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Research Article

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A Symmetric Interior Penalty Method for an Elliptic Distributed Optimal Control Problem with Pointwise State Constraints

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Abstract: We construct a symmetric interior penalty method for an elliptic distributed optimal control problem with pointwise state constraints on general polygonal domains. The resulting discrete problems are quadratic programs with simple box constraints that can be solved efficiently by a primal-dual active set algorithm. Both theoretical analysis and corroborating numerical results are presented.

Keywords: Elliptic Distributed Optimal Control Problems, Pointwise State Constraints, Symmetric Interior Penalty Methods

MSC 2010: 49J40, 49M41, 65K15, 65N30

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $y_d \in L_2(\Omega)$, $g \in H^4(\Omega)$, and let β be a positive constant. The optimal control problem (cf. [31]) is to find

$$(\bar{y}, \bar{u}) = \operatorname*{argmin}_{(y,u)\in\mathbb{K}_g} \frac{1}{2}[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|u\|^2_{L_2(\Omega)}], \tag{1.1}$$

where $(y, u) \in H^1(\Omega) \times L_2(\Omega)$ belongs to $\mathbb{K}_g$ if and only if

$$\int_\Omega \nabla y \cdot \nabla z \, dx = \int_\Omega uz \, dx \quad \text{for all } z \in H^1_0(\Omega), \tag{1.2}$$

$$y = g \quad \text{on } \partial\Omega, \tag{1.3}$$

$$y \le \psi \quad \text{a.e. in } \Omega. \tag{1.4}$$

We assume that the function

$$\psi \text{ belongs to } W^{3,p}(\Omega) \text{ for } p > 2 \text{ and } \psi > g \text{ on } \partial\Omega. \tag{1.5}$$

Remark 1.1. Throughout this paper, we will follow the standard notation for differential operators, function spaces and norms that can be found for example in [1, 23, 36].

Observe that (1.2)–(1.3) is equivalent to $y \in g + \mathring{E}(\Delta; L_2(\Omega))$, where

$$\mathring{E}(\Delta; L_2(\Omega)) = \{z \in H^1_0(\Omega) : \Delta z \in L_2(\Omega)\}$$

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and Δz is understood in the sense of distributions. It is well known (cf. [37, 46]) that

$$\mathring{E}(\Delta; L_2(\Omega)) \subset H^{1+\alpha}(\Omega) \quad \text{for some } \alpha \in (1/2, 1] \tag{1.6}$$

and

$$\|y\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|\Delta y\|_{L_2(\Omega)} \quad \text{for all } y \in \mathring{E}(\Delta; L_2(\Omega)). \tag{1.7}$$

Hence functions in $\mathring{E}(\Delta; L_2(\Omega))$ are continuous by the Sobolev inequality (cf. [1]).

We can reformulate the optimal control problem (1.1)–(1.4) as the following minimization problem that only involves y: find

$$\bar{y} = \operatorname*{argmin}_{y \in K_g} \frac{1}{2}[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|\Delta y\|^2_{L_2(\Omega)}], \tag{1.8}$$

where

$$K_g = \{y \in g + \mathring{E}(\Delta; L_2(\Omega)) : y \leq \psi \text{ in } \Omega\}. \tag{1.9}$$

Our goal is to solve the optimal control problem (1.1)–(1.4) by a symmetric interior penalty (SIP) method (cf. [3, 61]) that is based on the reformulation (1.8)–(1.9). This reformulation was discussed in [56], and the first numerical scheme based on this idea appeared in [53], where the analysis was carried out under certain ad hoc assumptions on the free boundary from [8]. These assumptions were later removed in the new convergence analysis in [24] by exploiting the regularity results in [30, 43, 44] for fourth-order elliptic variational inequalities. Various finite element methods based on this new approach have appeared in [19–22, 26–29].

Comparing with the more traditional approach in [32, 35, 49, 52, 54, 55] that is based on reducing the optimal control problem (1.1)–(1.4) to a problem that only involves the control, a distinct feature of the new approach is that the convergence of the state can also be established in the L_∞ norm. Another useful feature is that the discrete problems are quadratic programs with simple box constraints where the system matrices are available and consequently they can be solved efficiently by many optimization algorithms. Moreover, general polygonal/polyhedral domains can also be handled by the new approach (cf. [20, 22, 26]). These features are also shared by the SIP method in this paper. Compared to the P_1 finite element methods in [22, 26], the inverse of the block diagonal mass matrix of the SIP method of any order can be evaluated exactly, and hence the system matrix is available without mass lumping (cf. Remark 2.4 below).

The rest of the paper is organized as follows. We recall relevant results of the continuous problem and set up the discrete problem in Section 2. Technical tools for the error analysis are collected in Section 3, followed by an abstract error estimate in Section 4 and concrete error estimates in Section 5. Numerical results are presented in Section 6, and we end with some concluding remarks in Section 7. A heuristic justification of some of the numerical results is given in Appendix A.

Throughout the paper, we use C (with or without subscripts) to denote a generic positive constant independent of the mesh size h. We also use $A \lesssim B$ to represent the statement that $A \leq (\text{constant})B$, where the positive constant is independent of h.

2 Continuous and Discrete Problems

In this section, we recall relevant results of the continuous problem and construct the SIP method.

2.1 The Continuous Problem

It follows from the classical theory of calculus of variations (cf. [40, 51]) that (1.8)–(1.9) has a unique solution $\bar{y} \in g + \mathring{E}(\Delta; L_2(\Omega))$ characterized by the variational inequality

$$\int_\Omega (\bar{y} - y_d)(y - \bar{y})\, dx + \beta \int_\Omega (\Delta\bar{y})\Delta(y - \bar{y})\, dx \geq 0 \quad \text{for all } y \in g + \mathring{E}(\Delta; L_2(\Omega)). \tag{2.1}$$

Remark 2.1. In the case where Ω is convex, the space $\mathring{E}(\Delta; L_2(\Omega))$ coincides with $H^2(\Omega) \cap H_0^1(\Omega)$ and (2.1) is a fourth-order variational inequality.

