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Limit theorems of additive functionals for regime-switching diffusions with infinite delay

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Abstract

This paper focuses on a class of regime-switching functional diffusion processes with infinite delay and develops a central limit theorem (CLT) for additive functionals under uniform mixing conditions. In addition, a law of iterated logarithm (LIL) for the additive functionals is also established by using the square integrable martingale difference sequences. Finally, two examples are given to illustrate our results.

MSC: 60F05; 60B12; 60H20

Keywords: Regime-switching diffusion; Infinite delay; Central limit theorem; Law of the iterated logarithm

1. Introduction

Switching diffusions also known as regime-switching diffusions have drawn much attention from researchers and practitioners lately. This is largely because of their wide range of applications and potential applications. The use of switching random processes much extended the

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applicability of diffusion processes, which is used conveniently to depict random environment that cannot be formulated as usual in typical diffusion systems; see [19,32] and references therein. Continuing on the lines of works, switching diffusions in which the generator of the switching process depends on the past and takes values in a countable state space was considered in [20]. In recent years, we have seen success in using switching diffusions for control and optimization of stochastic systems, financial engineering, biological and ecological systems, multi-agent systems, queueing theory, networked systems, and social network modeling, among other. The vast applications demand that we have deeper understanding of such systems. A distinct feature of such processes is the coexistence of continuous dynamics and discrete events, which much extended the applicability of diffusion processes. This feature facilitates the flexibility and versatility of the models, but it also brings up new challenges and difficulties. For example, even two linear systems being both stable, joined together by use of a modulating random switching process may lead to instability of the system with switching. We refer the reader to the observation in [27] and the analysis in [31, pp. 229–233]; likewise for similar behavior of recurrence for switching diffusions [22].

This work continues our quest on regime-switching diffusions. In contrast to the existing work, this paper focuses on regime-switching functional stochastic differential equations. Our main effort is devoted to obtaining CLTs and LILs. For a fixed but otherwise arbitrary $r > 0$, consider

$$\mathcal{C}_r = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^d) : \lim_{\theta \to -\infty} e^{r\theta} \phi(\theta) \text{ exists in } \mathbb{R}^d \right\}$$

with the norm $\|\phi\|_r = \sup_{-\infty<\theta\le 0} e^{r\theta}|\phi(\theta)|$ for $r > 0$, and $\mathbf{S} = \{1, 2, \ldots, N\}$, $1 < N < \infty$. This paper is concerned with the regime-switching diffusion processes with infinite delay described by the following stochastic functional differential equations with infinite delay and Markovian switching

$$dX(t) = b(X_t, \alpha(t))dt + \sigma(X_t, \alpha(t))dW(t) \tag{1.1}$$

with initial data $(X_0, \alpha(0)) = (\phi, i) \in \mathcal{C}_r \times \mathbf{S}$, where $b : \mathcal{C}_r \times \mathbf{S} \mapsto \mathbb{R}^d$ and $\sigma : \mathcal{C}_r \times \mathbf{S} \mapsto \mathbb{R}^{d\times m}$ are continuous functionals, $W(t)$ is an m-dimensional Brownian motion, $\alpha(t)$ is a continuous-time Markovian chain on $\mathbf{S}$ with the transition rate satisfying

$$\mathbb{P}\left\{\alpha(t+\Delta) = j \middle| \alpha(t) = i\right\} = \begin{cases} q_{ij}\Delta + o(\Delta), & j \neq i \\ 1 + q_{ii}\Delta + o(\Delta), & j = i, \end{cases} \tag{1.2}$$

provided $\Delta \downarrow 0$. In this paper, $\alpha(t)$ is assumed to be independent of $W(t)$ and the Q-matrix $Q = (q_{ij})$ is assumed to be irreducible and conservative, this is, $q_{kk} = -\sum_{j\neq k} q_{kj}$. Note that in (1.1), $X(t)$ the solution process belonging to $\mathbb{R}^d$, where $X_t \in \mathcal{C}_r$ is called segment process or solution map, which is a function process. The solution process $X(t)$ and even the pair $(X(t), \alpha(t))$ are not Markov processes because of the time delay involved, but $(X_t, \alpha(t))$ is.

In our recent work [23], we showed that the Markovian process $(X_t, \alpha(t))$ determined by (1.1) and (1.2) is exponentially ergodic with a unique invariant probability measure. Based on the exponential ergodicity, the current paper examines further asymptotic behaviors of the functional $\int_0^t f(X_s, \alpha(s))ds$ as $t \to \infty$, for $f : \mathcal{C}_r \times \mathbf{S} \to \mathbb{R}$ in a class of reference Borel measurable functionals. More precisely, this paper will establish the CLT and the LIL for additive functionals of the regime-switching functional diffusion system determined by (1.1) and (1.2).

The central limit theorem for additive functionals of Markovian processes is a fundamental concept in classical probability theory. It can be traced back to Doeblin's seminal work [10] in 1938, in which the strong law of large numbers and the central limit theorem were established for discrete time Markov chains with countable state spaces. Since then, CLTs for additive functionals of Markovian processes have been extensively studied in the literature under different settings; see, for example, [13] for stationary reversible ergodic Markovian chains, [7,12] for ergodic Markovian processes in total variational metric, [2,9,15,17,24,26,28] for ergodic Markovian processes in a Wasserstein distance, and the references therein. For regime-switching diffusions, [21] discussed the CLT of scaled regime-switching diffusions, and [4] established the CLT for additive functionals of regime-switching diffusions. These two papers only considered the regime-switching diffusions without delay and their proofs depend on the infinitesimal generator of corresponding Markovian processes (diffusions). These methods cannot be used to examine the present process described by (1.1) and (1.2) since the infinitesimal generator is not available for the diffusion with delay. Motivated by these observations, this paper aims to establish the CLT for additive functionals of regime-switching diffusions with infinite delay.

Moreover, this paper also investigates laws of iterated logarithms for additive functionals of regime-switching diffusions with infinite delay. Classical strong law of invariance principle was first established for a sequence of independent and identically distributed random variables by V. Strassen [25], which is also referred to as the Strassen invariance principle. Also please refer to [2,4–6,8,11,14,16,28,29] and the references therein for the LIL under different settings. In particular, [4] considered the LIL for regime-switching diffusions without delay. [2,28] established the LIL for stochastic functional differential equations with finite and infinite delay in different assumptions, respectively. This paper aims to further this line of research by studying the LIL for additive functionals of regime-switching diffusion processes with infinite delay.

Note that neither the CLT nor the LIL for additive functionals of regime-switching diffusions with infinite delay can be obtained from existing methods. The main difficulties stem from the infinite delay and Markovian chain in these systems. Since the Markovian process $(X_t, \alpha(t))$ is only right continuous, to obtain the CLT, the method developed in [24] cannot be used directly. Some new estimations (see, for example, Lemma 3.6), and more analyses in the proof of Theorem 3.2 are needed. Compared with our previous result in [28, Theorem 3.1], this paper removes the non-degeneracy requirement on diffusion term and uses the "averaging" dissipative condition (see, Remark 2.2) to obtain ergodicity. For the LIL, we present a continuous-time version and our conditions allow some subsystems to be non-dissipative. This also generalizes our previous results in [28].

This paper is organized as follows. Section 2 begins with the formulation, notation, and definitions. Also included are preliminary results from [23]. The CLT and the LIL for additive functionals of the regime-switching functional diffusions determined by (1.1) and (1.2) are presented in Sections 3 and 4, respectively. Finally, two examples are given to illustrate our results.

2. Notation and preliminaries

To facilitate presentation to follow, we introduce some notation and definitions. Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space and $|\cdot|$ the Euclidean norm. Let $\|\cdot\|_{\mathrm{HS}}$ be the Hilbert–Schmidt norm, that is, $\|\sigma\|_{\mathrm{HS}}^2 = \sum_{k=1}^d \sum_{l=1}^m \sigma_{kl}^2$ for any matrix $\sigma = (\sigma_{kl}) \in \mathbb{R}^{d\times m}$, $\mathbf{1}_G$ be the indicator function of the set G, and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a complete filtered probability space. For $\kappa, q > 0$, define

$$d_{\kappa,q}\big((\xi,i),(\eta,j)\big) = \big(\|\xi-\eta\|_r^\kappa \wedge 1 + 1_{\{i\neq j\}}\big)\sqrt{1+\|\xi\|_r^{2q}+\|\eta\|_r^{2q}}, \quad \xi,\eta\in\mathcal{C}_r,\ i,j\in\mathbf{S}.$$

Note that $d_{\kappa,q}\big((\xi,i),(\eta,j)\big)$ is a quasi-metric (or a distance-like function), that is, it is symmetric, lower semi-continuous and $d_{\kappa,q}\big((\xi,i),(\eta,j)\big)=0$ if and only if $(\xi,i)=(\eta,j)$, but the triangle inequality may not hold. Denote $\mathbf{E}=\mathcal{C}_r\times\mathbf{S}$ and $C_{\kappa,q}(\mathbf{E})$ the family of continuous functionals on $\mathbf{E}$ such that

$$\|f\|_{\kappa,q}:=\sup_{(\xi,i)\in\mathbf{E}}\frac{|f(\xi,i)|}{1+\|\xi\|_r^q}+\sup_{(\xi,i)\neq(\eta,j)}\frac{|f(\xi,i)-f(\eta,j)|}{d_{\kappa,q}\big((\xi,i),(\eta,j)\big)}<\infty.$$

Denote by $\mathcal{P}(\mathbf{E})$ the set of probability measures on $\mathbf{E}$. Let $\mathcal{P}_{\kappa,q}(\mathbf{E})$ be the family of probability measures μ on $\mathbf{E}$ with $(\mu\times\mu)(d_{\kappa,q}(\cdot,\cdot))<\infty$. The $\mathscr{C}(\mu,\nu)$ denotes the collection of all couplings of μ and ν. For the quasi-metric $d_{\kappa,q}$, the associated Wasserstein (or Kantorovich) distance between two probability measures $\mu,\nu\in\mathcal{P}_{\kappa,q}(\mathbf{E})$ is defined as follows:

$$\mathbb{W}_{\kappa,q}(\mu,\nu)=\inf_{\Pi\in\mathscr{C}(\mu,\nu)}\int_{\mathbf{E}\times\mathbf{E}}d_{\kappa,q}(\mathbf{x},\mathbf{y})\Pi(d\mathbf{x},d\mathbf{y}).$$

Denote by $\mathcal{M}_0$ the set of probability measures on $(-\infty,0]$. For any $k>0$, let us further define $\mathcal{M}_k$, the subset of $\mathcal{M}_0$, by

$$\mathcal{M}_k=\Big\{\mu\in\mathcal{M}_0:\mu^{(k)}:=\int_{-\infty}^0e^{-k\theta}\mu(d\theta)<\infty\Big\}.$$

To ensure the existence and uniqueness of the solution and establish limit theorems for additive functionals of regime-switching functional diffusion processes determined by (1.1) and (1.2), let us impose the following assumptions.

Assumption 2.1. For any $i\in\mathbf{S}$, $b(\cdot,i)$ is bounded on bounded subset of $\mathcal{C}_r$. There exist a probability measure $\mu_1\in\mathcal{M}_{2r}$ and a positive constant $K>0$ such that for any $\phi,\psi\in\mathcal{C}_r$,

$$\begin{aligned}&2\big\langle\phi(0)-\psi(0),b(\phi,i)-b(\psi,i)\big\rangle_++\|\sigma(\phi,i)-\sigma(\psi,i)\|_{\mathrm{HS}}^2\\&\quad\le K|\phi(0)-\psi(0)|^2+K\int_{-\infty}^0|\phi(\theta)-\psi(\theta)|^2\mu_1(d\theta),\end{aligned}\tag{2.1}$$

where $a_+:=\max\{0,a\}$ for any $a\in\mathbb{R}$.

Remark 2.1. To highlight the dependence on the initial data for the regime-switching functional diffusions given by (1.1) and (1.2), we write $X^{\phi,i}(t)$, $X_t^{\phi,i}$, and $\alpha^i(t)$, respectively, with $(X_0,\alpha(0))=(\phi,i)\in\mathbf{E}$ in this paper. It follows from Assumption 2.1 that the system given by (1.1) and (1.2) has a unique strong solution $X(t)$ and $(X_t,\alpha(t))$ is a strong Markovian process (see, for example, [18]). Moreover, for any $T>0$, there exist a constant $C_\phi>0$ and an increasing function $A(t):\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\mathbb{E}\sup_{0\le t\le T}\|X_t^{\phi,i}\|_r^2\le C_\phi A(T).\tag{2.2}$$

Let C^2 be the family of twice continuously differentiable functions on $\mathbb{R}^d$. For any $i\in\mathbf{S}$ and $V\in C^2$, define $\mathcal{L}_iV$ the mapping from $\mathcal{C}_r$ to $\mathbb{R}$ by

$$\mathcal{L}_iV(\phi)=\langle V_x(\phi(0)),b(\phi,i)\rangle+\frac{1}{2}\mathrm{trace}\{\sigma^\top(\phi,i)V_{xx}(\phi(0))\sigma(\phi,i)\},\quad\phi\in\mathcal{C}_r,$$

and for $U \in C^2$ define also $\mathbb{L}_i U$ the mapping from $\mathcal{C}_r \times \mathcal{C}_r$ to $\mathbb{R}$ by

$$\begin{aligned}\mathbb{L}_i U(\phi, \psi) = & \langle U_x(\phi(0) - \psi(0)), b(\phi, i) - b(\psi, i)\rangle \\ & + \frac{1}{2}\text{trace}\{(\sigma^\top(\phi, i) - \sigma^\top(\psi, i))U_{xx}(\phi(0) \\ & -\psi(0))(\sigma(\phi, i) - \sigma(\psi, i))\}, \quad \phi, \psi \in \mathcal{C}_r.\end{aligned}$$

Assumption 2.2. Let $U : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a C^2 function satisfying $c_1|x|^2 \le U(x) \le c_2|x|^2$ for some constants $c_2 \ge c_1 > 0$. For any $\phi, \psi \in \mathcal{C}_r$ and $i \in \mathbf{S}$, there exist constants $\overline{a}(i) \in \mathbb{R}$ and $\overline{b}(i) \in \mathbb{R}_+$ with $\gamma := \min_{i\in\mathbf{S}}\{\overline{a}(i) + \overline{b}(i)\} < 0$ and $\gamma_0 < \gamma$, and a probability measure $\mu_2 \in \mathcal{M}_{(2r)\vee(-\gamma_0)}$ such that

$$\mathbb{L}_i U(\phi, \psi) \le \overline{a}(i)U(\phi(0) - \psi(0)) + \overline{b}(i)\int_{-\infty}^{0} U(\phi(\theta) - \psi(\theta))\mu_2(d\theta). \tag{2.3}$$

