\boldsymbol{v}_\theta$ converges weakly to $\boldsymbol{v}$ as $\theta\to 0$. As an application of [29, Lemma 3.4], the desired result "$\lim_{\theta\to 0}\lambda_\theta = \lambda$" follows directly from this convergence together with (2.6) and the linear growth rate of $g(s+\beta v,0,k)$. □

Now, we state our main results.

Theorem 2.2. *If $\lambda > 0$, there exists a unique invariant measure $\boldsymbol{v}^*$ of $(\mathbf{Z}(t),\xi(t))$ on $\mathbb{R}^{4,\circ}_+\times\mathcal{M}$. In addition, the transition probability converges to the unique invariant measure exponentially fast. That is, there is a constant $m>0$ such that*

$$\lim_{t\to\infty} e^{mt}\|P(t,(\mathbf{z},k),\cdot) - \boldsymbol{v}^*(\cdot)\|_{TV} = 0,\ \forall(\mathbf{z},k)\in\mathbb{R}^{4,*}_+\times\mathcal{M},$$

where $P(t,(\mathbf{z},k),\cdot)$ is the transition probability of the joint-process $(\mathbf{Z}(t),\xi(t))$.

Theorem 2.3. *If $\lambda<0$, $I(t)$ converges to 0 exponentially fast and $R(t)$ converges to 0 almost surely. More precisely,*

$$\mathbb{P}_{\mathbf{z},k}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda,\ \text{and}\ \lim_{t\to\infty}R(t) = 0\right\} = 1.\tag{2.10}$$

In addition, $(S(t),V(t))$ converges (weakly) to the solution of (2.4).

Algebraic representation of λ. Before proving the main results, let us provide some computable representation of the threshold λ. When $g(s+\beta v,0,k)$ is a linear function, that is, $g(s+\beta v,0,k) = b(k)(s+\beta v)$ with $b(\cdot):\mathcal{M}\to(0,\infty)$, we will construct an algebraic representation for λ. This representation is very useful from a computational point of view since we can compute λ by solving a system of linear equations.

Lemma 2.3. *Assume function g is a linear function, i.e., $g(s+\beta v,0,k) = b(k)(s+\beta v)$. Let $C := \big(c_1(1),c_1(2),\dots,c_1(m_0),c_2(1),c_2(2),\dots,c_2(m_0)\big)^\top$ be the unique solution to the linear system:*

$$\begin{cases} b(k) - \big(\alpha_0(k)+\widetilde{p}(k)\big)c_1(k) + \widetilde{p}(k)c_2(k) + \sum_{l\in\mathcal{M}}\gamma_{kl}c_1(l) = 0,\\ \beta b(k) + \gamma(k)c_1(k) - \big(\alpha_0(k)+\gamma(k)\big)c_2(k) + \sum_{l\in\mathcal{M}}\gamma_{kl}c_2(l) = 0,\quad k=1,2,\dots,m_0.\end{cases}\tag{2.11}$$

We have:

$$\lambda = \sum_{k\in\mathcal{M}}\left[c_1(k)(1-\widetilde{q}(k))A(k) - \left(\alpha_0(k)+\alpha_3(k)+\alpha_1(k)+\frac{\sigma_3^2(k)}{2}\right)\right]\pi_k.$$

Proof. The system (2.11) can be written in the following form

$$\mathbf{A}C = B,\tag{2.12}$$

where $B = \big(b(1), b(2), \dots, b(m_0), \beta b(1), \beta b(2) \dots, \beta b(m_0)\big)^\top$, and

$$\mathbf{A} = \begin{bmatrix} \alpha_0(1)+\widetilde{p}(1)-\gamma_{11} & \dots & -\gamma_{1m_0} & -\widetilde{p}(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma_{m_0 1} & \dots & \alpha_0(m_0)+\widetilde{p}(m_0)-\gamma_{m_0m_0} & 0 & \dots & -\widetilde{p}(m_0) \\ -\gamma(1) & \dots & 0 & \alpha_0(1)+\gamma(1)-\gamma_{11} & \dots & -\gamma_{1m_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -\gamma(m_0) & -\gamma_{m_01} & \dots & \alpha_0(m_0)+\gamma(m_0)-\gamma_{m_0m_0} \end{bmatrix}. \tag{2.13}$$

Since $-\gamma_{ii} = \sum_{j\neq i}\gamma_{ij}$, it is obvious that matrix $\mathbf{A} = [a_{ij}]_{2m_0\times 2m_0}$ is diagonally dominant, i.e., $|a_{ii}| \geq \sum_{j\neq i, j=1}^{2m_0}|a_{ij}|$ for any $i = 1,\dots,m_0$. It is well-known that a diagonally dominant matrix is non-singular. Thus, there exists a unique solution C to (2.12).

Denote by $\mathcal{L}_0$ the operator associated with the solution process of (2.4) and $(\widetilde{S}^0, \widetilde{V}^0)$ be the solution to (2.4). It is readily seen that if we let $U((s,v),k) = c_1(k)s + c_2(k)v$ then

$$\begin{aligned}\mathcal{L}_0 U((s,v),k) =& c_1(k)\left[(1-\widetilde{q}(k))A(k) + \alpha_3(k)\theta - \big(\alpha_0(k)+\widetilde{p}(k)\big)\, s + \gamma(k)v\right] \\ &+ c_2(k)\left[\widetilde{q}(k)A(k) + \widetilde{p}(k)s - \big(\alpha_0(k)+\gamma(k)\big)\, v\right] \\ &+ \Big(\sum_l \gamma_{kl}c_1(l)\Big)s + \Big(\sum_l \gamma_{kl}c_2(l)\Big)v \\ =& c_1(k)\left[(1-\widetilde{q}(k))A(k)\right] - b(k)s - \beta b(k)v.\end{aligned}$$

Then, by Dynkin's formula [30], we have

$$\begin{aligned}&\mathbb{E}_{(s,v),k}\ U((\widetilde{S}^0(t),\widetilde{V}^0(t)),\xi(t)) - U((\widetilde{S}^0(0),\widetilde{V}^0(0)),\xi(0)) \\ &\quad = \mathbb{E}_{(s,v),k}\int_0^t \mathcal{L}_0 U((\widetilde{S}^0(u),\widetilde{V}^0(u)),\xi(u))du \\ &\quad = \mathbb{E}_{(s,v),k}\int_0^t c_1(\xi(u))\left[(1-\widetilde{q}(\xi(u)))A(\xi(u))\right]du - \mathbb{E}_{(s,v),k}\int_0^t b(\xi(u))\left(\widetilde{S}^0(u)+\beta\widetilde{V}^0(u)\right)du.\end{aligned} \tag{2.14}$$

Moreover, one has

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{(s,v),k}\int_0^t c_1(\xi(u))\left[(1-\widetilde{q}(\xi(u)))A(\xi(u))\right]du = \sum_{k\in\mathcal{M}} c_1(k)(1-\widetilde{q}(k))A(k)\pi_k, \tag{2.15}$$

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{(s,v),k}\int_0^t b(\xi(u))\left(\widetilde{S}^0(u)+\beta\widetilde{V}^0(u)\right)du = \sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+} b(k)(s+\beta v)\nu(ds,dv), \tag{2.16}$$

$$\lim_{t\to\infty}\frac{\mathbb{E}_{(s,v),k}U(\widetilde{S}^0(t),\widetilde{V}^0(t),\xi(t)) - U(\widetilde{S}^0(0),\widetilde{V}^0(0),\xi(0))}{t} = 0. \tag{2.17}$$

Plugging (2.15), (2.16), and (2.17) in (2.14), we have

$$\sum_{k\in\mathcal{M}} c_1(k)(1-\widetilde{q}(k))A(k)\pi_k = \sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+} b(k)(s+\beta v)\boldsymbol{\nu}_0(ds,dv,k),$$

which and the formula (2.7) of λ imply

$$\lambda = \lambda_0 = \sum_{k\in\mathcal{M}} c_1(k)(1-\widetilde{q}(k))A(k)\pi_k - \sum_{k\in\mathcal{M}}\left(\alpha_0(k)+\alpha_3(k)+\alpha_1(k)+\frac{\sigma_3^2(k)}{2}\right)\pi_k.$$

The proof is complete. □

2.3. *Proof of Theorem 2.2*

Since $g(u,i,k)$ is continuous at $i=0$ uniformly in u, there exists $\delta_1 > 0$ such that

$$\sup_{u\geq 0, k\in\mathcal{M}, 0\leq i\leq\delta_1}|g(u,i,k) - g(u,0,k)| < \frac{1}{4}\lambda. \tag{2.18}$$

Since the function $g(u,0,k)$ is increasing in u, and

$$\sum_{k\in\mathcal{M}}\int_{\mathbb{R}^2_+}\left(g(s+\beta v,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_2^2(k)}{2}\right)\boldsymbol{\nu}(ds,dv,k) = \lambda,$$

there exists $H_1 > 0$ such that

$$\begin{aligned}&\sum_{k\in\mathcal{M}}\pi_k\left(g(H_1,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_3^2(k)}{2}\right) \\ &= \inf_{u\geq H_1}\sum_{k\in\mathcal{M}}\pi_k\left(g(u,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_3^2(k)}{2}\right) > \frac{3}{4}\lambda.\end{aligned} \tag{2.19}$$

In view of (2.18) and (2.19), we have

$$\sum_{k\in\mathcal{M}} \widehat{h}_k\pi_k \ge \frac{1}{2}\lambda \text{ where } \widehat{h}_k := \inf_{u\ge H_1, 0\le i\le\delta_1}\left(g(u,i,k)-(\alpha_0(k)+\alpha_3(k)+\alpha_1(k))-\frac{\sigma_3^2(k)}{2}\right). \tag{2.20}$$

Let

$$\widehat{\lambda}_1 := \frac{1}{3}\sum_{k\in\mathcal{M}}\widehat{h}_k\pi_k. \tag{2.21}$$

Because $\sum_{k\in\mathcal{M}}(3\widehat{\lambda}_1-\widehat{h}_k)\pi_k=0$, we deduce from the Fredholm alternative that there exist $\widetilde{c}_k\ge 0, k\in\mathcal{M}$ such that

$$\sum_{\ell\in\mathcal{M}} q_{k\ell}\widetilde{c}_\ell = 3\widehat{\lambda}_1-\widehat{h}_k. \tag{2.22}$$

We have the following lemma.