The variational inequality (2.1) is equivalent to the following generalized Karush–Kuhn–Tucker conditions:

$$\int_\Omega (\bar{y} - y_d)z\,dx + \beta \int_\Omega (\Delta\bar{y})(\Delta z)\,dx = \int_\Omega z\,d\mu \quad \text{for all } z \in \mathring{E}(\Delta; L_2(\Omega)), \tag{2.2}$$

where

$$\mu \text{ is a nonpositive finite Borel measure} \tag{2.3}$$

and

$$\int_\Omega (\bar{y} - \psi)\,d\mu = 0. \tag{2.4}$$

Details for the derivation of (2.2)–(2.4) and for the regularity results below can be found in [24, 26] and the references therein.

Remark 2.2. It follows from the complementarity condition (2.4) that μ is supported on the active set $\mathscr{A}$ for the constraint (1.4) given by $\mathscr{A} = \{x \in \bar{\Omega} : \bar{y}(x) = \psi(x)\}$. Note that $\mathscr{A}$ is a compact subset of Ω by the assumption that $\psi > g$ on $\partial\Omega$.

We have

$$\bar{u} = -\Delta\bar{y} \in H_0^1(\Omega) \tag{2.5}$$

and

$$\bar{y} \in H^3_{\mathrm{loc}}(\Omega) \cap W^{2,\infty}_{\mathrm{loc}}(\Omega) \cap H^{1+\alpha}(\Omega), \tag{2.6}$$

where $\alpha \in (1/2, 1]$ is the index of elliptic regularity as in (1.6).

Remark 2.3. In the case where Ω is convex, we can replace α in (2.6) with some $\tilde{\alpha} \in (1, 2]$ by exploiting (2.5).

Finally, the Lagrange multiplier μ belongs to $H^{-1}(\Omega)$, i.e.,

$$\int_\Omega z\,d\mu \leq C|z|_{H^1(\Omega)} \quad \text{for all } z \in H^1(\Omega). \tag{2.7}$$

2.2 The Discrete Problem

Let $\mathcal{T}_h$ be a triangulation of Ω, let k be a positive integer and

$$Y_h = \{y \in L_2(\Omega) : y_T = y|_T \in \mathbb{P}_k(T) \text{ for all } T \in \mathcal{T}_h\}$$

the discontinuous finite element space of degree at most k. The set of the edges of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$, $\mathcal{E}_h^b (\subset \mathcal{E}_h)$ is the set of the boundary edges and $|e|$ is the diameter of an edge e.

The discrete Laplace operator $\Delta_h\colon (g + \mathring{E}(\Delta; L_2(\Omega))) + Y_h \to Y_h$ is defined by

$$\int_\Omega (\Delta_h\zeta)z_h\,dx = -a_h(\zeta, z_h) \quad \text{for all } z_h \in Y_h, \tag{2.8}$$

where, following the convention for jumps and averages in [4],

$$a_h(y, z) = \sum_{T\in\mathcal{T}_h} \int_T \nabla y \cdot \nabla z\,dx - \sum_{e\in\mathcal{E}_h} \int_e (\{\!\{\nabla y\}\!\} \cdot [\![z]\!] + \{\!\{\nabla z\}\!\} \cdot [\![y]\!])\,ds + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e [\![y]\!] \cdot [\![z]\!]\,ds$$

is the bilinear form for the SIP method with a sufficiently large penalty parameter σ.

The SIP method is consistent (cf. [57]) in the sense that

$$a_h(\zeta, z_h) = \int_\Omega (-\Delta\zeta) z_h \, dx + \sum_{e \in \mathcal{E}_h^b} \frac{\sigma}{|e|} \int_e g z_h \, ds - \sum_{e \in \mathcal{E}_h^b} \int_e g(\nabla z_h \cdot n) \, ds \tag{2.9}$$

for all $\zeta \in g + \mathring{E}(\Delta; L_2(\Omega))$ and $z_h \in Y_h$.

Let $g_h \in Y_h$ be defined by

$$\int_\Omega g_h z_h \, dx = \sum_{e \in \mathcal{E}_h^b} \int_e g(\nabla z_h \cdot n) \, ds - \sum_{e \in \mathcal{E}_h^b} \frac{\sigma}{|e|} \int_e g z_h \, ds \quad \text{for all } z_h \in Y_h. \tag{2.10}$$

Then (2.8), (2.9) and (2.10) imply

$$g_h + \Delta_h y = Q_h(\Delta y) \quad \text{for all } y \in g + \mathring{E}(\Delta; L_2(\Omega)), \tag{2.11}$$

where Q_h is the orthogonal projection from $L_2(\Omega)$ onto Y_h.

The discrete problem is to find

$$\bar{y}_h = \operatorname*{argmin}_{y_h \in K_h} \frac{1}{2} [\|y_h - y_d\|^2_{L_2(\Omega)} + \beta \|g_h + \Delta_h y_h\|^2_{L_2(\Omega)}], \tag{2.12}$$

where

$$K_h = \{y \in Y_h : y_T(p) \le \psi(p) \text{ for all } p \in \mathcal{V}_T \text{ and all } T \in \mathcal{T}_h\}. \tag{2.13}$$

Here $\mathcal{V}_T$ is the set of the three vertices of T.

