Stability of Stochastic Functional Differential Equations With Past-Dependent Random Switching Involving Countably Infinite States

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***Abstract*—This work is devoted to stochastic functional differential equations with past-dependent random switching, in which the switching process is allowed to take values in a countable state space. Such processes are rather versatile and arise in a wide variety of applications. A central issue considered in this article is stability. The conditions provided are more general than the existing work. After getting the desired stability results, several applications, including linear systems, linearization of nonlinear systems, multi-agent systems consentability, and networked control systems are examined. Numerical results are also provided to demonstrate our results.**

***Index Terms*—Past-dependent switching, stability, stochastic functional differential equation (SFDE).**

I. Introduction

THIS work develops novel stability results for systems of functional stochastic differential equations (FSDEs) with random switching. The systems under consideration are hybrid switching diffusions that are modeled by stochastic functional differential equations (SFDEs) under random environments. It is widely understood that, intuitively, a functional differential equation is a differential equation for which derivatives of the solution of the equation depend not only on some functions of the solution itself, but also depends on some functions of past history. A SFDE is similar in spirit but has the noise perturbations (e.g., due to the Brownian motion). In this article, our attention is devoted to SFDEs originating from switching diffusions. A switching diffusion has two components $X(t)$ and $\alpha(t)$, in which $X(t)$ is termed a continuous component whereas $\alpha(t)$ is a discrete component. Note that by a discrete component, we mean that $\alpha(t)$ takes values in a discrete set. Compared with the existing results of switching diffusions in the literature, not only does the switching process depend on the continuous dynamics but also it depends on the past history of $X(t)$. The dependence on the past history of the continuous process is in a most general form. In addition, the discrete component $\alpha(t)$ is allowed to take values in a countable set.

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A. Motivation

Why should we be interested in such switching diffusions? There are numerous applications in a wide spectrum of areas in engineering, physical, biological, ecological, social dynamic systems, large-scale optimization, wired and wireless communications, and among others. In biology and ecology, many systems can be formulated under a big roof of the so-called Kolmogorov systems [31], [32]. Moreover, as studied in these references, very often for practical concerns, one needs to look into SFDEs that are equations with delay of the general form. In the aforementioned references, our primary goals are to answer a long-standing question, namely, finding the necessary and sufficient conditions under which the populations are persistent or extinct. It turns out that with more effort regime-switching Kolmogorov systems can be treated as well. In fact, in [30], we treated regime-switching systems of chemostat models with applications to wastewater treatment and obtained the corresponding optimal controls. In everyday life, practical systems are often running for a long time. We remark that many issues, such as permanence, extinction, and persistence, etc., of species in population dynamics and ecology are all linked to the stability issues.

The study of switching diffusions has been much tied-up with recent advances in automatic control and optimization. The quest of jump-linear systems can be found in the work of Mariton [25] for treating linear in the continuous-component processes. Further in-depth investigation on such systems was done by Costa et al. [4], in which quadratic criteria, H_2 and H_∞ controls were treated in detail. The work of Mao and Yuan [24] is devoted to nonlinear systems with Markov switching taking values in a finite state space. Then, Yin and Zhu [40], [46] treated switching diffusions, in which the switching process allows to be dependent on the continuous component. This line of work was substantially extended by Nguyen and Yin [27] with past-dependent switching with a countable state space. One of the centerpieces of the studies is around the issue of long-term asymptotic behavior, in particular, stability. Taking into consideration the coexistence and interactions of discrete and continuous states, regime-switching diffusion systems, which are stochastic hybrid systems and have drawn resurgent and increasing attention; see e.g., [18], [23], [35], [41], [46] and references therein. There is a long history of studying hybrid systems. Feedback controls were considered in [5], modeling and analysis of networked control systems (NCSs) can be found in [12], stability and stabilizability were considered in [14], [15], [19], [38], [41], [42], [43], and [45], consensus formation, control, and stability of stochastic multi-agent systems with or without time delays were treated in [16], [47], [48], and [49]; delay tolerance for stability of stochastic systems was treated in [50].

In light of the recent development, this article focuses on a crucially important issue, namely, stability. It is well-recognized that regime-switching diffusion systems have many interesting properties, which are somewhat counterintuitive. For example, the continuous component can be stable for each fixed regime of the discrete component, but the hybrid system may not be stable, and vice versa; see, [39].

B. Main Innovations of This Article

In this article, we consider the stability of stochastic delay differential equations under past-dependent random switching taking countably infinite states. In contrast to the existing works in the literature, the regime-switching diffusion we considered in this work has following distinct features:

1) not only does the continuous component depend on the discrete component, but also it depends on its past history;
2) in addition, the discrete component depends on the past information of the continuous component; and
3) the discrete component takes values in a countably infinite states space.

The aforementioned setting enlarges the application of switching processes in many perspectives. There are numerous applications in which the switching component is not a Markov chain, but the generator of the switching process depends on the history of the continuous component. As mentioned in [27], application examples may include linear quadratic control problems with random switching that depend on the past history, queueing networks, and evolution of two interacting species, in which the reproduction process is noninstantaneous resulting in past-dependent switching. The examples in Section IV of this article provide additional examples, such as consensus and NCSs, in which it is necessary to consider past-dependent switching. By allowing the two components to be fully coupled, our formulation enables us to treat many systems, in which subsystems and/or components are intertwined and highly coupled. Our formulation also enables us to account for past dependence in a most general form. It is clear that delay or past dependence is unavoidable. Finally, because we allow the switching to take values in an infinite state space, our formulation enlarges the applicability of previous considerations of finite-state space cases.

In contrast to the existing works in the literature, we highlight our contributions and novelties of this work as follows. In particular, we illustrate below why mode-independent Lyapunov function and pth-moment stability with $p < 2$ should be a natural choice.

1) While stochastic systems with past-dependent switching and systems with infinite jumps have been studied in [21], [27], [28], [29], [35], and [44], to the best of authors' knowledge, our article is one of the first providing stability results for stochastic functional (delay) differential equations under past-dependent random switching having infinite states. Combining the two levels of complexity makes the analysis more difficult, especially when the two components are fully coupled. We introduced novel techniques to overcome the difficulty arising. Moreover, if we remove delays in the continuous component, our results generalize the stability results of stochastic differential equations with (infinite states) past-dependent switching in literature, such as [28] and [35] and references therein. If we let switching be independent of the continuous component and have finite values, our results add to the research of stochastic delay systems under Markovian switching with finite states space in the literature, such as [9], [23], [42], [43], and [45] and references therein.
2) When studying this two-component system, we purposely do not work with a mode-dependent Lyapunov function, because it is very difficult in practice to find efficient mode-dependent Lyapunov functions. Normally, the dynamics of the continuous component in different switching modes should not be drastically different, but they should have something in common because it describes the nature of one component. Based on this observation, it is more practical to seek a mode-independent Lyapunov function, which is seemingly less general but it is much easier to select and use in practice.
3) When working with stochastic delay systems, most studies aim for second (or higher) moment stability. The main reason is that functions of the form $|x^\top Qx|^p$ are not used as a Lyapunov function for delay systems. For nondelay systems, they can be used because a solution not starting at 0 will not reach 0 at any finite time (under some mild conditions). It is not the case for delay systems. The pth-moment stability with $p < 2$ may be seemingly weaker than the second moment stability but the conditions for pth-moment stability are much weaker. Because of the technical issue mentioned above, pth-moment stability for $p < 2$ is a natural choice, but the results are very scarce in the literature. This article focuses on tackling this kind of stability with new analysis machinery.
4) While coupling techniques were used in probability theory and stochastic processes, in this article, we bring it into the picture to treat the stability of systems arising in control and systems theory. It brings in a powerful technique with further applications in a wide variety of applications in systems theory and control.

C. Outline of the Article

The rest of this article is organized as follows. Section II provides the precise formation. Section III presents the main results of this article together with the detailed proofs. Section IV, then, presents a number of applications. They include stability of linearized systems, multi-agent systems and related issues, and NCSs. Finally, Section V concludes this article.

II. Formulation and Assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and $W(t)$ be a d-dimensional standard Brownian motion. We denote by X_t the segment function of $X(t)$. That is, $X_t := \{X(t+s) : s \in [-r, 0]\} \in \mathcal{C}$, where $\mathcal{C} := \mathcal{C}([-r, 0], \mathbb{R}^n)$. Let, $b(\cdot,\cdot) : \mathcal{C} \times \mathbb{Z}_+ \to \mathbb{R}^n, \sigma(\cdot,\cdot) : \mathcal{C} \times \mathbb{Z}_+ \to \mathbb{R}^{n\times d}$ be measurable functions. Now, we consider the SFDE under random switching given by

$$\begin{cases} dX(t) = b\left(X_t, \alpha(t)\right) dt + \sigma\left(X_t, \alpha(t)\right) dW(t), \ X(t) \in \mathbb{R}^n \\ X_0 = \phi \in \mathcal{C}, \ \alpha(0) = i \in \mathbb{Z}_+ \end{cases} \tag{1}$$

where $\alpha(t)$ is a switching process taking value in $\mathbb{Z}_+ := \{1, 2, \ldots\}$, depending on X_t with invariant probability measure $\{\nu_k : k \in \mathbb{Z}_+\}$, and generator $Q = (q_{kl}(\cdot))_{k,l\in\mathbb{Z}_+}$, where $q_{ij}(\cdot) : \mathcal{C} \to \mathbb{R}$, and $\alpha(t)$ is formulated as follows. We assume that if $\alpha(t-) := \lim_{s\to t^-} \alpha(s) = i$, then, it can switch to j at t with intensity $q_{ij}(X_t)$. When $q_i(\phi) := \sum_{j=1,j\neq i}^{\infty} q_{ij}(\phi)$ is uniformly bounded in $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$, and $q_i(\cdot)$ and $q_{ij}(\cdot)$ are continuous, one may view the assumption as

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta) = j | \alpha(t) = i, X(s), \alpha(s), s \leq t\} \\ &= q_{ij}(X_t)\Delta + o(\Delta) \text{ if } i \neq j \text{ and} \\ &\mathbb{P}\{\alpha(t+\Delta) = i | \alpha(t) = i, X(s), \alpha(s), s \leq t\} \\ &= 1 - q_i(X_t)\Delta + o(\Delta). \end{aligned} \tag{2}$$

Note that $\alpha(t)$ is not a Markov chain because $Q(\cdot)$ depends on ϕ; it is only a Markov chain for each fixed ϕ.

Denote by $(X^\phi(t), \alpha^i(t))$ the solution of (1) and (2) with the initial data $X_0 := \phi \in \mathcal{C}$ and $\alpha(0) := i \in \mathbb{Z}_+$. In what follows, we denote the solution by $(X(t), \alpha(t))$ for simplifying notation purpose. Denote by $\mathbb{P}_{\phi,i}$ and $\mathbb{E}_{\phi,i}$, respectively, the probability and expectation corresponding to the initial data (ϕ, i).

Remark 2.1: It is noted that depending on how functions b and σ depend on the segment function X_t, we have different types of delays. For example, if b and σ depend on X_t through discrete points, we have discrete delays. If these functions depend on the whole path of X_t in an integral form, we have distributed delay. One way or another, as soon as delay comes into the system, the solutions of the FSDEs are no longer having the Markov property. Only the segment process is Markov. This makes things more difficult, because we have to work with infinite-dimensional systems.

Denote by $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^+)$, where $\mathbb{R}^+ := [0, \infty)$, the space of twice continuously differentiable functions. Suppose $V(x) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^+)$, and $V(x) = 0$ only if $x = 0$. Define

$$(\mathcal{L}_i V)(\phi) = V_x(\phi(0)) b(\phi, i) + \frac{1}{2}\mathrm{tr}\left(V_{xx}(\phi(0))\sigma(\phi, i)\sigma^\top(\phi, i)\right)$$

for $\phi \in \mathcal{C}$, where $\mathrm{tr}(A)$ denotes the trace of A. To proceed, we make the following assumptions on $b(\cdot,\cdot), \sigma(\cdot,\cdot), V(\cdot)$, and $\mathcal{L}_i V$, which will be used through out the rest of this article.

Assumption 2.1: The functions $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are locally Lipschitz continuous, which have linear growth with respect to the first variable.