Assumption 2.3. Let $V : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a C^2 function satisfying $c_3|x|^2 \le V(x) \le c_4|x|^2$ for some constants $c_4 \ge c_3 > 0$. For any $\phi, \psi \in \mathcal{C}_r$ and $i \in \mathbf{S}$, there exist constants $L > 0$, $\widetilde{a}(i) \in \mathbb{R}$, $\widetilde{b}(i) \in \mathbb{R}_+$, and a probability measure $\mu_3 \in \mathcal{M}_{2r}$ such that

$$\mathcal{L}_i V(\phi) \le L + \widetilde{a}(i)V(\phi(0)) + \widetilde{b}(i)\int_{-\infty}^{0} V(\phi(\theta))\mu_3(d\theta). \tag{2.4}$$

For $\overline{a}(i)$, $\widetilde{a}(i)$, $\overline{b}(i)$, and $\widetilde{b}(i)$ given in Assumptions 2.2 and 2.3, set

$$\widetilde{a}^* = \max_{i\in\mathbf{S}}\widetilde{a}(i), \quad \widetilde{a}_* = \min_{i\in\mathbf{S}}\widetilde{a}(i), \quad \widetilde{b}^* = \max_{i\in\mathbf{S}}\widetilde{b}(i), \quad \widetilde{b}_* = \min_{i\in\mathbf{S}}\widetilde{b}(i),$$

$$\widetilde{\gamma}_* = \min_{i\in\mathbf{S}}\{\widetilde{a}(i) + \widetilde{b}(i)\}, \quad \widetilde{\gamma}^* = \max_{i\in\mathbf{S}}\{\widetilde{a}(i) + \widetilde{b}(i)\}, \quad \epsilon = (\widetilde{b}^*)^{-1}\widetilde{b}_*\int_{-\infty}^{0} e^{\widetilde{\gamma}^*\theta}\mu_3(d\theta),$$

$$Q_p = Q + p\text{diag}\Big(\overline{a}(1) + \overline{b}(1)\int_{-\infty}^{0} e^{\gamma\theta}\mu_2(d\theta), \dots, \overline{a}(N) + \overline{b}(N)\int_{-\infty}^{0} e^{\gamma\theta}\mu_2(d\theta)\Big),$$

$$\begin{aligned}\widetilde{Q} = Q + \text{diag}\Big(&\widetilde{a}(1) + (1 - \epsilon\wedge 1)\widetilde{b}(1) + \widetilde{b}(1)\int_{-\infty}^{0} e^{\widetilde{\gamma}_*\theta}\mu_3(d\theta), \dots, \\ &\widetilde{a}(N) + (1 - \epsilon\wedge 1)\widetilde{b}(N) + \widetilde{b}(N)\int_{-\infty}^{0} e^{\widetilde{\gamma}_*\theta}\mu_3(d\theta)\Big),\end{aligned}$$

$$\widehat{Q} = Q + \text{diag}\Big(\widetilde{a}(1) + \widetilde{b}(1)\int_{-\infty}^{0} e^{\widetilde{a}_*\theta}\mu_3(d\theta), \dots, \widetilde{a}(N) + \widetilde{b}(N)\int_{-\infty}^{0} e^{\widetilde{a}_*\theta}\mu_3(d\theta)\Big),$$

$$\eta_p = -\max_{\lambda\in\text{Spec}(Q_p)}\text{Re}(\lambda), \quad \widetilde{\eta}_1 = -\max_{\lambda\in\text{Spec}(\widetilde{Q})}\text{Re}(\lambda), \quad \widehat{\eta}_1 = -\max_{\lambda\in\text{Spec}(\widehat{Q})}\text{Re}(\lambda),$$

where $\text{Spec}(Q_p)$, $\text{Spec}(\widetilde{Q})$, and $\text{Spec}(\widehat{Q})$ denote the spectrum of matrices Q_p, $\widetilde{Q}$, and $\widehat{Q}$, respectively, and $\text{Re}(\lambda)$ denotes the real part of λ.

Remark 2.2. Let $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ be the stationary distribution of the Markovian chain $\{\alpha(t)\}_{t\ge 0}$, and $\bar{a}(i), \bar{b}(i), \gamma$ introduced in Assumption 2.2. If

$$\sum_{i=1}^{N}\pi_i\left(\bar{a}(i) + \bar{b}(i)\int_{-\infty}^{0} e^{\gamma\theta}\mu_2(d\theta)\right) < 0, \tag{2.5}$$

then [3, Proposition 4.2] shows that there exists $\eta_p > 0$ for some $p \in (0, 1)$. And the regime-switching functional diffusion system determined by (1.1) and (1.2) is said to be dissipative "in average" if (2.5) holds. This also shows that $\eta_p > 0$ does not need $\bar{a}(i)+\bar{b}(i)\int_{-\infty}^{0} e^{\gamma\theta}\mu_2(d\theta) < 0$ for all $i \in \mathbf{S}$, which implies that some subsystems may be non-dissipative.

Let us first present some preliminary results given in [23].

Lemma 2.3 (*Lemma 3.1 in [23]*)**.** *Let Assumptions 2.1 and 2.2 hold, and $\eta_p > 0$ for some $p \in (0, 1]$. Then there exist constants $C_1 > 0$ and $\eta_\varepsilon \in (0, \eta_p)$ such that for any $(\phi, i) \in \mathbf{E}$ and $t \geq 0$,*

$$\mathbb{E}\|X_t^{\phi,i} - X_t^{\psi,i}\|^{2p} \leq C_1\|\phi - \psi\|_r^{2p} e^{-\eta_\varepsilon t}, \tag{2.6}$$

Lemma 2.4 (*Lemma 3.2 in [23]*)**.** *Let Assumptions 2.1 and 2.3 hold. For some $\bar{\varepsilon} > 0$, if one of the following two conditions holds:*

(i) $\widetilde{\eta}_1 > 0,\ \widetilde{\gamma}_* < 0,\ \mu_3 \in \mathcal{M}_{(2r)\vee(-\widetilde{\gamma}_*+\bar{\varepsilon})}$,
(ii) $\widehat{\eta}_1 > 0,\ \widetilde{a}_* < 0,\ \mu_3 \in \mathcal{M}_{(2r)\vee(-\widetilde{a}_*+\bar{\varepsilon})}$,

then there exist constants $C_2 > 0$ and $\rho > 0$ such that for any $(\phi, i) \in \mathbf{E}$

$$\mathbb{E}\|X_t^{\phi,i}\|_r^2 \leq C_2\Big(1 + e^{-\rho t}\|\phi\|_r^2\Big), \quad t \geq 0. \tag{2.7}$$

With these two lemmas at hand, it is readily seen that the Markovian semigroup associated with Markovian processes $(X_t, \alpha(t))$ determined by (1.1) and (1.2) admits a unique ergodic invariant probability measure with the use of a Wasserstein distance; see, [23, Theorem 2.2]. Based on this ergodicity, we further obtain CLT and LIL for additive functional in this paper.

3. Central limit theorem

To establish the CLT, a stronger result on moment estimation is needed, which together with Lemma 2.3, implies an improved ergodicity compared with [23]. The result is stated in the following proposition.

Proposition 3.1. *Let Assumptions 2.1–2.3 hold with $V(x) = |x|^2$. Let $\eta_p > 0$ for some $p \in (0, 1]$ and $\widetilde{a}^* + \widetilde{b}^* < 0$, and $\|\sigma\|_\infty := \sup_{(\phi,i)\in\mathbf{E}} \|\sigma(\phi, i)\|_{\mathrm{HS}} < \infty$. Then for any $q > 0$, there exist constants C_3 and $\bar{\rho} > 0$ such that for any $(\phi, i) \in \mathbf{E}$ and $t \geq 0$,*

$$\mathbb{E}\|X_t^{\phi,i}\|_r^q \leq C_3(1 + e^{-\bar{\rho}t}\|\phi\|_r^q). \tag{3.1}$$

Moreover, for any $\mu, \nu \in \mathcal{P}_{p,q}(\mathbf{E})$, there exist constants C and $\varrho > 0$ such that $t \geq 0$,

$$\mathbb{W}_{p,q}(\mu P_t, \nu P_t) \leq C e^{-\varrho t}\mathbb{W}_{p,q}(\mu, \nu), \quad \forall q > 0. \tag{3.2}$$

Furthermore, P_t has a unique invariant probability measure μ_* satisfying $\mu_*(\|\cdot\|_r^q) < \infty$ and

$$\mathbb{W}_{p,q}(\mu P_t, \mu_*) \leq C e^{-\varrho t}\mathbb{W}_{p,q}(\mu, \mu_*), \quad \forall q > 0,\ t \geq 0. \tag{3.3}$$

Proof. We first prove (3.1). For any $\lambda \in (0, r)$, the Itô formula, Assumption 2.3, and $V(x) = |x|^2$ yield that

$$
\begin{aligned}
e^{2\lambda t}|X^{\phi,i}(t)|^2 \le |\phi(0)|^2 &+ \int_0^t e^{2\lambda s}\big(L + (2\lambda + \tilde{a}^*)|X^{\phi,i}(s)|^2\big)ds \\
&+ \tilde{b}^* \int_0^t e^{2\lambda s} \int_{-\infty}^0 |X^{\phi,i}(s+\theta)|^2 \mu_3(d\theta)ds \\
&+ 2\int_0^t e^{2\lambda s}\langle X^{\phi,i}, \sigma(X^{\phi,i}(s), \alpha^i(s))dW(s)\rangle. \qquad (3.4)
\end{aligned}
$$

By using the Tonelli theorem and a substitution technique, we get

$$
\begin{aligned}
&\tilde{b}^* \int_0^t e^{2\lambda s} \int_{-\infty}^0 |X^{\phi,i}(s+\theta)|^2 \mu_3(d\theta)ds \\
&\le \tilde{b}^* \int_0^t e^{2\lambda s} \int_{-\infty}^{-s} |X^{\phi,i}(s+\theta)|^2 \mu_3(d\theta)ds + \tilde{b}^* \int_0^t e^{2\lambda s} \int_{-s}^0 |X^{\phi,i}(s+\theta)|^2 \mu_3(d\theta)ds \\
&\le \frac{1}{2(r-\lambda)}\tilde{b}^* \mu_3^{(2r)} \|\phi\|_r^2 + \tilde{b}^* \mu_3^{(2\lambda)} \int_0^t e^{2\lambda s}|X^{\phi,i}(s)|ds. \qquad (3.5)
\end{aligned}
$$

Recalling that $\tilde{a}^* + \tilde{b}^* < 0$ and $\lim_{\lambda\to 0}\mu_3^{(\lambda)} = 1$, we can choose $\lambda \in (0, r)$ small enough such that $2\lambda + \tilde{a}^* + \tilde{b}^*\mu_3^{(2\lambda)} \le 0$. Then substituting (3.5) into (3.4) yields that

$$
\begin{aligned}
e^{2\lambda t}|X^{\phi,i}(t)|^2 \le &\Big(1 + \frac{1}{2(r-\lambda)}\tilde{b}^*\mu_3^{(2r)}\Big)\|\phi\|_r^2 \\
&+ \frac{L}{2\lambda}e^{2\lambda t} + 2\int_0^t e^{2\lambda s}\langle X^{\phi,i}, \sigma(X^{\phi,i}(s), \alpha^i(s))dW(s)\rangle.
\end{aligned}
$$

By using the Burkholder–Davis–Gundy inequality, and the Young inequality, there exists a constant $C_4 > 0$ such that

$$
\begin{aligned}
&\mathbb{E} \sup_{0\le s\le t\wedge\tau_n} e^{q\lambda s}|X^{\phi,i}(s)|^q \\
&\le C_4\|\phi\|_r^q + C_4 e^{q\lambda(t\wedge\tau_n)} + C_4\mathbb{E}\Big(\int_0^{t\wedge\tau_n} e^{4\lambda s}|X^{\phi,i}(s)|^2 \|\sigma(X_s^{\phi,i}, \alpha(s)^i)\|_{\mathrm{HS}}^2 ds\Big)^{q/4} \\
&\le C_4\|\phi\|_r^q + C_4 e^{q\lambda(t\wedge\tau_n)} + C_4\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2\lambda s}|X^{\phi,i}(s)|^2 \int_0^{t\wedge\tau_n} e^{2\lambda s}\|\sigma\|_\infty^2 ds\Big)^{q/4} \\
&\le C_4\|\phi\|_r^q + C_4 e^{q\lambda(t\wedge\tau_n)} + \frac{1}{2}\mathbb{E}\sup_{0\le s\le t\wedge\tau_n} e^{q\lambda s}|X^{\phi,i}(s)|^q + \frac{C_4^2}{2}\Big(\frac{1}{2\lambda}\Big)^{q/2}\|\sigma\|_\infty^q e^{q\lambda(t\wedge\tau_n)},
\end{aligned}
$$

where $\tau_n := \inf\{t \ge 0 : \|X_t^{\phi,i}\|_r \ge n\}$ for $n \ge \|\phi\|_r$. Note that $\tau_n \to \infty$ as $n \to \infty$. Then by Fatou's lemma, we arrive at

$$
\mathbb{E}\sup_{0\le s\le t} e^{q\lambda s}|X^{\phi,i}(s)|^q \le 2C_4\|\phi\|_r^q + C_4\Big(2 + \Big(\frac{1}{2\lambda}\Big)^{q/2}\|\sigma\|_\infty^q\Big)e^{q\lambda t}.
$$

Since $\lambda < r$, from the definition of the norm $\|\cdot\|_r$, it follows that

$$
\|X_t^{\phi,i}\|_r \le e^{-\lambda t}\Big(\|\phi\|_r \vee \sup_{0\le s\le t} e^{\lambda s}|X^{\phi,i}(s)|\Big).
$$

Then we have

$$
\mathbb{E}\|X_t^{\phi,i}\|_r^q \le (1 + 2C_4)\|\phi\|_r^q e^{-q\lambda t} + C_4\Big(2 + \Big(\frac{1}{2\lambda}\Big)^{q/2}\|\sigma\|_\infty^q\Big),
$$

which implies that (3.1) holds for $C_3 = (1 + 2C_4) \vee C_4\Big(2 + \big(\frac{1}{2\lambda}\big)^{q/2}\|\sigma\|_\infty^q\Big)$ and $\bar{\rho} = q\lambda$.

