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null form semilinear wave equations

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We revisit the proof of small-data global existence for semilinear wave equations that satisfy a null condition. This new approach relies on a weighted local energy estimate that is akin to those of Dafermos and Rodnianski. Using weighted Sobolev estimates to obtain spatial decay and arguing in the spirit of the work of Keel, Smith, and Sogge, we are able to obtain global existence while only relying on translational and (spatial) rotational symmetries.

1. Introduction

We shall examine systems of semilinear wave equations in (1+3)-dimensions of the form

$$\begin{cases} \Box u^I := (\partial_t^2 - \Delta)u^I = Q^I(\partial u), \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad I = 1, 2, \ldots, M, \\ u^I(0, \cdot) = f^I, \quad \partial_t u^I(0, \cdot) = g^I. \end{cases} \tag{1-1}$$

Here $\partial u = (\partial_t u, \nabla u)$ is the space-time gradient, and each component Q^I is a smooth function that vanishes to second order at the origin. As we shall only consider small data, the long-time behavior is dictated by the lowest-order terms, and, as such, we will truncate Q to the quadratic level.

As the linear wave equation decays like $t^{-(n-1)/2}$ in n-spatial dimensions and as this factor is integrable at infinity when $n \geq 4$, it has long been known that global existence of solutions to (1-1) for sufficiently small initial data is guaranteed in these dimensions. When $n = 3$, however, a logarithmic blow up is instead encountered, and only almost global existence, which states that the lifespan of the solution grows exponentially as the size of the initial data shrinks, is available generically; see, e.g., [Sogge 2008].

When the nonlinearity is assumed to satisfy a null condition, it was discovered in [Christodoulou 1986; Klainerman 1986] that sufficiently small initial data always produce global solutions in three dimensions. In the current setting, assuming that our quadratic nonlinearity is of the form

$$Q^I(\partial u) = A_{JK}^{\alpha\beta,I}\,\partial_\alpha u^J\,\partial_\beta u^K,$$

the null condition requires that

$$A_{JK}^{\alpha\beta,I}\xi_\alpha\xi_\beta = 0, \quad \text{when } \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0. \tag{1-2}$$

Here we are using the summation convention with α, β running from 0 to 3 and the common conventions that $\partial_0 u = \partial_t u$, $\partial_j u = \partial_{x_j} u$. We are also allowing repeated capital indices to sum from 1 to M.

A common approach for establishing such long-time existence results relies on the method of invariant vector fields and the Klainerman–Sobolev inequality [Klainerman 1985]. Due to the unbounded normal component on the boundary, the Lorentz boosts $x_k\partial_t + t\partial_k$ are inappropriate when studying such nonlinear equations, say, exterior to a compact obstacle with Dirichlet boundary conditions. In response, [Keel et al. 2002] developed a method of establishing long-time existence for three-dimensional semilinear wave equations that only relies upon the generators of translations and spatial rotations:

$$\Omega_{ij} = x_i\,\partial_j - x_j\,\partial_i, \quad Z = (\partial_1, \partial_2, \partial_3, \Omega_{23}, \Omega_{13}, \Omega_{12}).$$

Here the authors depended on the integrated local energy estimate, which will be introduced in Section 2, and a weighted Sobolev estimate [Klainerman 1986] that provided decay in $|x|$ rather than t but only requires the vector fields Z. This method was adapted to the quasilinear setting in [Metcalfe and Sogge 2006] by exploring local energy estimates for perturbations of the d'Alembertian. The desire for a method that did not necessitate the use of the Lorentz boosts was also motivated by wanting to understand multiple speed systems of wave equations and the equations of elasticity; see, e.g., [Klainerman and Sideris 1996; Sideris 2000].

Here we shall explore small-data global existence for null-form wave equations. Many approaches exist for establishing such global existence, see, e.g., [Klainerman 1986; Christodoulou 1986; Sideris and Tu 2001; Metcalfe and Sogge 2007; Katayama and Kubo 2008; Lindblad et al. 2013]. Unlike many of the preceding results, our method shall only rely on the time-independent vector fields Z.

The key to our argument is to replace the use of the local energy estimate with a variant, specifically a type of r^p-weighted local energy estimate of [Dafermos and Rodnianski 2010]. See [Moschidis 2016] for some generalizations of this method. This estimate has been applied in a number of nonlinear settings such as [Luk 2013; Yang 2015a; 2015b; Keir 2018]. Typically it is used to derive decay in t. Such decay is then used to control the integral within the energy inequality and thus

provides long-time existence. We believe our approach to be more straightforward, though those preceding results were all in much more complicated settings.