For each $i, j \in \mathbb{Z}$, $q_{ij}(\cdot)$ is a continuous function on $\mathcal{C}$ and

$$\sup_{i\in\mathbb{N}^*, \phi\in\mathcal{C}} \sum_{j\neq i} q_{ij}(\phi) < \infty.$$

Assumption 2.2: There is a function $V(\cdot) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^+)$ satisfying the following conditions:

1) there exist $c_1, c_2 > 0$, such that
$$c_1|x|^2 \leq V(x) \leq c_2|x|^2 \quad \forall x \in \mathbb{R}^n$$
2) there exist $\Delta_0 > 0, m(i) \geq 0, a(i) \in \mathbb{R}$, and a probability measure μ on $[-r, 0]$, such that if $\|\phi\| \leq \Delta_0$ then
$$(\mathcal{L}_i V)(\phi) \leq a(i) V\left(\phi(0)\right) + m(i) \int_{-r}^{0} V\left(\phi(s)\right) \mu(ds). \tag{3}$$

Assumption 3.3: There exist constants $\tilde{C} > 0, \tilde{q} \in [0, 1]$, and a probability measure $\tilde{\mu}$ on $[-r, 0]$, such that

$$\begin{aligned} &|b(\phi, i)|^2 + \mathrm{tr}(\sigma(\phi, i)\sigma^\top(\phi, i)) \\ &\leq \tilde{C}\left(|\phi(0)|^2 + \int_{-r}^{0} |\phi(0)|^{2\tilde{q}} |\phi(s)|^{2(1-\tilde{q})} \tilde{\mu}(ds)\right). \end{aligned}$$

Remark 2.2: These conditions are natural. A simple and promising candidate function satisfying Assumptions 2.1, 2.2, and 2.3 is $V(x) = |x|^2$. Compared with the conditions in classical results, our conditions are in a milder form. First, we allow the coefficients $(a(i), m(i))$ in the Lyapunov condition depend on i, i.e., we do not need uniform estimates. It is often verifiable, but also more appropriate to reflect the nature of switching systems. Second, appearances of measures $\mu, \tilde{\mu}$ in conditions also enlarge the applicability of our results. By choosing $\mu, \tilde{\mu}$ to be Dirac measures, it recovers classical conditions.

III. Main Results

Our main purpose is to provide conditions for stability in the case that $\alpha(t)$ depends on the past information X_t of the continuous component $X(t)$. The arrangement of this section is as follows. In Section III-A, we first establish sufficient conditions for stability when $\alpha(t)$ is a Markov chain itself. That is, the Q is independent of the continuous component and the past history. Even in this case, our results are new because stability conditions of stochastic delay systems under random switching were given for the switching taking values in finite state space only so far. Later, in Section III-B, we consider the case of past-dependent switching with the help of coupling method to handle the past dependence and the results developed in Section III-A.

A. Markovian Switching

In this section, we assume $q_{k,l}$ with $k, l \in \mathbb{Z}_+$ is a constant independent of t and the continuous states. The resulting switching process is a Markov chain with stationary transition probabilities and the switching process $\alpha(t)$ is independent of $W(t)$. Denote by $p_{ij}(t)$ the transition probabilities from state i to state j at time t. Let, $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ be the filtrations generated by $W(t)$ and $\alpha(t)$, respectively. We recall the following definitions.

Definition 3.1: The Markov chain $\alpha(t)$ is said to be

1) *ergodic* if it has an invariant probability measure $\nu = (\nu_1, \nu_2, \ldots)$ (a row vector) satisfying

$$\lim_{t\to\infty} p_{ij}(t) = \nu_j \geq 0 \quad \forall i, j \in \mathbb{Z}_+$$

or equivalently

$$\lim_{t\to\infty} \sum_{j\in\mathbb{Z}_+} |p_{ij}(t) - \nu_j| = 0 \quad \forall i \in \mathbb{Z}_+$$

2) *strongly ergodic* if

$$\lim_{t\to\infty} \sup_{i\in\mathbb{Z}_+} \left\{ \sum_{j\in\mathbb{Z}_+} |p_{ij}(t) - \nu_j| \right\} = 0.$$

Let, V and Δ_0 be as in Assumption 2.2. For each, $\Delta \in (0, \Delta_0]$, define

$$\tau_\Delta := \inf\{t \geq 0 : V(X(t)) \geq \Delta\}. \tag{4}$$

We also let

$$\gamma := 0 \wedge \inf_{i\in\mathbb{Z}_+} \{a(i) + m(i)\}$$

and

$$\overline{\mu} := \int_{-r}^{0} e^{\gamma s} \mu(ds).$$

Remark 3.1: Note that another equivalent way of stating the ergodicity is that the system of equations

$$\nu_i \geq 0 \text{ for each } i,\ \nu Q = \nu,\ \nu\mathbb{1} = 1 \text{ with } \mathbb{1} = (1, 1, \ldots)^\top$$

has a unique solution, where $\nu_i > 0$ for each i, and $\mathbb{1}$ is a column vector with all component being 1.

Assumption 3.1: Either

1) $\alpha(t)$ is strongly ergodic; or
2) $\alpha(t)$ is ergodic with an additional condition that $\limsup_{i\to\infty}(a(i) + m(i)\overline{\mu}) < 0$.

With the condition given above, we proceed to obtain the main results of this section. It gives a stability criterion.

Theorem 3.1: Let, Assumptions 2.1, 2.2, 2.3, and 3.1 be satisfied, and assume $a(i) + m(i)\overline{\mu}$ bounded in $i \in \mathbb{Z}_+$ and

$$\sum_{i\in\mathbb{Z}_+} (a(i) + m(i)\overline{\mu})\, \nu_i < 0. \tag{5}$$

Then, for any $\epsilon > 0$, with τ_Δ defined in (4), there exists a $\delta > 0$, such that

$$\mathbb{P}_{\phi,i}\{\tau_\Delta = \infty\} \geq 1 - \epsilon \text{ if } \|\phi\| \leq \delta$$

and

$$\mathbb{P}_{\phi,i}\left\{\frac{\ln\|X_t\|}{t} < -\lambda\right\} \geq 1 - \epsilon \text{ if } \|\phi\| \leq \delta \tag{6}$$

for some $\lambda > 0$ independent of ϵ, δ.

Proof of Theorem 3.1: We begin with the following auxiliary results. Lemma 3.1 indicates the relation between the segment function X_t and the solution of the FSDE $X(t)$, whereas Lemma 3.2 is an estimate of log-Laplace transform of a random variable.

Lemma 3.1: ([9, Lemma 5]) There exists a constant $0 < H_1 < \infty$ independent of Δ, such that

$$\begin{aligned}&\mathbb{E}(\mathbf{1}_{\{\tau_\Delta > t+r\}}\|X_{t+r}\|^2|\mathcal{F}_\infty^2)\\ &\quad\leq H_1 \max_{s\in[t-r,t+r]} \mathbb{E}(\mathbf{1}_{\{\tau_\Delta > s\}}|X(s)|^2|\mathcal{F}_\infty^2) \quad \forall t \geq 0.\end{aligned}$$

Lemma 3.2: ([28, Lemma 3.4]) Let, Y and $\theta_0 > 0$ be a random variable and a constant, respectively. Suppose, that there is a $K_1 > 0$, such that

$$\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}(-\theta_0 Y) \leq K_1.$$

Then, the log-Laplace transform $\phi(\theta) = \ln\mathbb{E}\exp(\theta Y)$ is twice differentiable on $[0, \frac{\theta_0}{2})$, and

$$\frac{d\phi}{d\theta}(0) = \mathbb{E}Y, \text{ and } 0 \leq \frac{d^2\phi}{d\theta^2}(\theta) \leq K_2, \quad \text{for any } \theta \in \left[0, \frac{\theta_0}{2}\right)$$

for some $K_2 > 0$. Moreover,

$$\phi(\theta) \leq \theta\mathbb{E}Y + \theta^2 K_2 \quad \text{for all } \theta \in \left[0, \frac{\theta_0}{2}\right).$$

To proceed, we establish a moment bound for the segment function by the following Propositions 3.1 and 3.2, in which parts (i) and (ii) of Assumption 3.1 are assumed, respectively.

Proposition 3.1: Suppose, the Markov chain $\alpha(t)$ is strongly ergodic with invariant probability measure $\nu := (\nu_1, \nu_2, \ldots)$, and (5) holds.

There exist $p^* > 0$, $C^* > 0$, $m^* > 0$, such that

$$\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta > t\}}\|X_t\|^{2p^*} \leq C^*\exp\{-m^* t\}\|\phi\|^{2p^*}$$

for any $t \geq 0$ and $\|\phi\| \leq \Delta_0$.

Proof: Define

$$\gamma := 0 \wedge \min_{i\in\mathbb{Z}_+}\{a(i) + m(i)\} \text{ and}$$

$$\overline{\mu} := \max_{t\geq 0} \int_{-r}^{0} e^{\gamma s}\mu_t(ds).$$

Then, $\sum_{i\in\mathbb{Z}_+}(a(i) + m(i)\overline{\mu})\nu_i < 0$ yields $\gamma < 0$.

Let, $c(i) := a(i) + \epsilon_0 + m(i)\overline{\mu}$, where ϵ_0 small enough, such that $\lambda := \sum_{i\in\mathbb{Z}} c(i)\nu_i < 0$ and $c(i)$ is bounded. For each $X_0 = \phi$, $\alpha(0) = i$, for $s \in [-r, 0]$ letting $\alpha(s) = i$, for any $t \geq -r$, we consider the function

$$G(t) := \exp\left\{-\int_0^t c(\alpha(s))ds\right\}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta > t\}}V(X(t))\,|\mathcal{F}_\infty^2].$$

Let, $H(t) := \sup_{s\in[t-r,t]} G(s)$. It was proved in [9, Proposition 6], that $H(\cdot)$ is nonincreasing. Therefore,

$$\begin{aligned}&\exp\left\{-\int_0^t c(\alpha(s))\,ds\right\}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta > t\}}V(X(t))\,|\mathcal{F}_\infty^2]\\ &\leq \sup_{u\in[0,r]}\left\{\exp\left\{-\int_0^u c(\alpha(s))\,ds\right\}\right.\\ &\left.\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta > t\}}V(X(u))\,|\mathcal{F}_\infty^2]\right\} \leq K\|\phi\|^2 \quad \forall t \geq 0\end{aligned}$$

where $K := c_2 e^{r \max_{i\in\mathbb{Z}_+} c(i)}$. This implies that

$$\mathbb{E}_{\phi,i}[V(X(t))|\mathcal{F}_\infty^2] \leq K \exp\left\{\int_0^t c(\alpha(s))ds\right\} \sup_{u\in[-r,0]} \|V(\phi)\|^2.$$

Thus,

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^2|\mathcal{F}_\infty^2] \leq K' \exp\left\{\int_0^t c(\alpha(s))ds\right\}\|\phi\|^2$$

where $K' := \frac{H_1 K c_2}{c_1}$.

By virtue of Jensen's inequality for $p \in (0,1)$, we obtain

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^{2p}|\mathcal{F}_\infty^2] \leq K'\|\phi\|^{2p} \exp\left\{p\int_0^t c(\alpha(s))ds\right\}. \tag{7}$$

Now, since $\alpha(t)$ is strongly ergodic, that is, $\lim_{t\to\infty} \sup_{i\in\mathbb{Z}_+}\{\sum_{j\in\mathbb{Z}_+}|p_{ij}(t)-\nu_j|\} = 0$. Thus, $\frac{1}{t}\mathbb{E}_i\int_0^t c(\alpha(s))ds = \frac{1}{t}\int_0^t \sum_{j\in\mathbb{Z}_+} c(j)p_{ij}(s)ds$ converges uniformly in $i \in \mathbb{Z}_+$ to $\lambda = \sum_{j\in\mathbb{Z}_+} c(j)\nu_j$. As a result, there exists a $T > 0$, such that for any $t \geq T$ and $i \in \mathbb{Z}_+$, we have

$$\mathbb{E}_i\int_0^t c(\alpha(s))\,ds \leq -\frac{\lambda}{2}t < 0.$$

By virtue of Lemma 3.2, for $\int_0^t c(\alpha(s))ds$, we have

$$\begin{aligned}\ln \mathbb{E}_i \exp\left\{p\int_0^t c(\alpha(s))\,ds\right\} &\leq p\mathbb{E}_i\int_0^t c(\alpha(s))\,ds + p^2K \\ &\leq -p\frac{\lambda t}{2} + p^2K\end{aligned}$$

for some $K > 0$. We choose $p \in (0,1)$, such that $pK < \dfrac{\lambda t}{4}$ for $t \geq T$. As a consequence, we obtain

$$\ln \mathbb{E}_i \exp\left\{p\int_0^t c(\alpha(s))ds\right\} \leq \frac{-p\lambda t}{4}, t \geq T$$

which means

$$\mathbb{E}_i \exp\left\{p\int_0^t c(\alpha(s))ds\right\} \leq \exp\left\{\frac{-p\lambda t}{4}\right\}.$$

This implies that

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^{2p}|\mathcal{F}_\infty^2] \leq K'\|\phi\|^{2p}\exp\left\{-\frac{p\lambda t}{4}\right\}, t \geq T.$$

The proof is complete with $m^* = \frac{p\lambda}{4}$ and $C^* = K' \exp\{T\max_{j\in\mathbb{Z}_+}\{c(j)\} + \frac{p\lambda T}{4}\}$. □

Proposition 3.2: Suppose, the Markov chain $\alpha(t)$ is ergodic, $\limsup_{i\to\infty}(a(i) + m(i)\overline{\mu}) < 0$ and (5) holds. Then, the conclusion of Proposition 3.1 holds.