With the help of (3.1) and (2.6), by using a similar argument as in the proof of [23, Theorem 2.2], (3.2) and (3.3) hold for any $q > 0$. The proof is complete. □

To proceed, for $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$, define

$$R_f(\phi, i) = \int_0^\infty P_t f(\phi, i)dt \quad \text{and} \quad \Phi_f(\phi, i) = \mathbb{E}\Big|\int_0^1 f(X_s^{\phi,i}, \alpha^i(s))ds + R_f(X_1^{\phi,i}, \alpha^i(1)) - R_f(\phi, i)\Big|^2.$$

For any $\Sigma \in (0, \infty)$, let $F_\Sigma(\cdot)$ be the normal distribution function with mean zero and variance Σ^2. Also denote $F_0(x) := 1_{[0,\infty)}(x)$.

Theorem 3.2. *Let Assumptions 2.1–2.3 hold with $V(x) = |x|^2$. Let $\eta_p > 0$ for some $p \in (0, 1]$ and $\widetilde{a}^* + \widetilde{b}^* < 0$, and $\|\sigma\|_\infty := \sup_{(\phi,i)\in\mathbf{E}} \|\sigma(\phi, i)\|_{\rm HS} < \infty$. For any $q > 0$, $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$,*

$$\Sigma_f^2 := \lim_{t\to\infty} \sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \mathbb{E}\big[t^{-\frac{1}{2}} A_t^f(\phi, i)\big]^2 \mu_*(d\phi, i) = \mu_*(\Phi_f) < \infty, \tag{3.6}$$

where

$$A_t^f(\phi, i) := \int_0^t f\big(X_s^{\phi,i}, \alpha^i(s)\big)ds. \tag{3.7}$$

In addition, the following two assertions hold:

(i) *When $\Sigma_f > 0$, for any $\varepsilon \in (0, 1/4)$, there exists an increasing continuous function $h_\varepsilon : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\phi \in \mathcal{C}_r, i \in \mathbf{S}$,*

$$\sup_{x\in\mathbb{R}} \Big|\mathbb{P}\big(t^{-\frac{1}{2}} A_t^f(\phi, i) \le x\big) - F_{\Sigma_f}(x)\Big| \le h_\varepsilon(\|\phi\|_r, \|f\|_{p,q})t^{-\frac{1}{4}+\varepsilon}, \quad t \ge 1. \tag{3.8}$$

(ii) *When $\Sigma_f = 0$, there exists a continuous increasing function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\phi \in \mathcal{C}_r, i \in \mathbf{S}$,*

$$\sup_{x\in\mathbb{R}} \Big\{(1 \wedge |x|)\Big|\mathbb{P}\big(t^{-\frac{1}{2}} A_t^f(\phi, i) \le x\big) - F_{\Sigma_f}(x)\Big|\Big\} \le h(\|\phi\|_r, \|f\|_{p,q})t^{-\frac{1}{4}}, \quad t \ge 1.$$

Remark 3.3. Compared with our previous result [28, Theorem 3.1], the non-degenerated assumption on the diffusion coefficient is removed in Theorem 3.2. We provide an example to illustrate this result; see Example 5.1 in what follows.

To prove Theorem 3.2, we first derive the following three crucial lemmas.

Lemma 3.4. *Let assumptions of Theorem 3.2 hold. Then there exist constants δ and $c > 0$ such that for any $k \ge 0$ and $(\phi, i) \in \mathbf{E}$*

$$\mathbb{E}\Big(\sup_{t\in[k,k+1]} e^{\delta\|X_t^{\phi,i}\|_r^2}\Big) \le e^{c(1+\|\phi\|_r^2)}. \tag{3.9}$$

Proof. Noting that $\sup_{t\in[k,k+1]} \|X_t^{\phi,i}\|_r \le e^r \|X_{k+1}^{\phi,i}\|_r$ due to the definition of norm $\|\cdot\|_r$, we have

$$\mathbb{E}\Big(\sup_{t\in[k,k+1]} e^{\delta\|X_t^{\phi,i}\|_r^2}\Big) \le \mathbb{E}e^{e^{2r}\delta\|X_{k+1}^{\phi,i}\|_r^2}.$$

Hence, in order to prove (3.9), it suffices to show that there exist positive constants δ_0 and c_0 such that

$$\mathbb{E}e^{\delta_0\|X_t^{\phi,i}\|_r^2} \le e^{c_0(1+\|\phi\|_r^2)}. \tag{3.10}$$

To this end, let us consider the following regime-switching diffusions with infinite delay

$$dX^n(t) = b(X_t^n, \alpha^i(t))1_{[0,\tau_n]}(t)dt + \frac{1}{2}\widetilde{a}(\alpha^i(t))X^n(t)1_{(\tau_n,\infty)}dt + \sigma(X_t^n, \alpha^i(t))dW(t), \tag{3.11}$$

with initial data $X_0^n = \phi \in \mathcal{C}_r, \alpha^i(0) = i \in \mathbf{S}$, where $\tau_n = \inf\{t \ge 0 : \|X_t^{\phi,i}\|_r \ge n\}$ for $n \ge \|\phi\|_r$, and $\widetilde{a}(\cdot)$ comes from Assumption 2.3. Note that the regime-switching functional diffusion processes determined by (1.2) and (3.11) have a unique solution and $X^n(t) = X^{\phi,i}(t)$ for $t \le \tau_n$. Since $\tau_n \to \infty$ as $n \to \infty$, for any $t > 0$,

$$\lim_{n\to\infty} \|X_t^n - X_t^{\phi,i}\|_r = 0, \qquad \mathbb{P}\text{-}a.s. \tag{3.12}$$

By virtue of Assumption 2.3 and the boundedness of $\sigma(\cdot)$, we obtain

$$\begin{aligned}
&2\Big\langle X^n(t), b(X_t^n, \alpha^i(t))1_{[0,\tau_n]} + \frac{1}{2}\widetilde{a}(\alpha^i(t))X^n(t)1_{(\tau_n,\infty)}\Big\rangle + \|\sigma(X_t^n, \alpha^i(t))\|_{\mathrm{HS}} \\
&\quad \le L + \widetilde{a}(\alpha^i(t))|X^n(t)|^2 + \widetilde{b}(\alpha^i(t))\int_{-\infty}^0 |X^n(t+\theta)|^2\mu_3(d\theta)1_{[0,\tau_n]} + \|\sigma\|_\infty \\
&\quad \le L(n) + \widetilde{a}(\alpha^i(t))|X^n(t)|^2,
\end{aligned}$$

where $L(n) := L + \widetilde{b}^* n^2 + \|\sigma\|_\infty$. Then applying Itô's formula gives

$$|X^n(t)|^2 \le |\phi(0)|^2 + \int_0^t L(n) + \widetilde{a}(\alpha^i(s))|X^n(s)|^2 ds + 2\int_0^t \big\langle X^n(s), \sigma(X_s^n, \alpha^i(s))dW(s)\big\rangle. \tag{3.13}$$

Define $\widetilde{\tau}_m = \inf\{t \ge 0 : |X^n(t)| \ge m\}$ for $m \ge \|\phi\|_r$. Clearly, $\widetilde{\tau}_m \to \infty$ as $m \to \infty$. Letting $\alpha = (\widetilde{a}^*)^2/8\|\sigma\|_\infty^2$. Noting that $\widetilde{a}^* < 0$, then from (3.13),

$$\begin{aligned}
&\mathbb{E}\exp\left\{\alpha\int_0^{t\wedge\widetilde{\tau}_m} |X^n(s)|^2 ds\right\} \\
&\quad \le \exp\left\{\frac{\alpha}{-\widetilde{a}^*}(|\phi(0)|^2 + L(n)t)\right\}\mathbb{E}\exp\left\{\frac{2\alpha}{-\widetilde{a}^*}\int_0^{t\wedge\widetilde{\tau}_m}\big\langle X^n(s), \sigma(X_s^n, \alpha^i(s))dW(s)\big\rangle\right\} \\
&\quad \le \exp\left\{\frac{\alpha}{-\widetilde{a}^*}(|\phi(0)|^2 + L(n)t)\right\}\left(\mathbb{E}\exp\left\{\frac{8\alpha^2\|\sigma\|_\infty^2}{(\widetilde{a}^*)^2}\int_0^{t\wedge\widetilde{\tau}_m}|X^n(s)|^2 ds\right\}\right)^{1/2},
\end{aligned}$$

where the last inequality follows from the inequality $\mathbb{E}e^{N(s)} \le \big(\mathbb{E}e^{2\langle N\rangle_s}\big)^{1/2}$, where $N(s)$ is a $\mathbb{P}$-martingale and $\langle N\rangle_s$ denotes its quadratic variation. This, together with $\alpha = (\widetilde{a}^*)^2/8\|\sigma\|_\infty^2$ and Fatou's lemma, implies that for any fixed $n > 0$ and $t > 0$

$$\mathbb{E}\exp\left\{\alpha\int_0^t |X^n(s)|^2 ds\right\} \le \exp\left\{\frac{2\alpha}{-\widetilde{a}^*}\big(|\phi(0)|^2 + L(n)t\big)\right\} < \infty. \tag{3.14}$$

Take $\beta = -\widetilde{a}^*/8\|\sigma\|_\infty^2 = \sqrt{\alpha}/2\sqrt{2}\|\sigma\|_\infty$. Then it follows from (3.13) and (3.14) that

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\exp\left\{\beta|X^n(s)|^2\right\}\\
&\le \exp\big\{\beta|\phi(0)|^2+\beta L(n)t\big\}\mathbb{E}\sup_{0\le s\le t}\exp\left\{2\beta\int_0^s\big\langle X^n(u),\sigma(X^n_u,\alpha^i(u))dW(u)\big\rangle\right\}\\
&\le \exp\big\{\beta|\phi(0)|^2+\beta L(n)t\big\}e\mathbb{E}\exp\left\{2\beta\int_0^t\big\langle X^n(u),\sigma(X^n_u,\alpha^i(u))dW(u)\big\rangle\right\}\\
&\le \exp\big\{\beta|\phi(0)|^2+\beta L(n)t+1\big\}\left(\mathbb{E}\exp\left\{8\beta^2\|\sigma\|_\infty^2\int_0^t|X^n(s)|^2ds\right\}\right)^{1/2}\\
&\le \exp\left\{\Big(\beta+\frac{\alpha}{-\widetilde{a}^*}\Big)(|\phi(0)|^2+L(n)t)+1\right\},
\end{aligned}
\tag{3.15}
$$

where in the second step, we have used the fact that $\mathbb{E}\sup_{0\le s\le t}e^{N(s)} \le e\mathbb{E}e^{N(t)}$ for a $\mathbb{P}$-submartingale $N(t)$. Recalling the definition of norm $\|\cdot\|_r$, for $\forall\ n\ge 1$, $t\ge 0$, we arrive at

$$
\begin{aligned}
\mathbb{E}\exp\big\{\beta\|X^n_t\|_r^2\big\} &\le \mathbb{E}\exp\left\{\beta\big(\|\phi\|_r^2+\sup_{0\le s\le t}|X^n(s)|^2\big)\right\}\\
&\le \exp\left\{\Big(2\beta+\frac{\alpha}{-\widetilde{a}^*}\Big)(|\phi(0)|^2+L(n)t)+1\right\}<\infty.
\end{aligned}
\tag{3.16}
$$

By Assumption 2.3 with $V(x)=|x|^2$ and Itô's formula, for any $\lambda\in(0,2r)$, we get

$$
\begin{aligned}
e^{\lambda t}|X^n(t)|^2 \le\ & |\phi(0)|^2+\int_0^t e^{\lambda s}\Big(L+(\lambda+\widetilde{a}(\alpha^i(s)))|X^n(s)|^2\Big)ds\\
&+\int_0^t\int_{-\infty}^0 e^{\lambda s}\widetilde{b}(\alpha^i(s))|X^n(s+\theta)|^2\mu_3(d\theta)ds+\overline{M}(t),
\end{aligned}
\tag{3.17}
$$

where $\overline{M}(t):=2\int_0^t e^{\lambda s}\big\langle X^n(s),\sigma(X^n_s,\alpha^i(s))dW(s)\big\rangle$. Similar to (3.5),

$$
\begin{aligned}
&\int_0^t\int_{-\infty}^0 e^{\lambda s}\widetilde{b}(\alpha^i(s))|X^n(s+\theta)|^2\mu_3(d\theta)ds\\
&\quad\le \frac{1}{2r-\lambda}\|\phi\|_r^2\widetilde{b}^*\mu_3^{(2r)}+\widetilde{b}^*\mu_3^{(\lambda)}\int_0^t e^{\lambda s}|X^n(s)|^2ds.
\end{aligned}
\tag{3.18}
$$

Noting that $\widetilde{a}^*+\widetilde{b}^*<0$ and $\lim_{\lambda\to 0}\mu_3^{(\lambda)}=1$, there exists a constant $\lambda<2r$ small enough such that $\lambda+\widetilde{a}^*+\widetilde{b}^*\mu_3^{(\lambda)}\le 0$. Then substituting (3.18) into (3.17) gives

$$
\begin{aligned}
e^{\lambda t}|X^n(t)|^2 &\le \Big(1+\frac{1}{2r-\lambda}\widetilde{b}^*\mu_3^{(2r)}\Big)\|\phi\|_r^2\\
&\qquad+\int_0^t e^{\lambda s}\Big(L+\big(\lambda+\widetilde{a}^*+\widetilde{b}^*\mu_3^{(\lambda)}\big)|X^n(s)|^2\Big)ds+\overline{M}(t)\\
&\le \Big(1+\frac{1}{2r-\lambda}\widetilde{b}^*\mu_3^{(2r)}\Big)\|\phi\|_r^2+\frac{1}{\lambda}Le^{\lambda t}+\overline{M}(t).
\end{aligned}
\tag{3.19}
$$

For some $\delta_0 > 0$ to be determined later, by using a similar argument to derive (3.15), we have

$$\mathbb{E}\exp\left\{\delta_0 e^{-\lambda t}\sup_{0\le s\le t}\overline{M}(s)\right\} \le e\mathbb{E}\exp\left\{\delta_0 e^{-\lambda t}\overline{M}(t)\right\} \le e\left(\mathbb{E}\exp\left\{8\delta_0^2\|\sigma\|_\infty^2\int_0^t e^{-2\lambda(t-s)}|X^n(s)|^2 ds\right\}\right)^{1/2}. \tag{3.20}$$