Lemma 2.4. *For any $\rho_1\in(0,1)$ satisfying*

$$\rho_1\left(\frac{\sigma_3^2(k)}{2}(1-\rho_1\widetilde{c}_k)+(-3\widehat{\lambda}_1+\widehat{h}_k)\right)<1 \text{ and } \frac{1}{2}\le 1-\rho_1\widetilde{c}_k, \text{ for all } k\in\mathcal{M}, \tag{2.23}$$

define

$$V_1(i,k)=(1-\rho_1\widetilde{c}_k)i^{-\rho_1}.$$

We have the following conclusions.

- $[\mathcal{L}V_1](\mathbf{z},k)\le -\rho_1\widehat{\lambda}_1V_1(i,k)$ *for any* $\mathbf{z}=(s,v,i,r)$ *with* $s\ge H_1,\ i\le\delta_1$.
- *Let* $\widehat{c}_1 := \max_{k\in\mathcal{M}}\left\{\alpha_0(k)+\alpha_3(k)+\alpha_1(k)+\frac{\sigma_3^2(k)}{2}+2\sum_{\ell\in\mathcal{M}}|q_{k\ell}|\right\}$*, then* $[\mathcal{L}V_1](\mathbf{z},k)\le\rho_1\widehat{c}_1V_1(i,k)$ *for any* $(\mathbf{z},k)\in\mathbb{R}_+^{4,*}\times\mathcal{M}$*. As a result,* $\mathbb{E}_{\mathbf{z},k}V_1(I(\tau),\xi(\tau))e^{-\rho_1\widehat{c}_1\tau}\le V_1(i,k)$ *for any initial condition* $(\mathbf{z},k)\in\mathbb{R}_+^{4,*}\times\mathcal{M}$ *and any bounded stopping time* τ.

Proof. We have the following estimate:

$$\begin{aligned}(\mathcal{L}V_1)(\mathbf{z},k)=&-\rho_1V_1(i,k)\left(g(s+\beta v,i,k)-(\alpha_0(k)+\alpha_3(k)+\alpha_1(k))-\frac{\sigma_3^2(k)}{2}\right)\\&+\rho_1^2\frac{\sigma_3^2(k)}{2}V_1(i,k)+\sum_{\ell\in\mathcal{M}}q_{k\ell}V_1(i,\ell)\\ \le&-\rho_1V_1(i,k)\widehat{h}_k+\rho_1^2\frac{\sigma_3^2(k)}{2}V_1(i,k)+\sum_{\ell\in\mathcal{M}}q_{k\ell}V_1(i,\ell),\end{aligned} \tag{2.24}$$

where $\widehat{h}_k$ is defined as in (2.20). Note that

$$\begin{aligned}\sum_{\ell\in\mathcal{M}}q_{k\ell}V_1(i,\ell)=&\sum_{\ell\in\mathcal{M}}q_{k\ell}(1-\rho_1\widetilde{c}_\ell)i^{-\rho_1}\\=&-\rho_1 i^{-\rho_1}\sum_{\ell\in\mathcal{M}}q_{k\ell}\widetilde{c}_\ell\\=&(-3\widehat{\lambda}_1+\widehat{h}_k)\rho_1 i^{-\rho_1}\\=&(-3\widehat{\lambda}_1+\widehat{h}_k)\rho_1V_1(i,k)+(-3\widehat{\lambda}_1+\widehat{h}_k)\rho_1^2 i^{-\rho_1}.\end{aligned} \tag{2.25}$$

Plugging (2.25) into (2.24) and applying (2.23), we get

$$\begin{aligned}(\mathcal{L}V_1)(\mathbf{z},k)\le&\rho_1^2\left(\frac{\sigma_3^2(k)}{2}(1-\rho_1\widetilde{c}_k)+(-3\widehat{\lambda}_1+\widehat{h}_k)\right)i^{-\rho_1}-3\widehat{\lambda}_1\rho_1V_1(i,k)\\ \le&\widehat{\lambda}_1\rho_1 i^{-\rho_1}-3\widehat{\rho}_1\lambda_1V_1(i,k)\\ \le&\widehat{\lambda}_1\rho_1\left(\frac{1}{1-\rho_1\widetilde{c}_k}-3\right)V_1(i,k)\\ \le&-\rho_1\widehat{\lambda}_1V_1(i,k)\text{ for any }\mathbf{z}=(s,i,v,r)\text{ with }s\ge H_1, i\le\delta_1.\end{aligned} \tag{2.26}$$

Therefore, the first assertion is proved.

On the other hand, because

$$\sum_{\ell\in\mathcal{M}}q_{k\ell}V_1(i,\ell)=\sum_{\ell\in\mathcal{M}}q_{k\ell}\frac{1-\rho_1\widetilde{c}_k}{1-\rho_1\widetilde{c}_\ell}V_1(i,k)\le 2\sum_{\ell\in\mathcal{M}}|q_{k\ell}|V_1(i,k),$$

it follows from (2.24) that

$$[\mathcal{L}V_1](\mathbf{z},k)\le\rho_1\widehat{c}_1V_1(i,k)\text{ for any }(\mathbf{z},k)\in\mathbb{R}_+^{4,*}\times\mathcal{M}. \tag{2.27}$$

We deduce from (2.27) and Dynkin's formula that

$$\mathbb{E}_{\mathbf{z},k} V_1(I(\tau),\xi(\tau)) e^{-\rho_1 \hat{c}_1 \tau} \le V_1(i,k) \text{ for any } (\mathbf{z},k) \in \mathbb{R}_+^{4,*} \times \mathcal{M} \text{ and any bounded stopping time } \tau. \tag{2.28}$$

The proof is complete. □

Let $n^* > 0$ be such that $\hat{c}_1 - (n^*-1)\hat{\lambda}_1 < 0$, where $\hat{c}_1$ is as in Lemma 2.4 and $\hat{\lambda}_1$ is defined in (2.21).

Lemma 2.5. *For any $H > 0$, the set $B_H := [0,H]^2 \times [H^{-1},H] \times [0,H] \times \mathcal{M}$ is petite with respect to the Markov chain $\{(\mathbf{Z}(nn^*T^*), \xi(nn^*T^*)), n \in \mathbb{Z}_+\}$. That is, there exists a nontrivial measure μ on $\mathbb{R}^{4,*} \times \mathcal{M}$ and a nonnegative sequence $\{a_n\}_{n=1}^{\infty}$ such that*

$$\sum_{n=1}^{\infty} a_n = 1 \text{ and } \sum_{n=1}^{\infty} a_n \mathbb{P}_{\mathbf{z},k}\left\{(\mathbf{Z}(nn^*T^*), \xi(nn^*T^*)) \in A\right\} \ge \mu(A),$$

for any Borel set $A \subset \mathbb{R}_+^{4,*} \times \mathcal{M}$ and $(\mathbf{z},k) \in B_H$.

Proof. The proof is omitted because it is very similar to the proof of [31, Lemma 5.4]. □

Lemma 2.6. *There are $T^* > \max_{k\in\mathcal{M}} \frac{8\tilde{c}_k}{\lambda}$ sufficiently large and $\rho_1 > 0$, $\delta_2 > 0$ sufficiently small such that for any $\mathbf{z} = (s,v,i,r) \in \mathbb{R}_+^4$ with $0 \le i \le \delta_2$, $0 \le s + \beta v \le H_1$,*

$$\mathbb{E}_{\mathbf{z},k} V_1(I(T),\xi(T)) \le e^{-\frac{1}{8}\rho_1 \lambda T} V_1(i,k),\ \forall T \in [T^*, n^*T^*];$$

where V_1 is defined as in Lemma 2.4.