Proof: Note that (7) in Proposition 3.1 holds for any Markov chain $\alpha(t)$.

We still use $c(i) := a(i) + \epsilon_0 + m(i)\overline{\mu}$, with ϵ_0 being chosen small enough, such that $\lambda = \sum_{j\in\mathbb{Z}_+} c(j)\nu_j < 0$ and $\limsup_{j\to\infty} c(j) < 0$. With that choice, there exists a k_0, such that for all $i > k_0$

$$c(i) < -\varepsilon_1 < 0$$

for some $\varepsilon_1 > 0$. Moreover, from the fact that $\{c(j)\}_{j\in\mathbb{Z}_+}$ is bounded, we obtain

$$\lim_{k\to\infty}\sum_{j\leq k} c(j)\nu_j = \sum_{j\in\mathbb{Z}_+} c(j)\nu_j < 0.$$

Let $\overline{c} := \sup_{i\in\mathbb{Z}_+}|c(i)|$, $\lambda := -\sum_{i\in\mathbb{Z}_+} c(i)\nu_i > 0$, and n_* be a positive integer, such that

$$n_* \geq \max\left\{\frac{\lambda}{3\varepsilon_1} - 1, \frac{1}{\varepsilon_1}\left(\frac{\lambda}{3} + \overline{c}\right)\right\}.$$

Since $\alpha(t)$ is ergodic, there exists a $T > 0$ depending on k_0, such that for any $t \geq T$ and $i \leq k_0$

$$\mathbb{E}_i\int_0^t c(\alpha(s))ds \leq -\frac{3\lambda}{4}t.$$

Now, let

$$\tau := \inf\{s : \alpha(s) \leq k_0\}.$$

Then, for any $i \geq k_0$

$$\begin{aligned}
&\mathbb{E}_i\int_0^{(n_*+1)T} c(\alpha(s))ds \\
&= \mathbb{E}_i\left(\int_0^{\tau\wedge(n_*+1)T} c(\alpha(s))ds + \int_{\tau\wedge(n_*+1)T}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&= \mathbb{E}_i\mathbf{1}_{\{\tau\leq n_*T\}}\left(\int_0^{\tau\wedge(n_*+1)T} c(\alpha(s))ds\right. \\
&\quad \left.+ \int_{\tau\wedge(n_*+1)T}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&\quad + \mathbb{E}_i\mathbf{1}_{\{n_*T\leq\tau\leq(n_*+1)T\}}\left(\int_0^{\tau\wedge(n_*+1)T} c(\alpha(s))ds\right. \\
&\quad \left.+ \int_{\tau\wedge(n_*+1)T}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&\quad + \mathbb{E}_i\mathbf{1}_{\{\tau\geq(n_*+1)T\}}\left(\int_0^{\tau\wedge(n_*+1)T} c(\alpha(s))ds\right. \\
&\quad \left.+ \int_{\tau\wedge(n_*+1)T}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&= \mathbb{E}_i\left(\int_0^{\tau} c(\alpha(s))ds + \int_{\tau}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&\quad + \mathbb{E}_i\left(\int_0^{n_*T} c(\alpha(s))ds + \int_{n_*T}^{(n_*+1)T} c(\alpha(s))ds\right) \\
&\quad + \mathbb{E}_i\left(\int_0^{(n_*+1)T} c(\alpha(s))ds\right) \\
&\leq \mathbb{E}_i\left(-\varepsilon_1\tau + \int_{\tau}^{(n_*+1)T} c(\alpha(s))ds\right)
\end{aligned}$$

$$+\mathbb{E}_i\left(-\varepsilon_1(n_*T)+T\overline{c}\right)+\mathbb{E}_i\left(-\varepsilon_1(n_*+1)T\right)$$
$$\leq\left(-\frac{\lambda}{9}T\right)+\left(-\frac{\lambda}{9}T\right)+\left(-\frac{\lambda}{9}T\right)\leq-\frac{\lambda}{3}T.$$

Applying Lemma 3.2 for $\int_0^T c(\alpha(s))ds$ yields

$$\ln\mathbb{E}_i\exp\left\{p\int_0^T c(\alpha(s))ds\right\}\leq p\mathbb{E}_i\int_0^T c(\alpha(s))ds+p^2K$$
$$\leq -p\frac{\lambda T}{3}+p^2K$$

for some $K>0$. We choose $p\in(0,1)$, such that $pK<\dfrac{\lambda T}{4}$.

As a consequence, we obtain

$$\ln\mathbb{E}_i\exp\left\{p\int_0^T c(\alpha(s))ds\right\}\leq\frac{-p\lambda T}{12}$$

or

$$\mathbb{E}_i\exp\left\{p\int_0^T c(\alpha(s))ds\right\}\leq\exp\left\{\frac{-p\lambda T}{12}\right\}.$$

Applying Markov's property of $\alpha(t)$, we can easily imply

$$\mathbb{E}_i\exp\left\{p\int_0^{\ell T} c(\alpha(s))ds\right\}\leq\exp\left\{\frac{-p\lambda\ell T}{12}\right\},\ell\in\mathbb{Z}_+. \quad (8)$$

Since $\{c(i),i\in\mathbb{Z}_+\}$ is bounded, for each $t>0$, we deduce from (8) that

$$\mathbb{E}_i\exp\left\{p\int_0^t c(\alpha(s))ds\right\}\leq\check{K}\exp\left\{\frac{-p\lambda t}{12}\right\}$$

with $\check{K}=e^{pT\sup_{j\in\mathbb{Z}_+}\{|c(j)|\}}$. Applying this to (7) (which holds even if $\alpha(t)$ is not strongly ergodic) yields

$$\mathbb{E}_{\phi,i}[1_{\{\tau_\Delta>t\}}\|X_t\|^{2p}|\mathcal{F}^2_\infty]\leq K'\check{K}\|\phi\|^{2p}\exp\left\{-\frac{p\lambda t}{12}\right\}.$$

The proof is complete. □

Now, we have the following estimation for τ_Δ that will help us to show $\tau_\Delta=\infty$ with large probability later.

Lemma 3.3: Under the hypotheses of Theorem 3.1, for sufficiently small $p>0$ and any $T>0$, there exists a constant H_2 depending on p and T, such that

$$\mathbb{P}_{\phi,i}\{\tau_\Delta\leq T\}\leq H_2\frac{\|\phi\|^{2p}}{\Delta^p},\quad(\phi,i)\in\mathcal{C}\times\mathbb{Z}_+.$$

Proof: Let $\widetilde{c}=\sup\{|a_i|+b_i,i\in\mathbb{Z}_+\}$. Define

$$\widetilde{G}(t)=e^{-\widetilde{c}t}\mathbb{E}_{\phi,i}V(X_{t\wedge\tau_\Delta})\text{ and }\widetilde{H}(t)=\sup_{s\in[t-r,t]}\widetilde{G}(s).$$

Similar with the proof of Proposition 3.1 we obtain that $\widetilde{H}(t)$ is a nonincreasing function. Therefore,

$$e^{-\widetilde{c}t}\mathbb{E}_{\phi,i}V(X_{t\wedge\tau_\Delta})\leq\sup_{u\in[0,r]}\left\{e^{-\widetilde{c}u}\mathbb{E}_{\phi,i}V(X_{t\wedge\tau_\Delta})\right\}\leq H_1\|\phi\|^2.$$

Hence, for any $T>0$, one has

$$\mathbb{E}_{\phi,i}V(X_{T\wedge\tau_\Delta},T\wedge\tau_\Delta)\leq H_2\|\phi\|^2 \quad (9)$$

for some constant $H_2>0$. As a consequence

$$\mathbb{P}_{\phi,i}\{\tau_\Delta\leq T\}\leq\frac{\mathbb{E}_{\phi,i}V^p(X_{T\wedge\tau_\Delta},T\wedge\tau_\Delta)}{\Delta^p}\leq H_2\frac{\|\phi\|^{2p}}{\Delta^p} \quad (10)$$

for $(\phi,i)\in\mathcal{C}\times\mathbb{Z}_+$. □

Now, with the estimates in Proposition 3.1 (or Proposition 3.2) and Lemma 3.3, by the arguments as in the proof of [9, Theorem 8], we obtain

$$\mathbb{P}_{\phi,i}\{\tau_\Delta=\infty\}\geq1-\epsilon\text{ if }\|\phi\|\leq\delta$$

which yields the stability result (6). □

B. Past-Dependent Switching

Now, by combining the results in the previous section and a coupling argument, we are in a position to provide conditions for stability when $\alpha(t)$ is past-dependent.

In this section, we generalize the results obtained thus far by allowing the switching process $\alpha(t)$ to be past dependent. That is, the generator of $\alpha(t)$ is $Q(X_t)=\{q_{ij}(X_t)\}$ depending on X_t, i.e.,

$$\mathbb{P}\{\alpha(t+\Delta)=j|\alpha(t)=i,X_s,\alpha(s),s\leq t\}$$
$$=q_{ij}(X_t)\Delta+o(\Delta)\text{ if }i\neq j\text{ and}$$
$$\mathbb{P}\{\alpha(t+\Delta)=i|\alpha(t)=i,X_s,\alpha(s),s\leq t\}$$
$$=1-q_i(X_t)\Delta+o(\Delta). \quad (11)$$

Assumption 3.2: $q_{ij}(\phi)$ is bounded uniformly in i,j,ϕ satisfying

$$\sum_{j\in\mathbb{Z}_+,j\neq k}|q_{kj}(\phi)-q_{kj}(\mathbf{0})|\leq c_0\|\phi\|^\lambda,\quad k\in\mathbb{Z}_+$$

for some constant $c_0>0$ and $\lambda\in(0,1)$. Suppose further that a Markov chain $\widetilde{\alpha}(t)$ with generator $Q(\mathbf{0})$ is ergodic with the invariant measure $(\nu_1,\nu_2,\ldots)$.

Consider, $\widetilde{X}(t)$ a hybrid diffusion satisfies an equation similar to $X(t)$ but with $\alpha(t)$ replaced by the Markov chain $\widetilde{\alpha}(t)$. Now, consider the joint process $(X(t),\widetilde{X}(t),\alpha(t),\widetilde{\alpha}(t))$ with the basic coupling method (see e.g., [2, p. 11]), that is $(X(t),\widetilde{X}(t))$ satisfies

$$\begin{cases}dX(t)=b(X_t,\alpha(t),t)dt+\sigma(X_t,\alpha(t),t)dW(t)\\ d\widetilde{X}(t)=b(\widetilde{X}_t,\widetilde{\alpha}(t),t)dt+\sigma(\widetilde{X}_t,\widetilde{\alpha}(t),t)dW(t)\end{cases} \quad (12)$$

and the coupled pair $(\alpha(t), \widetilde{\alpha}(t))$ has the generator $\widetilde{Q}(X_t) = \{\widetilde{q}_{(kl)(ij)}(X_t)\}$, which is defined by

$$\widetilde{q}_{(kl)(ij)}(\phi) = \begin{cases} q_{ki}(\phi) \wedge q_{li}(\mathbf{0}) & \text{if } i = j \,\&\, (i,j) \neq (k,l) \\ [q_{ki}(\phi) - q_{li}(\mathbf{0})]^+ & \text{if } i \neq k \,\&\, l = j \\ [q_{lj}(\mathbf{0}) - q_{kj}(\phi)]^+ & \text{if } l \neq j \,\&\, i = k \\ 0 & \text{if } k \neq i \,\&\, l \neq j \\ -\sum_{(i',j')\neq(k,l)} q_{(k,l)(i',j')} & \text{if } (k,l) = (i,j) \end{cases} \tag{13}$$

where $[q_{ki}(\phi) - q_{li}(\mathbf{0})]^+ = \max\{|q_{ki}(\phi) - q_{li}(\mathbf{0})|, 0\}$, $[q_{lj}(\mathbf{0}) - q_{kj}(\phi)]^+ = \max\{|q_{lj}(\mathbf{0}) - q_{kj}(\phi)|, 0\}$.