The r^p-weighted local energy estimate only controls the "good" derivatives $\bar{\partial} = (\partial_t + \partial_r, \bar{\nabla})$, where $\bar{\nabla} = \nabla - (x/r)\partial_r$ are the angular derivatives. These are the directions that are tangent to the light cone and for which better decay is known. The r^p-weighted estimate is particularly well-suited to null form wave equations as the algebraic cancellation condition (1-2) precisely guarantees that in each quadratic term of $Q(\partial u)$ one of the two factors is a good derivative.

Our main result is:

Theorem 1.1. *Suppose that* $f, g \in (C^\infty(\mathbb{R}^3))^M$. *And let* $0 < p < 1$. *Then, for any* $\varepsilon > 0$ *sufficiently small, if*

$$\|(1+r)^{p/2} Z^{\leq 10} f\|_{L^2(\mathbb{R}^3)} + \|(1+r)^{p/2} Z^{\leq 9} g\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \tag{1-3}$$

then (1-1) *with nonlinearity satisfying* (1-2) *has a unique global solution* $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$.

Here, and throughout, we shall use the abbreviation $Z^{\leq N} u = \sum_{|\alpha| \leq N} Z^\alpha u$.

To keep the exposition as accessible as possible, we have only focused on semilinear equations on Minkowski space. We expect that the argument can readily be extended to, e.g., quasilinear equations and equations on exterior domains, and these topics will be explored subsequently.

Our proof of Theorem 1.1 most resembles [Lindblad et al. 2013]. There an alternate local energy estimate that relies upon $t - r$ weights, which is from [Lindblad and Rodnianski 2005; Alinhac 2001], was used. In order to achieve the decay in $t - r$, the authors called upon decay estimates of [Klainerman and Sideris 1996], but these in turn required the use of the time-dependent vector fields. The current argument is much more directly reminiscent of [Keel et al. 2002].

2. Integrated local energy estimates

The integrated local energy estimate first appeared in [Morawetz 1968]. Through subsequent refinements, on $\mathbb{R}_+ \times \mathbb{R}^n$, $n \geq 3$, we know that

$$\|\partial u\|^2_{L^\infty_t L^2_x} + \sup_{R \geq 1} R^{-1} \|\partial u\|^2_{L^2_t L^2_x(\mathbb{R}_+ \times \{\langle x \rangle \approx R\})} + \sup_{R \geq 1} R^{-3} \|u\|^2_{L^2_t L^2_x(\mathbb{R}_+ \times \{\langle x \rangle \approx R\})}$$

$$\lesssim \|\partial u(0, \cdot)\|^2_{L^2} + \int_0^\infty \int |\Box u|(|\partial u| + \langle x \rangle^{-1}|u|)\, dx\, dt. \tag{2-1}$$

The most robust proof of this estimate pairs the equation $\Box u$ with a multiplier of the form

$$C \partial_t u + \frac{r}{r+R} \partial_r u + \frac{n-1}{2} \frac{1}{r+R} u$$

and follows from integration by parts; see, e.g., [Sterbenz 2005; Metcalfe and Sogge 2006]. Related estimates are known to hold for stationary, nontrapping perturbations

and for sufficiently small nonstationary perturbations. See [Metcalfe et al. 2020] for a more complete history and the most general results in the nontrapping setting.

Our first task will be to prove the following r^p-weighted estimate, which first appeared in [Dafermos and Rodnianski 2010].

Proposition 2.1. *Suppose $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and that for every T there is an R so that $u(t, x) = 0$ for $t \in [0, T]$ and $|x| > R$. Then, for $0 < p < 1$,*

$$\|r^{(p-1)/2} \not{\partial} u\|^2_{L^2_t L^2_x} + \|r^{(p-3)/2} u\|^2_{L^2_t L^2_x} + \sup_t \widetilde{E}[u](t)$$
$$\lesssim \widetilde{E}[u](0) + \|r^{(p+1)/2} \Box u\|^2_{L^2_t L^2_x}, \quad (2\text{-}2)$$

where

$$\widetilde{E}[u](t) = \frac{1}{2} \int r^{p-2} |\not{\partial}(ru(t, x))|^2 \, dx + \frac{p}{2} \int r^{p-2} u^2(t, x) \, dx.$$

The local energy estimate (2-1) has an ℓ^∞-summation over the annuli, which we may take to be dyadic, in the left side. In [Keel et al. 2002], the difference between this and having ℓ^2-summability accounts for a logarithm, which in turn corresponds to the exponential within the notion of almost global existence. While restricted only to the good directions, the above estimate has the desired ℓ^2-summability, and as such, it will yield global existence so long as the equation permits its application on each term, which the null condition exactly provides.