Proof. To simply notation, let

$$h(s+\beta v,0,k) := g(s+\beta v,0,k) - (\alpha_0(k)+\alpha_3(k)+\alpha_1(k)) - \frac{\sigma_3^2(k)}{2}, \tag{2.29}$$

Because of

$$\lambda = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}_+^2} h(s+\beta v,0,k)\boldsymbol{\nu}_0(ds,dv,k),$$

there exists $H_2 > 0$ such that

$$\sum_{k\in\mathcal{M}} \int_{\mathbb{R}_+^2} \hat{h}_2(s+\beta v,0,k)\boldsymbol{\nu}_0(ds,dv,k) > \frac{3}{4}\lambda, \tag{2.30}$$

where

$$\hat{h}_2(s+\beta v,i,k) := H_2 \wedge h(s+\beta v,i,k).$$

Define the occupation measure in $\mathbb{R}_+^4 \times \mathcal{M}$

$$\Pi^t_{\mathbf{z},k}(\cdot) := \frac{1}{t}\mathbb{E}_{\mathbf{z},k} \int_0^t \mathbf{1}_{\{(\mathbf{Z}(s),\xi(s))\in\cdot\}} ds, \tag{2.31}$$

then [29, Lemma 3.4] claims that for a bounded sequence $\{\mathbf{z}_n\} \subset \mathbb{R}_+^4$, and sequence $\{k_n\} \subset \mathcal{M}$ and an increasing unbounded sequence $\{t_n\}$, we have that any weak limit of $\{\Pi^{t_n}_{\mathbf{z}_n,k_n}\}$ as $n \to \infty$ must be an invariant probability measure of $\{(\mathbf{Z}(t),\xi(t))\}$. Then, from (2.30), by a simple contradiction argument (e.g., see the details in [29, Lemma 4.1]), we can find $T^* = T^*(H) > \max_{k\in\mathcal{M}} \frac{8c_k}{\lambda}$ such that for any $T \ge T^*$, we have that for any $\mathbf{z} \in \mathbb{R}_+^4$ with $0 \le s+\beta v \le H_1$, $i = 0$, and $k \in \mathcal{M}$,

$$\mathbb{E}_{\mathbf{z},k} \frac{1}{T}\int_0^T \hat{h}_2(S(t)+\beta V(t),I(t),\xi(t))dt \ge \frac{1}{2}\lambda.$$

Because the process $\{(\mathbf{Z}(t),\xi(t))\}$ is a Markov–Feller process on $\mathbb{R}_+^4 \times \mathcal{M}$, we can find $\delta_2 \in (0,\delta_1)$ such that for any $\mathbf{z} \in \mathbb{R}_+^4$ with $0 \le s+\beta v \le H_1$, $0 < i < \delta_2$, $k \in \mathcal{M}$, and $T \in [T^*, n^*T^*]$,

$$\mathbb{E}_{\mathbf{z},k} \frac{1}{T}\int_0^T \hat{h}_2(S(t)+\beta V(t),I(t),\xi(t))dt \ge \frac{\lambda}{4}. \tag{2.32}$$

Consider $Y(T) := -\int_0^T \hat{h}_2(S(t)+\beta V(t),I(t),\xi(t))dt - \int_0^T \sigma_3(\xi(t))dW_3(t)$. We have

$$\mathbb{E}_{\mathbf{z},k} Y(T) \le -\frac{1}{4}\lambda T \text{ for any } 0 \le s,v \le H, 0 < i < \delta_2, k \in \mathcal{M}, T \in [T^*, n^*T^*]. \tag{2.33}$$

Since the function $\hat{h}_2$ is bounded, we have from a property of the Laplace's transform ([29, Lemma 3.5]) that

$$\frac{d \ln \mathbb{E}_{\mathbf{z},k} e^{\rho Y(T)}}{d\rho} = \mathbb{E}_{\mathbf{z},k} Y(T) \le -\frac{1}{4}\lambda T \text{ and } 0 \le \frac{d^2 \ln \mathbb{E}_{\mathbf{z},k} e^{\rho Y(T)}}{d\rho^2} \le K_{2,\rho},$$

for some constant $K_{2,\rho}$ uniformly in $\mathbf{z}$ with $0 \le s+\beta v \le H$, $0 < i < \delta_2$, $k \in \mathcal{M}$, and $T \in [T^*, n^*T^*]$. By Taylor's expansion, we have that for all $\mathbf{z} \in \mathbb{R}^4_+$ with $0 \le s+\beta v \le H_1$, $0 < i < \delta_2$, $k \in \mathcal{M}$, and $T \in [T^*, n^*T^*]$

$$\ln \mathbb{E}_{\mathbf{z},k} e^{\rho Y(T)} \le 0 - \frac{3}{8}\lambda T\rho + K_{2,\rho}\rho^2 \le -\frac{1}{4}\lambda T\rho, \tag{2.34}$$

whenever $\rho \le \frac{\lambda}{8K_{2,\rho}n^*T^*}$.

Now, pick out a $\rho_1 > 0$ satisfying (2.23) and

$$\rho_1 \le \frac{\lambda}{8K_{2,\rho}n^*T^*}.$$

We have from (2.34) that for all $\mathbf{z} \in \mathbb{R}^4_+$ with $0 \le s+\beta v \le H_1$, $0 < i < \delta_2$, $k \in \mathcal{M}$, and $T \in [T^*, n^*T^*]$

$$\mathbb{E}_{\mathbf{z},k} e^{\rho_1 Y(T)} \le \exp\{-\frac{1}{4}\lambda T\rho_1\}.$$

Subsequently, one has

$$\begin{aligned}\mathbb{E}_{\mathbf{z},k}\frac{I^{-\rho_1}(T)}{i^{-\rho_1}} =&\mathbb{E}_{\mathbf{z},k} e^{\rho_1\left(-\int_0^T h(S(u)+\beta V(u),I(u),\xi(u))du - \int_0^T \sigma_3(\xi(u))dW_3(u)\right)}\\ \le&\mathbb{E}_{\mathbf{z},k} e^{\rho_1 Y(T)} \le \exp\{-\frac{\lambda\rho_1 T}{4}\}.\end{aligned}$$

On the other hand, we have

$$(1-\rho_1\widetilde{c}_k) \ge e^{-\frac{\rho_1\lambda T}{8}} \text{ if } \frac{\lambda T}{8} \ge \widetilde{c}_k.$$

Therefore, we deduce that

$$\begin{aligned}\mathbb{E}_{\mathbf{z},k} V_1(I(T),\xi(T)) \le&\mathbb{E}_{\mathbf{z},k} I^{-\rho_1}(T) \le i^{-\rho_1}\exp\{-\frac{\lambda\rho_1 T}{4}\}\\ \le&(1-\rho_1\widetilde{c}_k)i^{-\rho_1}\exp\{-\frac{\lambda\rho_1 T}{8}\} = V_1(i,k)\exp\{-\frac{\lambda\rho_1 T}{8}\},\end{aligned} \tag{2.35}$$

for all $\mathbf{z} \in \mathbb{R}^4_+$ with $0 \le s+\beta v \le H_1$, $0 < i < \delta_2$, $k \in \mathcal{M}$, and $T \in [T^*, n^*T^*]$. The proof is complete. □

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Define $\eta_1 = \inf\{t \ge 0 : S(t)+\beta V(t) \le H_1\} \wedge n^*T^*$ and $\eta_2 = \inf\{t \ge 0 : I(t) \ge \delta_2\} \wedge n^*T^*$.

Let ρ_1 and V_1 be as in Lemma 2.6. By the Markov–Feller property of $\{(\mathbf{Z}(t),\xi(t))\}$ and Lemma 2.6, we have the following estimates. First,

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ &\le\mathbb{E}_{\mathbf{z},k}\left[\exp\left\{-\frac{\lambda\rho_1}{8}(n^*T^*-\eta_1)\right\}\mathbf{1}_{\{\eta_1\le T^*\}}V_1(I(\eta_1),\xi(\eta_1))\right]\\ &\le\exp\left\{-\frac{\lambda\rho_1}{8}T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}V_1(I(\eta_1),\xi(\eta_1))\right]\\ &\le\exp\left\{-\frac{\lambda\rho_1}{8}T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}V_1(I(\eta_1),\xi(\eta_1))\right]\\ &\le\exp\left\{-\frac{\lambda\rho_1}{8}T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[\exp\left\{\rho_1\widehat{\lambda}_1(\eta_1\wedge\eta_2)\right\}\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}V_1(I(\eta_1),\xi(\eta_1))\right].\end{aligned} \tag{2.36}$$

Second, due to the strong Markov property of $\mathbf{Z}(t)$

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ &=\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}\mathbb{E}_{(\mathbf{Z}(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))}[V_1(I(n^*T^*-\eta_1\wedge\eta_2),\xi(n^*T^*-\eta_1\wedge\eta_2))]\right];\end{aligned}$$

and due to (2.27) and as a consequence of Dynkin's formula

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}\mathbb{E}_{(\mathbf{Z}(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))}[V_1(I(n^*T^*-\eta_1\wedge\eta_2),\xi(n^*T^*-\eta_1\wedge\eta_2))]\right]\\ &\le\mathbb{E}_{\mathbf{z},k}\left[\exp\left\{\rho_1\widehat{c}_1(n^*T^*-\eta_1\wedge\eta_2)\right\}\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\\ &\le\exp\left\{\rho_1\widehat{c}_1T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[1_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\\ &\le\exp\left\{(\widehat{c}_1-(n^*-1)\widehat{\lambda}_1)\rho_1T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[1_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}e^{\rho_1\widehat{\lambda}_1\eta_1\wedge\eta_2}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ &\le\exp\left\{(\widehat{c}_1-(n^*-1)\widehat{\lambda}_1)\rho_1T^*\right\}\mathbb{E}_{\mathbf{z},k}\left[1_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}e^{\rho_1\widehat{\lambda}_1\eta_1\wedge\eta_2}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right].\end{aligned} \tag{2.37}$$

Third, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_2\le(n^*-1)T^*\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ \le&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_2\le(n^*-1)T^*\}}\exp\left\{n^*T^*-\eta_2\right\}V_1(I(\eta_2),\xi(\eta_2))\le\exp\{n^*T^*\}\delta_2^{\rho_1}.\end{aligned}\tag{2.38}$$