The unique solution $\bar{y}_h \in K_h$ of (2.12)–(2.13) is characterized by the discrete variational inequality

$$\int_\Omega (\bar{y}_h - y_d)(y_h - \bar{y}_h) \, dx + \beta \int_\Omega (g_h + \Delta_h \bar{y}_h) \Delta_h (y_h - \bar{y}_h) \, dx \ge 0 \quad \text{for all } y_h \in K_h. \tag{2.14}$$

Note that we can express the constraints in (2.13) concisely as

$$I_T y_T \le I_T(\psi|_T) \quad \text{for all } T \in \mathcal{T}_h, \tag{2.15}$$

where I_T is the nodal interpolation operator for the Lagrange $\mathbb{P}_1$ finite element on T.

Remark 2.4. The discrete problem is a quadratic program with simple box constraints. Let $\mathbf{M}_h$ be the mass matrix that represents the L_2 inner product on Y_h, and let $\mathbf{A}_h$ be the stiffness matrix that represents the bilinear form $a_h(\cdot, \cdot)$ on Y_h. The system matrix for the quadratic program is then given by $\mathbf{M}_h + \beta \mathbf{A}_h \mathbf{M}_h^{-1} \mathbf{A}_h$, and $\mathbf{M}_h^{-1}$ is available because $\mathbf{M}_h$ is a block diagonal matrix. Therefore, the discrete problem can be solved efficiently by the primal-dual active set algorithm in [9, 48].

For simplicity, we will focus on the analysis of the discrete problem for the case where $g = 0$. The extension to the case of general g is straightforward.

3 Some Technical Tools

We collect the results for some finite element tools in this section.

3.1 Mesh-Dependent Norms

Let D be a subdomain of Ω. We will denote by $\mathcal{T}_h(D)$ the collection of elements in $\mathcal{T}_h$ that have a nonempty intersection with D, i.e.,

$$\mathcal{T}_h(D) = \{T \in \mathcal{T}_h : T \cap D \ne \emptyset\}. \tag{3.1}$$

The mesh-dependent semi-norm $|||\cdot|||_{h,D}$ is defined by

$$|||z|||_{h,D} = \Big(\sum_{T\in\mathcal{T}_h(D)} \Big[|z|^2_{H^1(T)} + \sum_{e\subset\partial T} (\sigma^{-1}|e|\|\{\{\nabla z\}\}\cdot n_e\|^2_{L_2(e)} + \sigma|e|^{-1}\|[\![z]\!]\|^2_{L_2(e)})\Big]\Big)^{\frac{1}{2}}, \tag{3.2}$$

where n_e is a unit vector normal to e, and $|||\cdot|||_{h,\Omega}$ is simply denoted by $|||\cdot|||_h$.

Note that

$$a_h(y,z) \le |||y|||_h|||z|||_h \quad \text{for all } y,z \in \mathring{E}(\Delta; L_2(\Omega)) + Y_h \tag{3.3}$$

by the Cauchy–Schwarz inequality, and

$$|||y_h|||_h^2 \le C a_h(y_h, y_h) \quad \text{for all } y_h \in Y_h \tag{3.4}$$

provided that σ is sufficiently large (cf. [57]).

There is also a bound for $|||y|||_h$.

Lemma 3.1. *We have*

$$|||y|||_h \le C(|y|_{H^1(\Omega)} + h^\alpha\|\Delta y\|_{L_2(\Omega)}) \quad \textit{for all } y \in \mathring{E}(\Delta; L_2(\Omega)). \tag{3.5}$$

Proof. Let $y \in \mathring{E}(\Delta; L_2(\Omega))$. It follows from (3.2) that

$$|||y|||_h^2 = |y|^2_{H^1(\Omega)} + \sum_{T\in\mathcal{T}_h}\sum_{e\subset\partial T} \sigma^{-1}|e|\|\{\{\nabla y\}\}\cdot n_e\|^2_{L_2(e)}, \tag{3.6}$$

and we have a trace inequality with scaling,

$$|e|\|\{\{\nabla y\}\}\cdot n_e\|^2_{L_2(e)} \le C\sum_{T\in\mathcal{T}_e}(|y|^2_{H^1(T)} + h^{2\alpha}|y|^2_{H^{1+\alpha}(T)}), \tag{3.7}$$

where $\mathcal{T}_e$ is the set of the triangles in $\mathcal{T}_h$ that share e as a common edge.

Estimate (3.5) follows from (1.7), (3.6) and (3.7). □

3.2 Triangulations

We will consider both quasi-uniform triangulations and triangulations that are graded around the reentrant corners (cf. [2, 5, 6, 12, 45]).

Let D be a subdomain such that $D \Subset \Omega$, i.e., the closure of D is a compact subset of Ω. Note that $\mathcal{T}_h$ is quasi-uniform around D for both types of triangulations. Therefore, we have

$$|||y_h|||_{h,D} \le Ch^{-1}\|y_h\|_{L_2(\Omega)} \quad \text{for all } y_h \in Y_h \tag{3.8}$$

by applying standard inverse estimates (cf. [23, 36]) to (3.2).

Moreover, we have

$$\|y_h\|_{L_\infty(\Omega)} \le C(1 + |\ln h|)^{\frac{1}{2}}|||y_h|||_h \quad \text{for all } y_h \in Y_h \tag{3.9}$$

by the discrete Sobolev inequality in [17].

Remark 3.2. Estimate (3.9) was established in [17] for quasi-uniform triangulations. But the proof in [17] is also valid for meshes graded around the reentrant corners since the discrete Sobolev inequality for conforming Lagrange elements holds for such meshes (cf. [59, Lemma 6.4] and [23, Lemma 4.9.2]).

