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K. R. Goodearl  
M. T. Yakimov

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# Abstract

Various coordinate rings of varieties appearing in the theory of Poisson Lie groups and Poisson homogeneous spaces belong to the large, axiomatically defined class of symmetric Poisson nilpotent algebras, e.g. coordinate rings of Schubert cells for symmetrizable Kac–Moody groups, affine charts of Bott–Samelson varieties, coordinate rings of double Bruhat cells (in the last case after a localization). We prove that every symmetric Poisson nilpotent algebra satisfying a mild condition on certain scalars is canonically isomorphic to a cluster algebra which coincides with the corresponding upper cluster algebra, without additional localizations by frozen variables. The constructed cluster structure is compatible with the Poisson structure in the sense of Gekhtman, Shapiro and Vainshtein. All Poisson nilpotent algebras are proved to be equivariant Poisson Unique Factorization Domains. Their seeds are constructed from sequences of Poisson-prime elements for chains of Poisson UFDs; mutation matrices are effectively determined from linear systems in terms of the underlying Poisson structure. Uniqueness, existence, mutation, and other properties are established for these sequences of Poisson-prime elements.

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K.R.G. is affiliated with the Department of Mathematics, University of California, Santa Barbara, CA 93106.

M.T.Y. is affiliated with the Department of Mathematics, Northeastern University, Boston, MA, 02115.

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## CHAPTER 1

# Introduction

### 1.1. Cluster algebras and coordinate rings

Cluster algebras comprise a large, axiomatically defined class of algebras, introduced by Fomin and Zelevinsky [16]. They play a key role in representation theory, combinatorics, mathematical physics, algebraic and Poisson geometry. We refer the reader to the books [21, 37] and the papers [14, 33, 35, 45] for overviews of some of the applications of cluster algebras.

A *cluster algebra* (of *geometric type*) is defined by iteratively mutating generating sets of a rational function field  $\mathcal{F}$  in  $N$  variables over a base field  $\mathbb{K}$ . The generating sets  $(y_1, \dots, y_N)$  of  $\mathcal{F}$  are called *clusters*, and the process depends on a mutation matrix  $\tilde{B}$  associated to an initial cluster. The cluster algebra  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$  (namely, the integral cluster algebra  $\mathcal{A}(\tilde{B})_{\mathbb{Z}}$  with scalars extended to  $\mathbb{K}$ ) is the  $\mathbb{K}$ -subalgebra of  $\mathcal{F}$  generated by all cluster variables (both mutable and frozen), none of which are automatically inverted. We will write  $\mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$  for the localization of  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$  obtained by inverting a subset  $\mathbf{inv}$  of the frozen variables. By the Fomin-Zelevinsky Laurent Phenomenon [17], the algebra  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$  is a subalgebra of the *upper cluster algebra*  $\overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}}$ , obtained by intersecting in  $\mathcal{F}$  the mixed polynomial/Laurent polynomial rings of all clusters, in which the cluster variables are inverted but the frozen variables are not. The analogous intersection of polynomial/Laurent polynomial rings in which a set  $\mathbf{inv}$  of frozen variables is inverted will be denoted  $\overline{\mathcal{A}}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$ . One of the main goals of the program of Fomin and Zelevinsky [16] is

**PROBLEM A.** *Prove that the coordinate rings  $\mathbb{K}[V]$  of algebraic varieties  $V$  that appear in Lie theory have canonical cluster algebra structures, i.e.,  $\mathbb{K}[V] = \mathcal{A}(\tilde{B})_{\mathbb{K}}$  or, more generally,  $\mathbb{K}[V] = \mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$ .*

The more general use of  $\mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$  is necessary due to the concrete properties of the varieties  $V$  in question. The consequences of this are that one can apply the cluster algebra structure of  $\mathbb{K}[V]$  to the study of the totally nonnegative part of  $V$  and the canonical basis of  $\mathbb{K}[V]$ , in the cases when they are defined. After the work of Berenstein, Fomin and Zelevinsky [2], it became clear that another desirable property of cluster algebras is their equality with upper cluster algebras (when it holds), thus prompting

**PROBLEM B.** *Classify the coordinate rings  $\mathbb{K}[V]$  from Problem A for which the corresponding cluster algebras have the property  $\mathcal{A}(\tilde{B})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}}$  or, more generally,  $\mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$  for the same set  $\mathbf{inv}$  that is used in the solution of Problem A (equality of cluster and upper cluster algebras).*

Here is an illustration of Problems A and B: Berenstein, Fomin and Zelevinsky [2] proved that the coordinate rings of all double Bruhat cells  $G^{w,u}$  in complex simple Lie groups  $G$  possess upper cluster algebra structures:  $\mathbb{C}[G^{w,u}] = \overline{\mathcal{A}}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$ . The problem of whether  $\mathbb{C}[G^{w,u}] = \mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$ , or equivalently of whether Problem B has a positive solution in this case, was left open.

Gekhtman, Shapiro and Vainshtein [20, 21] proved that under mild assumptions a cluster algebra  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$  has a Poisson algebra structure with respect to which all clusters are *log-canonical*, which means that

$$\{y_l, y_j\} = \lambda_{lj} y_l y_j, \quad \lambda_{lj} \in \mathbb{K}^*$$

for the cluster variables  $y_1, \dots, y_N$  in each cluster of  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$ . Gekhtman, Shapiro and Vainshtein set up a program of proving that the coordinate rings of Poisson varieties  $V$  in Lie theory possess natural cluster structures by constructing sufficiently many log-canonical clusters on  $V$  (connected by mutations) and using their coordinates as cluster variables [21, 22]. This has had remarkable success, and led to the construction of compatible upper cluster algebra structures for the Belavin-Drinfeld Poisson structures on  $GL_n$  and their doubles, [22, 23]. However, similarly to [2], the methods of [20–23] rely on codimension 2 arguments and only show that certain  $\mathbb{K}[V]$ 's are isomorphic to upper cluster algebras rather than to cluster algebras.

There are three instances of positive solutions to Problem B. For acyclic cluster algebras this was shown in [2]. Muller [39] proved that Problem B has a positive solution for the larger class of locally acyclic cluster algebras, but only in the form  $\mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$  where  $\mathbf{inv}$  includes all frozen variables. (This extra localization does not allow to deduce that the coordinate rings of double Bruhat cells of type  $A_n$  are cluster algebras.) Geiss, Leclerc and Schröer [18] proved that the coordinate rings of Schubert cells in symmetric Kac-Moody groups possess cluster algebra structures such that  $\mathcal{A}(\tilde{B})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}}$ , using the representation theory of the related reprojective algebra.

## 1.2. Overview of results

The results obtained in this paper and the sequel [30] are as follows:

(I) In the present paper we show that, modulo a mild condition on scalars, every Poisson algebra  $R$  in the very large axiomatically defined class of symmetric Poisson nilpotent algebras possesses a natural cluster algebra structure and for it  $R = \mathcal{A}(\tilde{B})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}}$ . For each such cluster algebra we construct a large family of clusters and explicit mutations between them.

(II) In the sequel [30], we prove that the coordinate rings of all Schubert cells and double Bruhat cells for a symmetrizable Kac-Moody group, equipped with their standard Poisson structures, are symmetric Poisson nilpotent algebras or localizations thereof. We then establish that, for the double Bruhat cells of every complex simple Lie group, the Berenstein-Fomin-Zelevinsky cluster algebras equal the corresponding upper cluster algebras. We also prove such a result for the upper cluster algebra structures on the double Bruhat cells of all symmetrizable Kac-Moody groups, constructed by Williams [44].

Compared to the approaches in [18, 20–22] to Problems A and B, our general results do not rely on concrete initial combinatorial data and apply to algebras coming from symmetrizable Kac-Moody groups, not only symmetric Kac-Moody

groups. Compared to [39], the above results prove equalities of cluster algebras and upper cluster algebras without an additional localization by frozen variables.

### 1.3. Poisson nilpotent algebras and Main Theorem

Fix a base field  $\mathbb{K}$  throughout, with  $\text{char } \mathbb{K} = 0$ , and consider everything to be done over  $\mathbb{K}$ . In particular, the term “algebra” will always mean a  $\mathbb{K}$ -algebra, all automorphisms are assumed to be  $\mathbb{K}$ -algebra automorphisms, and all derivations are assumed to be  $\mathbb{K}$ -linear.

The Poisson nilpotent algebras (or Poisson-CGL extensions) with which we work are iterated *Poisson-Ore extensions*  $B[x; \sigma, \delta]_p$ , where  $B$  is a Poisson algebra,  $B[x; \sigma, \delta]_p = B[x]$  is a polynomial ring,  $\sigma$  and  $\delta$  are suitable Poisson derivations on  $B$ , and

$$\{x, b\} = \sigma(b)x + \delta(b), \quad \forall b \in B$$

(see §2.2). For an iterated Poisson-Ore extension

$$R := \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_N; \sigma_N, \delta_N]_p$$

and  $k \in [0, N]$ , denote

$$R_k := \mathbb{K}[x_1, \dots, x_k] = \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_k; \sigma_k, \delta_k]_p.$$

**DEFINITION.** An iterated Poisson-Ore extension  $R$  as above will be called a *Poisson-CGL extension* if it is equipped with a rational Poisson action of a torus  $\mathcal{H}$  such that

- (i) The elements  $x_1, \dots, x_N$  are  $\mathcal{H}$ -eigenvectors.
- (ii) For every  $k \in [2, N]$ , the map  $\delta_k$  on  $R_{k-1}$  is locally nilpotent.
- (iii) For every  $k \in [1, N]$ , there exists  $h_k \in \text{Lie } \mathcal{H}$  such that  $\sigma_k = (h_k \cdot)|_{R_{k-1}}$  and the  $h_k$ -eigenvalue of  $x_k$ , to be denoted by  $\lambda_k$ , is nonzero.

There is a unique choice of  $\mathcal{H}$ , maximal among tori acting faithfully and rationally on  $R$  by Poisson automorphisms such that the above conditions hold, as we prove in §6.2.

We say that the Poisson-CGL extension  $R$  is *symmetric* if it has the above properties (with respect to the same torus action) when the generators  $x_1, \dots, x_N$  are adjoined in reverse order. (This condition is of a different nature than the symmetry condition for Kac-Moody groups; all symmetrizable Kac-Moody groups give rise to symmetric Poisson CGL extensions in different ways.)

Poisson-CGL extensions are semiclassical analogs of the *Cauchon-Goodearl-Letzter extensions*, appearing in the theory of quantum groups. They have been extensively studied in recent years beginning with the papers [8, 26, 34]. CGL extensions provide a natural axiomatic framework for *quantum nilpotent algebras*: deformations of universal enveloping algebras of nilpotent Lie algebras at nonroots of unity [27, p. 9697]. Because of this, we view Poisson-CGL extensions as an axiomatic framework for *Poisson nilpotent algebras*, i.e., *Poisson algebra structures on polynomial rings which are semiclassical limits of quantum nilpotent algebras*. We note that a Poisson CGL extension is not a nilpotent Lie algebra itself under the Poisson bracket (Poisson algebras with such a property are very rare in the theory of Poisson Lie groups and Poisson homogeneous spaces).

Coordinate rings of varieties that appear in the theory of Poisson Lie groups and Poisson homogeneous spaces provide numerous examples of Poisson-CGL extensions, see for example [13] for coordinate rings of Schubert cells and affine charts of Bott-Samelson varieties. The coordinate rings of such varieties are semiclassical limits of quantized algebras of functions and the above axioms are Poisson incarnations of the Levendorskii-Soibelman straightening law for quantum groups [5, Proposition I.6.10]. A simple example of such a Poisson algebra, which is used as a running example through the paper, is the coordinate ring of the matrix affine Poisson space  $M_{m,n}(\mathbb{K})$  of  $m \times n$  matrices over  $\mathbb{K}$ . It is the polynomial algebra

$$\mathcal{O}(M_{m,n}(\mathbb{K})) = \mathbb{K}[t_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$$

with the Poisson bracket

$$\{t_{ij}, t_{kl}\} = (\text{sign}(k - i) + \text{sign}(l - j))t_{il}t_{kj},$$

see Example 5.3 for details. This Poisson structure arises when the space  $M_{m,n}(\mathbb{K})$  is identified with the open Schubert cell of the Grassmannian  $\text{Gr}(m, m+n)$  equipped with the standard Poisson structure, [6, 20].

The next theorem summarizes the main result of the paper:

**MAIN THEOREM.** *Every symmetric Poisson-CGL extension  $R$  with the property that  $\lambda_l/\lambda_j \in \mathbb{Q}_{>0}$  for all  $l, j$  has a canonical structure of cluster algebra which coincides with its upper cluster algebra.*

A detailed formulation of this result is given in Theorem 11.1. Additional features of the full formulation of the theorem are:

- (1) For each such Poisson-CGL extension  $R$ , the theorem constructs a large family of explicit seeds and mutations between them.
- (2) The Poisson structure on  $R$  is compatible with the cluster structure in the sense of Gekhtman, Shapiro and Vainshtein.
- (3) The cluster variables in the constructed seeds are the unique homogeneous Poisson-prime elements of Poisson-CGL (sub)extensions that do not belong to smaller subextensions.
- (4) The mutation matrices of the constructed seeds can be effectively computed using linear systems of equations, coming from the Poisson structure.
- (5) For each generator  $x_k$ , an appropriate rescaling of it is a cluster variable in one of our seeds. In particular, the cluster variables in our finite collection of seeds generate the cluster algebra in question.
- (6) The above features also hold for the localization of  $R$  with respect to any set **inv** of frozen variables; in particular,  $R = \mathcal{A}(\tilde{B}, \mathbf{inv})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B}, \mathbf{inv})_{\mathbb{K}}$ .

#### 1.4. Poisson-prime elements, Poisson-UFDs and proof of the Main Theorem

Our proofs rely on arguments with Unique Factorization Domains in the Poisson algebra setting. Let  $R$  be a noetherian Poisson domain (i.e., a Poisson algebra which is a noetherian integral domain). An element  $p \in R$  is called a *Poisson-prime element* if any of the following equivalent conditions is satisfied:

- (1) The ideal  $(p)$  is a prime ideal and a Poisson ideal.
- (2)  $p$  is a prime element of  $R$  (in the commutative sense) such that  $p \mid \{p, -\}$ .

- (3) [Assuming  $\mathbb{K} = \mathbb{C}$ .]  $p$  is a prime element of  $R$  and the zero locus  $V(p)$  is a union of symplectic leaves of the maximal spectrum of  $R$ ,

see §4.1 for details. The importance of the third formulation is that one can use Poisson geometry to classify Poisson-prime elements.

Recall that by Nagata's lemma, a noetherian integral domain  $R$  is a UFD if and only if every nonzero prime ideal contains a prime element. We call a Poisson domain  $R$  a *Poisson-UFD* if every nonzero Poisson-prime ideal contains a Poisson-prime element. If  $R$  is equipped with a Poisson action of a group  $\mathcal{H}$ , we say that  $R$  is an  $\mathcal{H}$ -*Poisson-UFD* if every nonzero  $\mathcal{H}$ -Poisson-prime ideal of  $R$  contains a Poisson-prime  $\mathcal{H}$ -eigenvector. In the case of a  $\mathbb{K}$ -torus  $\mathcal{H}$  acting rationally on  $R$ , the  $\mathcal{H}$ -eigenvectors are precisely the nonzero homogeneous elements of  $R$  with respect to its grading by the character group of  $\mathcal{H}$ , and we will use the second terminology.

Our proof of the Main Theorem is based on the following steps. The results in the individual steps are of independent interest and often admit wider generality than that of the Main Theorem.

STEP 1. Let  $B$  be a noetherian  $\mathcal{H}$ -Poisson-UFD for a  $\mathbb{K}$ -torus  $\mathcal{H}$ , and  $B[x; \sigma, \delta]_p$  a Poisson-Ore extension with  $\sigma$  and  $\delta$  satisfying the properties (i)-(iii) in the definition of a Poisson-CGL extension. We prove that  $B[x; \sigma, \delta]_p$  is also a noetherian  $\mathcal{H}$ -Poisson-UFD and give an explicit classification of its homogeneous Poisson-prime elements in terms of those of  $B$ . These results appear in Theorems 4.7 and 4.13. Section 4 contains other details on the Poisson-UFD properties of the extension  $B \subset B[x; \sigma, \delta]_p$ .

STEP 2. For each Poisson-CGL extension  $R$  and  $k \in [1, N]$ , we prove that the Poisson algebra  $R_k$  has a unique (up to rescaling) homogeneous Poisson-prime element  $y_k$  that does not belong to  $R_{k-1}$ . For the sequence

$$(1.1) \quad y_1, \dots, y_N \text{ of homogeneous Poisson-prime elements of}$$

$$\text{the algebras in the chain } R_1 \subset \dots \subset R_N,$$

we prove that each  $y_k$  is linear in  $x_k$  with leading term  $y_j x_k$  for some  $j < k$  if  $\delta_k \neq 0$ , and  $y_k = x_k$  if  $\delta_k = 0$ . These facts are proved in Theorem 5.5 and Section 5 contains additional facts for the sequences of Poisson-primes (1.1).

STEP 3. Let  $R$  be a symmetric Poisson-CGL extension. Each permutation in the set

$$(1.2) \quad \Xi_N := \{\tau \in S_N \mid \tau([1, k]) \text{ is an interval for all } k \in [2, N]\}$$

gives rise to a presentation of  $R$  as a Poisson-CGL extension with the generators of  $R$  adjoined in the order  $x_{\tau(1)}, \dots, x_{\tau(N)}$ . The associated sequence of Poisson-prime elements from Step 2 will be denoted by

$$(1.3) \quad y_{\tau,1}, \dots, y_{\tau,N}.$$

For a symmetric Poisson-CGL extension  $R$  and  $j, k \in [1, N]$ , the “interval subalgebra”

$$R_{[j,k]} := \mathbb{K}[x_i \mid j \leq i \leq k]$$

is an  $\mathcal{H}$ -stable Poisson subalgebra of  $R$  which is a Poisson-CGL extension in its own right with respect to the same torus action of  $\mathcal{H}$ .

In Theorem 8.3 we express the elements of each sequence (1.3) in terms of certain “final Poisson-prime elements” of the interval subalgebras  $R_{[j,k]}$ . Sections 6

and 8 contain further facts on symmetric Poisson CGL extensions and their interval subalgebras.

STEP 4. Next we link the clusters from the previous step by mutations. In Theorems 7.2 and 7.3 we prove that if a Poisson-CGL extension has two presentations by adjoining the generators in the orders  $x_1, \dots, x_N$  and  $x_1, \dots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \dots, x_N$ , then the corresponding sequences of Poisson-prime elements (1.1) are either a permutation of each other or a one-step “almost mutation” of each other, meaning that the new elements  $y$  are given by a mutation formula in terms of the old ones where the coefficient of one of the two monomials is not necessarily 1 but a nonzero scalar.

In Theorem 9.5 we prove that the generators  $x_1, \dots, x_N$  of any symmetric Poisson-CGL extension can be rescaled so that the sequences of Poisson-prime elements (1.3) corresponding to any pair  $\tau, \tau' \in \Xi_N$  such that  $\tau' = \tau(i, i+1)$  are either a permutation or a one-step mutation of each other. One of the upshots of Steps 2 and 4 is that the rescaling of each of the generators  $x_k$  equals the cluster variable  $y_{\tau,1}$ , for all  $\tau \in \Xi_N$  with  $\tau(1) = k$ . Sections 7 and 9 contain further details on the rescaling and mutations.

STEP 5. In Section 10 we describe a method to control the size of the involved upper cluster algebras from above. In Theorems 10.3 and 10.5 we prove that the intersection inside the fraction field of  $R$  of all mixed polynomial/Laurent polynomial rings associated to the clusters indexed by  $\Xi_N$  (without the frozen variables inverted) equals  $R$ .

STEP 6. Using the mutations from Step 4, we upgrade the clusters (1.3) to seeds of cluster algebras and prove that the corresponding mutation matrices can be effectively computed using linear systems of equations, coming from the Poisson structure (Theorem 11.1(a)). Using Steps 3–5 and the Laurent Phenomenon [17] we get the chain of inclusions for the constructed cluster algebras  $\mathcal{A}(\tilde{B})_{\mathbb{K}}$

$$R \subseteq \mathcal{A}(\tilde{B})_{\mathbb{K}} \subseteq \overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}} \subseteq R,$$

which forces the equalities  $R = \mathcal{A}(\tilde{B})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{B})_{\mathbb{K}}$ ; this is carried out in Section 11.

One of the consequences of the Main Theorem is that every symmetric Poisson-CGL extension with  $\lambda_i/\lambda_j \in \mathbb{Q}_{>0}$  possesses a canonical quantization given by the quantum cluster algebra [3] associated to the constructed Poisson cluster algebra. Recently, Y. Mi [38] independently constructed a quantization of each integral symmetric Poisson CGL extension that is a symmetric CGL extension, and proved that in a certain sense this quantization is unique. (Here integrality means that the elements  $h_k$  in the definition in §1.3 belong to the cocharacter lattice of  $\mathcal{H}$ .) The two quantizations can then be linked by the quantum cluster algebra structure on the latter constructed in [29, Theorem 8.2].

We note that, although the main results in this paper resemble the quantum results in [29], the proofs use substantially different methods based on Poisson algebra and related geometric structures.

### 1.5. Notation and conventions

We write  $X(\mathcal{H})$  for the (rational) character group of a  $\mathbb{K}$ -torus  $\mathcal{H}$ , and we view  $X(\mathcal{H})$  as an additive group. Rational actions of  $\mathcal{H}$  on an algebra  $R$  are

equivalent to  $X(\mathcal{H})$ -gradings [5, Lemma II.2.11], and we will use the terminology of gradings whenever convenient. In particular, when  $R$  is equipped with a rational  $\mathcal{H}$ -action, its homogeneous elements (with respect to the  $X(\mathcal{H})$ -grading) are the  $\mathcal{H}$ -eigenvectors together with 0. The degree of a nonzero homogeneous element  $r \in R$  will be denoted  $\chi_r$ ; this is just its  $\mathcal{H}$ -eigenvalue.

For any positive integer  $N$ , we view the elements of  $\mathbb{Z}^N$  as column vectors, and we write  $\{e_1, \dots, e_N\}$  for the standard basis of  $\mathbb{Z}^N$ . The row vector corresponding to any  $f \in \mathbb{Z}^N$  is the transpose of  $f$ , denoted  $f^T$ . If  $f = (m_1, \dots, m_N)^T$ , denote

$$(1.4) \quad [f]_+ := \sum_{j=1}^N \max(m_j, 0)e_j \quad \text{and} \quad [f]_- := \sum_{j=1}^N \min(m_j, 0)e_j.$$

We identify permutations in  $S_N$  with their corresponding permutation matrices in  $GL_N(\mathbb{Z})$ , so that each  $\tau \in S_N$  acts on  $\mathbb{Z}^N$  by  $\tau(e_i) = e_{\tau(i)}$ , for all  $i \in [1, N]$ . Corresponding to any skew-symmetric matrix  $\mathbf{q} = (q_{kj}) \in M_N(\mathbb{K})$  is a skew-symmetric bicharacter

$$(1.5) \quad \Omega_{\mathbf{q}} : \mathbb{Z}^N \times \mathbb{Z}^N \rightarrow \mathbb{K}, \quad \text{given by} \quad \Omega_{\mathbf{q}}(e_k, e_j) = q_{kj}, \quad \forall j, k \in [1, N].$$

Given  $f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N$  and an  $N$ -tuple  $\mathbf{x} = (x_1, \dots, x_N)$  of elements from a commutative ring, we denote

$$(1.6) \quad \mathbf{x}^f := \prod_{j=1}^N x_j^{m_j}.$$

We recall definitions and establish some auxiliary results for Poisson algebras and cluster algebras in Sections 2 and 3, respectively. Here we just mention that since our aim is to produce cluster algebra structures on  $\mathbb{K}$ -algebras, we build cluster algebras  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})_{\mathbb{K}}$  over the field  $\mathbb{K}$  directly, rather than first building cluster algebras over an integral (semi)group ring and then extending scalars to  $\mathbb{K}$ . Moreover, we work exclusively with cluster algebras of geometric type, as semiclassical limits of quantum cluster algebras are of geometric type, and as the cluster algebra structures we construct on Poisson nilpotent algebras are of geometric type.



## CHAPTER 2

# Poisson algebras

### 2.1. Poisson algebras and Poisson ideals

We recall that a *Poisson algebra* is a commutative algebra  $R$  equipped with a Poisson bracket  $\{-, -\}$ , that is, a Lie bracket which is also a derivation in each variable (for the associative product). For  $a \in R$ , the derivation  $\{a, -\}$  is called the *Hamiltonian associated to  $a$* . The Poisson bracket on  $R$  induces unique Poisson brackets on any quotient of  $R$  modulo a *Poisson ideal*, meaning an ideal  $I$  such that  $\{R, I\} \subseteq I$ , and on any localization of  $R$  with respect to a multiplicative set (e.g., [36, Proposition 1.7]). A *Poisson automorphism* of  $R$  is any algebra automorphism which preserves the Poisson bracket. We use the term *Poisson action* to refer to an action of a group on  $R$  by Poisson automorphisms.

EXAMPLES 2.1. A *Poisson-Weyl algebra* is a polynomial algebra  $\mathbb{K}[x_1, \dots, x_{2n}]$  equipped with the Poisson bracket such that

$$\{x_i, x_j\} = \{x_{n+i}, x_{n+j}\} = 0 \text{ and } \{x_i, x_{n+j}\} = \delta_{ij}, \forall i, j \in [1, n].$$

Given any skew-symmetric matrix  $(q_{kj}) \in M_N(\mathbb{K})$ , there are compatible Poisson brackets on the polynomial algebra  $\mathbb{K}[x_1, \dots, x_N]$  and the Laurent polynomial algebra  $\mathbb{K}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$  such that  $\{x_k, x_j\} = q_{kj}x_kx_j$  for all  $k, j$ . The Poisson algebras  $\mathbb{K}[x_1, \dots, x_N]$  and  $\mathbb{K}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$  are known as a *Poisson affine space algebra* and a *Poisson torus*, respectively.

The *Poisson center* of a Poisson algebra  $R$  is the subalgebra

$$Z_p(R) := \{z \in R \mid \{z, -\} = 0\}.$$

Its elements are sometimes called *Casimirs*, in which case  $Z_p(R)$  is denoted  $\text{Cas}(R)$ . An element  $c \in R$  is said to be *Poisson-normal* if the principal ideal  $Rc$  is a Poisson ideal, that is, if  $\{c, R\} \subseteq Rc$ . If also  $c$  is a non-zero-divisor, then  $\{c, -\}$  determines a derivation  $\partial_c$  on  $R$  such that

$$(2.1) \quad \{c, a\} = \partial_c(a)c, \forall a \in R.$$

Moreover, it follows from the Jacobi identity for  $\{-, -\}$  that

$$\partial_c(\{a, b\}) = \{\partial_c(a), b\} + \{a, \partial_c(b)\}, \forall a, b \in R,$$

so that  $\partial_c$  is also a derivation with respect to the Poisson bracket.

Given an arbitrary ideal  $J$  in  $R$ , there is a largest Poisson ideal contained in  $J$ , which, following [7], we call the *Poisson core of  $J$* ; we shall denote it  $\text{P.core}(J)$ . The *Poisson primitive ideals* of  $R$  are the Poisson cores of the maximal ideals of  $R$  (in [41], these are called *symplectic ideals*). A *Poisson-prime* ideal is any proper Poisson ideal  $P$  of  $R$  such that  $(IJ \subseteq P \implies I \subseteq P \text{ or } J \subseteq P)$  for all Poisson ideals  $I$  and  $J$  of  $R$ . The above concepts are related by the following Poisson version of [24, Lemma 1.1]; we repeat the short arguments for convenience.

LEMMA 2.2. *Let  $R$  be a Poisson algebra.*

- (a)  *$\text{P.core}(P)$  is prime for all prime ideals  $P$  of  $R$ .*
- (b) *Every Poisson primitive ideal of  $R$  is prime.*
- (c) *Every prime ideal minimal over a Poisson ideal is a Poisson ideal.*
- (d) *If  $R$  is noetherian, every Poisson-prime ideal of  $R$  is prime.*
- (e) *If  $R$  is affine over  $\mathbb{K}$ , every Poisson-prime ideal of  $R$  is an intersection of Poisson primitive ideals.*

REMARK. If  $R$  is a noetherian Poisson algebra, Lemma 2.2(d) implies that the Poisson-prime ideals in  $R$  are precisely the ideals which are both Poisson ideals and prime ideals; in that case, the hyphen in the term “Poisson-prime” becomes unnecessary.

PROOF. (a) Since  $\text{P.core}(P)$  is the largest ideal contained in  $P$  and stable under all the derivations  $\{a, -\}$ , this follows from [11, Lemma 3.3.2].

(b) and (c) are immediate from (a).

(d) Let  $Q$  be a Poisson-prime ideal of  $R$ . There exist prime ideals  $Q_1, \dots, Q_n$  minimal over  $Q$  such that  $Q_1 Q_2 \cdots Q_n \subseteq Q$ . The  $Q_i$  are Poisson ideals by (c), so the Poisson-primeness of  $Q$  implies that some  $Q_i = Q$ .

(e) Let  $Q$  be a Poisson-prime ideal of  $R$ . Since  $Q$  is prime (by (d)), the Nullstellensatz implies that  $Q$  is an intersection of maximal ideals  $M_i$ . Consequently,  $Q = \bigcap_i \text{P.core}(M_i)$ .  $\square$

If  $\mathbb{K} = \overline{\mathbb{K}}$  and the Poisson algebra  $R$  is the coordinate ring of an affine algebraic variety  $V$ , then  $V$  may be partitioned into *symplectic cores* [7, §3.3] parametrized by the Poisson primitive ideals of  $R$ , where the symplectic core corresponding to a Poisson primitive ideal  $P$  is the set

$$\{x \in V \mid \text{P.core}(\mathfrak{m}_x) = P\}.$$

(Here  $\mathfrak{m}_x$  denotes the maximal ideal of  $R$  corresponding to  $x$ .) In case  $\mathbb{K} = \mathbb{C}$ , the variety  $V$  is a union of smooth complex Poisson manifolds and, as such, is partitioned into symplectic leaves (see [7, §3.5]). In this case, it follows from [7, Lemma 3.5] that the Zariski closure of any symplectic leaf  $L$  of  $V$  is a union of symplectic cores. In fact, the lemma shows that the defining ideal  $I(\overline{L})$  is a Poisson primitive ideal  $P$ , equal to  $\text{P.core}(\mathfrak{m}_x)$  for any  $x \in L$ . Consequently,  $\overline{L}$  is the union of the symplectic cores corresponding to the Poisson primitive ideals containing  $P$ .

These observations lead to geometric ways to verify that certain ideals are Poisson ideals, as follows.

PROPOSITION 2.3. *Let  $\mathbb{K} = \mathbb{C}$  and  $R$  the coordinate ring of an affine algebraic variety  $V$ . Suppose  $R$  is a Poisson algebra and  $I$  is a radical ideal of  $R$ . Then  $I$  is a Poisson ideal if and only if the subvariety  $V(I)$  is a union of symplectic leaves, if and only if  $V(I)$  is a union of Zariski closures of symplectic leaves.*

PROOF. Suppose first that  $V(I)$  is a union of closures of symplectic leaves  $L_j$ . By [7, Lemma 3.5],  $I(\overline{L_j})$  is a Poisson ideal of  $R$ , and therefore  $I = \bigcap_j I(\overline{L_j})$  is a Poisson ideal.

Now assume that  $I$  is a Poisson ideal. By Lemma 2.2(e),  $I$  equals the intersection of the Poisson primitive ideals  $P_j$  that contain  $I$ . If  $x \in V(I)$ , then from  $\mathfrak{m}_x \supseteq I$  we get  $\text{P.core}(\mathfrak{m}_x) \supseteq I$ , whence  $\text{P.core}(\mathfrak{m}_x) = P_j$  for some  $j$ . By [7, Lemma

3.5], the closure of the symplectic leaf  $L$  containing  $x$  satisfies  $I(\overline{L}) = P_j$ , whence  $L \subseteq V(P_j) \subseteq V(I)$ . Therefore  $V(I)$  is a union of symplectic leaves.

Finally, if  $V(I)$  is a union of symplectic leaves, then, being closed in  $V$ , it is also a union of closures of symplectic leaves.  $\square$

A version of Proposition 2.3 over more general fields, with symplectic leaves replaced by symplectic cores, is available under additional hypotheses, such as uncountability of the base field, as we now show.

LEMMA 2.4. *Let  $T$  be an affine Poisson algebra. Assume that  $\mathbb{K}$  is uncountable,  $T$  is a domain, and  $Z_p(\text{Fract } T) = \mathbb{K}$ . Then there is a countable set  $\mathcal{S}$  of nonzero Poisson prime ideals of  $T$  such that every nonzero Poisson prime ideal of  $T$  contains a member of  $\mathcal{S}$ .*

PROOF. Let  $\mathcal{S}$  be the set of those nonzero Poisson prime ideals of  $T$  which do not properly contain any nonzero Poisson prime ideal. Due to the descending chain condition on prime ideals in  $T$ , every nonzero Poisson prime ideal of  $T$  contains a member of  $\mathcal{S}$ . The argument in the proof of [1, Theorem 3.2] shows that  $\mathcal{S}$  is countable. (The assumption  $\mathbb{K} = \mathbb{C}$  in that theorem is not used in this part of the proof.)  $\square$

The key argument of the following result was communicated to us by Jason Bell; we thank him for permission to include it here.

LEMMA 2.5. *If  $R$  is an affine Poisson algebra and  $\mathbb{K}$  is uncountable, then*

$$(2.2) \quad P = \bigcap \{ \mathfrak{m} \in \max R \mid \text{P.core}(\mathfrak{m}) = P \}, \quad \forall \text{ Poisson primitive ideals } P \text{ of } R.$$

PROOF. It suffices to consider the case where the Poisson primitive ideal  $P$  is zero. Hence, there is a maximal ideal  $\mathfrak{m}_0$  of  $R$  such that  $\text{P.core}(\mathfrak{m}_0) = 0$ .

Set  $\overline{T} := R \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ , which is an affine Poisson  $\overline{\mathbb{K}}$ -algebra, identify  $R$  with its image in  $\overline{T}$ , and observe that  $\overline{T}$  is integral over  $R$ . We will use several standard results about prime ideals in integral extensions, such as [32, Theorems 44, 46, 48]. In particular, since  $R$  and  $\overline{T}$  have the same finite Krull dimension, it follows that any non-minimal prime ideal of  $\overline{T}$  has nonzero contraction to  $R$ .

There is a prime ideal  $Q_0$  of  $\overline{T}$  such that  $Q_0 \cap R = 0$ . After possibly shrinking  $Q_0$ , we may assume that it is a minimal prime, and thus a Poisson prime (Lemma 2.2(c)). Hence, the domain  $T := \overline{T}/Q_0$  is an affine Poisson  $\overline{\mathbb{K}}$ -algebra. We identify  $R$  with its image in  $T$ .

There is a maximal ideal  $\mathfrak{m}_0^* \in \max T$  such that  $\mathfrak{m}_0^* \cap R = \mathfrak{m}_0$ . Since  $\text{P.core}(\mathfrak{m}_0^*)$  is a Poisson prime ideal of  $T$ , the contraction  $\text{P.core}(\mathfrak{m}_0^*) \cap R$  is a Poisson prime ideal of  $R$  contained in  $\mathfrak{m}_0$ , whence  $\text{P.core}(\mathfrak{m}_0^*) \cap R = 0$ . Consequently,  $\text{P.core}(\mathfrak{m}_0^*) = 0$ , and so 0 is a Poisson primitive ideal of  $T$ .

By [41, Proposition 1.10], 0 is a Poisson-rational ideal of  $T$ , meaning that the Poisson center of  $\text{Fract } T$  is (algebraic over)  $\overline{\mathbb{K}}$ . Hence, Lemma 2.4 provides us with a countable set  $\{Q_1, Q_2, \dots\}$  of nonzero Poisson prime ideals of  $T$  such that every nonzero Poisson prime ideal of  $T$  contains some  $Q_j$ . Moreover, the ideals  $P_j := Q_j \cap R$  are nonzero Poisson prime ideals of  $R$ .

We claim that any nonzero Poisson prime ideal  $P$  of  $R$  contains some  $P_j$ . There is a prime ideal  $Q$  of  $T$  lying over  $P$ , and after possibly shrinking  $Q$ , we may assume that  $Q$  is minimal over  $PT$ . Since  $PT$  is a Poisson ideal, so is  $Q$ . Then  $Q \supseteq Q_j$  for some  $j$ , whence  $P \supseteq P_j$ , validating the claim.

Choose a nonzero element  $f_j \in P_j$  for each  $j$ , let  $X$  be the multiplicative set generated by the  $f_j$ , and set  $R' := R[X^{-1}]$ . Since  $R'$  is a countably generated  $\mathbb{K}$ -algebra and  $\mathbb{K}$  is uncountable,  $R'$  satisfies the Nullstellensatz over  $\mathbb{K}$  (e.g., [5, Proposition II.7.16]). This means that the Jacobson radical of  $R'$  is zero and  $R'/\mathfrak{m}'$  is algebraic over  $\mathbb{K}$  for all  $\mathfrak{m}' \in \max R'$ . Consequently,  $\mathfrak{m}' \cap R \in \max R$  for any  $\mathfrak{m}' \in \max R'$ . Moreover,  $\mathfrak{m}' \cap R$  cannot contain any  $P_j$ , and so the Poisson prime ideal  $\text{P.core}(\mathfrak{m}' \cap R)$  must be zero. Since  $\bigcap \{\mathfrak{m}' \cap R \mid \mathfrak{m}' \in \max R'\} = 0$ , (2.2) is proved.  $\square$

The condition (2.2) also holds if  $R$  is affine and all Poisson primitive ideals of  $R$  are locally closed points of the Poisson-prime spectrum of  $R$ . This follows from the proof of [24, Theorem 1.5].

**PROPOSITION 2.6.** *Let  $\mathbb{K} = \overline{\mathbb{K}}$  and  $R$  the coordinate ring of an affine algebraic variety  $V$ . Suppose  $R$  is a Poisson algebra satisfying (2.2), and let  $I$  be a radical ideal of  $R$ . Then  $I$  is a Poisson ideal if and only if  $V(I)$  is a union of symplectic cores.*

**PROOF.** Suppose first that  $I$  is a Poisson ideal. By Lemma 2.2(e),  $I$  equals the intersection of the Poisson primitive ideals  $P_j$  that contain  $I$ . Obviously any point in the symplectic core  $C_j$  corresponding to  $P_j$  lies in  $V(I)$ . Conversely, if  $x \in V(I)$ , then from  $\mathfrak{m}_x \supseteq I$  we get  $\text{P.core}(\mathfrak{m}_x) \supseteq I$ , whence  $\text{P.core}(\mathfrak{m}_x) = P_j$  and  $x \in C_j$ , for some  $j$ . Thus,  $V(I) = \bigcup_j C_j$ .

Now assume that  $V(I)$  is a union of symplectic cores  $C_j$  corresponding to Poisson primitive ideals  $P_j$ . Then

$$I = \bigcap_j \left( \bigcap \{\mathfrak{m}_x \mid x \in C_j\} \right).$$

Because of (2.2),  $\bigcap \{\mathfrak{m}_x \mid x \in C_j\} = P_j$  for all  $j$ . Therefore  $I = \bigcap_j P_j$ , whence  $I$  is a Poisson ideal.  $\square$

## 2.2. Poisson polynomial rings

**DEFINITION 2.7.** A *Poisson derivation* on a Poisson algebra  $B$  is any  $\mathbb{K}$ -linear map  $\sigma$  on  $B$  which is a derivation with respect to both the associative multiplication and the Poisson bracket, that is,

$$(2.3) \quad \sigma(ab) = \sigma(a)b + a\sigma(b) \quad \text{and} \quad \sigma(\{a, b\}) = \{\sigma(a), b\} + \{a, \sigma(b)\}, \quad \forall a, b \in B.$$

For instance, the maps  $\partial_c$  defined in (2.1) are Poisson derivations.

Assume that  $\sigma$  is a Poisson derivation on  $B$ . Following the terminology of [12, §1.1.2], a *Poisson  $\sigma$ -derivation* on  $B$  is any derivation  $\delta$  (with respect to the associative multiplication) such that

$$(2.4) \quad \delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \sigma(a)\delta(b) - \delta(a)\sigma(b), \quad \forall a, b \in B.$$

For any  $c \in B$ , the map

$$a \mapsto \{c, a\} - \sigma(a)c, \quad \forall a \in B,$$

is a Poisson  $\sigma$ -derivation, and we refer to the ones of this form as *inner Poisson  $\sigma$ -derivations* of  $B$ .

LEMMA 2.8. [42, Theorem 1.1] *Let  $B$  be a Poisson algebra,  $\sigma$  a Poisson derivation on  $B$ , and  $\delta$  a Poisson  $\sigma$ -derivation on  $B$ . The Poisson bracket on  $B$  extends uniquely to a Poisson bracket on the polynomial algebra  $B[x]$  such that*

$$(2.5) \quad \{x, b\} = \sigma(b)x + \delta(b), \quad \forall b \in B.$$

DEFINITION 2.9. Suppose that  $B$  is a Poisson algebra,  $\sigma$  a Poisson derivation on  $B$ , and  $\delta$  a Poisson  $\sigma$ -derivation on  $B$ . The polynomial algebra  $B[x]$ , equipped with the Poisson bracket of Lemma 2.8, is just called a *Poisson polynomial ring* in [42]. Here we will say that  $B[x]$  is a *Poisson-Ore extension* of  $B$ , as in [12]. The Poisson bracket on  $B[x]$  extends uniquely to one on the Laurent polynomial algebra  $B[x^{\pm 1}]$ , and we will call the latter Poisson algebra a *Poisson-Laurent extension* of  $B$ . To express that  $B[x]$  and  $B[x^{\pm 1}]$  are Poisson algebras of the types just defined, we use notation analogous to that for Ore and skew-Laurent extensions, namely

$$B[x; \sigma, \delta]_p \quad \text{and} \quad B[x^{\pm 1}; \sigma, \delta]_p.$$

In case  $\delta$  is identically zero, we omit it from the notation, writing just  $B[x; \sigma]_p$  and  $B[x^{\pm 1}; \sigma]_p$ .

The converse of Lemma 2.8 is worth noting; it may be phrased as follows: If a polynomial ring  $R = B[x]$  supports a Poisson bracket such that  $B$  is a Poisson subalgebra and  $\{x, B\} \subseteq Bx + B$ , then  $R = B[x; \sigma, \delta]_p$  for suitable  $\sigma$  and  $\delta$ .

Basic examples of iterated Poisson-Ore extensions of the form

$$\mathbb{K}[x_1][x_2; \sigma_2, \delta_2]_p \cdots [x_N; \sigma_N, \delta_N]_p$$

are Poisson-Weyl algebras and Poisson affine space algebras (Examples 2.1).

### 2.3. Differentials of torus actions

Suppose that  $R$  is an algebra equipped with a rational action of a torus  $\mathcal{H}$  (by algebra automorphisms). The differential of this action provides an action of  $\mathfrak{h} := \text{Lie } \mathcal{H}$  on  $R$  by derivations, as discussed in [25, §1.2], where it is also noted that the  $\mathcal{H}$ -stable  $\mathbb{K}$ -subspaces of  $A$  coincide with the  $\mathfrak{h}$ -stable subspaces. It follows from this discussion and [25, Lemma 1.3] that

(2.6) The  $\mathcal{H}$ -eigenspaces of  $R$  coincide with the  $\mathfrak{h}$ -eigenspaces;

(2.7) The  $\mathcal{H}$ -eigenvectors in  $R$  coincide with the  $\mathfrak{h}$ -eigenvectors;

(2.8) The  $\mathfrak{h}$ -action on  $R$  commutes with the  $\mathcal{H}$ -action;

(2.9)  $\mathfrak{h}.a = 0$ ,  $\forall$  homogeneous  $a \in A$  of degree 0.

LEMMA 2.10. [25, Lemma 1.4] *Suppose a Poisson algebra  $R$  is equipped with a rational Poisson action of a torus  $\mathcal{H}$ . Then  $\text{Lie } \mathcal{H}$  acts on  $R$  by Poisson derivations.*



## CHAPTER 3

# Cluster algebras and Poisson cluster algebras

In this chapter, we collect basic definitions and notation concerning cluster algebras and compatible Poisson structures on them, together with some auxiliary results. We consider exclusively cluster algebras of geometric type, recall [16, Definition 5.7], and we do not invert frozen variables unless specifically indicated.

### 3.1. Cluster algebras

Fix positive integers  $n \leq N$  and a subset  $\mathbf{ex} \subset [1, N]$  of cardinality  $n$ . The indices in  $\mathbf{ex}$  are called *exchangeable*, and those in  $[1, N] \setminus \mathbf{ex}$  *frozen*. Matrices indexed by  $[1, N] \times \mathbf{ex}$  will be called  $N \times \mathbf{ex}$  matrices. Fix a purely transcendental field extension  $\mathcal{F} \supset \mathbb{K}$  of transcendence degree  $N$ . We define cluster algebras over  $\mathbb{K}$  directly, rather than first defining them over  $\mathbb{Z}$  and then extending scalars. Moreover, we follow [16] rather than [2] in that frozen variables in cluster algebras are not automatically inverted. In our notation, we combine the mutation matrices for cluster and frozen variables as in [16, p. 515] and [2, p. 6].