On the other hand, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{z},k}\left[\left(\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}+\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T^*\}}\right)\exp\left\{\rho_1\hat{\lambda}_1(\eta_1\wedge\eta_2)\right\}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\\ \le&\mathbb{E}_{\mathbf{z},k}\left[\exp\left\{\rho_1\hat{\lambda}_1(\eta_1\wedge\eta_2)\right\}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\le V_1(i,k),\end{aligned}\tag{2.39}$$

where the last inequality is due to (2.26) and an application of Dynkin's formula. Since $\Omega=\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}\cup\{\eta_1\wedge\eta_2\ge(n^*-1)T\}\cup\{\eta_2\le(n^*-1)T^*\}$, putting

$$\kappa=\max\left\{\exp\left\{-\frac{\lambda\rho}{8}T^*\right\},\exp\left\{(\hat{c}_1-(n^*-1)\hat{\lambda}_1)\rho_1T^*\right\}\right\}<1,$$

we have from (2.36), (2.37), (2.38) and (2.39) that

$$\begin{aligned}\mathbb{E}_{\mathbf{z},k}V_1(I(n^*T^*),\xi(n^*T^*))=&\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ &+\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ &+\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_2\le(n^*-1)T^*\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ \le&\kappa\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\le\eta_2\wedge(n^*-1)T^*\}}\exp\left\{\rho_1\hat{\lambda}_1(\eta_1\wedge\eta_2)\right\}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\\ &+\kappa\mathbb{E}_{\mathbf{z},k}\left[\mathbf{1}_{\{\eta_1\wedge\eta_2\ge(n^*-1)T^*\}}\exp\left\{\rho_1\hat{\lambda}_1(\eta_1\wedge\eta_2)\right\}V_1(I(\eta_1\wedge\eta_2),\xi(\eta_1\wedge\eta_2))\right]\\ &+\mathbb{E}_{\mathbf{z},k}\mathbf{1}_{\{\eta_2\le(n^*-1)T^*\}}V_1(I(n^*T^*),\xi(n^*T^*))\\ \le&\kappa V_1(i,k)+\exp\{n^*T^*\}\delta_2^{\rho_1}.\end{aligned}\tag{2.40}$$

The Lyapunov-type inequality (2.40) together with Lemma 2.5 will imply the desired result. Similar arguments to obtain the exponential convergence in total variation can be found in the proof of [29, Theorem 4.1]. □

2.4. *Proof of Theorem 2.3*

By Lemma 2.3, it is readily seen that λ_θ is continuous in θ. Therefore, there exists $\theta_0>0$ such that

$$\lambda_{\theta_0}<\frac{\lambda_0}{2}=\frac{\lambda}{2}.\tag{2.41}$$

It is noted that we are considering the case $\lambda<0$, so $\lambda_{\theta_0}<0$.

We will establish the following lemma, which shows that if the solution starts from initial points very close to the boundary, $I(t)\to0$ (exponentially fast) and $R(t)\to0$ with high probability. To keep the flow of presentation, its proof is postponed to Appendix.

Lemma 2.7. *Let $\theta_0>0$ be as in (2.41). For any $\varepsilon>0$, $H>0$, there is a constant $\theta_2>0$ such that*

$$\mathbb{P}_{\mathbf{z},k}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t}=\lambda<0\text{ and }\lim_{t\to\infty}R(t)=0\right\}\ge1-\varepsilon,\ \forall(\mathbf{z},k)\in[0,H]^2\times(0,\theta_2]^2\times\mathcal{M}.\tag{2.42}$$

Proof of Theorem 2.3. Because of Lemma 2.7, the process $\{(\mathbf{Z}(t),\xi(t))\}$ is transient in $\mathbb{R}^{4,\circ}_+\times\mathcal{M}$. This fact leads to that the process has no invariant probability measure in $\mathbb{R}^{4,\circ}_+\times\mathcal{M}$. As a result, $\boldsymbol{\nu}_0$ is the unique invariant probability measure of $\{(\mathbf{Z}(t),\xi(t))\}$ in $[0,\infty)^2\times\{0\}^2\times\mathcal{M}$. Let $H>0$ be sufficiently large such that $\boldsymbol{\nu}_0(\{s,v\in(0,H)\})>1-\varepsilon$. On the other hand, the process $\{(\mathbf{Z}(t),\xi(t))\}$ is tight (due to (2.2)). Therefore, for any initial condition $(\mathbf{z},k)\in\mathbb{R}^4_+\times\mathcal{M}$ the family of occupation measures $\Pi^t_{\mathbf{z},k}(\cdot)$, which is defined in (2.31), is tight in $\mathbb{R}^4_+\times\mathcal{M}$. Since any weak-limit of $\Pi^t_{\mathbf{z},k}$ as $t\to\infty$ must be an invariant probability measure of $(\mathbf{Z}(t),\xi(t))$, we have that $\Pi^t_{\mathbf{z},k}$ converges weakly to $\boldsymbol{\nu}_0$ as $t\to\infty$. As a result, for any $\delta>0$, there is a constant $\hat{T}>0$ satisfying that

$$\Pi^{\hat{T}}_{\mathbf{z},k}((0,H)^2\times(0,\delta)^2\times\mathcal{M})>1-\varepsilon,$$

or equivalently,

$$\frac{1}{\hat{T}}\int_0^{\hat{T}}\mathbb{P}_{\mathbf{z},k}\{(\mathbf{Z}(t),\xi(t))\in(0,H)^2\times(0,\delta)^2\times\mathcal{M}\}dt>1-\varepsilon.$$

Therefore,

$$\mathbb{P}_{\mathbf{z},k}\{\hat{\zeta}\le\hat{T}\}>1-\varepsilon,$$

where

$$\hat{\zeta}=\inf\{t\ge0:(\mathbf{Z}(t),\xi(t))\in(0,H)^2\times(0,\delta)^2\times\mathcal{M}\}.$$

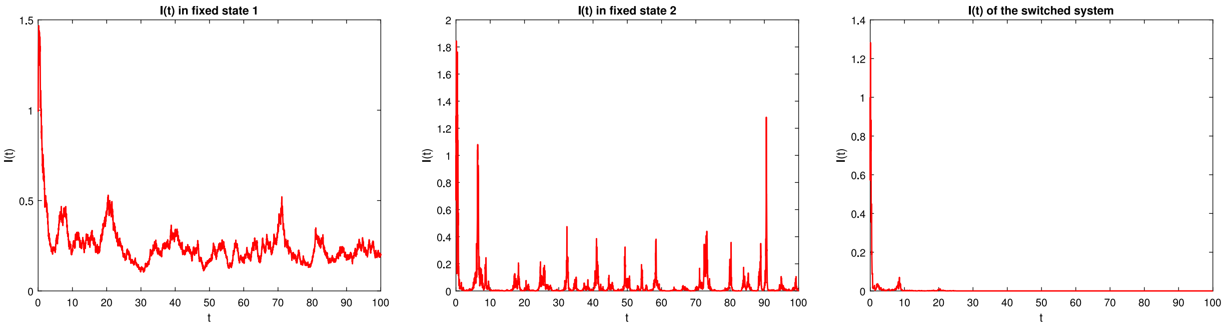

Fig. 1. Sample paths of $I(t)$ in Example 3.2. From left to right: in system with $\xi(t) = 1$, in system with $\xi(t) = 2$, and in the switched system, respectively.

We obtain from the strong Markov property and Lemma 2.7 that

$$\mathbb{P}_{\mathbf{z},k}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda < 0 \text{ and } \lim_{t\to\infty} R(t) = 0\right\} \geq 1-\varepsilon,\ \forall(\mathbf{z},k)\in\mathbb{R}_+^{4,*}\times\mathcal{M}.$$

Because $\varepsilon > 0$ is arbitrary, we obtain (2.10). The proof of Theorem 2.3 is complete. □

3. Numerical examples

In this section, we present some numerical examples to illustrate our theoretical results. These examples will also demonstrate interesting effects of discrete events process $\xi(t)$. It will be shown that random switching can reverse persistence to extinction, and vice versa.

Example 3.1. We examine system (1.1) with two switching states $\mathcal{M} = \{1,2\}$. Suppose $g(s+\beta v,i,1) = 4(s+0.01v)$ and $g(s+\beta v,i,2) = 8(s+0.01v)$. The other parameters are $\widetilde{q}(1) = 0.8$, $\widetilde{q}(2) = 0.4$, $A(1) = 2.5$, $A(2) = 1$, $\alpha_0(1) = 1$, $\alpha_0(2) = 0.5$, $\widetilde{p}(1) = 1$, $\widetilde{p}(2) = 0.3$, $\alpha_1(1) = 0.2$, $\alpha_1(2) = 3$, $\alpha_2(1) = 0.1, \alpha_2(2) = 0.2$, $\gamma(1) = 0.2$, $\gamma(2) = 0.5$, $\sigma_1(1) = 0.2, \sigma_1(2) = 1, \alpha_3(1) = 0.1, \alpha_3(2) = 3, \sigma_3(1) = 0.2, \sigma_3(2) = 2, \sigma_2(1) = 0.2, \sigma_2(2) = 1, \sigma_4(1) = \sigma_4(2) = 0.1$.