Proof. For any $0 \le p \le 2$, we first consider

$$\int_0^T \int \Box u \cdot r^p \left(\partial_t u + \partial_r u + \frac{1}{r} u \right) dx \, dt$$
$$= \int_0^T \iint r^p (\partial_t^2 - \partial_r^2 - \not{\nabla} \cdot \not{\nabla})(ru)(\partial_t + \partial_r)(ru) \, d\sigma \, dr \, dt.$$

Using integration by parts and the fact that $[\not{\nabla}, \partial_r] = (1/r)\not{\nabla}$, the right side is equal to

$$\frac{1}{2} \int_0^T \iint r^p (\partial_t - \partial_r)[(\partial_t + \partial_r)(ru)]^2 \, d\sigma \, dr \, dt$$
$$+ \frac{1}{2} \int_0^T \iint r^p (\partial_t + \partial_r)|\not{\nabla}(ru)|^2 \, d\sigma \, dr \, dt + \int_0^T \iint r^{p-1} |\not{\nabla}(ru)|^2 \, d\sigma \, dr \, dt.$$

Further integrating by parts yields

$$\int_0^T \int \Box u \cdot r^p \left(\partial_t u + \partial_r u + \frac{1}{r} u \right) dx \, dt$$
$$= \frac{1}{2} \iint r^p \{[(\partial_t + \partial_r)(ru)]^2 + |\not{\nabla}(ru)|^2\} \, d\sigma \, dr \Big|_{t=0}^T$$
$$+ \frac{p}{2} \int_0^T \iint r^{p-1}[(\partial_t + \partial_r)(ru)]^2 \, d\sigma \, dr \, dt$$
$$+ \left(1 - \frac{p}{2}\right) \int_0^T \iint r^{p-1} |\not{\nabla}(ru)|^2 \, d\sigma \, dr \, dt. \quad (2\text{-}3)$$

For simplicity, we now restrict to $0 \le p < 1$. We then observe that

$$\frac{p}{2}\int_0^T \iint r^{p-1}[(\partial_t + \partial_r)(ru)]^2 \, d\sigma \, dr \, dt = \frac{p}{2}\int_0^T \int r^{p-1}(\partial_t u + \partial_r u)^2 \, r^2 \, d\sigma \, dr \, dt$$
$$+ \frac{p}{2}\int_0^T \iint r^p(\partial_t + \partial_r)u^2 \, d\sigma \, dr \, dt$$
$$+ \frac{p}{2}\int_0^T \iint r^{p-1}u^2 \, d\sigma \, dr \, dt,$$

which upon a last integration by parts and reverting back to rectangular coordinates gives

$$\frac{p}{2}\int r^{p-2}u^2 \, dx \Big|_{t=0}^T + \frac{p}{2}\int_0^T \int r^{p-1}(\partial_t u + \partial_r u)^2 \, dx \, dt + \frac{p(1-p)}{2}\int_0^T \int r^{p-3}u^2 \, dx \, dt.$$

Making this replacement in (2-3) and applying the Schwarz inequality gives

$$\frac{p}{2}\|r^{(p-1)/2}(\partial_t + \partial_r)u\|_{L_t^2 L_x^2}^2 + \frac{2-p}{2}\|r^{(p-1)/2}\slashed{\nabla}u\|_{L_t^2 L_x^2}^2$$
$$+ \frac{p(1-p)}{2}\|r^{(p-3)/2}u\|_{L_t^2 L_x^2}^2 + \widetilde{E}[u](T)$$
$$\le \widetilde{E}[u](0) + \|r^{(p+1)/2}\Box u\|_{L_t^2 L_x^2}\big(\|r^{(p-1)/2}(\partial_t + \partial_r)u\|_{L_t^2 L_x^2} + \|r^{(p-3)/2}u\|_{L_t^2 L_x^2}\big).$$

Bootstrapping the last factor of the last term completes the proof. We moreover note that the implicit constant is independent of T, and thus we may take the supremum over all T to obtain (2-2). $\qquad\square$

In the sequel, we shall require a version of (2-2) that permits the application of the invariant vector fields, which is presented in the next proposition.