DEFINITION 3.1. A *seed* (of geometric type) in  $\mathcal{F}$  is a pair  $(\tilde{\mathbf{x}}, \tilde{B})$  where

- (i)  $\tilde{\mathbf{x}} = (x_1, \dots, x_N) \in \mathcal{F}^N$  such that  $\{x_1, \dots, x_N\}$  is algebraically independent over  $\mathbb{K}$  and generates  $\mathcal{F}$  as a field extension of  $\mathbb{K}$ .
- (ii)  $\tilde{B} = (b_{ij})$  is an  $N \times \mathbf{ex}$  integer matrix.
- (iii) The  $\mathbf{ex} \times \mathbf{ex}$  submatrix  $B$  of  $\tilde{B}$  is *skew-symmetrizable*: there exist  $d_i \in \mathbb{Z}_{>0}$  such that  $d_i b_{ij} = -d_j b_{ji}$  for all  $i, j \in \mathbf{ex}$ .

Note that (iii) implies  $b_{kk} = 0$  for all  $k \in \mathbf{ex}$ .

The set  $\mathbf{x} := \{x_j \mid j \in \mathbf{ex}\} \subseteq \tilde{\mathbf{x}}$  is a *cluster*, and its elements are *cluster variables*. Elements of  $\tilde{\mathbf{x}} \setminus \mathbf{x}$  are *frozen variables*. The submatrix  $B$  of  $\tilde{B}$  is the *principal part* of  $\tilde{B}$ , and is called the *exchange matrix* of the seed  $(\tilde{\mathbf{x}}, \tilde{B})$ .

For quantum cluster algebras and cluster algebras with compatible Poisson structure, the matrix  $\tilde{B}$  is automatically of full rank, see Proposition 3.10 below for the Poisson case and [3, Definition 4.5 and Proposition 3.3] for the quantum case.

DEFINITION 3.2. Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed, with  $\tilde{\mathbf{x}} = (x_1, \dots, x_N)$ . For  $k \in \mathbf{ex}$ , define

$$(3.1) \quad x'_k := \frac{\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{x_k} \in \mathcal{F}$$

and  $\mu_k(\tilde{\mathbf{x}}) := (x_1, \dots, x_{k-1}, x'_k, x_{k+1}, \dots, x_N)$ . (Then  $\mu_k(\tilde{\mathbf{x}})$  is another sequence of algebraically independent elements generating  $\mathcal{F}$  over  $\mathbb{K}$ .)

DEFINITION 3.3. Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed,  $\tilde{B} = (b_{ij})$ , and  $k \in \mathbf{ex}$ . Define  $\tilde{B}' = \mu_k(\tilde{B}) = (b'_{ij})$  by

$$(3.2) \quad b'_{ij} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

The matrix  $\mu_k(\tilde{B})$  is the *mutation of  $\tilde{B}$  in direction  $k$* .

If the principal part of  $\tilde{B}$  is  $B$ , then the principal part of  $\mu_k(\tilde{B})$  is  $\mu_k(B)$ . Moreover,  $\mu_k(\tilde{B})$  preserves the condition from Definition 3.1 that its principal part is skew-symmetrizable [16, Proposition 4.5]. If  $\tilde{B}$  has full rank, then  $\mu_k(\tilde{B})$  also has full rank [2, Lemma 3.2].

The *mutation of  $(\tilde{\mathbf{x}}, \tilde{B})$  in direction  $k$*  is the seed  $\mu_k(\tilde{\mathbf{x}}, \tilde{B}) := (\mu_k(\tilde{\mathbf{x}}), \mu_k(\tilde{B}))$ . Mutation is involutive:  $\mu_k(\mu_k(\tilde{\mathbf{x}}, \tilde{B})) = (\tilde{\mathbf{x}}, \tilde{B})$ .

In general, seeds  $(\tilde{\mathbf{x}}, \tilde{B})$  and  $(\tilde{\mathbf{x}}', \tilde{B}')$  are *mutation-equivalent*, written  $(\tilde{\mathbf{x}}, \tilde{B}) \sim (\tilde{\mathbf{x}}', \tilde{B}')$ , if  $(\tilde{\mathbf{x}}', \tilde{B}')$  can be obtained from  $(\tilde{\mathbf{x}}, \tilde{B})$  by a finite sequence of seed mutations. Mutation-equivalent seeds share the same set of frozen variables.

DEFINITION 3.4. Upper bounds and upper cluster algebras associated to the seed  $(\tilde{\mathbf{x}}, \tilde{B})$  are intersections of mixed polynomial-Laurent polynomial algebras generated by the variables from some cluster along with inverses of some of them. For any set  $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$  of frozen indices, let us define

$$(3.3) \quad \mathcal{TA}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv}) := \mathbb{K}[x_i \mid i \in [1, N] \setminus (\mathbf{ex} \sqcup \mathbf{inv})][x_j^{\pm 1} \mid j \in \mathbf{ex} \sqcup \mathbf{inv}].$$

The *upper bound* for  $(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})$  is

$$(3.4) \quad \mathcal{U}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} := \mathcal{TA}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv}) \cap \bigcap_{k \in \mathbf{ex}} \mathcal{TA}(\mu_k(\tilde{\mathbf{x}}), \mu_k(\tilde{B}), \mathbf{inv}),$$

and the corresponding *upper cluster algebra* is

$$(3.5) \quad \overline{\mathcal{A}}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} := \bigcap_{(\tilde{\mathbf{x}}', \tilde{B}') \sim (\tilde{\mathbf{x}}, \tilde{B})} \mathcal{TA}(\tilde{\mathbf{x}}', \tilde{B}', \mathbf{inv}).$$

The proof of the following theorem can be found in [2, Corollary 1.9] and [23, Theorem 4.1 and Lemma 4.3].

THEOREM 3.5. *If the exchange matrix  $\tilde{B}$  has full rank, then  $\mathcal{U}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} = \overline{\mathcal{A}}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  for any seed  $(\tilde{\mathbf{x}}, \tilde{B})$  in  $\mathcal{F}$  and any  $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$ .*

Following this result, we refer to  $\mathcal{U}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  as the upper cluster algebra corresponding to  $(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})$ .

DEFINITION 3.6. The *cluster algebra* associated to a seed  $(\tilde{\mathbf{x}}, \tilde{B})$  in  $\mathcal{F}$  is the  $\mathbb{K}$ -subalgebra  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})_{\mathbb{K}}$  of  $\mathcal{F}$  generated by the cluster and frozen variables from all the seeds mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B})$ . More generally, for any set  $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$ , we define

$$\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} := \mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})_{\mathbb{K}}[x_l^{-1} \mid l \in \mathbf{inv}].$$

These algebras are also called *cluster algebras*.

The *Laurent Phenomenon* for cluster algebras may be expressed as follows; see [2, Corollary 1.12] for the version in terms of upper cluster algebras.

**THEOREM 3.7.** [16, Theorem 3.1] *Every cluster algebra is contained in the corresponding upper cluster algebra:  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} \subseteq \mathcal{U}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  for all seeds  $(\tilde{\mathbf{x}}, \tilde{B})$  in  $\mathcal{F}$  and all  $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$ . Equivalently,*

$$\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} \subseteq \mathcal{TA}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})$$

for all  $(\tilde{\mathbf{x}}, \tilde{B})$  and  $\mathbf{inv}$ .

If  $G$  is a group acting on  $\mathcal{F}$  by automorphisms, we denote by  $\mathcal{F}^G$  the fixed field (or Galois subfield) of this action. For a  $G$ -eigenvector  $u$ , we denote its  $G$ -eigenvalue by  $\chi_u$ , or by  $\chi[u]$  in case  $u$  is a lengthy expression. The character lattice  $X(G)$  is viewed as an additive group.

The following equivariance of mutations of seeds is probably well known, but we could not locate a reference. Recall (1.6) for the notation  $\tilde{\mathbf{x}}^{b^j}$ .

**LEMMA 3.8.** *Let  $G$  be a group acting on  $\mathcal{F}$  by  $\mathbb{K}$ -algebra automorphisms, and let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed in  $\mathcal{F}$ . Assume that all entries of  $\tilde{\mathbf{x}}$  are  $G$ -eigenvectors, and that  $\tilde{\mathbf{x}}^{b^j} \in \mathcal{F}^G$  for all columns  $b^j$  of  $\tilde{B}$ . Then all seeds  $(\tilde{\mathbf{x}}', \tilde{B}')$  mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B})$  have the same properties.*

**PROOF.** It suffices to prove the lemma when  $(\tilde{\mathbf{x}}', \tilde{B}') = \mu_k(\tilde{\mathbf{x}}, \tilde{B})$  for some  $k \in \mathbf{ex}$ . Denote the entries of  $\tilde{\mathbf{x}}, \tilde{\mathbf{x}}', \tilde{B}$ , and  $\tilde{B}'$  by  $x_j, x'_j, b_{lj}$ , and  $b'_{lj}$ , respectively, and denote the columns of  $\tilde{B}'$  by  $(b')^j$ .

We may write  $\tilde{\mathbf{x}}^{b^k}$  in the form

$$\tilde{\mathbf{x}}^{b^k} = \tilde{\mathbf{x}}^{[b^k]_+} \tilde{\mathbf{x}}^{[b^k]_-}$$

(recall (1.4)). As  $\tilde{\mathbf{x}}^{[b^k]_{\pm}}$  are  $G$ -eigenvectors and  $\chi[\tilde{\mathbf{x}}^{b^k}] = 0$  by hypothesis, we see that

$$\chi[\tilde{\mathbf{x}}^{[b^k]_-}] = -\chi[\tilde{\mathbf{x}}^{[b^k]_+}].$$

Rewriting (3.1) in the form

$$x'_k = x_k^{-1} (\tilde{\mathbf{x}}^{[b^k]_+} + \tilde{\mathbf{x}}^{-[b^k]_-}),$$

we find that  $x'_k$  is a  $G$ -eigenvector with

$$(3.6) \quad \chi[x'_k] = -\chi[x_k] + \chi[\tilde{\mathbf{x}}^{[b^k]_+}] = -\chi[x_k] - \chi[\tilde{\mathbf{x}}^{[b^k]_-}].$$

Of course,  $x'_j = x_j$  is a  $G$ -eigenvector for all  $j \neq k$ , by assumption.

Turning to the second condition, we have  $(\tilde{\mathbf{x}}')^{(b')^k} = \tilde{\mathbf{x}}^{-b^k}$  because  $b'_{kk} = 0$  and  $b'_{ik} = -b_{ik}$  for  $i \neq k$ , whence  $(\tilde{\mathbf{x}}')^{(b')^k} \in \mathcal{F}^G$ . Now let  $j \neq k$ . If  $b_{kj} = 0$ , then  $(b')^j = b^j$  and  $(\tilde{\mathbf{x}}')^{(b')^j} = \tilde{\mathbf{x}}^{b^j}$ , yielding  $(\tilde{\mathbf{x}}')^{(b')^j} \in \mathcal{F}^G$ . Finally, if  $b_{kj} \neq 0$ , set  $\epsilon = \text{sign}(b_{kj})$ , and observe from (3.2) that

$$(b')^j = b^j + \epsilon b_{kj} [b^k]_{\epsilon} - 2b_{kj} e_k.$$

Since the  $k$ -th entry of  $b^j + \epsilon b_{kj} [b^k]_{\epsilon} - b_{kj} e_k$  is zero,

$$(\tilde{\mathbf{x}}')^{(b')^j} = \tilde{\mathbf{x}}^{b^j + \epsilon b_{kj} [b^k]_{\epsilon} - b_{kj} e_k} (x'_k)^{-b_{kj}},$$

and using (3.6) we obtain

$$\chi[(\tilde{\mathbf{x}}')^{(b')^j}] = \chi[\tilde{\mathbf{x}}^{b^j}] + \epsilon b_{kj} \chi[\tilde{\mathbf{x}}^{b^k}] - b_{kj} \chi[x_k] - b_{kj} (-\chi[x_k] + \epsilon \chi[\tilde{\mathbf{x}}^{b^k}]) = 0,$$

completing the proof.  $\square$

### 3.2. Compatible pairs

Fix  $n$ ,  $N$ ,  $\mathbf{ex}$ , and  $\mathcal{F}$  as in §3.1. We use a slight generalization of the notion of compatible pairs from [3, Definition 3.1].

DEFINITION 3.9. Let  $\tilde{B} = (b_{kj}) \in M_{N \times \mathbf{ex}}(\mathbb{Z})$ , and let  $\mathbf{r} = (r_{ij}) \in M_N(\mathbb{K})$  be a skew-symmetric scalar matrix. We say that the pair  $(\mathbf{r}, \tilde{B})$  is *compatible* if the following two conditions are satisfied:

$$(3.7) \quad \sum_{i=1}^N b_{ik} r_{ij} = 0, \quad \forall k \in \mathbf{ex}, j \in [1, N], k \neq j \quad \text{and}$$

$$(3.8) \quad \sum_{i=1}^N b_{ik} r_{ik} \neq 0, \quad \forall k \in \mathbf{ex}.$$

Note that the scalars appearing in (3.7) and (3.8) are just the entries of the matrix  $\tilde{B}^T \mathbf{r}$ . Moreover, due to the skew-symmetry of  $\mathbf{r}$  and the bicharacter  $\Omega_{\mathbf{r}}$  associated to  $\mathbf{r}$  as in (1.5), we have

$$(3.9) \quad \sum_{i=1}^N b_{ik} r_{ij} = \Omega_{\mathbf{r}}(b^k, e_j) \quad \forall k \in \mathbf{ex}, j \in [1, N],$$

where  $b^k$  denotes the  $k$ -th column of  $\tilde{B}$ .

PROPOSITION 3.10. *If the pair  $(\mathbf{r}, \tilde{B})$  is compatible, then  $\tilde{B}$  has full rank, and the nonzero scalars  $\beta_k := (\tilde{B}^T \mathbf{r})_{kk}$  satisfy*

$$(3.10) \quad \beta_k b_{kj} = -\beta_j b_{jk}, \quad \forall k, j \in \mathbf{ex}.$$

PROOF. This is proved just as [3, Proposition 3.3].  $\square$

Unlike the case in [3, Proposition 3.3], compatibility of  $(\mathbf{r}, \tilde{B})$  does not in general imply that the principal part of  $\tilde{B}$  is skew-symmetrizable. The next lemma describes an instance when this condition appears naturally.

LEMMA 3.11. *Assume that  $\tilde{B} = (b_{kj}) \in M_{N \times \mathbf{ex}}(\mathbb{Z})$  and  $\mathbf{r} \in M_N(\mathbb{K})$  form a compatible pair, and set  $\beta_k := (\tilde{B}^T \mathbf{r})_{kk}$  for  $k \in \mathbf{ex}$ . If there exist positive integers  $d_k$ , for  $k \in \mathbf{ex}$ , such that*

$$(3.11) \quad d_j \beta_k = d_k \beta_j, \quad \forall j, k \in \mathbf{ex},$$

*then the principal part of  $\tilde{B}$  is skew-symmetrizable via these  $d_k$ , that is,  $d_k b_{kj} = -d_j b_{jk}$  for all  $k, j \in \mathbf{ex}$ .*

PROOF. By Proposition 3.10,  $\beta_k b_{kj} = -\beta_j b_{jk}$  for all  $j, k \in \mathbf{ex}$ . Combining this with (3.11) leads to

$$d_k \beta_j b_{kj} = d_j \beta_k b_{kj} = -d_j \beta_j b_{jk}.$$

Thus  $d_k b_{kj} = -d_j b_{jk}$  because of (3.8).  $\square$

REMARK 3.12. The condition (3.11) is satisfied if and only if there exists  $q \in \mathbb{K}^*$  such that each  $\beta_k = m_k q$  for some  $m_k \in \mathbb{Z}_{>0}$ . Then one can set  $d_k := m_k$ .

We define mutations of compatible pairs as in [3, Definition 3.5].

DEFINITION 3.13. Let  $(\mathbf{r}, \tilde{B})$  be a compatible pair, with  $\tilde{B} = (b_{lj}) \in M_{N \times \mathbf{ex}}(\mathbb{Z})$ , and let  $k \in \mathbf{ex}$ . By [2, Eq. (3.2)], the mutated matrix  $\mu_k(\tilde{B})$  can be expressed as

$$\mu_k(\tilde{B}) = E_\epsilon \tilde{B} F_\epsilon$$

for both choices of sign  $\epsilon = \pm$ , where  $E_\epsilon = E_\epsilon^{\tilde{B}, k}$  and  $F_\epsilon = F_\epsilon^{\tilde{B}, k}$  are the  $N \times N$  and  $\mathbf{ex} \times \mathbf{ex}$  matrices with entries given by

$$(3.12) \quad \begin{aligned} (E_\epsilon)_{ij} &:= \begin{cases} \delta_{ij}, & \text{if } j \neq k \\ -1, & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}), & \text{if } i \neq j = k \end{cases} \\ (F_\epsilon)_{ij} &:= \begin{cases} \delta_{ij}, & \text{if } i \neq k \\ -1, & \text{if } i = j = k \\ \max(0, \epsilon b_{kj}), & \text{if } i = k \neq j. \end{cases} \end{aligned}$$

We define the mutation in direction  $k$  of the matrix  $\mathbf{r}$  by

$$(3.13) \quad \mu_k(\mathbf{r}) := E_\epsilon^T \mathbf{r} E_\epsilon.$$

PROPOSITION 3.14. Let  $(\mathbf{r}, \tilde{B})$  be a compatible pair and  $k \in \mathbf{ex}$ .

(a) The matrix  $\mu_k(\mathbf{r})$ , defined in (3.13), does not depend on the choice of sign  $\epsilon = \pm$ . It is skew-symmetric, and the pair  $(\mu_k(\mathbf{r}), \mu_k(\tilde{B}))$  is compatible.

(b) Assume also that the principal part of  $\tilde{B}$  is skew-symmetrizable. Then the principal part of  $\mu_k(\tilde{B})$  is skew-symmetrizable, and

$$(3.14) \quad \mu_k(\tilde{B})^T \mu_k(\mathbf{r}) = \tilde{B}^T \mathbf{r}.$$

We define the *mutation in direction*  $k \in \mathbf{ex}$  of the compatible pair  $(\mathbf{r}, \tilde{B})$  to be the compatible pair  $(\mu_k(\mathbf{r}), \mu_k(\tilde{B}))$ .

PROOF OF PROPOSITION 3.14. Part (a) is proved as [3, Proposition 3.4].

(b) The principal part of  $\mu_k(\tilde{B})$ , namely  $\mu_k(B)$ , is skew-symmetrizable (for the same choice of positive integers  $d_j$ ,  $j \in \mathbf{ex}$  as for  $B$ ), by the observations in [15, Proposition 4.5].

We have

$$\mu_k(\tilde{B})^T \mu_k(\mathbf{r}) = F_\epsilon^T \tilde{B}^T E_\epsilon^T E_\epsilon^T \mathbf{r} E_\epsilon = F_\epsilon^T \tilde{B}^T \mathbf{r} E_\epsilon.$$

The second statement in Proposition 3.10 and the fact that for all  $i, j \in \mathbf{ex}$ ,  $b_{ij}$  and  $-b_{ji}$  have the same signs (which follows from the skew-symmetrizability assumption) imply

$$\tilde{B}^T \mathbf{r} E_\epsilon = F_\epsilon^T \tilde{B}^T \mathbf{r}.$$

Therefore  $\mu_k(\tilde{B})^T \mu_k(\mathbf{r}) = F_\epsilon^T \tilde{B}^T \mathbf{r} E_\epsilon = \tilde{B}^T \mathbf{r}$ .  $\square$

### 3.3. Poisson cluster algebras

Continue with  $n$ ,  $N$ ,  $\mathbf{ex}$ , and  $\mathcal{F}$  as in §3.1. We first slightly extend two concepts from [20, Introduction and §1.3], [21, §4.1.1].

DEFINITION 3.15. Suppose we have a Poisson  $\mathbb{K}$ -algebra structure on  $\mathcal{F}$ . An  $N$ -tuple  $\tilde{\mathbf{x}} = (x_1, \dots, x_N) \in \mathcal{F}^N$  is *log-canonical* (with respect to the given Poisson structure) if  $\{x_l, x_j\} \in \mathbb{K}x_lx_j$  for all  $l, j \in [1, N]$ . We say that a cluster algebra  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})_{\mathbb{K}} \subset \mathcal{F}$  is *Poisson-compatible* in case  $\tilde{\mathbf{x}}$  is log-canonical for all seeds  $(\tilde{\mathbf{x}}', \tilde{B}')$  mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B})$ . The same terminology is used with the localized cluster algebras  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$ . In the Poisson-compatible case, the upper cluster algebra  $\mathcal{U}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  is a Poisson subalgebra of  $\mathcal{F}$ .

PROPOSITION 3.16. Fix a  $\mathbb{K}$ -algebra Poisson bracket on  $\mathcal{F}$ . Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed in  $\mathcal{F}$  and  $k \in \mathbf{ex}$ , and write  $\tilde{\mathbf{x}} = (x_1, \dots, x_N)$  and  $\mu_k(\tilde{\mathbf{x}}) = (x'_1, \dots, x'_N)$ . Suppose that  $\mathbf{r} \in M_N(\mathbb{K})$  is a matrix such that  $(\mathbf{r}, \tilde{B})$  is a compatible pair. If

$$(3.15) \quad \{x_l, x_j\} = \Omega_{\mathbf{r}}(e_l, e_j)x_lx_j, \quad \forall l, j \in [1, N],$$

then

$$(3.16) \quad \{x'_l, x'_j\} = \Omega_{\mu_k(\mathbf{r})}(e_l, e_j)x'_lx'_j, \quad \forall l, j \in [1, N].$$

PROOF. Let  $\mathbf{r}' := \mu_k(\mathbf{r}) = E_{\epsilon}^T \mathbf{r} E_{\epsilon}$ , where  $E_{\epsilon} = E_{\epsilon}^{\tilde{B}, k}$ . Then

$$(3.17) \quad \Omega_{\mathbf{r}'}(f, g) = f^T E_{\epsilon}^T \mathbf{r} E_{\epsilon} g = \Omega_{\mathbf{r}}(E_{\epsilon} f, E_{\epsilon} g), \quad \forall f, g \in \mathbb{Z}^N, \quad \epsilon = \pm.$$

Write  $b^k$  for the  $k$ -th column of  $\tilde{B}$ . From the definitions (3.12) and (3.1), we see that

$$(3.18) \quad \begin{aligned} E_{\epsilon} e_l &= e_l, \quad \forall \epsilon = \pm, \quad l \in [1, N], \quad l \neq k \\ E_{+} e_k &= -e_k - [b^k]_{-} \\ E_{-} e_k &= -e_k + [b^k]_{+}, \end{aligned}$$

and so

$$(3.19) \quad x'_k = \tilde{\mathbf{x}}^{E_{-} e_k} + \tilde{\mathbf{x}}^{E_{+} e_k}.$$

If  $l, j \in [1, N]$  with  $l, j \neq k$ , then (3.17) and (3.18) imply that  $\Omega_{\mathbf{r}'}(e_l, e_j) = \Omega_{\mathbf{r}}(e_l, e_j)$ , whence

$$\{x'_l, x'_j\} = \{x_l, x_j\} = \Omega_{\mathbf{r}}(e_l, e_j)x_lx_j = \Omega_{\mathbf{r}'}(e_l, e_j)x'_lx'_j.$$

When  $l \neq k$ , we have  $\Omega_{\mathbf{r}'}(e_l, e_k) = \Omega_{\mathbf{r}}(e_l, E_{\epsilon} e_k)$  for  $\epsilon = \pm$ , and so, taking account of (3.19), we have

$$\{x'_l, x'_k\} = \Omega_{\mathbf{r}}(e_l, E_{-} e_k)x_l \tilde{\mathbf{x}}^{E_{-} e_k} + \Omega_{\mathbf{r}}(e_l, E_{+} e_k)x_l \tilde{\mathbf{x}}^{E_{+} e_k} = \Omega_{\mathbf{r}'}(e_l, e_k)x'_lx'_k.$$

The remaining cases of (3.16) follow via the skew-symmetry of  $\{-, -\}$  and  $\Omega_{\mathbf{r}'}$ .  $\square$

THEOREM 3.17. Suppose that  $\mathcal{F}$  is a Poisson  $\mathbb{K}$ -algebra. Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed in  $\mathcal{F}$ , and  $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$ . Assume there exists a skew-symmetric matrix  $\mathbf{r} \in M_N(\mathbb{K})$  such that  $(\mathbf{r}, \tilde{B})$  is a compatible pair and the entries  $x_i$  of  $\tilde{\mathbf{x}}$  satisfy

$$\{x_l, x_j\} = \Omega_{\mathbf{r}}(e_l, e_j)x_lx_j, \quad \forall l, j \in [1, N].$$

Then the cluster algebra  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  is Poisson-compatible.

PROOF. It suffices to prove the theorem in the case  $\mathbf{inv} = \emptyset$ .

Let  $\tilde{T}$  denote the collection of all triples  $(\tilde{\mathbf{x}}', \tilde{B}', \mathbf{r}')$  such that  $(\tilde{\mathbf{x}}', \tilde{B}')$  is a seed in  $\mathcal{F}$  and  $(\mathbf{r}', \tilde{B}')$  is a compatible pair. For any such triple and any  $k \in \mathbf{ex}$ , we have a mutated seed  $(\mu_k(\tilde{\mathbf{x}}'), \mu_k(\tilde{B}'))$  and a mutated compatible pair  $(\mu_k(\mathbf{r}'), \mu_k(\tilde{B}'))$  (by

Proposition 3.14), so that  $\mu_k(\tilde{\mathbf{x}}', \tilde{B}', \mathbf{r}') := (\mu_k(\tilde{\mathbf{x}}'), \mu_k(\tilde{B}'), \mu_k(\mathbf{r}'))$  is another triple in  $\tilde{T}$ . This defines mutation operations  $\mu_k$  on  $\tilde{T}$ .

Let  $\tilde{T}_0$  be the collection of all triples in  $\tilde{T}$  mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B}, \mathbf{r})$ . It follows from Proposition 3.16 that for each  $(\tilde{\mathbf{x}}', \tilde{B}', \mathbf{r}') \in \tilde{T}_0$ , the entries  $x'_i$  of  $\tilde{\mathbf{x}}'$  satisfy

$$\{x'_l, x'_j\} = \Omega_{\mathbf{r}'}(e_l, e_j)x'_l x'_j, \quad \forall l, j \in [1, N].$$

In particular,  $\tilde{\mathbf{x}}'$  is log-canonical.

For any seed  $(\tilde{\mathbf{x}}', \tilde{B}') = \mu_{k_n} \cdots \mu_{k_1}(\tilde{\mathbf{x}}, \tilde{B})$  mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B})$ , we have a matrix  $\mathbf{r}' := \mu_{k_n} \cdots \mu_{k_1}(\mathbf{r})$  such that  $(\tilde{\mathbf{x}}', \tilde{B}', \mathbf{r}') \in \tilde{T}_0$ , and thus  $\tilde{\mathbf{x}}'$  is log-canonical. Therefore  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})_{\mathbb{K}}$  is Poisson-compatible.  $\square$



## Poisson-primes in Poisson-Ore extensions

### 4.1. Equivariant Poisson unique factorization domains

Let  $R$  be a Poisson domain (i.e., a Poisson algebra which is also an integral domain). A *Poisson-prime element* of  $R$  is any Poisson-normal prime element  $p \in R$ , that is,  $p$  is a nonzero Poisson-normal element and  $R/\langle p \rangle$  is a domain (where  $\langle p \rangle$  denotes the principal ideal  $Rp$ ). It follows from Proposition 2.3 that for every affine Poisson domain  $R$  over  $\mathbb{C}$ , the following are equivalent for  $p \in R$ :

- (1)  $p$  is a Poisson-prime.
- (2) The ideal  $\langle p \rangle$  is a prime ideal and a Poisson ideal.
- (3)  $p$  is a prime element of  $R$  and the zero locus  $V(p)$  is a union of symplectic leaves of the spectrum of  $R$ .
- (4)  $p$  is a prime element of  $R$  and  $V(p)$  is a union of Zariski closures of symplectic leaves of  $\text{Spec } R$ .

In the special case when the spectrum of  $R$  is smooth, the equivalence of (1) and (3) was proved in [40, Remark 2.4(iii)]. In case  $\mathbb{K}$  is uncountable and algebraically closed, Lemma 2.5 and Proposition 2.6 show that  $p$  is Poisson-prime if and only if

- (5)  $p$  is a prime element of  $R$  and  $V(p)$  is a union of symplectic cores of  $\text{Spec } R$ .

We shall say that  $R$  is a *Poisson unique factorization domain*, abbreviated *Poisson-UFD* or *P-UFD*, if each nonzero Poisson-prime ideal of  $R$  contains a Poisson-prime element. The hyphen is important here – a Poisson algebra which is also a UFD need not be a P-UFD. For example, equip the polynomial algebra  $R = \mathbb{C}[x, y, z]$  with the Poisson bracket such that

$$\{x, y\} = 0, \quad \{z, y\} = x + y^2, \quad \{z, x\} = 2y.$$

As shown in [31, Example 5.12],  $\langle x, y \rangle$  is a Poisson prime ideal of  $R$  which does not properly contain any nonzero Poisson prime ideal. Hence,  $\langle x, y \rangle$  does not contain any Poisson-prime element.

We shall also need an equivariant version of this concept, as in [28, 29]. Suppose that  $R$  is a Poisson domain equipped with a Poisson action of a group  $\mathcal{H}$ . On replacing “Poisson ideal” by “ $\mathcal{H}$ -stable Poisson ideal” in the definition of a Poisson-prime ideal, we obtain the definition of an  $\mathcal{H}$ -Poisson-prime ideal of  $R$ . We shall say that  $R$  is an  $\mathcal{H}$ -Poisson unique factorization domain, abbreviated  $\mathcal{H}$ -P-UFD, if each nonzero  $\mathcal{H}$ -Poisson-prime ideal of  $R$  contains a Poisson-prime  $\mathcal{H}$ -eigenvector. The arguments of [28, Proposition 2.2] and [46, Proposition 6.18 (ii)] are easily adapted to give the following equivariant Poisson version of results of Chatters and Jordan [9, Proposition 2.1], [10, p. 24].

**PROPOSITION 4.1.** *Let  $R$  be a noetherian  $\mathcal{H}$ -Poisson-UFD. Every Poisson-normal  $\mathcal{H}$ -eigenvector in  $R$  is either a unit or a product of Poisson-prime  $\mathcal{H}$ -eigenvectors. The factors are unique up to reordering and taking associates.*

REMARK 4.2. A Poisson version of [34, Proposition 1.6 and Theorem 3.6], as adapted in [28, Theorem 2.4], shows that if  $R$  is a noetherian  $\mathcal{H}$ -Poisson-UFD where  $\mathcal{H}$  is a  $\mathbb{K}$ -torus acting rationally on  $R$  by Poisson automorphisms, then  $R$  is a Poisson-UFD. We will not need this result, however.

The next lemma clarifies the picture of  $\mathcal{H}$ -Poisson-prime ideals in the case of a rational torus action.

LEMMA 4.3. *Let  $R$  be a noetherian Poisson algebra, equipped with a rational Poisson action of a torus  $\mathcal{H}$ . Then any prime ideal of  $R$  minimal over an  $\mathcal{H}$ -stable Poisson ideal is itself an  $\mathcal{H}$ -stable Poisson ideal, and the  $\mathcal{H}$ -Poisson-prime ideals of  $R$  are exactly the  $\mathcal{H}$ -stable, Poisson, prime ideals of  $R$ .*

PROOF. The first statement follows from [24, Lemma 1.1(d)] and [5, Proposition II.2.9]. Obviously  $\mathcal{H}$ -stable, Poisson, prime ideals are  $\mathcal{H}$ -Poisson-prime. Conversely, let  $P$  be an  $\mathcal{H}$ -Poisson-prime ideal of  $R$ . Since  $R$  is noetherian, there are prime ideals  $Q_1, \dots, Q_m$  minimal over  $P$  such that  $Q_1 Q_2 \cdots Q_m \subseteq P$ . Each  $Q_i$  is an  $\mathcal{H}$ -stable Poisson ideal. The  $\mathcal{H}$ -Poisson-primeness of  $P$  thus implies that some  $Q_j \subseteq P$ , whence  $Q_j = P$ . Therefore  $P$  is a prime ideal.  $\square$

Lemma 4.3 allows a shortening of terminology: In the setting of the lemma,

$$(4.1) \quad \{\mathcal{H}\text{-Poisson-prime ideals of } R\} = \{\text{Poisson } \mathcal{H}\text{-prime ideals of } R\}.$$

This is immediate from the lemma together with the fact that the  $\mathcal{H}$ -prime ideals of  $R$  coincide with the  $\mathcal{H}$ -stable prime ideals [5, Proposition II.2.9].

## 4.2. Poisson-Cauchon extensions

In this section, we study a Poisson analog of the *Cauchon extensions* defined in [34, Definition 2.5], and develop a version of [34, Proposition 2.9] which allows us to prove that suitable Poisson-Cauchon extensions are  $\mathcal{H}$ -P-UFDs.

DEFINITION 4.4. A *Poisson-Cauchon extension* of a Poisson algebra  $B$  is a Poisson-Ore extension  $R = B[x; \sigma, \delta]_p$  which is equipped with a rational Poisson action of a torus  $\mathcal{H}$  such that

- (i) The subalgebra  $B$  is  $\mathcal{H}$ -stable, and  $x$  is an  $\mathcal{H}$ -eigenvector.
- (ii)  $\delta$  is locally nilpotent.
- (iii) There is some  $h_\circ \in \mathfrak{h} = \text{Lie } \mathcal{H}$  such that  $\sigma = (h_\circ \cdot)|_B$  and the  $h_\circ$ -eigenvalue of  $x$ , to be denoted  $\lambda_\circ$ , is nonzero.

Condition (iii) relies on  $x$  being an  $\mathfrak{h}$ -eigenvector, which follows from (i) and (2.7). (This definition is not a precise parallel to [34, Definition 2.5], since we have not assumed that  $B$  is a noetherian domain, while we have restricted  $\mathcal{H}$  to being a torus.)

Whenever we refer to a Poisson-Cauchon extension  $R = B[x; \sigma, \delta]_p$ , we take the associated torus and its Lie algebra to be denoted  $\mathcal{H}$  and  $\mathfrak{h}$  unless otherwise specified.

Since  $\mathcal{H}$ -stable subspaces of  $B$  are also  $\mathfrak{h}$ -stable (§2.3), it follows from (iii) that

$$(4.2) \quad \text{Every } \mathcal{H}\text{-stable subspace of } B \text{ is } \sigma\text{-stable.}$$

Our assumptions also imply that

$$(4.3) \quad \sigma \circ \delta = \delta \circ (\sigma + \lambda_\circ)$$

$$(4.4) \quad (h \cdot)|_B \circ \delta = \chi_x(h) \delta \circ (h \cdot)|_B, \quad \forall h \in \mathcal{H}.$$

Namely, for  $b \in B$  and  $h \in \mathcal{H}$  we compute that

$$\begin{aligned}\sigma\delta(b) &= h_\circ \cdot (\{x, b\} - \sigma(b)x) = \{\lambda_\circ x, b\} + \{x, \sigma(b)\} - \sigma^2(b)x - \sigma(b)\lambda_\circ x \\ &= \lambda_\circ \sigma(b)x + \lambda_\circ \delta(b) + \sigma^2(b)x + \delta\sigma(b) - \sigma^2(b)x - \sigma(b)\lambda_\circ x = \delta(\sigma(b) + \lambda_\circ b) \\ h \cdot \delta(b) &= h \cdot (\{x, b\} - \sigma(b)x) = \{\chi_x(h)x, h \cdot b\} - h \cdot \sigma(b)\chi_x(h)x \\ &= \chi_x(h)(\{x, h \cdot b\} - \sigma(h \cdot b)x) = \chi_x(h)\delta(h \cdot b),\end{aligned}$$

where we have used (2.8) to commute  $(h \cdot)|_B$  and  $\sigma$ . Further, this commutativity, together with (4.4), implies that for any nonzero homogeneous element  $b \in B$ , the elements  $\sigma(b)$  and  $\delta(b)$  are homogeneous, with

$$(4.5) \quad \chi_{\sigma(b)} = \chi_b \text{ if } \sigma(b) \neq 0, \text{ and } \chi_{\delta(b)} = \chi_b + \chi_x \text{ if } \delta(b) \neq 0.$$

There is a Poisson version of Cauchon's derivation-deleting map [8, Section 2], as follows.

PROPOSITION 4.5. *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension.*

(a) *There is a Poisson algebra isomorphism  $\theta : B[y^{\pm 1}; \sigma]_p \rightarrow \widehat{R} := B[x^{\pm 1}; \sigma, \delta]_p$  such that  $\theta(y) = x$  and*

$$(4.6) \quad \theta(b) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-1}{\lambda_\circ} \right)^n \delta^n(b) x^{-n}, \quad \forall b \in B.$$

(b)  *$\{x, \theta(b)\} = \theta\sigma(b)x$  for all  $b \in B$ .*

(c) *The given  $\mathcal{H}$ -action on  $B$  extends uniquely to a rational action of  $\mathcal{H}$  on  $B[y^{\pm 1}; \sigma]_p$  by Poisson automorphisms such that  $\chi_y = \chi_x$ . With respect to this action, the isomorphism  $\theta$  is  $\mathcal{H}$ -equivariant.*

PROOF. Parts (a) and (b) are proved in [25, Lemmas 3.4, 3.6, Theorem 3.7].

(c) It is clear that the  $\mathcal{H}$ -action on  $B$  extends uniquely to a rational action of  $\mathcal{H}$  on  $B[y^{\pm 1}; \sigma]_p$  by  $\mathbb{K}$ -algebra automorphisms such that  $\chi_y = \chi_x$ .

Any  $h \in \mathcal{H}$  already acts on  $B$  by Poisson automorphisms. Moreover,  $(h \cdot)|_B$  commutes with  $(h_\circ \cdot)|_B = \sigma$  by (2.8), whence

$$h \cdot \{y, b\} = h \cdot (\sigma(b)y) = \sigma(h \cdot b)\chi_y(h)y = \chi_y(h)\{y, h \cdot b\} = \{h \cdot y, h \cdot b\}, \quad \forall b \in B.$$

It follows that  $h$  acts on  $B[y^{\pm 1}; \sigma]_p$  by a Poisson automorphism.

The  $\mathcal{H}$ -action on  $B[y^{\pm 1}; \sigma]_p$  has been chosen so that  $\theta(h \cdot y) = h \cdot x = h \cdot \theta(y)$  for any  $h \in \mathcal{H}$ . From (4.4), we obtain  $(h \cdot)|_B \delta^n = \chi_x(h)^n \delta^n(h \cdot)|_B$  for all  $h \in \mathcal{H}$  and  $n \in \mathbb{Z}_{\geq 0}$ , and consequently  $h \cdot (\delta^n(b)x^{-n}) = \delta^n(h \cdot b)x^{-n}$  for all  $b \in B$ . This implies that  $h \cdot \theta(b) = \theta(h \cdot b)$  for all  $b \in B$ , and we conclude that  $(h \cdot)\theta = \theta(h \cdot)$ .  $\square$

COROLLARY 4.6. *Let  $R = B[x; \sigma, \delta]_p$ ,  $\theta$ , and  $\widehat{R}$  be as in Proposition 4.5.*

(a)  *$\theta$  restricts to a Poisson algebra isomorphism of  $B[y; \sigma]_p$  onto the subalgebra  $\theta(B)[x; \alpha]_p$  of  $\widehat{R}$ , where  $\alpha$  is the Poisson derivation on  $\theta(B)$  such that  $\alpha\theta(b) = \theta\sigma(b)$  for  $b \in B$ .*

(b)  *$\alpha = (h_\circ \cdot)|_{\theta(B)}$  and  $\theta(B)[x; \alpha]_p$  is a Poisson-Cauchon extension of  $\theta(B)$ .*

(c)  *$\widehat{R} = \theta(B)[x^{\pm 1}; \alpha]_p$  is a Poisson-Laurent extension of  $\theta(B)$ .*

(d) *Every  $\mathcal{H}$ -stable Poisson prime ideal of  $\widehat{R}$  is induced from an  $\mathcal{H}$ -stable Poisson prime ideal of  $\theta(B)$ .*

PROOF. (a) Obviously  $\theta$  must map the polynomial ring  $B[y]$  isomorphically onto a polynomial ring  $\theta(B)[\theta(y)] = \theta(B)[x]$ . In view of Proposition 4.5,  $\theta(B)$  is a

Poisson subalgebra of  $\widehat{R}$ , and  $\{x, a\} = \alpha(a)x$  for all  $a \in \theta(B)$ , by definition of  $\alpha$ . Thus,  $\theta(B)[x]$  is a Poisson-Ore extension of the form  $\theta(B)[x; \alpha]_p$ .

(b) It is clear that the subalgebra  $T := \theta(B)[x; \alpha]_p$  is  $\mathcal{H}$ -stable, and that the  $\mathcal{H}$ -action on  $T$  is rational. Once we show that  $\alpha = (h_\circ \cdot) |_{\theta(B)}$ , we will have that  $T$  is a Poisson-Cauchon extension.

If  $b \in B$  is homogeneous and nonzero, then  $\theta(b)$  is homogeneous of the same degree, so  $b$  and  $\theta(b)$  lie in the same  $\mathcal{H}$ -eigenspace of  $\widehat{R}$ . Since this  $\mathcal{H}$ -eigenspace is also an  $\mathfrak{h}$ -eigenspace,  $b$  and  $\theta(b)$  are  $\mathfrak{h}$ -eigenvectors with the same  $\mathfrak{h}$ -eigenvalue. Hence, there is some  $\mu \in \mathbb{K}$  such that  $h_\circ \cdot b = \mu b$  and  $h_\circ \cdot \theta(b) = \mu \theta(b)$ , and consequently

$$h_\circ \cdot \theta(b) = \theta(\mu b) = \theta \sigma(b) = \alpha \theta(b).$$

Therefore  $(h_\circ \cdot) |_{\theta(B)} = \alpha$ , as required.

(c) This is clear from the isomorphism  $\theta$  and part (a).

(d) Since  $\alpha = (h_\circ \cdot) |_{\theta(B)}$  and  $h_\circ \cdot x = \lambda_\circ x$  with  $\lambda_\circ \neq 0$ , part (d) follows from [25, Lemma 1.1].  $\square$

**THEOREM 4.7.** *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension. If  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD, then so is  $R$ .*

**PROOF.** Let  $\theta$  and  $\widehat{R}$  be as Proposition 4.5 and  $\alpha$  as in Corollary 4.6. Set  $A := \theta(B)$  and  $T := A[x; \alpha]_p$ , and recall from Corollary 4.6(c) that  $\widehat{R} = A[x^{\pm 1}; \alpha]_p$ . Since  $B$  is an  $\mathcal{H}$ -P-UFD, so is  $A$ . We first establish the following

**Claim:** Suppose  $u$  is a homogeneous Poisson-prime element of  $A$  and  $s \geq 0$  is minimal such that  $v := ux^s \in R$ . Then  $v$  is a homogeneous Poisson-prime element of  $R$ .

Homogeneity of  $v$  is clear from the homogeneity of  $u$  and  $x$ . Now  $u = \theta(u_0)$  for some homogeneous element  $u_0 \in B$ , and  $u_0$  is an  $\mathfrak{h}$ -eigenvector by (2.7), hence also a  $\sigma$ -eigenvector. Thus,  $u$  is an  $\alpha$ -eigenvector, and so  $\{x, u\} = \alpha(u)x \in \mathbb{K}ux$ . Since  $u$  is a Poisson-normal element of  $A$ , it follows that  $u$  is a Poisson-normal element of  $T$ . Moreover,  $T/uT \cong (A/uA)[x]$ , which is a domain. Therefore  $u$  is a homogeneous Poisson-prime element of  $T$ .

Since the Poisson  $\mathcal{H}$ -prime ideal  $uT$  of  $T$  does not contain  $x$ , we see that  $u\widehat{R}$  is a Poisson  $\mathcal{H}$ -prime ideal of  $\widehat{R}$ . But  $u\widehat{R} = v\widehat{R}$ , so  $I := v\widehat{R} \cap R$  is a Poisson  $\mathcal{H}$ -prime ideal of  $R$ . We show that  $I = vR$ , which yields the claim.

Obviously  $vR \subseteq I$ , so consider an element  $w \in I$ . Then  $w \in v\widehat{R}$ . Choose  $t \geq 0$  minimal such that  $wx^t \in vR$ , and write  $wx^t = vr$  with  $r \in R$ . Suppose  $t > 0$ , whence  $x \mid vr$  in  $R$ . We cannot have  $r = r_1x$  with  $r_1 \in R$ , since that would imply  $wx^{t-1} = vr_1 \in vR$ , contradicting the minimality of  $t$ . Thus  $x \nmid r$  in  $R$ , and consequently  $x \mid v$  in  $R$ . If  $s > 0$ , this would imply  $ux^{s-1} = vx^{-1} \in R$ , contradicting the minimality of  $s$ . Hence,  $s = 0$ , and so  $u \in R$  with  $x \mid u$ . On the other hand, since  $u \in \theta(B)$ , (4.6) then implies  $u \in B$ . This is impossible, because  $u \neq 0$  and  $x \mid u$  in  $R$ . Therefore  $t = 0$  and  $w \in vR$ , completing the proof of the claim.

To prove that  $R$  is an  $\mathcal{H}$ -P-UFD, consider a nonzero Poisson  $\mathcal{H}$ -prime ideal  $J$  of  $R$ .