Assume that $\alpha(0) = \widetilde{\alpha}(0) = i, X_0 = \widetilde{X}_0 = \phi$ and define

$$\xi = \inf\{t \geq 0 : \alpha(t) \neq \widetilde{\alpha}(t)\}. \tag{14}$$

Lemma 3.4: Assume Assumptions 2.1, 2.2, and 2.3. Let 3.2 and inequality (5) be satisfied with $(\nu_1, \nu_2, \ldots)$ as in Assumption 3.2. Assume further that $\widetilde{\alpha}(t)$ is strongly ergodic or $\limsup_{i\to\infty} c(i) < 0$. There is an $H_4 > 0$, independent of i, ϕ, and Δ, such that

$$\mathbb{P}_{\phi,i}\{\xi < \infty\} < \frac{H_4}{\Delta^{p^*}}\|\phi\|^{2p^*}.$$

Proof: Pick any $T^* > 0$ (which can be 1 for instance). Because $\widetilde{\alpha}(t)$ is an ergodic Markov chain, we can apply Proposition 3.2, Lemma 3.3, and Theorem 3.1 to the process $(\widetilde{X}(t), \widetilde{\alpha}(t))$. In particular, there is a $p^* \in (0, \lambda/2)$, such that

$$\mathbb{E}_{\phi,i}\mathbf{1}_{\{\widetilde{\tau}_\Delta > t\}}\|\widetilde{X}_t\|^{2p^*} \leq C^* \exp\{-m^* t\}\|\phi\|^{2p^*} \tag{15}$$

and

$$\mathbb{P}_{\phi,i}\{\widetilde{\tau}_\Delta \leq T^*\} \leq H_2 \frac{\|\phi\|^{2p^*}}{\Delta^{p^*}}, (\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ \tag{16}$$

where $\widetilde{\tau}_\Delta := \inf\{t \geq 0 : \|\widetilde{X}_t\| \geq \Delta\}$ and H_2 depending only on p^* and T^*. To obtain the desired result for $(X(t), \alpha(t))$, we show that

$$\mathbb{P}_{\phi,i}\{\alpha(t) = \widetilde{\alpha}(t), t \geq 0\} > 1 - \epsilon$$

if $\|\phi\|$ is sufficiently small. This property is usually true only for a finite time interval. But, using the exponentially convergence of $\widetilde{X}(t)$ to 0, we can get a bound for $[0, \infty)$.

Since $\alpha(t)$ and $\widetilde{\alpha}(t)$ are the same up to ξ, we have $X_t = \widetilde{X}_t$ for $t \leq \xi$. We also have that

$$\mathbb{E}_{\phi,i} \sup_{0 \leq t \leq T \wedge \tau_\Delta \wedge \xi} \{|X(t) - \widetilde{X}(t)|^2\} \leq K|x - \widetilde{x}|^2. \tag{17}$$

Let $\widetilde{f}(k,l) = \mathbf{1}_{\{k \neq l\}}$, where $\mathbf{1}_A$ is the indicator of the set A. By the definition of the function $\widetilde{f}$ and (13), we have

$$\begin{aligned} \widetilde{Q}(x)\widetilde{f}(k,k) &= \sum_{j \in \mathbb{Z}_+, j \neq k} [q_{kj}(x) - q_{kj}(\mathbf{0})]^+ \\ &\quad + \sum_{j \in \mathbb{Z}_+, j \neq k} [q_{kj}(\mathbf{0}) - q_{kj}(x)]^+ \\ &= \sum_{j \in \mathbb{Z}_+, j \neq k} |q_{kj}(x) - q_{kj}(\mathbf{0})| =: \rho(x, k). \end{aligned} \tag{18}$$

Applying the generalized Itô formula to (12) and noting that $X(t) = \widetilde{X}(t), \alpha(t) = \widetilde{\alpha}(t)$ for $t < \xi$, we obtain that

$$\begin{aligned} &\mathbb{P}_{\phi,i}\{\xi \wedge \widetilde{\tau}_\Delta \leq T^*\} \\ &= \mathbb{E}_{\phi,i}\widetilde{f}\left(\alpha(T \wedge \widetilde{\tau}_\Delta \wedge \xi), \widetilde{\alpha}(T^* \wedge \widetilde{\tau}_\Delta \wedge \xi)\right) \\ &= \mathbb{E}_{\phi,i}\int_0^{T^* \wedge \widetilde{\tau}_\Delta \wedge \xi} \widetilde{Q}(X_t, \widetilde{X}_t)\widetilde{f}(\alpha(t), \widetilde{\alpha}(t))dt \\ &= \mathbb{E}_{\phi,i}\int_0^{T \wedge \widetilde{\tau}_\Delta \wedge \xi} \rho(X(t), \widetilde{X}(t), \alpha(t))dt \\ &\leq \mathbb{E}_{\phi,i}\int_0^{\widetilde{\tau}_\Delta \wedge T^* \wedge \xi} \|X_t\|^\lambda dt \leq \mathbb{E}_{\phi,i}\int_0^{T^*} \mathbf{1}_{\{\widetilde{\tau}_\Delta > t\}}\|\widetilde{X}_t\|^\lambda dt \\ &\leq \mathbb{E}_{\phi,i}\int_0^{T^*} \mathbf{1}_{\{\widetilde{\tau}_\Delta > t\}}\|\widetilde{X}_t\|^{2p^*} dt \\ &\leq C^*\|\phi\|^{2p^*}\int_0^{T^*} \exp(-m^* t)dt \text{ (due to (15))} \\ &\leq \frac{C^*}{m^*}\|\phi\|^{2p^*}. \end{aligned} \tag{19}$$

As a consequence of (19) and (16),

$$\mathbb{P}_{\phi,i}\{\xi \wedge \widetilde{\tau}_\Delta \leq T^*\} \leq \left(\frac{C^*}{m^*} + \frac{H_2}{\Delta^{p^*}}\right)\|\phi\|^{2p^*} \leq \frac{H_3}{\Delta^{p^*}}\|\phi\|^{2p^*} \tag{20}$$

for $0 < \Delta < 1$ and some $H_3 = H_3(T^*, p^*)$ that is independent of ϕ and Δ. We have the estimate

$$\begin{aligned} &\mathbb{P}_{\phi,i}\{kT^* < \xi \wedge \widetilde{\tau}_\Delta \leq (k+1)T^*\} \\ &= \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\xi \wedge \widetilde{\tau}_\Delta > kT^*\}}\mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\xi \wedge \widetilde{\tau}_\Delta \leq (k+1)T^*\}}\middle|\mathcal{F}_{kT^*}\right]\right] \\ &= \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\xi \wedge \widetilde{\tau}_\Delta > kT^*\}}\mathbb{E}_{\widetilde{X}_{kT^*}, \widetilde{\alpha}(kT^*)}\mathbf{1}_{\{\xi \wedge \widetilde{\tau}_\Delta \leq T^*\}}\right] \\ &\leq \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\xi \wedge \widetilde{\tau}_\Delta > kT^*\}}\frac{H_3}{\Delta^{p^*}}\|\widetilde{X}_{kT^*}\|^{2p^*}\right] \\ &\leq \frac{H_3}{\Delta^{p^*}}\mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\widetilde{\tau}_\Delta > kT^*\}}\|\widetilde{X}_{kT^*}\|^{2p^*}\right] \\ &\leq \frac{H_3\|\phi\|^{2p^*}}{\Delta^{p^*}}C^* \exp\{-m^* kT^*\}. \end{aligned} \tag{21}$$

In (21), the second equality is due to the strong Markov property of the coupled process $(X(t), \widetilde{X}(t), \alpha(t), \widetilde{\alpha}(t))$; the first and third inequalities are because of (20) and (15) respectively. Hence,

$$\begin{aligned} \mathbb{P}_{\phi,i}\{\xi < \infty\} &\leq \mathbb{P}_{\phi,i}\{\xi \wedge \widetilde{\tau}_\Delta < \infty\} \\ &= \sum_{k=0}^{\infty}\mathbb{P}_{\phi,i}\{kT < \xi \wedge \widetilde{\tau}_\Delta \leq (k+1)T\} \\ &\leq \frac{H_3\|\phi\|^{2p^*}}{\Delta^{p^*}}\sum_{k=0}^{\infty}C^* \exp\{-m^* kT^*\} \\ &\leq \frac{H_4\|\phi\|^{2p^*}}{\Delta^{p^*}} \end{aligned}$$

for any $0 < \Delta < 1$ with $H_4 = H_3\sum_{k=0}^{\infty}C^* \exp\{-m^* kT^*\}$, the lemma is proved. $\square$

With the preparation above, we obtain the following result. It bridges the result of Theorem 3.1 under Markovian switching to the past-dependent case.

Theorem 3.2: Let Assumptions 2.1, 2.2, 2.3, and 3.2 be satisfied, and inequality (5) hold with $(\nu_1, \nu_2, \ldots)$ as in Assumption 3.2. Assume further that $\widetilde{\alpha}(t)$ is strongly ergodic or $\limsup_{i\to\infty} c(i) < 0$. Then, for any $\epsilon > 0$, with τ_Δ defined in (4), there exists a $\delta > 0$, such that

$$\mathbb{P}_{\phi,i}\{\tau_\Delta = \infty\} \geq 1 - \epsilon \text{ if } \|\phi\| \leq \delta$$

and

$$\mathbb{P}_{\phi,i}\left\{\frac{\ln \|X_t\|}{t} < -\lambda\right\} \geq 1 - \epsilon \text{ if } \|\phi\| \leq \delta \tag{22}$$

for some $\lambda > 0$ independent of ϵ, δ.

Proof: The proof follows directly from the fact that the conclusion holds for $(\widetilde{X}(t))$ due to Theorem 3.1 and Lemma 3.4, which shows that $X(t) = \widetilde{X}(t) \forall t$ with a probability no less that $1 - \frac{H_4}{\Delta^{p^*}}\|\phi\|^{2p^*}$. □

IV. Applications

A. Linear Systems and Linearization

By linear systems and linearization we are referring to linear w.r.t. the continuous variable x. Such linear systems or switching linear systems and/or jump linear systems and the corresponding controlled systems have been the central focus for a long period of time; see [4] and [25] among others. In this section, we assume that $W(t) = (W_1(t), \ldots, W_d(t))^\top$ is a d-dimensional Brownian motion; $\alpha(t)$ is a switching process taking values in $\mathbb{Z}_+$ and having the generator $Q(X(t)) = (q_{k,l}(X(t)))_{k,l\in\mathbb{Z}_+}$ satisfying Assumption 3.2.

1) Linear Stochastic Differential-Difference Equations With Random Switching: *Example 4.1:* Consider a linear stochastic delay differential equation with regime-switching of the form

$$dX(t) = A(\alpha(t))X(t)dt + \sum_{j=1}^{d} C_j(\alpha(t))X(t-r)dW_j(t) \tag{23}$$

where $X(t) \in \mathbb{R}^n$; $A(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$; $C_j(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$, and $j = 1, \ldots, d$. Let, $\Sigma(i) := \sum_{j=1}^{d} C(i)C^\top(i)$. For a symmetric matrix $U \in \mathbb{R}^{n\times n}$, let

$$\Lambda^M(U) := \sup\{x^\top Ux : x \in \mathbb{R}^n, |x| = 1\}$$

and

$$a(i) := \Lambda^M(A^\top(i) + A(i)),\ m(i) := \Lambda^M(\Sigma(i)).$$

Define $V(x) = |x|^2$. Then, we have

$$\begin{aligned}
&(\mathcal{L}_i V)(\phi)\\
&= (\phi(0))^\top(A^\top(i) + A(i))\phi(0) + (\phi(-r))^\top\Sigma(i)\phi(-r)\\
&\leq \left(\Lambda^M(A^\top(i) + A(i))\right)|\phi(0)|^2 + m(i)|\phi(-r)|^2.
\end{aligned}$$

If we assume $\widetilde{\alpha}(t)$, a Markov chain with generator $Q(0)$, is strongly ergodic and Assumption 3.2 is satisfied, by Theorem 3.2 with condition (5) with $a(i), m(i)$ defined above and μ, $\tilde{\mu}$ being measures concentrating on $\{-r\}$, (23) is exponentially stable in probability.

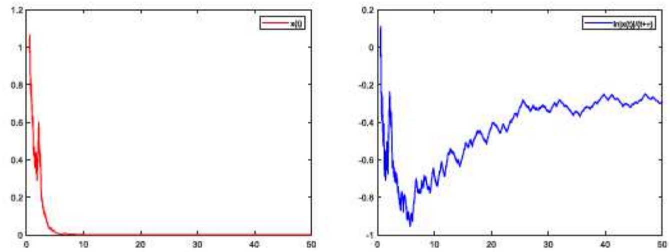


Fig. 1. Sample paths of $x(t)$ (left) and $\frac{\ln x(t)}{t}$ (right) in Case I.