In this example, if there is no random switching, the thresholds for the system in state 1 (i.e., $\xi(t) = 1$ for all t) and in state 2 (i.e., $\xi(t) = 2$ for all t) are $\lambda_1 = 0.5800$ and $\lambda_2 = 1.4077$, respectively. It then follows that without switching, the disease will persist in either fixed state. However, with switching rates $\gamma_{12} = \gamma_{21} = 20$, we have $\lambda = -0.3087$ for system (2.2), which implies that the disease will eventually disappear. This example shows that the random switching can reverse persistence into extinction, see Fig. 1.

With the algebraic representation of λ, we can view λ as a function of the parameters. For example, with $\gamma_{12} = \gamma_{21} = y$ and $\widetilde{q}(1) = \widetilde{q}(2) = x$ while the other parameters receive values as above, we have Fig. 2 for λ as a function of x, y and Fig. 3 for λ as a function of x or y given some fixed values of the other. We will discuss biological interpretation of these relationship later in Section 4.

Example 3.2. Consider system (1.1) with two random switching states $\mathcal{M} = \{1,2\}$. Suppose $g(s+\beta v,i,1) = 5(s+0.02v)$ and $g(s+\beta v,i,2) = (s+0.02v)$. The other parameters are $\widetilde{q}(1) = 0.8, \widetilde{q}(2) = 0.8, A(1) = 1.2, A(2) = 5, \alpha_0(1) = 0.8, \alpha_0(2) = 0.8, \widetilde{p}(1) = 0.5, \widetilde{p}(2) = 0.5, \alpha_1(1) = 2, \alpha_1(2) = 2, \gamma(1) = 0.2, \gamma(2) = 0.5, \alpha_2(1) = 0.1, \alpha_2(2) = 0.1, \alpha_3(1) = 0.11, \alpha_3(2) = 0.11, \sigma_1(1) = \sigma_1(2) = \sigma_3(1) = \sigma_3(2) = \sigma_2(1) = \sigma_2(2) = 0.1$.

In this example, the system in state 1 (i.e., $\xi(t) = 1$ for all t) and the system in fixed state 2 (i.e., $\xi(t) = 2$ for all t) are $\lambda_1 = -1.0010$ and $\lambda_2 = -0.5442$. Thus, when switching is not involved, the disease will die out in each of the two fixed states. However, with switching rates $\gamma_{12} = \gamma_{21} = 20$, we have $\lambda = 1.0261$, which implies the disease persists; see Fig. 4.

With $\gamma_{12} = \gamma_{21} = y$ and $\widetilde{q}(1) = \widetilde{q}(2) = x$ while the other parameters taking values as above, we have Fig. 5 for λ as a function of x, y and Fig. 6 for λ as a function of x or y given some fixed values of the other.

4. Discussion and interpretation

From our analysis, we can explore the effect of vaccination to the disease. Recall that our theoretical results show that threshold λ given by (2.7) will determine whether or not the pandemic will go away. In fact, we have shown that if $\lambda < 0$ then the pandemic will end in the future and if $\lambda > 0$ the disease always persists. As an application to real world problems, to control the disease, we would try to reduce λ as much as possible. In this section, we will consider the dependence of λ on parameters in epidemic systems, e.g., vaccinated rate in newborns and susceptible groups, etc. From these interpretations, we can answer some questions such as how much vaccination is sufficient to control the disease, and what percentage of newborns should get the vaccine, etc.

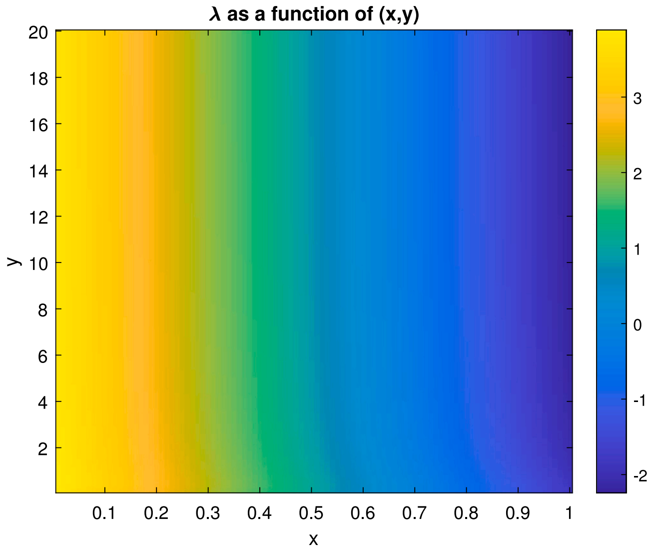

Fig. 2. λ as a function of $\widetilde{q}(1) = \widetilde{q}(2) = x$ and $\gamma_{12} = \gamma_{21} = y$.

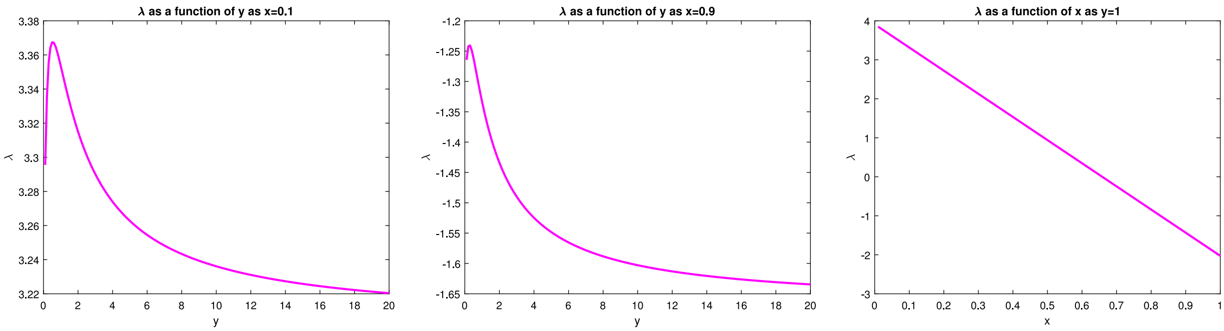

Fig. 3. From left to right: the graphs of $\lambda(x = 0.1, y)$, $\lambda(x = 0.9, y)$, and $\lambda(x, y = 1)$, respectively.

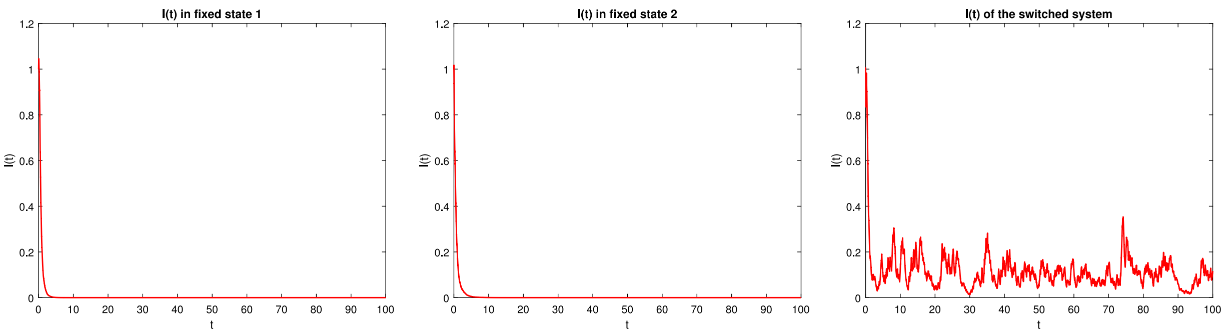

Fig. 4. Sample paths of $I(t)$. From left to right: in system with $\xi(t) = 1$, in system with $\xi(t) = 2$, and in the switched system, respectively in Example 3.1.

First, to have more concrete discussion, we consider the case that the incidence rate is linear. In this case, by Lemma 2.3 we have the following explicit formula for λ

$$\lambda = \sum_{k\in\mathcal{M}} \left[c_1(k)(1 - \widetilde{q}(k))A(k) - \left(\alpha_0(k) + \alpha_3(k) + \alpha_1(k) + \frac{\sigma_3^2(k)}{2} \right) \right] \pi_k.$$

This shows that λ depends "linearly" on $\widetilde{q}$, the proportion of vaccinated newborns. Moreover, if $\widetilde{q} = 1$, $\lambda < 0$; if $\widetilde{q}$ is increasing then λ is decreasing. This observation coincides with the natural intuition that the more newborns are vaccinated, the higher chance

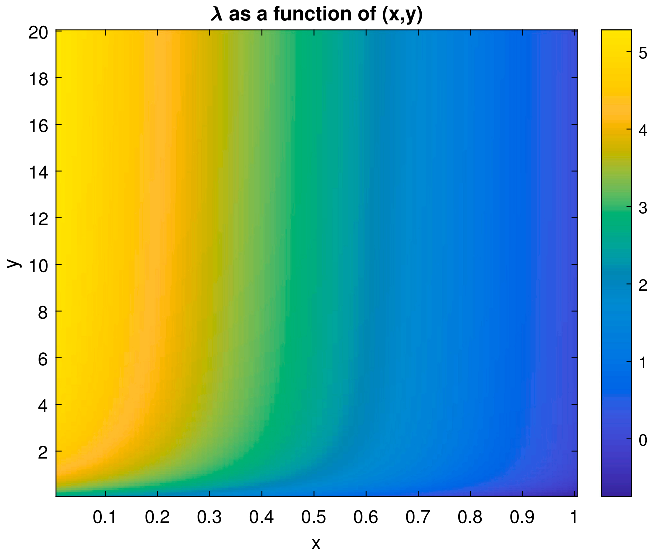

Fig. 5. λ as a function of $\widetilde{q}(1) = \widetilde{q}(2) = x$ and $\gamma_{12} = \gamma_{21} = y$.