Consider the following probability measure ν on $[-1, t]$

$$\nu(ds) = \frac{2\lambda}{1-e^{-2\lambda(t+1)}}e^{-2\lambda(t-s)}ds.$$

Then by (3.20) and Jensen's inequality, we obtain

$$\begin{aligned}\mathbb{E}\exp\left\{\delta_0 e^{-\lambda t}\sup_{0\le s\le t}\overline{M}(s)\right\} &\le e\left(\mathbb{E}\exp\left\{8\delta_0^2\|\sigma\|_\infty^2\int_0^t e^{-2\lambda(t-s)}|X^n(s)|^2 ds\right\}\right)^{1/2}\\ &\le e\left(\mathbb{E}\exp\left\{8\delta_0^2\|\sigma\|_\infty^2\frac{1-e^{-2\lambda(t+1)}}{2\lambda}\int_{-1}^t |X^n(s)|^2\nu(ds)\right\}\right)^{1/2}\\ &\le e\left(\mathbb{E}\int_{-1}^t \exp\left\{\frac{4\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}\frac{2\lambda}{1-e^{-2\lambda(t+1)}}e^{-2\lambda(t-s)}ds\right)^{1/2}\\ &\le e + \frac{2\lambda e}{1-e^{-2\lambda}}\mathbb{E}\int_{-1}^t \exp\left\{\frac{4\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}e^{-2\lambda(t-s)}ds.\end{aligned}$$

This, together with (3.19), means that

$$\begin{aligned}&\mathbb{E}\exp\left\{\delta_0\sup_{0\le s\le t}e^{-\lambda(t-s)}|X^n(s)|^2\right\}\\ &\le \exp\left\{\delta_0\left(\left(1+\frac{1}{2r-\lambda}\widetilde{b}^*\mu_3^{(2r)}\right)\|\phi\|_r^2 + L\lambda^{-1}\right)\right\}\mathbb{E}\exp\left\{\delta_0 e^{-\lambda t}\sup_{0\le s\le t}M(s)\right\}\\ &\le A_1 + \frac{2A_1\lambda}{1-e^{-2\lambda}}\mathbb{E}\int_{-1}^t \exp\left\{\frac{4\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}e^{-2\lambda(t-s)}ds,\end{aligned} \tag{3.21}$$

where $A_1 := \exp\{\delta_0((1+(2r-\lambda)^{-1}\widetilde{b}^*\mu_3^{(2r)})\|\phi\|_r^2 + L\lambda^{-1}) + 1\}$. Then by the Cauchy inequality,

$$\begin{aligned}&\mathbb{E}\exp\{\delta_0\|X_t^n\|_r^2\}\\ &\le \mathbb{E}\exp\left\{\delta_0\|\phi\|_r^2 + \delta_0\sup_{0\le s\le t}e^{-\lambda(t-s)}|X^n(s)|^2\right\}\\ &\le A_1^2 + \frac{2A_1^2\lambda}{1-e^{-2\lambda}}\mathbb{E}\int_{-1}^t \exp\left\{\frac{4\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}e^{-2\lambda(t-s)}ds\\ &\le A_1^2 + \frac{A_1^4\lambda}{(1-e^{-2\lambda})^2}\int_{-1}^t e^{-2\lambda(t-s)}ds + \lambda\int_{-1}^t \mathbb{E}\exp\left\{\frac{8\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}e^{-2\lambda(t-s)}ds\\ &\le A_1^2 + \frac{A_1^4}{(1-e^{-2\lambda})^2} + \lambda\int_{-1}^t \mathbb{E}\exp\left\{\frac{8\delta_0^2\|\sigma\|_\infty^2}{\lambda}|X^n(s)|^2\right\}e^{-2\lambda(t-s)}ds.\end{aligned} \tag{3.22}$$

Using $\beta = -\widetilde{a}_*/8\|\sigma\|_\infty^2$, take $\delta_0 \le \big((-\widetilde{a}_*)\wedge\lambda\big)/8\|\sigma\|_\infty^2 \le \beta$. Then from (3.22) we see that

$$\mathbb{E}\exp\left\{\delta_0\|X_t^n\|_r^2\right\} \le A_1^2 + \frac{A_1^4}{(1-e^{-2\lambda})^2} + \lambda\int_{-1}^t \mathbb{E}\exp\left\{\delta_0\|X_s^n\|_r^2\right\}e^{-2\lambda(t-s)}ds.$$

By using the Gronwall inequality,

$$\begin{aligned}\mathbb{E}\exp\{\delta_0\|X_t^n\|_r^2\} &\le A_1^2+\frac{A_1^4}{(1-e^{-2\lambda})^2}+\Big(A_1^2+\frac{A_1^4}{(1-e^{-2\lambda})^2}\Big)\lambda\int_{-1}^t e^{-\lambda(t-s)}ds\\ &\le 2\Big(A_1^2+\frac{A_1^4}{(1-e^{-2\lambda})^2}\Big).\end{aligned} \tag{3.23}$$

Note that $A_1:=\exp\{\delta_0\big((1+(2r-\lambda)^{-1}\widetilde{b}^*\mu_3^{(2r)})\|\phi\|_r^2+L\lambda^{-1}\big)+1\}$ is independent of n. By using the Fatou Lemma, it follows from (3.12) and (3.23) that (3.10) holds. This further implies (3.9), which completes the proof. □

Lemma 3.5. *Let assumptions of Theorem 3.2 hold. Then for any $q>0$ and $f\in C_{p,q}(\mathbf{E})$ with $\mu_*(f)=0$, there exists a constant $C_0>0$ such that*

$$\|\Phi_f\|_{p,2q}\le C_0\|f\|_{p,q}^2<\infty, \tag{3.24}$$

that is, $\Phi_f\in C_{p,2q}(\mathbf{E})$, and

$$0\le\mu_*(\Phi_f)=2\mu_*(fR_f)<\infty. \tag{3.25}$$

Proof. For any $f\in C_{p,q}(\mathbf{E})$, in light of (3.3), applying a standard argument gives that

$$\begin{aligned}|P_tf(\phi,i)-\mu_*(f)| &\le \inf_{\pi\in\mathscr{C}(P_t(\phi,i;\cdot,\cdot),\mu_*)}\int_{\mathbf{E}\times\mathbf{E}}d_{p,q}((\psi,j),(\zeta,k))\|f\|_{p,q}\pi(d\psi,j;d\zeta,k)\\ &\le 2C(1+\mu_*(\|\cdot\|_r^q))\|f\|_{p,q}(1+\|\phi\|_r^q)e^{-\varrho t},\end{aligned} \tag{3.26}$$

that is, the Markov process $(X_t^{\phi,i},\alpha^i(t))$ is uniformly mixing for the class $C_{p,q}(\mathbf{E})$, (see, [24, Definition 2.5]). Then for any $f\in C_{p,q}(\mathbf{E})$ with $\mu_*(f)=0$, we obtain

$$\begin{aligned}\big|R_f(\phi,i)\big| &= \Big|\int_0^\infty\big[P_tf(\phi,i)-\mu_*(f)\big]dt\Big|\\ &\le \Big|\int_0^\infty 2C(1+\mu_*(\|\cdot\|_r^q))\|f\|_{p,q}(1+\|\phi\|_r^q)e^{-\varrho t}dt\Big|\\ &\le 2C\varrho^{-1}(1+\mu_*(\|\cdot\|_r^q))\|f\|_{p,q}(1+\|\phi\|_r^q),\quad\forall\, q>0,\end{aligned} \tag{3.27}$$

which implies that $R_f(\phi,i)$ is well defined for all $f\in C_{p,q}(\mathbf{E})$. The Markovian property of $(X_t^{\phi,i},\alpha^i(t))$ shows

$$P_tR_f(\phi,i)=R_f(\phi,i)-\int_0^tP_sf(\phi,i)ds,\quad t\ge0, \tag{3.28}$$

which further implies that

$$\begin{aligned}&\int_0^1\mathbb{E}\big[f(X_s^{\phi,i},\alpha^i(s))R_f(X_1^{\phi,i},\alpha^i(1))\big]ds=\int_0^1P_s(fP_{1-s}R_f)(\phi,i)ds\\ &=\int_0^1P_s(fR_f)(\phi,i)ds-\int_0^1\int_0^{1-s}P_s(fP_tf)(\phi,i)dtds.\end{aligned} \tag{3.29}$$

By using the property of conditional expectation and a substitution technique, one has

$$\begin{aligned}\mathbb{E}\bigg(\int_0^1 f(X_t^{\phi,i},\alpha^i(t))dt\bigg)^2 &= 2\mathbb{E}\int_0^1\int_s^1 f(X_s^{\phi,i},\alpha^i(s))f(X_t^{\phi,i},\alpha^i(t))dtds\\ &= 2\int_0^1\int_s^1 P_s(fP_{t-s}f)(\phi,i)dtds\\ &= 2\int_0^1\int_0^{1-s} P_s(fP_tf)(\phi,i)dtds. \end{aligned}\tag{3.30}$$

Then according to the definition of Φ_f, it follows from (3.28)–(3.30) that

$$\begin{aligned}\Phi_f(\phi,i) &= \mathbb{E}\bigg(\int_0^1 f(X_t^{\phi,i},\alpha^i(t))dt\bigg)^2 + \mathbb{E}R_f^2(X_1^{\phi,i},\alpha^i(1)) + R_f^2(\phi,i)\\ &\quad +2\int_0^1 \mathbb{E}\big[f(X_t^{\phi,i},\alpha^i(t))R_f(X_1^{\phi,i},\alpha^i(1))\big]dt - 2R_f(\phi,i)\mathbb{E}R_f(X_1^{\phi,i},\alpha^i(1))\\ &\quad -2R_f(\phi,i)\int_0^1 \mathbb{E}f(X_t^{\phi,i},\alpha^i(t))dt\\ &= P_1(R_f^2)(\phi,i) - R_f^2(\phi,i) + 2\int_0^1 P_s(fR_f)(\phi,i)ds. \end{aligned}\tag{3.31}$$

Hence, in order to prove (3.24), with Lemmas 2.3 and 3.4 at hand, it suffices to show that $R_f^2(\cdot), (fR_f)(\cdot) \in C_{p,2q}(\mathbf{E})$ for any $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$. To this end, applying (3.2) with $\mu = \delta_{(\phi,i)}$ and $\nu = \delta_{(\psi,j)}$ yields

$$|P_tf(\phi,i) - P_tf(\psi,j)| \le C\|f\|_{p,q}d_{p,q}\big((\phi,i),(\psi,j)\big)e^{-\varrho t},\tag{3.32}$$

which, together with the definition of R_f, yields

$$\begin{aligned}&\big|R_f(\phi,i) - R_f(\psi,j)\big|\\ &\quad\le \int_0^\infty |P_tf(\phi,i) - P_tf(\psi,j)|dt \le C\varrho^{-1}\|f\|_{p,q}d_{p,q}\big((\phi,i),(\psi,j)\big).\end{aligned}\tag{3.33}$$

Combining this with (3.27), there exists a constant $C(\varrho) > 0$ such that

$$\begin{aligned}|R_f^2(\phi,i) - R_f^2(\psi,j)| &= |R_f(\phi,i) + R_f(\psi,j)||R_f(\phi,i) - R_f(\psi,j)|\\ &\le C(\varrho)(1+\mu_*(\|\cdot\|_r^q))\|f\|_{p,q}^2 d_{p,2q}\big((\phi,i),(\psi,j)\big),\end{aligned}\tag{3.34}$$

which, together with (3.27), implies that $R_f^2(\cdot) \in C_{p,2q}(\mathbf{E})$. Similarly, it is easy to verify that $(fR_f)(\cdot) \in C_{p,2q}(\mathbf{E})$. Then (3.24) follows. Noting that μ_* is invariant and $\mu_*(\|\cdot\|_r^{2q}) < \infty$, it follows from (3.31) that $\mu_*(\Phi_f) = 2\mu_*(fR_f)$, that is, (3.25) holds. This proof is complete. □

Lemma 3.6. *Under assumptions of Theorem 3.2, for any $n \in \mathbb{Z}_+$, $q > 0$ and $f \in C_{p,q}(\mathbf{E})$, there exists a continuous increasing function $h_0(\cdot)$ such that*

$$\mathbb{E}\left|\frac{1}{m}\sum_{k=0}^{m-1} f(X_k^{\phi,i},\alpha^i(k)) - \mu_*(f)\right|^{2n} \le h_0(\|\phi\|_r)\|f\|_{p,q}^{2n}m^{-n}.\tag{3.35}$$

Proof. Fix an arbitrary functional $f \in C_{p,q}(\mathbf{E})$. Without loss of generality, assume that $\mu_*(f) = 0$. Then by the Hölder inequality, we have

$$
\begin{aligned}
\mathbb{E}\left|\frac{1}{m}\sum_{k=0}^{m-1} f(X_k^{\phi,i},\alpha^i(k))\right|^{2n} &= \frac{1}{m^{2n}}\mathbb{E}\prod_{j=1}^{2n}\Big[\sum_{k_j=0}^{m-1} f(X_{k_j}^{\phi,i},\alpha^i(k_j))\Big] \\
&\le \frac{(2n)!}{m^{2n}}\mathbb{E}\Big[\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{2n}=0}^{k_{2n-1}} f(X_{k_1}^{\phi,i},\alpha^i(k_1))f(X_{k_2}^{\phi,i},\alpha^i(k_2))\ldots f(X_{k_{2n}}^{\phi,i},\alpha^i(k_{2n}))\Big] \\
&= \frac{(2n)!}{m^{2n}}\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\mathbb{E}\Big[\Big(\sum_{k_3=0}^{k_2}\cdots\sum_{k_{2n}=0}^{k_{2n-1}} f(X_{k_3}^{\phi,i},\alpha^i(k_3))\cdots f(X_{k_{2n}}^{\phi,i},\alpha^i(k_{2n}))\Big)g(k_1,k_2)\Big] \\
&\le \frac{(2n)!}{m^{2n}}\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\mathbb{E}\Big[\Big|\sum_{k=0}^{m-1} f(X_k^{\phi,i},\alpha^i(k))\Big|^{2n-2} g(k_1,k_2)\Big] \\
&\le \left(\mathbb{E}\left|\frac{1}{m}\sum_{k=0}^{m-1} f(X_k^{\phi,i},\alpha^i(k))\right|^{2n}\right)^{\frac{n-1}{n}} \frac{(2n)!}{m^2}\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\Big(\mathbb{E}g^n(k_1,k_2)\Big)^{\frac{1}{n}},
\end{aligned} \tag{3.36}
$$

where $g(k_1,k_2) := f(X_{k_2}^{\phi,i},\alpha^i(k_2))\mathbb{E}\big[f(X_{k_1}^{\phi,i},\alpha^i(k_1))\big|\mathcal{F}_{k_2}\big]$. This implies that

$$
\mathbb{E}\left|\frac{1}{m}\sum_{k=0}^{m-1} f(X_k^{\phi,i},\alpha^i(k))\right|^{2n} \le \frac{[(2n)!]^n}{m^{2n}}\left(\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\Big(\mathbb{E}g^n(k_1,k_2)\Big)^{\frac{1}{n}}\right)^n. \tag{3.37}
$$