Similarly, if $\widetilde{\alpha}(t)$ is ergodic and $\limsup_{i\to\infty} c(i) < 0$ where $c(i) := a(i) + m(i)\overline{\mu}$. Then, by Theorem 3.2, (23) is exponentially stable in probability.

Example 4.2: Consider a general linear stochastic delay differential equation with regime-switching of the form

$$\begin{aligned}
dX(t) =\ & [A(\alpha(t))X(t) + B(\alpha(t))X(t-r)]\,dt\\
&+ \sum_{j=1}^{d}[C_j(\alpha(t))X(t) + D_j(\alpha(t))X(t-r)]\,dW_j(t)
\end{aligned} \tag{24}$$

where $X(t) \in \mathbb{R}^n$; $A(\cdot), B(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$, $C_j(\cdot)$, $D_j(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$, $j = 1, \ldots, d$. Moreover, we assume that for each $i \in \mathbb{Z}_+$, $B(i)$ is positive definite. It is clear that

$$2x^\top B(i)y \leq x^\top B(i)x + y^\top B(i)y \text{ for all } x, y \in \mathbb{R}^n \text{ and } i \in \mathbb{Z}_+.$$

Hence, similar to Example 4.1, let

$$a(i) = \Lambda^M\left((A^\top(i) + A(i)) + B(i) + \sum_{j=1}^{d} C_j(i)C_j^\top(i)\right) \text{ and}$$

$$m(i) = \Lambda^M\left(B(i) + \sum_{j=1}^{d} D_j(i)D_j^\top(i)\right)$$

and assume that $a(i)$ and $m(i)$ are bounded in i and satisfy condition (5) with $\mu, \tilde{\mu}$ being the Dirac measure whose mass is on $\{-r\}$. If $\widetilde{\alpha}(t)$, a Markov chain with generator $Q(0)$, is strongly ergodic and Assumption 3.2 is satisfied, then (24) is exponentially stable in probability. Likewise, the (24) is exponentially stable in probability if $\widetilde{\alpha}(t)$ is ergodic and $\limsup_{i\to\infty} c(i) < 0$.

Next, we run simulation for a scalar equation

$$dx(t) = a_{\alpha(t)}x(t) + b_{\alpha(t)}x(t-\tau) + \sigma X(t)dW(t)$$

where $a_1 = 1, b_1 = 0.5, a_2 = -2, b_2 = 0.3$, and $q_{12} = q_{21} = 5$. We have $a(1) = 2.5, a(2) = -3.7$; $m(1) = 0.5$, and $m(2) = 0.3$, so it is easy to check that the equation is stable when τ is small. We consider two cases. Case I, when $\tau = 0.5$, the Lyapunov exponent is negative and the trivial solution decays exponentially fast.

See Fig. 1. Case II, when $\tau = 10$, the moment Lyapunov exponent is positive leading to trivial solution being not exponentially stable; see Fig. 2.

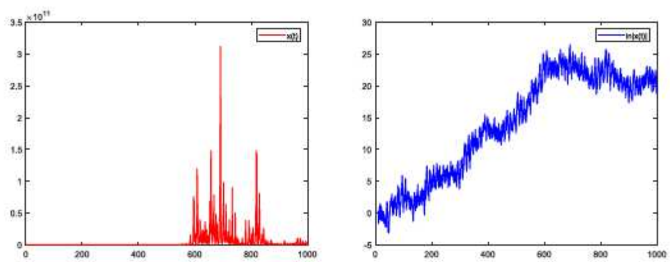


Fig. 2. Sample paths of $x(t)$ (left) and $\frac{\ln x(t)}{t}$ (right) in Case II.

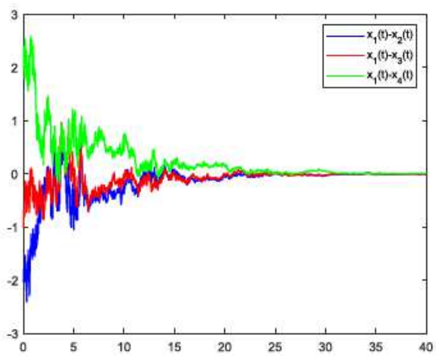


Fig. 3. Sample paths of $x_1(t) - x_2(t)$, $x_1(t) - x_3(t)$, and $x_1(t) - x_3(t)$.

2) Linearization for Nonlinear SFDEs With Random Switching: This section is devoted to the stability of linearized systems. In the study of dynamic systems, nonlinear systems and associated linearization have long been regarded as important, which goes back to the celebrated Hartman–Grobman theorem. For the related study, we refer the reader to [17] for systems given by stochastic differential equations, and [18] for switching diffusions. In this section, we consider a nonlinear functional system of equations of the form

$$dX(t) = b(\zeta(t, X_t), \alpha(t))dt + \sigma(\zeta(t, X_t), \alpha(t))dW(t) \quad (25)$$

where $X(t) \in \mathbb{R}^n, X_t := \{X(t+s) : s \in [-r, 0]\} \in \mathcal{C}([-r, 0], \mathbb{R}^n)$, $b(y, i) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^n$, $\sigma(y, i) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{n\times d}$, $W(t)$ is a d-dimensional Brownian motion. $\zeta(t, \phi) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is an M-grid "approximation operator" (for some fixed M), i.e., for each fixed t, $\zeta(t, \phi)$ is defined as follows:

$$\zeta(t, \phi) = \sum_{k=1}^{M} c_k \phi(r_k) \quad \forall \phi \in \mathcal{C}$$

for some "M-grid" points $r_1, \dots, r_M \in [-r, 0]$ and weights $c_1, \dots, c_M \in \mathbb{R}$ (depending on t). The "approximation operator" acting on segment function is in a remedy from a computational point of view. Mainly, one cannot store an infinite-dimensional vector in a computer. Thus, a finite dimensional approximation is used.

Now, we assume that the drift and diffusion terms $b(y, i)$ and $\sigma(y, i)$ can be linearized as follows:

$$b(y, i) = \widetilde{b}(i, t)y + o(|y|)$$
$$\sigma(y, i) = (\widetilde{\sigma}_1(i, t)y, \dots, \widetilde{\sigma}_d(i, t)y) + o(|y|)$$

where $o(|y|)$ represents a (vector or matrix)-valued function $h(y, i)$ satisfying $\lim_{y\to 0} \frac{h(y,i)}{|y|} = 0$ uniformly in i. We examine the stability of the linearized approximation system of (25) using the linear system

$$dX(t) = \widetilde{b}(\alpha(t), t)\zeta(t, X_t)dt + \sum_{j=1}^{d} \widetilde{\sigma}_j(\alpha(t), t)\zeta(t, X_t)dW_j(t). \quad (26)$$

Assume that $\widetilde{b}(i, t)$ and $\widetilde{\sigma}_j(i, t)$ $(j = 1, \dots, d)$ are uniformly bounded in (i, t), and that there exist $m(i) \geq 0$, $a(i) \in \mathbb{R}$, such that $m(i), a(i)$ are bounded in i and that for all t

$$\begin{aligned} &2\phi(0)^\top \widetilde{b}(i, t)\zeta(t, \phi) \\ &+ \frac{1}{2}\sum_{j=1}^{d} \operatorname{tr}\left(V_{xx}(\phi(0))[\widetilde{\sigma}_j(i, t)\zeta(t, \phi)][\widetilde{\sigma}_j(i, t)\zeta(t, \phi)]^\top\right) \\ &\leq a(i)|\phi(0)|^2 + \frac{m(i)}{M}\sum_{k=1}^{M}|\phi(r_k)|^2 \end{aligned} \quad (27)$$

and

$$\sum_{i\in\mathbb{Z}_+} (a(i) + m(i)\bar{\mu})\, \nu_i < 0 \quad (28)$$

where $\{r_k\}_{k=1}^M$ depending on t are determined as in definition of $\zeta(t, \phi)$, $\gamma := 0 \wedge \min_{i\in\mathbb{Z}_+}\{a(i) + m(i)\}$, and $\bar{\mu} := \max_{\{t\geq 0\}} \frac{1}{M}\sum_{i=1}^{M} e^{\gamma r_i}$. Hence, Assumption 2.2 is satisfied, i.e.,

$$(\mathcal{L}_i V)(\phi, t) \leq a(i)|\phi(0)|^2 + m(i)\int_{-r}^{0} [V(\phi(0))]\, \mu_t(ds) \quad (29)$$

with $V(x) = |x|^2$, μ_t is a probability measure concentrated on $\{r_k\}$, $\mu_t(r_k) = \frac{1}{M}, k = 1, \dots, M$. Moreover, it is easily seen that Assumption 2.1 and 2.3 also hold. Using Theorem 3.2, we have that system (26) is exponentially stable in probability if either the Markov chain with generator $Q(0)$ is strongly ergodic or it is ergodic and $\limsup_{i\to\infty} c(i) < 0$. Next, we show that under these conditions, the nonlinear system (25) is also exponentially stable in probability.

Indeed, because of (28), there is a sufficiently small $\epsilon > 0$, such that (28) still holds with $a(i)$ and $m(i)$ being replaced by $\bar{a}(i) := a(i) + \epsilon$ and $\bar{m}(i) := m(i) + \epsilon$, respectively. On the other hand, since $|o(|\zeta(t, \phi)|)|/|\zeta(t, \phi)| \to 0$ as $|\zeta(t, \phi)| \to 0$ uniformly in i, in a small neighbor of the origin, we have that

$$\begin{aligned} &2\phi(0)^\top \left[\widetilde{b}(i, t)\zeta(t, \phi) + o(|\zeta(t, \phi)|)\right] \\ &+ \frac{1}{2}\sum_{j=1}^{d} \operatorname{tr}\left(V_{xx}(\phi(0))\left[\widetilde{\sigma}_j(i, t)\zeta(t, \phi)\right.\right. \\ &\left.\left.+ o(|\zeta(t, \phi)|)\right]\left[\widetilde{\sigma}_j(i, t)\zeta(t, \phi) + o(|\zeta(t, \phi)|)\right]^\top\right) \\ &\leq (a(i) + \epsilon)|\phi(0)|^2 + \frac{(m(i) + \epsilon)}{M}\sum_{k=1}^{M}|\phi(r_k)|^2 \end{aligned}$$

which leads to that Assumption 2.2 is also satisfied for system (25). As a consequence, the nonlinear system is also exponentially stable if either the Markov chain with generator $Q(0)$ is strongly ergodic or it is ergodic and $\limsup_{i\to\infty} c(i) < 0$.

3) Linearization for Stochastic Differential-Difference Equations With Random Switching: This section is specialized to stochastic differential-difference systems. We carry out the linearization for such equations. Because of its structure, we do not need to use pull back mapping $\zeta(\cdot)$. Assume that $W(t) = (W_1(t), \dots, W_d(t))^\top$ is a d-dimensional Brownian motion; $\alpha(t)$ is a switching process taking values in $\mathbb{Z}_+$ and having the generator $Q(X(t)) = (q_{k,l}(X(t)))_{k,l\in\mathbb{Z}_+}$, and $\lim_{t\to\infty} p_{ij}(t) = \nu_j$ for any $i, j \in \mathbb{Z}_+$.

Consider a nonlinear stochastic differential-difference system as follows:

$$\begin{aligned} dX(t) &= b(X(t), X(t-r), \alpha(t))dt \\ &\quad + \sigma(X(t), X(t-r), \alpha(t))dW(t) \end{aligned} \tag{30}$$

where $X(t) \in \mathbb{R}^n$, $b(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{n\times d}$, $W(t) = (W_1(t), \dots, W_d(t))$ is a d-dimensional Brownian motion, and its associated linearized system

$$\begin{aligned} dX(t) &= [A(\alpha(t))X(t) + B(\alpha(t))X(t-r)]\, dt \\ &\quad + \sum_{j=1}^{d} [C_j(\alpha(t))X(t) + D_j(\alpha(t))X(t-r)]\, dW_j(t) \end{aligned} \tag{31}$$

where $A(\cdot), B(\cdot), C_j(\cdot), D_j(\cdot) \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, d)$. In the above, we assume $b(y_1, y_2, i)$ and $\sigma(y_1, y_2, i)$ can be linearized by

$$b(y_1, y_2, i) = A(i)y_1 + B(i)y_2 + o(|y_1|) + o(|y_2|)$$

and

$$\begin{aligned} \sigma(y_1, y_2, i) = &\ (C_1(i)y_1, \dots, C_d(i)y_1) \\ &+ (D_1(i)y_2, \dots, D_d(i)y_2) + o(|y_1|) + o(|y_2|). \end{aligned}$$

As in Example 4.2, recall system (31) is exponentially stable under the condition $a(i), m(i)$ are bounded in i and

$$\sum_{i\in\mathbb{Z}_+} \left(a(i) + m(i)e^{-\gamma r}\right) \nu_i < 0$$

where

$$a(i) = \Lambda^M \left((A^\top(i) + A(i)) + B(i) + \sum_{j=1}^{d} C_j(i)C_j^\top(i) \right) \text{ and}$$

$$m(i) = \Lambda^M \left(B(i) + \sum_{j=1}^{d} D_j(i)D_j^\top(i) \right)$$

and $\gamma := 0 \wedge \inf_{i\in\mathbb{Z}_+}\{a(i) + m(i)\}$. Then, if $\widetilde{\alpha}(t)$ is strongly ergodic, system (30) is stable. In case $\widetilde{\alpha}(t)$ is just ergodic, we need one more condition for stability of (30) that $\limsup_{i\to\infty}(a(i) + m(i)e^{-\gamma r}) < 0$.