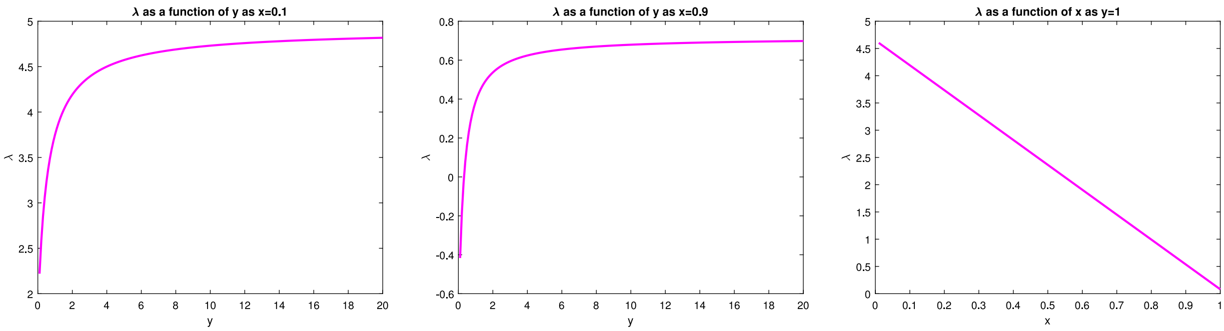

Fig. 6. From left to right: the graphs of $\lambda(y)$ with $x = 0.1$, $\lambda(y)$ with $x = 0.9$, and $\lambda(x)$ with $y = 1$, respectively.

of ending the pandemic we have. If all newborns are vaccinated (i.e., $\widetilde{q} = 1$), the disease will eventually disappear in the future. However, getting $\widetilde{q} = 1$ is often impossible because of many reasons such as religion, economics, budgets, etc. One would like to find the best possible $\widetilde{q}$ that still guarantees to end the pandemic in the future. From our formula of λ, we can easily to find this best value for $\widetilde{q}$. If we assume that there is no random switching, i.e., $\mathcal{M}$ has only one value, then the best $\widetilde{q}$ making $\lambda < 0$ is

$$\widetilde{q} > 1 - \frac{\alpha_0 + \alpha_3 + \alpha_1 + \sigma_3^2/2}{c_1 A}.$$

(It is noted again that $\xi(t)$ can be used to represent the changes in discrete event such as season, infection status of other diseases, etc. The discussion of the effects of such random switching are presented in numerical examples in Section 3.)

The effects of $\widetilde{p}$, the rate of vaccination of susceptible group is similar to $\widetilde{q}$, but less straightforward. Note that $\widetilde{p}$ influences λ through $c_1(k)$, which is the solution of (2.11). For simplicity of computation, let us assume that there is no switching. In this case, solving (2.11) gives us

$$c_1 = \frac{b(\widetilde{p}\beta + \alpha_0 + \gamma)}{\widetilde{p}\alpha_0 + \alpha_0^2 + \alpha_0\gamma}.$$

Therefore, $\frac{\partial c_1}{\partial \widetilde{p}} = (\beta - 1)b(\alpha_0^2 + \alpha_0\gamma)(\widetilde{p}\alpha_0 + \alpha_0^2 + \alpha_0\gamma)^{-2} < 0$ due to $\beta < 1$. That means increasing $\widetilde{p}$ will decrease c_1, and thus, will decrease λ. As a result, better rate of vaccination for the susceptible community gives us higher chance for ending the disease in the long future. The dependence of λ on $\widetilde{p}$ is illustrated numerically in Figs. 7 and 8.

Example 4.1. We revisit Examples 3.1 and 3.2. Since λ depends explicitly on $\widetilde{q}$ through algebraic representation of λ and that is also illustrated in Section 3, we will not simulate this relationship. Fig. 7 shows the dependence of λ on $\widetilde{p}$ in the case that other

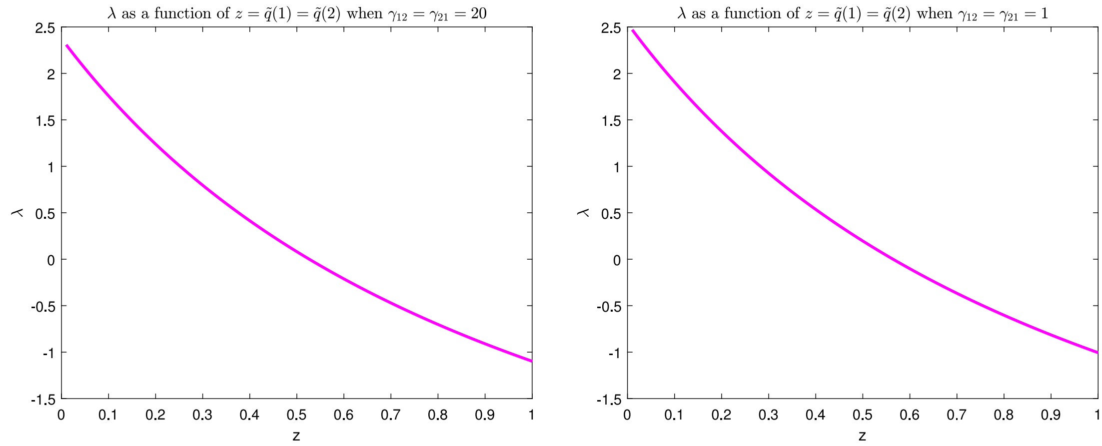

Fig. 7. From left to right: the graphs of $\lambda(z)$ with $\widetilde{p}(1) = \widetilde{p}(2) = z$; $\gamma_{12} = \gamma_{21} = 20$ and $\gamma_{12} = \gamma_{21} = 1$ respectively. The other parameters have values in Example 3.1.

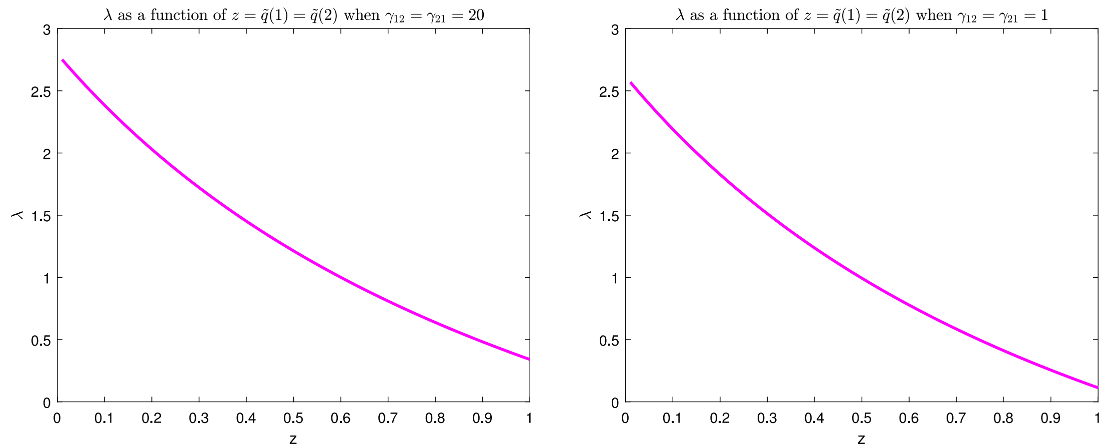

Fig. 8. From left to right: the graphs of $\lambda(x)$ with $\widetilde{p}(1) = \widetilde{p}(2) = x$; $\gamma_{12} = \gamma_{21} = 20$ and $\gamma_{12} = \gamma_{21} = 1$ respectively. The other parameters have values in Example 3.2.

parameters have values as in Example 3.1; and Fig. 8 shows the dependence of λ on $\widetilde{p}$ in the case that other parameters have values as in Example 3.2.

Second, if the incidence rate does not have linear forms, but is a general function, it is difficult to provide a rigorous discussion in how $\widetilde{p}$, $\widetilde{q}$ influence λ as we could not provide explicit calculations. We provide some numerical examples about relationship between λ and $\widetilde{q}$, $\widetilde{p}$ (see Fig. 9).