3.3 The Interpolation Operator Π_h

Let Π_T be the nodal interpolation operator for the $\mathbb{P}_k$ Lagrange finite element on the triangle T. The following estimates (cf. [23, 36]) are standard:

$$\|\zeta - \Pi_T\zeta\|_{L_2(T)} + h_T|\zeta - \Pi_T\zeta|_{H^1(T)} + h_T^2|\zeta - \Pi_h\zeta|_{H^2(T)} \le Ch_T^2|\zeta|_{H^2(T)} \qquad \text{for all } \zeta \in H^2(T), \tag{3.10}$$

$$\|\zeta - \Pi_T\zeta\|_{L_\infty(T)} + h_T|\zeta - \Pi_T\zeta|_{W^{1,\infty}(T)} \le Ch_T^2|\zeta|_{W^{2,\infty}(T)} \qquad \text{for all } \zeta \in W^{2,\infty}(T). \tag{3.11}$$

Let $V_h = Y_h \cap H_0^1(\Omega)$ be the conforming $\mathbb{P}_k$ Lagrange finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$. The operator $\Pi_h : \mathring{E}(\Delta; L_2(\Omega)) \to V_h$ is defined by

$$(\Pi_h \zeta)|_T = \Pi_T(\zeta|_T) \quad \text{for all } T \in \mathcal{T}_h.$$

It follows from (3.10) (cf. [18, 57]) that

$$\|\zeta - \Pi_h \zeta\|_{L_2(\Omega)} + h\|\zeta - \Pi_h \zeta\|_{L_\infty(\Omega)} + h|\!|\!|\zeta - \Pi_h \zeta|\!|\!|_h \le Ch^{1+\tau}\|\Delta\zeta\|_{L_2(\Omega)} \tag{3.12}$$

for all $\zeta \in \mathring{E}(\Delta; L_2(\Omega))$, where

$$\tau = \begin{cases} \alpha & \text{if } \mathcal{T}_h \text{ is quasi-uniform,} \\ 1 & \text{if } \mathcal{T}_h \text{ is graded around the reentrant corners,} \end{cases} \tag{3.13}$$

and α is the index of elliptic regularity as in (1.6).

In the case where $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\|\zeta - \Pi_h \zeta\|_{L_2(\Omega)} + h\|\zeta - \Pi_h \zeta\|_{L_\infty(\Omega)} + h|\!|\!|\zeta - \Pi_h \zeta|\!|\!|_h \le Ch^2|\zeta|_{H^2(\Omega)} \tag{3.14}$$

for both quasi-uniform and graded meshes. In particular, for a subdomain $D \Subset \Omega$, interior elliptic regularity (cf. [42]) and (3.14) imply

$$\|\zeta - \Pi_h \zeta\|_{L_2(D)} + h\|\zeta - \Pi_h \zeta\|_{L_\infty(D)} + h|\!|\!|\zeta - \Pi_h \zeta|\!|\!|_{h,D} \le Ch^2\|\Delta\zeta\|_{L_2(\Omega)} \tag{3.15}$$

for all $\zeta \in \mathring{E}(\Delta; L_2(\Omega))$.

The following result provides an estimate of $\Delta_h \circ \Pi_h$ on smooth functions.

Lemma 3.3. *Let ϕ be a C^∞ function with compact support in Ω. There exists a positive constant C independent of h such that $\|\Delta_h(\Pi_h \phi)\|_{L_2(\Omega)} \le C\|\phi\|_{H^2(\Omega)}$.*

Proof. Let $D \Subset \Omega$ be an open neighborhood of the support of ϕ, and let $z_h \in Y_h$ be arbitrary. It follows from (2.8), (3.3), (3.8) and (3.14) that

$$\begin{aligned} \int_\Omega [\Delta_h(\phi - \Pi_h \phi)] z_h \, dx &= -a_h(\phi - \Pi_h \phi, z_h) \\ &\le |\!|\!|\phi - \Pi_h \phi|\!|\!|_h |\!|\!|z_h|\!|\!|_{h,D} \le Ch|\phi|_{H^2(\Omega)} |\!|\!|z_h|\!|\!|_{h,D} \le C|\phi|_{H^2(\Omega)}\|z_h\|_{L_2(\Omega)}, \end{aligned}$$

which implies $\|\Delta_h(\phi - \Pi_h \phi)\|_{L_2(\Omega)} \le C|\phi|_{H^2(\Omega)}$. Consequently, we have, in view of (2.11) (with $g = 0 = g_h$),

$$\|\Delta_h(\Pi_h \phi)\|_{L_2(\Omega)} \le \|\Delta_h(\phi - \Pi_h \phi)\|_{L_2(\Omega)} + \|\Delta_h \phi\|_{L_2(\Omega)} \le C\|\phi\|_{H^2(\Omega)}.$$ □

3.4 The Ritz Operator R_h

The operator $R_h : \mathring{E}(\Delta; L_2(\Omega)) \to Y_h$ is defined by

$$a_h(R_h \zeta, z_h) = a_h(\zeta, z_h) \quad \text{for all } z_h \in Y_h, \tag{3.16}$$

and we have the following well-known error estimates for the SIP method (cf. [18, 57]):

$$|\!|\!|\zeta - R_h \zeta|\!|\!|_h \le Ch^\tau\|\Delta\zeta\|_{L_2(\Omega)} \quad \text{for all } \zeta \in \mathring{E}(\Delta; L_2(\Omega)), \tag{3.17}$$

$$\|\zeta - R_h \zeta\|_{L_2(\Omega)} \le Ch^{2\tau}\|\Delta\zeta\|_{L_2(\Omega)} \quad \text{for all } \zeta \in \mathring{E}(\Delta; L_2(\Omega)). \tag{3.18}$$

Note that (2.8), (2.11) (with $g = 0 = g_h$) and (3.16) imply

$$\Delta_h(R_h \zeta) = \Delta_h \zeta = Q_h(\Delta\zeta) \quad \text{for all } \zeta \in \mathring{E}(\Delta; L_2(\Omega)). \tag{3.19}$$