B. Application to Multiagent Systems

Starting in the early 2000, multi-agent systems have attracted considerable attention. It stems from the study of consensus in physics, computer graphics to mimic animal behaviors, and cooperative control systems, among others, which provided insights and benefited the designing, and implementing distributed controllers. Along with the intensive study, the concept of consentability comes into being. It is concerned with conditions on system parameters, which enables the existence of consensus protocol. It is proven to be import, both theoretically and practically, for cooperative control protocol development, such as flocking behavior, agent rendezvous, and robot coordination, among others. In this section, we consider consentability of nonlinear multi-agent systems consisting of N agents. Each of the agents is described by a dynamic system

$$\dot{y}_k(t) = f(\alpha(t), y_k(t), y_k(t-\tau_1), t) + u_k(t), \quad k = 1, \dots, N \tag{32}$$

where $y_k(t) \in \mathbb{R}^n$ is the state of the kth agent, τ_1 is a delay, and $u = [u_1, \dots, u_N]^\top$ $(u_i \in \mathbb{R}^n, i = 1, \dots, N)$ is the control to be designed, $\alpha(t)$ is a Markov switching with countable state space $\mathbb{Z}_+$, $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a nonlinear function. Now, we model the information flow structure among different agents as follows. Let, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a connected undirected graph, where $\mathcal{V} = \{1, \dots, N\}$ denotes the set of nodes with k representing the kth agent, $\mathcal{E}$ is the set of undirected edges, $\mathcal{A} = [a_{kl}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with the element $a_{kl} = 1$ or 0 indicating whether or not there is a directed information flow between agent l and agent k. Also, denote by $\mathcal{N}_k$ the set of the node k's neighbors, i.e., $\mathcal{N}_k := \{l \in \{1, \dots, N\} : a_{kl} = 1\}$, and $\deg_k := \sum_{l=1}^{N} a_{kl}$ the degree of k th agent. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{H} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \operatorname{diag}(\deg_1, \dots, \deg_N)$. It is clear that $\mathcal{H}$ is a symmetric matrix with N eigenvalues denoted by $0 = \lambda_1 < \lambda_2 < \dots < \lambda_N$; see e.g., [47].

Consider the control

$$u_k(t) = P \sum_{l\in N_k} z_{lk}(t) \tag{33}$$

where a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is the control gain to be designed,

$$z_{lk}(t) = y_l(t) - y_k(t) + g_{lk}(\alpha(t), y_l(t-\tau_2) - y_k(t-\tau_2))\xi_{lk}(t)$$

is the measurement of the agent k from its neighbor agent l, τ_2 is the delay, $\xi_{lk}(t)$ is a scalar independent Gaussian white noise, and $g_{lk} : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the noise intensity function. Denote by

$$\begin{aligned} \mathcal{U} = \{u(t) = (u_1(t), \dots, u_N(t)) | u_i(t) &\text{ is given by (33)} \\ t \geq 0, \text{ and } i &= 1, \dots, N\} \end{aligned}$$

the collection of all admissible distributed protocols. We refer the reader to [47], [48], and [49] and the references therein for the detailed motivation of the above protocol. The following Assumption states some usual conditions for this problem.

Assumption 4.1: We assume that

1) For each $i \in \mathbb{Z}_+$, $f(i,0,0,t) = g_{lk}(i,0) = 0$ and there exist C_i and σ_{ilk}, such that
$$|f(i,x,y,t) - f(i,\bar{x},\bar{y},t)| \le C_i\left(|x-\bar{x}| + |y-\bar{y}|\right)$$
and
$$|g_{lk}(i,x) - g(i,\bar{x})| \le \sigma_{ilk}\,|x-\bar{x}|$$
for all $x,\bar{x},y,\bar{y} \in \mathbb{R}^n$. We assume that $\sup_{i\in\mathbb{Z}_+,l,k\in\{1,\ldots,N\}}\{C_i,\sigma_{ilk}\} < \infty$.
2) The noise processes $\xi_{lk}(t) \in \mathbb{R}$ satisfy $\int_0^t \xi_{lk}(s)ds = w_{lk}(t)$, $t \ge 0$, $k = 1,\ldots,N$, and $l \in N_k$, where $w_{lk}(t)$ are independent standard Brownian motions defined on the complete probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathcal{P})$.
3) The Markov switching $\alpha(t)$ is independent of $w_{lk}(t)$ and strongly ergodic with a unique invariant measure $\{\nu_i\}_{i\in\mathbb{Z}_+}$.

Definition 4.1: System (32) is said to be exponentially consentable in probability with respect to $\mathcal{U}$ if there exists a protocol $u \in \mathcal{U}$, so that for any $\epsilon > 0$, there exists $\delta > 0$, such that for all $i \neq j \in \mathcal{V}$,
$$\mathbb{P}\left\{|y_i(t) - y_j(t)| \text{ converges exponentially fast to } 0\right\} \ge 1-\epsilon$$
whenever the initial values $\varphi = (\varphi_1,\ldots,\varphi_N) \in \mathcal{C}([\tau_1 \vee \tau_2, 0];\mathbb{R}^{nN})$ of (32) satisfying that
$$\sum_{k=1}^N \left\|\varphi_k - \frac{1}{N}\sum_{j=1}^N \varphi_j\right\|^2 < \delta.$$

Assumption 4.2: Assume the Laplacian matrix $\mathcal{H}$ satisfies
$$\lambda_2^2 - 2\lambda_N \frac{(N-1)}{N}\left(\hat{\sigma}^M\right)^2\left(C^M + 3\sum_{i\in\mathbb{Z}_+} C_i\nu_i\right) > 0$$
where $\hat{\sigma}_i := \sup_{k,l\in\{1,\ldots,N\}} \sigma_{ijk}$, for $i \in \mathbb{Z}_+$, $\hat{\sigma}^M = \sup_i\{\hat{\sigma}_i\}$, $C^M := \sup_i\{C_i\}$.

We proceed to present our result in this section.

Theorem 4.1: Under Assumptions 4.1 and 4.2, the multiagent system (32) is exponentially consentable in probability with respect to $\mathcal{U}$. In particular, the consensus problem can be solved by the protocol (33) with $P = pI_n$, where p is a positive number satisfying
$$-2p\lambda_2 + \sum_{i\in\mathbb{Z}_+} 3C_i\nu_i + C^M + 2p^2\lambda_N\frac{(N-1)\hat{\sigma}_i^2}{N} < 0.$$
Under Assumption 4.2, such p exists.

Proof: For simplicity of notation, denote by $\mathbf{1}_N$ the N-dimensional column vector with all ones; $J_N = \frac{1}{\cdot}N\mathbf{1}_N\mathbf{1}_N^\top$; $\eta_{N,k}$ the N-dimension column vector with the kth element being 1 and others being 0, I_n the n-dimensional identity matrix. Let
$$y(t) := \left[y_1^\top(t),\ldots,y_N^\top(t)\right]^\top$$
$$F(y(t),\alpha(t),t) := \left[f^\top(\alpha(t),y_1(t),y_1(t-\tau_1),t)\right.$$
$$\left.\ldots, f^\top(\alpha(t),y_N(t),y_N(t-\tau_1),t)\right]^\top.$$

Also, let
$\delta(t) := [(I_N - J_N)\otimes I_n]y(t)$, $\delta(t) =: [\delta_1^\top(t),\ldots,\delta_N^\top(t)]^\top$.
Hence, we have
$$dy(t) = -\left[p(\mathcal{H}\otimes I_n)y(t) + F(y(t),\alpha(t),t)\right]dt + dM_1(t)$$
where
$$M_1(t) := p\sum_{k,l=1}^N a_{kl}\int_0^t [\eta_{N,k}$$
$$\otimes\left(\bar{g}_{lk}(\alpha(s),\delta(s-\tau_2))\right)]\,dw_{lk}(s)$$
and
$$\bar{g}_{lk}(\alpha(s),\delta(s-\tau_2)) := g_{lk}\left(\alpha(s),\delta_l(s-\tau_2) - \delta_k(s-\tau_2)\right).$$
Let $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^N y_j(t)$. Then, $\delta_k(t) = y_k(t) - \bar{y}(t)$ and $\delta_l(t) - \delta_k(t) = y_l(t) - y_k(t)$. Thus,
$$d\delta(t) = -p\left(\mathcal{H}\otimes I_n\right)\delta(t)dt + \bar{F}(y(t),\alpha(t),t)dt + dM_2(t)$$
where
$$\bar{F}(y(t),\alpha(t),t) = \left[\bar{f}_1^\top(y(t),\alpha(t),t),\ldots,\bar{f}_N^\top(y(t),\alpha(t),t)\right]^\top$$
$$:= \left[(I_N - J_N)\otimes I_n\right]F(y(t),\alpha(t),t)$$
i.e.,
$$\bar{f}_k(y(t),\alpha(t),t) = f\left(\alpha(t),y_k(t),y_k(t-\tau_1),t\right)$$
$$-\frac{1}{N}\sum_{j=1}^N f\left(\alpha(t),y_j(t),y_j(t-\tau_1),t\right)$$
and
$$M_2(t) = p\sum_{k,l=1}^N\int_0^t a_{kl}\left\{[(I_N - J_N)\eta_{N,k}]\right.$$
$$\left.\otimes(\bar{g}_{lk}(\alpha(s),\delta(s-\tau_2)))\right\}dw_{lk}(s).$$
Define the unitary matrix
$$Q = \left[\frac{\mathbf{1}_N}{\sqrt{N}}, v_2,\ldots,v_N\right] := \left[\frac{\mathbf{1}_N}{\sqrt{N}},\widetilde{Q}\right]$$
where v_i is the unit eigenvector of $\mathcal{H}$ corresponding to the eigenvalue λ_i for $i = 2,\ldots,N$ and $\Lambda = \mathrm{diag}(\lambda_2,\ldots,\lambda_N)$. Denote by
$$\widetilde{\delta}(t) = \left[\widetilde{\delta}_1^\top(t),\ldots,\widetilde{\delta}_N^\top(t)\right]^\top := \left(Q^{-1}\otimes I_n\right)\delta(t)$$
$$\widetilde{\delta}_k \in \mathbb{R}^n$$
$$\bar{\delta}(t) := \left[\widetilde{\delta}_2^\top(t),\ldots,\widetilde{\delta}_N^\top(t)\right]^\top$$
it is easy to verify that $\widetilde{\delta}_1(t) = 0$. By the definition of Q, we obtain
$$d\bar{\delta}(t) = (\widetilde{Q}^\top\otimes I_n)\bar{F}(y(t),\alpha(t),t)dt$$
$$-p\left(\Lambda\otimes I_n\right)\bar{\delta}(t)dt + dM_3(t) \qquad (34)$$

where

$$M_3(t) = p\sum_{k,l=1}^{N} a_{kl}\int_0^t \{[\widetilde{Q}^\top (I_N - J_N)\eta_{N,k}] \otimes (\bar{g}_{lk}(\alpha(s), \delta(s-\tau_2)))\}dw_{lk}(s) = p\sum_{k,l=1}^{N} a_{kl}\int_0^t G_{kl}(\alpha(s),\delta(s-\tau_2))dw_{lk}(s).$$