If  $x \notin J$ , then  $J$  is disjoint from  $\{x^n \mid n \in \mathbb{Z}_{\geq 0}\}$ , and so  $J\widehat{R}$  is a nonzero Poisson  $\mathcal{H}$ -prime ideal of  $\widehat{R}$ , with  $J\widehat{R} \cap R = J$ . In view of Corollary 4.6(d),  $J\widehat{R} \cap A$  is a nonzero Poisson  $\mathcal{H}$ -prime ideal of  $A$ . Since  $A$  is an  $\mathcal{H}$ -P-UFD,  $J\widehat{R} \cap A$  contains a homogeneous Poisson-prime element  $u$  of  $A$ . Let  $s \geq 0$  be minimal such that

$v := ux^s \in R$ , and note that  $v \in J\widehat{R} \cap R = J$ . By the Claim,  $v$  is a homogeneous Poisson-prime element of  $R$ , so we are done in this case.

Now assume that  $x \in J$ . If  $\delta = 0$ , then we are done because  $x$  is a homogeneous Poisson-prime element of  $R$ , so we may assume that  $\delta \neq 0$ . Since  $\delta(b) = \{x, b\} - \sigma(b)x$  for  $b \in B$ , we see that  $\delta(B) \subseteq J$ . Consequently,  $J \cap B \neq 0$ .

Note that we have an  $\mathcal{H}$ -equivariant Poisson algebra homomorphism

$$\psi : A \xrightarrow{\theta^{-1}} B \xrightarrow{\text{incl}} R \xrightarrow{\text{quo}} R/J.$$

Hence,  $J' := \ker \psi$  is a Poisson  $\mathcal{H}$ -prime ideal of  $A$ . Since  $J \cap B \neq 0$ , we see that  $J' \neq 0$ . Thus,  $J'$  contains a homogeneous Poisson-prime element  $u$  of  $A$ . Let  $s \geq 0$  be minimal such that  $v := ux^s \in R$ . Then  $v$  is a homogeneous Poisson-prime element of  $R$ , and we will be done if  $v \in J$ .

Set  $z := \theta^{-1}(u) \in B$ , and let  $d \geq 0$  be maximal such that  $\delta^d(z) \neq 0$ . Then

$$u = \theta(z) = \sum_{n=0}^d \frac{1}{n!} \left( \frac{-1}{\lambda_o} \right)^n \delta^n(z) x^{-n},$$

and since  $\delta^d(z) \neq 0$  we see that  $s = d$ . Consequently,

$$(4.7) \quad v = ux^d = \sum_{n=0}^d \frac{1}{n!} \left( \frac{-1}{\lambda_o} \right)^n \delta^n(z) x^{d-n}.$$

As  $u \in \ker \psi$ , we have  $z \in J$ . Since also  $\delta(B) \subseteq J$ , we conclude from (4.7) that  $v \in J$ . This completes the proof.  $\square$

### 4.3. Poisson-prime elements in Poisson-Cauchon extensions

LEMMA 4.8. *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension, where  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD. Let  $E \subseteq B$  be the multiplicative set generated by all homogeneous Poisson-prime elements of  $B$ , and assume there exist proper nonzero  $\mathcal{H}$ -stable Poisson ideals in  $R[E^{-1}]$ . Then there exists a unique homogeneous element  $d \in B[E^{-1}]$  such that  $d = 0$  or  $d$  has the same degree as  $x$  and*

$$(4.8) \quad \delta(b) = \{d, b\} - \sigma(b)d, \quad \forall b \in B[E^{-1}].$$

Moreover,  $R[E^{-1}](x - d)$  is the unique nonzero Poisson  $\mathcal{H}$ -prime ideal of  $R[E^{-1}]$ .

PROOF. Lemma 4.3 ensures that  $R[E^{-1}]$  has at least one nonzero Poisson  $\mathcal{H}$ -prime ideal  $P$ . Now  $P \cap B$  is a Poisson  $\mathcal{H}$ -prime ideal of  $B$ , disjoint from  $E$ . Since  $B$  is an  $\mathcal{H}$ -P-UFD,  $P \cap B = 0$ , whence  $P \cap B[E^{-1}] = 0$ .

Following the proof of [25, Proposition 1.2], we obtain an element  $d \in B[E^{-1}]$  such that  $h \cdot d = \chi_x(h)d$  for all  $h \in \mathcal{H}$  and (4.8) holds. Now  $R[E^{-1}] = B[E^{-1}][y; \sigma]_p$ , where  $y := x - d$ . Since  $B[E^{-1}][y^{\pm 1}; \sigma]_p$  has no nonzero Poisson  $\mathcal{H}$ -prime ideals (by [25, Lemma 1.1]), we conclude that  $P = R[E^{-1}]y$ . Thus,  $R[E^{-1}]y$  is the unique nonzero Poisson  $\mathcal{H}$ -prime ideal of  $R[E^{-1}]$ .

If  $d'$  is any homogeneous element of  $B[E^{-1}]$  satisfying the stated properties of  $d$ , then  $R[E^{-1}] = B[E^{-1}][x - d'; \sigma]_p$  and  $P = R[E^{-1}](x - d')$ . Consequently,  $d - d' \in P$ , and therefore  $d - d' = 0$ .  $\square$

COROLLARY 4.9. *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension, where  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD. Then all homogeneous Poisson-prime elements of  $R$  have degree at most 1 in  $x$ . Up to taking associates, there is at most one*

homogeneous Poisson-prime element of  $R$  which does not lie in  $B$  (i.e., has degree 1 in  $x$ ).

PROOF. If  $v$  is a homogeneous Poisson-prime element of  $R$  such that  $v \notin B$ , then  $Rv$  is disjoint from the multiplicative set  $E$  of Lemma 4.8. Consequently,  $v$  is a homogeneous Poisson-prime element of  $R[E^{-1}]$ , and so  $R[E^{-1}]v$  is a nonzero Poisson  $\mathcal{H}$ -prime ideal of  $R[E^{-1}]$ . Hence, there exists  $d \in B[E^{-1}]$  as in Lemma 4.8, and  $v$  is an associate of the prime element  $x - d \in R[E^{-1}]$  (as prime elements of  $R[E^{-1}]$ ). This implies that  $v$  has degree at most 1 in  $x$ . Since  $Rv$  is a prime ideal of  $R$  disjoint from  $E$ , we have

$$Rv = (R[E^{-1}]v) \cap R = (R[E^{-1}](x - d)) \cap R.$$

If  $w$  is any other homogeneous Poisson-prime element of  $R$  that is not in  $B$ , the same argument as above shows that  $Rw = (R[E^{-1}](x - d)) \cap R$ , and therefore  $Rw = Rv$ .  $\square$

For an ideal  $Q$  of a polynomial ring  $R = B[x]$ , denote the ideal of its leading coefficients

$$\text{lc}(Q) := \{b \in B \mid \exists a \in Q, m \in \mathbb{Z}_{\geq 0} \text{ such that } a - bx^m \in Bx^{m-1} + \cdots + B\}.$$

LEMMA 4.10. *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension, where  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD, let  $\theta : B[y^{\pm 1}; \sigma]_p \rightarrow \widehat{R}$  be the Poisson isomorphism of Proposition 4.5, and let  $J$  be a Poisson  $\mathcal{H}$ -prime ideal of  $B$ . If  $J$  is a height one prime ideal of  $B$ , then  $\text{lc}(\theta(J)\widehat{R} \cap R) = J$ , and  $\theta(J)\widehat{R} \cap R$  is a Poisson  $\mathcal{H}$ -prime ideal of  $R$  as well as a height one prime ideal of  $R$ .*

PROOF. Set  $Q := \theta(J)\widehat{R} \cap R$ , and note that  $Q$  is a nonzero prime ideal of  $R$ . Since  $\theta$  is an  $\mathcal{H}$ -equivariant Poisson isomorphism, we see that  $Q$  is  $\mathcal{H}$ -stable and that  $\{\theta(b), \theta(J)\widehat{R}\} \subseteq \theta(J)\widehat{R}$  for all  $b \in B$ . Moreover,  $\{x, \theta(J)\} \subseteq \theta(J)x$  because of Proposition 4.5 and (4.2). It follows that  $\theta(J)\widehat{R}$  is a Poisson ideal of  $\widehat{R}$ , whence  $Q$  is a Poisson ideal of  $R$ . Thus,  $Q$  is a Poisson  $\mathcal{H}$ -prime ideal of  $R$ .

We have  $J = uB$  for some homogeneous Poisson-prime element  $u \in B$ , and so

$$\theta(J)\widehat{R} = \theta(u)\widehat{R} = \sum_{n \in \mathbb{Z}} \theta(u)Bx^n.$$

Since  $\theta(u) \in u + \sum_{i < 0} Bx^i$ , it is clear that  $\text{lc}(Q) = uB = J$ .

Because  $R$  is an  $\mathcal{H}$ -P-UFD (Theorem 4.7),  $Q$  contains a homogeneous Poisson-prime element  $p$  of  $R$ , and hence it contains the nonzero Poisson  $\mathcal{H}$ -prime ideal  $P := pR$ . Obviously  $x \notin Q$ , since  $1 \notin J = \text{lc}(Q)$ . As a result, we have prime ideals

$$(4.9) \quad 0 \subsetneq P\widehat{R} \subseteq Q\widehat{R} = \theta(J)\widehat{R}$$

in  $\widehat{R}$ . By Corollary 4.6(d),  $P\widehat{R} = P'\widehat{R}$  for some nonzero Poisson  $\mathcal{H}$ -prime ideal  $P'$  of  $\theta(B)$ , and (4.9) implies that  $P' \subseteq \theta(J)$ . But  $\theta(J)$  is a height one prime ideal of  $\theta(B)$ , so we must have  $P' = \theta(J)$ , whence  $P\widehat{R} = Q\widehat{R}$ . We conclude that  $Q = P$ , and therefore  $Q$  has height one.  $\square$

LEMMA 4.11. (a) *Let  $B$  be a domain of characteristic zero and  $\delta$  a locally nilpotent derivation on  $B$ . If  $a \in B$  and  $\delta(a) \in aB$  or  $\delta(a) \in Ba$ , then  $\delta(a) = 0$ .*

(b) *Assume that  $R = B[x; \sigma, \delta]_p$  is a Poisson-Cauchon extension of a domain  $B$ , and that  $\delta$  is inner as a Poisson  $\sigma$ -derivation, that is, there exists  $c \in B$  such that  $\delta(b) = \{c, b\} - \sigma(b)c$  for all  $b \in B$ .*

Then  $\delta = 0$ . If, in addition,  $c$  is a homogeneous element of  $B$  of the same  $X(\mathcal{H})$ -degree as  $x$ , then  $c = 0$ .

PROOF. (a) is proved by the argument of [43, Lemme 7.2.3.2], as follows. Suppose that  $\delta(a) = ab$  for some  $b \in B$  and that  $\delta(a) \neq 0$ . Let  $m, n \in \mathbb{Z}_{>0}$  be minimal such that  $\delta^m(a) = \delta^n(b) = 0$ . By the Leibniz Rule,

$$\begin{aligned} \delta^{m+n-1}(a) &= \delta^{m+n-2}(ab) = \sum_{i=0}^{m+n-2} \binom{m+n-2}{i} \delta^{m+n-2-i}(a) \delta^i(b) \\ &= \binom{m+n-2}{n-1} \delta^{m-1}(a) \delta^{n-1}(b) \neq 0, \end{aligned}$$

because  $\delta^i(b) = 0$  for  $i \geq n$  and  $\delta^{m+n-2-i}(a) = 0$  for  $i \leq n-2$ . This forces  $m+n-1 \leq m-1$  and so  $n = 0$ , contradicting our assumptions. Thus,  $\delta(a) = 0$ .

(b) Write  $c = c_0 + \cdots + c_m$  where the  $c_i$  are homogeneous elements (possibly zero) of  $B$  with distinct  $X(\mathcal{H})$ -degrees  $\mu_i$  and  $\mu_0 = \chi_x$ . If  $b \in B$  is homogeneous of degree  $\rho$ , then  $\delta(b)$  is homogeneous of degree  $\chi_x + \rho$  by (4.5). On the other hand, each  $\{c_i, b\} - \sigma(b)c_i$  is homogeneous of degree  $\mu_i + \rho$ , so  $\delta(b) = \{c_0, b\} - \sigma(b)c_0$ . Since this holds for all homogeneous  $b \in B$ , it holds for all  $b \in B$ .

The elements  $c_0$  and  $x$  lie in the same  $\mathcal{H}$ -eigenspace of  $R$ , so they also lie in the same  $\mathfrak{h}$ -eigenspace by (2.6). Hence,  $\sigma(c_0) = h_\circ \cdot c_0 = \lambda_\circ c_0$ . Now  $\delta(c_0) = \{c_0, c_0\} - \sigma(c_0)c_0 = -\lambda_\circ c_0^2$ . Part (a) implies that  $\delta(c_0) = 0$ , and so  $c_0 = 0$ . Thus,  $\delta = 0$  and the lemma is proved.  $\square$

THEOREM 4.12. Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension, where  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD. Let  $u$  be a homogeneous Poisson-prime element of  $B$ , and let  $\mu \in \mathbb{K}$  such that

$$\sigma(u) = h_\circ \cdot u = \mu u.$$

Then exactly one of the following two situations occurs:

(i) The element  $u$  remains a Poisson-prime element of  $R$ .

In this case,  $\{u, x\} = -\mu ux$  and  $\delta(u) = 0$ .

(ii) There exists a unique element  $c_\circ \in B$  such that  $v := ux - c_\circ$  is a homogeneous Poisson-prime element of  $R$  (in particular,  $u \nmid c_\circ$ ). Moreover,  $c_\circ$  is homogeneous.

In this case,  $\delta$  is given by

$$(4.10) \quad \delta(b) = \{u^{-1}c_\circ, b\} - \sigma(b)(u^{-1}c_\circ), \quad \forall b \in B,$$

and  $\{v, -\}$  is determined by

$$(4.11) \quad \{v, x\} = -\mu xv, \quad \{v, b\} = (\partial_u + \sigma)(b)v, \quad \forall b \in B,$$

where  $\partial_u$  is the derivation on  $B$  given in (2.1). Furthermore,

$$\{u, c_\circ\} = -(\mu + \lambda_\circ)c_\circ u, \quad \delta(u) = \lambda_\circ c_\circ \neq 0, \quad \delta(c_\circ) = 0.$$

We note that by Corollary 4.9, the situation (ii) cannot simultaneously occur for two homogeneous Poisson-prime elements  $v$  of  $R$ , since such elements  $v$  must be associates of each other.

PROOF. We first show that  $\delta(u) = 0$  in situation (i) while  $\delta(u) \neq 0$  in situation (ii), and so the two situations cannot occur simultaneously.

In situation (i),  $u$  is Poisson-normal in  $R$ , whence the element  $\{x, u\} = \mu ux + \delta(u)$  is divisible by  $u$  in  $R$ . Consequently,  $\delta(u)$  is divisible by  $u$  in  $B$ , and so  $\delta(u) = 0$  by Lemma 4.11(a).

In situation (ii), the element  $\{x, v\}$  is divisible by  $v$  in  $R$ , and we obtain

$$(\mu ux + \delta(u))x - (\sigma(c_\circ)x + \delta(c_\circ)) = \{x, v\} = (ax + b)(ux - c_\circ)$$

for some  $a, b \in B$ , whence

$$au = \mu u, \quad bc_\circ = \delta(c_\circ), \quad bu - ac_\circ = \delta(u) - \sigma(c_\circ).$$

Then  $a = \mu$  and  $b = \delta(c_\circ) = 0$  by Lemma 4.11(a), so

$$\delta(u) = \sigma(c_\circ) - \mu c_\circ.$$

Since  $v$  and  $ux$  are homogeneous, they have the same  $\mathcal{H}$ -eigenvalue, and  $c_\circ$  is homogeneous with that  $\mathcal{H}$ -eigenvalue. Hence, by (2.7),  $v$ ,  $ux$ , and  $c_\circ$  are  $h_\circ$ -eigenvectors with the same  $h_\circ$ -eigenvalue, namely  $\mu + \lambda_\circ$ , and so  $\sigma(c_\circ) = h_\circ \cdot c_\circ = (\mu + \lambda_\circ)c_\circ$ . Thus,

$$\delta(u) = \lambda_\circ c_\circ \neq 0.$$

This verifies that situations (i) and (ii) are disjoint.

We now show that the cases  $\delta(u) = 0$  and  $\delta(u) \neq 0$  lead to situations (i) and (ii), respectively. Let  $\theta : B[y^{\pm 1}; \sigma]_p \rightarrow \widehat{R}$  be the Poisson isomorphism of Proposition 4.5.

If  $\delta(u) = 0$ , then  $\{x, u\} = \mu ux$ , and the Poisson-normality of  $u$  in  $B$  implies that  $u$  is Poisson-normal in  $R$ . Moreover,  $\theta(u) = u$ , whence

$$\theta(uB)\widehat{R} \cap R = uB[x^{\pm 1}] \cap B[x] = uB[x] = uR.$$

Lemma 4.10 now implies that  $uR$  is a prime ideal of  $R$ . Therefore  $u$  is a Poisson-prime element of  $R$ , and situation (i) holds.

Assume that  $\delta(u) \neq 0$  for the remainder of the proof. By Lemma 4.10, the set  $\theta(uB)\widehat{R} \cap R$  is a height one Poisson  $\mathcal{H}$ -prime ideal of  $R$ , and by Theorem 4.7,  $R$  is an  $\mathcal{H}$ -P-UFD. Therefore there exists a homogeneous Poisson-prime element  $v$  of  $R$  such that

$$\theta(uB)\widehat{R} \cap R = vR.$$

Denote the leading coefficient of  $v$  (as a polynomial in  $B[x]$ ) by  $u'$ . Then Lemma 4.10 implies that

$$uB = \text{lc}(\theta(uB)\widehat{R} \cap R) = \text{lc}(vR) = u'B.$$

Consequently,  $u'$  is a homogeneous Poisson-prime element of  $B$  which is an associate of  $u$  (in  $B$ ). Thus, after multiplying  $v$  by a homogeneous unit of  $B$  we can assume that

$$u' = u.$$

By what we proved at the beginning,  $v \neq u$ , since that would imply  $\delta(u) = 0$ . On the other hand,  $v$  has degree at most 1 in  $x$ , by Corollary 4.9. Therefore

$$v = ux - c_\circ$$

for some  $c_\circ \in B$ . Uniqueness of  $c_\circ$  holds by the comment ahead of the proof.

We are now in situation (ii), and it remains to verify the associated conditions stated in the theorem. We have already seen that  $\delta(c_\circ) = 0$  and  $\delta(u) = \lambda_\circ c_\circ \neq 0$ , while  $\{x, v\} = \mu xv$ .

For any  $b \in B$ , the element

$$\{v, b\} = \partial_u(b)ux + u(\sigma(b)x + \delta(b)) - \{c_\circ, b\}$$

is divisible by  $v$  in  $R$ , whence

$$\{v, b\} = (\partial_u(b) + \sigma(b))v \quad \text{and} \quad u\delta(b) - \{c_\circ, b\} = -(\partial_u(b) + \sigma(b))c_\circ.$$

The first equation gives us the remainder of (4.11), and the second equation together with a straightforward calculation yields (4.10).

We saw above that  $\sigma(c_\circ) = (\mu + \lambda_\circ)c_\circ$ . Applying (4.10) to  $c_\circ$ , we obtain

$$0 = \delta(c_\circ) = \{u^{-1}c_\circ, c_\circ\} - \sigma(c_\circ)u^{-1}c_\circ = -u^{-2}\{u, c_\circ\}c_\circ - \sigma(c_\circ)u^{-1}c_\circ,$$

and consequently  $\{u, c_\circ\} = -\sigma(c_\circ)u = -(\mu + \lambda_\circ)c_\circ u$ . This completes the proof.  $\square$

**THEOREM 4.13.** *Let  $R = B[x; \sigma, \delta]_p$  be a Poisson-Cauchon extension, where  $B$  is a noetherian  $\mathcal{H}$ -Poisson-UFD, and let  $\{u_i \mid i \in I\}$  be a list of the homogeneous Poisson-prime elements of  $B$  up to taking associates. Then there are the following three possibilities for a list of the homogeneous Poisson-prime elements of  $R$  up to taking associates:*

(i)  $\{u_i \mid i \in I, i \neq i_0\} \sqcup \{v_{i_0} := u_{i_0}x - c_\circ\}$ , for some  $i_0 \in I$  and  $c_\circ \in B$  such that  $u_{i_0}^{-1}c_\circ$  is a nonzero homogeneous element of  $B[u_{i_0}^{-1}]$  with the same  $X(\mathcal{H})$ -degree as  $x$ .

(ii)  $\{u_i \mid i \in I\} \sqcup \{x\}$ .

(iii)  $\{u_i \mid i \in I\}$ .

Before proving Theorem 4.13, we record the following information for the three cases of the theorem. This proposition is a direct consequence of Theorem 4.12.

**PROPOSITION 4.14.** *Assume the setting of Theorem 4.13. For  $i \in I$ , let  $\mu_i \in \mathbb{K}$  such that  $\sigma(u_i) = h_\circ \cdot u_i = \mu_i u_i$ .*

(a) *In case (i) of Theorem 4.13, we have*

$$\delta(b) = \{u_{i_0}^{-1}c_\circ, b\} - \sigma(b)(u_{i_0}^{-1}c_\circ), \quad \forall b \in B,$$

$$\begin{aligned} \delta(u_i) &= 0, \quad \forall i \in I \setminus \{i_0\}, & \delta(u_{i_0}) &= \lambda_\circ c_\circ \neq 0, \\ \delta(c_\circ) &= 0, & \{u_{i_0}, c_\circ\} &= -(\mu_{i_0} + \lambda_\circ)c_\circ u_{i_0}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \{u_i, x\} &= -\mu_i u_i x, \quad \forall i \in I \setminus \{i_0\}, & \{v_{i_0}, x\} &= -\mu_{i_0} v_{i_0} x, \\ \{v_{i_0}, b\} &= (\partial_{u_{i_0}} + \sigma)(b)v_{i_0}, \quad \forall b \in B. \end{aligned}$$

(b) *In case (ii) of Theorem 4.13, we have  $\delta = 0$ ,*

$$\{x, b\} = \sigma(b)x, \quad \forall b \in B, \quad \text{and} \quad \{u_i, x\} = -\mu_i u_i x, \quad \forall i \in I.$$

(c) *In case (iii) of Theorem 4.13, we have  $\delta \neq 0$ ,*

$$\delta(u_i) = 0 \quad \text{and} \quad \{u_i, x\} = -\mu_i u_i x, \quad \forall i \in I.$$

PROOF OF THEOREM 4.13. Corollary 4.9 implies that there is at most one index  $i \in I$  for which  $u_i$  falls into situation (ii) of Theorem 4.12. Thus, we have two cases:

- (1) For all  $i \in I$ , the element  $u_i$  remains Poisson-prime in  $R$ .
- (2) There is an index  $i_0 \in I$  such that  $u_{i_0}$  remains Poisson-prime in  $R$  for all  $i \neq i_0$  and there exists  $c_o \in B$  such that  $v_{i_0} := u_{i_0}x - c_o$  is a homogeneous Poisson-prime element of  $R$ .

Note that it is possible that  $I = \emptyset$ , in which case (1) holds.

Case (1). In this case,  $\delta(u_i) = 0$  for all  $i \in I$ , by Theorem 4.12. Any homogeneous Poisson-prime element of  $R$  which has degree 0 in  $x$  is clearly Poisson-prime in  $B$  and so is an associate of one of the elements  $u_i$ . If there are no other homogeneous Poisson-prime elements in  $R$ , we are in the situation (iii).

Suppose that  $R$  has a homogeneous Poisson-prime element  $v$  that is not an associate of any of the  $u_i$ . By Corollary 4.9,  $v$  has degree 1 in  $x$  and any homogeneous Poisson-prime element of  $R$  is an associate of either  $v$  or one of the  $u_i$ . Write  $v = ux - c_o$  for some homogeneous elements  $u, c_o \in B$ , with  $u \neq 0$ . For any  $b \in B$ , the element

$$\{v, b\} = \{u, b\}x + u(\sigma(b)x + \delta(b)) - \{c_o, b\}$$

is divisible by  $v$  in  $R$ , whence  $\{u, b\} + u\sigma(b)$  is divisible by  $u$  in  $B$ , and so  $\{u, b\}$  is divisible by  $u$ . Thus,  $u$  is a Poisson-normal element of  $B$ .

By Proposition 4.1,  $u = wu'$  where  $w$  is a homogeneous unit in  $B$  and  $u'$  is either 1 or a product  $u_{i_1}u_{i_2} \cdots u_{i_m}$  for some  $i_j \in I$ . After replacing  $v$  by  $w^{-1}v$ , we may assume that  $u = u'$ . Since  $\delta(u_i) = 0$  for all  $i \in I$ , we now have  $\delta(u) = 0$ . Let  $\mu \in \mathbb{K}$  be such that  $\sigma(u) = h_o \cdot u = \mu u$ , and observe that  $v$  is an  $h_o$ -eigenvector with eigenvalue  $\mu + \lambda_o$ , whence  $\sigma(c_o) = h_o \cdot c_o = (\mu + \lambda_o)c_o$ . Now the element

$$\{x, v\} = \sigma(u)x^2 - \sigma(c_o)x - \delta(c_o) = \mu ux^2 - (\mu + \lambda_o)c_o x - \delta(c_o)$$

is divisible by  $v$  in  $R$ , and hence

$$\mu ux^2 - (\mu + \lambda_o)c_o x - \delta(c_o) = (ux - c_o)(\mu x + c)$$

for some  $c \in B$ . Consequently,  $-\lambda_o c_o = uc$  and so  $u \mid c_o$  in  $B$ , whence  $u \mid v$  in  $R$ . It follows that  $u$  must be a unit in  $B$ . By our choices above,  $u = 1$ .

Therefore  $v = x - c_o$ . The homogeneity of  $v$  implies  $\chi_{c_o} = \chi_x$ . For any  $b \in B$ , the element

$$\{v, b\} = \sigma(b)x + \delta(b) - \{c_o, b\}$$

is divisible by  $v$ , whence  $\{v, b\} = \sigma(b)(x - c_o)$  and consequently

$$\delta(b) = \{c_o, b\} - \sigma(b)c_o.$$

Lemma 4.11(b) implies that  $c_o = 0$  and  $\delta = 0$ . Therefore  $v = x$ , and we are in the situation (ii).

Case (2). As before, any homogeneous Poisson-prime element of  $R$  which has degree 0 in  $x$  is an associate of one of the elements  $u_i$ . By Corollary 4.9, any other homogeneous Poisson-prime element of  $R$  is an associate of  $v_{i_0}$ . It follows from Theorem 4.12 that  $u_{i_0}$  cannot be Poisson-prime in  $R$ , since the cases in that theorem are mutually exclusive. Thus, in this case we are in situation (i).  $\square$

## CHAPTER 5

### Iterated Poisson-Ore extensions

We now introduce the class of iterated Poisson-Ore extensions that we call *Poisson nilpotent algebras*, and we classify the homogeneous Poisson-prime elements in such algebras.

#### 5.1. Poisson nilpotent algebras

We focus on iterated Poisson-Ore extensions

$$(5.1) \quad R := \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_N; \sigma_N, \delta_N]_p,$$

where it is taken as implied that  $\mathbb{K}[x_1]_p = \mathbb{K}[x_1; \sigma_1, \delta_1]_p$  with  $\sigma_1 = \delta_1 = 0$ . The integer  $N$  will be called the *length* of the extension  $R$ . For  $k \in [0, N]$ , set

$$R_k := \mathbb{K}[x_1, \dots, x_k] = \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_k; \sigma_k, \delta_k]_p.$$

In particular,  $R_0 = \mathbb{K}$ .

The Poisson analog of the concept of a CGL extension introduced in [34, Definition 3.1] is defined as follows.

**DEFINITION 5.1.** An iterated Poisson-Ore extension (5.1) is called a *Poisson nilpotent algebra* or a *Poisson-CGL extension* (abbreviated to *P-CGL extension*) if it is equipped with a rational Poisson action of a torus  $\mathcal{H}$  such that

- (i) The elements  $x_1, \dots, x_N$  are  $\mathcal{H}$ -eigenvectors.
- (ii) For every  $k \in [2, N]$ ,  $\delta_k$  is a locally nilpotent Poisson  $\sigma_k$ -derivation of the algebra  $R_{k-1}$ .
- (iii) For every  $k \in [1, N]$ , there exists  $h_k \in \mathfrak{h} = \text{Lie } \mathcal{H}$  such that  $\sigma_k = (h_k \cdot)|_{R_{k-1}}$  and the  $h_k$ -eigenvalue of  $x_k$ , to be denoted by  $\lambda_k$ , is nonzero.

Note that each  $x_k$  is an  $\mathfrak{h}$ -eigenvector, by virtue of (i) and (2.7).

Whenever we refer to a Poisson-CGL extension  $R$ , we take the associated torus and its Lie algebra to be denoted  $\mathcal{H}$  and  $\mathfrak{h}$  unless otherwise specified. For any iterated Poisson-Ore extension (5.1), there is a canonical (and maximal in a suitable sense) choice of a  $\mathbb{K}$ -torus acting rationally on  $R$  by Poisson automorphisms such that the variables  $x_k$  are eigenvectors. (See §6.2.)

Conditions (i) and (iii) above imply that

$$(5.2) \quad \sigma_k(x_j) = h_k \cdot x_j = \lambda_{kj} x_j \text{ for some } \lambda_{kj} \in \mathbb{K}, \forall 1 \leq j < k \leq N.$$

We then set  $\lambda_{kk} := 0$  and  $\lambda_{jk} := -\lambda_{kj}$  for  $j < k$ . This gives rise to a skew-symmetric matrix  $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K})$  and the corresponding skew-symmetric bicharacter  $\Omega_{\boldsymbol{\lambda}} : \mathbb{Z}^N \times \mathbb{Z}^N \rightarrow \mathbb{K}$  from (1.5).

For each  $k \in [1, N]$ , the algebra  $R_k$  is a Poisson-Cauchon extension of  $R_{k-1}$ . In particular, it follows from (4.3) and (4.4) that

$$(5.3) \quad \sigma_k \circ \delta_k = \delta_k \circ (\sigma_k + \lambda_k), \quad \forall k \in [2, N]$$

$$(5.4) \quad (h \cdot)|_{R_{k-1}} \circ \delta_k = \chi_{x_k}(h) \delta_k \circ (h \cdot)|_{R_{k-1}}, \quad \forall h \in \mathcal{H}, \quad k \in [2, N].$$

By induction on Theorem 4.7, we obtain

**THEOREM 5.2.** *Every Poisson-CGL extension  $R$  is a noetherian  $\mathcal{H}$ -Poisson-UFD.*

Any Poisson affine space algebra  $R = \mathbb{K}[x_1, \dots, x_N]$  (Examples 2.1) is a Poisson-CGL extension with all  $\delta_k = 0$ , relative to the standard action of  $\mathcal{H} = (\mathbb{K}^*)^N$  on  $R$ . Many examples of Poisson-CGL extensions occur as semiclassical limits of non-commutative CGL extensions, as shown, for instance, in [25, §§2.2–2.7, 4.1–4.6]. The semiclassical limit of the standard (uniparameter) quantized coordinate ring of  $m \times n$  matrices yields the following.

**EXAMPLE 5.3.** For positive integers  $m$  and  $n$ , let  $\mathcal{O}(M_{m,n}(\mathbb{K}))$  be the ring of polynomial functions on the matrix variety  $M_{m,n}(\mathbb{K})$ , written as a polynomial ring

$$\mathcal{O}(M_{m,n}(\mathbb{K})) = \mathbb{K}[t_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n].$$

There is a standard Poisson structure on  $\mathcal{O}(M_{m,n}(\mathbb{K}))$  such that

$$\begin{aligned} \{t_{ij}, t_{kj}\} &= t_{ij}t_{kj}, & \text{for } i < k, \\ \{t_{ij}, t_{il}\} &= t_{ij}t_{il}, & \text{for } j < l, \\ \{t_{ij}, t_{kl}\} &= 0, & \text{for } i < k, \quad j > l, \\ \{t_{ij}, t_{kl}\} &= 2t_{il}t_{kj}, & \text{for } i < k, \quad j < l. \end{aligned}$$

This Poisson algebra is an iterated Poisson-Ore extension

$$\mathcal{O}(M_{m,n}(\mathbb{K})) = \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_N; \sigma_N, \delta_N]_p,$$

where  $N = mn$  and  $x_{(r-1)n+c} = t_{rc}$ . It is easy to write explicit formulas for the maps  $\sigma_k$  and  $\delta_k$ , and to check that each  $\delta_k$  is locally nilpotent. For later reference, we note that the scalars  $\lambda_{kj}$  are given as follows:

$$\lambda_{(r-1)n+c, (r'-1)n+c'} = \begin{cases} \text{sign}(r' - r), & \text{if } c = c', \\ \text{sign}(c' - c), & \text{if } r = r', \\ 0, & \text{otherwise,} \end{cases} \quad \forall r, r' \in [1, m], \quad c, c' \in [1, n].$$

There is a rational Poisson action of the torus  $\mathcal{H} = (\mathbb{K}^*)^{m+n}$  on  $\mathcal{O}(M_{m,n}(\mathbb{K}))$  such that

$$(\xi_1, \dots, \xi_{m+n}) \cdot t_{rc} = \xi_r \xi_{m+c}^{-1} t_{rc}$$

for all  $(\xi_1, \dots, \xi_{m+n}) \in \mathcal{H}$ ,  $r \in [1, m]$ ,  $c \in [1, n]$ . Define

$$h_{rc} := (0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots, 0) \in \mathfrak{h} = \mathbb{K}^{m+n} = \text{Lie } \mathcal{H},$$

where the entries  $-1$  and  $1$  reside in positions  $r$  and  $m+c$ , respectively. Then  $\sigma_{(r-1)n+c} = (h_{rc} \cdot)$  and  $h_{rc} \cdot t_{rc} = -2t_{rc}$ . In other words, for  $k = (r-1)n+c$  we have  $h_k = h_{rc}$  and the  $h_k$ -eigenvalue of  $x_k$  is  $\lambda_k = -2$ . Thus,  $\mathcal{O}(M_{m,n}(\mathbb{K}))$  is a P-CGL extension.

### 5.2. Poisson-prime elements in Poisson-CGL extensions

The next theorem describes the homogeneous Poisson-prime elements of a Poisson-CGL extension  $R = R_N$  iteratively from those of  $R_{N-1}$ . It shows that the situation (iii) from Theorem 4.13 never arises in the framework of Poisson-CGL extensions. Note that the group of units of  $R$  is reduced to scalars. Thus, two prime elements of  $R$  are associates if and only if they are scalar multiples of each other.

**THEOREM 5.4.** *Let  $R$  be an arbitrary Poisson-CGL extension of length  $N$  as in (5.1).*

(a) *Let  $\{u_i \mid i \in I\}$  be a list of the homogeneous Poisson-prime elements of  $R_{N-1}$  up to scalar multiples. There are two possibilities for a list of the homogeneous Poisson-prime elements of  $R$  up to scalar multiples:*

- (i)  $\{u_i \mid i \in I \setminus \{i_0\}\} \sqcup \{u_{i_0}x_N - c_0\}$  for some  $i_0 \in I$  and  $c_0 \in R_{N-1}$  such that  $u_{i_0}^{-1}c_0$  is a nonzero homogeneous element of  $R_{N-1}[u_{i_0}^{-1}]$  with the same  $X(\mathcal{H})$ -degree as  $x_N$ .
- (ii)  $\{u_i \mid i \in I\} \sqcup \{x_N\}$ .

(b) *Let  $\delta$  be a locally nilpotent derivation on  $R$ . If*

$$(5.5) \quad \delta(u) = 0 \text{ for all homogeneous Poisson-prime elements } u \text{ of } R,$$

*then  $\delta = 0$ .*

The proof of Theorem 5.4 will be given in section 5.3. We next derive a number of consequences from part (a) of Theorem 5.4.

It follows from Theorem 4.13 (or Theorem 5.4) that a Poisson-CGL extension  $R$  has only a finite number of pairwise nonproportional homogeneous Poisson-prime elements. We will call this number the *rank* of  $R$ . It also equals the number of Poisson  $\mathcal{H}$ -prime ideals of height 1 in  $R$ . For each  $k \in [1, N]$ , Theorem 5.4 in combination with Proposition 4.14 implies that  $\text{rank } R_k = \text{rank } R_{k-1}$  if  $\delta_k \neq 0$ , while  $\text{rank } R_k = \text{rank } R_{k-1} + 1$  if  $\delta_k = 0$ . Thus,

$$(5.6) \quad \text{rank } R_k = |\{j \in [1, k] \mid \delta_j = 0\}|, \quad \forall k \in [1, N].$$

Using Theorem 5.4, we can describe the homogeneous Poisson-prime elements in each stage of any Poisson-CGL extension  $R$ , as follows. The statement of the result involves the standard predecessor and successor functions,  $p = p_\eta$  and  $s = s_\eta$ , for the level sets of a function  $\eta : [1, N] \rightarrow \mathbb{Z}$ , defined as follows:

$$(5.7) \quad \begin{aligned} p(k) &= \max\{j < k \mid \eta(j) = \eta(k)\}, \\ s(k) &= \min\{j > k \mid \eta(j) = \eta(k)\}, \end{aligned}$$

where  $\max \emptyset = -\infty$  and  $\min \emptyset = +\infty$ . Define corresponding order functions  $O_\pm : [1, N] \rightarrow \mathbb{Z}_{\geq 0}$  by

$$(5.8) \quad \begin{aligned} O_-(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid p^m(k) \neq -\infty\}, \\ O_+(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid s^m(k) \neq +\infty\}. \end{aligned}$$

**THEOREM 5.5.** *Let  $R$  be a Poisson-CGL extension of length  $N$  as in (5.1). There exist a function  $\eta : [1, N] \rightarrow \mathbb{Z}$  and homogeneous elements*

$$c_k \in R_{k-1} \text{ for all } k \in [2, N] \text{ with } p(k) \neq -\infty$$

such that the elements  $y_1, \dots, y_N \in R$ , recursively defined by

$$(5.9) \quad y_k := \begin{cases} y_{p(k)}x_k - c_k, & \text{if } p(k) \neq -\infty \\ x_k, & \text{if } p(k) = -\infty, \end{cases}$$

are homogeneous and have the property that for every  $k \in [1, N]$ ,

$$(5.10) \quad \{y_j \mid j \in [1, k], s(j) > k\}$$

is a list of the homogeneous Poisson-prime elements of  $R_k$  up to scalar multiples. In particular,  $y_k$  is a homogeneous Poisson-prime element of  $R_k$  as well as a prime element of  $R$ , for all  $k \in [1, N]$ .

The elements  $y_1, \dots, y_N \in R$  with these properties are unique. The function  $\eta$  satisfying the above conditions is not unique, but the partition of  $[1, N]$  into the disjoint union of the level sets of  $\eta$  is uniquely determined by  $R$ , as are the predecessor and successor functions  $p$  and  $s$ . The function  $p$  has the property that  $p(k) = -\infty$  if and only if  $\delta_k = 0$ .

The uniqueness statements in the final paragraph of the theorem are dependent on the given P-CGL extension presentation (5.1) of  $R$ , not on  $R$  as an algebra. Typically,  $R$  will have many different P-CGL extension presentations, as we discuss in section 6.1.

Uniqueness of the elements  $y_k$ ,  $k \in [1, N]$ , and of the level sets of  $\eta$ , follows at once from Theorem 5.4. This uniqueness immediately implies the uniqueness of  $p$  and  $s$ . From the final statement of the theorem together with (5.6), we see that the rank of  $R$  is given by

$$(5.11) \quad \text{rank } R = |\eta([1, N])| = |\{j \in [1, N] \mid p(j) = -\infty\}| = |\{j \in [1, N] \mid s(j) = +\infty\}|.$$

PROOF OF THEOREM 5.5. We define  $\eta : [1, k] \rightarrow \mathbb{Z}$  and elements  $c_k \in R_{k-1}$  (when  $p(k) \neq -\infty$ ) for  $k = 1, \dots, N$ . At each step, the new function  $\eta$  will be an extension of the previous one, and so the corresponding new predecessor function will also be an extension of the previous one. However, the successor functions may change, so we will write  $s_k$  for the successor function going with  $\eta$  on  $[1, k]$ .

To start, set  $\eta(1) := 1$ . Note that  $p(1) = -\infty$  and  $s_1(1) = +\infty$ . Moreover,  $y_1 := x_1$  is the unique homogeneous Poisson-prime element of  $R_1$  up to scalar multiples. This relies on the fact that the homogeneous elements of  $R_1$  are just the scalar multiples of the monomials  $x_1^m$ , which follows from (2.7) and the assumption that the  $h_1$ -eigenvalue of  $x_1$  is nonzero.

Now let  $1 < k \leq N$ , and assume that  $\eta$  has been defined on  $[1, k-1]$ , together with elements  $c_j \in R_{j-1}$  for  $j \in [1, k-1]$  with  $p(j) \neq -\infty$  and homogeneous elements  $y_j \in R_j$  for  $j \in [1, k-1]$  such that (5.9) and (5.10) hold. In particular,

$$\{y_j \mid j \in [1, k-1], s_{k-1}(j) \geq k\}$$

is a list of the homogeneous Poisson-prime elements of  $R_{k-1}$  up to scalar multiples. There are two cases to consider, corresponding to situations (i), (ii) of Theorem 5.4(a).

In the first case, there is some  $j_0 \in [1, k-1]$  such that  $s_{k-1}(j_0) \geq k$  and some homogeneous  $c_k \in R_{k-1}$  such that

$$(5.12) \quad \{y_j \mid j \in [1, k-1], j \neq j_0, s_{k-1}(j) \geq k\} \sqcup \{y_{j_0}x_k - c_k\}$$

is a list of the homogeneous Poisson-prime elements of  $R_k$  up to scalar multiples. In this case, we extend  $\eta$  to a function  $[1, k] \rightarrow \mathbb{Z}$  by setting  $\eta(k) = \eta(j_0)$ . Since  $s_{k-1}(j_0) \geq k$ , we see that  $p(k) = j_0$ , and so  $y_k := y_{j_0}x_k - c_k$ . It is easily checked that the set (5.12) equals (5.10).

In the second case,  $\{y_j \mid j \in [1, k-1], s_{k-1}(j) \geq k\} \sqcup \{x_k\}$  is a list of the homogeneous Poisson-prime elements of  $R_k$  up to scalar multiples. In this case, we set  $\eta(k)$  equal to any integer not in  $\eta([1, k-1])$  and readily check the desired properties.

It remains to verify the final statement of the theorem. If  $k \in [1, N]$  and  $p(k) = -\infty$ , then  $x_k = y_k$  is a Poisson-prime element of  $R_k$ . Consequently, the elements  $\sigma_k(b)x_k + \delta_k(b) = \{x_k, b\}$  are divisible by  $x_k$  for all  $b \in R_{k-1}$ , which forces  $\delta_k = 0$ . On the other hand, if we have  $\delta_k = 0$ , then  $x_k$  is a homogeneous Poisson-prime element of  $R_k$ , whence  $y_k$  must be a scalar multiple of  $x_k$ . That is not possible when  $p(k) \neq -\infty$ , by (5.9), so we must have  $p(k) = -\infty$ . This concludes the proof.  $\square$

To illustrate Theorem 5.5, we continue Example 5.3.

EXAMPLE 5.6. Let  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$  be the P-CGL extension in Example 5.3. For any two subsets  $I \subseteq [1, m]$  and  $J \subseteq [1, n]$  of the same cardinality  $d$ , let  $\Delta_{I,J}$  denote the  $d \times d$  minor in  $R$  with row index set  $I$  and column index set  $J$ . The sequence of homogeneous Poisson-prime elements of  $R$  from Theorem 5.5 consists of solid minors; more precisely,

$$y_{(r-1)n+c} = \Delta_{[r-\min(r,c)+1, r], [c-\min(r,c)+1, c]}, \quad \forall r \in [1, m], \quad c \in [1, n].$$

Furthermore, the function  $\eta : [1, N] \rightarrow \mathbb{Z}$  can be chosen as

$$\eta((r-1)n+c) := c-r, \quad \forall r \in [1, m], \quad c \in [1, n].$$

It is easily checked that for the P-CGL extension presentation of  $R$  in Example 5.3, we have  $\delta_k = 0$  if and only if  $k \in [1, n]$  or  $k = (r-1)n+1$  for some  $r \in [2, m]$ . Hence,  $\text{rank } R = m+n-1$ , by (5.6).

The next result provides a constructive method for finding the sequence of Poisson-prime elements  $y_1, \dots, y_N$  for a given Poisson-CGL extension  $R$ . Note that the elements  $y'_k$  in the statement of the proposition are not a priori assumed to be prime.

PROPOSITION 5.7. *Let  $R$  be a Poisson-CGL extension of length  $N$ . Assume that  $y'_1, \dots, y'_N$  and  $c'_1, \dots, c'_N$  are two sequences of elements of  $R$  such that*

- (i)  *$y'_1, \dots, y'_N$  are homogeneous Poisson-normal elements of  $R_1, \dots, R_N$ , respectively.*
- (ii)  *$c'_k \in R_{k-1}$ , for all  $k \in [1, N]$ .*
- (iii) *For every  $k \in [1, N]$ , either  $y'_k = x_k - c'_k$  or there exists  $j \in [1, k-1]$  such that  $y'_k = y'_j x_k - c'_k$ .*
- (iv) *If  $k \in [1, N]$  and  $p(k) = -\infty$ , then the first equality in (iii) holds.*

*Then  $y'_1, \dots, y'_N$  is precisely the sequence  $y_1, \dots, y_N$  of homogeneous Poisson-prime elements from Theorem 5.5, and the function  $p$  satisfies  $p(k) = j$  if the second equality in (iii) holds and  $p(k) = -\infty$  otherwise.*

PROOF. The given assumptions imply that  $c'_1 \in \mathbb{K}$  and  $y'_1 = x_1 - c'_1$ . Since  $y'_1$  is homogeneous, we must have  $y'_1 = x_1 = y_1$ .