3.5 Interior Estimates

Let D_1 and D_2 be subdomains of Ω such that $D_1 \Subset D_2 \Subset \Omega$. We have an interior energy norm error estimate for the SIP method (cf. [25, 34])

$$\|\!|\zeta - R_h\zeta|\!\|_{h,D_1} \le C\big(\min_{z_h\in Y_h} \|\!|\zeta - z_h|\!\|_{h,D_2} + \|\zeta - R_h\zeta\|_{L_2(D_2)}\big) \quad \text{for all } \zeta \in \mathring{E}(\Delta; L_2(\Omega)), \tag{3.20}$$

and also an interior maximum norm error estimate (cf. [25])

$$\|\zeta - R_h\zeta\|_{L_\infty(D_1)} \le C\big(\|\zeta - \Pi_h\zeta\|_{L_\infty(D_2)} + h(1 + |\ln h|)\|\zeta - \Pi_h\zeta\|_{W_h^{1,\infty}(D_2)} + \|\zeta - R_h\zeta\|_{L_2(D_2)} + h\|\!|\zeta - \Pi_h\zeta|\!\|_h\big) \tag{3.21}$$

that is valid for all $\zeta \in \mathring{E}(\Delta; L_2(\Omega))$, where the norm $\|\cdot\|_{W_h^{1,\infty}(D)}$ is given by

$$\|w\|_{W_h^{1,\infty}(D)} = \max_{T\in\mathcal{T}_h(D)} \big[\|\nabla w\|_{L_\infty(T)} + \max_{e\subset\partial T}(\|\{\!\{\nabla w\}\!\} \cdot n_e\|_{L_\infty(e)} + |e|^{-1}\|[\![w]\!]\|_{L_\infty(e)})\big] \tag{3.22}$$

for all $w \in \mathring{E}(\Delta; L_2(\Omega)) + Y_h$.

3.6 The Connection Operator $\mathfrak{C}_h$

Recall $V_h = Y_h \cap H_0^1(\Omega)$ is the conforming $\mathbb{P}_k$ Lagrange finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$. We can construct an operator $\mathfrak{C}_h\colon Y_h \to V_h$ by averaging,

$$(\mathfrak{C}_h y_h)(p) = \frac{1}{|\mathcal{T}_h(p)|} \sum_{T\in\mathcal{T}_h(p)} (y_h|_T)(p) \tag{3.23}$$

for any node p of the $\mathbb{P}_k$ Lagrange finite element space interior to Ω, where $\mathcal{T}_h(p)$ is the set of the triangles in $\mathcal{T}_h$ that share the node p.

It follows from (2.13) and (3.23) that

$$\mathfrak{C}_h v_h = v_h \quad \text{for all } v_h \in V_h, \tag{3.24}$$

$$\mathfrak{C}_h y_h \in K_h \quad \text{for all } y_h \in K_h, \tag{3.25}$$

and for any subdomain D of Ω, we have (cf. [13, 14, 16])

$$h^{-2}\|y_h - \mathfrak{C}_h y_h\|_{L_2(D)}^2 + \sum_{T\in\mathcal{T}_h(D)} |y_h - \mathfrak{C}_h y_h|_{H^1(T)}^2 \le C \sum_{T\in\mathcal{T}_h^*(D)} \sum_{e\subset\partial T} |e|^{-1}\|[\![y_h]\!]\|_{L_2(e)}^2, \tag{3.26}$$

where

$$T \in \mathcal{T}_h \text{ belongs to } \mathcal{T}_h^*(D) \text{ if and only if } S_T \cap D \ne \emptyset. \tag{3.27}$$

Here S_T (the star of T) is the union of all the triangles in $\mathcal{T}_h$ that share a common vertex with T.

Moreover, it follows from (3.23) and a scaling argument that

$$\|\mathfrak{C}_h y_h\|_{L_\infty(T)} \le C\|y_h\|_{L_\infty(S_T)} \quad \text{for all } T \in \mathcal{T}_h. \tag{3.28}$$

3.7 The Smoothing Operator E_h

The operator $E_h\colon Y_h \to \mathring{E}(\Delta; L_2(\Omega))$ is defined by

$$\Delta(E_h y_h) = \Delta_h y_h. \tag{3.29}$$

It follows from (2.9) (with $g = 0$), (3.16) and (3.29) that

$$y_h = R_h(E_h y_h). \tag{3.30}$$

Consequently, we have, by (3.17), (3.18) and (3.29),

$$|||E_h y_h - y_h|||_h \le Ch^\tau \|\Delta_h y_h\|_{L_2(\Omega)} \quad \text{for all } y_h \in Y_h, \tag{3.31}$$

$$\|E_h y_h - y_h\|_{L_2(\Omega)} \le Ch^{2\tau} \|\Delta_h y_h\|_{L_2(\Omega)} \quad \text{for all } y_h \in Y_h. \tag{3.32}$$

Let $G(\mathscr{A}) \Subset \Omega$ be an open neighborhood of the active set $\mathscr{A}$. Putting (3.15), (3.18), (3.20) and (3.29) together, we find

$$|||E_h y_h - y_h|||_{h,G(\mathscr{A})} \le Ch \|\Delta_h y_h\|_{L_2(\Omega)} \quad \text{for all } y_h \in Y_h. \tag{3.33}$$

Note that (3.33) also holds if $G(\mathscr{A})$ is replaced by a subdomain D such that $G(\mathscr{A}) \Subset D \Subset \Omega$. It then follows from (3.2) that

$$\sum_{T \in \mathfrak{T}_h^*(G(\mathscr{A}))} \sum_{e \subset \partial T} |e|^{-1} \|[\![y_h]\!]\|^2_{L_2(e)} \le Ch^2 \|\Delta_h y_h\|^2_{L_2(\Omega)} \quad \text{for all } y_h \in Y_h. \tag{3.34}$$

3.8 The Operator I_h

The operator $I_h : \mathring{E}(\Delta; L_2(\Omega)) \to V_h$ is defined by $(I_h \zeta)|_T = I_T(\zeta|_T)$ for all $T \in \mathfrak{T}_h$, where I_T is the nodal interpolation operator for the Lagrange $\mathbb{P}_1$ finite element on T. It follows from the case $k = 1$ in Section 3.3 that all the estimates for Π_h in Section 3.3 can be applied to I_h.