The definition of $\widetilde{\delta}(t)$ implies that the consensus of (32) is equivalent to the stability of (34). Now, we verify Assumptions 2.1, 2.2, and 2.3 under Assumptions 4.1 and 4.2. It is clear that Assumption 2.1 is satisfied. Let, $\widetilde{\mu}$ be a probability measure concentrated on $[-r,0]$ satisfying $\widetilde{\mu}(-\tau) = \widetilde{\mu}(-\tau_1) = \widetilde{\mu}(-\tau_2) = \frac{1}{3}$, it is easy to confirm that Assumption 2.3 holds. Therefore, it is sufficient to verify Assumption 2.2. Consider $V(\phi) := |\phi(0)|^2$. Hence, by directed calculation, we obtain

$$\begin{aligned}(\mathcal{L}_i V)(\bar{\delta}_t) =& -2p\bar{\delta}^\top(t)\left(\Lambda\otimes I_n\right)\bar{\delta}(t) \\ &+ 2\bar{\delta}^\top(t)\left(\widetilde{Q}^\top\otimes I_n\right)\bar{F}(y(t),i,t) \\ &+ p^2\sum_{k,l=1}^{N} a_{kl}G_{kl}(i,\delta(t-\tau_2))G_{kl}^\top(i,\delta(t-\tau_2)) \\ =& I_1(i,\bar{\delta}_t) + I_2(i,\bar{\delta}_t) + I_3(i,\bar{\delta}_t).\end{aligned} \tag{35}$$

By noting that $\widetilde{Q}\widetilde{Q}^\top = I_N - J_N$, $\eta_{N,k}^\top(I_N - J_N) = \frac{N-1}{N}$, $(I_N - J_N)^2 = I_N - J_N$; the Lipschitz properties of $\bar{g}$ and the identity

$$\sum_{k,l=1}^{N} a_{kl}\left|\delta_l(t-\tau_2) - \delta_k(t-\tau_2)\right|^2 = \bar{\delta}^\top(t-\tau_2)\left[\left(\Lambda+\Lambda^\top\right)\otimes I_n\right]\bar{\delta}(t-\tau_2)$$

we obtain

$$I_3(i,\bar{\delta}_t) \le p^2\frac{(N-1)\hat{\sigma}_i^2}{N}\bar{\delta}^\top(t-\tau_2)\left[\left(\Lambda+\Lambda^\top\right)\otimes I_n\right]\bar{\delta}(t-\tau_2).$$

As a consequence, we have

$$I_3(i,\bar{\delta}_t) \le 2p^2\lambda_N\frac{(N-1)\hat{\sigma}_i^2}{N}\left|\bar{\delta}(t-\tau_2)\right|^2. \tag{36}$$

On the other hand, we have

$$I_1(i,\bar{\delta}_t) \le -2p\lambda_2\left|\bar{\delta}(t)\right|^2. \tag{37}$$

Moreover, by definition of δ, $\bar{\delta}$ and the orthogonality of Q, it is easily seen that $|\delta(t)|^2 = |\bar{\delta}(t)|^2$. This together with Assumption 4.1 of F imply that

$$\begin{aligned}I_2(i,\bar{\delta}_t) &= 2\delta^\top(t)\left(\widetilde{Q}\widetilde{Q}^\top\otimes I_n\right)\left[(I_N - J_N)\otimes I_n\right]F(y(t),i,t) \\ &\le 3C_i\left|\bar{\delta}(t)\right|^2 + C_i\left|\bar{\delta}(t-\tau_1)\right|^2.\end{aligned} \tag{38}$$

Thus, combining (36), (37), and (38) yields that

$$\begin{aligned}(\mathcal{L}_i V)(\bar{\delta}_t) \le& -2p\lambda_2\left|\bar{\delta}(t)\right|^2 + 3C_i\left|\bar{\delta}(t)\right|^2 \\ &+ C_i\left|\bar{\delta}(t-\tau_1)\right|^2 \\ &+ 2p^2\lambda_N\frac{(N-1)\hat{\sigma}_i^2}{N}\left|\bar{\delta}(t-\tau_2)\right|^2.\end{aligned} \tag{39}$$

Under condition

$$\lambda_2^2 - 2\lambda_N\frac{(N-1)}{N}\left(\hat{\sigma}^M\right)^2\left(C^M + 3\sum_{i\in\mathbb{Z}_+} C_i\nu_i\right) > 0$$

there exists a $p > 0$, such that

$$-2p\lambda_2 + \sum_{i\in\mathbb{Z}_+} 3C_i\nu_i + C^M + 2p^2\lambda_N\frac{(N-1)\hat{\sigma}_i^2}{N} < 0.$$

For such a p, Assumption 2.2 is satisfied with $a(i) = -2p\lambda_2 + 3C_i$, $m(i) := C^M + 2p^2\lambda_N\frac{(N-1)(\hat{\sigma}^M)^2}{N}$, and μ being the probability measure concentrated on $[\tau_1\vee\tau_2, 0]$ satisfying

$$\mu(\{\tau_1\}) = \frac{C^M}{C^M + 2p^2\lambda_N\frac{(N-1)(\hat{\sigma}^M)^2}{N}} \text{ and}$$

$$\mu(\{\tau_2\}) = \frac{2p^2\lambda_N\frac{(N-1)(\hat{\sigma}^M)^2}{N}}{C^M + 2p^2\lambda_N\frac{(N-1)(\hat{\sigma}^M)^2}{N}}.$$

Then we obtain the stability of (34).

Example 4.3: Consider a system with four agents satisfying (32) under control (33) with: $\tau_1 = 10, \tau_2 = 5, g_{lk}(i,x) = 0.1x$, and

$$f(i,y,y') = \begin{cases}0.1y + 0.3y' & \text{if } i = 1\\ -0.3y + 0.2y' & \text{if } i = 2\end{cases}, \quad Q_\alpha = \begin{pmatrix}-2 & 2\\ 1 & -1\end{pmatrix}.$$

Suppose, the information flow structure has a star graph $\mathcal{E} = \{(1,2);(1,3);(1,4)\}$. Then, we can obtain the eigenvalues of $\mathcal{H}$, that are $\{0,1,1,4\}$. In this example, $C^M = 0.3, \nu_1 = 1/3, \nu_2 = 2/3$, and $(\hat{\sigma}^M)^2 = 0.1$ with $p = 1$,

$$-2p\lambda_2 + \sum_{i\in\mathbb{Z}_+} 3C_i\nu_i + C^M + 2p^2\lambda_N\frac{(N-1)\hat{\sigma}_i^2}{N} = -0.2 < 0.$$

□

C. Application to NCSs

In the new era, control systems have become increasingly more complex. Isolated systems, single machines or systems are often replaced by NCSs with many components and subsystems. In this section, we consider applications of our results to the NCSs. Here, the NCSs we consider consist of a number of components, which interact with each other as follows.

1) The controlled plant P: its state $x_P \in \mathbb{R}^{n_p}$ is controlled by an updated control $\hat{u} \in \mathbb{R}^{n_u}$, which is constructed by updating the control $u \in \mathbb{R}^{n_u}$ of the controller by a scheduling protocol, and samples output $y \in \mathbb{R}^{n_y}$ to send to the controller.
2) The controller C: its state $x_C \in \mathbb{R}^{n_c}$ is established by the updated output $\hat{y} \in \mathbb{R}^{n_y}$, which is obtained by updating the output y of the plant by a scheduling protocol; and sends a control u to the plant.
3) A stochastic protocol: it is designed as a Markov switching process $\alpha(t)$, whose state space is $\mathcal{M} = \{i = (i_1, i_2) : i_1 \in \{1,\dots,n_y\}, i_2 \in \{1,\dots,n_u\}\}$, and admits an invariant measure $\{\nu_i, i = (i_1,i_2) \in \mathcal{M}\}$. At the time t, if $\alpha(t) = (i_1, i_2)$, only components of y_{i_1}

of y and u_{i_2} of u are updated whereas the others are unchanged.

4) In above, sending and receiving the information of y and u are delayed in delay times τ_1 and τ_2, respectively. Moreover, in the literature, the noises are not often used. In our setting, we allow the system to be observed under noises.

The NCS we consider is shown as the following system of stochastic differential equation

$$\begin{cases} dx_P(t) = (A_P x_P(t) + B_P \hat{u}(t))\, dt + g_P(x_P, \hat{u}) dW_1(t) \\ y(t) = C_P x_P(t) \\ d\hat{y}(t) = (U_P(\alpha(t)) y(t-\tau_1) + [I - U_P(\alpha(t))]\, \hat{y}(t-\tau_1))\, dt \\ dx_C(t) = (A_C x_C(t) + B_C \hat{y}(t))\, dt + g_C(x_C, \hat{y}) dW_2(t) \\ u(t) = C_C x_C(t) + D_C \hat{y}(t) \\ d\hat{u}(t) = (U_C(\alpha(t)) u(t-\tau_2) + [I - U_C(\alpha(t))]\, \hat{u}(t-\tau_2))\, dt \end{cases} \tag{40}$$

where $A_P \in \mathbb{R}^{n_p \times n_p}$, $B_P \in \mathbb{R}^{n_p \times n_u}$, $C_P \in \mathbb{R}^{n_y \times n_p}$, $A_C \in \mathbb{R}^{n_c \times n_c}$, $B_C \in \mathbb{R}^{n_c \times n_y}$, $C_C \in \mathbb{R}^{n_u \times n_c}$, and $D_C \in \mathbb{R}^{n_u \times n_y}$; $W_1(t)$ and $W_2(t)$ are independent d-dimensional Brownian motions, $g_P(\cdot,\cdot) : \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_p \times d}$, $g_C(\cdot,\cdot) : \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_c \times d}$, and $U_P((i_1, i_2)) := \mathrm{diag}(\delta_{1i_1}, \ldots, \delta_{n_y i_1})$, $U_C((i_1, i_2)) := \mathrm{diag}(\delta_{1i_2}, \ldots, \delta_{n_u i_2})$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. By setting $x := (x_P^\top, x_C^\top)^\top$; and $e_y := \hat{y} - y$, $e_u := \hat{u} - u$, $e := (e_y^\top, e_u^\top)^\top$; $\bar{x} := (x^\top, e^\top)^\top = (x_P^\top, x_C^\top, e_y^\top, e_u^\top)^\top$, and $W(t) = (W_1^\top(t), W_2^\top(t))^\top$, (40) is written as

$$\begin{aligned} d\bar{x}(t) = \Gamma_0 \bar{x}(t) dt + \Gamma_1(\alpha(t)) \bar{x}(t-\tau_1) dt \\ + \Gamma_2(\alpha(t)) \bar{x}(t-\tau_2) dt + G(\bar{x}(t)) dW(t) \end{aligned} \tag{41}$$

where

$$\Gamma_0 =$$

$$\begin{bmatrix} A_P + B_P D_C C_P & B_P C_C & B_P D_C & B_P \\ B_C C_P & A_C & B_C & 0 \\ -C_P(A_P + B_P D_C C_P) & -C_P B_P C_C & -C_P B_P D_C & -C_P B_P \\ -C_C B_C C_P & -C_C A_C & -C_C B_C & 0 \end{bmatrix}$$

$$\Gamma_1(\alpha(t)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_P & 0 & I - U_P(\alpha(t)) & 0 \\ -D_C C_P & 0 & -D_C(I - U_P(\alpha(t))) & 0 \end{bmatrix}$$

$$\Gamma_2(\alpha(t)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D_C C_P & C_C & D_C & I - U_C(\alpha(t)) \end{bmatrix}$$

$$G(\bar{x}) = \begin{bmatrix} g_P(x_P, \hat{u}) & 0 \\ 0 & g_C(x_C, \hat{y}) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the above, I denotes the identity matrix with suitable dimension, and 0 denotes matrix whose entries are 0 with compatible dimension. In what follows, we sometimes do not specify the dimension of matrices and vectors. However, it is understood that they have compatible dimensions. In the literature, many of the NCSs were presented as discrete-time systems, whereas our formulation examines the continuous-time counterpart; see e.g., [13] and [33] for similar formulations.

Definition 4.2: System (41) is exponentially stable in probability if for any $\epsilon > 0$, there is $\delta > 0$, such that $\mathbb{P}\{|\bar{x}(t)| \text{ converges exponentially fast to } 0\} \geq 1 - \epsilon$, whenever $\|\bar{x}_0\| < \delta$.