Proposition 2.2. *Let $0 < p < 1$ and fix any $N \in \mathbb{N}$. Suppose $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and that, for every T, there is an R so that $u(t, x) = 0$ for $t \in [0, T]$ and $|x| > R$. Then,*

$$\|Z^{\le N}\partial u\|_{L_t^\infty L_x^2}^2 + \|(1+r)^{(p-1)/2}Z^{\le N}\slashed{\partial}u\|_{L_t^2 L_x^2}^2 + \|(1+r)^{(p-3)/2}Z^{\le N}u\|_{L_t^2 L_x^2}^2$$
$$\lesssim \|(1+r)^{p/2}Z^{\le N}\partial u(0, \cdot)\|_{L^2}^2 + \|(1+r)^{(p+1)/2}Z^{\le N}\Box u\|_{L_t^2 L_x^2}^2. \quad (2\text{-}4)$$

Proof. We first note that

$$\int_0^\infty \int |\Box u|(|\partial u| + \langle x \rangle^{-1}|u|) \, dx \, dt$$
$$\le \|(1+r)^{(p+1)/2}\Box u\|_{L_t^2 L_x^2}\big(\|(1+r)^{-(p+1)/2}\partial u\|_{L_t^2 L_x^2} + \|(1+r)^{-(p+3)/2}u\|_{L_t^2 L_x^2}\big),$$

and that

$$\|(1+r)^{-(p+1)/2}\partial u\|_{L_t^2 L_x^2} + \|(1+r)^{-(p+3)/2}u\|_{L_t^2 L_x^2}$$
$$\lesssim \sup_{j\ge 0} 2^{-j/2}\|\partial u\|_{L_t^2 L_x^2(\mathbb{R}_+ \times \{\langle x \rangle \approx 2^j\})} + \sup_{j\ge 0} 2^{-3j/2}\|u\|_{L_t^2 L_x^2(\mathbb{R}_+ \times \{\langle x \rangle \approx 2^j\})}.$$

Thus, by bootstrapping this factor into the left side of (2-1), we see from (2-1) that

$$\|\partial u\|_{L_t^\infty L_x^2} \lesssim \|\partial u(0, \cdot)\|_{L^2} + \|(1+r)^{(p+1)/2} \square u\|_{L_t^2 L_x^2}.$$

Since $[\square, Z] = 0$ and since $[Z, \partial] \in \mathrm{span}(\partial)$, the bound for the first term in (2-4) follows by replacing u by $Z^{\leq N} u$.

Since

$$[\partial_i, \partial_t + \partial_r] = \frac{1}{r} \slashed{\nabla}_i, \quad [\partial_i, \slashed{\nabla}_j] = \frac{1}{r}\left(-\delta_{ij} + \frac{x_i x_j}{r^2}\right)\partial_r - \frac{1}{r}\frac{x_j}{r}\slashed{\nabla}_i,$$

$$[\Omega_{ij}, \slashed{\nabla}_k] = \delta_{jk}\slashed{\nabla}_i - \delta_{ik}\slashed{\nabla}_j, \quad [\Omega_{ij}, \partial_t + \partial_r] = 0$$

and since $|\slashed{\nabla} u| \leq (1/r)|\Omega u|$, we have that $|[Z, \slashed{\partial}]u| \leq (1/r)|Zu|$. Thus the remainder of the proof follows upon replacing u by $Z^{\leq N} u$ in (2-2). We may readily replace r by $1+r$ in the $L_t^2 L_x^2$-terms since the powers in the left are negative, while powers in the right are positive. We also note that, due to a Hardy-type inequality,

$$\widetilde{E}[u](t) \lesssim \|(1+r)^{p/2}\partial u(t, \cdot)\|_{L^2}^2. \qquad \square$$

3. Proof of Theorem 1.1

The decay that we require will be obtained from the following weighted Sobolev estimate of [Klainerman 1986]. This estimate only provides decay in $|x|$, but simultaneously it does not necessitate the use of any time-dependent vector fields.