Now let  $k \in [2, N]$ . We will prove that if  $y'_i = y_i$  for all  $i \in [1, k-1]$ , then  $y'_k = y_k$ . This implies the first statement of the proposition by induction. By Proposition 4.1 and Theorems 5.2, 5.5,

$$y'_k = \xi \prod \{y_i^{m_i} \mid i \in [1, k], s(i) > k\}$$

for some  $\xi \in \mathbb{K}^*$  and  $m_i \in \mathbb{Z}_{\geq 0}$ . Comparing the coefficients of  $x_k$  and using the form of  $y_k$  from Theorem 5.5, we obtain that  $m_k = 1$ .

The  $x_k$ -coefficient of  $y'_k$  is either 1 or  $y'_j$ , hence either a unit or a prime element, by induction. Thus, one of the following three situations holds:

- (a)  $y_k = x_k$  and  $y'_k = \xi y_k$ .
- (b)  $y_k = x_k$  and  $y'_k = \xi y_j y_k$ , for some  $j \in [1, k-1]$ .
- (c)  $y_k = y_{p(k)} x_k - c_k$  and  $y'_k = \xi y_k$ .

Because of assumption (iv), the situation (b) cannot occur. In the situation (a), we have  $\xi = 1$ ,  $c'_k = 0$ , and  $y'_k = y_k$ . In the situation (c), we obtain  $\xi = 1$ ,  $j = p(k)$ ,  $c'_k = c_k$ , and  $y'_k = y_k$ . This argument also proves the second statement of the proposition.  $\square$

We next pin down the scalars involved in certain leading coefficients and Poisson brackets. Assume that  $R$  is a Poisson-CGL extension of length  $N$ , with elements  $y_1, \dots, y_N$  as in Theorem 5.5.

The algebra  $R$  has the  $\mathbb{K}$ -basis

$$(5.13) \quad \{x^f := (x_1, \dots, x_N)^f \mid f \in \mathbb{Z}_{\geq 0}^N\}$$

(recall (1.6)). Denote by  $\prec$  the reverse lexicographic order on  $\mathbb{Z}_{\geq 0}^N$ ; namely,

$$(m'_1, \dots, m'_N)^T \prec (m_1, \dots, m_N)^T \iff \exists j \in [1, N] \text{ such that } m'_j < m_j \text{ and } m'_k = m_k \forall k \in [j+1, N].$$

We will say that  $b \in R \setminus \{0\}$  has *leading term*  $\text{lt}(b) := \xi x^f$ , where  $\xi \in \mathbb{K}^*$  and  $f \in \mathbb{Z}_{\geq 0}^N$ , if

$$b = \xi x^f + \sum_{g \in \mathbb{Z}_{\geq 0}^N, g \prec f} \xi_g x^g$$

for some  $\xi_g \in \mathbb{K}$ . It follows from (5.2) that

$$(5.14) \quad \text{lt}(\{x^f, x^{f'}\}) = \left( \sum_{k > j} m_k m'_j \lambda_{kj} \right) x^{f+f'} = \Omega_{\lambda}(f, f') x^{f+f'},$$

$$\forall f = (m_1, \dots, m_N)^T, f' = (m'_1, \dots, m'_N)^T \in \mathbb{Z}_{\geq 0}^N,$$

recall (1.5). Denote

$$(5.15) \quad \bar{e}_k := \sum_{l=0}^{O_-(k)} e_{p^l(k)}, \quad \forall k \in [1, N].$$

Equation (5.9) implies

$$(5.16) \quad \text{lt}(y_k) = x^{\bar{e}_k}, \quad \forall k \in [1, N].$$

For  $k, j \in [1, N]$ , set

$$(5.17) \quad \alpha_{kj} := \Omega_{\lambda}(e_k, \bar{e}_j) = \sum_{m=0}^{O_-(j)} \lambda_{k,p^m(j)} \in \mathbb{K},$$

$$(5.18) \quad q_{kj} := \Omega_{\lambda}(\bar{e}_k, \bar{e}_j) = \sum_{l=0}^{O_-(k)} \sum_{m=0}^{O_-(j)} \lambda_{p^l(k), p^m(j)} \in \mathbb{K},$$

$$(5.19) \quad \alpha_{k,-\infty} = q_{k,-\infty} := 0.$$

It will also be useful to set

$$(5.20) \quad y_{-\infty} := 1.$$

Since  $\lambda$  is a skew-symmetric matrix, so is  $\mathbf{q} := (q_{kj}) \in M_N(\mathbb{K})$ . It follows from (5.16), the homogeneity of  $y_j$ , and (2.7) that

$$(5.21) \quad \sigma_k(x^{\bar{e}_j}) = \alpha_{kj} x^{\bar{e}_j} \quad \text{and} \quad \sigma_k(y_j) = \alpha_{kj} y_j, \quad \forall 1 \leq j < k \leq N.$$

PROPOSITION 5.8. *Let  $R$  be a Poisson-CGL extension of length  $N$ . The elements  $y_1, \dots, y_N$  from Theorem 5.5 are algebraically independent over  $\mathbb{K}$ , and*

$$(5.22) \quad \mathbb{K}[y_1, \dots, y_N] \subseteq R \subseteq \mathbb{K}[y_1^{\pm 1}, \dots, y_N^{\pm 1}] \subset \mathbb{K}(x_1, \dots, x_N) = \mathbb{K}(y_1, \dots, y_N).$$

*The algebras  $\mathbb{K}[y_1, \dots, y_N]$  and  $\mathbb{K}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$  are Poisson algebras, with*

$$(5.23) \quad \{y_k, y_j\} = q_{kj} y_j y_k, \quad \forall k, j \in [1, N],$$

*and the inclusions in (5.22) are inclusions of Poisson algebras.*

PROOF. Algebraic independence of  $y_1, \dots, y_N$  is clear from the fact that each  $y_k$  is a polynomial in  $\mathbb{K}[x_1, \dots, x_k]$  of degree 1 in  $x_k$ . For  $k \in [1, N]$ , either  $x_k = y_k$  or  $x_k = y_{p(k)}^{-1}(y_k + c_k)$  with  $c_k \in R_{k-1}$ . By induction, we obtain  $R_k \subseteq \mathbb{K}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$  for all  $k \in [1, N]$ , which yields the second inclusion of (5.22). The first and third inclusions are clear, and the final equality follows.

From Theorem 5.5 and equations (5.14), (5.16), we obtain

$$\text{lt}(\{y_k, y_j\}) = q_{kj} \text{lt}(y_j y_k), \quad \forall j, k \in [1, N].$$

On the other hand, using Theorem 5.5 and Proposition 4.14 (or Proposition 5.9 below), we see by induction on  $k$  that  $\{y_k, y_j\}$  is a scalar multiple of  $y_j y_k$  for  $1 \leq j < k \leq N$ . Therefore, (5.23) holds for  $1 \leq j < k \leq N$ . The remaining cases follow because of the skew-symmetry of the matrix  $(q_{kj})$ .  $\square$

Applying Proposition 4.14 to the situation of Theorem 5.5 leads to the following facts.

PROPOSITION 5.9. *Keep the assumptions and notation of Theorem 5.5. Let  $k \in [2, N]$ .*

(a) *If  $p(k) = -\infty$ , then  $\delta_k = 0$ , and the derivations of  $R_k$  corresponding to its homogeneous Poisson-prime elements are determined by*

$$\{y_j, x_k\} = -\alpha_{kj} y_j x_k, \quad \forall j \in [1, k-1], \quad \{y_k, a\} = \sigma_k(a) y_k, \quad \forall a \in R_{k-1},$$

*together with the actions of  $\partial_{y_j}$  on  $R_{k-1}$  for  $j \in [1, k-1]$  such that  $s(j) \geq k$  (obtainable by recursion).*

(b) *If  $p(k) \neq -\infty$ , then the derivation  $\delta_k$  is nonzero and is given by*

$$\delta_k(a) = \{y_{p(k)}^{-1} c_k, a\} - \sigma_k(a) y_{p(k)}^{-1} c_k, \quad \forall a \in R_{k-1}.$$

This derivation also satisfies

$$(5.24) \quad \begin{aligned} \delta_k(y_{p(k)}) &= \lambda_k c_k \neq 0, & \delta_k(c_k) &= 0, \\ \delta_k(y_j) &= 0, \quad \forall j \in [1, k-1] \text{ such that } s(j) > k. \end{aligned}$$

The derivations of  $R_k$  corresponding to its homogeneous Poisson-prime elements are determined by

$$(5.25) \quad \begin{aligned} \{y_j, x_k\} &= -\alpha_{kj} y_j x_k, \quad \forall j \in [1, k-1] \text{ such that } s(j) > k, \\ \{y_k, x_k\} &= -\alpha_{kp(k)} y_k x_k, \\ \{y_k, a\} &= (\partial_{y_{p(k)}} + \sigma_k)(a) y_k, \quad \forall a \in R_{k-1}, \end{aligned}$$

together with the actions of  $\partial_{y_j}$  on  $R_{k-1}$  for  $j \in [1, k-1]$  such that  $s(j) \geq k$  (obtainable by recursion). Furthermore, the components  $y_{p(k)}$  and  $c_k$  of  $y_k$  satisfy

$$\{y_{p(k)}, c_k\} = -(\alpha_{kp(k)} + \lambda_k) c_k y_{p(k)}.$$

COROLLARY 5.10. *If  $R$  is a Poisson-CGL extension of length  $N$ , then every homogeneous Poisson-prime element of  $R$  quasi-Poisson-commutes with  $x_1, \dots, x_N$ . More precisely,*

$$(5.26) \quad \{y_j, x_k\} = -\alpha_{kj} y_j x_k, \quad \forall j, k \in [1, N] \text{ with } s(j) > k.$$

PROOF. We proceed by induction on  $l \in [1, N]$ , to prove that (5.26) holds for  $j, k \in [1, l]$ . The case  $l = 1$  is clear, since  $y_1 = x_1$  and  $\alpha_{11} = \lambda_{11} = 0$ .

Now let  $l > 1$ , and assume (5.26) holds for  $j, k \in [1, l-1]$ . If  $j \in [1, l-1]$  and  $s(j) > l$ , then both cases of Proposition 5.9 yield  $\{y_j, x_l\} = -\alpha_{lj} y_j x_l$ . Hence, it just remains to consider  $\{y_l, -\}$ .

If  $p(l) = -\infty$ , then  $y_l = x_l$  and  $\delta_l = 0$ . In this case,

$$\{y_l, x_k\} = \sigma_l(x_k) y_l = \lambda_{lk} y_l x_k = -\alpha_{kl} y_l x_k$$

for  $k \in [1, l-1]$ , while  $\{y_l, x_l\} = -\alpha_{ll} y_l x_l$  because  $\alpha_{ll} = \lambda_{ll} = 0$ . Finally, suppose that  $p(l) \neq -\infty$ , and note that  $\alpha_{ll} = \lambda_{ll} + \alpha_{l, p(l)} = \alpha_{l, p(l)}$ . Hence, it follows from Proposition 5.9(b) that  $\{y_l, x_l\} = -\alpha_{ll} y_l x_l$ . Since  $s(p(l)) = l$ , our induction hypothesis implies that  $\{y_{p(l)}, x_k\} = -\alpha_{k, p(l)} y_{p(l)} x_k$  for  $k \in [1, l-1]$ . Appealing again to Proposition 5.9(b), we conclude that

$$\{y_l, x_k\} = (\partial_{y_{p(l)}} + \sigma_l)(x_k) y_l = (-\alpha_{k, p(l)} + \lambda_{lk}) y_l x_k = -\alpha_{kl} y_l x_k$$

for  $k \in [1, l-1]$ .

This completes the induction. □

From the first equation in (5.24), we immediately obtain the following.

COROLLARY 5.11. *Keep the assumptions and notation of Theorem 5.5. The homogeneous Poisson-prime elements  $y_k$  are explicitly given by*

$$(5.27) \quad y_k = \begin{cases} x_k & \text{if } p(k) = -\infty \\ y_{p(k)} x_k - \lambda_k^{-1} \delta_k(y_{p(k)}) & \text{if } p(k) \neq -\infty \end{cases}$$

for  $k \in [1, N]$ . □

### 5.3. Proof of Theorem 5.4

For any positive integer  $L$  and statement  $X$  about Poisson-CGL extensions, we denote by  $X[L]$  the validity of the statement  $X$  for all Poisson-CGL extensions  $R$  of length at most  $L$ . The work of section 5.2 shows that

$$(5.28) \quad \text{Theorem 5.4(a)}[N] \implies \text{Theorem 5.5}[N] \text{ and Proposition 5.8}[N].$$

The proof of Theorem 5.4 will be completed once the following implications are established:

$$(5.29) \quad \text{Theorem 5.4(b)}[N-1] \implies \text{Theorem 5.4(a)}[N],$$

$$(5.30) \quad \text{Theorem 5.4(a)}[N] \implies \text{Theorem 5.4(b)}[N].$$

PROOF OF THE IMPLICATION (5.29). We apply Theorem 4.13 to the Poisson-Ore extension  $R = R_{N-1}[x_N; \sigma_N, \delta_N]$ . All we need to show is that in this setting, the situation (iii) of Theorem 4.13 can never occur. Suppose to the contrary that situation (iii) does obtain. Then by Proposition 4.14,  $\delta_N \neq 0$  but  $\delta_N(u) = 0$  for all homogeneous Poisson-prime elements  $u$  of  $R_{N-1}$ . However, this contradicts Theorem 5.4(b)[ $N-1$ ].  $\square$

Our proof of (5.30) involves some analysis of derivations on the Laurent polynomial ring  $\mathcal{T} := \mathbb{K}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$ . For  $f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N$ , define the Laurent monomial

$$(5.31) \quad y^f := y_1^{m_1} \cdots y_N^{m_N} \in \mathcal{T}.$$

The algebra  $\mathcal{T}$  is  $\mathbb{Z}^N$ -graded, with

$$\deg y^f := f, \quad \forall f \in \mathbb{Z}^N.$$

We will say that a  $\mathbb{K}$ -linear map  $\varphi : \mathcal{T} \rightarrow \mathcal{T}$  is  $\mathbb{Z}^N$ -homogeneous of degree  $g \in \mathbb{Z}^N$  if  $\varphi(y^f) \in \mathbb{K}y^{f+g}$  for all  $f \in \mathbb{Z}^N$ , where we use the term  $\mathbb{Z}^N$ -homogeneous to distinguish the above condition from homogeneity with respect to the  $X(\mathcal{H})$ -grading. Given a general  $\mathbb{K}$ -linear map  $\varphi$  on  $\mathcal{T}$ , there are unique  $\mathbb{Z}^N$ -homogeneous  $\mathbb{K}$ -linear maps  $\varphi^g$  of degree  $g$  on  $\mathcal{T}$  such that

$$\varphi = \sum_{g \in \mathbb{Z}^N} \varphi^g.$$

If  $\delta$  is a derivation of  $\mathcal{T}$ , then each component  $\delta^g$ , for  $g \in \mathbb{Z}^N$ , is a derivation of  $\mathcal{T}$ . Since  $\mathcal{T}$  is a finitely generated algebra,  $\delta^g \neq 0$  for at most finitely many  $g \in \mathbb{Z}^N$ .

Let  $\prec$  be the reverse lexicographic order on  $\mathbb{Z}^N$  (defined as it was above on  $\mathbb{Z}_{\geq 0}^N$ ). Any nonzero element  $u \in \mathcal{T}$  can be uniquely written in the form

$$(5.32) \quad u = \zeta_1 y^{f_1} + \cdots + \zeta_r y^{f_r} \text{ where } f_1 \prec \cdots \prec f_r \text{ in } \mathbb{Z}^N \text{ and } \zeta_1, \dots, \zeta_r \in \mathbb{K}^*.$$

We will say that  $\zeta_r y^{f_r}$  is the *leading term* of  $u$  and denote it  $\text{lt}_{\mathcal{T}}(u)$ , to distinguish it from our previous usage of leading terms. For future reference, observe that

$$(5.33) \quad \text{lt}_{\mathcal{T}}(ay_k + b) = \text{lt}_{\mathcal{T}}(a)y_k, \quad \forall k \in [2, N], \quad a, b \in \mathbb{K}[y_1^{\pm 1}, \dots, y_{k-1}^{\pm 1}], \quad a \neq 0.$$

If  $\delta$  is a nonzero derivation of  $\mathcal{T}$ , we have

$$(5.34) \quad \delta = \delta^{g_1} + \cdots + \delta^{g_t} \text{ with } g_1 \prec \cdots \prec g_t \text{ in } \mathbb{Z}^N, \text{ all } \delta^{g_i} \neq 0.$$

For  $m > 0$  and  $u$  as in (5.32), the component of  $\delta^m(u)$  in degree  $f_r + mg_t$  is  $\zeta_r(\delta^{g_t})^m(y^{f_r})$ . Hence,

$$(5.35) \quad \delta^m(u) = 0 \implies (\delta^{g_t})^m(\text{lt}_{\mathcal{T}}(u)) = 0$$

for all  $m > 0$  and nonzero  $u \in \mathcal{T}$ .

LEMMA 5.12. *Let  $\mathcal{T}$  be a domain of characteristic zero and  $\delta$  a derivation on  $\mathcal{T}$ . Suppose  $y \in \mathcal{T}$  is a unit such that  $\delta^m(y) = \delta^m(y^{-1}) = 0$  for some  $m > 0$ . Then  $\delta(y) = 0$ .*

PROOF. Let  $r, s \geq 0$  be maximal such that  $\delta^r(y) \neq 0$  and  $\delta^s(y^{-1}) \neq 0$ . Then

$$\delta^{r+s}(yy^{-1}) = \sum_{l=0}^{r+s} \binom{r+s}{l} \delta^l(y) \delta^{r+s-l}(y^{-1}) = \binom{r+s}{r} \delta^r(y) \delta^s(y^{-1}) \neq 0.$$

Since  $\delta(1) = 0$ , we must have  $r + s = 0$ . Hence,  $r = 0$ , and therefore  $\delta(y) = 0$ .  $\square$

PROOF OF THE IMPLICATION (5.30). Let  $R$  be a Poisson-CGL extension of length  $N$ . By (5.28), the statements of Theorem 5.5 and Proposition 5.8 hold for  $R$ . Let  $\delta$  be a locally nilpotent derivation of  $R$  satisfying (5.5), and suppose that  $\delta \neq 0$ . The assumption (5.5) implies

$$(5.36) \quad \delta(y_k) = 0, \forall k \in [1, N] \text{ such that } s(k) = +\infty.$$

Continue to denote by  $\delta$  the canonical (unique) extension of  $\delta$  to a derivation of the quotient field  $\mathbb{K}(x_1, \dots, x_N)$ . Set  $\mathcal{T} := \mathbb{K}[y_1^{\pm 1}, \dots, y_N^{\pm 1}]$  as above. Since  $\delta(y_k) \in R \subseteq \mathcal{T}$  for all  $k \in [1, N]$ , we see that  $\delta(\mathcal{T}) \subseteq \mathcal{T}$ . Now view  $\delta$  as a derivation of  $\mathcal{T}$ , and decompose  $\delta$  as in (5.34). Since  $\delta^{g_t} \neq 0$ , we must have  $\delta^{g_t}(y_j) \neq 0$  for some  $j \in [1, N]$ . Moreover,  $\delta(y_j) \neq 0$ , and so  $s(j) \neq +\infty$ . We will prove the following implication:

$$(5.37) \quad \text{if } \delta^{g_t}(y_k) = 0 \text{ for some } k \in [1, N] \text{ with } p(k) \neq -\infty, \text{ then } \delta^{g_t}(y_{p(k)}) = 0.$$

By (5.36) and (5.35),  $\delta^{g_t}(y_k) = 0$  for all  $k \in [1, N]$  such that  $s(k) = +\infty$ . A downward recursive application of (5.37) leads to  $\delta^{g_t}(y_k) = 0$  for all  $k \in [1, N]$ , contradicting the fact that  $\delta^{g_t}(y_j) \neq 0$ . This contradiction proves the implication (5.30).

We are left with proving (5.37). Assume that  $\delta^{g_t}(y_k) = 0$  for some  $k \in [1, N]$  with  $p(k) \neq -\infty$ . There exists  $m > 0$  such that

$$0 = \delta^m(x_k) = \delta^m(y_{p(k)}^{-1}y_k + y_{p(k)}^{-1}c_k).$$

Since  $c_k \in R_{k-1}$ , the elements  $c_k$  and  $y_{p(k)}^{-1}c_k$  belong to  $\mathbb{K}[y_1^{\pm 1}, \dots, y_{k-1}^{\pm 1}]$ . Observations (5.33) and (5.35) then imply  $(\delta^{g_t})^m(y_{p(k)}^{-1}y_k) = 0$ . But  $\delta^{g_t}(y_k) = 0$ , so

$$(5.38) \quad 0 = (\delta^{g_t})^m(y_{p(k)}^{-1}y_k) = ((\delta^{g_t})^m(y_{p(k)}^{-1}))y_k, \text{ i.e., } (\delta^{g_t})^m(y_{p(k)}^{-1}) = 0.$$

On the other hand, the restriction of  $\delta$  to  $R$  is locally nilpotent. Hence,  $\delta^{m'}(y_{p(k)}) = 0$  for some  $m' > 0$ . It follows from (5.35) that  $(\delta^{g_t})^{m'}(y_{p(k)}) = 0$ . We combine this and the last equality in (5.38), and apply Lemma 5.12 to obtain  $\delta^{g_t}(y_{p(k)}) = 0$ , as required.  $\square$

## Symmetry and maximal tori for Poisson-CGL extensions

We first introduce *symmetric* Poisson-CGL extensions, which are ones that also have a Poisson-CGL extension presentation with the variables in reverse order. We then analyze the torus action on an arbitrary Poisson-CGL extension  $R$ , and show that there is a unique torus which is maximal among tori acting faithfully on  $R$  by rational Poisson actions.

### 6.1. Symmetric Poisson-CGL extensions

If  $R = \mathbb{K}[x_1, \dots, x_N]$  is a polynomial ring, with the variables fixed in the order  $x_1, \dots, x_N$ , we denote

$$(6.1) \quad R_{[j,k]} := \mathbb{K}[x_i \mid j \leq i \leq k], \quad \forall j, k \in [1, N].$$

So,  $R_{[j,k]} = \mathbb{K}$  if  $j \not\leq k$ .

DEFINITION 6.1. We call a Poisson-CGL extension  $R$  of length  $N$  as in Definition 5.1 *symmetric* if the following conditions hold:

- (i)  $\delta_k(x_j) \in R_{[j+1, k-1]}$ , for all  $1 \leq j < k \leq N$ .
- (ii) For all  $j \in [1, N]$ , there exists  $h_j^* \in \mathfrak{h}$  such that

$$h_j^* \cdot x_k = -\lambda_{kj} x_k = \lambda_{jk} x_k, \quad \forall k \in [j+1, N]$$

$$\text{and } h_j^* \cdot x_j = \lambda_j^* x_j \text{ for some nonzero } \lambda_j^* \in \mathbb{K}.$$

Under these conditions, set

$$\sigma_j^* := (h_j^*), \quad \forall j \in [1, N-1].$$

Then  $\sigma_j^*$  is a Poisson derivation on  $R$ , and there is an inner Poisson  $\sigma_j^*$ -derivation  $\delta_j^*$  on  $R$  given by  $\delta_j^*(a) := \{x_j, a\} - \sigma_j^*(a)x_j$ . The restriction of  $\delta_j^*$  to  $R_{[j+1, N]}$  is determined by

$$\delta_j^*(x_k) = \{x_j, x_k\} - \lambda_{jk} x_j x_k = -\delta_k(x_j), \quad \forall k \in [j+1, N].$$

For all  $1 \leq j < k \leq N$ , the maps  $\sigma_k$  and  $\delta_k$  preserve  $R_{[j, k-1]}$ , and  $\sigma_j^*$  and  $\delta_j^*$  preserve  $R_{[j+1, N]}$ . This gives rise to the Poisson-Ore extensions

$$(6.2) \quad R_{[j,k]} = R_{[j, k-1]}[x_k; \sigma_k, \delta_k]_p \quad \text{and} \quad R_{[j,k]} = R_{[j+1, k]}[x_j; \sigma_j^*, \delta_j^*]_p.$$

In particular, it follows that  $R$  has an iterated Poisson-Ore extension presentation with the variables  $x_k$  in descending order:

$$(6.3) \quad R = \mathbb{K}[x_N]_p[x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*]_p \cdots [x_1; \sigma_1^*, \delta_1^*]_p,$$

which is the reason for the name “symmetric”. The presentation (6.3) is, in fact, a Poisson-CGL extension presentation of  $R$ ; we will record this as part of Proposition 6.4.

EXAMPLE 6.2. For example, the Poisson-CGL extension  $\mathcal{O}(M_{m,n}(\mathbb{K}))$  of Example 5.3 is symmetric. It is clear that in this example, condition (i) of Definition 6.1 holds. Condition (ii) can be verified for the following elements  $h_j^* \in \mathfrak{h}$ . Given  $j \in [1, N]$ , write  $j = (r-1)n + c$  for some  $r \in [1, m]$  and  $c \in [1, n]$ , and take

$$h_j^* := (0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$$

where the entries 1 and  $-1$  reside in positions  $r$  and  $m+c$ , respectively. In particular,  $h_j^* \cdot x_j = 2x_j$ , so that  $\lambda_j^* = 2$ .

DEFINITION 6.3. Denote the following subset of the symmetric group  $S_N$ :

$$(6.4) \quad \Xi_N := \{\tau \in S_N \mid \tau(k) = \max \tau([1, k-1]) + 1 \text{ or} \\ \tau(k) = \min \tau([1, k-1]) - 1, \forall k \in [2, N]\}.$$

In other words,  $\Xi_N$  consists of those  $\tau \in S_N$  such that  $\tau([1, k])$  is an interval for all  $k \in [2, N]$ . For each  $\tau \in \Xi_N$ , we have the iterated Poisson-Ore extension presentation

$$(6.5) \quad R = \mathbb{K}[x_{\tau(1)}]_p [x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}]_p \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}]_p,$$

where  $\sigma''_{\tau(k)} := \sigma_{\tau(k)}$  and  $\delta''_{\tau(k)} := \delta_{\tau(k)}$  if  $\tau(k) = \max \tau([1, k-1]) + 1$ , while  $\sigma''_{\tau(k)} := \sigma_{\tau(k)}^*$  and  $\delta''_{\tau(k)} := \delta_{\tau(k)}^*$  if  $\tau(k) = \min \tau([1, k-1]) - 1$ .

The following proposition is straightforward.

PROPOSITION 6.4. *For every symmetric P-CGL extension  $R$  of length  $N$  and  $\tau \in \Xi_N$ , the iterated Poisson-Ore extension presentation (6.5) of  $R$  is a Poisson-CGL extension presentation for the same choice of torus  $\mathcal{H}$ , and the associated elements  $h''_{\tau(1)}, \dots, h''_{\tau(N)} \in \mathfrak{h}$  required by Definition 5.1(iii) are given by  $h''_{\tau(k)} = h_{\tau(k)}$  if  $\tau(k) = \max \tau([1, k-1]) + 1$  and  $h''_{\tau(k)} = h_{\tau(k)}^*$  if  $\tau(k) = \min \tau([1, k-1]) - 1$ .*

Theorem 6.8 implies that the group  $\mathcal{H}_{\max}(R)$  does not change in passing from the Poisson-CGL extension presentation (5.1) to (6.5), modulo identifying  $(\mathbb{K}^*)^N$  with the copy obtained by permuting coordinates with  $\tau$ . Consequently, the theorem shows that the rank of  $R$  does not depend on the choice of Poisson-CGL extension presentation (6.5).

When describing permutations  $\tau \in S_N$  as functions, we will use the one-line notation,

$$(6.6) \quad \tau = [\tau(1), \tau(2), \dots, \tau(N)] := \begin{bmatrix} 1 & 2 & \cdots & N \\ \tau(1) & \tau(2) & \cdots & \tau(N) \end{bmatrix}.$$

A special role is played by the longest element of  $S_N$ ,

$$(6.7) \quad w_{\circ} := [N, N-1, \dots, 1].$$

The corresponding Poisson-CGL extension presentation described in Proposition 6.4, namely (6.3), is symmetric, while the ones for the other elements of  $\Xi_N$  do not possess this property in general.

## 6.2. Maximal tori

Let  $R$  be a Poisson-CGL extension of length  $N$  as in (5.1). Equip  $R$  with the rational action of the torus  $(\mathbb{K}^*)^N$  by  $\mathbb{K}$ -algebra automorphisms such that

$$(6.8) \quad (\alpha_1, \dots, \alpha_N) \cdot x_i = \alpha_i x_i, \quad \forall i \in [1, N].$$

The differential of this action is the derivation action of  $\mathbb{K}^N = \text{Lie}(\mathbb{K}^*)^N$  on  $R$  given by

$$(6.9) \quad (\beta_1, \dots, \beta_N) \cdot (x_1^{m_1} \cdots x_N^{m_N}) = \left( \sum_{i=1}^N m_i \beta_i \right) x_1^{m_1} \cdots x_N^{m_N}, \quad \forall (m_1, \dots, m_N) \in \mathbb{Z}_{\geq 0}^N.$$

The given action of  $\mathcal{H}$  on  $R$  factors through the above  $(\mathbb{K}^*)^N$ -action via a morphism (of algebraic groups)  $\mathcal{H} \rightarrow (\mathbb{K}^*)^N$ . Since nothing is lost by reducing  $\mathcal{H}$  modulo the kernel of its action, we may assume that the action of  $\mathcal{H}$  is faithful and then identify  $\mathcal{H}$  with its image in  $(\mathbb{K}^*)^N$ . Thus,

$$(6.10) \quad \text{Without loss of generality, } \mathcal{H} \text{ is a closed subgroup of } (\mathbb{K}^*)^N.$$

For the fact that the image of  $\mathcal{H}$  in  $(\mathbb{K}^*)^N$  must be closed, see, e.g., [4, Corollary 1.4].

Next, set

$$(6.11) \quad G := \{\psi \in (\mathbb{K}^*)^N \mid (\psi \cdot), \text{ acting as in (6.8), is a Poisson automorphism of } R\},$$

and observe that  $G$  is a closed subgroup of  $(\mathbb{K}^*)^N$ . Since  $G$  is diagonalizable, its connected component of the identity,  $G^\circ$ , is a torus (e.g., [4, Corollary 8.5]). This subgroup is the unique maximal torus of  $G$ , and so it contains  $\mathcal{H}$ . Let us set

$$(6.12) \quad \mathcal{H}_{\max}(R) := G^\circ.$$

(The definition of this torus, and its position within  $(\mathbb{K}^*)^N$ , depend on the Poisson-CGL extension presentation (5.1) of  $R$ . However, we do not indicate this dependence in the notation.) Since  $\mathcal{H}_{\max}(R)$  contains  $\mathcal{H}$ , the algebra  $R$  is also a Poisson-CGL extension with respect to  $\mathcal{H}_{\max}(R)$ . We shall prove that  $G$  is connected and so  $\mathcal{H}_{\max}(R) = G$ .

The torus  $\mathcal{H}_{\max}(R)$  is unique (up to isomorphism) because of the following universal property, assuming that we fix the Poisson-CGL extension presentation (5.1). If  $\mathcal{H}_1$  is any torus with a rational Poisson action on  $R$  such that  $(R, \mathcal{H}_1)$  is Poisson-CGL for the presentation (5.1), then the action of  $\mathcal{H}_1$  on  $R$  factors uniquely through the action of  $\mathcal{H}_{\max}(R)$ , via an algebraic group morphism  $\mathcal{H}_1 \rightarrow \mathcal{H}_{\max}(R)$ .

Analogously to  $G$  and  $G^\circ$ , define the Lie subalgebra

$$(6.13) \quad \mathfrak{h}_{\max}(R) := \{t \in \mathbb{K}^N \mid (t \cdot), \text{ acting as in (6.9), is a Poisson derivation of } R\},$$

and observe that  $\mathfrak{h} = \text{Lie } \mathcal{H} \subseteq \mathfrak{h}_{\max}(R)$ . We shall prove that  $G$  is connected and  $\text{Lie } G = \mathfrak{h}_{\max}(R)$ . To do so, we first pin down  $\mathfrak{h}_{\max}(R)$ .

**LEMMA 6.5.** *Let  $R$  be a P-CGL extension of length  $N$  as in (5.1). For any  $j \in [1, N]$  with  $\delta_j = 0$ , there exists  $t \in \mathfrak{h}_{\max}(R)$  such that  $t \cdot x_j = x_j$  and  $t \cdot x_k = 0$  for all  $k \in [1, j-1]$ .*

**PROOF.** By Theorem 5.5,  $p(j) = -\infty$ . There is some  $l \in [1, N]$  with  $s(l) = +\infty$  and  $p^{O-(l)}(l) = j$ . Then  $y_l$  is Poisson-normal in  $R$ , and we have a Poisson derivation  $\partial_{y_l}$  on  $R$  (recall (2.1)) such that  $\{y_l, a\} = \partial_{y_l}(a)y_l$  for all  $a \in R$ . In view of Corollary 5.10,  $\partial_{y_l}(x_k) = -\alpha_{kl}x_k$  for all  $k \in [1, N]$ .

Set  $\theta := -\partial_{y_l} + \sum_{m=0}^{O-(l)} (h_{p^m(l)} \cdot)$ . Since  $\theta$  is a Poisson derivation of  $R$  for which all the  $x_k$  are eigenvectors,  $\theta = (t' \cdot)$  for some  $t' \in \mathfrak{h}_{\max}(R)$ . Since  $h_{p^m(l)} \cdot x_k =$

$\lambda_{p^m(l),k}x_k$  when  $k < p^m(l)$ , we find that  $t' \cdot x_k = 0$  for  $k < j$  and  $t' \cdot x_j = \lambda_j x_j$ . Thus, the element  $t := \lambda_j^{-1}t'$  of  $\mathfrak{h}_{\max}(R)$  has the desired properties.  $\square$

PROPOSITION 6.6. *Let  $R$  be a Poisson-CGL extension of length  $N$  and rank  $n$  as in Definition 5.1, and define  $\mathfrak{h}_{\max}(R)$  as in (6.13). Set*

$$D' := \{k \in [2, N] \mid \delta_k \neq 0\} = \{k \in [2, N] \mid p(k) \neq -\infty\}.$$

*For each  $k \in D'$ , choose  $j_k \in [1, k-1]$  such that  $\delta_k(x_{j_k}) \neq 0$ , and choose  $f_k \in \mathbb{Z}_{\geq 0}^{k-1}$  such that the monomial  $x^{f_k}$  appears in  $\delta_k(x_{j_k})$  when  $\delta_k(x_{j_k})$  is expressed in the basis (5.13). Then*

$$(6.14) \quad \mathfrak{h}_{\max}(R) = \left\{ \xi \in \mathbb{K}^N \mid \xi_k = -\xi_{j_k} + \sum_{i=1}^{k-1} f_{ki} \xi_i, \forall k \in D' \right\},$$

*and  $\dim \mathfrak{h}_{\max}(R) = n$ .*

PROOF. Let  $\mathfrak{h}_1$  denote the subspace of  $\mathbb{K}^N$  described on the right hand side of (6.14), and note that  $\dim \mathfrak{h}_1 = N - |D'| = n$ . For  $j \in [1, N] \setminus D'$ , Lemma 6.5 provides us with an element  $t_j \in \mathfrak{h}_{\max}(R)$  such that  $t_j \cdot x_j = x_j$  and  $t_j \cdot x_k = 0$  for all  $k \in [1, j-1]$ . These  $t_j$  are obviously linearly independent, and therefore  $\dim \mathfrak{h}_{\max}(R) \geq n$ .

For any  $\xi \in \mathfrak{h}_{\max}(R)$ , the assumption that  $(\xi \cdot)$  is a Poisson derivation implies

$$\xi \cdot \{x_k, x_{j_k}\} = \{\xi_k x_k, x_{j_k}\} + \{x_k, \xi_{j_k} x_{j_k}\} = (\xi_k + \xi_{j_k})(\lambda_{k,j_k} x_{j_k} x_k + \delta_k(x_{j_k}))$$

for  $k \in D'$ . On the other hand,

$$\xi \cdot \{x_k, x_{j_k}\} = \xi \cdot (\lambda_{k,j_k} x_{j_k} x_k + \delta_k(x_{j_k})) = \lambda_{k,j_k} (\xi_{j_k} + \xi_k) x_{j_k} x_k + \xi \cdot \delta_k(x_{j_k}),$$

from which we see that  $\xi \cdot \delta_k(x_{j_k}) = (\xi_k + \xi_{j_k})\delta_k(x_{j_k})$ , and hence  $\xi_k + \xi_{j_k} = \sum_{i=1}^{k-1} f_{ki} \xi_i$ . Since this holds for all  $k \in D'$ , we conclude that  $\xi \in \mathfrak{h}_1$ .

Thus,  $\mathfrak{h}_{\max}(R) \subseteq \mathfrak{h}_1$ . Comparison of dimensions then forces  $\mathfrak{h}_{\max}(R) = \mathfrak{h}_1$  and  $\dim \mathfrak{h}_{\max}(R) = n$ .  $\square$

In order to establish an analog of Proposition 6.6 for  $\mathcal{H}_{\max}(R)$ , we will need to transport Poisson properties from  $\mathfrak{h}_{\max}(R)$  to a torus whose Lie algebra is  $\mathfrak{h}_{\max}(R)$ . The following lemma accomplishes this.

LEMMA 6.7. *Let  $R$  be a Poisson algebra equipped with a rational action of a torus  $\mathcal{H}$  by  $\mathbb{K}$ -algebra automorphisms, and let  $\mathfrak{h} = \text{Lie } \mathcal{H}$  act on  $R$  by the differential of the  $\mathcal{H}$ -action. Then  $\mathcal{H}$  acts on  $R$  by Poisson automorphisms if and only if  $\mathfrak{h}$  acts on  $R$  by Poisson derivations.*

PROOF. We have already noted, in Lemma 2.10, that if  $\mathcal{H}$  acts by Poisson automorphisms, then  $\mathfrak{h}$  acts by Poisson derivations [25, Lemma 1.4]. Now assume that  $\mathfrak{h}$  acts by Poisson derivations.

The algebra  $R$  is graded by the character group  $X(\mathcal{H})$ , which we will write additively in this proof. We may assume that  $\mathcal{H} = (\mathbb{K}^*)^n$  and  $\mathfrak{h} = \mathbb{K}^n$ , where  $n := \text{rank } \mathcal{H}$ . Then  $X(\mathcal{H})$  is naturally identified with  $\mathbb{Z}^n$ , so that

$$u(h) = h_1^{u_1} h_2^{u_2} \cdots h_n^{u_n}, \quad \forall u = (u_1, \dots, u_n) \in \mathbb{Z}^n, \quad h = (h_1, \dots, h_n) \in \mathcal{H}.$$

For  $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ , let  $R_u$  denote the  $\mathcal{H}$ -eigenspace of  $R$  for the  $\mathcal{H}$ -eigenvalue  $u$ , and define  $u' \in \mathfrak{h}^*$  by

$$u'(\xi) = u_1 \xi_1 + \cdots + u_n \xi_n, \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathfrak{h}.$$

The map  $u \mapsto u'$  is an injective group homomorphism  $\mathbb{Z}^n \rightarrow \mathfrak{h}^*$ .

Recall from (2.6) that the  $\mathcal{H}$ -eigenspaces of  $R$  coincide with the  $\mathfrak{h}$ -eigenspaces. As indicated in [25, Lemma 1.3], the action of  $\mathfrak{h}$  on the  $\mathcal{H}$ -eigenspaces of  $R$  is given by

$$(6.15) \quad \xi \cdot a = u'(\xi)a, \quad \forall \xi \in \mathfrak{h}, \quad a \in R_u, \quad u \in \mathbb{Z}^n.$$

Let  $h \in \mathcal{H}$ . To prove that  $(h \cdot)$  is a Poisson automorphism of  $R$ , it is enough to show that it preserves Poisson brackets of homogeneous elements. Thus, let  $u, v \in \mathbb{Z}^n$  and  $a \in R_u, b \in R_v$ . Then

$$\{h \cdot a, h \cdot b\} = \{u(h)a, v(h)b\} = u(h)v(h)\{a, b\}.$$

If  $\{a, b\} = 0$ , then immediately  $h \cdot \{a, b\} = \{h \cdot a, h \cdot b\}$ , so we may assume that  $\{a, b\} \neq 0$ . For any  $\xi \in \mathfrak{h}$ , we have

$$(6.16) \quad \xi \cdot \{a, b\} = \{\xi \cdot a, b\} + \{a, \xi \cdot b\} = (u'(\xi) + v'(\xi))\{a, b\}.$$

Then  $\{a, b\}$  is an  $\mathfrak{h}$ -eigenvector, hence also an  $\mathcal{H}$ -eigenvector, so  $\{a, b\} \in R_w$  for some  $w \in \mathbb{Z}^n$ . Moreover, it follows from (6.15) and (6.16) that  $w' = u' + v'$ , whence  $w = u + v$ . Therefore

$$h \cdot \{a, b\} = w(h)\{a, b\} = u(h)v(h)\{a, b\} = \{h \cdot a, h \cdot b\},$$

as required.  $\square$

**THEOREM 6.8.** *Let  $R$  be a Poisson-CGL extension of length  $N$  and rank  $n$  as in Definition 5.1, and define the group  $G \subseteq (\mathbb{K}^*)^N$  as in (6.11). Then*

$$\mathcal{H}_{\max}(R) = G, \quad \text{rank } \mathcal{H}_{\max}(R) = n, \quad \text{and} \quad \text{Lie } \mathcal{H}_{\max}(R) = \mathfrak{h}_{\max}(R).$$

Now set  $D' := \{k \in [2, N] \mid \delta_k \neq 0\} = \{k \in [2, N] \mid p(k) \neq -\infty\}$ . For each  $k \in D'$ , choose  $j_k \in [1, k-1]$  such that  $\delta_k(x_{j_k}) \neq 0$ , and choose  $f_k \in \mathbb{Z}_{\geq 0}^{k-1}$  such that the monomial  $x^{f_k}$  appears in  $\delta_k(x_{j_k})$  when  $\delta_k(x_{j_k})$  is expressed in the basis (5.13). Then

$$(6.17) \quad \mathcal{H}_{\max}(R) = \left\{ \psi \in (\mathbb{K}^*)^N \mid \psi_k = \psi_{j_k}^{-1} \prod_{i=1}^{k-1} \psi_i^{f_{ki}}, \quad \forall k \in D' \right\}.$$

**PROOF.** Throughout the proof, assume that  $\mathcal{H} = \mathcal{H}_{\max}(R)$ . Let  $G_2$  denote the closed subgroup of  $(\mathbb{K}^*)^N$  described on the right hand side of (6.17), and note that  $G_2$  is a torus of rank  $N - |D'| = n$ , recalling (5.6). We shall prove that  $\mathcal{H} = G = G_2$  and  $\text{Lie } G_2 = \mathfrak{h}_{\max}(R)$ .

By construction,  $\mathcal{H} \subseteq G$ . Let  $\psi \in G$ , and let  $k \in D'$ . On applying the Poisson automorphism  $(\psi \cdot)$  to the relation

$$\{x_k, x_{j_k}\} = \lambda_{k, j_k} x_{j_k} x_k + \delta_k(x_{j_k}),$$

we see that  $\psi \cdot \delta_k(x_{j_k}) = \psi_k \psi_{j_k} \delta_k(x_{j_k})$ . Consequently, all the monomials appearing in the expansion of  $\delta_k(x_{j_k})$  in the basis (5.13) must have  $\psi$ -eigenvalue  $\psi_k \psi_{j_k}$ . One of these monomials is  $x^{f_k}$ , whose  $\psi$ -eigenvalue also equals  $\prod_{i=1}^{k-1} \psi_i^{f_{ki}}$ . Hence,  $\psi_k = \psi_{j_k}^{-1} \prod_{i=1}^{k-1} \psi_i^{f_{ki}}$ . This proves that  $\mathcal{H} \subseteq G \subseteq G_2$ .

Write  $L := \mathcal{O}((\mathbb{K}^*)^N)$  as a Laurent polynomial ring  $L = \mathbb{K}[Z_1^{\pm 1}, \dots, Z_N^{\pm 1}]$ , where  $Z_k$  is the  $k$ -th coordinate projection  $(\mathbb{K}^*)^N \rightarrow \mathbb{K}$ . The standard identification

of  $\mathbb{K}^N$  with the Lie algebra of  $(\mathbb{K}^*)^N$  is via the map

$$(\xi_1, \dots, \xi_N) \mapsto \sum_{k=1}^N \xi_k \frac{\partial}{\partial Z_k} \Big|_e,$$

where  $e := (1, \dots, 1)$ . Now  $\mathcal{O}(G_2) = L/I_2$  where

$$I_2 := \left\langle Z_k - Z_{j_k}^{-1} \prod_{i=1}^{k-1} Z_i^{f_{ki}} \mid k \in D' \right\rangle.$$

Observe that a point derivation  $\sum_{k=1}^N \xi_k \frac{\partial}{\partial Z_k} \Big|_e$  vanishes on  $I_2$  if and only if

$$\xi_k + \xi_{j_k} - \sum_{i=1}^{k-1} f_{ki} \xi_i = 0, \quad \forall k \in D'.$$

Thus,  $\text{Lie } G_2 = \mathfrak{h}_{\max}(R)$ , by (6.14).