Example 4.2. In this example, we assume that three is no switching. Suppose $g(s+\beta v, 1) = \frac{s+0.02v}{0.5+s+0.02v}$. The other parameters are $A = 2.5$, $\alpha_0 = 1$, $\alpha_1 = .2$, $\alpha_2 = 0.1$, $\gamma = .2$, $\sigma_1 = .2$, $\alpha_3 = .1, \sigma_3 = .2$, $\sigma_2 = 0.2$, $\sigma_4 = 0.1$ Since the incidence rate g is not linear, we do not have an explicit formula for λ. Therefore, we run stochastic simulation to approximate the value λ. The figures below show approximately the graph of λ as a function of $\widetilde{q}$ and $\widetilde{p}$. The curves are rough due to the randomness of the approximation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Lemma 2.7

Proof of Lemma 2.7. We obtain from the ergodicity of $(\widetilde{S}^{\theta_0}(t), \widetilde{V}^{\theta_0}(t), \xi(t))$ that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t h(\widetilde{S}^{\theta_0}(u) + \beta\widetilde{V}^{\theta_0}(u), 0, \xi(u))du = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} h(s+\beta v, 0, k)\boldsymbol{\nu}_{\theta_0}(ds, dv, k) = \lambda_{\theta_0} \text{ a.s. }, \tag{A.1}$$

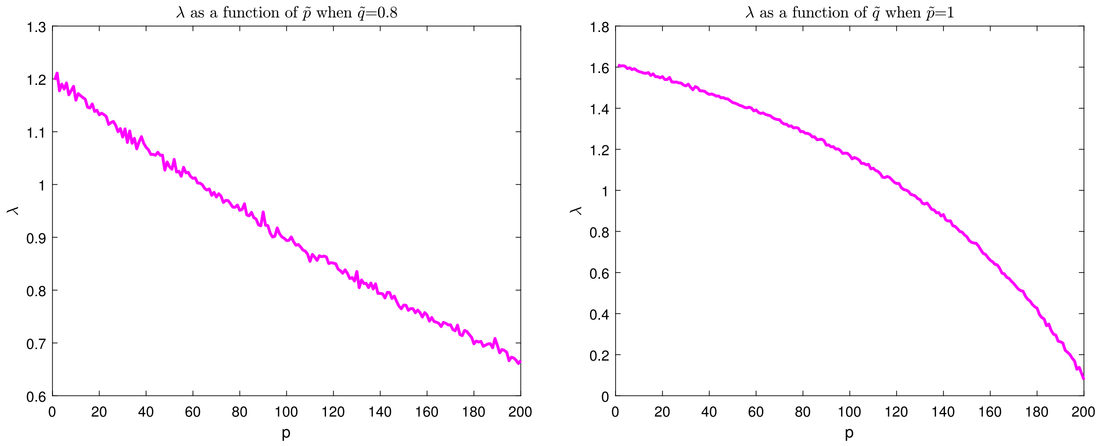

Fig. 9. From left to right: the graphs of $\lambda(\widetilde{p})$ when $\widetilde{q}=0.8$ and $\lambda(\widetilde{q})$ when $\widetilde{p}=1$ respectively.

where h as in (2.29), and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\alpha_0(\xi(u))+\alpha_2(\xi(u))+\frac{\sigma_4^2(\xi(u))}{2}\right)du=\sum_{k\in\mathcal{M}}(\alpha_0(k)+\alpha_2(k)+\frac{\sigma_4(k)}{2})\pi_k=:\widehat{c}_2. \tag{A.2}$$

By the strong law of large numbers for martingales,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_\ell(\xi(u))dW_\ell(u)=0\quad \ell=1,2,3,4. \tag{A.3}$$

Now, denote

$$\widehat{\lambda}_2:=\min\{\frac{1}{9}|\lambda_{\theta_0}|,\frac{1}{3}\widehat{c}_2\}. \tag{A.4}$$

It follows from (A.1) that for any $\varepsilon>0$, there exists a $\widetilde{T}_1=\widetilde{T}_1(H,\varepsilon)>0$ such that $\mathbb{P}_{(H,H),k}(\Omega_1)\geq 1-\frac{\varepsilon}{5}$, where

$$\Omega_1=\left\{\omega:\frac{1}{t}\int_0^t h(\widetilde{S}^{\theta_0}_{(H,H),k}(u)+\beta\widetilde{V}^{\theta_0}_{(H,H),k}(u),0,\xi(u))du\leq\lambda_{\theta_0}+\widehat{\lambda}_2\leq-8\widehat{\lambda}_2,\quad\text{for all } t\geq\widetilde{T}_1\right\}. \tag{A.5}$$

In the above, the subscript in $\mathbb{P}_{(H,H),k}$, $(\widetilde{S}^{\theta_0}_{(H,H),k}(t),\ \widetilde{V}^{\theta_0}_{(H,H),k}(t))$ shows the initial value of $(\widetilde{S}^{\theta_0}(t),\widetilde{V}^{\theta_0}(t),\xi(t))$.

Because of the comparison theorem [30], we have $S_{\mathbf{z},k}(u)\leq\widetilde{S}^{\theta}_{(H,H),k}(u)$, $V_{\mathbf{z},k}(u)\leq\widetilde{V}^{\theta}_{(H,H),i}(u)$, $\forall 0\leq u\leq\widetilde{\tau}$ almost surely if $\mathbf{z}=(s,v,i,r)$ satisfies $s\leq H,v\leq H$. This and (A.2), (A.3) deduce that there exists $\widetilde{T}_2=\widetilde{T}_2(\varepsilon)>0$ such that $\mathbb{P}(\Omega_2)\geq 1-\frac{\varepsilon}{5}$ and $\mathbb{P}(\Omega_3)\geq 1-\frac{\varepsilon}{5}$ where

$$\Omega_2=\left\{\omega\in\Omega:\frac{1}{t}\left|\int_0^t\sigma_\ell(\xi(u))dW_\ell(u)\right|\leq\widehat{\lambda}_2,\ \ell=1,2,3,4\quad\text{for all } t\geq\widetilde{T}_2\right\}, \tag{A.6}$$

and

$$\Omega_3=\left\{\omega\in\Omega:\widehat{c}_2-\widehat{\lambda}_2<\frac{1}{t}\int_0^t\left(\alpha_0(\xi(u)+\alpha_2(\xi(u)))+\frac{\sigma_4(\xi(u))}{2}\right)du<\widehat{c}_2+\widehat{\lambda}_2\quad\text{for all } t\geq\widetilde{T}_2\right\}. \tag{A.7}$$

Let $\widetilde{T}=\max\{\widetilde{T}_1,\widetilde{T}_2\}$. On the finite interval $[0,\widetilde{T}]$, thanks to (2.3), we can find a sufficiently large $M>0$ satisfying that $\mathbb{P}(\Omega_4)\geq 1-\frac{\varepsilon}{5}$, if $(\mathbf{z},k)\in[0,H]^2\times[0,1]^2\times\mathcal{M}$ where

$$\Omega_4=\left\{\omega\in\Omega:\int_0^{\widetilde{T}}h(S(u)+\beta V(u),I(u),\xi(u))du+\sup_{t\in[0,\widetilde{T}],\ell=3,4}\left|\int_0^t\sigma_\ell(\xi(u))W_\ell(u)du\right|<M\right\}. \tag{A.8}$$

It is noted that Ω_4 (and other Ω_i) depends on initial values of the solution, and it would be labeled as $\Omega_4^{\mathbf{z},k}$ to indicate the corresponding initial values. However, to simplify notation, we will remove this superscript. Moreover, Doob's inequality allows us to choose M sufficiently large such that

$$\mathbb{P}(\Omega_5)\geq 1-\frac{\varepsilon}{5},\ \text{where } \Omega_5=\left\{\omega\in\Omega:\left|\int_0^t\sigma_\ell(\xi(u))W_\ell(u)du\right|\leq\frac{M}{2},\ \text{for all } t\in[0,\widetilde{T}]\text{ and }\ell=1,2,3,4\right\}. \tag{A.9}$$

Denote

$$\check{\gamma}:=\max_{k\in\mathcal{M}}\alpha_3(k)\text{ and }\widehat{c}_3:=\max_{k\in\mathcal{M}}\{\alpha_0(k)+\alpha_2(k)+\frac{\sigma_4^2(k)}{2}\}.$$

Since $h(u,i,k) = g(u,i,k) - (\alpha_0(k) + \alpha_3(k) + \alpha_1(k)) - \frac{\sigma_3^2(k)}{2}$ is continuous at $i = 0$ uniformly in u, we can pick $\theta_1 \in \left(0, \frac{\theta_0}{M(1+\check{\gamma}Me^{\hat{c}_3 T + M}T)}\right)$ such that

$$|h(u,i,k) - h(u,0,k)| \le \hat{\lambda}_2 \text{ whenever } 0 \le i \le \theta_1. \tag{A.10}$$

Next, we pick θ_2 satisfying $0 < \theta_2 < \min\left\{\theta_1 e^{-M}, \theta_0\left(1 + \frac{\check{\gamma}}{\hat{c}_2 - 4\hat{\lambda}_2}\right)^{-1}\right\}$.

Combining the second equation of (1.1), (A.9), and (A.8) implies that for all $\omega \in \Omega_3$, we have

$$\begin{aligned} I(t) =& I(0)\exp\left\{\int_0^t h(S(u)+\beta V(u), I(u), \xi(u))du + \int_0^t \sigma_3(\xi(u))dW_3(u)\right\} \\ \le& I(0)e^M \le \theta_2 e^M < \theta_1 \text{ for any } t \in [0,\widetilde{T}], \text{ if } I(0) \le \theta_2. \end{aligned} \tag{A.11}$$

On the other hand, by a variation of constants formula for linear stochastic differential equations (e.g., [32, Chapter 3]), we have

$$R(t) = \frac{1}{\Phi(t)}\left(R(0) + \int_0^t \alpha_3(\xi(u))\Phi(u)I(u)du\right), \tag{A.12}$$

where

$$\Phi(t) = \exp\left\{\int_0^t \left(\alpha_0(\xi(u)) + \alpha_2(\xi(u)) + \frac{\sigma_4^2(\xi(u))}{2}\right)du - \int_0^t \sigma_4(\xi(u))dW_4(u)\right\}.$$