Lemma 3.1. *For $h \in C^\infty(\mathbb{R}^3)$ and $R \geq 1$,*

$$\|h\|_{L^\infty(\{R/2 < \langle x \rangle < R\})} \lesssim R^{-1}\|Z^{\leq 2}h\|_{L^2(\{R/4 < \langle x \rangle < 2R\})}. \tag{3-1}$$

The bound (3-1) follows, after localizing appropriately, from applying Sobolev estimates in the r- and ω-variables separately and comparing the volume element $dr\,d\sigma(\omega)$ with that of $\mathbb{R}^3$ in spherical coordinates: $r^2\,dr\,d\sigma(\omega)$.

As mentioned earlier, the null condition (1-2) guarantees that at least one of the two factors in each nonlinear term is a "good" derivative. In fact, using a product rule argument, we have

$$|Z^{\leq 10}Q(\partial u)| \lesssim |Z^{\leq 5}\partial u||Z^{\leq 10}\slashed{\partial}u| + |Z^{\leq 5}\slashed{\partial}u||Z^{\leq 10}\partial u|. \tag{3-2}$$

This is well known; we refer the reader to, e.g., [Lindblad et al. 2013, Lemma 2.3].

We will use an iteration to solve (1-1). We let $u_{-1} \equiv 0$ and let u_k solve

$$\begin{cases} \square u_k = Q(\partial u_{k-1}), \\ u_k(0, \cdot) = f, \quad \partial_t u_k(0, \cdot) = g. \end{cases}$$

Boundedness: Our first step is to show an appropriate boundedness of this iteration. To this end, we shall set

$$M_k = \|Z^{\leq 10}\partial u_k\|_{L_t^\infty L_x^2} + \|(1+r)^{(p-1)/2}Z^{\leq 10}\slashed{\partial}u_k\|_{L_t^2 L_x^2} + \|(1+r)^{(p-3)/2}Z^{\leq 10}u_k\|_{L_t^2 L_x^2}.$$

Due to (2-4) and (1-3), there is a constant C_0 so that

$$M_0 \leq C_0 \varepsilon.$$

We shall argue inductively that for every k

$$M_k \leq 2 C_0 \varepsilon. \tag{3-3}$$

To show (3-3), we use (2-4), which provides the bound

$$M_k \leq C_0 \varepsilon + C \| (1+r)^{(p+1)/2} Z^{\leq 10} Q(\partial u_{k-1}) \|_{L_t^2 L_x^2}.$$

Applying (3-2) and (3-1) we obtain

$$\| (1+r)^{(p+1)/2} Z^{\leq 10} Q(\partial u_{k-1}) \|_{L_t^2 L_x^2}$$
$$\lesssim \| (1+r)^{(p+1)/2} | Z^{\leq 5} \partial u_{k-1} | | Z^{\leq 10} \not\partial u_{k-1} | \|_{L_t^2 L_x^2}$$
$$+ \| (1+r)^{(p+1)/2} | Z^{\leq 5} \not\partial u_{k-1} | | Z^{\leq 10} \partial u_{k-1} | \|_{L_t^2 L_x^2}$$
$$\lesssim \| Z^{\leq 7} \partial u_{k-1} \|_{L_t^\infty L_x^2} \| (1+r)^{(p-1)/2} Z^{\leq 10} \not\partial u_{k-1} \|_{L_t^2 L_x^2}$$
$$+ \| (1+r)^{(p-1)/2} Z^{\leq 7} \not\partial u_{k-1} \|_{L_t^2 L_x^2} \| Z^{\leq 10} \partial u_{k-1} \|_{L_t^\infty L_x^2}.$$

Thus, using the inductive hypothesis, it follows that

$$M_k \leq C_0 \varepsilon + C (M_{k-1})^2 \leq C_0 \varepsilon + C \cdot C_0^2 \varepsilon^2.$$

And if $\varepsilon < 1/(C \cdot C_0)$, (3-3) results as desired.

Cauchy: We complete the proof by showing that the sequence is Cauchy in an appropriate norm. By completeness, the sequence must converge and by standard results the limiting function solves (1-1) as desired.