By definition of  $\mathfrak{h}_{\max}(R)$ , we now have that  $\text{Lie } G_2$  acts on  $R$  by Poisson derivations. Lemma 6.7 then implies that  $G_2$  acts on  $R$  by Poisson automorphisms. Therefore  $G_2 \subseteq G$ . Since  $G_2$  is a torus, it follows that  $\mathcal{H} = G = G_2$ . This verifies (6.17), and also yields  $\text{Lie } \mathcal{H} = \mathfrak{h}_{\max}(R)$ . Finally,  $\text{rank } \mathcal{H} = \dim \mathfrak{h}_{\max}(R) = n$  by Proposition 6.6.  $\square$

EXAMPLE 6.9. In the case of  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$  (Example 5.3), Theorem 6.8 can be applied as follows. First,  $D' = \{(r-1)n + c \mid r \in [2, m], c \in [2, n]\}$ . For  $k = (r-1)n + c \in D'$ , we can choose  $j_k := 1$  and  $f_k := e_c + e_{(r-1)n+1}$ . Then

$$\mathcal{H}_{\max}(R) = \{\psi \in (\mathbb{K}^*)^{mn} \mid \psi_{(r-1)n+c} = \psi_1^{-1} \psi_c \psi_{(r-1)n+1} \quad \forall r \in [2, m], c \in [2, n]\}.$$

The relation of this torus to the torus  $\mathcal{H} = (\mathbb{K}^*)^{m+n}$  of Example 5.3 is as follows. There is a surjective morphism of algebraic groups  $\pi : \mathcal{H} \rightarrow \mathcal{H}_{\max}(R)$  given by

$$\pi(\xi)_{(r-1)n+c} = \xi_r \xi_{m+c}^{-1}, \quad \forall r \in [1, m], c \in [1, n],$$

and  $\pi$  transports the given action of  $\mathcal{H}$  to that of  $\mathcal{H}_{\max}(R)$ , that is,  $\pi(\xi) \cdot r = \xi \cdot r$  for all  $\xi \in \mathcal{H}$  and  $r \in R$ .

## CHAPTER 7

### One-step mutations in Poisson-CGL extensions

In this chapter we obtain a very general way of constructing mutations of potential cluster variables in Poisson-CGL extensions. The key idea is that, if an algebra  $R$  has two different P-CGL extension presentations obtained by reversing the order in which two adjacent variables  $x_k$  and  $x_{k+1}$  are adjoined, then the corresponding sequences of Poisson-prime elements from Theorem 5.5 are obtained by a type of mutation formula. This is realized in §7.1. In Chapter 9, we treat more general mutations. One problem arises along the way: In the mutation-type formula (7.15) for the current situation, the last term has a nonzero coefficient which does not equal 1 in general. For a one-step mutation such a coefficient can be always made 1 after rescaling, but for the purposes of constructing cluster algebras one needs to be able to synchronize those rescalings to obtain a chain of mutations. This delicate issue is resolved in the next two sections.

We investigate a P-CGL extension

$$(7.1) \quad R := \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_k; \sigma_k, \delta_k]_p[x_{k+1}; \sigma_{k+1}, \delta_{k+1}]_p \cdots [x_N; \sigma_N, \delta_N]_p$$

of length  $N$  as in Definition 5.1 such that, for some  $k \in [1, N-1]$ ,  $R$  has a second P-CGL extension presentation of the form

$$(7.2) \quad R := \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}]_p \\ [x_{k+1}; \sigma'_k, \delta'_k]_p[x_k; \sigma'_{k+1}, \delta'_{k+1}]_p[x_{k+2}; \sigma_{k+2}, \delta_{k+2}]_p \cdots [x_N; \sigma_N, \delta_N]_p.$$

Corresponding to the presentation (7.2), we write  $y'_i$  for the elements from Theorem 5.5,  $\lambda'_{ij}$ ,  $\alpha'_{ij}$ ,  $q'_{ij}$  for the scalars from (5.2), (5.17), (5.18), and  $h'_i$ ,  $\lambda'_i$  for the elements of  $\mathfrak{h}$  and  $\mathbb{K}^*$  from Definition 5.1(iii).

#### 7.1. A general mutation formula

LEMMA 7.1. *Assume that  $R$  is a P-CGL extension as in (7.1), and that  $R$  has a second P-CGL extension presentation of the form (7.2). Then  $\delta_{k+1}$  and  $\delta'_{k+1}$  map  $R_{k-1}$  to itself, and*

$$(7.3) \quad \begin{aligned} \sigma'_k &= \sigma_{k+1}|_{R_{k-1}} & \delta'_k &= \delta_{k+1}|_{R_{k-1}} \\ \sigma'_{k+1}|_{R_{k-1}} &= \sigma_k & \delta'_{k+1}|_{R_{k-1}} &= \delta_k. \end{aligned}$$

Moreover,

$$(7.4) \quad \sigma'_{k+1}(x_{k+1}) = \lambda_{k,k+1}x_{k+1} \quad \delta'_{k+1}(x_{k+1}) = -\delta_{k+1}(x_k).$$

PROOF. Note first that  $R_{k-1}$  is stable under  $\sigma_{k+1}$  and  $\sigma'_{k+1}$ . For  $a \in R_{k-1}$ , we have  $\{x_{k+1}, a\} = \sigma'_k(a)x_{k+1} + \delta'_k(a)$  with  $\sigma'_k(a), \delta'_k(a) \in R_{k-1}$ . Comparing this relation with  $\{x_{k+1}, a\} = \sigma_{k+1}(a)x_{k+1} + \delta_{k+1}(a)$ , and using the fact that 1,  $x_{k+1}$  are linearly independent over  $R_k$ , we conclude that  $\sigma'_k(a) = \sigma_{k+1}(a)$  and  $\delta'_k(a) = \delta_{k+1}(a)$ . Thus,  $R_{k-1}$  is stable under  $\delta_{k+1}$ , and the first line of (7.3) holds.

By symmetry (since we may view (7.2) as the initial P-CGL extension presentation of  $R$  and (7.1) as the second one),  $R_{k-1}$  is stable under  $\delta'_{k+1}$ , and the second line of (7.3) holds.

Now  $\{x_k, x_{k+1}\} = \lambda'_{k+1,k} x_{k+1} x_k + \delta'_{k+1}(x_{k+1})$ , and so we have

$$-\lambda'_{k+1,k} x_k x_{k+1} - \delta'_{k+1}(x_{k+1}) = \{x_{k+1}, x_k\} = \lambda_{k+1,k} x_k x_{k+1} + \delta_{k+1}(x_k),$$

with  $\delta'_{k+1}(x_{k+1}) \in R'_k = \bigoplus_{l=0}^{\infty} R_{k-1} x_{k+1}^l$  and  $\delta_{k+1}(x_k) \in R_k = \bigoplus_{l=0}^{\infty} R_{k-1} x_k^l$ . Moreover,

$$(7.5) \quad R_{k+1} \text{ is a free } R_{k-1}\text{-module with basis } \{x_k^{l_k} x_{k+1}^{l_{k+1}} \mid l_k, l_{k+1} \in \mathbb{Z}_{\geq 0}\}.$$

Hence, we conclude that  $-\lambda'_{k+1,k} = \lambda_{k+1,k}$  and  $-\delta'_{k+1}(x_{k+1}) = \delta_{k+1}(x_k) \in R_{k-1}$ , from which (7.4) follows.  $\square$

**THEOREM 7.2.** *Assume that  $R$  is a Poisson-CGL extension of length  $N$  as in (7.1), and  $k \in [1, N-1]$  such that  $\delta_{k+1}(x_k) \in \mathbb{K}$ . Denote by  $y_1, \dots, y_N$  and  $\eta : [1, N] \rightarrow \mathbb{Z}$  the sequence and function from Theorem 5.5. Assume that  $R$  has a second Poisson-CGL extension presentation of the form (7.2), and let  $y'_1, \dots, y'_N$  be the corresponding sequence from Theorem 5.5.*

- (a) *If  $\eta(k) \neq \eta(k+1)$ , then  $y'_j = y_j$  for  $j \neq k, k+1$  and  $y'_k = y_{k+1}$ ,  $y'_{k+1} = y_k$ .*
- (b) *If  $\eta(k) = \eta(k+1)$ , then*

$$y_k y'_k - y_{p(k)} y_{k+1}$$

*is a homogeneous Poisson-normal element of  $R_{k-1}$ , recall (5.20). It Poisson-normalizes the elements of  $R_{k-1}$  in exactly the same way as  $y_{p(k)} y_{k+1}$ . Furthermore,*

$$(7.6) \quad y'_j = y_j, \quad \forall j \in [1, N], \quad j \neq k.$$

*In both cases (a) and (b), the function  $\eta' : [1, N] \rightarrow \mathbb{Z}$  from Theorem 5.5 associated to the second presentation can be chosen to be  $\eta' = \eta(k, k+1)$ , where  $(k, k+1)$  denotes a transposition in  $S_N$ . In particular, the ranges of  $\eta$  and  $\eta'$  coincide and the rank of  $R$  is the same for both CGL extension presentations.*

**REMARK.** The assumption  $\delta_{k+1}(x_k) \in \mathbb{K}$  holds in case  $R$  is a symmetric P-CGL extension, as part of condition (i) in Definition 6.1. In any case, it is equivalent to  $\delta'_{k+1}(x_{k+1}) \in \mathbb{K}$  in view of (7.4).

**PROOF.** Let  $R'_j$  be the  $j$ -th algebra in the chain (7.2) for  $j \in [0, N]$ . Obviously  $R'_j = R_j$  for  $j \neq k$  and  $y'_j = y_j$  for  $j \in [1, k-1]$ , and we may choose  $\eta'(j) = \eta(j)$  for  $j \in [1, k-1]$ . Since  $R_{k+1}$  is a free  $R_{k-1}$ -module with basis as in (7.5), and  $R_k$  and  $R'_k$  equal the  $R_{k-1}$ -submodules with bases  $\{x_k^{l_k} \mid l_k \in \mathbb{Z}_{\geq 0}\}$  and  $\{x_{k+1}^{l_{k+1}} \mid l_{k+1} \in \mathbb{Z}_{\geq 0}\}$  respectively, we have

$$(7.7) \quad R_k \cap R'_k = R_{k-1}.$$

In particular,  $y'_k \notin R_k$ .

Denote by  $L$  the number of homogeneous Poisson-prime elements of  $R_{k+1}$  that do not belong to  $R_{k-1}$ , up to taking associates.

(a) The condition  $\eta(k) \neq \eta(k+1)$  implies  $L = 2$  and thus  $\eta'(k) \neq \eta'(k+1)$ . Moreover,  $y'_k$  and  $y'_{k+1}$  are scalar multiples of either  $y_k$  and  $y_{k+1}$  or  $y_{k+1}$  and  $y_k$ . Since  $y'_k \notin R_k$ , we must have  $y'_k = \xi_{k+1} y_{k+1}$  and  $y'_{k+1} = \xi_k y_k$  for some  $\xi_k, \xi_{k+1} \in \mathbb{K}^*$ . Invoking Theorem 5.5 and looking at leading terms gives  $\xi_k = \xi_{k+1} = 1$  and allows us to choose  $\eta'(k) = \eta(k+1)$  and  $\eta'(k+1) = \eta(k)$ . It follows from  $R'_j = R_j$

for  $j > k + 1$  that  $y'_j = y_j$  for such  $j$ , and that we can choose  $\eta'(j) = \eta(j)$  for all  $j > k + 1$ . With these choices,  $\eta' = \eta(k, k + 1)$ .

(b) In this case,  $L = 1$  and  $y'_{k+1}$  is a scalar multiple of  $y_{k+1}$ . Furthermore,  $\eta'(k)$  and  $\eta'(k + 1)$  must agree, and they need to equal  $\eta'(p(k))$  if  $p(k) \neq -\infty$ , so we can choose  $\eta'(k) = \eta'(k + 1) = \eta(k) = \eta(k + 1)$ . By Theorem 5.5,

$$(7.8) \quad \begin{aligned} y_k &= \begin{cases} y_{p(k)}x_k - c_k, & \text{if } p(k) \neq -\infty \\ x_k, & \text{if } p(k) = -\infty \end{cases} & y_{k+1} &= y_kx_{k+1} - c_{k+1} \\ y'_k &= \begin{cases} y_{p(k)}x_{k+1} - c'_k, & \text{if } p(k) \neq -\infty \\ x_{k+1}, & \text{if } p(k) = -\infty \end{cases} & y'_{k+1} &= y'_kx_k - c'_{k+1} \end{aligned}$$

for some  $c_k, c'_k \in R_{k-1}$ ,  $c_{k+1} \in R_k$ , and  $c'_{k+1} \in R'_k$ . Write the above elements in terms of the basis in (7.5). The coefficients of  $x_kx_{k+1}$  in  $y_{k+1}$  and  $y'_{k+1}$  are both equal to  $y_{p(k)}$ , recall (5.20). Since  $y'_{k+1}$  is a scalar multiple of  $y_{k+1}$ , we thus see that

$$(7.9) \quad y'_{k+1} = y_{k+1}.$$

Eq. (7.6) and the fact that we may choose  $\eta'(j) = \eta(j)$  for  $j > k + 1$  now follow easily from Theorem 5.5. In particular,  $\eta' = \eta = \eta(k, k + 1)$ .

Next, we verify that

$$(7.10) \quad y_ky'_k - y_{p(k)}y_{k+1} \in R_{k-1}.$$

Assume first that  $p(k) = -\infty$ . By (5.24),  $c_{k+1}$  is a nonzero scalar multiple of  $\delta_{k+1}(x_k)$ , and so  $c_{k+1} \in \mathbb{K}^*$  by hypothesis. Then from (7.8) we obtain

$$(7.11) \quad y_ky'_k - y_{p(k)}y_{k+1} = x_kx_{k+1} - (x_kx_{k+1} - c_{k+1}) = c_{k+1} \in \mathbb{K}^*.$$

This verifies (7.10) (and also shows that  $y_ky'_k - y_{p(k)}y_{k+1}$  is a homogeneous Poisson-normal element of  $R_{k-1}$ ) in the present case.

Now assume that  $p(k) \neq -\infty$ . First, we obtain

$$(7.12) \quad \begin{aligned} y_ky'_k - y_{p(k)}y_{k+1} &= y_k(y_{p(k)}x_{k+1} - c'_k) - y_{p(k)}(y_kx_{k+1} - c_{k+1}) \\ &= -y_kc'_k + y_{p(k)}c_{k+1} \in R_k. \end{aligned}$$

Using (7.9), we obtain

$$\begin{aligned} y_ky'_k - y_{p(k)}y_{k+1} &= (y_{p(k)}x_k - c_k)y'_k - y_{p(k)}(y'_kx_k - c'_{k+1}) \\ &= -c_ky'_k + y_{p(k)}c'_{k+1} \in R'_k. \end{aligned}$$

This equation, combined with (7.12) and (7.7), yields (7.10).

We now use (7.12) to verify that  $y_ky'_k - y_{p(k)}y_{k+1}$  is homogeneous. Note first that  $y_kc'_k$  and  $y_{p(k)}c_{k+1}$  are homogeneous. By (5.24),  $c'_k$  and  $c_{k+1}$  are scalar multiples of  $\delta'_k(y_{p(k)}) = \delta_{k+1}(y_{p(k)})$  and  $\delta_{k+1}(y_k)$ , respectively. Hence, it follows from (5.4) that

$$X(\mathcal{H})\text{-deg}(y_kc'_k) = \chi_{y_k} + \chi_{x_{k+1}} + \chi_{y_{p(k)}} = X(\mathcal{H})\text{-deg}(y_{p(k)}c_{k+1}).$$

Thus,  $-y_kc'_k + y_{p(k)}c_{k+1}$  is homogeneous, as desired.

Finally, whether  $p(k) = -\infty$  or not, it follows from (5.25) that

$$(7.13) \quad \begin{aligned} \{y_ky'_k, x_j\} &= -(\alpha_{jk} + \alpha'_{jk})x_j(y_ky'_k), \\ \{y_{p(k)}y_{k+1}, x_j\} &= -(\alpha_{j,p(k)} + \alpha_{j,k+1})x_j(y_{p(k)}y_{k+1}), \end{aligned} \quad \forall j \in [1, k-1].$$

Since

$$\begin{aligned}\alpha_{jk} + \alpha'_{jk} &= \sum_{m=0}^{O_-(k)} (\lambda_{j,p^m(k)} + \lambda'_{j,p^m(k)}) \\ &= \lambda_{j,k+1} + \sum_{m=0}^{O_-(k)} \lambda_{j,p^m(k)} + \sum_{m=1}^{O_-(k)} \lambda_{j,p^m(k)} = \alpha_{j,k+1} + \alpha_{j,p(k)},\end{aligned}$$

we obtain  $\{y_k y'_k - y_{p(k)} y_{k+1}, x_j\} = -(\alpha_{j,p(k)} + \alpha_{j,k+1}) x_j (y_k y'_k - y_{p(k)} y_{k+1})$ . This shows that  $y_k y'_k - y_{p(k)} y_{k+1}$  is a Poisson-normal element of  $R_{k-1}$  which Poisson-normalizes the elements of  $R_{k-1}$  in exactly the same way as  $y_{p(k)} y_{k+1}$ , and completes the proof of the theorem.  $\square$

Our next result turns the conclusion of Theorem 7.2(b) into a cluster mutation statement. We define

$$(7.14) \quad P(k) := \{j \in [1, k] \mid s(j) > k\}, \quad \forall k \in [1, N].$$

Then  $\{y_j \mid j \in P(k)\}$  is a list of the homogeneous Poisson-prime elements of  $R_k$  up to scalar multiples, and  $|P(N)| = \text{rank } R$  (5.11).

**THEOREM 7.3.** *In the setting of Theorem 7.2(b), there exist  $\kappa \in \mathbb{K}^*$  and a collection of nonnegative integers  $\{m_i \mid i \in P(k-1), i \neq p(k)\}$  such that*

$$(7.15) \quad y'_k = y_k^{-1} \left( y_{p(k)} y_{k+1} + \kappa \prod_{i \in P(k-1), i \neq p(k)} y_i^{m_i} \right).$$

If  $p(k) = -\infty$ , then all  $m_i = 0$ .

**PROOF.** By Theorem 7.2(b),  $y_k y'_k - y_{p(k)} y_{k+1}$  is a homogeneous Poisson-normal element of  $R_{k-1}$ . Applying Proposition 4.1 and Theorems 5.2, 5.5, we obtain

$$y_k y'_k - y_{p(k)} y_{k+1} = \kappa \prod_{i \in P(k-1)} y_i^{m_i}$$

for some  $\kappa \in \mathbb{K}$  and a collection of nonnegative integers  $\{m_i \mid i \in P(k-1)\}$ . Recall from (7.11) that if  $p(k) = -\infty$ , then  $y_k y'_k - y_{p(k)} y_{k+1}$  is a nonzero scalar. Hence,  $\kappa \neq 0$  and  $m_i = 0$  for all  $i \in P(k-1)$  in this case. We need to prove in general that  $\kappa \neq 0$ , and that  $m_{p(k)} = 0$  if  $p(k) \neq -\infty$ .

Suppose that  $\kappa = 0$ . Then

$$y_k y'_k - y_{p(k)} y_{k+1} = 0,$$

which is a contradiction since  $y_{k+1}$  is a prime element of  $R_{k+1}$  which does not divide either  $y_k$  or  $y'_k$  (for  $y_{k+1} \nmid y'_k$ , recall (7.8)).

Now suppose that  $p(k) \neq -\infty$  and  $m_{p(k)} \neq 0$ . Then  $y_{p(k)}$  is a prime element of  $R_{k-1}$  and

$$y_k y'_k - y_{p(k)} y_{k+1} \in y_{p(k)} R_{k-1}.$$

Hence,

$$y_k y'_k \in y_{p(k)} R_{k+1}.$$

Furthermore, by Theorem 5.5,  $y_k = y_{p(k)} x_k - c_k$  and  $y'_k = y_{p(k)} x_{k+1} - c'_k$  for some  $c_k, c'_k \in R_{k-1}$  such that  $y_{p(k)}$  does not divide  $c_k$  or  $c'_k$ . But since

$$c_k c'_k = y_k y'_k - y_{p(k)} x_k y'_k + c_k y_{p(k)} x_{k+1} \in y_{p(k)} R_{k+1} \cap R_{k-1} = y_{p(k)} R_{k-1},$$

this contradicts the fact that  $y_{p(k)}$  is a prime element of  $R_{k-1}$ .  $\square$

## 7.2. Almost cluster mutations between Poisson-CGL extension presentations

Let  $R$  be a  $\mathbb{K}$ -algebra with a P-CGL extension presentation as in (7.1). It is easy to see that the assumption that (7.2) is a second P-CGL extension presentation of  $R$  is equivalent to the following condition:

- (i)  $\delta_{k+1}(x_j) \in R_{k-1}$  for all  $j \in [1, k]$ , and there exists  $h'_{k+1} \in \text{Lie } \mathcal{H}$  satisfying  $h'_{k+1} \cdot x_k = \lambda'_{k+1} x_k$  for some  $\lambda'_{k+1} \in \mathbb{K}^*$  and  $h'_{k+1} \cdot x_j = \lambda_{kj} x_j$  for all  $j \in [1, k-1] \cup \{k+1\}$ .

The rest of the data for the second P-CGL extension presentation of  $R$  ( $\mathbb{K}$ -torus  $\mathcal{H}'$ , scalars  $\lambda'_{kj} \in \mathbb{K}$ , and elements  $h'_k \in \text{Lie } \mathcal{H}'$ ) is given by Lemma 7.1 and the following:

- (ii) The torus  $\mathcal{H}'$  acting on the second P-CGL extension presentation can be taken as the original torus  $\mathcal{H}$ . The corresponding elements  $h'_j \in \text{Lie } \mathcal{H}$  are given by (i) for  $j = k+1$  and  $h'_j = h_j$  for  $j \neq k+1$ .
- (iii)  $\sigma'_{k+1} = (h'_{k+1} \cdot \cdot)|_{R'_k}$ , where  $R'_k$  is the unital  $\mathbb{K}$ -subalgebra of  $R$  generated by  $x_1, \dots, x_{k-1}$  and  $x_{k+1}$ .
- (iv)  $\lambda'_{lj} = \lambda_{(k,k+1)(l),(k,k+1)(j)}$  for  $l, j \in [1, N]$ .

Here and below,  $(k, k+1)$  denotes a transposition in the symmetric group  $S_N$ , and  $S_N$  is embedded in  $GL_N(\mathbb{Z})$  via permutation matrices.

We continue with the hypotheses and notation of Theorems 7.2, 7.3. Define maps  $Y, Y' : \mathbb{Z}^N \rightarrow \text{Fract}(R)$  by

$$(7.16) \quad Y(f) := \prod_{i=1}^N y_i^{f_i} \text{ and } Y'(f) := \prod_{i=1}^N (y'_i)^{f_i}, \quad \forall f = (f_1, \dots, f_N)^T \in \mathbb{Z}^N.$$

We also adopt the convention that

$$(7.17) \quad e_{-\infty} = \bar{e}_{-\infty} := 0.$$

**THEOREM 7.4.** *Assume the setting of Theorem 7.2.*

(a) *If  $\eta(k) \neq \eta(k+1)$ , then  $y'_j = y_{(k,k+1)j}$  for all  $j \in [1, N]$ , i.e.,  $Y' = Y(k, k+1)$ .*

(b) *If  $\eta(k) = \eta(k+1)$ , then  $Y'(e_j) = y'_j = y_j = Y(e_j)$  for all  $j \neq k$  and*

$$(7.18) \quad Y'(e_k) = y'_k = Y(-e_k + e_{p(k)} + e_{k+1}) + \kappa Y\left(-e_k + \sum_{i \in P(k-1), i \neq p(k)} m_i e_i\right)$$

*for the collection of nonnegative integers  $\{m_i \mid i \in P(k-1), i \neq p(k)\}$  and the scalar  $\kappa \in \mathbb{K}^*$  from Theorem 7.3.*

**PROOF.** This follows from Theorems 7.2 and 7.3. □

## 7.3. Scalars associated to mutations of Poisson-prime elements

Next, we derive formulas for certain scalars which will appear in the seeds which we construct in Section 11.

Recall the skew-symmetric matrix  $\mathbf{q} = (q_{kj}) \in M_N(\mathbb{K})$  from (5.18), and define the corresponding skew-symmetric bicharacter  $\Omega_{\mathbf{q}} : \mathbb{Z}^N \times \mathbb{Z}^N \rightarrow \mathbb{K}$  by (1.5), so that

$$(7.19) \quad \Omega_{\mathbf{q}}(e_k, e_j) = q_{kj} = \Omega_{\lambda}(\bar{e}_k, \bar{e}_j), \quad \forall j, k \in [1, N].$$

In view of Proposition 5.8, we have

$$(7.20) \quad \{Y(f), Y(g)\} = \Omega_{\mathbf{q}}(f, g)Y(f)Y(g), \quad \forall f, g \in \mathbb{Z}^N.$$

Recall from §1.5 that for an  $\mathcal{H}$ -eigenvector  $u \in R$ ,  $\chi_u \in X(\mathcal{H})$  denotes its eigenvalue.

THEOREM 7.5. *Assume the setting of Theorem 7.3. Then*

$$(7.21) \quad \Omega_{\mathbf{q}}\left(e_{p(k)} + e_{k+1} - \sum_{i \in P(k-1), i \neq p(k)} m_i e_i, e_j\right) = 0, \quad \forall j \neq k$$

and

$$(7.22) \quad \Omega_{\mathbf{q}}\left(e_{p(k)} + e_{k+1} - \sum_{i \in P(k-1), i \neq p(k)} m_i e_i, e_k\right) = -\lambda_{k+1} = \lambda'_{k+1}$$

for the collection of nonnegative integers  $\{m_i \mid i \in P(k-1), i \neq p(k)\}$  from Theorems 7.3 and 7.4(b).

PROOF. Denote for brevity the elements

$$g := \sum_{i \in P(k-1), i \neq p(k)} m_i e_i \quad \text{and} \quad \bar{g} := \sum_{i \in P(k-1), i \neq p(k)} m_i \bar{e}_i \quad \text{in } \mathbb{Z}^N.$$

For  $j \in [1, N]$  with  $j \neq k$ , we have  $y_j = y'_j$  by (7.6), and so

$$\begin{aligned} q'_{kj} y'_k y_j &= q'_{kj} (Y(-e_k + e_{p(k)} + e_{k+1}) + \kappa Y(-e_k + g)) y_j \\ q'_{kj} y'_k y_j &= \{y'_k, y_j\} = \{Y(-e_k + e_{p(k)} + e_{k+1}) + \kappa Y(-e_k + g), y_j\} \\ &= \Omega_{\mathbf{q}}(-e_k + e_{p(k)} + e_{k+1}, e_j) Y(-e_k + e_{p(k)} + e_{k+1}) y_j \\ &\quad + \kappa \Omega_{\mathbf{q}}(-e_k + g, e_j) Y(-e_k + g) y_j. \end{aligned}$$

Consequently,

$$\Omega_{\mathbf{q}}(-e_k + e_{p(k)} + e_{k+1}, e_j) = \Omega_{\mathbf{q}}(-e_k + g, e_j) = q'_{kj},$$

which implies (7.21).

Applying (7.21) for  $j = k + 1$  leads to

$$\begin{aligned} (7.23) \quad &\Omega_{\mathbf{q}}(e_{p(k)} + e_{k+1} - g, e_k) \\ &= \Omega_{\mathbf{q}}(e_{p(k)} + e_{k+1} - g, e_k) - \Omega_{\mathbf{q}}(e_{p(k)} + e_{k+1} - g, e_{k+1}) \\ &= \Omega_{\lambda}(\bar{e}_{p(k)} + \bar{e}_{k+1} - \bar{g}, \bar{e}_k) - \Omega_{\lambda}(\bar{e}_{p(k)} + \bar{e}_{k+1} - \bar{g}, \bar{e}_{k+1}) \\ &= -\Omega_{\lambda}(\bar{e}_{p(k)} + \bar{e}_{k+1} - \bar{g}, e_{k+1}) = -\Omega_{\lambda}(\bar{e}_{p(k)} + \bar{e}_k - \bar{g}, e_{k+1}). \end{aligned}$$

Since  $y'_k$  is an  $\mathcal{H}$ -eigenvector, it follows from (7.15) that  $Y(e_{p(k)} + e_{k+1})$  and  $Y(g)$  have the same  $X(\mathcal{H})$ -degree, whence  $\chi_{Y(e_{p(k)} + e_{k+1} - g)} = 0$ . Using the fact that  $\chi_{Y(e_{p(k)} + e_{k+1} - g)} = \chi_{Y(e_{p(k)} + e_k - g)} + \chi_{x_{k+1}}$ , we obtain

$$h_{k+1} \cdot (Y(e_{p(k)} + e_k - g)x_{k+1}) = 0$$

from (2.9). Eq. (5.21) and the definition of  $\lambda_{k+1}$  then yield

$$\begin{aligned} 0 &= \alpha_{k+1, p(k)} + \alpha_{k+1, k} - \sum_{i \in P(k-1), i \neq p(k)} m_i \alpha_{k+1, i} + \lambda_{k+1} \\ &= \Omega_{\lambda}(e_{k+1}, \bar{e}_{p(k)} + \bar{e}_k - \bar{g}) + \lambda_{k+1}. \end{aligned}$$

Combining this with (7.23) proves the first equality in (7.22).

It follows from (7.21) and (7.20) that

$$\{Y(e_{p(k)} + e_{k+1} - g), y_j\} = 0, \quad \forall j \neq k,$$

whence  $Y(e_{p(k)} + e_{k+1} - g)$  Poisson-commutes with  $Y(g)$ , and so  $Y(e_{p(k)} + e_{k+1} - g)$  also Poisson-commutes with  $Y(e_{p(k)} + e_{k+1}) = y_{p(k)}y_{k+1}$ . Since  $y_k y'_k$  is a linear combination of  $y_{p(k)}y_{k+1}$  and  $Y(g)$ , we get

$$(7.24) \quad \{Y(e_{p(k)} + e_{k+1} - g), y_k y'_k\} = 0.$$

By (7.20) and the first equality in (7.22),

$$\{Y(e_{p(k)} + e_{k+1} - g), y_k\} = -\lambda_{k+1}Y(e_{p(k)} + e_{k+1} - g)y_k.$$

Interchanging the roles of the P-CGL extension presentations (7.1) and (7.2) and using the symmetric nature of the assumptions of Theorem 7.2(b) shows that

$$\{Y'(e_{p(k)} + e_{k+1} - g), y'_k\} = -\lambda'_{k+1}Y'(e_{p(k)} + e_{k+1} - g)y'_k.$$

Since  $Y'(e_{p(k)} + e_{k+1} - g) = Y(e_{p(k)} + e_{k+1} - g)$ , we obtain

$$\{Y(e_{p(k)} + e_{k+1} - g), y_k y'_k\} = -(\lambda_{k+1} + \lambda'_{k+1})Y(e_{p(k)} + e_{k+1} - g)y_k y'_k.$$

Therefore  $\lambda_{k+1} + \lambda'_{k+1} = 0$  because of (7.24), which proves the final equality in (7.22).  $\square$



## Homogeneous Poisson-prime elements for subalgebras of symmetric Poisson-CGL extensions

Each symmetric Poisson-CGL extension  $R$  of length  $N$  has many different P-CGL extension presentations given by (6.5). They are parametrized by the elements of the subset  $\Xi_N$  of  $S_N$ , cf. (6.4). In order to phrase Theorem 7.3 into a mutation statement between cluster variables associated to the elements of  $\Xi_N$  and to make the scalars  $\kappa$  from Theorem 7.3 equal to one, we need a good picture of the sequences of homogeneous Poisson-prime elements  $y_1, \dots, y_N$  from Theorem 5.5 associated to each presentation (6.5). This is obtained in Theorem 8.3. Theorem 8.1 contains a description of the homogeneous Poisson-prime elements that enter into this result. Those Poisson-prime elements (up to rescaling) comprise the cluster variables that will be used in Chapter 11 to construct cluster algebra structures on symmetric P-CGL extensions. Along the way, we explicitly describe the elements of  $\Xi_N$  and prove an invariance property of the scalars  $\lambda_l$  and  $\lambda_l^*$  from Definitions 5.1 and 6.1. Theorem 8.13, which appears at the end of the section, contains a key result used in the next section to normalize the generators  $x_j$  of symmetric P-CGL extensions so that all scalars  $\kappa$  in Theorem 7.3 become equal to one.

Throughout the section,  $\eta$  will denote a function  $[1, N] \rightarrow \mathbb{Z}$  satisfying the conditions of Theorem 5.5, with respect to the original P-CGL extension presentation (5.1) of  $R$ , and  $p$  and  $s$  will denote the corresponding predecessor and successor functions. We will repeatedly use the one-line notation (6.6) for permutations.

### 8.1. The elements $y_{[i, s^m(i)]}$

Recall from §6.1 that for a symmetric P-CGL extension  $R$  of rank  $N$  and  $1 \leq j \leq k \leq N$ ,  $R_{[j, k]}$  denotes the unital subalgebra of  $R$  generated by  $x_j, \dots, x_k$ . It is a Poisson-Ore extension of both  $R_{[j, k-1]}$  and  $R_{[j+1, k]}$  (6.2). All such subalgebras are (symmetric) P-CGL extensions and Theorem 5.5 applies to them.

For  $i \in [1, N]$  and  $0 \leq m \leq O_+(i)$ , recall (5.8) (i.e.,  $s^m(i) \in [1, N]$ ), set

$$(8.1) \quad e_{[i, s^m(i)]} := e_i + e_{s(i)} + \dots + e_{s^m(i)} \in \mathbb{Z}^N.$$

The vectors (5.15) are special cases of these:

$$\bar{e}_k = e_{[p^{O_-(k)}(k), k]}, \quad \forall k \in [1, N].$$

We also set  $e_\emptyset := 0$ . The next theorem treats the Poisson-prime elements that will appear as cluster variables for symmetric CGL extensions. It will be proved in §8.4.

**THEOREM 8.1.** *Assume that  $R$  is a symmetric Poisson-CGL extension of length  $N$ , and  $i \in [1, N]$  and  $m \in \mathbb{Z}_{\geq 0}$  are such that  $s^m(i) \in [1, N]$ , i.e.,  $s^m(i) \neq +\infty$ . Then the following hold:*

(a) All homogeneous Poisson-prime elements of  $R_{[i,s^m(i)]}$  that do not belong to  $R_{[i,s^m(i)-1]}$  are associates of each other.

(b) All homogeneous Poisson-prime elements of  $R_{[i,s^m(i)]}$  that do not belong to  $R_{[i+1,s^m(i)]}$  are associates of each other. In addition, the set of these homogeneous Poisson-prime elements coincides with the set of homogeneous Poisson-prime elements in part (a).

(c) The homogeneous Poisson-prime elements in (a) and (b) have leading terms of the form

$$\xi x^{e_{[i,s^m(i)]}} = \xi x_i x_{s(i)} \dots x_{s^m(i)}$$

for some  $\xi \in \mathbb{K}^*$ , see §5.2. For each  $\xi \in \mathbb{K}^*$ , there is a unique homogeneous Poisson-prime element of  $R_{[i,s^m(i)]}$  with such a leading term. Denote by  $y_{[i,s^m(i)]}$  the homogeneous Poisson-prime element of  $R_{[i,s^m(i)]}$  with leading term  $x_i x_{s(i)} \dots x_{s^m(i)}$ . Let  $y_\emptyset := 1$ .

(d) We have

$$y_{[i,s^m(i)]} = y_{[i,s^{m-1}(i)]} x_{s^m(i)} - c_{[i,s^m(i)-1]} = x_i y_{[s(i),s^m(i)]} - c'_{[i+1,s^m(i)]}$$

for some  $c_{[i,s^m(i)-1]} \in R_{[i,s^m(i)-1]}$  and  $c'_{[i+1,s^m(i)]} \in R_{[i+1,s^m(i)]}$ .

(e) For all  $k \in [1, N]$  such that  $p(i) < k < s^{m+1}(i)$ , we have

$$\{y_{[i,s^m(i)]}, x_k\} = \Omega_\lambda(e_{[i,s^m(i)]}, e_k) x_k y_{[i,s^m(i)]}.$$

The case  $m = 0$  of this theorem is easy to verify. In that case,  $y_{[i,i]} = x_i$ , and statement (e) follows by applying (5.26) in  $R_{[\min\{i,k\}, \max\{i,k\}]}$ .

EXAMPLE 8.2. In the case of  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$ , the elements  $y_{[i,s^m(i)]}$  of Theorem 8.1 are solid minors, just as in Example 5.6. More precisely, if  $i = (r-1)n + c$  for some  $r \in [1, m]$  and  $c \in [1, n]$  and  $s^l(i) \neq +\infty$ , then  $s^l(i) = (r+l-1)n + c + l$  with  $r+l \leq m$  and  $c+l \leq n$ , and  $y_{[i,s^l(i)]} = \Delta_{[r,r+l],[c,c+l]}$ .

The following theorem describes the  $y$ -sequences from Theorem 5.5 associated to the P-CGL extension presentations (6.5) in terms of the Poisson-prime elements from Theorem 8.1. It will be proved in §8.4. Recall that for every  $\tau \in \Xi_N$  and  $k \in [1, N]$ , the set  $\tau([1, k])$  is an interval.

THEOREM 8.3. Assume that  $R$  is a symmetric Poisson-CGL extension of length  $N$  and  $\tau$  an element of the subset  $\Xi_N$  of  $S_N$ , cf. (6.4). Let  $y_{\tau,1}, \dots, y_{\tau,N}$  be the sequence in  $R$  from Theorem 5.5 applied to the P-CGL extension presentation (6.5) of  $R$  corresponding to  $\tau$ . Let  $k \in [1, N]$ .

(a) If  $\tau(k) \geq \tau(1)$ , then  $y_{\tau,k} = y_{[p^m(\tau(k)), \tau(k)]}$ , where

$$(8.2) \quad m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k)) \in \tau([1, k])\}.$$

(b) If  $\tau(k) \leq \tau(1)$ , then  $y_{\tau,k} = y_{[\tau(k), s^m(\tau(k))]}$ , where

$$(8.3) \quad m = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(\tau(k)) \in \tau([1, k])\}.$$

In both cases, the predecessor and successor functions are with respect to the original P-CGL extension presentation (5.1) of  $R$ .

## 8.2. The elements of $\Xi_N$

In this and the next section, we investigate the elements of the subset  $\Xi_N$  of  $S_N$  defined in (6.4). It follows from (6.4) that every element  $\tau \in \Xi_N$  has the property that either  $\tau(N) = 1$  or  $\tau(N) = N$ . This implies the following recursive description of  $\Xi_N$ .

LEMMA 8.4. *For each  $\tau \in \Xi_N$ , there exists  $\tau' \in \Xi_{N-1}$  such that either*

$$\tau(i) = \tau'(i), \quad \forall i \in [1, N-1] \quad \text{and} \quad \tau(N) = N$$

or

$$\tau(i) = \tau'(i) + 1, \quad \forall i \in [1, N-1] \quad \text{and} \quad \tau(N) = 1.$$

For all  $\tau' \in \Xi_{N-1}$ , the above define elements of  $\Xi_N$ .

Given  $k \in [1, N]$  and a sequence  $k \leq j_k \leq \dots \leq j_1 \leq N$ , define

$$(8.4) \quad \tau_{(j_k, \dots, j_1)} := (k(k+1) \dots j_k) \dots (23 \dots j_2)(12 \dots j_1) \in S_N,$$

where in the right hand side we use the standard notation for cycles in  $S_N$ . Using Lemma 8.4, the elements of  $\Xi_N$  are easily characterized as follows.

LEMMA 8.5. [29, Lemma 5.4] *The subset  $\Xi_N \subset S_N$  consists of the elements of the form  $\tau_{(j_k, \dots, j_1)}$ , where  $k \in [1, N]$  and  $k \leq j_k \leq \dots \leq j_1 \leq N$ .*

The representation of an element of  $\Xi_N$  in the form (8.4) is not unique. As observed in [29], one way to visualize  $\tau_{(j_k, \dots, j_1)}$  is that the sequence  $\tau(1), \dots, \tau(N)$  is obtained from the sequence  $1, \dots, N$  by the following procedure:

(\*) *The number 1 is pulled to the right to position  $j_1$  (preserving the order of the other numbers), then the number 2 is pulled to the right to position  $j_2 - 1, \dots$ , at the end the number  $k$  is pulled to the right to position  $j_k - k + 1$ .*

For example, for  $k = 2$  the following illustrates how  $\tau_{(j_2, j_1)}$  is obtained from the identity permutation:

$$\begin{aligned} & [\textcircled{1}, \textcircled{2}, 3, 4, \dots, j_2, j_2 + 1, \dots, j_1, j_1 + 1, \dots, N] \mapsto \\ & [\textcircled{2}, 3, 4, \dots, j_2, j_2 + 1, \dots, j_1, \textcircled{1}, j_1 + 1, \dots, N] \mapsto \\ & [3, 4, \dots, j_2, \textcircled{2}, j_2 + 1, \dots, j_1, \textcircled{1}, j_1 + 1, \dots, N], \end{aligned}$$

where the numbers that are pulled (1 and 2) are circled.

If we perform the above procedure one step at a time, so in each step we only interchange the positions of two adjacent numbers, then the elements of  $S_N$  from all intermediate steps will belong to  $\Xi_N$ . For example, this requires factoring the cycle  $(1, 2, \dots, j_1)$  as  $(1, j_1)(1, j_1 - 1) \dots (1, 2)$  rather than as  $(1, 2)(2, 3) \dots (j_1 - 1, j_1)$ . This implies at once the first part of the next corollary [29, Corollary 5.5(a)].

COROLLARY 8.6. *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$  and  $\tau \in \Xi_N$ .*

(a) *There exists a sequence  $\tau_0 = \text{id}, \tau_1, \dots, \tau_n = \tau$  in  $\Xi_N$  such that for all  $l \in [1, n]$ ,*

$$\tau_l = (\tau_{l-1}(k_l), \tau_{l-1}(k_l + 1))\tau_{l-1} = \tau_{l-1}(k_l, k_l + 1)$$

*for some  $k_l \in [1, N-1]$  such that  $\tau_{l-1}(k_l) < \tau_{l-1}(k_l + 1)$ .*

(b) *If  $\eta : [1, N] \rightarrow \mathbb{Z}$  is a function satisfying the conditions of Theorem 5.5 for the original P-CGL presentation of  $R$ , then*

$$(8.5) \quad \eta_\tau := \eta \tau : [1, N] \rightarrow \mathbb{Z}$$

*satisfies the conditions of Theorem 5.5 for the  $\eta$ -function of the P-CGL extension presentation (6.5) of  $R$  corresponding to  $\tau$ .*

The sequence described in part (a) of the corollary for the element  $\tau_{(j_k, \dots, j_1)} \in \Xi_N$  has length  $n = j_1 + \dots + j_k - k(k+1)/2$ . The second part of the corollary follows by recursively applying Theorem 7.2 to the P-CGL extension presentations (6.5) for the elements  $\tau_{l-1}$  and  $\tau_l$ . Corollary 8.6(b) gives a second proof of the fact that the rank of a symmetric P-CGL extension does not depend on the choice of P-CGL extension presentation of the form (6.5), see §6.1.

**COROLLARY 8.7.** *Assume that  $R$  is a symmetric Poisson-CGL extension of length  $N$  and  $\eta : [1, N] \rightarrow \mathbb{Z}$  is a function satisfying the conditions of Theorem 5.5. Then for all  $1 \leq j \leq k \leq N$ , the function  $\eta_{[j,k]} : [1, k-j+1] \rightarrow \mathbb{Z}$  given by*

$$\eta_{[j,k]}(l) = \eta(j+l-1), \quad \forall l \in [1, k-j+1]$$

*satisfies the conditions of Theorem 5.5 for the symmetric P-CGL extension  $R_{[j,k]}$ .*

Thus, the  $\eta$ -functions for “interval subalgebras” of symmetric P-CGL extensions can be chosen to be restrictions of the original  $\eta$ -function up to shifts.