In addition, we have an obvious estimate

$$\|I_h \zeta\|_{L_\infty(T)} \le \|\zeta\|_{L_\infty(T)} \quad \text{for all } \zeta \text{ continuous on } \bar{T}, \tag{3.35}$$

and, by standard inverse and interpolation estimates (cf. [23, 36]),

$$|q - I_h q|_{H^1(T)} \le Ch_T^{-1} |q - I_h q|_{L_2(T)} \le C|q|_{H^1(T)} \quad \text{for all } q \in \mathbb{P}_k(T), \tag{3.36}$$

where the positive constant C only depends on the shape regularity of T and k.

3.9 Estimates for $\bar{y}$

It follows from (2.5) and (3.19) and a standard interpolation error estimate (cf. [23, 36]) that

$$\|\Delta_h(R_h \bar{y}) - \Delta \bar{y}\|_{L_2(\Omega)} = \|Q_h(\Delta \bar{y}) - \Delta \bar{y}\|_{L_2(\Omega)} \le Ch|\Delta \bar{y}|_{H^1(\Omega)}. \tag{3.37}$$

Let $D \Subset \Omega$. We have, by (2.6), (3.1), (3.11), (3.22) and (3.27),

$$\|\bar{y} - \Pi_h \bar{y}\|_{L_\infty(D)} \le Ch^2 \max_{T \in \mathfrak{T}_h(D)} |\bar{y}|_{W^{2,\infty}(T)}, \tag{3.38}$$

$$\|\bar{y} - \Pi_h \bar{y}\|_{W_h^{1,\infty}(D)} \le Ch \max_{T \in \mathfrak{T}_h^*(D)} |\bar{y}|_{W^{2,\infty}(T)}. \tag{3.39}$$

Combining (3.12), (3.18), (3.21), (3.38) and (3.39), we find

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(D)} \le C\big((1 + |\ln h|)h^2 + h^{2\tau}\big) \tag{3.40}$$

for any subdomain $D \Subset \Omega$.

The following lemma provides a simple global error estimate in the maximum norm.

Lemma 3.4. *We have*

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)} \le C(1 + |\ln h|)^{\frac{1}{2}} h^\tau. \tag{3.41}$$

Proof. According to (3.2) and (3.9),

$$\begin{aligned} \|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)} &\le \|\bar{y} - \Pi_h \bar{y}\|_{L_\infty(\Omega)} + \|\Pi_h \bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)} \\ &\lesssim \|\bar{y} - \Pi_h \bar{y}\|_{L_\infty(\Omega)} + (1 + |\ln h|)^{\frac{1}{2}} |||\Pi_h \bar{y} - R_h \bar{y}|||_h, \end{aligned} \tag{3.42}$$

which together with (3.12) and (3.17) implies (3.41). □

Remark 3.5. More sophisticated maximum norm error estimates for discontinuous Galerkin methods under stronger assumptions can be found in [33, 34, 47].

4 An Abstract Error Estimate

We will measure the error in terms of the mesh-dependent norm $\|\cdot\|_h$ defined by

$$\|y\|_h^2 = \|y\|_{L_2(\Omega)}^2 + \beta\|\Delta_h y\|_{L_2(\Omega)}^2. \tag{4.1}$$

From here on, we also use $(\cdot,\cdot)$ to denote the inner product on $L_2(\Omega)$.

Our goal is to establish the abstract error estimate in the following theorem, where $\bar{y} \in \mathring{E}(\Delta; L_2(\Omega))$ is the solution of (2.1) (with $g = 0$) and $\bar{y}_h$ is the solution of (2.14) (with $g_h = 0$).

Remark 4.1. For simplicity, we have absorbed various norms of y involved in the error analysis into the generic constant C that appears in this section and Section 5.

Theorem 4.2. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_h \le C\big(h + \inf_{y_h \in K_h} [\|\bar{y} - y_h\|_h + \|I_h(\bar{y} - \mathfrak{C}_h y_h)\|_{L_\infty(\mathscr{A})}^{\frac{1}{2}}]\big). \tag{4.2}$$

Proof. Let $y_h \in K$ be arbitrary. It follows from (2.14), (4.1) and the Cauchy–Schwarz inequality that

$$\begin{aligned}\|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta\big(\Delta_h(y_h - \bar{y}), \Delta_h(y_h - \bar{y}_h)\big) + (\bar{y} - y_d, y_h - \bar{y}_h) \\ &\quad + \beta\big(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)\big) - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta\big(\Delta\bar{y}_h, \Delta(y_h - \bar{y}_h)\big) \\ &\le \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta\big(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)\big).\end{aligned} \tag{4.3}$$

Since $E_h(y_h - \bar{y}_h) \in \mathring{E}(\Delta; L_2(\Omega))$, we can use (2.2) and (3.29) to write

$$\begin{aligned}(\bar{y} - y_d, y_h - \bar{y}_h) + \beta\big(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)\big) &= \big(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)\big) + \big(\bar{y} - y_d, E_h(y_h - \bar{y}_h)\big) \\ &\quad + \beta\big(\Delta\bar{y}, \Delta(E_h(y_h - \bar{y}_h))\big) \\ &= \big(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)\big) + \int_\Omega E_h(y_h - \bar{y}_h)\, d\mu,\end{aligned} \tag{4.4}$$

and we have, by (3.32),

$$\big(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)\big) \le Ch^{2\tau}\|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}. \tag{4.5}$$