To this end, we set

$$A_k = \| Z^{\leq 10} \partial(u_k - u_{k-1}) \|_{L_t^\infty L_x^2} + \| (1+r)^{(p-1)/2} Z^{\leq 10} \not\partial(u_k - u_{k-1}) \|_{L_t^2 L_x^2}$$
$$+ \| (1+r)^{(p-3)/2} Z^{\leq 10} (u_k - u_{k-1}) \|_{L_t^2 L_x^2}.$$

We note that

$$Q^I(\partial u_{k-1}) - Q^I(\partial u_{k-2})$$
$$= A_{JK}^{\alpha\beta,I} \partial_\alpha (u_{k-1}^J - u_{k-2}^J) \partial_\beta u_{k-1}^K + A_{JK}^{\alpha\beta,I} \partial_\alpha u_{k-2}^J \partial_\beta (u_{k-1}^K - u_{k-2}^K).$$

Thus, as in (3-2), we obtain

$$|Z^{\leq 10} Q(\partial u_{k-1}) - Z^{\leq 10} Q(\partial u_{k-2})|$$
$$\lesssim |Z^{\leq 5} \not\partial(u_{k-1} - u_{k-2})| (|Z^{\leq 10} \partial u_{k-1}| + |Z^{\leq 10} \partial u_{k-2}|)$$
$$+ (|Z^{\leq 5} \partial u_{k-1}| + |Z^{\leq 5} \partial u_{k-2}|) |Z^{\leq 10} \not\partial(u_{k-1} - u_{k-2})|$$
$$+ |Z^{\leq 5} \partial(u_{k-1} - u_{k-2})| (|Z^{\leq 10} \not\partial u_{k-1}| + |Z^{\leq 10} \not\partial u_{k-2}|)$$
$$+ (|Z^{\leq 5} \not\partial u_{k-1}| + |Z^{\leq 5} \not\partial u_{k-2}|) |Z^{\leq 10} \partial(u_{k-1} - u_{k-2})|. \tag{3-4}$$

As above, we apply (3-1) to the lower-order factor in each term to see that

$$\|(1+r)^{(p+1)/2}(Z^{\leq 10}Q(\partial u_{k-1}) - Z^{\leq 10}Q(\partial u_{k-2}))\|_{L_t^2 L_x^2}$$

$$\lesssim \|(1+r)^{(p-1)/2}Z^{\leq 7}\partial\!\!\!/(u_{k-1}-u_{k-2})\|_{L_t^2 L_x^2}$$
$$\times \left(\|Z^{\leq 10}\,\partial u_{k-1}\|_{L_t^\infty L_x^2} + \|Z^{\leq 10}\,\partial u_{k-2}\|_{L_t^\infty L_x^2}\right)$$

$$+\left(\|Z^{\leq 7}\,\partial u_{k-1}\|_{L_t^\infty L_x^2} + \|Z^{\leq 7}\,\partial u_{k-2}\|_{L_t^\infty L_x^2}\right)$$
$$\times \|(1+r)^{(p-1)/2}Z^{\leq 10}\partial\!\!\!/(u_{k-1}-u_{k-2})\|_{L_t^2 L_x^2}$$

$$+\|Z^{\leq 7}\,\partial(u_{k-1}-u_{k-2})\|_{L_t^\infty L_x^2}$$
$$\times \left(\|(1+r)^{(p-1)/2}Z^{\leq 10}\partial\!\!\!/u_{k-1}\|_{L_t^2 L_x^2} + \|(1+r)^{(p-1)/2}Z^{\leq 10}\partial\!\!\!/u_{k-2}\|_{L_t^2 L_x^2}\right)$$

$$+\left(\|(1+r)^{(p-1)/2}Z^{\leq 7}\partial\!\!\!/u_{k-1}\|_{L_t^2 L_x^2} + \|(1+r)^{(p-1)/2}Z^{\leq 7}\partial\!\!\!/u_{k-2}\|_{L_t^2 L_x^2}\right)$$
$$\times \|Z^{\leq 10}\,\partial(u_{k-1}-u_{k-2})\|_{L_t^\infty L_x^2}.$$

From (2-4) it then follows that

$$A_k \leq C(M_{k-1} + M_{k-2})A_{k-1} \leq C \cdot C_0 \varepsilon A_{k-1}.$$

So long as, say, $\varepsilon < 1/(2C \cdot C_0)$, we obtain

$$A_k \leq \tfrac{1}{2}A_{k-1} \quad \text{for all } k,$$

which implies that the sequence is Cauchy and completes the proof.

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