In light of (3.1) and (3.26), there exists a continuous increasing function $h_1(\cdot)$ such that

$$
\begin{aligned}
\Big(\mathbb{E}g^n(k_1,k_2)\Big)^{\frac{1}{n}} &\le C\|f\|_{p,q}^2\Big(\mathbb{E}(1+\|X_{k_2}^{\phi,i}\|^q)^{2n}e^{-n\varrho(k_1-k_2)}\Big)^{\frac{1}{n}} \\
&\le h_1(\|\phi\|_r)\|f\|_{p,q}^2 e^{-\varrho(k_1-k_2)}.
\end{aligned}
$$

Substituting this inequality into (3.37) yields the desired result (3.35). The proof is complete. □

To derive the functional CLT, we first introduce some notation. For $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$ and $(\phi, i) \in \mathbf{E}$,

$$
\begin{aligned}
M_t^{f,\phi,i} &:= \int_0^\infty \big[\mathbb{E}\big(f(X_s^{\phi,i},\alpha^i(s))|\mathcal{F}_t\big) - P_s f(\phi,i)\big]ds \\
&= \int_0^t f(X_s^{\phi,i},\alpha^i(s))ds + \int_t^\infty P_{s-t}f(X_t^{\phi,i},\alpha^i(t))ds - \int_0^\infty P_s f(\phi,i)ds \\
&= \int_0^t f(X_s^{\phi,i},\alpha^i(s))ds + R_f(X_t^{\phi,i},\alpha^i(t)) - R_f(\phi,i).
\end{aligned} \tag{3.38}
$$

For any fixed $(\phi, i) \in \mathbf{E}$ and $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$, the stochastic process $\{M_t^{f,\phi,i}\}_{t\ge 0}$ forms a well-defined zero-mean martingale. Define a conditional variance for $M_m^{f,\phi,i}$, $m \in \mathbf{Z}_+$ by the following formula

$$
\langle M^{f,\phi,i}\rangle_m = \sum_{k=1}^m \mathbb{E}\Big[\Big(M_k^{f,\phi,i} - M_{k-1}^{f,\phi,i}\Big)^2\Big|\mathcal{F}_{k-1}\Big].
$$

Recalling the definitions of $M_t^{f,\phi,i}$ and $R_f(\phi,i)$, it can be observed that

$$M_k^{f,\phi,i} = M_{k-1}^{f,\phi,i} + \int_{k-1}^{k} f(X_s^{\phi,i}, \alpha^i(s))ds + R_f(X_k^{\phi,i}, \alpha^i(k)) - R_f(X_{k-1}^{\phi,i}, \alpha^i(k-1)). \quad (3.39)$$

Then the Markovian property of $(X_t^{\phi,i}, \alpha^i(t))$ implies

$$\mathbb{E}\Big[\Big(M_k^{f,\phi,i} - M_{k-1}^{f,\phi,i}\Big)^2\Big|\mathcal{F}_{k-1}\Big] = \Phi_f(X_{k-1}^{\phi,i}, \alpha^i(k-1)),$$

and hence

$$\langle M^{f,\phi,i}\rangle_m = \sum_{k=0}^{m-1} \Phi_f(X_k^{\phi,i}, \alpha^i(k)), \quad m \in \mathbf{Z}_+. \quad (3.40)$$

Proof of Theorem 3.2. This proof is divided into three steps.

Step 1: Proof of (3.6). By virtue of the invariance of μ_*, applying the techniques similar to that used in (3.30) yields

$$\begin{aligned}
&\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \mathbb{E}\Big[t^{-\frac{1}{2}} A_t^f(\phi,i)\Big]^2 \mu_*(d\phi,i) \\
&= \frac{2}{t}\int_0^t \int_0^{t-u} \Big(\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} P_u(fP_sf)(\phi,i)\mu_*(d\phi,i)\Big) ds du \\
&= 2\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \int_0^t f(\phi,i)P_sf(\phi,i) ds \mu_*(d\phi,i) \\
&\quad - \frac{2}{t}\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \int_0^t s f(\phi,i)P_sf(\phi,i) ds \mu_*(d\phi,i). \quad (3.41)
\end{aligned}$$

Note that $\mu_*(\|\cdot\|_r^q) < \infty$ for $\forall q > 0$. For $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$, by virtue of (3.26), we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \Big|\frac{2}{t}\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \int_0^t s f(\phi,i)P_sf(\phi,i) ds \mu_*(d\phi,i)\Big| \\
&\quad \le \lim_{t\to\infty} \frac{2}{t}\int_0^t s \sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \big|f(\phi,i)P_sf(\phi,i)\big|\mu_*(d\phi,i) ds \\
&\quad \le \lim_{t\to\infty} \frac{8C\|f\|_{p,q}^2\big(1+\mu_*(\|\cdot\|_r^{2q})\big)^2}{t}\int_0^t s e^{-\varrho s} ds = 0,
\end{aligned}$$

which, together with (3.41) and (3.25), implies that

$$\begin{aligned}
\Sigma_f^2 &= \lim_{t\to\infty}\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \mathbb{E}\Big[t^{-\frac{1}{2}} A_t^f(\phi,i)\Big]^2 \mu_*(d\phi,i) \\
&= 2\sum_{i\in\mathbf{S}} \int_{\mathcal{C}_r} \int_0^\infty f(\phi,i)P_sf(\phi,i) ds \mu_*(d\phi,i) \\
&= 2\mu_*(fR_f) = \mu_*(\Phi_f).
\end{aligned}$$

Step 2: Proof of (i). Let $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$ and $\Sigma_f > 0$. By using [24, Lemma 2.9] for $\varepsilon = t^{-\frac{1}{4}}$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left\{t^{-\frac{1}{2}} A_t^f(\phi, i) \le x\right\} - F_{\Sigma_f}(x)\right| \\
&\quad\le \frac{1}{\Sigma_f\sqrt{2\pi}} t^{-\frac{1}{4}} + \sup_{x\in\mathbb{R}} \left|\mathbb{P}\left\{\lfloor t\rfloor^{-\frac{1}{2}} M_{\lfloor t\rfloor}^{f,\phi,i} \le x\right\} - F_{\Sigma_f}(x)\right| \\
&\qquad+ \mathbb{P}\left\{\left|t^{-\frac{1}{2}} A_t^f(\phi, i) - \lfloor t\rfloor^{-\frac{1}{2}} M_{\lfloor t\rfloor}^{f,\phi,i}\right| > t^{-\frac{1}{4}}\right\} \\
&\quad=: \frac{1}{\Sigma_f\sqrt{2\pi}} t^{-\frac{1}{4}} + \Gamma_1(t) + \Gamma_2(t),
\end{aligned} \tag{3.42}$$

where $\lfloor t\rfloor$ denotes the integer part of t. According to definitions of $A_t^f(\phi, i)$ and $M_t^{f,\phi,i}$, it follows from [23, Lemma 4.1], (3.1) and (3.27) that there exists a constant $A_2 > 0$ such that

$$\begin{aligned}
\mathbb{E}\left|t^{-\frac{1}{2}} A_t^f(\phi, i) - \lfloor t\rfloor^{-\frac{1}{2}} M_{\lfloor t\rfloor}^{f,\phi,i}\right| &\le (\lfloor t\rfloor)^{-\frac{1}{2}}\mathbb{E}\left|A_t^f(\phi, i)\right| + \lfloor t\rfloor^{-\frac{1}{2}} \int_{\lfloor t\rfloor}^t \mathbb{E}\left|f(X_s^{\phi,i}, \alpha^i(s))\right| ds \\
&\quad+ \lfloor t\rfloor^{-\frac{1}{2}}\mathbb{E}\left|R_f(X_{\lfloor t\rfloor}^{\phi,i}, \alpha^i(\lfloor t\rfloor)) - R_f(\phi, i)\right| \\
&\le A_2\|f\|_{p,q}(1 + \|\phi\|_r^q)\lfloor t\rfloor^{-\frac{1}{2}} \\
&\le 2A_2\|f\|_{p,q}(1 + \|\phi\|_r^q)t^{-\frac{1}{2}},
\end{aligned} \tag{3.43}$$

where $t \ge 1$. This, together with the Chebyshev inequality, gives

$$\Gamma_2(t) \le 2A_2\|f\|_{p,q}(1 + \|\phi\|_r^q)t^{-\frac{1}{4}}. \tag{3.44}$$

Now let us estimate $\Gamma_1(t)$. By virtue of (3.27), (3.39) and Lemma 3.4, there exists a positive constant A_3 such that

$$\begin{aligned}
\left|M_k^{f,\phi,i} - M_{k-1}^{f,\phi,i}\right| &= \left|\int_{k-1}^k f(X_s^{\phi,i}, \alpha^i(s))ds + R_f(X_k^{\phi,i}, \alpha^i(k)) - R_f(X_{k-1}^{\phi,i}, \alpha^i(k-1))\right| \\
&\le A_3\|f\|_{p,q}\left(1 + \sup_{s\in[k-1,k]} \delta^{\frac{q}{2}}\|X_s^{\phi,i}\|_r^q\right),
\end{aligned}$$

whence it follows that

$$\begin{aligned}
\mathbb{E}\exp\left\{\left|M_k^{f,\phi,i} - M_{k-1}^{f,\phi,i}\right|^{\frac{1}{q}}\right\} &\le \exp\left\{1 + 4^{(\frac{1}{q}-1)\vee 0} A_3^{\frac{2}{q}}\|f\|_{p,q}^{\frac{2}{q}}\right\}\mathbb{E}\sup_{s\in[k-1,k]} e^{\delta\|X_s^{\phi,i}\|_r^2} \\
&\le \exp\left\{1 + 4^{(\frac{1}{q}-1)\vee 0} A_3^{\frac{2}{q}}\|f\|_{p,q}^{\frac{2}{q}} + c(1 + \|\phi\|_r^2)\right\}.
\end{aligned}$$

Then it follows from [24, Proposition 2.10], (3.40), (3.6), (3.35), and (3.24) that for any $\varepsilon \in (0, 1/4)$ and $\kappa > 0$ there exists a positive constant $A_\varepsilon(\Sigma_f)$ such that

$$\begin{aligned}
\Gamma_1(t) &\le A_\varepsilon(\Sigma_f)\lfloor t\rfloor^{-\frac{1}{4}+\varepsilon} + \Sigma_f^{-4\kappa}\lfloor t\rfloor^{\kappa(1-4\varepsilon)}\mathbb{E}\left|\lfloor t\rfloor^{-1}\langle M^{f,\phi,i}\rangle_{\lfloor t\rfloor} - \Sigma_f^2\right|^{2\kappa} \\
&\le A_\varepsilon(\Sigma_f)\lfloor t\rfloor^{-\frac{1}{4}+\varepsilon} + \Sigma_f^{-4\kappa}\lfloor t\rfloor^{\kappa(1-4\varepsilon)}\mathbb{E}\left|\lfloor t\rfloor^{-1}\sum_{k=0}^{\lfloor t\rfloor-1} \Phi_f(X_k^{\phi,i}, \alpha^i(k)) - \mu_*(\Phi_f)\right|^{2\kappa} \\
&\le A_\varepsilon(\Sigma_f)\lfloor t\rfloor^{-\frac{1}{4}+\varepsilon} + 2C_0 h_0(\|\phi\|_r)\|f\|_{p,q}^{4\kappa}\Sigma_f^{-4\kappa}\lfloor t\rfloor^{-4\kappa\varepsilon}.
\end{aligned} \tag{3.45}$$

For any given $\varepsilon \in (0, 1/4)$, take $\kappa > 0$ such that $16\kappa\varepsilon \geq 1$. Then substituting (3.44) and (3.45) into (3.42) leads to the desired estimation (3.8).

Step 3: Proof of (ii). Let $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$ and $\Sigma_f = 0$. Then we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\Big\{(1\wedge|x|)\Big|\mathbb{P}\Big\{m^{-\frac{1}{2}}M_m^{f,\phi,i} \leq x\Big\} - F_0(x)\Big|\Big\}\\
&= \sup_{x\in\mathbb{R}}\Big\{(1\wedge|x|)\Big(\mathbb{P}\Big\{m^{-\frac{1}{2}}M_m^{f,\phi,i} \leq x\Big\}1_{\{x<0\}} + \mathbb{P}\Big\{m^{-\frac{1}{2}}M_m^{f,\phi,i} > x\Big\}1_{\{x\geq 0\}}\Big)\Big\}\\
&\leq \sup_{x>0}\Big\{(1\wedge x)\mathbb{P}\Big\{\Big|m^{-\frac{1}{2}}M_m^{f,\phi,i}\Big| \geq x\Big\}\Big\}.
\end{aligned} \tag{3.46}$$

According to the martingale property of $M_t^{f,\phi,i}$, we have $\mathbb{E}\big(M_m^{f,\phi,i}\big)^2 = \mathbb{E}\langle M^{f,\phi,i}\rangle_m$. Then for any $x > 0$, by using the Chebyshev inequality, it follows from (3.40), (3.35) and (3.24) that

$$\begin{aligned}
\mathbb{P}\Big\{\Big|m^{-\frac{1}{2}}M_m^{f,\phi,i}\Big| \geq x\Big\} &\leq x^{-1}m^{-\frac{1}{2}}\mathbb{E}\Big|M_m^{f,\phi,i}\Big| \leq x^{-1}m^{-\frac{1}{2}}\big(\mathbb{E}\langle M^{f,\phi,i}\rangle_m\big)^{\frac{1}{2}}\\
&= x^{-1}\bigg(\mathbb{E}\bigg[\frac{1}{m}\sum_{k=0}^{m-1}\Phi_f(X_k^{\phi,i}, \alpha^i(k))\bigg]\bigg)^{\frac{1}{2}}\\
&\leq 2C_0x^{-1}h_0(\|\phi\|_r)\|f\|_{p,q}m^{-\frac{1}{4}}.
\end{aligned}$$

Combining this with (3.46) gives

$$\sup_{x\in\mathbb{R}}\Big\{(1\wedge|x|)\Big|\mathbb{P}\big\{m^{-\frac{1}{2}}M_m^{f,\phi,i} \leq x\big\} - F_0(x)\Big|\Big\} \leq 2C_0h_0(\|\phi\|_r)\|f\|_{p,q}m^{-\frac{1}{4}}.$$

Then by using [24, Lemma 2.9] for $\sigma = 0$, the desired assertion follows. □

4. Law of iterated logarithm

This section establishes an LIL for additive functional of the regime-switching diffusion process with infinite delay described by (1.1) and (1.2). To proceed, we need some necessary notation. When $\Sigma_f > 0$, for any $(\phi, i) \in \mathbf{E}$ and $n > e$, define a sequence of $C([0, 1]; \mathbb{R})$-valued random variable as follows:

$$H_n^{f,\phi,i}(t) = \frac{1}{\Sigma_f\sqrt{2n\log\log n}}\int_0^{nt} f(X_s^{\phi,i}, \alpha^i(s))ds, \quad t \in [0, 1].$$