For a matrix $A \in \mathbb{R}^{\bar{n} \times \bar{n}}$, where $\bar{n} := n_p + n_y + n_c + n_u$, we define

$$\begin{aligned} A_0 &:= (A + A^\top)\Gamma_0, \\ A_1(i) &:= (A + A^\top)\Gamma_1(i), \\ A_2(i) &:= (A + A^\top)\Gamma_2(i), \\ A_G(x) &= \frac{1}{2}\mathrm{trace}\left[(G(x))^\top (A + A^\top) G(x) \right], \text{ and} \\ \lambda(A) &:= \sup\{x^\top (A + A^\top) x : \|x\| = 1\}. \end{aligned}$$

Assumption 4.3: Assume that there exists a matrix $A \in \mathbb{R}^{\bar{n} \times \bar{n}}$ satisfying that there exists positive real numbers $c_0, c_1(i), \bar{c}_1(i), c_2(i), \bar{c}_2(i)$, such that $-\infty < \lambda(A_0) \leq -2c_0$; and for any $x, y \in \mathbb{R}^{\bar{n}}$, $x^\top A_1(i) y \leq c_1(i)|x|^2 + \bar{c}_1(i)|y|^2$, $x^\top A_2(i) y \leq c_2(i)|x|^2 + \bar{c}_2(i)|y|^2$, $A_G(x) \leq c_G |x|^2$, and $c_0 > \sum_{i \in \mathbb{Z}_+} (c_1(i) + c_2(i) + 2\max\{\bar{c}_1(i), \bar{c}_2(i)\}) \nu_i$.

Theorem 4.2: Assume that Assumption 4.3 holds. Then, system (41) is exponentially stable in probability.

Proof: Let $V(\bar{x}(t)) = (\bar{x}(t))^\top A \bar{x}(t)$. By directed calculations, we obtain

$$\begin{aligned} &(L_i V)(\bar{x}(t)) \\ &= (\bar{x}(t))^\top A_0 \bar{x}(t) + (\bar{x}(t))^\top A_1(i) \bar{x}(t-\tau_1) \\ &\quad + (\bar{x}(t))^\top A_2(i) \bar{x}(t-\tau_2) + A_G(\bar{x}(t)) \\ &\leq (-c_0 + c_1(i) + c_2(i) + c_G) |\bar{x}(t)|^2 + \bar{c}_1(i) \bar{x}(t-\tau_1) \\ &\quad + \bar{c}_2(i) \bar{x}(t-\tau_2). \end{aligned}$$

Our assumptions in the main results are satisfied if we let $a(i) = -c_0 + c_1(i) + c_2(i) + c_G$, $m(i) = 2\max\{\bar{c}_1(i), \bar{c}_2(i)\}$, and μ be the probability measure on $[-\tau_1 \vee -\tau_2, 0]$, such that $\mu(\{-\tau_1\}) = \mu(\{-\tau_2\}) = \frac{1}{2}$. □

Remark 4.1: For the NCSs with different settings, we refer the reader to [6], [7], [8], [10], [12], [20], and [38]. Some differences can be emphasized as follows. To begin, under our formulation, the systems are observed under noises. In addition, we allow the transmissions to take place in continuous time instead of assuming that they happen only at specific times epoch.

V. Conclusion

This article has been devoted to stability of SFDEs with past-dependent random switching. In contrast to the work in the literature, models presented in this article are most general. New methods of analysis are presented using coupling techniques. A number of application examples are also presented. We hope that not only are the models of interest, but also the techniques are useful to the control and systems community. Further study on systems with additional Poisson-type jumps (see [3] for related references) can be considered in future study and investigation.

REFERENCES

[1] M. Athans, "Command and control (C2) theory: A challenge to control science," *IEEE Trans. Autom. Control*, vol. 32, no. 87, pp. 286–293, Apr. 1987.

[2] M. F. Chen, *From Markov Chains to Non-Equilibrium Particle Systems*. River Edge, NJ, USA: World Scientific, 2004.

[3] X. Chen, Z.-Q. Chen, K. Tran, and G. Yin, "Properties of switching jump diffusions: Maximum principles and Harnack inequalities," *Bernoulli*, vol. 25, pp. 1045–1075, 2019.

[4] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-time Markov Jump Linear Systems*. Berlin, Germany: Springer, 2013.

[5] O. L. V. Costa, E. O. Assumpcao, E. K. Boukas, and R. P. Marques, "Constrained quadratic state feedback control of discrete-time Markovian jump linear systems," *Automatica*, vol. 35, no. 4, pp. 617–626, 1999.

[6] L. Dongsong, W. Zhiwen, and G. Ge, "Modeling and stability analysis of MIMO networked control systems," in *Proc. 27th Chin. Control Conf.*, 2008, pp. 240–242.

[7] M. C. F. Donkers, W. P. M. H. Heemels, N. van de Wouw, and L. Hetel, "Stability analysis of networked control systems using a switched linear systems approach," *IEEE Trans. Autom. Control*, vol. 56, no. 9, pp. 2101–2115, Sep. 2011.

[8] M. C. F. Donkers, W. P. M. H. Heemels, D. Bernardini, A. Bemporad, and V. Shneer, "Stability analysis of stochastic networked control systems," *Automatica*, vol. 48, no. 5, pp. 917–925, 2012.

[9] N. H. Du, D. H. Nguyen, N. N. Nguyen, and G. Yin, "Stability of stochastic functional differential equations with random switching and applications," *Automatica*, vol. 125, 2021, Art. no. 109410.

[10] S. Heijmans, R. Postoyan, D. Nesić, N. Noroozi, and M. Heemels, "Stability analysis of networked linear control systems with direct-feed through terms," *Automatica*, vol. 96, pp. 186–200, 2018.

[11] A. Hening and D. Nguyen, "Coexistence and extinction for stochastic Kolmogorov systems," *Ann. Appl. Probability*, vol. 28, no. 3, pp. 1893–1942, 2018.

[12] J. P. Hespanha, "Modeling and analysis of networked control systems using stochastic hybrid systems," *Annu. Rev. Control*, vol. 38, pp. 155–170, 2014.

[13] B. Hu and T. Tamba, "Optimal transmission power and controller design for networked control systems under state-dependent Markovian channels," *IEEE Trans. Autom. Control*, vol. 67, no. 10, pp. 5669–5676, Oct. 2022.

[14] Y. Ji and H. J. Chizeck, "Controllability, stabilizability and continuous time Markovian jump linear quadratic control," *IEEE Trans. Autom. Control*, vol. 35, no. 7, pp. 777–788, Jul. 1990.

[15] Y. Ji, Y. H. J. Chizeck, X. Feng, and K. A. Loparo, "Stability and control of discrete time jump linear systems," *Control Theory Adv. Technol.*, vol. 7, pp. 247–270, 1991.

[16] R. Jia and X. Zong, "Time-varying formation control of linear multiagent systems with time delays and multiplicative noises," *Int. J. Robust Nonlinear Control*, vol. 31, no. 18, pp. 9008–9025, 2021.

[17] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed. Berlin, Germany: Springer, 2012.

[18] R. Z. Khasminskii, C. Zhu, and G. Yin, "Stability of regime-switching diffusions," *Stochastic Process. Appl.*, vol. 117, no. 8, pp. 1037–1051, 2007.

[19] K. Liu and E. Fridman, "Wirtinger's inequality and Lyapunov-based sampled-data stabilization," *Automatica*, vol. 48, no. 1, pp. 102–108, 2012.

[20] M. Lješnjanin, D. Nesić, and D. E. Quevedo, "Robust stability of a class of networked control systems," *Automatica*, vol. 73, pp. 117–124, 2016.

[21] S. Luo, F. Deng, B. Zhang, Z. Hu, and L. Shixian, "Almost sure stability of hybrid stochastic systems under asynchronous Markovian switching," *Syst. Control Lett.*, vol. 133, 2019, Art. no. 104556.

[22] X. Mao, "Stochastic functional differential equations with Markovian switching," *Funct. Differ. Equ.*, vol. 6, no. 3–4, pp. 375–396, 1999.

[23] X. Mao, "Exponential stability of stochastic delay interval systems with Markovian switching," *IEEE Trans. Autom. Control*, vol. 47, no. 10, pp. 1604–1612, Oct. 2002.

[24] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*. London, U.K.: Imperial College Press, 2006.

[25] M. Mariton, *Jump Linear Systems in Automatic Control*, New York, NY, USA: Marcel Dekker, Inc., 1990.

[26] S.-E. A. Mohammed, *Stochastic Functional Differential Equations*, New York, NY, USA: Longman, 1986.

[27] D. H. Nguyen and G. Yin, "Modeling and analysis of switching diffusion systems: Past-dependent switching with a countable state space," *SIAM J. Control Optim.*, vol. 54, no. 5, pp. 2450–2477, 2016.

[28] D. H. Nguyen and G. Yin, "Stability of regime-switching diffusion systems with discrete states belonging to a countable set," *SIAM J. Control Optim.*, vol. 56, no. 5, pp. 3893–3917, 2018.

[29] D. H. Nguyen and G. Yin, "Recurrence and ergodicity of switching diffusions with past-dependent switching having a countable state space," *Potential Anal.*, vol. 48, no. 4, pp. 405–435, 2018.

[30] D. Nguyen, N. Nguyen, and G. Yin, "General nonlinear stochastic systems motivated by chemostat models: Complete characterization of long-time behavior, optimal controls, and applications to wastewater treatment," *Stochastic Processes Appl.*, vol. 130, pp. 4608–4642, 2020.

[31] D. Nguyen, N. Nguyen, and G. Yin, "Stochastic functional Kolmogorov equations I: Persistence," *Stochastic Process Appl.*, vol. 142, pp. 319–364, 2021.

[32] D. Nguyen, N. Nguyen, and G. Yin, "Stochastic functional Kolmogorov equations II: Extinction," *J. Differ. Equ.*, vol. 294, pp. 1–39, 2021.

[33] A. Sargolzaei, F. Zegers, A. Abbaspour, C. Crane, and W. Dixon, "Secure control design for networked control systems with nonlinear dynamics under time-delay-switch attacks," *IEEE Trans. Autom. Control*, vol. 68, no. 2, pp. 798–811, Feb. 2023.

[34] S. P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems*, Cambridge, MA, USA: Birkhäuser,1994.

[35] J. Shao and F. Xi, "Stability and recurrence of regime-switching diffusion processes," *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3496–3516, 2014.

[36] L. Shaikhet, "Stability of stochastic hereditary systems with Markov switching," *Theory Stochastic Processes*, vol. 2, no. 18, pp. 180–184, 1996.

[37] L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Functional Differential Equations*. Berlin, Germany: Springer, 2013.

[38] C. Tan, L. Li, and H. Zhang, "Stabilization of networked control systems with both network-induced delay and packet dropout," *Automatica*, vol. 59, pp. 194–199, 2015.

[39] L. Y. Wang, P. P. Khargonekar, and A. Beydoun, "Robust control of hybrid systems: Performance guided strategies," in *Proc. Int. Hybrid Syst. Workshop*, 1999, pp. 356–390.

[40] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*. Berlin, Germany: Springer, 2010.

[41] C. Yuan and J. Lygeros, "On the exponential stability of switching diffusion processes," *IEEE Trans. Autom. Control*, vol. 50, no. 9, pp. 1422–1426, Sep. 2005.

[42] C. Yuan and J. Lygeros, "Asymptotic stability and boundedness of delay switching diffusions," *IEEE Trans. Autom. Control*, vol. 51, no. 1, pp. 171–175, Sep. 2006.

[43] D. Yue and Q. Han, "Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 217–222, Feb. 2005.

[44] L. Zhang and X. Bai, "Robust control for stochastic nonlinear delay systems with jumps," *Math. Problems Eng.*, vol. 2023, 2023, Art. no. 5748824.

[45] T. Zhang and H. Chen, "The stability with a general decay of stochastic delay differential equations with Markovian switching," *Appl. Math. Comput.*, vol. 359, pp. 294–307, 2019.

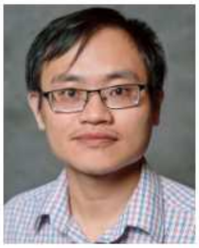
[46] C. Zhu and G. Yin, "Asymptotic properties of hybrid diffusion systems," *SIAM J. Control Optim.*, vol. 46, no. 4, pp. 1155–1179, 2007.

[47] X. Zong, T. Li, G. Yin, L. Y. Wang, and J. Zhang, "Stochastic consentability of linear systems with time delays and multiplicative noises," *IEEE Trans. Autom. Control*, vol. 63, no. 4, pp. 1059–1074, Apr. 2018.

[48] X. Zong, G. Yin, L. Y. Wang, T. Li, and J. F. Zhang, "Stability of stochastic functional differential systems using degenerate Lyapunov functionals and applications," *Automatica*, vol. 91, pp. 197–207, 2018.

[49] X. Zong, T. Li, and J. F. Zhang, "Consensus conditions of continuous-time multi-agent systems with time-delays and measurement noises," *Automatica*, vol. 99, pp. 412–419, 2019.

[50] X. Zong, T. Li, G. Yin, and J.-F. Zhang, "Delay tolerance for stable stochastic systems and extensions," *IEEE Trans. Autom. Control*, vol. 66, no. 6, pp. 2604–2619, Jun. 2021.



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