We can rewrite the last term on the right-hand side of (4.4) as

$$\begin{aligned}\int_\Omega E_h(y_h - \bar{y}_h)\, d\mu &= \int_\Omega [E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)]\, d\mu + \int_\Omega I_h(\psi - \mathfrak{C}_h\bar{y}_h)\, d\mu \\ &\quad + \int_\Omega [I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)]\, d\mu + \int_\Omega I_h(\bar{y} - \psi)\, d\mu + \int_\Omega I_h(\mathfrak{C}_h y_h - \bar{y})\, d\mu \\ &= T_1 + T_2 + T_3 + T_4 + T_5.\end{aligned} \tag{4.6}$$

The estimates for T_2, T_4 and T_5 are straightforward. In view of (2.3), (2.15) and (3.25), we immediately have

$$T_2 = \int_\Omega I_h(\psi - \mathfrak{C}_h\bar{y}_h)\, d\mu \le 0. \tag{4.7}$$

Using (1.5), (2.3), (2.4), (2.6) and (3.11) (applied to I_T), we obtain

$$T_4 = \int_\Omega [I_h(\bar{y} - \psi) - (\bar{y} - \psi)]\, d\mu \le \|I_h(\bar{y} - \psi) - (\bar{y} - \psi)\|_{L_\infty(\mathscr{A})}|\mu(\Omega)| \le Ch^2, \tag{4.8}$$

$$T_5 = \int_\Omega I_h(\mathfrak{C}_h y_h - \bar{y})\, d\mu \le \|I_h(\bar{y} - \mathfrak{C}_h y_h)\|_{L_\infty(\mathscr{A})}|\mu(\Omega)|. \tag{4.9}$$

Next we turn to T_1. Let $G(\mathscr{A}) \Subset \Omega$ be an open neighborhood of the active set $\mathscr{A}$. We have, by (2.7), (3.1), (3.2) and (3.26),

$$
\begin{aligned}
T_1 &= \int_\Omega [E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)]\,d\mu \\
&\lesssim |E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)|_{H^1(G(\mathscr{A}))} \\
&\lesssim \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)|^2_{H^1(T)}\Big)^{\frac{1}{2}} + \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\
&\lesssim |||E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)|||_{h,G(\mathscr{A})} + \Big(\sum_{T\in\mathfrak{T}^*_h(G(\mathscr{A}))} \sum_{e\subset\partial T} |e|^{-1} \|[\![y_h - \bar{y}_h]\!]\|^2_{L_2(e)}\Big)^{\frac{1}{2}},
\end{aligned}
$$

and hence

$$T_1 \le Ch\|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \tag{4.10}$$

by (3.33) and (3.34).

Finally, we consider T_3. Let $w_h = I_h E_h(\bar{y}_h - y_h)$. We have, by (2.7), (3.1), (3.2), (3.12) (applied to I_h), (3.26), (3.33), (3.34) and (3.36),

$$
\begin{aligned}
T_3 &= \int_\Omega [I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)]\,d\mu \\
&\lesssim |I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)|_{H^1(G(\mathscr{A}))} \\
&= |I_h[\mathfrak{C}_h(\bar{y}_h - y_h) - w_h] - [\mathfrak{C}_h(\bar{y}_h - y_h) - w_h]|_{H^1(G(\mathscr{A}))} \\
&\lesssim \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |\mathfrak{C}_h(\bar{y}_h - y_h) - w_h|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |\mathfrak{C}_h(\bar{y}_h - y_h) - (\bar{y}_h - y_h)|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |(\bar{y}_h - y_h) - E_h(\bar{y}_h - y_h)|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\sum_{T\in\mathfrak{T}_h(G(\mathscr{A}))} |E_h(\bar{y}_h - y_h) - I_hE_h(\bar{y}_h - y_h)|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\
&\lesssim h\|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}.
\end{aligned} \tag{4.11}
$$

Putting (4.6)–(4.11) together, we find

$$\int_\Omega E_h(y_h - \bar{y}_h)\,d\mu \le C(h^2 + h\|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} + \|I_h(\bar{y} - \mathfrak{C}_h y_h)\|_{L_\infty(\mathscr{A})}),$$

which together with the inequality of arithmetic and geometric means, (4.1) and (4.3)–(4.5) implies

$$\|y_h - \bar{y}_h\|_h \le C(h + \|y_h - \bar{y}\|_h + \|I_h(\bar{y} - \mathfrak{C}_h y_h)\|^{\frac{1}{2}}_{L_\infty(\mathscr{A})}) \quad \text{for all } y_h \in K_h. \tag{4.12}$$

Estimate (4.2) follows from (4.12) and the triangle inequality. □

An Improved Abstract Error Estimate

Estimate (4.2), which is established under assumption (1.5), implies that $\|\bar{y} - \bar{y}_h\|_h$ is at most $O(h)$ in general. However, it can be improved under the following additional regularity assumptions:

$$\psi \in H^4(\Omega), \tag{4.13}$$

$$\mathscr{A} \text{ has a smooth boundary,} \tag{4.14}$$

$$\bar{y} \text{ belongs to } H^4(D \setminus \mathscr{A}), \tag{4.15}$$

for any $D \Subset \Omega$ that is an open neighborhood of $\mathscr{A}$. Note that it follows from the interior elliptic regularity for the biharmonic operator and (2.4) that it suffices to assume (4.15) for just one such D.