Let $\dot{x}$ denote the derivative of x and

$$\mathcal{H} = \Big\{ x \in C([0, 1]; \mathbb{R}) : x \text{ is absolutely continuous such that } x(0) = 0 \text{ and } \int_0^1 |\dot{x}(s)|^2 ds \leq 1 \Big\}.$$

Theorem 4.1. *Let Assumptions 2.1–2.3 hold, and $\eta_p > 0$ for some $p \in (0, 1]$. For some $\bar{\varepsilon} > 0$, if one of the following two conditions holds:*

(i) $\widetilde{\eta}_1 > 0$, $\widetilde{\gamma}_* < 0$, $\mu_3 \in \mathcal{M}_{(2r)\vee(-\widetilde{\gamma}_*+\bar{\varepsilon})}$,
(ii) $\widehat{\eta}_1 > 0$, $\widetilde{a}_* < 0$, $\mu_3 \in \mathcal{M}_{(2r)\vee(-\widetilde{a}_*+\bar{\varepsilon})}$,

then for any $q \in (0, 1/2)$, $(\phi, i) \in \mathbf{E}$, $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$ and $\Sigma_f > 0$, $\{H_n^{f,\phi,i}(\cdot), n > e\}$ is almost surely relatively compact in $C([0, 1]; \mathbb{R})$ and the set of its limit *points coincides with $\mathcal{H}$. Consequently, $\mathbb{P}$-a.s.*

$$\limsup_{t\to\infty}\frac{\int_0^t f(X_s^{\phi,i},\alpha^i(s))ds}{\sqrt{2t\log\log t}}=\Sigma_f,\qquad \liminf_{t\to\infty}\frac{\int_0^t f(X_s^{\phi,i},\alpha^i(s))ds}{\sqrt{2t\log\log t}}=-\Sigma_f. \tag{4.1}$$

Remark 4.2. Under assumptions of Theorem 4.1, (3.2) and (3.3) hold for $q\in(0,2]$, that is, Theorem 2.2 in [23] holds (in [23, Theorem 2.2], we wrote it as $q\in(0,1]$, which is a typographical error). In addition, to prove the LIL, we further assume $q\in(0,1/2)$ in Theorem 4.1. This implies we can use Lemma 3.5 and some estimates in its proof for $q\in(0,1/2)$ in what follows.

Remark 4.3. Compared with [2, Theorem 1.3] and our previous result [28, Theorem 4.3], this paper considers a Markovian process $(X_t^{\phi,i},\alpha^i(t))$ determined by (1.1) and (1.2). It presents a continuous-time version LIL for additive functionals. To this end, we modify the reference function sequence $H_n^{f,\phi,i}(t)$. In addition, our assumptions in Theorem 4.1 allow some subsystems to be fully non-dissipative (see Example 5.2 in what follows), and this paper removes the boundedness assumption on the diffusion and its inverse in [2]. Theorem 3.4 in [6] and Theorem 4.1.5 in [16] consider the LIL for additive functionals for the Markov chain and the stationary Markov processes, respectively. They cannot be directly used to deal with the present non-stationary setting since $(X_t^{\phi,i},\alpha^i(t))$ is a non-stationary Markovian process in the Polish space $\mathcal{C}_r\times\mathbf{S}$.

Remark 4.4. Under some additional growth conditions on the drift to ensure the solution to be locally Hölder continuous, if $f\not\equiv 0$ on the support of the invariant probability measure μ_* (supp(μ_*)), then $\Sigma_f>0$. Assume, by contradiction, that $\Sigma_f=0$. Applying a similar approach to derive $\Sigma_f^2=\mu_*(\Phi_f)$ yields that $\Sigma_f^2=T^{-1}\mu_*(\tilde\Phi_{f,T})$ for any $T>0$, where

$$\tilde\Phi_{f,T}(\phi,i):=\mathbb{E}\Big|\int_0^T f(X_s^{\phi,i},\alpha^i(s))ds+R_f(X_T^{\phi,i},\alpha^i(T))-R_f(\phi,i)\Big|^2.$$

Then $\Sigma_f=0$ implies that $\mu_*(\tilde\Phi_{f,T})=0$, which, in turn, implies

$$\int_0^T f(X_s^{\phi,i},\alpha^i(s))ds+R_f(X_T^{\phi,i},\alpha^i(T))-R_f(\phi,i)=0,\quad \mathbb{P}_{\mu_*}\text{-}a.s. \tag{4.2}$$

Under $\mathbb{P}_{\mu_*}$, $(X_t,\alpha(t))$ is a stationary strong Markovian process with initial distribution μ_*. Recalling that $f\in C_{p,q}(\mathbf{E})$, $f\not\equiv 0$ on supp(μ_*) and $\mu_*(f)=0$, there exist bounded measurable subsets $B_0\subset B$ of $\mathcal{C}_r$ with $\mu_*(B_0\times\mathbf{S})>0$ and $\varepsilon>0$ such that $f(\xi,i)\ge\varepsilon$ (or $-f(\xi,i)\ge\varepsilon$), $\forall\xi\in B, i\in\mathbf{S}$ and $\mathrm{dist}(B_0,\partial B):=\inf_{\xi\in B_0,\eta\in\partial B}\|\xi-\eta\|_r>\varepsilon_0$ for some $\varepsilon_0>0$. In addition, there exists a compact set $K_c\subset\mathcal{C}_r$ such that $\mathbb{P}_{\mu_*}\{(X_s^{\phi,i},\alpha^i(s))\in K_c\times\mathbf{S}\}=\mu_*(K_c\times\mathbf{S})>1-\mu_*(B_0\times\mathbf{S})/2$ for any $s\ge 0$ since μ_* is invariant and tight. Hence $\mu_*((K_c\cap B_0)\times\mathbf{S})>0$.

Under appropriate growth conditions, it is easy to see that the solution process is locally Hölder continuous, that is,

$$\mathbb{P}_{\mu_*}\left\{\omega:\sup_{0<t-s<h(\omega),s,t\in[0,T_0]}\frac{|X^{\phi,i}(t,\omega)-X^{\phi,i}(s,\omega)|}{|t-s|^\gamma}<\delta\right\}=1,$$

where $T_0,\gamma,\delta>0$ are constants and $h(\omega)>0$, $\mathbb{P}_{\mu_*}$-a.e. Then by the definition of the norm $\|\cdot\|_r$, there exists $t_0>0$ small enough such that $\sup_{0<t-s\le t_0,s,t\in[0,T_0]}|X^{\phi,i}(t,\omega)-X^{\phi,i}(s,\omega)|\le\varepsilon_0/4$ on the event $\{h(\omega)\ge t_0\}$, and $\sup_{0\le s\le t_0}\|X_s^{\phi,i}-X_0^{\phi,i}\|_r\le\varepsilon_0/2$ on $\{X_0^{\phi,i}\in K_c\cap\bar B_0,\ h(\omega)\ge t_0\}$

where $\bar{B}_0$ is the closure of B_0. In addition, noting that $\mathbb{P}_{\mu_*}\{h(\omega) \geq t_0\} \to 1$ as $t_0 \to 0$. Hence we can take $t_0 > 0$ small enough such that for any $s \geq 0$, $\mathbb{P}_{\mu_*}\{(X_s^{\phi,i}, \alpha^i(s)) \in (K_c \cap B_0) \times \mathbf{S}, h(\omega) \geq t_0\} > 0$ due to $\mathbb{P}_{\mu_*}\{(X_s^{\phi,i}, \alpha^i(s)) \in (K_c \cap B_0) \times \mathbf{S}\} = \mu_*((K_c \cap B_0) \times \mathbf{S}) > 0$. Then we arrive at

$$\begin{aligned} &\mathbb{P}_{\mu_*}\left\{(X_s^{\phi,i}, \alpha^i(s)) \in B \times \mathbf{S}, \forall s \in [0, t_0]\right\} \\ &\geq \mathbb{P}_{\mu_*}\left\{(X_0^{\phi,i}, \alpha^i(0)) \in (K_c \cap B_0) \times \mathbf{S}, h(\omega) \geq t_0\right\} > 0. \end{aligned}$$

Furthermore, by the Markovian property and stationarity of $(X_s^{\phi,i}, \alpha^i(s))$ under $\mathbb{P}_{\mu_*}$, we have

$$\mathbb{P}_{\mu_*}\left\{(X_s^{\phi,i}, \alpha^i(s)) \in B \times \mathbf{S}, \forall s \in [0, T]\right\} > 0, \quad \forall T > 0.$$

Then we have

$$\mathbb{P}_{\mu_*}\left\{\int_0^T f(X_s^{\phi,i}, \alpha^i(s))ds \geq T\varepsilon\right\} \geq \mathbb{P}_{\mu_*}\{(X_s^{\phi,i}, \alpha^i(s)) \in B \times \mathbf{S}, s \in [0, T]\} > 0. \tag{4.3}$$

In addition, since $R_f(\cdot, \cdot)$ is bounded on bounded set of $\mathcal{C}_r \times \mathbf{S}$ (see (3.27)), then we have $|R_f(\xi, i)| < K_b, \forall \xi \in B, i \in \mathbf{S}$ for some $K_b > 0$. Hence,

$$\mathbb{P}_{\mu_*}\{|R_f(X_T^{\phi,i}, i) - R_f(\phi, i)| < 2K_b\} \geq \mathbb{P}_{\mu_*}\{X_T^{\phi,i}, X_0^{\phi,i} \in B\} > 0.$$

Choosing $T > 0$ large enough such that $T\varepsilon > 2K_b$, this, together with (4.3), implies that (4.2) cannot hold with probability one. This is a contradiction. Hence, $\Sigma_f > 0$ if $f \not\equiv 0$ on supp(μ_*) and the solution process is locally Hölder continuous. As for Examples in this paper, it is easy to verify that their solution processes are locally Hölder continuous

However, it is not clear to us how to determine the support of the invariant probability measure μ_*. Note that $(X_t^{\phi,i}, \alpha^i(t))$ is a highly degenerate Markovian process, that is, $(X_t^{\phi,i}, \alpha^i(t))$ is an infinite-dimensional Markovian process with finite-dimensional noises $W(t)$. Therefore, even though the diffusion coefficient is non-degenerate, the support of the invariant probability measure μ_* cannot be the whole space $\mathcal{C}_r \times \mathbf{S}$. We only know that supp$(\mu_*) \subseteq \mathcal{C}_r^0 \times \mathbf{S}$, where $\mathcal{C}_r^0 = \{\phi \in \mathcal{C}_r : \lim_{\theta \to -\infty} e^{r\theta}\phi(\theta) = 0\}$, which follows from our previous work [30, Remark 3.6].

To prove this theorem, let us first present a crucial lemma.

Lemma 4.5. *Under assumptions of Theorem 4.1,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Z_k^2(\phi, i) = \Sigma_f^2, \qquad \mathbb{P}\text{-}a.s.,$$

where $Z_k(\phi, i) := M_k^{f,\phi,i} - M_{k-1}^{f,\phi,i}$ *and* $M_k^{f,\phi,i}$ *is given by (3.39).*

Proof. As observed in the proof of [14, Proposition 3.1], to obtain the desired result, it is sufficient to prove the continuity of the two maps

$$\Psi_1(\phi, i) := \mathbb{E}\left(\left|\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} Z_k^2(\phi, i)\right) - \Sigma_f^2\right| \wedge 1\right),$$

$$\Psi_2(\phi, i) := \mathbb{E}\left(\left|\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} Z_k^2(\phi, i)\right) - \Sigma_f^2\right| \wedge 1\right),$$

with respect to $(\phi, i) \in \mathbf{E}$. Because $\mathbf{S}$ is a finite set with discrete metric, it is enough to show that $\Psi_j(\phi, i)$, $j = 1, 2$ is continuous with respect to $\phi \in \mathcal{C}_r$. We first show the continuity of

$\Psi_1(\phi, i)$ with respect to ϕ. Note that for any $n, K_0 \in \mathbf{Z}_+$ with $n \geq K_0 \geq 1$

$$\frac{1}{n}\sum_{k=K_0}^{n} Z_k^2(\phi, i) = \frac{1}{n}\sum_{k=1}^{n} Z_k^2(\phi, i) - \frac{1}{n}\sum_{k=1}^{K_0-1} Z_k^2(\phi, i).$$

Recalling the definition of $M_t^{f,\phi,i}$ and $Z_k(\phi, i)$, (2.7) implies $\mathbb{E}|Z_k(\phi, i)|^2 < \infty$. This in turn implies $Z_k^2(\phi, i) < \infty$, $\mathbb{P}$-$a.s.$. Then we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=K_0}^{n} Z_k^2(\phi, i) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Z_k^2(\phi, i), \quad \mathbb{P}\text{-}a.s.$$

Therefore, for fixed $K_0 \in \mathbf{Z}_+$ and $K_0 > 1$, we arrive at

$$\begin{aligned}
\Psi_1(\phi, i) &= \mathbb{E}\left(\left|\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=K_0}^{n} Z_k^2(\phi, i)\right) - \Sigma_f^2\right| \wedge 1\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\left|\sup_{m\geq n}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\phi, i)\right) - \Sigma_f^2\right| \wedge 1\right) \\
&= \lim_{n\to\infty}\lim_{N\to\infty} \mathbb{E}\left(\left|\sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\phi, i)\right) - \Sigma_f^2\right| \wedge 1\right) \\
&= \lim_{n\to\infty}\lim_{N\to\infty} \mathbb{E}\left(\left|\sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\phi, i)\right) \wedge \left(1+\Sigma_f^2\right) - \Sigma_f^2\right| \wedge 1\right) \\
&=: \lim_{n\to\infty}\lim_{N\to\infty} \Psi_{n,N}(\phi, i). \qquad (4.4)
\end{aligned}$$