Provided $(\mathbf{z},k) \in [0,H]^2 \times [0,1]^2 \times \mathcal{M}$, for $t \le \widetilde{T}$ and $\omega \in \Omega_4$, we have from (A.8) that $\frac{1}{\Phi(t)} \le M$ (where M is defined as in (A.8)) and $\Phi(t) \le Me^{\hat{c}_3 t}$, which in combination with (A.11), (A.12), and the definition of θ_1 implies

$$R(t) \le M\left(r + Me^{\hat{c}_3\widetilde{T}}\check{\gamma}\int_0^t I(u)du\right) \le M(\theta_1 + Me^{\hat{c}_3\widetilde{T}}\check{\gamma}\widetilde{T}\theta_1) < \theta_0. \tag{A.13}$$

Define the stopping time

$$\widetilde{\zeta} := \inf\left\{t \ge 0 : R(t) \ge \theta_0 \text{ or } I(t) \ge \theta_1\right\}. \tag{A.14}$$

Because of (A.11) and (A.13),

$$\widetilde{\zeta} > \widetilde{T} \text{ if } (\mathbf{z},k) \in [0,H]^2 \times [0,\theta_2]^2 \times \mathcal{M}, \text{ and } \omega \in \Omega_4. \tag{A.15}$$

It is noted again that $S_{\mathbf{z},k}(t) \le \widetilde{S}^{\theta_0}_{(H,H),k}(t), V_{\mathbf{z},k}(t) \le \widetilde{V}^{\theta_0}_{(H,H),k}(t)$ for any $t \le \widetilde{\zeta}$ given that $\mathbf{z} = (s,v,i,r)$ with $s \le H, v \le H$ (thanks to the comparison theorem [30]). Thus, from (A.11), (A.10) and non-decreasing in u property of $h(u,x,k)$, if $t \le \widetilde{\zeta}$, one has

$$\begin{aligned} h(S(t)+\beta V(t), I(t), \xi(t)) &\le h(S(t)+\beta V(t), 0, \xi(t)) + \hat{\lambda}_2 \\ &\le h(\widetilde{S}^{\theta_0}_{(H,H),k}(t) + \beta\widetilde{V}^{\theta_0}_{(H,H),k}(t), 0, \xi(t)) + \hat{\lambda}_2, \end{aligned}$$

given the initial value of $(\mathbf{Z}(t),\xi(t))$ is in $[0,H]^2 \times [0,\theta_2] \times [0,\infty) \times \mathcal{M}$. As a result,

$$I(t) \le I(0)\exp\left\{\int_0^t \left(h(\widetilde{S}^{\theta_0}_{(H,H),k}(u) + \beta\widetilde{V}^{\theta_0}_{(H,H),k}(u), 0, \xi(u))\right)du + \hat{\lambda}_2 t + \int_0^t \sigma_3(\xi(u))dW_2(u)\right\}. \tag{A.16}$$

Combining (A.5), (A.6), (A.11), (A.15) and (A.16) for $\omega \in \bigcap_{j=1}^5 \Omega_j$ and $(\mathbf{z},k) \in [0,H]^2 \times [0,\theta_2]^2 \times \mathcal{M}$, we have $\widetilde{\zeta} > \widetilde{T}$ and that

$$\begin{aligned} I(t) &\le I(0)\exp\left\{\int_0^t g(\widetilde{S}^{\theta_0}_{(H,H),k}(u) + \beta\widetilde{V}^{\theta_0}_{(H,H),k}(u), 0, \xi(u))du + \hat{\lambda}_2 t + \int_0^t \sigma_3(\xi(u))dW_3(u)\right\} \\ &\le I(0)\exp\left\{-8\hat{\lambda}_2 t + 2\hat{\lambda}_2 t\right\} \le I(0)\exp\{-6\hat{\lambda}_2 t\} < \theta_0, \ t \in [\widetilde{T},\widetilde{\zeta}). \end{aligned} \tag{A.17}$$

We also have that for any $\widetilde{T} < t \le \widetilde{\zeta}$ and $\omega \in \bigcap_{j=1}^5 \Omega_j$ and $(\mathbf{z},k) \in [0,H]^2 \times [0,\theta_2]^2 \times \mathcal{M}$ that

$$e^{(\hat{c}_2 - 2\hat{\lambda}_2)t} \le \Phi(t) \le e^{(\hat{c}_2 + 2\hat{\lambda}_2)t},$$

which implies

$$\begin{aligned} R(t) =& \frac{1}{\Phi(t)}\left(R(0) + \int_0^t \alpha_3(\xi(u))\Phi(u)I(u)du\right) \\ \le& e^{(-\hat{c}_2 + 2\hat{\lambda})t}\left(r + \check{\gamma}\int_0^t e^{(\hat{c}_2 + 2\hat{\lambda}_2)u} ie^{-6\hat{\lambda}_2 u}du\right) \\ \le& e^{(-\hat{c}_2 + 2\hat{\lambda}_2)t}\left(\theta_2 + \theta_2\frac{\check{\gamma}}{\hat{c}_2 - 2\hat{\lambda}_2}e^{(\hat{c}_2 - 4\hat{\lambda}_2)t}\right) \\ \le& \theta_2 + \theta_2\frac{\check{\gamma}}{\hat{c}_2 - 4\hat{\lambda}_2} < \theta_0. \end{aligned} \tag{A.18}$$

Thanks to (A.17) and (A.18), we must have $\widetilde{\zeta} = \infty$ for all $\omega \in \bigcap_{j=1}^{5} \Omega_j$, $(\mathbf{z}, k) \in [0, H]^2 \times [0, \theta_2]^2 \times \mathcal{M}$. Because of $\widetilde{\zeta} = \infty$, we have from (A.17) and (A.18) again that

$$I(t) \le I(0)e^{-6\hat{\lambda}_2} \text{ and } R(t) \le e^{(-\hat{c}_2+2\hat{\lambda}_2)t} r + i \frac{\check{\gamma}}{\hat{c}_2 - 4\hat{\lambda}_2} e^{-2\hat{\lambda}_2 t}, \text{ for any } t \ge \widetilde{T}, \omega \in \bigcap_{j=1}^{5} \Omega_j, I(0) = i \le \theta_2.$$

This clearly implies that $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = 0$ $\forall \omega \in \bigcap_{j=1}^{5} \Omega_j$, provided $(\mathbf{z}, k) \in [0, H]^2 \times [0, \theta_2]^2 \times \mathcal{M}$.

Next, we define a randomized occupation measure

$$\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot) := \frac{1}{t} \int_0^t \mathbf{1}_{\{(\mathbf{Z}(u),\xi(u))\in\cdot\}} du, \quad t > 0,$$

in which, the subscript in $\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot)$ indicates the initial condition. As an application of Lemma 2.7 and the comparison $S(t) \le \widetilde{S}^{\theta_0}_{(H,H),k}(t), V(t) \le \widetilde{V}^{\theta_0}_{(H,H),k}(t), t \ge 0$, the family of measures $\{\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot;\omega), t > 0, \omega \in \cap_{j=1}^{5}\Omega_j\}$ is tight in $\mathbb{R}^4_+ \times \mathcal{M}$ and any weak limit of $\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot)$ as $t \to \infty$ must have a support that is on $[0,\infty)^2 \times \{0\}^2 \times \mathcal{M}$. On the other hand, with probability 1, any weak-limit of $\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot)$ as $t \to \infty$ is an invariant probability measure of the process $\{(\mathbf{Z}(t), \xi(t))\}$ on $\mathbb{R}^4_+ \times \mathcal{M}$; see e.g., [29,33]. Moreover, it is readily seen that $\boldsymbol{\nu}_0$, when regarded as an invariant measure of $\{(\mathbf{Z}(t), \xi(t))\}$, is the unique invariant probability measure on $[0,\infty)^2 \times \{0\}^2 \times \mathcal{M}$. Therefore, for almost every $\omega \in \cap_{j=1}^{5}\Omega_j$, $\widetilde{\Pi}^t_{\mathbf{z},k}(\cdot)$ converges weakly to $\boldsymbol{\nu}_0$ as $t \to \infty$. As a result, we obtain from the weak convergence that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t h(S(u) + \beta V(u), I(u), \xi(u)) du = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} h(s + \beta v, 0, k) \boldsymbol{\nu}_0(ds, dv, k) = \lambda, \tag{A.19}$$

for almost every $\omega \in \cap_{j=1}^{5}\Omega_j$. It is noted that the limit (A.19) is valid because of the weak convergence and the uniform integrability

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \int_0^t (S(u) + V(u))^{1+p} du &\le \lim_{t\to\infty} \frac{1}{t} \int_0^t (\widetilde{S}^{\theta_0}(u) + \widetilde{V}^{\theta_0}(u))^{1+p} du \\ &= \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} (s + v)^{1+p} \boldsymbol{\nu}_{\theta_0}(ds, dv, k) < \infty \text{ for some small } p > 0;\end{aligned}$$

see e.g., [29, Lemma 5.6]. By (A.11), one has

$$\frac{\ln I(t)}{t} = \frac{\ln I(0)}{t} + \frac{1}{t} \int_0^t h(S(u) + \beta V(u), 0, \xi(u)) du + \frac{1}{t} \int_0^t \sigma_2(\xi(u)) dW_2(u). \tag{A.20}$$

Therefore, letting $t \to \infty$ in (A.20) and because of (A.3) and (A.19), we have that for almost every $\omega \in \cap_{j=1}^{5}\Omega_j$,

$$\lim_{t\to\infty} \frac{\ln I(t)}{t} = \lambda \text{ and } \lim_{t\to\infty} R(t) = 0.$$

As a result, by noticing that $\mathbb{P}(\cap_{j=1}^{5}\Omega_j) \ge 1 - \varepsilon$, the lemma is proved. □

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