**PROOF.** Apply Corollary 8.6(b) to

$$\tau = [j, \dots, k, j-1, \dots, 1, k+1, \dots, N] \in \Xi_N$$

and consider the  $(k-j+1)$ -st subalgebra of the corresponding Poisson-Ore extension, which equals  $R_{[j,k]}$ .  $\square$

### 8.3. A subset of $\Xi_N$

We will work with a certain subset  $\Gamma_N$  of  $\Xi_N$  which was introduced in [29, §5.3] and which will play an important role in Chapter 11. Recall (6.7) that  $w_\circ = [N, N-1, \dots, 1]$  denotes the longest element of  $S_N$ . For  $1 \leq i \leq j \leq N$ , define the following elements of  $\Xi_N$ :

$$(8.6) \quad \tau_{i,j} := [i+1, \dots, j, i, j+1, \dots, N, i-1, i-2, \dots, 1] \in \Xi_N.$$

They satisfy

$$\tau_{1,1} = \text{id}, \quad \tau_{i,N} = \tau_{i+1,i+1}, \quad \forall i \in [1, N-1], \quad \tau_{N,N} = w_\circ.$$

Denote by  $\Gamma_N$  the subset of  $\Xi_N$  consisting of all  $\tau_{i,j}$ ’s and consider the following linear ordering on it

$$(8.7) \quad \Gamma_N := \{\text{id} = \tau_{1,1} \prec \dots \prec \tau_{1,N} = \tau_{2,2} \prec \dots \prec \tau_{2,N} = \tau_{3,3} \prec \dots \prec \tau_{3,N} = \tau_{4,4} \prec \dots \prec \tau_{N-2,N} = \tau_{N-1,N-1} \prec \tau_{N-1,N} = \tau_{N,N} = w_\circ\}.$$

In the notation of (8.4), the elements of  $\Gamma_N$  are given by

$$\tau_{i,j} := \tau_{(j,N,\dots,N)}, \quad \text{where } N \text{ is repeated } i-1 \text{ times, } \forall 1 \leq i \leq j \leq N.$$

The sequence of elements (8.7) is nothing but the sequence of the intermediate steps of the procedure  $(*)$  from §8.2 applied to the longest element  $w_\circ \in S_N$  (in which case  $k = N-1$  and  $j_1 = j_2 = \dots = j_{N-1} = N$ ). Note that

$$\tau_{i,l+1} = \tau_{i,l}(l+2-i, l+1-i), \quad \forall 1 \leq i \leq l < N.$$

Assume that  $R$  is a symmetric P-CGL extension. To each element of  $\Gamma_N$ , Proposition 6.4 associates a P-CGL extension presentation of  $R$ . Each two consecutive

presentations are associated to a pair  $\tau_{i,j}, \tau_{i,j+1} \in \Xi_N$  for some  $1 \leq i \leq j < N$ . They have the forms

$$(8.8) \quad R = \mathbb{K}[x_{i+1}]_p \cdots [x_j; \sigma_j, \delta_j]_p [x_i; \sigma_i^*, \delta_i^*]_p [x_{j+1}; \sigma_{j+1}, \delta_{j+1}]_p \cdots [x_N; \sigma_N, \delta_N]_p [x_{i-1}; \sigma_{i-1}^*, \delta_{i-1}^*]_p \cdots [x_1; \sigma_1^*, \delta_1^*]_p$$

and

$$(8.9) \quad R = \mathbb{K}[x_{i+1}]_p \cdots [x_j; \sigma_j, \delta_j]_p [x_{j+1}; \sigma_{j+1}, \delta_{j+1}]_p [x_i; \sigma_i^*, \delta_i^*]_p [x_{j+2}; \sigma_{j+2}, \delta_{j+2}]_p \cdots [x_N; \sigma_N, \delta_N]_p [x_{i-1}; \sigma_{i-1}^*, \delta_{i-1}^*]_p \cdots [x_1; \sigma_1^*, \delta_1^*]_p,$$

respectively. These two presentations satisfy the assumptions of Theorem 7.2. Recall from Definitions 5.1 and 6.1 that  $\sigma_l = (h_l \cdot)$ ,  $\sigma_l^* = (h_l^* \cdot)$  and  $h_l \cdot x_l = \lambda_l x_l$ ,  $h_l^* \cdot x_l = \lambda_l^* x_l$  with  $h_l, h_l^* \in \text{Lie } \mathcal{H}$  and  $\lambda_l, \lambda_l^* \in \mathbb{K}^*$ . By Corollary 8.6(b), the  $\eta$ -function for the P-CGL extension presentation (8.8) can be taken to be  $\eta_{\tau_{i,j}}$ . In particular, the values of this function on  $j+1-i$  and  $j+2-i$  are  $\eta(i)$  and  $\eta(j+1)$ , respectively. If  $\eta(i) = \eta(j+1)$ , then Eq. (7.22) of Theorem 7.5 applied to the presentations (8.8) and (8.9) implies

$$(8.10) \quad \lambda_i^* + \lambda_{j+1} = 0.$$

**PROPOSITION 8.8.** *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$  and  $a \in \mathbb{Z}$  be such that  $|\eta^{-1}(a)| > 1$ . Denote*

$$\eta^{-1}(a) = \{l, s(l), \dots, s^m(l)\}$$

where  $l \in [1, N]$  and  $m = O_+(l) \in \mathbb{Z}_{>0}$ . Then

$$(8.11) \quad \lambda_l^* = \lambda_{s(l)}^* = \cdots = \lambda_{s^{m-1}(l)}^* = -\lambda_{s(l)} = -\lambda_{s^2(l)} = \cdots = -\lambda_{s^m(l)}.$$

**PROOF.** It follows from (8.10) that

$$\lambda_{s^{m_1}(l)}^* = -\lambda_{s^{m_2}(l)}, \quad \forall 0 \leq m_1 < m_2 \leq m$$

which is equivalent to the statement of the proposition.  $\square$

#### 8.4. Sequences of homogeneous Poisson-prime elements

We proceed with the proofs of the two theorems formulated in §8.1.

**PROOF OF THEOREM 8.1.** As already noted, the case  $m = 0$  is easily verified. Assume now that  $m > 0$ .

(a) Consider the P-CGL extension presentation of  $R$  associated to

$$(8.12) \quad \tau_{(s^m(i)-1, s^m(i), \dots, s^m(i))} = [i+1, \dots, s^m(i)-1, i, s^m(i), i-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N,$$

where  $s^m(i)$  is repeated  $i-1$  times. The  $(s^m(i)-i)$ -th and  $(s^m(i)-i+1)$ -st algebras in the chain are precisely  $R_{[i, s^m(i)-1]}$  and  $R_{[i, s^m(i)]}$ . Theorem 5.5 implies that the homogeneous Poisson-prime elements of  $R_{[i, s^m(i)]}$  that do not belong to  $R_{[i, s^m(i)-1]}$  are associates of each other. Denote by  $z_{[i, s^m(i)]}$  one such element. Set  $z_\emptyset = 1$ . (One can prove part (a) using the simpler presentation of  $R$  associated to the permutation

$$[i, i+1, \dots, s^m(i), i-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N$$

but the first presentation will also play a role in the proof of part (b).) The equivariance property of the  $\eta$ -function from Corollary 8.6(b) and Theorem 5.5 imply that

$$(8.13) \quad z_{[i, s^m(i)]} = \xi_{i,m} (z_{[i, s^{m-1}(i)]} x_{s^m(i)} - c_{[i, s^m(i)-1]})$$

for some  $c_{[i, s^m(i)-1]} \in R_{[i, s^m(i)-1]}$  and  $\xi_{i,m} \in \mathbb{K}^*$ .

(b) Now consider the P-CGL extension presentation of  $R$  associated to

$$(8.14) \quad \tau_{(s^m(i), \dots, s^m(i))} = [i+1, \dots, s^m(i), i, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N,$$

where  $s^m(i)$  is repeated  $i$  times. The  $(s^m(i)-i)$ -th and  $(s^m(i)-i+1)$ -st algebras in the chain are  $R_{[i+1, s^m(i)]}$  and  $R_{[i, s^m(i)]}$ . Theorem 5.5 implies that the homogeneous Poisson-prime elements of  $R_{[i, s^m(i)]}$  that do not belong to  $R_{[i+1, s^m(i)]}$  are associates of each other. They are also associates of  $z_{[i, s^m(i)]}$ . This follows from Eq. (7.6) of Theorem 7.2(b) applied to the P-CGL extension presentations of  $R$  associated to the elements (8.12) and (8.14). The fact that these two P-CGL extension presentations satisfy the assumptions of Theorem 7.2(b) follows from the equivariance of the  $\eta$ -function from Corollary 8.6(b). This equivariance and Theorem 5.5 applied to the presentation for (8.14) also imply that

$$(8.15) \quad z_{[i, s^m(i)]} = \xi'_{i,m} (x_i z_{[s(i), s^m(i)]} - c'_{[i+1, s^m(i)]})$$

for some  $c'_{[i+1, s^m(i)]} \in R_{[i+1, s^m(i)]}$  and  $\xi'_{i,m} \in \mathbb{K}^*$ .

Parts (c) and (d) follow at once by comparing the leading terms in Eqs. (8.13) and (8.15), and using the fact that the group of units of an iterated Poisson-Ore extension over  $\mathbb{K}$  is reduced to scalars.

For part (e), we apply (5.26) in  $R_{[\min\{k, i\}, \max\{k, s^m(i)\}]}$  with  $j = s^m(i)$ .  $\square$

PROOF OF THEOREM 8.3. We first show that in situation (a),  $y_{\tau, k}$  is a scalar multiple of  $y_{[p^m(\tau(k)), \tau(k)]}$ .

For  $k \in [0, N]$ , denote by  $R_{\tau, k}$  the  $k$ -th algebra in the chain (6.5). Since  $\tau([1, j])$  is an interval for all  $j \leq k$ ,

$$\tau([1, k]) = [\tau(i), \tau(k)] \quad \text{for some } i \in [1, k].$$

Therefore  $R_{\tau, k} = R_{[\tau(i), \tau(k)]}$  and  $R_{\tau, k-1} = R_{[\tau(i), \tau(k)-1]}$ . For  $m \in \mathbb{Z}_{\geq 0}$  given by (8.2) we have

$$\tau(i) \leq p^m(\tau(k)) \leq \tau(k).$$

Theorem 5.5 implies that  $y_{[p^m(\tau(k)), \tau(k)]}$  is a homogeneous Poisson-prime element of  $R_{[\tau(i), \tau(k)]} = R_{\tau, k}$ . It does not belong to  $R_{\tau, k-1} = R_{[\tau(i), \tau(k)-1]}$  because of Theorem 8.1(c). It follows from Theorem 5.5 that the homogeneous Poisson-prime elements of  $R_{\tau, k}$  that do not belong to  $R_{\tau, k-1}$  are associates of  $y_{\tau, k}$ . Hence,  $y_{\tau, k}$  is a scalar multiple of  $y_{[p^m(\tau(k)), \tau(k)]}$ .

Analogously, in situation (b),  $y_{\tau, k}$  is a scalar multiple of  $y_{[\tau(k), s^m(\tau(k))]}$ .

To prove the stated equalities, we proceed by induction on  $k$ . The case  $k = 1$  is clear, since  $y_{\tau, 1} = x_{\tau(1)} = y_{[\tau(1), \tau(1)]}$ . Now assume that  $k > 1$ , and suppose we are in situation (a). If  $m = 0$ , then  $y_{\tau, k} = x_{\tau(k)} = y_{[\tau(k), \tau(k)]}$ , so we may assume that  $m > 0$ .

Let  $l \in [1, k-1]$  be maximal such that  $\eta(\tau(l)) = \eta(\tau(k))$ , and note that

$$p(\tau(k)), \dots, p^m(\tau(k)) \in \tau([1, l]).$$

By construction of  $y_{\tau, k}$  and Theorem 8.1(d), we have

$$(8.16) \quad y_{\tau, k} = y_{\tau, l} x_{\tau(k)} - c \quad \text{and} \quad y_{[p^m(\tau(k)), \tau(k)]} = y_{[p^m(\tau(k)), p(\tau(k))]} x_{\tau(k)} - c'$$

for some  $c, c' \in R_{[1, \tau(k)-1]}$ . If  $\tau(l) \geq \tau(1)$ , then  $\tau([1, l]) \subseteq [1, \tau(l)]$ , whence  $\tau(l) = p(\tau(k))$  and

$$m - 1 = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(l)) \in \tau([1, l])\}.$$

By induction,  $y_{\tau, l} = y_{[p^{m-1}(\tau(l)), \tau(l)]} = y_{[p^m(\tau(k)), p(\tau(k))]}.$  On the other hand, if  $\tau(l) \leq \tau(1)$ , then  $\tau([1, l]) \subseteq [\tau(l), \tau(k) - 1]$ , whence  $\tau(l) = p^m(\tau(k))$  and

$$m - 1 = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(\tau(l)) \in \tau([1, l])\}.$$

In this case, induction yields  $y_{\tau, l} = y_{[\tau(l), s^{m-1}(\tau(l))]} = y_{[p^m(\tau(k)), p(\tau(k))]}.$  In both cases, we conclude from (8.16) and the equality  $y_{\tau, l} = y_{[p^m(\tau(k)), p(\tau(k))]}$  that  $y_{\tau, k} = y_{[p^m(\tau(k)), \tau(k)]}.$

Situation (b), when  $\tau(k) < \tau(1)$ , is analogous and is left to the reader.  $\square$

**COROLLARY 8.9.** *If  $R$  is a symmetric Poisson-CGL extension of length  $N$  and  $i, j \in [1, N]$ ,  $m, n \in \mathbb{Z}_{\geq 0}$  are such that*

$$i \leq j \leq s^n(j) \leq s^m(i) \leq N,$$

*then*

$$\{y_{[i, s^m(i)]}, y_{[j, s^n(j)]}\} = \Omega_{\lambda}(e_{[i, s^m(i)]}, e_{[j, s^n(j)]})y_{[j, s^n(j)]}y_{[i, s^m(i)]}.$$

**PROOF.** Apply Theorem 8.3 to the P-CGL extension presentation of  $R$  associated to

$$\tau' := [j, \dots, s^m(i), j-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N$$

and then use (5.18), (5.23). For this permutation, Theorem 8.3 implies that  $y_{\tau', s^n(j)-j+1}$  and  $y_{\tau', s^m(i)-i+1}$  are equal to  $y_{[j, s^n(j)]}$  and  $y_{[i, s^m(i)]}$ , respectively.  $\square$

### 8.5. An identity for Poisson-normal elements

For  $1 \leq j \leq l \leq N$ , set

$$(8.17) \quad O_-^j(l) = \max\{m \in \mathbb{Z}_{\geq 0} \mid p^m(l) \geq j\}.$$

The following fact follows directly from Theorems 5.5 and 8.3.

**COROLLARY 8.10.** *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$  and  $1 \leq j \leq k \leq N$ . Then*

$$\left\{ y_{[p^{O_-^j(i)}(i), i]} \mid i \in [j, k], s(i) > k \right\}$$

*is a list of the homogeneous Poisson-prime elements of  $R_{[j, k]}$  up to scalar multiples.*

**COROLLARY 8.11.** *Assume that  $R$  is a symmetric Poisson-CGL extension of length  $N$ , and  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  are such that  $s^m(i) \in [1, N]$ . Then*

$$(8.18) \quad u_{[i, s^m(i)]} := y_{[i, s^{m-1}(i)]}y_{[s(i), s^m(i)]} - y_{[s(i), s^{m-1}(i)]}y_{[i, s^m(i)]}$$

*is a nonzero homogeneous Poisson-normal element of  $R_{[i+1, s^m(i)-1]}$  which is not a multiple of  $y_{[s(i), s^{m-1}(i)]}$  if  $m \geq 2$ . It Poisson-normalizes the elements of the algebra  $R_{[i+1, s^m(i)-1]}$  in exactly the same way as  $y_{[s(i), s^{m-1}(i)]}y_{[i, s^m(i)]}$  does. Moreover,*

$$(8.19) \quad u_{[i, s^m(i)]} = \psi \prod_{k \in P} y_{[p^{O_-^{i+1}(k)}(k), k]}^{m_k}$$

where  $\psi \in \mathbb{K}^*$ ,

$$(8.20) \quad P = P_{[i, s^m(i)]} := \{k \in [i, s^m(i)] \setminus \{i, s(i), \dots, s^m(i)\} \mid s(k) > s^m(i)\},$$

and the integers  $m_k$  are those from Theorems 7.3, 7.4. Consequently, the leading term of  $u_{[i, s^m(i)]}$  has the form

$$\psi x^{n_{i+1}e_{i+1} + \dots + n_{s^m(i)-1}e_{s^m(i)-1}}$$

for some  $n_{i+1}, \dots, n_{s^m(i)-1} \in \mathbb{Z}_{\geq 0}$  such that  $n_{s(i)} = \dots = n_{s^{m-1}(i)} = 0$  and  $n_j = n_l$  for all  $i+1 \leq j \leq l \leq s^m(i)-1$  with  $\eta(j) = \eta(l)$ .

PROOF. Consider the P-CGL extension presentations (6.5) of  $R$  associated to the elements

$$\tau_{i, s^m(i)-1} = [i+1, \dots, s^m(i)-1, i, s^m(i), s^m(i)+1, \dots, N, i-1, \dots, 1]$$

$$\tau_{i, s^m(i)} = [i+1, \dots, s^m(i)-1, s^m(i), i, s^m(i)+1, \dots, N, i-1, \dots, 1]$$

of  $\Xi_N$ . Applied to these presentations, Theorems 7.3, 7.4, and 8.3 yield all but the last statement. That statement follows from (8.19), due to the choice of leading terms for the elements  $y_{[p^{O^{i+1}(k)}(k), k]}$ .  $\square$

Since  $y_{\emptyset} = 1$  and  $y_{[i, i]} = x_i$ , the case  $m = 1$  of the corollary states that

$$(8.21) \quad u_{[i, s(i)]} := x_i x_{s(i)} - y_{[i, s(i)]} = c_{[i, s(i)-1]} = c'_{[i+1, s(i)]}$$

is a nonzero homogeneous Poisson-normal element of  $R_{[i+1, s(i)-1]}$ , cf. Theorem 8.1(d). We set

$$u_{[i, i]} := 1.$$

EXAMPLE 8.12. In the case of  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$  (Example 5.3), the result of Example 8.2 shows how to express each  $u_{[i, s^l(i)]}$  as a difference of products of solid minors. In particular, if  $i = (r-1)n + c$  with  $s(i) \in [1, N]$ , then  $r < m$ ,  $c < n$ , and

$$(8.22) \quad \begin{aligned} u_{[i, s(i)]} &:= x_i x_{s(i)} - y_{[i, s(i)]} = t_{rc} t_{r+1, c+1} - \Delta_{[r, r+1], [c, c+1]} \\ &= t_{r, c+1} t_{r+1, c} = x_{(r-1)n+c+1} x_{rn+c}. \end{aligned}$$

More generally, if  $i = (r-1)n + c$  and  $l \in \mathbb{Z}_{\geq 0}$  with  $s^l(i) \neq +\infty$ , one can calculate that

$$u_{[i, s^l(i)]} = \Delta_{[r, r+l-1], [c+1, c+l]} \Delta_{[r+1, r+l], [c, c+l-1]}.$$

The next result, which is the main result in this section, describes relationships among the Poisson-normal elements in Corollary 8.11. It will play a key role in normalizing the scalars  $\kappa$  from Theorem 7.3. Recall §5.2 for the definition of leading terms  $\text{lt}(-)$ .

THEOREM 8.13. *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$ . For all  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  such that  $s^{m+1}(i) \in [1, N]$ ,*

$$(8.23) \quad \text{lt}(u_{[s(i), s^m(i)]} u_{[i, s^{m+1}(i)]}) = \text{lt}(u_{[i, s^m(i)]} u_{[s(i), s^{m+1}(i)]}).$$

PROOF. Consider congruences modulo the ideal

$$W_m := y_{[s(i), s^m(i)]} R_{[i, s^{m+1}(i)]}$$

of  $R_{[i, s^{m+1}(i)]}$ . By the definition of the elements  $u_{[*], [*]}$ ,

$$\begin{aligned} u_{[s(i), s^m(i)]} &\equiv y_{[s(i), s^{m-1}(i)]} y_{[s^2(i), s^m(i)]} & u_{[i, s^{m+1}(i)]} &\equiv y_{[i, s^m(i)]} y_{[s(i), s^{m+1}(i)]} \\ u_{[i, s^m(i)]} &\equiv -y_{[s(i), s^{m-1}(i)]} y_{[i, s^m(i)]} & u_{[s(i), s^{m+1}(i)]} &\equiv -y_{[s^2(i), s^m(i)]} y_{[s(i), s^{m+1}(i)]} \end{aligned}$$

modulo  $W_m$ , whence

$$u_{[s(i), s^m(i)]} u_{[i, s^{m+1}(i)]} \equiv u_{[i, s^m(i)]} u_{[s(i), s^{m+1}(i)]} \pmod{W_m}.$$

Consequently,

$$(8.24) \quad u_{[s(i), s^m(i)]} u_{[i, s^{m+1}(i)]} = u_{[i, s^m(i)]} u_{[s(i), s^{m+1}(i)]} + y_{[s(i), s^m(i)]} r$$

for some  $r \in R_{[i, s^{m+1}(i)]}$ . Since we are done if  $r = 0$ , we may assume that  $r \neq 0$ .

Note that the leading term of  $y_{[s(i), s^m(i)]} r$  has the form  $\xi x^f$  where  $\xi \in \mathbb{K}^*$  and  $f = \sum_{l=i}^{s^{m+1}(i)} n_l e_l$  with  $n_{s(i)} \geq 1$ . On the other hand, the last statement of Corollary 8.11 implies that the leading terms of the other two products in (8.24) do not contain positive powers of  $x_{s(i)}$ . Hence, neither of these leading terms is a scalar multiple of  $x^f$ . The validity of (8.23) follows.  $\square$



## CHAPTER 9

# Chains of mutations in symmetric Poisson-CGL extensions

For a given Poisson-CGL extension  $R$  one has the freedom of rescaling the generators  $x_j$  by elements of the base field  $\mathbb{K}$ . The prime elements  $y_j$  from Theorem 5.5 obviously depend (again up to rescaling) on the choice of  $x_j$ . In this section we prove that for each symmetric P-CGL extension  $R$  its generators can be rescaled in such a way that all scalars  $\kappa$  in Theorem 7.3 become equal to one. This implies a mutation theorem, proved in Section 11, associated to the elements of  $\Xi_N$  via the sequences of prime elements from Theorem 8.3.

Throughout this section we will assume that  $R$  is a symmetric P-CGL extension of length  $N$ .

### 9.1. The leading coefficients of the elements $u_{[i, s^m(i)]}$

Recall (8.18). For  $i \in [1, N]$  and  $m \in \mathbb{Z}_{\geq 0}$  such that  $s^m(i) \in [1, N]$ , let

$$\pi_{[i, s^m(i)]} \in \mathbb{K}^* \quad \text{and} \quad f_{[i, s^m(i)]} \in \sum_{j=i+1}^{s^m(i)-1} \mathbb{Z}_{\geq 0} e_j \subset \mathbb{Z}^N$$

be given by

$$(9.1) \quad \text{lt}(u_{[i, s^m(i)]}) = \pi_{[i, s^m(i)]} x^{f_{[i, s^m(i)]}}.$$

Note that  $\pi_{[i, i]} = 1$  and  $f_{[i, i]} = 0$ , because  $u_{[i, i]} = 1$ .

From (8.23), we obtain

$$(9.2) \quad \begin{aligned} \pi_{[s(i), s^m(i)]} \pi_{[i, s^{m+1}(i)]} &= \pi_{[i, s^m(i)]} \pi_{[s(i), s^{m+1}(i)]} \\ f_{[s(i), s^m(i)]} + f_{[i, s^{m+1}(i)]} &= f_{[i, s^m(i)]} + f_{[s(i), s^{m+1}(i)]} \end{aligned}$$

for  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  with  $s^{m+1}(i) \in [1, N]$ . A quick induction then shows that

$$(9.3) \quad \begin{aligned} \pi_{[i, s^m(i)]} &= \pi_{[i, s(i)]} \cdots \pi_{[s^{m-1}(i), s^m(i)]} \\ f_{[i, s^m(i)]} &= f_{[i, s(i)]} + \cdots + f_{[s^{m-1}(i), s^m(i)]} \end{aligned}$$

for  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  with  $s^m(i) \in [1, N]$ . Consequently, we have

**LEMMA 9.1.** *If  $\pi_{[i, s(i)]} = 1$  for all  $i \in [1, N]$  such that  $s(i) \neq +\infty$ , then  $\pi_{[i, s^m(i)]} = 1$  for all  $i \in [1, N]$  and  $m \in \mathbb{Z}_{\geq 0}$  such that  $s^m(i) \in [1, N]$ .  $\square$*

### 9.2. Rescaling of the generators of a symmetric Poisson-CGL extension

For  $\gamma := (\gamma_1, \dots, \gamma_N) \in (\mathbb{K}^*)^N$  and  $f := (n_1, \dots, n_N) \in \mathbb{Z}^N$  set

$$\gamma^f := \gamma_1^{n_1} \cdots \gamma_N^{n_N} \in \mathbb{K}^*.$$

Given  $\gamma \in (\mathbb{K}^*)^N$ , one can rescale the generators  $x_j$  of  $R$ ,

$$(9.4) \quad x_j \mapsto \gamma_j x_j, \quad \forall j \in [1, N],$$

by which we mean that one can use  $\gamma_1 x_1, \dots, \gamma_N x_N$  as a new sequence of generators and express  $R$  as a P-CGL extension of the form

$$R = \mathbb{K}[\gamma_1 x_1]_p [\gamma_2 x_2; \sigma_2, \gamma_2 \delta_2]_p \cdots [\gamma_N x_N; \sigma_N, \gamma_N \delta_N]_p.$$

This change of generators obviously does not effect the  $\mathcal{H}$ -action and the matrix  $\lambda$ , but one obtains a new set of elements  $y_k, y_{[i, s^m(i)]}, u_{[i, s^m(i)]}$  by applying Theorems 5.5, 8.1 and Corollary 8.11 for the new set of generators. (Note that this is not the same as substituting (9.4) in the formulas for  $y_k$  and  $y_{[i, s^m(i)]}$ ; those other kind of transformed elements may not be even prime because (9.4) does not determine an algebra automorphism.) The uniqueness part of Theorem 8.1 implies that the effect of (9.4) on the elements  $y_{[i, s^m(i)]}$  is that they are rescaled by the rule

$$y_{[i, s^m(i)]} \mapsto \gamma^{e_{[i, s^m(i)]}} y_{[i, s^m(i)]} = (\gamma_i \gamma_{s(i)} \cdots \gamma_{s^m(i)}) y_{[i, s^m(i)]}$$

for all  $i \in [1, N]$ ,  $m \in \mathbb{Z}_{\geq 0}$  such that  $s^m(i) \in [1, N]$ . Hence, the effect of (9.4) on the elements  $u_{[i, s^m(i)]}$  is that they are rescaled by

$$u_{[i, s^m(i)]} \mapsto \gamma^{e_{[i, s^m(i)]} + e_{[s(i), s^{m-1}(i)]}} u_{[i, s^m(i)]} = (\gamma_i \gamma_{s(i)}^2 \cdots \gamma_{s^{m-1}(i)} \gamma_{s^m(i)}) u_{[i, s^m(i)]}.$$

It follows from (9.1) that the effect of (9.4) on the scalars  $\pi_{[i, s^m(i)]}$  is that they are rescaled by

$$\begin{aligned} \pi_{[i, s^m(i)]} \mapsto \gamma^{e_{[i, s^m(i)]} + e_{[s(i), s^{m-1}(i)]} - f_{[i, s^m(i)]}} \pi_{[i, s^m(i)]} = \\ \gamma_i \gamma_{s^m(i)} \gamma^{2e_{[s(i), s^{m-1}(i)]} - f_{[i, s^m(i)]}} \pi_{[i, s^m(i)]}. \end{aligned}$$

(Note that the rescaling (9.4) has no effect on the integer vector  $f_{[i, s^m(i)]}$ .) This implies at once the following fact.

**PROPOSITION 9.2.** *Let  $R$  be a symmetric CGL extension of length  $N$  and rank  $\text{rk}(R)$ . Then there exist  $N$ -tuples  $\gamma \in (\mathbb{K}^*)^N$  such that after the rescaling (9.4) we have*

$$(9.5) \quad \pi_{[i, s(i)]} = 1, \quad \forall i \in [1, N] \text{ such that } s(i) \neq +\infty.$$

*The set of those  $N$ -tuples  $\gamma$  is parametrized by  $(\mathbb{K}^*)^{\text{rk}(R)}$  and the coordinates  $\gamma_1, \dots, \gamma_N$  of all such  $N$ -tuples are recursively determined by*

$$\gamma_i \text{ is arbitrary if } p(i) = -\infty$$

*and*

$$(9.6) \quad \gamma_i = \gamma_{p(i)}^{-1} \gamma^{f_{[p(i), i]}} \pi_{[p(i), i]}^{-1}, \quad \text{if } p(i) \neq -\infty,$$

*where on the right hand side the  $\pi$ -scalars are the ones for the original generators  $x_1, \dots, x_N$  of  $R$ .*

Note that the product of the first two terms of the right hand side of (9.6) is a product of powers of  $\gamma_{p(i)}, \dots, \gamma_{i-1}$  since  $f_{[p(i), i]} \in \mathbb{Z}_{\geq 0} e_{p(i)+1} + \cdots + \mathbb{Z}_{\geq 0} e_{i-1}$ .

**EXAMPLE 9.3.** In the case of  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$ , (8.22) shows that the elements  $u_{[i, s(i)]}$  all have leading coefficient 1, and so no rescaling is needed in this case.

### 9.3. Mutations in symmetric Poisson-CGL extensions

Continue to let  $R$  be a symmetric P-CGL extension of length  $N$ . Let  $1 \leq j \leq l \leq N$ , and recall the definition (8.17) of the function  $O_-^j$ . By Corollary 8.10,

$$\left\{ y_{[p^{O_-^j(k)}(k), k]} \mid k \in [j, l], s(k) > l \right\}$$

is a list of the the homogeneous prime elements of  $R_{[j, l]}$  up to scalar multiples.

Let  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  be such that  $s^m(i) \in [1, N]$ , and consider the P-CGL extension presentation of  $R_{[i, s^m(i)]}$  associated to the following order of adjoining the generators:

$$(9.7) \quad x_{i+1}, \dots, x_{s^m(i)-1}, x_i, x_{s^m(i)}.$$

The intermediate subalgebras for this presentation of  $R_{[i, s^m(i)]}$  are

$$R_{[i+1, i+1]}, \dots, R_{[i+1, s^m(i)-1]}, R_{[i, s^m(i)-1]}, R_{[i, s^m(i)]}.$$

We identify  $\mathbb{Z}^{s^m(i)-i+1} \cong \mathbb{Z}e_i + \dots + \mathbb{Z}e_{s^m(i)} \subseteq \mathbb{Z}^N$ , and we define a homomorphism

$$Y_{[i, s^m(i)]} : \mathbb{Z}^{s^m(i)-i+1} \rightarrow (\text{Fract}(R_{[i, s^m(i)]}))^*$$

by setting

$$(9.8) \quad Y_{[i, s^m(i)]}(e_k) = \begin{cases} y_{[p^{O_-^{i+1}(k)}(k), k]}, & \text{if } k \in [i+1, s^m(i)-1] \\ y_{[i, s^m(i)-1]}, & \text{if } k = i \\ y_{[i, s^m(i)]}, & \text{if } k = s^m(i). \end{cases}$$

Recall the definition (9.1) of the vectors  $f_{[i, s^m(i)]}$ . It follows from Corollary 8.11 that there exists a unique vector

$$(9.9) \quad g_{[i, s^m(i)]} = \sum \{m_k e_k \mid k \in P_{[i, s^m(i)]}\} \in \mathbb{Z}^{s^m(i)-i+1}$$

such that

$$f_{[i, s^m(i)]} = \sum \left\{ m_k e_{[p^{O_-^{i+1}(k)}(k), k]} \mid k \in P_{[i, s^m(i)]} \right\},$$

where the integers  $m_k$  are those from Theorems 7.3, 7.4.

Set  $t := s^m(i) - i + 1$ . As above, we identify  $\mathbb{Z}^t$  with  $\mathbb{Z}e_i + \dots + \mathbb{Z}e_{s^m(i)}$ , that is, with the sublattice of  $\mathbb{Z}^N$  with basis  $e_j$ ,  $j \in [i, s^m(i)]$ . Define  $\sigma \in M_t(\mathbb{Z})$  by

$$(9.10) \quad \sigma(e_k) := \begin{cases} e_{[p^{O_-^{i+1}(k)}(k), k]}, & \text{if } k \in [i+1, s^m(i)-1] \\ e_{[i, s^m(i)-1]}, & \text{if } k = i \\ e_{[i, s^m(i)]}, & \text{if } k = s^m(i). \end{cases}$$

for all  $k \in [i, s^m(i)]$ . It is easy to check that

$$\sigma \text{ is invertible and } \sigma(\mathbb{Z}_{\geq 0}^t) \subseteq \mathbb{Z}_{\geq 0}^t.$$

This choice of  $\sigma$  ensures that

$$(9.11) \quad \sigma(g_{[i, s^m(i)]}) = f_{[i, s^m(i)]}.$$

The definition of  $Y_{[i, s^m(i)]}$  also implies that

$$(9.12) \quad \text{lt}(Y_{[i, s^m(i)]}(e_k)) = x^{\sigma(e_k)}, \quad \forall k \in [i, s^m(i)].$$

LEMMA 9.4. For all  $g \in \mathbb{Z}_{\geq 0}^t$ ,

$$\text{lt}(Y_{[i, s^m(i)]}(g)) = x^{\sigma(g)}.$$

PROOF. This is immediate from (9.12), since  $Y_{[i,s^m(i)]}$  and  $\sigma$  are homomorphisms.  $\square$

THEOREM 9.5. *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$ . Assume that the generators  $x_1, \dots, x_N$  of  $R$  are normalized so that the condition (9.5) is satisfied. Then for all  $i \in [1, N]$  and  $m \in \mathbb{Z}_{>0}$  such that  $s^m(i) \in [1, N]$ ,*

$$(9.13) \quad \begin{aligned} & y_{[s(i), s^m(i)]} = \\ & \begin{cases} Y_{[i, s(i)]}(e_{s(i)} - e_i) + Y_{[i, s(i)]}(g_{[i, s(i)]} - e_i), & \text{if } m = 1 \\ Y_{[i, s^m(i)]}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i) + Y_{[i, s^m(i)]}(g_{[i, s^m(i)]} - e_i), & \text{if } m > 1. \end{cases} \end{aligned}$$

PROOF. Lemma 9.4 and Eq. (9.11) imply

$$\text{lt}(Y_{[i, s^m(i)]}(g_{[i, s^m(i)]})) = x_i^{f_{[i, s^m(i)]}}.$$

By Eq. (8.19),  $u_{[i, s^m(i)]}$  is a scalar multiple of  $Y_{[i, s^m(i)]}(g_{[i, s^m(i)]})$ , and so we find that

$$(9.14) \quad u_{[i, s^m(i)]} = Y_{[i, s^m(i)]}(g_{[i, s^m(i)]}),$$

taking account of (9.5) and Lemma 9.1.

In case  $m > 1$ , we observe using (9.14) that

$$\begin{aligned} & y_{[s(i), s^m(i)]} - Y_{[i, s^m(i)]}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i) \\ &= y_{[s(i), s^m(i)]} - y_{[i, s^{m-1}(i)]}^{-1} y_{[s(i), s^{m-1}(i)]} y_{[i, s^m(i)]} \\ &= y_{[i, s^{m-1}(i)]}^{-1} u_{[i, s^m(i)]} = Y_{[i, s^m(i)]}(g_{[i, s^m(i)]} - e_i). \end{aligned}$$

A similar argument shows that

$$y_{[s(i), s(i)]} - Y_{[i, s(i)]}(e_{s(i)} - e_i) = x_i^{-1} u_{[i, s(i)]} = Y_{[i, s(i)]}(g_{[i, s(i)]} - e_i)$$

in the case  $m = 1$ .  $\square$

REMARK 9.6. Because of Theorem 8.1, all instances when Theorem 7.4 is applicable to a symmetric P-CGL extension with respect to the P-CGL extension presentations associated to the elements of the set  $\Xi_N$  are covered by Theorem 9.5. We refer the reader to Proposition 11.8 for details.

## Division properties of mutations between Poisson-CGL extension presentations

We prove in this chapter that every symmetric Poisson-CGL extension is equal to the intersection of appropriate localizations with respect to sets of  $y$ -elements. This plays a key role in Chapter 11 where we prove that every symmetric P-CGL extension is a cluster algebra which equals the corresponding upper cluster algebra. In the process, we introduce auxiliary permutations  $\tau_\bullet$ , for  $\tau \in \Xi_N$ , which are used to re-order our sets of cluster variables.

### 10.1. One-step mutations

In this section we describe the intersection of the localizations of a P-CGL extension  $R$  by the two sets of  $y$ -elements in the setting of Theorem 7.2(b).

Assume now that  $R$  is a P-CGL extension of length  $N$  as in (5.1). Let  $y_1, \dots, y_N \in R$  be the sequence of elements from Theorem 5.5. For all  $I \subseteq [1, N]$ , the set

$$(10.1) \quad E_I := \mathbb{K}^* \left\{ \prod_{j \in I} y_j^{m_j} \mid m_j \in \mathbb{Z}_{\geq 0} \right\}$$

is a multiplicative set in  $R$ . Set  $E := E_{[1, N]}$ . We will say that  $y_1^{m_1} \dots y_N^{m_N}$  is a *minimal denominator* of a nonzero element  $v \in R[E^{-1}]$  if

$$v = y_1^{-m_1} \dots y_N^{-m_N} s$$

for some  $s \in R$  such that  $y_j \nmid s$  for all  $j \in [1, N]$  with  $m_j > 0$ .

For the rest of this section, we assume the setting of Theorem 7.2(b). Then for all  $I' \subseteq [1, N]$ , the set

$$(10.2) \quad E'_{I'} := \mathbb{K}^* \left\{ \prod_{j \in I'} (y'_j)^{m_j} \mid m_j \in \mathbb{Z}_{\geq 0} \right\}$$

is another multiplicative set in  $R$ . Recall from Theorem 7.2(b) that  $y'_j = y_j$  for  $j \neq k$ . Hence,  $E'_{I'} = E_{I'}$  if  $k \notin I'$ . Denote  $E' := E'_{[1, N]}$ .

**LEMMA 10.1.** *For any distinct  $j, l \in [1, N]$ , we have  $y_j \nmid y_l$  and  $y_l \nmid y_j$ . Moreover,  $y_j \nmid y'_k$  and  $y'_k \nmid y_j$  for all  $j \in [1, N]$ .*

**PROOF.** If  $1 \leq j < l \leq N$ , then  $y_l \nmid y_j$  because  $y_l$  and  $y_j$  are elements of  $R_l$  with  $x_l$ -degrees 1 and 0 respectively. Since  $y_l$  and  $y_j$  are prime elements of  $R$ , it follows that  $y_j \nmid y_l$  as well.

On replacing the  $y_j$  by the  $y'_j$ , we have  $y'_j \nmid y'_l$  and  $y'_l \nmid y'_j$  for all  $j \neq l$ , which yields  $y_j \nmid y'_k$  and  $y'_k \nmid y_j$  for all  $j \neq k$ . Since  $y_k$  and  $y'_k$  are elements of  $R_{k+1}$  with  $x_{k+1}$ -degrees 0 and 1 respectively, we conclude as above that  $y'_k \nmid y_k$  and  $y_k \nmid y'_k$ .  $\square$

LEMMA 10.2. *For all  $I, I' \subseteq [1, N]$ ,*

$$(10.3) \quad E_I \cap E'_{I'} = E_{(I \cap I') \setminus \{k\}} = E'_{(I \cap I') \setminus \{k\}}.$$

PROOF. Let  $e \in E_I \cap E'_{I'}$ , and write

$$e = \beta \prod_{j=1}^N y_j^{m_j} = \beta' \prod_{j=1}^N (y'_j)^{m'_j}$$

for some  $\beta, \beta' \in \mathbb{K}^*$  and  $m_j, m'_j \in \mathbb{Z}_{\geq 0}$  such that  $m_j = 0$  for  $j \notin I$  and  $m'_j = 0$  for  $j \notin I'$ . Since the  $y_j$  and  $y'_j$  are prime elements of  $R$ , with  $y_j = y'_j$  for all  $j \neq k$ , it follows from Lemma 10.1 that

$$m_k = m'_k = 0 \quad \text{and} \quad m_j = m'_j, \quad \forall j \in [1, N], \quad j \neq k,$$

which then implies  $e \in E_{(I \cap I') \setminus \{k\}} = E'_{(I \cap I') \setminus \{k\}}$ .  $\square$

The next theorem implies that each nonzero element of  $R[E^{-1}]$  has a unique minimal denominator. It was inspired by [19, Theorem 4.1].

THEOREM 10.3. *Assume the setting of Theorem 7.2(b). Let*

$$y_1^{m_1} \dots y_N^{m_N} \quad \text{and} \quad y_1^{m'_1} \dots y_{k-1}^{m'_{k-1}} (y'_k)^{m'_k} y_{k+1}^{m'_{k+1}} \dots y_N^{m'_N}$$

*be two minimal denominators of a nonzero element  $v \in R[E^{-1}] \cap R[(E')^{-1}]$  (with respect to the two different localizations). Then*

$$m_k = m'_k = 0 \quad \text{and} \quad m_j = m'_j, \quad \forall j \in [1, N], \quad j \neq k.$$

*In particular,*

$$R[(E_I)^{-1}] \cap R[(E'_{I'})^{-1}] = R[(E_I \cap E'_{I'})^{-1}] = R[(E_{(I \cap I') \setminus \{k\}})^{-1}]$$

*for all  $I, I' \subseteq [1, N]$ .*

PROOF. Write

$$v = y_1^{-m_1} \dots y_N^{-m_N} s = y_1^{-m'_1} \dots y_{k-1}^{-m'_{k-1}} (y'_k)^{-m'_k} y_{k+1}^{-m'_{k+1}} \dots y_N^{-m'_N} s'$$

for some nonzero elements  $s, s' \in R$ . Then

$$(10.4) \quad y_1^{m_1} \dots y_N^{m_N} s' = y_1^{m'_1} \dots y_{k-1}^{m'_{k-1}} (y'_k)^{m'_k} y_{k+1}^{m'_{k+1}} \dots y_N^{m'_N} s.$$

If  $j \in [1, N] \setminus \{k\}$  and  $m_j > 0$ , then  $y_j \nmid s$ , so it follows from Lemma 10.1 and the primeness of  $y_j$  that  $m_j \leq m'_j$ . The reverse inequality holds by symmetry, whence  $m_j = m'_j$ .

Equation (10.4) now reduces to  $y_k^{m_k} s' = (y'_k)^{m'_k} s$ . Since  $y_k \nmid s$  if  $m_k > 0$ , and  $y_k \nmid y'_k$  (by Lemma 10.1), we must have  $m_k = 0$ , and similarly  $m'_k = 0$ .

The final statement of the theorem follows.  $\square$

## 10.2. Permutations $\tau_\bullet$ .

If  $R$  is a symmetric P-CGL extension, then for each  $\tau \in \Xi_N$  we have a P-CGL extension presentation (6.5) for  $R$  and corresponding  $y$ -elements by applying Theorem 5.5 to this presentation. We label these  $y$ -elements  $y_{\tau,1}, \dots, y_{\tau,N}$ ; they will become a set of cluster and frozen variables in our main theorem. In order to connect these various sets of variables via mutations, we need to permute the order of each set of variables. This is done by applying a composition of  $\tau$  and a permutation  $\tau_\bullet \in \prod_{a \in \mathbb{Z}} S_{\eta^{-1}(a)}$ , where  $\eta: [1, N] \rightarrow \mathbb{Z}$  is a function satisfying the

conditions of Theorem 5.5 for the original P-CGL extension presentation of  $R$ . Note that all terms in the above product are trivial except for the terms coming from the range of  $\eta$ . For  $a$  in the range of  $\eta$ , denote for brevity  $|a| := |\eta^{-1}(a)|$ . Write the set  $\eta^{-1}(a)$  in the form

$$(10.5) \quad \eta^{-1}(a) = \{\tau(k_1), \dots, \tau(k_{|a|}) \mid k_1 < \dots < k_{|a|}\}$$

and order its elements in ascending order. Define  $\tau_\bullet \in S_N$  by setting  $\tau_\bullet(\tau(k_i))$  to be equal to the  $i$ -th element in the list (for all choices of  $a$  and  $i$ ). In other words,

$$(\tau_\bullet) \text{ For } l \in [1, N], \text{ let } \eta^{-1}\eta(l) = \{t_1 < \dots < t_m\} \text{ and} \\ i := |\tau^{-1}\eta^{-1}\eta(l) \cap [1, \tau^{-1}(l)]|.$$

Then  $\tau_\bullet(l) := t_i$ .

Thus, for each level set  $L$  of  $\eta$ , the permutation  $\tau_\bullet\tau$  gives an order-preserving bijection of  $\tau^{-1}(L)$  onto  $L$ .

The point of applying  $\tau$  is to match the indexing of the  $y$ -elements with that of the  $x$ -elements in (6.5). (Note that the order of the  $x$ -elements in (6.5) is  $x_{\tau(1)}, \dots, x_{\tau(N)}$ .) The application of  $\tau_\bullet$  then rearranges the  $\eta$ -preimages  $\tau(k_1), \dots, \tau(k_{|a|})$  from (10.5) in ascending order. This is needed because in the setting of Theorem 7.2(b) the element  $y_k$  (not  $y_{k+1}$ ) gets mutated. Clearly,  $\tau_\bullet$  preserves the level sets of  $\eta$ .

Recall that  $P(N) = \{k \in [1, N] \mid s(k) = +\infty\}$  parametrizes the set of homogeneous Poisson-prime elements of  $R$ , i.e.,

(10.6)  $\{y_k \mid k \in P(N)\}$  is a list of the homogeneous Poisson-prime elements of  $R$  up to associates. Define

$$(10.7) \quad \mathbf{ex} := [1, N] \setminus P(N) = \{k \in [1, N] \mid s(k) \neq +\infty\}.$$

Since  $|P(N)| = \text{rk}(R)$ , the cardinality of this set is  $|\mathbf{ex}| = N - \text{rk}(R)$ .

Some properties of the permutations  $\tau_\bullet\tau$  are given in the following lemma [29, Lemma 8.16].

LEMMA 10.4. (a) For any  $\tau \in \Xi_N$ , the permutation  $(\tau_\bullet\tau)^{-1}$  maps  $\mathbf{ex}$  bijectively onto the set  $\{k \in [1, N] \mid s_\tau(k) \neq +\infty\}$ .

(b) Suppose  $\tau, \tau' \in \Xi_N$  and  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$ . If  $\eta\tau(k) = \eta\tau(k+1)$ , then  $\tau'_\bullet\tau' = \tau_\bullet\tau$ .