Then, for any $(\phi, i), (\psi, i) \in \mathbf{E}$ and $n \geq K_0 > 1$, we have

$$\begin{aligned}
&\left|\Psi_{n,N}(\phi, i) - \Psi_{n,N}(\psi, i)\right| \\
&= \Bigg|\mathbb{E}\left(\left|\sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\phi, i)\right) \wedge \left(1+\Sigma_f^2\right) - \Sigma_f^2\right| \wedge 1\right) \\
&\quad - \mathbb{E}\left(\left|\sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\psi, i)\right) \wedge \left(1+\Sigma_f^2\right) - \Sigma_f^2\right| \wedge 1\right)\Bigg| \\
&\leq \mathbb{E}\Bigg|\sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\phi, i)\right) \wedge \left(1+\Sigma_f^2\right) \\
&\quad - \sup_{m\in\{n,n+1,\dots,n+N\}}\left(\frac{1}{m}\sum_{k=K_0}^{m} Z_k^2(\psi, i)\right) \wedge \left(1+\Sigma_f^2\right)\Bigg| \\
&\leq \mathbb{E}\Bigg|\sup_{m\in\{n,n+1,\dots,n+N\}} \frac{1}{m}\left(\sum_{k=K_0}^{m}\left(Z_k^2(\phi, i) \wedge m\left(1+\Sigma_f^2\right)\right) \wedge m\left(1+\Sigma_f^2\right)\right)
\end{aligned}$$

$$
-\sup_{m\in\{n,n+1,\ldots,n+N\}}\frac{1}{m}\Big(\sum_{k=K_0}^{m}\Big(Z_k^2(\psi,i)\wedge m\Big(1+\Sigma_f^2\Big)\Big)\wedge m\Big(1+\Sigma_f^2\Big)\Big)\Big|
$$

$$
\leq \mathbb{E}\Big|\sup_{m\in\{n,n+1,\ldots,n+N\}}\frac{1}{m}\sum_{k=K_0}^{m}\Big(Z_k^2(\phi,i)\wedge m\Big(1+\Sigma_f^2\Big)-Z_k^2(\psi,i)\wedge m\Big(1+\Sigma_f^2\Big)\Big)\Big|
$$

$$
\leq \mathbb{E}\sup_{m\in\{n,n+1,\ldots,n+N\}}\frac{2\sqrt{m(1+\Sigma_f^2)}}{m}\sum_{k=K_0}^{m}|Z_k(\phi,i)-Z_k(\psi,i)|. \tag{4.5}
$$

Then it follows from (4.4) and (4.5) that

$$
\begin{aligned}
|\Psi_1(\phi,i)-\Psi_1(\psi,i)| &= \lim_{n\to\infty}\lim_{N\to\infty}|\Psi_{n,N}(\phi,i)-\Psi_{n,N}(\psi,i)|\\
&\leq 2(1+\Sigma_f^2)\sum_{k=K_0}^{\infty}\mathbb{E}|Z_k(\phi,i)-Z_k(\psi,i)|.
\end{aligned}\tag{4.6}
$$

Recalling the definition of $Z_k(\phi,i)$, by virtue of (3.39) and (3.33), we derive

$$
\begin{aligned}
|Z_k(\phi,i)-Z_k(\psi,i)| \leq{}& \Big|\int_{k-1}^{k}\big[f(X_s^{\phi,i},\alpha^i(s))-f(X_s^{\psi,i},\alpha^i(s))\big]ds\Big|\\
&+\big|R_f(X_k^{\phi,i},\alpha^i(k))-R_f(X_k^{\psi,i},\alpha^i(k))\big|\\
&+\big|R_f(X_{k-1}^{\phi,i},\alpha^i(k-1))-R_f(X_{k-1}^{\psi,i},\alpha^i(k-1))\big|\\
\leq{}& \int_{k-1}^{k}\|f\|_{p,q}\|X_s^{\phi,i}-X_s^{\psi,i}\|_r^p\sqrt{1+\|X_s^{\phi,i}\|_r^{2q}+\|X_s^{\psi,i}\|_r^{2q}}ds\\
&+C\varrho^{-1}\|f\|_{p,q}\|X_k^{\phi,i}-X_k^{\psi,i}\|_r^p\sqrt{1+\|X_k^{\phi,i}\|_r^{2q}+\|X_k^{\psi,i}\|_r^{2q}}\\
&+C\varrho^{-1}\|f\|_{p,q}\|X_{k-1}^{\phi,i}-X_{k-1}^{\psi,i}\|_r^p\sqrt{1+\|X_{k-1}^{\phi,i}\|_r^{2q}+\|X_{k-1}^{\psi,i}\|_r^{2q}}.
\end{aligned}
$$

Then by using Hölder's inequality and Lemma 2.3, there exists a positive constant A_4 such that

$$
\begin{aligned}
\mathbb{E}|Z_k(\phi,i)-Z_k(\psi,i)| \leq{}& \|f\|_{p,q}\int_{k-1}^{k}\big(\mathbb{E}\|X_s^{\phi,i}-X_s^{\psi,i}\|_r^{2p}\big)^{\frac12}\\
&\times\sqrt{1+\mathbb{E}\|X_s^{\phi,i}\|_r^{2q}+\mathbb{E}\|X_s^{\psi,i}\|_r^{2q}}ds\\
&+C\varrho^{-1}\|f\|_{p,q}\big(\mathbb{E}\|X_k^{\phi,i}-X_k^{\psi,i}\|_r^{2p}\big)^{\frac12}\\
&\times\sqrt{1+\mathbb{E}\|X_k^{\phi,i}\|_r^{2q}+\mathbb{E}\|X_k^{\psi,i}\|_r^{2q}}\\
&+C\varrho^{-1}\|f\|_{p,q}\big(\mathbb{E}\|X_{k-1}^{\phi,i}-X_{k-1}^{\psi,i}\|_r^{2p}\big)^{\frac12}\\
&\times\sqrt{1+\mathbb{E}\|X_{k-1}^{\phi,i}\|_r^{2q}+\mathbb{E}\|X_{k-1}^{\psi,i}\|_r^{2q}}\\
\leq{}& A_4\|f\|_{p,q}\|\phi-\psi\|_r^p e^{-\frac12\eta_\varepsilon(k-1)}.
\end{aligned}
$$

Substituting this into (4.6) yields

$$
\Big|\Psi_1(\phi,i)-\Psi_1(\psi,i)\Big| \leq 2(1+\Sigma_f^2)A_4\|f\|_{p,q}\sum_{k=K_0}^{\infty}e^{-\frac12\eta_\varepsilon(k-1)}\|\phi-\psi\|_r^p.
$$

Since K_0 is arbitrary, this implies $\Psi_1(\phi, i)$ is a constant and in particular continuous with respect to ϕ. Similarly, we can show the continuity of $\Psi_2(\cdot, \cdot)$. Hence this proof is complete. □

Proof of Theorem 4.1. Noting that $\Phi_f \in C_{p,2q}(\mathbf{E})$ and $\Sigma_f^2 = \mu_*(\Phi_f)$, by using a similar argument to (3.26), there exists a constant $A_5 > 0$ such that

$$|P_k \Phi_f(\xi) - \Sigma_f^2| = |P_k \Phi_f(\xi) - \mu_*(\Phi_f)| \le A_5 e^{-\varrho k}, \quad \forall\, k \ge 0.$$

Let $S_n^2(f, \phi, i) = \mathbb{E}|M_n^{f,\phi,i}|^2$. Then it follows from (3.40) that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{S_n^2(f,\phi,i)}{n} - \Sigma_f^2\right| &= \lim_{n\to\infty}\left|\frac{\mathbb{E}|M_n^{f,\phi,i}|^2}{n} - \Sigma_f^2\right| \\
&= \lim_{n\to\infty}\left|\frac{\mathbb{E}\langle M^{f,\phi,i}\rangle_n}{n} - \Sigma_f^2\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{n}\sum_{k=0}^{n-1} P_k\Phi_f(\phi,i) - \Sigma_f^2\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{n}\sum_{k=0}^{n-1} (P_k\Phi_f(\phi,i) - \Sigma_f^2)\right| \\
&\le \lim_{n\to\infty}\frac{A_5}{n}\sum_{k=0}^{n-1} e^{-\varrho k} = 0.
\end{aligned} \tag{4.7}$$

This implies $S_n^2(f, \phi, i) = O(n)$. Recalling $q \in (0, 1/2)$ and the definition of $Z_n(\phi, i)$, it follows from (2.7), (3.27) and (3.39) that there is a constant $A_6 > 0$ independent of n such that

$$\begin{aligned}
\mathbb{E}\big|Z_n(\phi,i)\big|^4 &\le 27\mathbb{E}\int_{n-1}^{n}\big|f(X_s^{\phi,i},\alpha^i(s))\big|^4 ds + 27\mathbb{E}\big|R_f(X_n^{\phi,i},\alpha^i(n))\big|^4 \\
&\quad + 27\mathbb{E}\big|R_f(X_{n-1}^{\phi,i},\alpha^i(n-1))\big|^4 \\
&\le A_6\|f\|_{p,q}(1+\|\phi\|_r^{4q}),
\end{aligned}$$

which, together with the fact that $S_n^2(f, \phi, i) = O(n)$, yields

$$\sum_{n=1}^{\infty} S_n^{-4}(f,\phi,i)\mathbb{E}\big|Z_n(\phi,i)\big|^4 \le A_6\|f\|_{p,q}(1+\|\phi\|_r^{4q})\sum_{n=1}^{\infty} S_n^{-4}(f,\phi,i) < \infty. \tag{4.8}$$

Then it follows from Lemma 4.5 and (4.7) that

$$\lim_{n\to\infty}\frac{1}{S_n^2(f,\phi,i)}\sum_{k=1}^{n} Z_k^2(\xi) = \lim_{n\to\infty}\frac{n}{S_n^2(f,\phi,i)}\left(\frac{1}{n}\sum_{k=1}^{n} Z_k^2(\xi)\right) = 1 \tag{4.9}$$

and

$$\lim_{n\to\infty}\frac{\sqrt{2S_n^2(f,\phi,i)\log\log S_n^2(f,\phi,i)}}{\Sigma_f\sqrt{2n\log\log n}} = 1. \tag{4.10}$$

Then in light of (4.8)–(4.10) and the fact that $S_n^2(f, \phi, i) = O(n)$, it follows from [11, Corollary 1] that the sequence of real random functions $\{\widehat{H}_n^{f,\phi,i}(\cdot), n > e\}$ on $[0, 1]$ is almost surely relatively compact in $C([0, 1]; \mathbb{R})$ and the set of its limit points coincides with $\mathcal{H}$, where

$$\widehat{H}_n^{f,\phi,i}(t) := \frac{M_k^{f,\phi,i} + (S_n^2(f,\phi,i)t - S_k^2(f,\phi,i))(S_{k+1}^2(f,\phi,i) - S_k^2(f,\phi,i))^{-1}Z_{k+1}(\phi,i)}{\Sigma_f\sqrt{2n\log\log n}}$$

$$\text{for } t \in (0,1],\ S_k^2(f,\phi,i) \le S_n^2(f,\phi,i)t \le S_{k+1}^2(f,\phi,i),\ k=0,1,\ldots,n-1,$$
$$\widehat{H}_n^{f,\phi,i}(t) := 0, \qquad \text{for } t=0.$$

Furthermore, applying a similar argument as in the proof of [6, Theorem 1] (see also [8, Theorem 3.7]) gives that the sequence of real random functions $\{\widetilde{H}_n^{f,\phi,i}(\cdot), n > e\}$ on $[0,1]$ is almost surely relatively compact in $C([0,1];\mathbb{R})$ and the set of its limit points coincides with $\mathcal{H}$, where

$$\widetilde{H}_n^{f,\phi,i}(t) := \frac{M_k^{f,\phi,i} + (nt-k)Z_{k+1}(\phi,i)}{\Sigma_f\sqrt{2n\log\log n}} \qquad \text{for } t\in(0,1],\ k\le nt\le k+1,\ k=0,1,\ldots,n-1,$$
$$\widetilde{H}_n^{f,\phi,i}(t) := 0 \qquad \text{for } t=0.$$

Then by the Chebyshev inequality, (2.7), and the Borel–Cantelli lemma, for any $l \in (q/2, 1/2)$ there exists a $\mathbf{Z}_+$-value random variable $\tau(\phi,i)$ such that for almost sure $\omega \in \Omega$,

$$\sup_{t\in[k,k+1]} \|X_t^{\phi,i}(\omega)\|_r^q \le k^l - 1 \quad \text{for} \quad k \ge \tau(\phi,i)(\omega),$$

which, together with (2.2), implies that for any fixed k

$$\lim_{n\to\infty} \frac{\sup_{s\in[k,k+1]}\|X_s^{\phi,i}\|_r^q}{\Sigma_f\sqrt{2n\log\log n}} = 0, \quad \mathbb{P}\text{-}a.s.$$

Combining this with (3.39) and (3.27) yields that

$$\lim_{n\to\infty} \frac{1}{\Sigma_f\sqrt{2n\log\log n}}\left[\int_k^{k+1} \left|f(X_s^{\phi,i},\alpha^i(s))\right|ds + \left|R_f(X_k^{\phi,i},\alpha^i(k))\right|\right] = 0, \quad \mathbb{P}\text{-}a.s. \tag{4.11}$$

Hence, this, together with (3.38), gives

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\left|H_n^{f,\phi,i}(t) - \widetilde{H}_n^{f,\phi,i}(t)\right| = 0, \quad \mathbb{P}\text{-}a.s.,$$

which implies that the sequence of real random functions $\{H_n^{f,\phi,i}(\cdot), n > e\}$ on $[0,1]$ is also almost surely relatively compact in $C([0,1];\mathbb{R})$ and the set of its limit points coincides with $\mathcal{H}$.

Now it remains to prove (4.1). Note that for any $x \in \mathcal{H}$, $x(1) \le 1$. According to the definition of $H_n^{f,\phi,i}(\cdot)$,

$$\limsup_{n\to\infty} |H_n^{f,\phi,i}(1)| = \limsup_{n\to\infty} \frac{\int_0^n f(X_s^{\phi,i},\alpha^i(s))ds}{\Sigma_f\sqrt{2n\log\log n}} \le 1. \tag{4.12}$$

In addition, since the set of limit points of $\{H_n^{f,\phi,i}(\cdot), n \ge 0\}$ in $C([0,1];\mathbb{R})$ coincides with $\mathcal{H}$, then for $x(s) := s$, $s\in[0,1]$ ($x(\cdot)\in\mathcal{H}$) and any fixed $\omega\in\Omega$ (there may be another $\widetilde{\Omega}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$, but we still write it as Ω here without loss of generality), there exists a subsequence $n_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty}\sup_{s\in[0,1]} |H_{n_k}^{f,\phi,i}(s) - x(s)| = 0.$$

Then for the fixed $\omega \in \Omega$ above, we obtain

$$\lim_{k\to\infty} \frac{\int_0^{n_k} f(X_s^{\phi,i},\alpha^i(s))ds}{\Sigma_f\sqrt{2n_k\log\log n_k}} = \lim_{k\to\infty} H_{n_k}^{f,\phi,i}(1) = 1,$$

which together with (4.12) gives

$$\limsup_{n\to\infty} \frac{\int_0^n f(X_s^{\phi,i}, \alpha^i(s))ds}{\sqrt{2n\log\log n}} = \Sigma_f, \quad \mathbb{P}\text{-}a.s. \tag{4.13}$$

Note that $\Sigma_f = \Sigma_{-f}$. Hence, applying a similar argument for $-f$ yields

$$\liminf_{n\to\infty} \frac{\int_0^n f(X_s^{\phi,i}, \alpha^i(s))ds}{\sqrt{2n\log\log n}} = -\Sigma_f, \quad \mathbb{P}\text{-}a.s. \tag{4.14}$$

Applying a similar argument to derive (4.11), we arrive at

$$\lim_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}} \int_{[t]}^t \left| f(X_s^{\phi,i}, \alpha^i(s))\right| ds = 0, \quad \mathbb{P}\text{-}a.s.,$$

where $[t]$ denotes the integer part of t. This, together with (4.13), (4.14) and the fact that

$$\lim_{t\to\infty} \sqrt{\frac{2[t]\log\log[t]}{2t\log\log t}} = 1,$$

gives the desired results (4.1). The proof is thus complete. □

If $\sigma(\phi, i)$ is bounded, one can prove that the assertions in Theorem 4.1 hold for any $q > 0$ instead of $q \in (0, 1/2)$.

Corollary 4.6. *Let Assumptions 2.1–2.3 hold with $V(x) = |x|^2$. Let $\eta_p > 0$ for some $p \in (0, 1]$ and $\widetilde{a}^* + \widetilde{b}^* < 0$, and $\|\sigma\|_\infty := \sup_{(\phi,i)\in\mathbf{E}} \|\sigma(\phi, i)\|_{\mathrm{HS}} < \infty$. Then for any $q > 0$, $(\phi, i) \in \mathbf{E}$, $f \in C_{p,q}(\mathbf{E})$ with $\mu_*(f) = 0$ and $\Sigma_f > 0$, the assertions in Theorem 4.1 hold.*

Proof. A slightly modifying proof of Theorem 4.1 gives the desired assertions with the help of Proposition 3.1. □

5. Examples

In this section, two examples are given to illustrate our results.

Example 5.1. Let $d = 2$, $m = r = 1$, and $(\alpha(t))_{t\geq 0}$ be a Markovian chain taking values in $\mathbf{S} = \{1, 2\}$ with generator

$$Q = \begin{pmatrix} -1 & 1 \\ A & -A \end{pmatrix}, \tag{5.1}$$

where A is a constant. In this example, we choose $A > 14/5 + 8\sqrt{3}/5$. For a scalar Brownian motion $W(t)$ independent of $\alpha(t)$, consider the following regime-switching diffusion systems with infinite delay:

$$dX(t) = b(X_t, \alpha(t))dt + \sigma(X_t, \alpha(t))dW(t), \quad t \geq 0, \tag{5.2}$$

with

$$b(\phi, 1) = \begin{pmatrix} -\phi_1(0) + \frac{1}{2}\int_{-\infty}^0 \phi_2(\theta)\mu(d\theta) \\ -\phi_2(0) \end{pmatrix}, \quad \sigma(\phi, 1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$b(\phi, 2) = \begin{pmatrix} 1-\phi_1^3(0) \\ \phi_1(0)-\phi_2^3(0)+\int_{-\infty}^0 \phi_2(\theta)\mu(d\theta) \end{pmatrix}, \quad \sigma(\phi, 2) = \int_{-\infty}^0 g(\phi(\theta))\mu(d\theta),$$

where $\phi = (\phi_1, \phi_2) \in \mathcal{C}_r$, $\mu(d\theta) = 4e^{4\theta}d\theta \in \mathcal{M}_2$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded Lipschitz function with Lipschitz constant 1. It is obvious that the diffusion coefficient of Eq. (5.2) is degenerated when $\alpha(t) = 1$. Let us check the conditions of Theorem 3.2 and Corollary 4.6.

Clearly, Assumption 2.1 holds and $\sigma(\phi, i)$ is bounded. Let $U(x) = |x|^2$. Then we obtain

$$\begin{aligned}
\mathbb{L}_1 U(\phi-\psi) &= 2\big\langle \phi(0)-\psi(0), b(\phi,1)-b(\psi,1)\big\rangle + \|\sigma(\phi,1)-\sigma(\psi,1)\|_{\mathrm{HS}}^2 \\
&\le -\frac{3}{2}|\phi(0)-\psi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)-\psi(\theta)|^2\mu(d\theta), \\
\mathbb{L}_2 U(\phi-\psi) &= 2\big\langle \phi(0)-\psi(0), b(\phi,2)-b(\psi,2)\big\rangle + \|\sigma(\phi,2)-\sigma(\psi,2)\|_{\mathrm{HS}}^2 \\
&\le (1+\varepsilon)|\phi(0)-\psi(0)|^2 + \Big(1+\frac{1}{\varepsilon}\Big)\int_{-\infty}^0 |\phi(\theta)-\psi(\theta)|^2\mu(d\theta),
\end{aligned}$$

where $\varepsilon > 0$ is a constant. Hence Assumption 2.2 holds and

$$\overline{a}(1) = -\frac{3}{2}, \quad \overline{b}(1) = \frac{1}{2}, \quad \overline{a}(2) = 1+\varepsilon, \quad \overline{b}(2) = 1+\frac{1}{\varepsilon}, \quad \gamma = -1.$$

The invariant probability measure of $\alpha(t)$ is

$$\pi = (\pi_1, \pi_2) = \Big(\frac{A}{1+A}, \frac{1}{1+A}\Big).$$

We compute

$$\begin{aligned}
\Sigma(\varepsilon) &:= \pi_1\Big(\overline{a}(1)+\overline{b}(1)\int_{-\infty}^0 e^{\gamma\theta}\mu(d\theta)\Big) + \pi_2\Big(\overline{a}(2)+\overline{b}(2)\int_{-\infty}^0 e^{\gamma\theta}\mu(d\theta)\Big) \\
&= \frac{1}{1+A}\Big(-\frac{5}{6}A+\frac{7}{3}+\varepsilon+\frac{4}{3\varepsilon}\Big).
\end{aligned}$$

Recalling $A > 14/5 + 8\sqrt{3}/5$, there exists a $\varepsilon_0 > 0$ such that $\Sigma(\varepsilon_0) < 0$. Then, it follows from [3, Proposition 4.2] that there exists a $p \in (0, 1]$ such that $\eta_p > 0$. On the other hand, let $V(x) = |x|^2$. Then it follows that

$$\mathcal{L}_1 V(\phi) = 2\big\langle \phi(0), b(\phi.1)\big\rangle + \|\sigma(\phi,1)\|_{\mathrm{HS}}^2 \le 1 - \frac{3}{2}|\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2\mu(d\theta),$$

$$\mathcal{L}_2 V(\phi) = 2\big\langle \phi(0), b(\phi.2)\big\rangle + \|\sigma(\phi,2)\|_{\mathrm{HS}}^2 \le L - |\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2\mu(d\theta),$$

where

$$L := \sup_{x\in\mathbb{R}^2}\Big\{-2|x|^4 + |x|^2 + 2x_1x_2 + 2|x_2|^2 + 2x_1 + |g(x)|^2\Big\}.$$

Then

$$\widetilde{a}(1) = -\frac{3}{2}, \quad \widetilde{b}(1) = \frac{1}{2}, \quad \widetilde{a}(2) = -1, \quad \widetilde{b}(2) = \frac{1}{2}, \quad \tilde{a}^* = -1, \quad \tilde{b}^* = \frac{1}{2}.$$

Hence, $\tilde{a}^* + \tilde{b}^* = -1/2 < 0$, and all the conditions in Theorem 3.2 and Corollary 4.6 hold. Hence the results in Theorem 3.2 and Corollary 4.6 hold.

Example 5.2. Let $r = 1$, $h > 0$, and $\alpha(t)$ be the Markovian chain defined by (5.1). The following d-dimensional regime-switching diffusion system with infinite delay is a modification of [1, Example 1.2]:

$$dX(t) = \left\{a_{\alpha(t)}X(t) + c_{\alpha(t)}|X(t)|^h X(t) + b_{\alpha(t)} \int_{-\infty}^{0} X(t+\theta)\mu(d\theta)\right\} dt + \sigma(X_t, \alpha(t))dW(t), \quad (5.3)$$

where $a_1 = b_1 = 1/10, b_2 = 1, a_2 = -2$, $c_1 = 0, c_2 = -1$, $\mu(d\theta) = 4e^{4\theta}d\theta \in \mathcal{M}_2$, and $\sigma : \mathcal{C}_r \times \mathbf{S} \mapsto \mathbb{R}^{d\times m}$ satisfies

$$\|\sigma(\phi, i) - \sigma(\psi, i)\|_{\text{HS}}^2 \le \frac{1}{10}\int_{-\infty}^{0} |\phi(\theta) - \psi(\theta)|^2\mu(d\theta), \quad i = 1, 2,$$

and $W(t)$ is an m-dimensional Brownian motion independent of $\alpha(t)$. For $a_1 = b_1 = 1/10$ and $c_1 = 0$, it is obvious that the subsystem with coefficients $b(\phi, 1)$ and $\sigma(\phi, 1)$ is non-dissipative. For $A \in (0, 594/1955)$, let us check the conditions in Theorem 4.1 for $(X_t^{\phi,i}, \alpha^i(t))$ determined by (5.1) and (5.3).

For $i = 1, 2$ and $\phi \in \mathcal{C}_r$, let

$$b(\phi, i) = a_i\phi(0) + c_i|\phi(0)|^h\phi(0) + b_i \int_{-\infty}^{0} \phi(\theta)\mu(d\theta).$$

Let $f = |x|^h x$ for $x \in \mathbb{R}^d$. As [1, Example 1.2], it is easy to verify that for any $x, y \in \mathbb{R}^d$,

$$\langle x - y, f(x) - f(y)\rangle \ge 0.$$

Recalling that $c_1 = 0$ and $c_1 = -1$, for $U(x) = |x|^2$, we have

$$\begin{aligned}\mathbb{L}_i U(\phi - \psi) &= 2\big\langle\phi(0) - \psi(0), b(\phi, i) - b(\psi, i)\big\rangle + \|\sigma(\phi, i) - \sigma(\psi, i)\|_{\text{HS}}^2 \\ &\le (2a_i + b_i)|\phi(0) - \psi(0)|^2 + \left(b_i + \frac{1}{10}\right)\int_{-\infty}^{0} |\phi(\theta) - \psi(\theta)|^2\mu(d\theta).\end{aligned}$$

Hence

$$\bar{a}(1) = \frac{3}{10}, \quad \bar{a}(2) = -3, \quad \bar{b}(1) = \frac{1}{5}, \quad \bar{b}_2 = \frac{11}{10}, \quad \gamma = -\frac{19}{10}$$

in Assumptions 2.2, 2.1, and 2.2 hold. Furthermore, a direct calculation gives

$$\begin{aligned}\Sigma(\varepsilon) &:= \pi_1\Big(\overline{a}(1) + \overline{b}(1)\int_{-\infty}^{0} e^{\gamma\theta}\mu(d\theta)\Big) + \pi_2\Big(\overline{a}(2) + \overline{b}(2)\int_{-\infty}^{0} e^{\gamma\theta}\mu(d\theta)\Big) \\ &= \frac{1}{1+A}\Big(\frac{143}{210}A - \frac{19}{21}\Big),\end{aligned}$$

which, together with $A \in (0, 594/1955)$ and [3, Proposition 4.2], implies that there exists a $p \in (0, 1]$ such that $\eta_p > 0$. In addition, let $V(x) = |x|^2$. Then we arrive at

$$\mathcal{L}_1 V(\phi) = 2\big\langle\phi(0), b(\phi.1)\big\rangle + \|\sigma(\phi, 1)\|_{\text{HS}}^2 \le L_1 + \frac{3}{10}|\phi(0)|^2 + \frac{3}{10}\int_{-\infty}^{0} |\phi(\theta)|^2\mu(d\theta),$$

$$\mathcal{L}_2 V(\phi) = 2\big\langle\phi(0), b(\phi.2)\big\rangle + \|\sigma(\phi, 2)\|_{\text{HS}}^2 \le L_2 - \frac{13}{10}|\phi(0)|^2 + \frac{3}{10}\int_{-\infty}^{0} |\phi(\theta)|^2\mu(d\theta),$$

where

$$L_1 := 2\|\sigma(0, 1)\|_{\text{HS}}^2, \quad L_2 := \sup_{x\in\mathbb{R}^2}\Big\{-2|x|^{2+h} + \frac{73}{10}|x|^2 + 2\|\sigma(0, 2)\|_{\text{HS}}^2\Big\}.$$

Then Assumption 2.3 holds and

$$\widetilde{a}(1)=\widetilde{b}(1)=\frac{3}{10},\quad \widetilde{a}(2)=-\frac{13}{10},\quad \widetilde{b}(2)=\frac{3}{10},\quad \widetilde{\gamma}_*=-1,\quad \widetilde{\gamma}^*=\frac{3}{5},\quad \widetilde{b}^*=\widetilde{b}_*=\frac{3}{10},$$

and

$$\epsilon:=\frac{\widetilde{b}_*}{\widetilde{b}^*}\int_{-\infty}^{0}e^{\widetilde{\gamma}^*\theta}\mu(d\theta)=\frac{20}{23}.$$

Hence,

$$\widetilde{a}(1)+(1-\epsilon\wedge 1)\widetilde{b}(1)+\widetilde{b}(1)\int_{-\infty}^{0}e^{\widetilde{\gamma}_*\theta}\mu(d\theta)=\frac{17}{23},$$

$$\widetilde{a}(2)+(1-\epsilon\wedge 1)\widetilde{b}(2)+\widetilde{b}(2)\int_{-\infty}^{0}e^{\widetilde{\gamma}_*\theta}\mu(d\theta)=-\frac{99}{115}.$$

Then we have

$$\widetilde{Q}=Q+\mathrm{diag}\left\{\frac{17}{23},-\frac{171}{200}\right\}=\begin{pmatrix}-\frac{6}{23} & 1\\ A & -A-\frac{99}{115}\end{pmatrix}.$$

The determinant of $\widetilde{Q}-\lambda I_{n\times n}$ is given by

$$|\widetilde{Q}-\lambda I_{n\times n}|=\lambda^2+\left(A+\frac{258}{230}\right)\lambda-\frac{17}{23}A+\frac{594}{2645}.$$

Recalling $A\in(0,513/1700)$, we obtain

$$\widetilde{\eta}_1=-\max_{\lambda\in\mathrm{Spec}(\widetilde{Q})}\mathrm{Re}(\lambda)=\frac{\left(A+\frac{258}{230}\right)-\sqrt{\left(A+\frac{258}{230}\right)^2+\frac{68}{23}A-\frac{2376}{2645}}}{2}>0.$$

These show that all the conditions in Theorem 4.1 are satisfied. Hence the results in Theorem 4.1 hold.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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