(c) Suppose  $\tau, \tau' \in \Xi_N$  and  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$ . If  $\eta\tau(k) \neq \eta\tau(k+1)$ , then  $\tau'_\bullet\tau' = \tau_\bullet\tau(k, k+1)$ .

(d) Let  $j \in \mathbf{ex}$ . Then there exist  $\tau, \tau' \in \Gamma_N$  such that  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$  with  $\eta\tau(k) = \eta\tau(k+1)$  and  $\tau_\bullet\tau(k) = j$ .

### 10.3. Intersections of localizations

Throughout this section, we will assume that  $R$  is a symmetric P-CGL extension of rank  $N$  as in Definition 6.1, and we fix a function  $\eta : [1, N] \rightarrow \mathbb{Z}$  satisfying the conditions of Theorem 5.5. For each  $\tau \in \Xi_N$ , there is a P-CGL presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}]$$

as in (6.5). Let  $p_\tau$  and  $s_\tau$  denote the predecessor and successor functions for the level sets of  $\eta_\tau := \eta\tau$ , which by Corollary 8.6(b) can be chosen as the  $\eta$ -function

for the presentation (6.5). Let  $y_{\tau,1}, \dots, y_{\tau,N}$  be the corresponding sequence of homogeneous Poisson-prime elements of  $R$  from Theorem 5.5, and denote

$$\begin{aligned}\mathcal{A}_\tau &:= \text{the } \mathbb{K}\text{-subalgebra of } R \text{ generated by } \{y_{\tau,k} \mid k \in [1, N]\} \\ \mathcal{T}_\tau &:= \text{the } \mathbb{K}\text{-subalgebra of } \text{Fract}(R) \text{ generated by } \{y_{\tau,k}^{\pm 1} \mid k \in [1, N]\} \\ E_\tau &:= \text{the multiplicative subset of } \mathcal{A}_\tau \text{ generated by} \\ &\quad \mathbb{K}^* \sqcup \{y_{\tau,k} \mid k \in [1, N], s_\tau(k) \neq +\infty\}.\end{aligned}$$

By Proposition 5.8,  $\mathcal{T}_\tau$  is a Poisson torus with corresponding Poisson affine space algebra  $\mathcal{A}_\tau$ .

Recall the subset  $\Gamma_N$  of  $\Xi_N$  and its total ordering from (8.7).

**THEOREM 10.5.** *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$ .*

- (a)  $\mathcal{A}_\tau \subseteq R \subseteq \mathcal{A}_\tau[E_\tau^{-1}] \subseteq \mathcal{T}_\tau \subseteq \text{Fract}(R)$ , for all  $\tau \in \Xi_N$ .
- (b)  $R$  is generated as a  $\mathbb{K}$ -algebra by  $\{y_{\tau,k} \mid \tau \in \Gamma_N, k \in [1, N]\}$ .
- (c)  $R[E_\tau^{-1}] = \mathcal{A}_\tau[E_\tau^{-1}]$ , for all  $\tau \in \Xi_N$ .
- (d)  $R = \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}] = \bigcap_{\tau \in \Gamma_N} \mathcal{A}_\tau[E_\tau^{-1}]$ .
- (e) Let  $\mathbf{inv}$  be any subset of  $\{k \in [1, N] \mid s(k) = +\infty\}$ . Then

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}][y_k^{-1} \mid k \in \mathbf{inv}].$$

**PROOF.** (a) The first, third, and fourth inclusions are clear. For the second, it suffices to show that  $x_{\tau(k)} \in \mathcal{A}_\tau[E_\tau^{-1}]$  for all  $k \in [1, N]$ , which we do by induction on  $k$ . The case  $k = 1$  is immediate from the fact that  $x_{\tau(1)} = y_{\tau,1}$ .

Now let  $k \in [2, N]$ . If  $p_\tau(k) = -\infty$ , then  $x_{\tau(k)} = y_{\tau,k} \in \mathcal{A}_\tau[E_\tau^{-1}]$ . If  $p_\tau(k) = l \neq -\infty$ , then  $l < k$  and  $y_{\tau,k} = y_{\tau,l}x_{\tau(k)} - c_{\tau,k}$  for some element  $c_{\tau,k}$  in the  $\mathbb{K}$ -subalgebra of  $R$  generated by  $x_{\tau(1)}, \dots, x_{\tau(k-1)}$ . By induction,  $c_{\tau,k} \in \mathcal{A}_\tau[E_\tau^{-1}]$ . Further,  $y_{\tau,l} \in E_\tau$  because  $s_\tau(l) = k \neq +\infty$ , and thus

$$x_{\tau(k)} = y_{\tau,l}^{-1}(y_{\tau,k} + c_{\tau,k}) \in \mathcal{A}_\tau[E_\tau^{-1}]$$

in this case also.

(b) For each  $j \in [1, N]$ , we have  $\tau_{j,j} \in \Gamma_N$  with  $\tau_{j,j}(1) = j$ , and so  $y_{\tau_{j,j},1} = x_j$ . Thus, in fact,  $R$  is generated by  $\{y_{\tau,1} \mid \tau \in \Gamma_N\}$ .

(c) This is clear from part (a).

(d) The second equality follows from part (c), and one inclusion of the first equality is obvious.

Let  $v \in \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}]$  be a nonzero element. For each  $\tau \in \Gamma_N$ , let

$$\prod_{l \in [1, N]} y_{\tau,l}^{m_{\tau,l}}$$

be a minimal denominator of  $v$  with respect to the localization  $R[E_\tau^{-1}]$ , where all  $m_{\tau,l} \in \mathbb{Z}_{\geq 0}$  and  $m_{\tau,l} = 0$  when  $s_\tau(l) = +\infty$ . We first verify the following

**Claim.** Let  $\tau, \tau' \in \Gamma_N$  such that  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$ .

- (1)  $m_{\tau,(\tau_\bullet \tau)^{-1}(j)} = m_{\tau',(\tau'_\bullet \tau')^{-1}(j)}$ , for all  $j \in [1, N]$ .
- (2) If  $\eta\tau(k) = \eta\tau(k+1)$ , then  $m_{\tau,k} = 0$ .

If  $\eta\tau(k) = \eta\tau(k+1)$ , then (1) and (2) follow from Lemma 10.4(b), Theorem 10.3, and Theorem 7.2(b). If  $\eta\tau(k) \neq \eta\tau(k+1)$ , we obtain from Theorem 7.2(a) that  $y_{\tau',k} = y_{\tau,k+1}$  and  $y_{\tau',k+1} = y_{\tau,k}$ , while  $y_{\tau',l} = y_{\tau,l}$  for all  $l \neq k, k+1$ . As

a result,  $E_{\tau'} = E_{\tau}$  and we see that  $m_{\tau',k} = m_{\tau,k+1}$  and  $m_{\tau',k+1} = m_{\tau,k}$ , while  $m_{\tau',l} = m_{\tau,l}$  for all  $l \neq k, k+1$ . In this case, (1) follows from Lemma 10.4(c).

Since all the permutations in  $\Gamma_N$  appear in the chain (8.7), part (1) of the claim implies that

$$(10.8) \quad m_{\sigma,(\sigma \bullet \sigma)^{-1}(j)} = m_{\tau,(\tau \bullet \tau)^{-1}(j)}, \quad \forall \sigma, \tau \in \Gamma_N, \quad j \in [1, N].$$

For any  $j \in \mathbf{ex}$ , Lemma 10.4(d) shows that there exist  $\tau, \tau' \in \Gamma_N$  such that  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$  with  $\eta\tau(k) = \eta\tau(k+1)$  and  $\tau \bullet \tau(k) = j$ . Part (2) of the claim above then implies that  $m_{\tau,(\tau \bullet \tau)^{-1}(j)} = 0$ . From (10.8), we thus get

$$m_{\sigma,(\sigma \bullet \sigma)^{-1}(j)} = 0, \quad \forall \sigma \in \Gamma_N, \quad j \in \mathbf{ex}.$$

In particular,  $m_{\text{id},j} = 0$  for all  $j \in \mathbf{ex}$ , whence  $m_{\text{id},j} = 0$  for all  $j \in [1, N]$  and therefore  $v \in R$ , which completes the proof of part (d).

(e) Let  $v \in \bigcap_{\tau \in \Gamma_N} R[E_{\tau}^{-1}][y_k^{-1} \mid k \in \mathbf{inv}]$ . For each  $\tau \in \Gamma_N$ , we can write  $v$  as a fraction with numerator from  $R[E_{\tau}^{-1}]$  and denominator from the multiplicative set  $Y$  generated by  $\{y_k \mid k \in \mathbf{inv}\}$ . Hence, choosing a common denominator, we obtain  $y \in Y$  such that  $vy \in R[E_{\tau}^{-1}]$  for all  $\tau \in \Gamma_N$ . From part (d), we conclude that  $vy \in R$ , and thus  $v \in R[y_k^{-1} \mid k \in \mathbf{inv}]$ , as required.  $\square$



## Symmetric Poisson nilpotent algebras and cluster algebras

In this chapter we prove that every symmetric Poisson nilpotent algebra, i.e., Poisson-CGL extension, possesses a cluster algebra structure under a mild additional assumption on the scalars  $\lambda_k^*$ . The cluster algebra structure is constructed in an explicit fashion. We furthermore prove that each of these cluster algebras equals the corresponding upper cluster algebra.

Concerning notation: We continue to use  $x_1, \dots, x_N$  to denote polynomial variables in P-CGL extensions, and our cluster and frozen variables will be denoted  $y_\iota$  for various indices  $\iota$ .

### 11.1. Statement of the main result

Fix a symmetric P-CGL extension  $R$  of length  $N$  and rank  $\text{rk}(R)$  as in §6.1. Recall the map  $Y : \mathbb{Z}^N \rightarrow \text{Fract}(R)$  from (7.16), and note that  $Y$  can be viewed as a group homomorphism from  $\mathbb{Z}^N$  to  $\text{Fract}(R)^*$ . Set

$$(11.1) \quad \tilde{\mathbf{y}} := (Y(e_1), \dots, Y(e_N)) = (y_1, \dots, y_N) \in \text{Fract}(R)^N,$$

and recall from Proposition 5.8 that  $\tilde{\mathbf{y}}$  is a sequence of algebraically independent elements generating  $\text{Fract}(R)$  over  $\mathbb{K}$ . Recall also the skew-symmetric matrices  $\boldsymbol{\lambda}, \mathbf{q} \in M_N(\mathbb{K})$  from Definition 5.1 and (5.18), and the corresponding skew-symmetric bicharacters  $\Omega_{\boldsymbol{\lambda}}, \Omega_{\mathbf{q}}$  from Definition 5.1 and (7.19).

Consider an arbitrary element  $\tau \in \Xi_N \subset S_N$ , recall (6.4), and the associated auxiliary permutation  $\tau_\bullet$  from §10.2. By Proposition 6.4, we have the P-CGL extension presentation

$$(11.2) \quad R = \mathbb{K}[x_{\tau(1)}]_p[x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}]_p \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}]_p$$

with  $\sigma''_{\tau(k)} = (h''_{\tau(k)} \cdot)$  where  $h''_{\tau(k)} \in \mathcal{H}$ ,  $\forall k \in [1, N]$  and

$$\begin{aligned} h''_{\tau(k)} &= h_{\tau(k)}, \quad \delta''_{\tau(k)} = \delta_{\tau(k)}, & \text{if } \tau(k) = \max \tau([1, k-1]) + 1 \\ h''_{\tau(k)} &= h_{\tau(k)}^*, \quad \delta''_{\tau(k)} = \delta_{\tau(k)}^*, & \text{if } \tau(k) = \min \tau([1, k-1]) - 1. \end{aligned}$$

The  $\boldsymbol{\lambda}$ -matrix of the presentation (11.2) is  $\boldsymbol{\lambda}_\tau := \tau^{-1} \boldsymbol{\lambda} \tau$  (where as before we use the canonical embedding  $S_N \hookrightarrow GL_N(\mathbb{Z})$  via permutation matrices). In other words, the entries of  $\boldsymbol{\lambda}_\tau$  are given by  $(\boldsymbol{\lambda}_\tau)_{lj} := \lambda_{\tau(l), \tau(j)}$ . Let  $y_{\tau,1}, \dots, y_{\tau,N}$  be the sequence of homogeneous Poisson-prime elements from Theorem 5.5 for the presentation (11.2).

Let  $\mathbf{q}_\tau \in M_N(\mathbb{K})$  be the skew-symmetric matrix with entries

$$(\mathbf{q}_\tau)_{kj} := \sum_{l=0}^{O_-^\tau(k)} \sum_{m=0}^{O_-^\tau(j)} (\boldsymbol{\lambda}_\tau)_{p_\tau^l(k), p_\tau^m(j)}, \quad \forall k, j \in [1, N],$$

where  $O_-^\tau$  denotes the order function corresponding to  $p_\tau$ , and let  $\Omega_{\mathbf{q}_\tau}$  be the skew-symmetric bicharacter on  $\mathbb{Z}^N$  obtained from  $\mathbf{q}_\tau$  as in (1.5). Then  $\Omega_{\mathbf{q}_\tau}$  is the skew-symmetric bicharacter corresponding to  $\Omega_{\mathbf{q}}$  (recall (7.19)) for the presentation (11.2). Set

$$\mathbf{r}_\tau := (\tau_\bullet \tau) \mathbf{q}_\tau (\tau_\bullet \tau)^{-1},$$

and let  $\Omega_{\mathbf{r}_\tau}$  be the skew-symmetric bicharacter on  $\mathbb{Z}^N$  obtained from  $\mathbf{r}_\tau$  as in (1.5).

Finally, define group homomorphisms  $Y_\tau, \tilde{Y}_\tau : \mathbb{Z}^N \rightarrow \text{Fract}(R)^*$  such that

$$(11.3) \quad Y_\tau(e_k) := y_{\tau,k} \text{ and } \tilde{Y}_\tau(e_k) := y_{\tau,(\tau_\bullet \tau)^{-1}(k)}, \quad \forall k \in [1, N],$$

and set

$$(11.4) \quad \tilde{\mathbf{y}}_\tau := (\tilde{Y}_\tau(e_1), \dots, \tilde{Y}_\tau(e_N)) = (y_{\tau,(\tau_\bullet \tau)^{-1}(1)}, \dots, y_{\tau,(\tau_\bullet \tau)^{-1}(N)}) \in \text{Fract}(R)^N.$$

It follows from Theorem 10.5(a) that  $\tilde{\mathbf{y}}_\tau$  is a sequence of algebraically independent elements generating  $\text{Fract}(R)$  over  $\mathbb{K}$ . Since  $\text{id}_\bullet = \text{id}$ , we have  $\tilde{Y}_{\text{id}} = Y$  and  $\tilde{\mathbf{y}}_{\text{id}} = \tilde{\mathbf{y}}$ , as well as  $\mathbf{r}_{\text{id}} = \mathbf{q}_{\text{id}} = \mathbf{q}$ .

For a subset  $X \subseteq \mathbf{ex}$ , by an  $N \times X$  matrix we will mean a matrix of size  $N \times |X|$  whose columns are indexed by the set  $X$ . The set of such matrices with integral entries will be denoted by  $M_{N \times X}(\mathbb{Z})$ .

The next theorem contains the main result of the paper.

**THEOREM 11.1.** *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$  and rank  $\text{rk}(R)$  as in Definition 6.1. Define  $\mathbf{ex} \subset [1, N]$  by (10.7). Assume also that there exist positive integers  $d_i$ ,  $i \in \text{range}(\eta)$  such that*

$$(11.5) \quad d_{\eta(j)} \lambda_l^* = d_{\eta(l)} \lambda_j^*, \quad \forall j, l \in \mathbf{ex},$$

*recall the equality (8.11). Let the sequence of generators  $x_1, \dots, x_N$  of  $R$  be normalized (rescaled) so that (9.5) is satisfied (recall Proposition 9.2).*

*Then the following hold:*

(a) *For all  $\tau \in \Xi_N$  (see (6.4)) and  $l \in \mathbf{ex}$ , there exists a unique vector  $b_\tau^l \in \mathbb{Z}^N$  such that  $\chi_{\tilde{Y}_\tau(b_\tau^l)} = 0$  and*

$$(11.6) \quad \Omega_{\mathbf{r}_\tau}(b_\tau^l, e_j) = 0, \quad \forall j \in [1, N], \quad j \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_\tau}(b_\tau^l, e_l) = \lambda_l^*.$$

*Denote by  $\tilde{B}_\tau \in M_{N \times \mathbf{ex}}(\mathbb{Z})$  the matrix with columns  $b_\tau^l$ ,  $l \in \mathbf{ex}$ . Let  $\tilde{B} := \tilde{B}_{\text{id}}$ .*

(b) *For all  $\tau \in \Xi_N$ , the pair  $(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$  is a seed for  $\text{Fract}(R)$  and  $(\mathbf{r}_\tau, \tilde{B}_\tau)$  is a compatible pair. The principal part of  $\tilde{B}_\tau$  is skew-symmetrizable via the integers  $d_{\eta(k)}$ ,  $k \in \mathbf{ex}$ .*

(c) *All seeds as in (b) are mutation-equivalent to each other. More precisely, they are linked by the following one-step mutations. Let  $\tau, \tau' \in \Xi_N$  be such that*

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

*for some  $k \in [1, N-1]$ . If  $\eta(\tau(k)) \neq \eta(\tau(k+1))$ , then  $(\tilde{\mathbf{y}}_{\tau'}, \tilde{B}_{\tau'}) = (\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$ . If  $\eta(\tau(k)) = \eta(\tau(k+1))$ , then  $(\tilde{\mathbf{y}}_{\tau'}, \tilde{B}_{\tau'}) = \mu_{k_\bullet}(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$ , where  $k_\bullet = \tau_\bullet \tau(k)$ .*

(d) *We have the following equality between the P-CGL extension  $R$  and the cluster and upper cluster algebras associated to  $(\tilde{\mathbf{y}}, \tilde{B})$ :*

$$R = \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}} = \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \emptyset)_{\mathbb{K}}.$$

*In particular,  $\mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}}$  is affine and noetherian, and more precisely  $\mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}}$  is generated by the cluster and frozen variables in the seeds parametrized by the subset  $\Gamma_N$  of  $\Xi_N$ , recall (8.7).*

(e) The cluster algebra structure on  $R = \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}}$  is compatible with the given Poisson structure on  $R$ . Namely, the  $N$ -tuple of variables in any seed of  $\text{Fract}(R)$  mutation-equivalent to  $(\tilde{\mathbf{x}}, \tilde{B})$  is log-canonical. In particular, for each  $\tau \in \Xi_N$ , the variables  $\tilde{Y}_\tau(e_i)$  in the seed  $(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$  satisfy

$$\{\tilde{Y}_\tau(e_l), \tilde{Y}_\tau(e_j)\} = \Omega_{\mathbf{r}_\tau}(e_l, e_j) \tilde{Y}_\tau(e_l) \tilde{Y}_\tau(e_j), \quad \forall l, j \in [1, N].$$

(f) Let  $\mathbf{inv}$  be any subset of the set  $P(N)$  of frozen indices, cf. (10.6). Then

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} = \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}.$$

Moreover, this cluster algebra structure is compatible with the given Poisson structure on  $R[y_k^{-1} \mid k \in \mathbf{inv}]$ .

Theorem 11.1 is proved in §11.5–11.7. The strategy of the proof is summarized in §11.4. In §11.2 we derive an explicit formula for the cluster variables of the seeds that appear in the statement of Theorem 11.1.

EXAMPLE 11.2. Let  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$  with the Poisson-CGL extension presentation as in Example 5.3. Define  $\eta$  for  $R$  as in Example 5.6 and  $\mathbf{ex}$  as in (10.7). Then

$$\mathbf{ex} = \{(r-1)n + c \mid r \in [1, m-1], c \in [1, n-1]\}.$$

As shown in Example 6.2, the P-CGL extension presentation of  $R$  is symmetric, and all  $\lambda_j^* = 2$ . Hence, (11.5) is trivially satisfied with  $d_i := 1$  for all  $i \in \text{range}(\eta)$ . Finally, Example 9.3 shows that condition (9.5) holds.

Therefore all hypotheses of Theorem 11.1 are satisfied, and the theorem yields a cluster algebra structure on  $R$  compatible with the given Poisson structure. The initial mutation matrix  $\tilde{B} = (b_{ij})$  for  $R$  is easily computed:

$$b_{(r-1)n+c, (r'-1)n+c'} = \begin{cases} \pm 1, & \text{if } r = r', c' = c \pm 1 \\ & \text{or } c = c', r' = r \pm 1 \\ & \text{or } r = r' \pm 1, c = c' \pm 1, \\ 0, & \text{otherwise} \end{cases} \quad \forall r, r' \in [1, m], c, c' \in [1, n].$$

From Theorem 11.1(f), we also get Poisson-compatible cluster algebra structures on coordinate rings of various open subvarieties of  $M_{m,n}$  defined by non-vanishing of certain minors. For instance, assume  $m \leq n$ , choose

$$\mathbf{inv} = \{rn \mid r \in [1, m]\} \sqcup \{(m-1)n + c \mid c \in [1, m]\},$$

and let  $R' := R[y_k^{-1} \mid k \in \mathbf{inv}]$ . From Example 5.6,

$$y_{rn} = \Delta_{[1,r],[n+1-r,n]} \quad (r \in [1, m]), \quad y_{(m-1)n+c} = \Delta_{[m+1-c,m],[1,c]} \quad (c \in [1, m]).$$

It follows that  $R'$  is the coordinate ring of the generalized double Bruhat cell

$$B_m^+ w_1 B_n^+ \cap B_m^- w_2 B_n^-$$

in  $M_{m,n}$ , where  $B_k^\pm$  are the standard Borel subgroups of  $GL_k(\mathbb{K})$  and  $w_1, w_2$  are the partial permutation matrices

$$w_1 := \begin{bmatrix} w_\circ^m & 0_{m,n-m} \end{bmatrix}, \quad w_2 := \begin{bmatrix} 0_{m,n-m} & w_\circ^m \end{bmatrix}$$

with  $w_\circ^m$  the (matrix of the) longest element of  $S_m$ . To see that

$$\begin{aligned} B_m^+ w_1 B_n^+ \cap B_m^- w_2 B_n^- &= \{p \in M_{m,n} \mid \Delta_{[1,r],[n+1-r,n]}(p) \neq 0, \forall r \in [1, m] \\ &\quad \text{and } \Delta_{[m+1-c,m],[1,c]}(p) \neq 0, \forall c \in [1, m]\}, \end{aligned}$$

one can use [6, Proposition 4.1].

REMARK 11.3. The property (11.5) requires that for any particular  $j_0 \in \mathbf{ex}$ , all the scalars  $\lambda_l^*$  for  $l \in \mathbf{ex}$  are positive integer multiples of  $\lambda_{j_0}^*/d_{\eta(j_0)}$ . Conversely, if there is some  $\lambda_0^* \in \mathbb{K}^*$  such that  $\lambda_l^* \in \mathbb{Z}_{>0} \lambda_0^*$  for all  $l \in \mathbf{ex}$ , then (11.5) is satisfied by choosing  $d_{\eta(l)} := \lambda_l^*/\lambda_0^*$  for  $l \in \mathbf{ex}$ , which we can do because  $\lambda_k^* = \lambda_l^*$  for all  $k, l \in \mathbf{ex}$  with  $\eta(k) = \eta(l)$  (Proposition 8.8). This holds, of course, if all  $\lambda_l^* \in \mathbb{Q}_{>0}$ , which is the case for many P-CGL extensions, such as semiclassical limits of standard uniparameter quantum algebras.

Examples in which non-integer scalars appear among the  $\lambda_k^*$  include semiclassical limits of multiparameter quantum symplectic and euclidean spaces (e.g., see [25, §§2.4, 2.5]). For instance, there are symmetric P-CGL extensions (from [25, §2.4], rewritten in symmetric form) in which  $N = 2n$  and  $\mathbf{ex} = [1, n-1]$  for an integer  $n \geq 2$ , and  $\lambda_1^*, \dots, \lambda_{n-1}^*$  are arbitrary nonzero scalars. When  $n \geq 3$  and  $\lambda_2^*/\lambda_1^* \notin \mathbb{Q}_{>0}$ , property (11.5) fails.

REMARK 11.4. For applications, it is useful to determine the exchange matrix  $\tilde{B}$  before the generators  $x_k$  have been rescaled to satisfy (9.5). This is possible because the rescaling does not change  $\tilde{B}$ , as we next note.

Assume  $R$  is a symmetric P-CGL extension satisfying all the hypotheses of Theorem 11.1 *except* (9.5). Let the elements  $y_k$  and the map  $Y = \tilde{Y}_{\text{id}}$  be as above. Now suppose that we rescale the  $x_k$  according to Proposition 9.2, say with new generators  $x'_1, \dots, x'_N$ , to make (9.5) hold. Build new  $y$ -elements from Theorem 5.5 with respect to these generators, and call them  $y'_1, \dots, y'_N$ . Each  $y'_k$  is a scalar multiple of  $y_k$ . Since the scalars  $\lambda_{kj}$  and  $q_{kj}$  do not change, neither do the bicharacters  $\Omega_{\lambda}$  and  $\Omega_{\mathbf{q}}$ . Define  $Y'$  analogously to  $Y$ , using the  $y'_k$  in place of the  $y_k$ . For any vector  $b \in \mathbb{Z}^N$ , the element  $Y'(b)$  is a scalar multiple of  $Y(b)$ , and thus  $\chi_{Y'(b)} = \chi_{Y(b)}$ . Finally, the same elements  $h_k, h_k^* \in \mathfrak{h}$  which enter into the symmetric P-CGL conditions for the original generators are used with respect to the new generators, which means that the scalars  $\lambda_k$  and  $\lambda_k^*$  do not change under the rescaling.

Thus, the conditions in Theorem 11.1(a) which uniquely determine the columns of  $\tilde{B} = \tilde{B}_{\text{id}}$  are the same before and after rescaling. Therefore  $\tilde{B}$  does not change under the rescaling.

## 11.2. Cluster and frozen variables

The next result gives an explicit formula for the cluster and frozen variables that appear in Theorem 11.1.

PROPOSITION 11.5. *Assume the setting of Theorem 11.1.*

(a) *For  $\tau \in \Xi_N$  and  $k \in [1, N]$ ,*

$$\tilde{Y}_{\tau}(e_k) = \begin{cases} y_{[p^{O-(k)}(\tau_{\bullet}^{-1}(k)), \tau_{\bullet}^{-1}(k)]} & (\text{if } \tau_{\bullet}^{-1}(k) \geq \tau(1)) \\ y_{[\tau_{\bullet}^{-1}(k), s^{O+(k)}(\tau_{\bullet}^{-1}(k))]} & (\text{if } \tau_{\bullet}^{-1}(k) \leq \tau(1)), \end{cases}$$

*where the predecessor and successor functions are computed with respect to the original P-CGL extension presentation (5.1) of  $R$ .*

(b) *The cluster and frozen variables appearing in the tuples  $\tilde{\mathbf{y}}_{\tau}$  for  $\tau \in \Xi_N$  or in  $\Gamma_N$  are exactly the homogeneous Poisson-prime elements  $y_{[i,j]}$ . More precisely,*

$$\begin{aligned} \{\tilde{Y}_{\tau}(e_k) \mid \tau \in \Xi_N, k \in [1, N]\} &= \{\tilde{Y}_{\tau}(e_k) \mid \tau \in \Gamma_N, k \in [1, N]\} \\ &= \{y_{[i,j]} \mid 1 \leq i \leq j \leq N, \eta(i) = \eta(j)\}. \end{aligned}$$

REMARK 11.6. Theorem 11.1(d) and Proposition 11.5 imply that the cluster algebra  $R = \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}}$  coincides with the subalgebra of  $\text{Fract}(R)$  generated by the cluster and frozen variables from the (finite) set of seeds  $\{(\tilde{\mathbf{y}}_{\tau}, \tilde{B}_{\tau}) \mid \tau \in \Gamma_N\}$ .

PROOF OF PROPOSITION 11.5. (a) Recall that  $\tilde{Y}_{\tau}(e_k) = y_{\tau, (\tau_{\bullet}\tau)^{-1}(k)}$ . Set  $m := O_{-}(k)$  and  $m_{+} := O_{+}(k)$ , and let  $L := \eta^{-1}(\eta(k))$  be the level set of  $\eta$  containing  $k$ . Then

$$L = \{p^m(k) < p^{m-1}(k) < \cdots < k < s(k) < \cdots < s^{m+}(k)\}.$$

Now set  $k' := (\tau_{\bullet}\tau)^{-1}(k)$ ; then  $\tau(k') = \tau_{\bullet}^{-1}(k)$ . Since  $(\tau_{\bullet}\tau)^{-1}$  restricts to an order-preserving bijection of  $L$  onto the level set  $\tau^{-1}(L)$  of  $\eta_{\tau}$ , we see that

$$\tau^{-1}(L) = \{p_{\tau}^m(k') < \cdots < k' < s_{\tau}(k') < \cdots < s_{\tau}^{m+}(k')\}.$$

Consequently,

$$(11.7) \quad m+1 = |\{j' \in [1, k'] \mid \eta_{\tau}(j') = \eta_{\tau}(k')\}| = |\{j \in \tau([1, k']) \mid \eta(j) = \eta(\tau(k'))\}|.$$

If  $\tau(k') \geq \tau(1)$ , then  $\tau([1, k']) = [\tau(i'), \tau(k')]$  for some  $i' \in [1, k']$ , and it follows from (11.7) that  $m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k')) \in \tau([1, k'])\}$ . Theorem 8.3 then shows that

$$\tilde{Y}_{\tau}(e_k) = y_{\tau, k'} = y_{[p^m(\tau(k')), \tau(k')]} = y_{[p^{O_{-}(k)}(\tau_{\bullet}^{-1}(k)), \tau_{\bullet}^{-1}(k)]}$$

in this case. The case when  $\tau(k') \leq \tau(1)$  follows similarly.

(b) The first two of the displayed sets are contained in the third by part (a), and the second is contained in the first a priori. It remains to show that the third is contained in the second.

Thus, let  $1 \leq i \leq j \leq N$  with  $\eta(i) = \eta(j)$ . Then  $i = p^m(j)$  where

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(j) \in [i, j]\}.$$

Set  $k' := j - i + 1$ , and choose  $\tau \in \Gamma_N$  as follows:

$$\tau = \begin{cases} \tau_{1,1} = \text{id} & (\text{if } i = 1) \\ \tau_{i-1,j} = [i, \dots, j, i-1, j+1, \dots, N, i-1, \dots, 1] & (\text{if } i > 1). \end{cases}$$

Then  $\tau(1) = i \leq j = \tau(k')$  and  $\tau([1, k']) = [i, j]$ . Consequently,

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k')) \in \tau([1, k'])\},$$

and Theorem 8.3 shows that

$$y_{\tau, k'} = y_{[p^m(\tau(k')), \tau(k')]} = y_{[i, j]}.$$

Therefore  $y_{[i, j]} = y_{\tau, (\tau_{\bullet}\tau)^{-1}(k)} = \tilde{Y}_{\tau}(e_k)$  where  $k := \tau_{\bullet}\tau(k')$ .  $\square$

EXAMPLE 11.7. Let  $R = \mathcal{O}(M_{m,n}(\mathbb{K}))$  have the Poisson-compatible cluster algebra structure from Theorem 11.1 as in Example 11.2. Proposition 11.5 and Example 8.2 show that the cluster and frozen variables appearing in Theorem 11.1 are exactly the solid minors within  $[1, m] \times [1, n]$ . For if  $l \in \mathbb{Z}_{\geq 0}$  and  $r, r+l \in [1, m]$ ,  $c, c+l \in [1, n]$ , then

$$\Delta_{[r, r+l], [c, c+l]} = y_{[i, s^l(i)]}, \text{ where } i = (r-1)n + c$$

(recall Example 8.2), and the  $y_{[i, s^l(i)]}$  are exactly the cluster and frozen variables  $\tilde{Y}_{\tau}(e_k)$  for  $\tau \in \Xi_N$  and  $k \in [1, N]$ .

### 11.3. Auxiliary results

In this section we establish two results that will be needed for the proof of Theorem 11.1. The first one uses Theorems 9.5 and 8.3 to construct mutations between pairs of the tuples  $\tilde{\mathbf{y}}_\tau$  for  $\tau \in \Xi_N$ . The corresponding mutations of seeds (Theorem 11.1(c)) are constructed in §11.5–11.6.

For  $g = \sum_j g_j e_j \in \mathbb{Z}^N$  set

$$\text{supp}(g) := \{j \in [1, N] \mid g_j \neq 0\}.$$

**PROPOSITION 11.8.** *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$ . Assume that the generators of  $R$  are rescaled so that the condition (9.5) is satisfied. Let  $\tau, \tau' \in \Xi_N$  be such that*

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

for some  $k \in [1, N-1]$  such that  $\tau(k) < \tau(k+1)$ .

(a) If  $\eta(\tau(k)) \neq \eta(\tau(k+1))$ , then  $\mathbf{r}_{\tau'} = \mathbf{r}_\tau$  and  $\tilde{Y}_{\tau'} = \tilde{Y}_\tau$ .

(b) Let  $\eta(\tau(k)) = \eta(\tau(k+1))$ , and set  $k_\bullet := \tau_\bullet \tau(k)$ . Then  $k_\bullet = \tau'_\bullet \tau'(k)$  and

$$(11.8) \quad \tilde{Y}_{\tau'}(e_j) = \begin{cases} \tilde{Y}_\tau(e_j), & \text{if } j \neq k_\bullet \\ \tilde{Y}_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - e_{k_\bullet}) + \tilde{Y}_\tau(g - e_{k_\bullet}), & \text{if } j = k_\bullet \end{cases}$$

for some  $g \in \mathbb{Z}_{\geq 0}^N$  such that  $\text{supp}(g) \cap \eta^{-1}\eta(k_\bullet) = \emptyset$  and  $|\text{supp}(g) \cap \eta^{-1}(a)| \leq 1$  for all  $a \in \mathbb{Z}$ . Furthermore, the vector  $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g \in \mathbb{Z}^N$  satisfies the identities

$$(11.9) \quad \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_j) = 0, \quad \forall j \neq k_\bullet,$$

$$(11.10) \quad \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_{k_\bullet}) = \lambda_{k_\bullet}^*,$$

and

$$(11.11) \quad \chi_{\tilde{Y}_{\tau'}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g)} = 0.$$

Note that the condition  $\tau(k) < \tau(k+1)$  is not essential since, if  $\tau(k) > \tau(k+1)$  and all other conditions are satisfied, then one can interchange the roles of  $\tau$  and  $\tau'$ .

**PROOF.** (a) That  $\tilde{Y}_{\tau'} = \tilde{Y}_\tau$  follows from Theorem 7.2(a) applied to the P-CGL extension presentation (11.2) of  $R$  associated to  $\tau$ , using that  $\tau'_\bullet \tau' = \tau_\bullet \tau(k, k+1)$  (Lemma 10.4(c)). Now

$$\lambda_{\tau'} = (\tau')^{-1} \lambda_{\tau'} = (k, k+1) \tau^{-1} \lambda_\tau(k, k+1) = (k, k+1) \lambda_\tau(k, k+1),$$

from which it follows that  $\mathbf{q}_{\tau'} = (k, k+1) \mathbf{q}_\tau(k, k+1)$ . Using  $\tau'_\bullet \tau' = \tau_\bullet \tau(k, k+1)$  again, we conclude that

$$\begin{aligned} \mathbf{r}_{\tau'} &= (\tau'_\bullet \tau') \mathbf{q}_{\tau'} (\tau'_\bullet \tau')^{-1} = (\tau'_\bullet \tau')(k, k+1) \mathbf{q}_\tau(k, k+1) (\tau'_\bullet \tau')^{-1} \\ &= (\tau_\bullet \tau) \mathbf{q}_\tau (\tau_\bullet \tau)^{-1} = \mathbf{r}_\tau. \end{aligned}$$

(b) In this case,  $\tau'_\bullet \tau' = \tau_\bullet \tau$  by Lemma 10.4(b). Since  $\tau([1, j])$  is an interval for all  $j \leq k+1$  and  $\tau(k) < \tau(k+1)$ , we have  $\tau'([1, k+1]) = \tau([1, k+1]) = [\tau(i), \tau(k+1)]$  and  $\tau([1, k]) = [\tau(i), \tau(k+1) - 1]$  for some  $i \in [1, k]$ . On the other hand, the set

$$\tau'([1, k]) = \tau'([1, k+1]) \setminus \{\tau'(k+1)\} = \tau([1, k+1]) \setminus \{\tau(k)\}$$

must be also an interval, so  $\tau(k) = \tau(i)$  and  $i = k$ . Therefore,

$$(11.12) \quad \tau'([1, k+1]) = \tau([1, k+1]) = [\tau(k), \tau(k+1)], \\ \tau([1, k]) = [\tau(k), \tau(k+1) - 1] \quad \text{and} \quad \tau'([1, k]) = [\tau(k) + 1, \tau(k+1)].$$

This implies that

$$\tau(k+1) = s^m(\tau(k)) \text{ for some } m \in \mathbb{Z}_{>0}$$

and

$$(11.13) \quad \eta^{-1}(\eta\tau(k)) \cap \tau([1, k+1]) = \{\tau(k), s(\tau(k)), \dots, s^m(\tau(k)) = \tau(k+1)\}.$$

From the assumption  $\eta(\tau(k)) = \eta(\tau(k+1))$  and the fact that  $\tau(j) = \tau'(j)$  for  $j \neq k, k+1$  we infer

$$\tau^{-1}\eta^{-1}(\eta\tau(k)) = (\tau')^{-1}\eta^{-1}(\eta\tau(k)).$$

By the definition of the permutations  $\tau_\bullet$  and  $\tau'_\bullet$ ,

$$k_\bullet = \tau_\bullet\tau(k) = \tau'_\bullet\tau'(k) = s^{m-1}(\tau(k)).$$

From Theorem 7.4(b) we have  $y_{\tau',l} = y_{\tau,l}$  for  $l \neq k$ . Eq. (11.8) for  $j \neq k_\bullet$  follows from this.

The identities in (11.12) and Theorem 8.3 give

$$(11.14)$$

$$y_{\tau,k} = y_{[\tau(k), s^{m-1}(\tau(k))]}, \quad y_{\tau,k+1} = y_{[\tau(k), s^m(\tau(k))]}, \quad y_{\tau',k} = y_{[s(\tau(k)), s^m(\tau(k))]}.$$

Recall the definition of the maps  $Y_{[i, s^m(i)]}$  from (9.8). We will construct an isomorphism  $\dot{w}$  from  $\mathbb{Z}^{k+1}$  to  $\mathbb{Z}e_{\tau(k)} + \dots + \mathbb{Z}e_{s^m(\tau(k))}$  such that  $Y_{[\tau(k), s^m(\tau(k))]} \dot{w}$  and  $Y_\tau$  agree on suitable  $e_j$ .

Recall the definition of the set  $P_{[\tau(k), s^m(\tau(k))]}$  from (8.20), set

$$A := \eta(P_{[\tau(k), s^m(\tau(k))]})) = \eta(\tau([1, k+1])) \setminus \{\eta(\tau(k))\} \\ Q := \{j \in [1, k-1] \mid \eta(\tau(j)) \neq \eta(\tau(l)), \forall l \in [j+1, k+1]\},$$

and note that  $\eta\tau$  restricts to a bijection of  $Q$  onto  $A$ . Thus,

$$(11.15) \quad |Q| = |A| = |P_{[\tau(k), s^m(\tau(k))]}|.$$

The definition of  $Q$  also ensures that

$$(11.16) \quad \{t \in [\tau(k), s^m(\tau(k))] \mid \eta(t) = \eta(\tau(j))\} \subseteq \tau([1, j]), \quad \forall j \in Q.$$

If  $j \in Q$  and  $\tau(j) \geq \tau(1)$ , then  $\tau([1, j]) = [\tau(i_j), \tau(j)]$  for some  $i_j \in [1, j]$ , and we observe that  $\tau(j) \in P_{[\tau(k), s^m(\tau(k))]}$ . Moreover, the integer  $m_j$  corresponding to  $j$  in Theorem 8.3 equals  $O_-^{\tau(k)+1}(\tau(j))$ , and hence we obtain

$$(11.17) \quad y_{\tau,j} = y_{[p^{m_j}(\tau(j)), \tau(j)]} = Y_{[\tau(k), s^m(\tau(k))]}(e_{\tau(j)}), \quad \forall j \in Q \text{ with } \tau(j) \geq \tau(1).$$

On the other hand, if  $\tau(j) < \tau(1)$ , then  $\tau([1, j]) = [\tau(j), \tau(i_j)]$  for some  $i_j \in [1, j-1]$ . Let  $m_j$  denote the integer corresponding to  $j$  in Theorem 8.3, and observe that  $s^{m_j}(\tau(j)) = \tau(j^-) \in P_{[\tau(k), s^m(\tau(k))]}$  for some  $j^- \in [1, j]$ . Moreover,  $m_j = O_-^{\tau(k)+1}(\tau(j^-))$ , and so

$$(11.18) \quad y_{\tau,j} = y_{[p^{m_j}(\tau(j^-)), \tau(j^-)]} = Y_{[\tau(k), s^m(\tau(k))]}(e_{\tau(j^-)}), \quad \forall j \in Q \text{ with } \tau(j) < \tau(1).$$

Note also that since  $\eta\tau(j^-) = \eta\tau(j)$  and  $\eta\tau|_Q$  is injective,  $\tau(j^-) \neq \tau(i^-)$  for all  $i \in Q \setminus \{j\}$  with  $\tau(i) < \tau(1)$ , while  $\tau(j^-) \neq \tau(i)$  for all  $i \in Q \setminus \{j\}$  with  $\tau(i) \geq \tau(1)$ .

In case  $m > 1$ , we set

$$\begin{aligned} t &:= \max(\tau^{-1}\{s(\tau(k)), \dots, s^{m-1}(\tau(k))\}) \\ &= \max\{j \in [1, k-1] \mid \eta(\tau(j)) = \eta(\tau(k))\} = p_\tau(k) \end{aligned}$$

(see (11.13)). Then either  $\tau(t) = s^{m-1}(\tau(k))$  or  $\tau(t) = s(\tau(k))$ , and Theorem 8.3 yields

$$(11.19) \quad y_{\tau,t} = y_{[s(\tau(k)), s^{m-1}(\tau(k))]} = Y_{[\tau(k), s^m(\tau(k))]}(e_{s^{m-1}(\tau(k))}), \text{ if } m > 1.$$

Now choose a bijection  $w : [1, k+1] \rightarrow [\tau(k), s^m(\tau(k))]$  such that

$$\begin{aligned} w(k+1) &= s^m(\tau(k)) \\ w(k) &= \tau(k) \\ w(t) &= s^{m-1}(\tau(k)), \text{ if } m > 1 \\ w(j) &= \tau(j), \forall j \in Q \text{ with } \tau(j) \geq \tau(1) \\ w(j) &= \tau(j^-), \forall j \in Q \text{ with } \tau(j) < \tau(1), \end{aligned}$$

and let  $\dot{w}$  denote the isomorphism from  $\mathbb{Z}^{k+1}$  onto  $\sum_{i=\tau(k)}^{s^m(\tau(k))} \mathbb{Z}e_i$  such that  $\dot{w}(e_j) = e_{w(j)}$  for  $j \in [1, k+1]$ . In particular,  $\eta w(j) = \eta\tau(j)$  for  $j \in Q$ , so  $\eta w|_Q$  is injective. By construction,  $w(Q) \subseteq P_{[\tau(k), s^m(\tau(k))]}$ , and so we conclude from (11.15) that

$$(11.20) \quad w|_Q : Q \longrightarrow P_{[\tau(k), s^m(\tau(k))]} \text{ is a bijection.}$$

Combining Eqs. (11.14) and (11.17)–(11.19) with the definition of  $w$ , we see that

$$(11.21) \quad Y_\tau(e_j) = y_{\tau,j} = Y_{[\tau(k), s^m(\tau(k))]} \dot{w}(e_j), \quad \begin{cases} \forall j \in Q \cup \{t, k, k+1\} & (m > 1) \\ \forall j \in Q \cup \{k, k+1\} & (m = 1). \end{cases}$$

The next step is to apply Theorem 9.5. We do the case  $m > 1$  and leave the case  $m = 1$  to the reader. (In the latter case,  $p(k_\bullet) = -\infty$  and  $e_{p(k_\bullet)} = 0$ .) Observe that (11.21) implies that

$$(11.22) \quad Y_\tau(f) = Y_{[\tau(k), s^m(\tau(k))]} \dot{w}(f), \quad \forall f \in \mathbb{Z}^N \text{ with } \text{supp}(f) \subseteq Q \cup \{t, k, k+1\}.$$

Thus, taking account of (11.14), (11.20), and (11.21), Theorem 9.5 implies that

$$y_{\tau',k} = Y_\tau(e_t + e_{k+1} - e_k) + Y_\tau(g' - e_k),$$

where  $g' := \dot{w}^{-1}(g_{[\tau(k), s^m(\tau(k))]}]$ , recall (9.9). Since

$$\text{supp}(g_{[\tau(k), s^m(\tau(k))]}]) \subseteq P_{[\tau(k), s^m(\tau(k))]} = w(Q),$$

we have  $\text{supp}(g') \subseteq Q$ . By the definition of  $\tau_\bullet$  and Eq. (11.13),

$$\tau_\bullet \tau(t) = \tau_\bullet \tau(p_\tau(k)) = p(k_\bullet) \text{ and } \tau_\bullet \tau(k+1) = s(k_\bullet).$$

Therefore,

$$\tilde{Y}_{\tau'}(e_{k_\bullet}) = Y_{\tau'}(e_k) = y_{\tau',k} = \tilde{Y}_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - e_{k_\bullet}) + \tilde{Y}_\tau(\tau_\bullet \tau(g') - e_{k_\bullet}),$$

which implies the validity of (11.8) for  $j = k_\bullet$ , where  $g := \tau_\bullet \tau(g')$ .

Finally, the identities (11.9)–(11.10) follow from Theorem 7.5 (applied to the P-CGL presentation (11.2)), the definition of  $\mathbf{r}_\tau$ , and the fact that  $R_{\tau, k+1} = R_{[\tau(k), s^m(\tau(k))]}$ , see (11.12). We note that  $\eta(k_\bullet) = \eta(\tau(k))$  and  $s(\tau(k)) \neq +\infty$ ,  $s(k_\bullet) \neq +\infty$ , which follow from the definition of  $\tau_\bullet$  and Eq. (11.13). Because of this and Proposition 8.8,  $\lambda_{\tau'(k+1)}^* = \lambda_{\tau(k)}^* = \lambda_{k_\bullet}^*$ . The identity (11.11) follows from the fact that  $y_{\tau,j}$  and  $y_{\tau',j}$  are  $\mathcal{H}$ -eigenvectors for all  $j \in [1, N]$  and Eq. (11.8).  $\square$

Our second auxiliary result relies on a strong rationality property analogous to [5, Theorem II.6.4]. Given a ring  $A$  equipped with an action of a group  $\mathcal{H}$  by automorphisms, we write  $A^{\mathcal{H}}$  for the fixed subring  $\{a \in A \mid h \cdot a = a \text{ for all } h \in \mathcal{H}\}$ .

**THEOREM 11.9.** *Let  $R$  be a Poisson-CGL extension of length  $N$ , and  $P$  an  $\mathcal{H}$ -Poisson-prime ideal of  $R$ . Then*

$$(11.23) \quad Z_p(\text{Fract } R/P)^{\mathcal{H}} = \mathbb{K}.$$

**PROOF.** We induct on  $N$ , the case  $N = 0$  (when  $R = \mathbb{K}$ ) being trivial. Now let  $N > 0$ , and assume the theorem holds for  $R_{N-1}$ .

The contraction  $Q := P \cap R_{N-1}$  is an  $\mathcal{H}$ -Poisson-prime ideal of  $R_{N-1}$  and is stable under  $\sigma_N$  and  $\delta_N$ . Hence,  $R/QR$  is a Poisson polynomial ring of the form

$$R/QR = (R_{N-1}/Q)[x_N; \sigma_N, \delta_N]_p,$$

where we have omitted overbars on cosets and induced maps. Let  $S$  be the localization of  $R_{N-1}/Q$  with respect to the set of all  $\mathcal{H}$ -eigenvectors in  $R_{N-1}/Q$ , and note that  $S$  has no nonzero proper  $\mathcal{H}$ -stable ideals. The induced action of  $\mathcal{H}$  on  $R_{N-1}/Q$  extends to a Poisson action on  $S$ , and since  $\text{Fract } R_{N-1}/Q = \text{Fract } S$ , our induction hypothesis implies that

$$Z_p(\text{Fract } S)^{\mathcal{H}} = \mathbb{K}.$$

Our choice of  $S$  ensures that the  $\mathcal{H}$ -action on  $S$  is rational. There are unique extensions of  $\sigma_N$  and  $\delta_N$  to a Poisson derivation  $\sigma_N$  on  $S$  and a Poisson  $\sigma_N$ -derivation  $\delta_N$  on  $S$ . The differential of the  $\mathcal{H}$ -action gives an action of  $\text{Lie } \mathcal{H}$  on  $S$ , the action of  $(h_N \cdot)$  on  $S$  extends that on  $R_{N-1}/Q$ , and  $(h_N \cdot) = \sigma_N$  on  $S$ . Finally,  $R/QR$  extends to a Poisson polynomial ring

$$\widehat{R} := S[x_N; \sigma_N, \delta_N]_p,$$

and the  $\mathcal{H}$ -action on  $R/QR$  extends to a rational Poisson action on  $\widehat{R}$ . The ideal  $P/QR$  induces an  $\mathcal{H}$ -Poisson-prime ideal  $\widehat{P}$  of  $\widehat{R}$  and there is an  $\mathcal{H}$ -equivariant Poisson isomorphism  $\text{Fract } \widehat{R}/\widehat{P} \cong \text{Fract } R/P$ , so it suffices to prove that

$$(11.24) \quad Z_p(\text{Fract } \widehat{R}/\widehat{P})^{\mathcal{H}} = \mathbb{K}.$$

There are two situations to consider, depending on whether  $\widehat{R}$  has any proper nonzero  $\mathcal{H}$ -stable Poisson ideals. Assume first that it does not; in particular,  $\widehat{P} = 0$ .

Consider a nonzero element  $u \in Z_p(\text{Fract } \widehat{R})^{\mathcal{H}}$ , set  $I := \{r \in \widehat{R} \mid ru \in \widehat{R}\}$ , and observe that  $I$  is a nonzero  $\mathcal{H}$ -stable Poisson ideal of  $\widehat{R}$ . Our current assumption implies  $I = \widehat{R}$ , whence  $u \in \widehat{R}$ . Similarly,  $u^{-1} \in \widehat{R}$ , forcing  $u \in S$ . Thus,  $u \in Z_p(\text{Fract } S)^{\mathcal{H}} = \mathbb{K}$ .

Now assume that  $\widehat{R}$  does have proper nonzero  $\mathcal{H}$ -stable Poisson ideals. As in the proof of [25, Proposition 1.2],  $\delta_N$  is an inner Poisson  $\sigma_N$ -derivation of  $S$ , implemented by a homogeneous element  $d \in S$  with  $\chi_d = \chi_{x_N}$ , and the only  $\mathcal{H}$ -Poisson-prime ideals of  $\widehat{R}$  are 0 and  $z\widehat{R}$ , where  $z := x_N - d$ . Moreover,  $\widehat{R} = S[z; \sigma_N]_p$ . There is an  $\mathcal{H}$ -equivariant Poisson isomorphism  $\widehat{R}/z\widehat{R} \cong S$ , and so  $Z_p(\text{Fract } \widehat{R}/z\widehat{R})^{\mathcal{H}} = \mathbb{K}$ . This establishes (11.24) when  $\widehat{P} = z\widehat{R}$ .

The case  $\widehat{P} = 0$  remains. Consider a nonzero element  $u \in Z_p(\text{Fract } \widehat{R})^{\mathcal{H}}$ . As above,  $I := \{r \in \widehat{R} \mid ru \in \widehat{R}\}$  is a nonzero  $\mathcal{H}$ -stable Poisson ideal of  $\widehat{R}$ . If  $I \neq \widehat{R}$ , any prime ideal minimal over  $I$  is an  $\mathcal{H}$ -Poisson-prime ideal (Lemma 4.3) and so must equal  $z\widehat{R}$ , in which case  $\sqrt{I} = z\widehat{R}$ . Whether or not  $I = \widehat{R}$ , it follows that

$z^n \in I$  for some  $n \geq 0$ . Hence,  $u = vz^{-n}$  for some homogeneous element  $v \in \widehat{R}$ . Note that  $\{z, u\} = 0$  implies  $\{z, v\} = 0$ . Write  $v = v_0 + v_1z + \cdots + v_tz^t$  for some  $v_i \in S$ . Now

$$\begin{aligned} 0 &= \{z, v\} = \{z, v_0\} + \{z, v_1\}z + \cdots + \{z, v_t\}z^t \\ &= \sigma_N(v_0)z + \sigma_N(v_1)z^2 + \cdots + \sigma_N(v_t)z^{t+1}, \end{aligned}$$

whence  $h_N \cdot v_i = \sigma_N(v_i) = 0$  for  $i \in [0, t]$ . Then, since  $v = uz^n$  and  $h_N \cdot u = 0$  by (2.9), we get

$$n\lambda_N v = h_N \cdot v = \lambda_N v_1z + \cdots + t\lambda_N v_tz^t.$$

Since  $\lambda_N \neq 0$ , it follows that  $v_i = 0$  for all  $i \neq n$ , and so  $u = v_n \in S$ . Therefore  $u \in Z_p(\text{Fract } S)^{\mathcal{H}} = \mathbb{K}$ , yielding (11.24) in the case  $\widehat{P} = 0$ .  $\square$

The next lemma proves uniqueness of integral vectors satisfying bilinear identities of the form (11.9)–(11.10).

For  $\tau \in \Xi_N$ , applying (7.20) to the P-CGL extension (11.2) yields

$$\{Y_\tau(f), Y_\tau(g)\} = \Omega_{\mathbf{q}_\tau}(f, g)Y_\tau(f)Y_\tau(g), \quad \forall f, g \in \mathbb{Z}^N,$$

and consequently

$$(11.25) \quad \{\widetilde{Y}_\tau(f), \widetilde{Y}_\tau(g)\} = \Omega_{\mathbf{r}_\tau}(f, g)\widetilde{Y}_\tau(f)\widetilde{Y}_\tau(g), \quad \forall f, g \in \mathbb{Z}^N.$$

**LEMMA 11.10.** *Assume that  $R$  is a symmetric P-CGL extension of length  $N$ . For any  $\tau \in \Xi_N$ ,  $\theta \in X(\mathcal{H})$ , and  $\xi_1, \dots, \xi_N \in \mathbb{K}$ , there exists at most one vector  $b \in \mathbb{Z}^N$  such that  $\chi_{\widetilde{Y}_\tau(b)} = \theta$  and  $\Omega_{\mathbf{r}_\tau}(b, e_j) = \xi_j$  for all  $j \in [1, N]$ .*

**PROOF.** Let  $b_1, b_2 \in \mathbb{Z}^N$  be such that  $\chi_{\widetilde{Y}_\tau(b_1)} = \chi_{\widetilde{Y}_\tau(b_2)} = \theta$  and  $\Omega_{\mathbf{r}_\tau}(b_1, e_j) = \Omega_{\mathbf{r}_\tau}(b_2, e_j) = \xi_j$  for all  $j \in [1, N]$ . Taking account of (11.25), we find that

$$\{\widetilde{Y}_\tau(b_1 - b_2), \widetilde{Y}_\tau(e_j)\} = 0, \quad \forall j \in [1, N].$$

This implies that  $\widetilde{Y}_\tau(b_1 - b_2)$  belongs to the Poisson center of  $\text{Fract } R$ , because by Theorem 10.5(a),  $\widetilde{Y}_\tau(e_1), \dots, \widetilde{Y}_\tau(e_N)$  generate the field  $\text{Fract } R$  over  $\mathbb{K}$ . Furthermore,

$$\chi_{\widetilde{Y}_\tau(b_1 - b_2)} = 0,$$

so that  $\widetilde{Y}_\tau(b_1 - b_2)$  is fixed by  $\mathcal{H}$ . By Theorem 11.9,  $\widetilde{Y}_\tau(b_1 - b_2) \in \mathbb{K}$ . This is only possible if  $b_1 = b_2$ , because  $\widetilde{Y}_\tau(e_1), \dots, \widetilde{Y}_\tau(e_N)$  are algebraically independent over  $\mathbb{K}$ .  $\square$

#### 11.4. An overview of the proof of Theorem 11.1

In this section we give a summary of the strategy of our proof of Theorem 11.1.

In §11.1 we constructed  $N$ -tuples  $\tilde{\mathbf{y}}_\tau \in \text{Fract}(R)^N$  associated to the elements of the set  $\Xi_N$ . In order to extend them to seeds of  $\text{Fract}(R)$ , one needs to construct a compatible matrix  $\widetilde{B}_\tau \in M_{N \times \mathbf{ex}}(\mathbb{Z})$  for each of them. This will be first done for the subset  $\Gamma_N$  of  $\Xi_N$  in an iterative fashion with respect to the linear ordering (8.7). If  $\tau$  and  $\tau'$  are two consecutive elements of  $\Gamma_N$  in that linear ordering, then  $\tau' = \tau(k, k+1)$  for some  $k \in [1, N-1]$  such that  $\tau(k) < \tau(k+1)$ . If  $\eta(\tau(k)) \neq \eta(\tau(k+1))$  then  $\tilde{\mathbf{y}}_{\tau'} = \tilde{\mathbf{y}}_\tau$  by Proposition 11.8(a) and nothing happens at that step. If  $\eta(\tau(k)) = \eta(\tau(k+1))$ , then we use Proposition 11.8(b) to construct  $b_\tau^{k_\bullet}$  and  $b_{\tau'}^{k_\bullet}$  where  $k_\bullet := (\tau_\bullet, \tau)(k)$ . Up to  $\pm$  sign these vectors are equal to  $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g$ , where  $g \in \mathbb{Z}_{\geq 0}^N$  is the vector from Proposition 11.8(b). Then we use “reverse”

mutation to construct  $b_\sigma^{k_\bullet}$  for  $\sigma \in \Gamma_N$ ,  $\sigma \prec \tau$  in the linear order (8.7). Effectively this amounts to starting with a cluster algebra in which all variables are frozen and then recursively adding more exchangeable variables.

There are two things that can go wrong with this. Firstly, the reverse mutations from different stages might not be synchronized. Secondly, there are many pairs of consecutive elements  $\tau, \tau'$  for which  $k_\bullet$  is the same. So we need to prove that  $b_\sigma^{k_\bullet}$  is not *overdetermined*. We use strong rationality of P-CGL extensions to handle both via Lemma 11.10. This part of the proof (of parts (a) and (b) of Theorem 11.1) is carried out in §11.5.

Once  $\tilde{B}_{\text{id}}$  is (fully) constructed then the  $\tilde{B}_\tau$  are constructed inductively by applying Proposition 11.8 and using the sequences of elements of  $\Xi_N$  from Corollary 8.6(a). At each step Lemma 11.10 is applied to match columns of mutation matrices. This proves parts (a) and (b) of Theorem 11.1. Part (c) of the theorem is obtained in a somewhat similar manner from Proposition 11.8. This is done in §11.6. Once parts (a)–(c) have been proved, we will obtain the algebra equalities in parts (d) and (f) of Theorem 11.1 from Theorem 10.5 and the Poisson-compatibility statements in parts (e) and (f) from Theorem 3.17 and Proposition 3.16. This is done in §11.7.

### 11.5. Recursive construction of seeds for $\tau \in \Gamma_N$

Recall the linear ordering (8.7) on  $\Gamma_N \subset \Xi_N$ . We start by constructing a chain of subsets  $\mathbf{ex}_\tau \subseteq \mathbf{ex}$  indexed by the elements of  $\Gamma_N$  such that

$$\mathbf{ex}_{\text{id}} = \emptyset, \quad \mathbf{ex}_{w_\circ} = \mathbf{ex} \quad \text{and} \quad \mathbf{ex}_\sigma \subseteq \mathbf{ex}_\tau, \quad \forall \sigma, \tau \in \Gamma_N, \quad \sigma \prec \tau.$$

This is constructed inductively by starting with  $\mathbf{ex}_{\text{id}} = \emptyset$ . If  $\tau \prec \tau'$  are two consecutive elements in the linear ordering, then for some  $1 \leq i < j \leq N$ ,

$$\tau = \tau_{i,j-1} \quad \text{and} \quad \tau' = \tau_{i,j}$$

(recall (8.6) and the equalities in (8.7)). Assuming that  $\mathbf{ex}_\tau$  has been constructed, we define

$$\mathbf{ex}_{\tau'} := \begin{cases} \mathbf{ex}_\tau \cup \{p(j)\}, & \text{if } p(i) = -\infty, \eta(i) = \eta(j) \\ \mathbf{ex}_\tau, & \text{otherwise.} \end{cases}$$

It is clear that this process ends with  $\mathbf{ex}_{w_\circ} = \mathbf{ex}$ .

The following lemma provides an inductive procedure for establishing parts (a) and (b) of Theorem 11.1 for  $\tau \in \Gamma_N$ . In the proof, we allow seeds whose exchange matrices are empty.

**LEMMA 11.11.** *Assume that  $R$  is a symmetric P-CGL extension of length  $N$  satisfying (11.5). Assume also that the generators of  $R$  are rescaled so that the condition (9.5) is satisfied.*

*Let  $\tau \in \Gamma_N$ . For all  $\sigma \in \Gamma_N$  with  $\sigma \preceq \tau$ , there exists a unique matrix  $\tilde{B}_{\sigma,\tau}$  in  $M_{N \times \mathbf{ex}_\tau}(\mathbb{Z})$  whose columns  $b_{\sigma,\tau}^l \in \mathbb{Z}^N$  satisfy*

$$(11.26) \quad \chi_{\tilde{Y}_\sigma}(b_{\sigma,\tau}^l) = 0, \quad \Omega_{\mathbf{r}_\sigma}(b_{\sigma,\tau}^l, e_j) = 0, \quad \forall j \in [1, N], \quad j \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_\sigma}(b_{\sigma,\tau}^l, e_l) = \lambda_l^*$$

*for all  $l \in \mathbf{ex}_\tau$ . The matrix  $\tilde{B}_{\sigma,\tau}$  has full rank, and its principal part is skew-symmetrizable via the integers  $d_{\eta(k)}$ ,  $k \in \mathbf{ex}_\tau$ . Moreover,  $(\mathbf{r}_\sigma, \tilde{B}_{\sigma,\tau})$  is a compatible pair, for all  $\sigma \preceq \tau$ .*

PROOF. The uniqueness statement follows at once from Lemma 11.10. If a matrix  $\tilde{B}_{\sigma,\tau}$  with the properties (11.26) exists, then, in view of (3.9),  $(\mathbf{r}_\sigma, \tilde{B}_{\sigma,\tau})$  is a compatible pair. The scalars  $\beta_l^{\sigma,\tau} := (\tilde{B}_{\sigma,\tau}^T \mathbf{r}_\sigma)_l$  satisfy

$$\beta_l^{\sigma,\tau} = \sum_{i=1}^N (\tilde{B}_{\sigma,\tau})_{il} (\mathbf{r}_\sigma)_i = \sum_{i=1}^N (\tilde{B}_{\sigma,\tau})_{il} \Omega_{\mathbf{r}_\sigma}(e_i, e_l) = \Omega_{\mathbf{r}_\sigma}(b_{\sigma,\tau}^l, e_l) = \lambda_l^*, \quad \forall l \in \mathbf{ex}_\tau.$$

Consequently, the principal part of  $\tilde{B}_{\sigma,\tau}$  is skew-symmetrizable via the integers  $d_{\eta(k)}$  by Lemma 3.11 and the condition (11.5). Moreover, Proposition 3.10 implies that  $\tilde{B}_{\sigma,\tau}$  has full rank. Thus,  $(\tilde{\mathbf{y}}_\sigma, \tilde{B}_{\sigma,\tau})$  is a seed in  $\mathcal{F}$ .

What remains to be proved is the existence statement in the lemma. It trivially holds for  $\tau = \text{id}$  since  $\mathbf{ex}_{\text{id}} = \emptyset$ .

Let  $\tau < \tau'$  be two consecutive elements of  $\Gamma_N$  in the linear ordering (8.7). Assuming that the existence statement in the lemma holds for  $\tau$ , we will show that it holds for  $\tau'$ . The lemma will then follow by induction.

As noted above, for some  $1 \leq i < j \leq N$  we have  $\tau = \tau_{i,j-1}$  and  $\tau' = \tau_{i,j}$ . In particular,  $\tau' = (ij)\tau = \tau(j-i, j-i+1)$  and  $\tau(j-i) = i < j = \tau(j-i+1)$ , so Proposition 11.8 is applicable to the pair  $(\tau, \tau')$ , with  $k := j-i$ . Note that  $\tau(k) = i$  and  $\tau(k+1) = j$ .

If  $\eta(i) \neq \eta(j)$ , then  $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$ , and  $\tilde{Y}_{\tau'} = \tilde{Y}_\tau$  and  $\mathbf{r}_{\tau'} = \mathbf{r}_\tau$  by Proposition 11.8(a). So,  $\Omega_{\mathbf{r}_{\tau'}} = \Omega_{\mathbf{r}_\tau}$ . These identities imply that the following matrices have the properties (11.26) for the element  $\tau' \in \Gamma_N$ :  $\tilde{B}_{\sigma,\tau'} := \tilde{B}_{\sigma,\tau}$  for  $\sigma \preceq \tau$  and  $\tilde{B}_{\tau',\tau'} := \tilde{B}_{\tau,\tau}$ .

Next, we consider the case  $\eta(i) = \eta(j)$ . This implies that  $j = s^m(i)$  for some  $m \in \mathbb{Z}_{>0}$ . This fact and the definition of  $\tau_\bullet$  give that the element  $\tau_\bullet \tau(j-i)$  equals the  $m$ -th element of  $\eta^{-1}(\eta(i))$  when the elements in the preimage are ordered from least to greatest. Therefore this element is explicitly given by

$$(11.27) \quad \tau_\bullet \tau(j-i) = s^{m-1} p^{O-(i)}(i).$$

Now set

$$k_\bullet := \tau_\bullet \tau(j-i)$$

as in Proposition 11.8(b). There are two subcases: (1)  $p(i) \neq -\infty$  and (2)  $p(i) = -\infty$ .

**Subcase (1).** In this situation  $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$ , so we do not need to generate an “extra column” for each matrix. Set  $\tilde{B}_{\sigma,\tau'} := \tilde{B}_{\sigma,\tau}$  for  $\sigma \in \Gamma_N$ ,  $\sigma \preceq \tau$ . Eq. (11.26) for the pairs  $(\sigma, \tau')$  with  $\sigma \preceq \tau$  follows from the equality  $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$ .

Next we deal with the pair  $(\sigma, \tau') = (\tau', \tau')$ . Applying the inductive assumption (11.26) for  $\tilde{B}_{\tau,\tau}$  and Proposition 11.8(b) shows that the vector  $g \in \mathbb{Z}_{\geq 0}^N$  has the properties

$$\chi_{\tilde{Y}_\tau(b_{\tau,\tau}^{k_\bullet})} = 0 = \chi_{\tilde{Y}_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g)}$$

and

$$\Omega_{\mathbf{r}_\tau}(b_{\tau,\tau}^{k_\bullet}, e_t) = \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_t), \quad \forall t \in [1, N].$$

Lemma 11.10 implies that  $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g = b_{\tau,\tau}^{k_\bullet}$ . It follows from this and Eq. (11.8) that

$$(11.28) \quad \begin{aligned} \tilde{Y}_{\tau'}(e_j) &= \tilde{Y}_\tau(e_j), \quad \forall j \neq k_\bullet, \\ \tilde{Y}_{\tau'}(e_{k_\bullet}) &= \tilde{Y}_\tau(-e_{k_\bullet} + [b_{\tau,\tau}^{k_\bullet}]_+) + \tilde{Y}_\tau(-e_{k_\bullet} - [b_{\tau,\tau}^{k_\bullet}]_-), \end{aligned}$$

whence  $\tilde{\mathbf{y}}_{\tau'} = \mu_{k_\bullet}(\tilde{\mathbf{y}}_\tau)$ .

We set  $\tilde{B}_{\tau',\tau'} := \mu_{k_\bullet}(\tilde{B}_{\tau,\tau})$ , so that  $(\tilde{\mathbf{y}}_{\tau'}, \tilde{B}_{\tau',\tau'}) = \mu_{k_\bullet}(\tilde{\mathbf{y}}_\tau, \tilde{B}_{\tau,\tau})$ . Since the columns of  $\tilde{B}_{\tau,\tau}$  satisfy (11.26) and the entries  $\tilde{Y}_\tau(e_j)$  of  $\tilde{\mathbf{y}}_\tau$  are  $\mathcal{H}$ -eigenvectors, Lemma 3.8 implies that the columns of  $\tilde{B}_{\tau',\tau'}$  have the property  $\chi_{\tilde{Y}_{\tau'}(b_{\tau',\tau'}^l)} = 0$  for all  $l \in \mathbf{ex}_\tau = \mathbf{ex}_{\tau'}$ .

According to (11.25),

$$(11.29) \quad \{\tilde{Y}_\tau(e_l), \tilde{Y}_\tau(e_j)\} = \Omega_{\mathbf{r}_\tau}(e_l, e_j) \tilde{Y}_\tau(e_l) \tilde{Y}_\tau(e_j), \quad \forall l, j \in [1, N],$$

$$(11.30) \quad \{\tilde{Y}_{\tau'}(e_l), \tilde{Y}_{\tau'}(e_j)\} = \Omega_{\mathbf{r}_{\tau'}}(e_l, e_j) \tilde{Y}_{\tau'}(e_l) \tilde{Y}_{\tau'}(e_j), \quad \forall l, j \in [1, N].$$

On the other hand, since the  $\tilde{Y}_\tau(e_i)$  and  $\tilde{Y}_{\tau'}(e_i)$  are the entries of  $\tilde{\mathbf{y}}_\tau$  and  $\tilde{\mathbf{y}}_{\tau'} = \mu_{k_\bullet}(\tilde{\mathbf{y}}_\tau)$ , it follows from (11.29) and Proposition 3.16 that

$$(11.31) \quad \{\tilde{Y}_{\tau'}(e_l), \tilde{Y}_{\tau'}(e_j)\} = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}(e_l, e_j) \tilde{Y}_{\tau'}(e_l) \tilde{Y}_{\tau'}(e_j), \quad \forall l, j \in [1, N].$$

Comparing (11.30) and (11.31), we see that  $\Omega_{\mathbf{r}_{\tau'}} = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}$ , and therefore

$$(11.32) \quad \mathbf{r}_{\tau'} = \mu_{k_\bullet}(\mathbf{r}_\tau).$$

Now Proposition 3.14(b) implies that  $\tilde{B}_{\tau',\tau'}^T \mathbf{r}_{\tau'} = \tilde{B}_{\tau,\tau}^T \mathbf{r}_\tau$ . Consequently,

$$\Omega_{\mathbf{r}_{\tau'}}(b_{\tau',\tau'}^l, e_j) = e_l^T \tilde{B}_{\tau',\tau'}^T \mathbf{r}_{\tau'} e_j = e_l^T \tilde{B}_{\tau,\tau}^T \mathbf{r}_\tau e_j = \Omega_{\mathbf{r}_\tau}(b_{\tau,\tau}^l, e_j) = \delta_{jl} \lambda_l^*$$

for all  $l \in \mathbf{ex}_\tau$  and  $j \in [1, N]$ . Thus, the columns of  $\tilde{B}_{\tau',\tau'}$  satisfy the conditions (11.26), which completes the proof of the inductive step of the lemma in this subcase.

**Subcase (2).** In this case,  $k_\bullet = s^{m-1}(i) = p(j)$  and  $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau \sqcup \{k_\bullet\}$ . Define the matrix  $\tilde{B}_{\tau,\tau'}$  by

$$b_{\tau,\tau'}^l = \begin{cases} b_{\tau,\tau}^l, & \text{if } l \neq k_\bullet, \\ e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, & \text{if } l = k_\bullet, \end{cases}$$

where  $g \in \mathbb{Z}_{\geq 0}^N$  is the vector from Proposition 11.8(b). Applying the assumption (11.26) for  $\tilde{B}_{\tau,\tau}$  and Proposition 11.8(b), we obtain that the matrix  $\tilde{B}_{\tau,\tau'}$  has the properties (11.26). We set  $\tilde{B}_{\tau',\tau'} := \mu_{k_\bullet}(\tilde{B}_{\tau,\tau'})$ .

As in subcase (1), using Lemma 3.8 and Propositions 3.14(b) and 3.16, one derives that  $\tilde{B}_{\tau',\tau'}$  satisfies the properties (11.26).

We are left with constructing  $\tilde{B}_{\sigma,\tau'} \in M_{N \times \mathbf{ex}_{\tau'}}(\mathbb{Z})$  for  $\sigma \in \Gamma_N$ ,  $\sigma \prec \tau$ . We do this by a downward induction on the linear ordering (8.7) in a fashion that is similar to the proof of the lemma in the subcase (1). Assume that  $\sigma \prec \sigma'$  is a pair of consecutive elements of  $\Gamma_N$  such that  $\sigma' \preceq \tau$ . As in the beginning of the section, we have that for some  $1 \leq i^\vee < j^\vee \leq N$ ,

$$\sigma = \tau_{i^\vee, j^\vee-1} \text{ and } \sigma' = \tau_{i^\vee, j^\vee},$$

so

$$\tau' = (i^\vee j^\vee) \tau = \tau(j^\vee - i^\vee, j^\vee - i^\vee + 1).$$

Assume that there exists a matrix  $\tilde{B}_{\sigma',\tau'} \in M_{N \times \mathbf{ex}_{\tau'}}(\mathbb{Z})$  that satisfies (11.26). We define the matrix  $\tilde{B}_{\sigma,\tau'} \in M_{N \times \mathbf{ex}_{\tau'}}(\mathbb{Z})$  by

$$\tilde{B}_{\sigma,\tau'} := \begin{cases} \tilde{B}_{\sigma',\tau'}, & \text{if } \eta(i^\vee) \neq \eta(j^\vee) \\ \mu_{k_\bullet^\vee}(\tilde{B}_{\sigma,\tau'}), & \text{if } \eta(i^\vee) = \eta(j^\vee), \end{cases}$$

where

$$k_{\bullet}^{\vee} := \sigma_{\bullet} \sigma(j^{\vee} - i^{\vee}).$$

Analogously to the proof of the lemma in the subcase (1), using Propositions 11.8 and 3.14(b) and Lemmas 11.10 and 3.8, one proves that the matrix  $\tilde{B}_{\sigma, \tau'}$  has the properties (11.26). This completes the proof of the lemma.  $\square$

REMARK 11.12. It follows from Lemma 11.10 that the matrices  $\tilde{B}_{\sigma, \tau}$  in Lemma 11.11 have the following restriction property:

*For all triples  $\sigma \preceq \tau \prec \tau'$  of elements of  $\Gamma_N$ ,*

$$b_{\sigma, \tau}^l = b_{\sigma, \tau'}^l, \quad \forall l \in \mathbf{ex}_{\tau}.$$

*In other words,  $\tilde{B}_{\sigma, \tau}$  is obtained from  $\tilde{B}_{\sigma, \tau'}$  by removing all columns indexed by the set  $\mathbf{ex}_{\tau'} \setminus \mathbf{ex}_{\tau}$ .*

This justifies that Lemma 11.11 gradually enlarges a matrix  $\tilde{B}_{\sigma, \sigma} \in M_{N \times \mathbf{ex}_{\sigma}}(\mathbb{Z})$  to a matrix  $\tilde{B}_{\sigma, w_0} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$ , for all  $\sigma \in \Gamma_N$ . In the case of  $\sigma = \text{id}$ , we start with an empty matrix ( $\mathbf{ex}_{\text{id}} = \emptyset$ ) and obtain a matrix  $\tilde{B}_{\text{id}, w_0} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$  which will be the needed mutation matrix for the initial cluster variables  $\tilde{\mathbf{y}}$ .

PROOF OF THEOREM 11.1(a)(b) FOR  $\tau \in \Gamma_N$ . Change  $\tau$  to  $\sigma$  in these statements. These parts of the theorem for the elements of  $\Gamma_N$  follow from Lemma 11.11 applied to  $(\sigma, \tau) = (\sigma, w_0)$ . For all  $\sigma \in \Gamma_N$  we set  $\tilde{B}_{\sigma} := \tilde{B}_{\sigma, w_0}$  and use that  $\mathbf{ex}_{w_0} = \mathbf{ex}$ .  $\square$

### 11.6. Proofs of parts (a), (b) and (c) of Theorem 11.1

Next we establish Theorem 11.1(a)(b) in full generality. This will be done by using the result of §11.5 for  $\tau = \text{id}$  and iteratively applying the following proposition.

PROPOSITION 11.13. *Let  $R$  be a symmetric Poisson-CGL extension of length  $N$  satisfying (11.5). Assume also that the generators of  $R$  are rescaled so that the condition (9.5) is satisfied. Let  $\tau, \tau' \in \Xi_N$  be such that*

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

*for some  $k \in [1, N-1]$  with  $\tau'(k) < \tau'(k+1)$  and  $\eta(\tau(k)) = \eta(\tau(k+1))$ . Set  $k_{\bullet} := \tau_{\bullet} \tau(k)$ .*

*Assume that there exists an  $N \times \mathbf{ex}$  matrix  $\tilde{B}_{\tau}$  with integral entries whose columns  $b_{\tau}^l \in \mathbb{Z}^N$ ,  $l \in \mathbf{ex}$  satisfy*

$$(11.33) \quad \chi_{\tilde{Y}_{\tau}(b_{\tau}^l)} = 0, \quad \Omega_{\mathbf{r}_{\tau}}(b_{\tau}^l, e_j) = 0, \quad \forall j \in [1, N], \quad j \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_{\tau}}(b_{\tau}^l, e_l) = \lambda_l^*$$

*for all  $l \in \mathbf{ex}$ . Then its principal part is skew-symmetrizable and the columns  $b_{\tau'}^j \in \mathbb{Z}^N$ ,  $j \in \mathbf{ex}$  of the matrix  $\mu_{k_{\bullet}}(\tilde{B}_{\tau})$  satisfy*

$$(11.34) \quad \chi_{\tilde{Y}_{\tau'}(b_{\tau'}^l)} = 0, \quad \Omega_{\mathbf{r}_{\tau'}}(b_{\tau'}^l, e_j) = 0, \quad \forall j \in [1, N], \quad j \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_{\tau'}}(b_{\tau'}^l, e_l) = \lambda_l^*$$

*for all  $l \in \mathbf{ex}$ . Furthermore,*

$$(11.35) \quad \mathbf{r}_{\tau'} = \mu_{k_{\bullet}}(\mathbf{r}_{\tau})$$

*and*

$$(11.36) \quad b_{\tau}^{k_{\bullet}} = -b_{\tau'}^{k_{\bullet}} = e_{p(k_{\bullet})} + e_{s(k_{\bullet})} - g,$$

*where  $g \in \mathbb{Z}_{\geq 0}^N$  is the vector from Proposition 11.8.*

REMARK 11.14. The statements of Lemma 11.11 and Proposition 11.13 have many similarities and their proofs use similar ideas. However, we note that there is a principal difference between the two results. In the former case we have no mutation matrices to start with and we use Proposition 11.8(b) to gradually add columns. In the latter case we already have a mutation matrix for one seed and construct a mutation matrix for another seed.

PROOF OF PROPOSITION 11.13. The fact that the principal part of  $\tilde{B}_\tau$  is skew-symmetrizable follows from Lemma 3.11 and the condition (11.5), since  $(\mathbf{r}_\tau, \tilde{B}_\tau)$  is a compatible pair by (11.33). The assumptions on  $\tilde{B}_\tau$  and Proposition 11.8(b) imply

$$\chi_{\tilde{Y}_\tau(b_\tau^{k_\bullet})} = \chi_{\tilde{Y}_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g)}$$

and

$$\Omega_{\mathbf{r}_\tau}(b_\tau^{k_\bullet}, e_j) = \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_j), \quad \forall j \in [1, N].$$

By Lemma 11.10,  $b_\tau^{k_\bullet} = e_{p(k_\bullet)} + e_{s(k_\bullet)} - g$ . The mutation formula for  $\mu_{k_\bullet}(\tilde{B}_\tau)$  also gives that  $b_\tau^{k_\bullet} = -b_\tau^{k_\bullet}$ , so we obtain (11.36). Analogously to the proof of (11.32), one establishes (11.35). Finally, all identities in (11.34) follow from the general mutation facts in Proposition 3.14(b) and Lemma 3.8.  $\square$

PROOF OF THEOREM 11.1(a)(b) FOR ALL  $\tau \in \Xi_N$ . Similarly to the proof of Lemma 11.11, the uniqueness statement in part (a) follows from Lemma 11.10. We will prove the existence statement in part (a) by an inductive argument on  $\tau$ . Once the existence of the matrix  $\tilde{B}_\tau$  with the stated properties is established, the fact that the principal part of  $\tilde{B}_\tau$  is skew-symmetrizable follows from Lemma 3.11 and the condition (11.5), and the compatibility of the pair  $(\mathbf{r}_\tau, \tilde{B}_\tau)$  follows from (11.6) and (3.9). Hence,  $(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$  is a seed and this yields part (b) of the theorem for the given  $\tau \in \Xi_N$ .

For the existence statement in part (a) we fix  $\tau \in \Xi_N$ . By Corollary 8.6(a), there exists a sequence  $\tau_0 = \text{id}, \tau_1, \dots, \tau_n = \tau$  in  $\Xi_N$  with the property that for all  $l \in [1, n]$ ,

$$\tau_l = (\tau_{l-1}(k_l), \tau_{l-1}(k_l + 1))\tau_{l-1} = \tau_{l-1}(k_l, k_l + 1)$$

for some  $k_l \in [1, N - 1]$  such that  $\tau_{l-1}(k_l) < \tau_{l-1}(k_l + 1)$ . In §11.5 we established the validity of Theorem 11.1(a) for the identity element of  $S_N$ . By induction on  $l$  we prove the validity of Theorem 11.1(a) for  $\tau_l$ . If  $\eta(\tau_{l-1}(k_l)) \neq \eta(\tau_l(k_l))$ , then Proposition 11.8(a) implies that  $\tilde{Y}_{\tau_l} = \tilde{Y}_{\tau_{l-1}}$  and we can choose  $\tilde{B}_{\tau_l} = \tilde{B}_{\tau_{l-1}}$ . If  $\eta(\tau_{l-1}(k_l)) = \eta(\tau_l(k_l))$ , then Proposition 11.13 proves that the validity of Theorem 11.1(a) for  $\tau_{l-1}$  implies the validity of Theorem 11.1(a) for  $\tau_l$ . In this case  $\tilde{B}_{\tau_l} := \mu_{(k_l)_\bullet}(\tilde{B}_{\tau_{l-1}})$ , where  $(k_l)_\bullet = (\tau_{k_l})_\bullet \tau_{k_l}(k_l)$ . This completes the proof of Theorem 11.1(a)(b).  $\square$

PROOF OF THEOREM 11.1(c). The one-step mutation statement in part (c) of Theorem 11.1 and Corollary 8.6(a) imply that all seeds associated to the elements of  $\Xi_N$  are mutation-equivalent to each other.

In the rest we prove the one-step mutation statement in part (c) of the theorem. If  $\eta(\tau(k)) \neq \eta(\tau(k + 1))$ , then the statement follows from Proposition 11.8(a).

Now let  $\eta(\tau(k)) = \eta(\tau(k + 1))$ . We have that either  $\tau(k) < \tau(k + 1)$  or  $\tau'(k) = \tau(k + 1) < \tau'(k + 1) = \tau(k)$ . In the first case we apply Proposition 11.13 to the pair  $(\tau, \tau')$  and in the second case to the pair  $(\tau', \tau)$ . The one-step mutation statement

in Theorem 11.1(c) follows from this, the uniqueness statement in part (a) of the theorem, and the involutivity of mutations of seeds.  $\square$

### 11.7. Completion of the proof of Theorem 11.1

Recall the setting of §10.3. For all  $\tau \in \Xi_N$ , define the multiplicative subsets

$$E_\tau := \left\{ \alpha \tilde{Y}_\tau(f) \mid \alpha \in \mathbb{K}^*, f \in \sum_{j \in \mathbf{ex}} \mathbb{Z}_{\geq 0} e_j \right\} \subset R.$$

In view of Lemma 10.4(a) and the definition of  $\tilde{Y}_\tau$ , we see that  $E_\tau$  is generated (as a multiplicative set) by

$$\mathbb{K}^* \sqcup \{y_{\tau,k} \mid k \in [1, N], s_\tau(k) \neq +\infty\},$$

matching the definition used in §10.3.

PROOF OF THEOREM 11.1(d)–(f). By Theorem 11.1(c), for all  $\tau \in \Xi_N$  the seeds  $(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau)$  are mutation-equivalent to each other. For each  $j \in [1, N]$ , we have  $\tau_{j,j} \in \Gamma_N$  with  $\tau_{j,j}(1) = j$ , and so  $\bar{y}_{\tau_{j,j},1} = \bar{y}_{[j,j]} = x_j$  by Theorem 8.3. Thus, each generator  $x_j$  of  $R$  is a cluster or frozen variable for a seed associated to some  $\tau \in \Gamma_N$ . Hence,

$$(11.37) \quad R \subseteq \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}}.$$

The Laurent phenomenon (Theorem 3.7) implies that

$$(11.38) \quad \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}} \subseteq \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \emptyset)_{\mathbb{K}}.$$

For all  $\tau \in \Xi_N$ , we have  $\mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \emptyset)_{\mathbb{K}} = \mathcal{U}(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau, \emptyset)_{\mathbb{K}}$  in view of part (c) of the theorem and Theorem 3.5. Moreover,  $\mathcal{U}(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau, \emptyset)_{\mathbb{K}} \subseteq \mathcal{TA}(\tilde{\mathbf{y}}_\tau, \tilde{B}_\tau, \emptyset)_{\mathbb{K}} \subseteq R[E_\tau^{-1}]$  because  $y_{\tau,i} \in R$  for all  $i \in [1, N]$  and  $y_{\tau,(\tau_\bullet \tau)^{-1}(k)} \in E_\tau$  for all  $k \in \mathbf{ex}$ . Consequently,

$$(11.39) \quad \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \emptyset)_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}].$$

Combining the inclusions (11.37), (11.38), (11.39) and Theorem 10.5(d) leads to

$$R \subseteq \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B})_{\mathbb{K}} \subseteq \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \emptyset)_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}] = R,$$

which establishes all equalities in Theorem 11.1(d).

For part (f) we obtain inclusions

$$(11.40) \quad \begin{aligned} R[y_k^{-1} \mid k \in \mathbf{inv}] &\subseteq \mathcal{A}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} \\ &\subseteq \mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[y_k^{-1} \mid k \in \mathbf{inv}][E_\tau^{-1}] \end{aligned}$$

in the same way as above. Theorem 10.5(e) and (11.40) imply the validity of the first statement in part (f) of the theorem.

It remains to verify the Poisson-compatibility statements in parts (e) and (f).

By part (b) of the theorem,  $(\mathbf{r}_\tau, \tilde{B}_\tau)$  is a compatible pair, for each  $\tau \in \Xi_N$ . In particular,  $(\mathbf{q}, \tilde{B}) = (\mathbf{r}_{\text{id}}, \tilde{B}_{\text{id}})$  is a compatible pair. Eq. (5.23) in Proposition 5.8 says that

$$(11.41) \quad \{y_l, y_j\} = \Omega_{\mathbf{q}}(e_l, e_j) y_l y_j, \quad \forall l, j \in [1, N].$$

Consequently, Theorem 3.17 implies that the cluster algebras  $\mathcal{A}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}}$  are Poisson-compatible.

Given any  $\tau \in \Xi_N$ , we again appeal to Corollary 8.6(a) to obtain a sequence  $\tau_0 = \text{id}, \tau_1, \dots, \tau_n = \tau$  in  $\Xi_N$  with the property that for all  $i \in [1, n]$ ,

$$\tau_i = (\tau_{i-1}(k_i), \tau_{i-1}(k_i + 1))\tau_{i-1} = \tau_{i-1}(k_i, k_i + 1)$$

for some  $k_i \in [1, N - 1]$  such that  $\tau_{i-1}(k_i) < \tau_{i-1}(k_i + 1)$ . We prove by induction on  $i \in [1, n]$  that

$$(11.42) \quad \{\tilde{Y}_{\tau_i}(e_l), \tilde{Y}_{\tau_i}(e_j)\} = \Omega_{\mathbf{r}_{\tau_i}}(e_l, e_j)\tilde{Y}_{\tau_i}(e_l)\tilde{Y}_{\tau_i}(e_j), \quad \forall l, j \in [1, N].$$

The case  $i = 0$  is just (11.41). Now let  $i > 0$ . If  $\eta(\tau_{i-1}(k_i)) \neq \eta(\tau_i(k_i))$ , then  $\mathbf{r}_{\tau_i} = \mathbf{r}_{\tau_{i-1}}$  and  $\tilde{Y}_{\tau_i} = \tilde{Y}_{\tau_{i-1}}$  by Proposition 11.8(a), and so the case  $i$  of (11.42) is immediate from the case  $i - 1$ . If  $\eta(\tau_{i-1}(k_i)) = \eta(\tau_i(k_i))$ , then  $\mathbf{r}_{\tau_i} = \mu_{(k_i)\bullet}(\mathbf{r}_{\tau_{i-1}})$  by Proposition 11.13 and  $\tilde{Y}_{\tau_i} = \mu_{(k_i)\bullet}(\tilde{Y}_{\tau_{i-1}})$  by statement (c) of Theorem 11.1. In this case, Proposition 3.16 shows that the case  $i - 1$  of (11.42) implies the case  $i$ .  $\square$

The chain of embeddings (11.40) and Theorem 10.5(e) also imply the following description of the upper cluster algebra in Theorem 11.1 as a finite intersection of mixed polynomial-Laurent polynomial algebras.

COROLLARY 11.15. *In the setting of Theorem 11.1,*

$$\mathcal{U}(\tilde{\mathbf{y}}, \tilde{B}, \mathbf{inv})_{\mathbb{K}} = \bigcap_{\tau \in \Delta} \mathcal{TA}(\tilde{\mathbf{x}}_{\tau}, \tilde{B}_{\tau}, \mathbf{inv})$$

for every subset  $\Delta$  of  $\Gamma_N$  which is an interval with respect to the linear ordering (8.7) and has the property that for each  $k \in \mathbf{ex}$  there exist two consecutive elements  $\tau = \tau_{i,j-1} \prec \tau' = \tau_{i,j}$  of  $\Delta$  such that  $\eta(i) = \eta(j)$  and  $k = \tau_{\bullet}\tau(j - i)$ , recall §11.5.

In particular, this property holds for  $\Delta = \Gamma_N$ .



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