Relations (2.2), (2.4) and integration by parts imply

$$\int_\Omega z\,d\mu = \int_{\mathscr{A}} (\bar{y} - y_d) z\,dx + \beta\Big(\int_{\partial\mathscr{A}} [\![\partial(\Delta\bar{y})/\partial n]\!] z\,ds + \int_{\mathscr{A}} (\Delta^2\bar{y}) z\,dx\Big) \tag{4.16}$$

for all $z \in \mathring{E}(\Delta; L_2(\Omega))$, where n is the outer unit normal on $\partial\mathscr{A}$ and $[\![\partial(\Delta\bar{y})/\partial n]\!]$ equals $\partial(\Delta\bar{y})/\partial n$ from the outside of $\mathscr{A}$ minus $\partial(\Delta\bar{y})/\partial n$ from the inside of $\mathscr{A}$. Consequently, we have, by (4.16) and the trace theorem,

$$\Big|\int_\Omega z\,d\mu\Big| \le C_\epsilon \|z\|_{H^{\frac{1}{2}+\epsilon}(G(\mathscr{A}))} \quad \text{for all } z \in \mathring{E}(\Delta; L_2(\Omega)), \tag{4.17}$$

for any $\epsilon > 0$.

Moreover, assumptions (4.13)–(4.15) also imply the estimate

$$|\bar{y}(x) - \psi(x)| \le C_\epsilon d^{3-\epsilon}, \tag{4.18}$$

which holds for any $\epsilon > 0$ and any $x \in D$ whose distance to $\mathscr{A}$ is less than or equal to d (cf. [20, Lemma 5.5]).

For simplicity, we assume $\tau = 1$ in (3.13) in the discussion below.

In view of (4.18), we can improve estimate (4.8) to

$$T_4 \le C_\epsilon h^{3-\epsilon} \quad \text{for all } \epsilon > 0 \tag{4.19}$$

provided that we choose $G(\mathscr{A})$ so that

$$\operatorname{dist}(\Omega \setminus G(\mathscr{A}), \mathscr{A}) \le h. \tag{4.20}$$

For the term T_1, observe that we have the estimate

$$\begin{aligned}\|E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)\|_{L_2(G(\mathscr{A}))} &\le \|E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)\|_{L_2(G(\mathscr{A}))} \\ &\quad + \|(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)\|_{L_2(G(\mathscr{A}))} \\ &\le Ch^2 \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}\end{aligned} \tag{4.21}$$

by (3.26), (3.32) and (3.34). Consequently, it follows from (4.10), (4.17), (4.21) and interpolation between Sobolev spaces (cf. [58]) that

$$\begin{aligned}T_1 = \int_\Omega [E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)]\,d\mu &\le C_\epsilon \|E_h(y_h - \bar{y}_h) - \mathfrak{C}_h(y_h - \bar{y}_h)\|_{H^{\frac{1}{2}+\epsilon}(G(\mathscr{A}))} \\ &\le C_\epsilon h^{\frac{3}{2}-\epsilon} \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}.\end{aligned} \tag{4.22}$$

Note that we can assume $G(\mathscr{A})$ to be a smooth domain so that the interpolation between Sobolev spaces on $G(\mathscr{A})$ can be handled without difficulty.

Similarly, using (4.11) (where $w_h = I_h E_h(\bar{y}_h - y_h)$) and a standard interpolation error estimate, we find

$$\begin{aligned}\|I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)\|_{L_2(G(\mathscr{A}))} &\le \Big(\sum_{T\in\mathcal{T}_h(G(\mathscr{A}))} \|I_h[\mathfrak{C}_h(\bar{y}_h - y_h) - w_h] - [\mathfrak{C}_h(\bar{y}_h - y_h) - w_h]\|^2_{L_2(T)}\Big)^{\frac{1}{2}} \\ &\le Ch\Big(\sum_{T\in\mathcal{T}_h(G(\mathscr{A}))} |\mathfrak{C}_h(\bar{y}_h - y_h) - w_h|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\ &\le Ch^2 \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)},\end{aligned}$$

which together with (4.11), (4.17) and interpolation between Sobolev spaces yields

$$\begin{aligned}T_3 = \int_\Omega [I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)]\,d\mu &\le C_\epsilon \|I_h\mathfrak{C}_h(\bar{y}_h - y_h) - \mathfrak{C}_h(\bar{y}_h - y_h)\|_{H^{\frac{1}{2}+\epsilon}(G(\mathscr{A}))} \\ &\le C_\epsilon h^{\frac{3}{2}-\epsilon} \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}.\end{aligned} \tag{4.23}$$

Putting (4.1), (4.3), (4.4), (4.5) (with $\tau = 1$), (4.6), (4.7), (4.9), (4.19), (4.22) and (4.23) together, we arrive at the improved abstract error estimate

$$\|\bar{y} - \bar{y}_h\|_h \le C_\epsilon \big(h^{\frac{3}{2}-\epsilon} + \inf_{y_h\in K_h} [\|\bar{y} - y_h\|_h + \|I_h(\bar{y} - \mathfrak{C}_h y_h)\|^{\frac{1}{2}}_{L_\infty(\mathscr{A})}]\big). \tag{4.24}$$

5 Concrete Error Estimates

The key to derive concrete error estimates is to bound the infimum in (4.2) by constructing a function $y_h^* \in K_h$ with the desired properties.

Lemma 5.1. *There exists $y_h^* \in K_h$ for h sufficiently small such that*

$$\|\bar{y} - y_h^*\|_h + \|I_h(\bar{y} - \mathfrak{C}_h y_h^*)\|_{L_\infty(\mathscr{A})}^{\frac{1}{2}} \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^\tau\big). \tag{5.1}$$

Proof. Let $G(\mathscr{A}) \Subset \Omega$ be an open neighborhood of the active set $\mathscr{A}$. According to (3.40), we have

$$\epsilon_h = \|R_h\bar{y} - \bar{y}\|_{L_\infty(G(\mathscr{A}))} \le C\big((1 + |\ln h|)h^2 + h^{2\tau}\big). \tag{5.2}$$

Let ϕ be a nonnegative C^∞ function with compact support in Ω such that $\phi = 1$ on $G(\mathscr{A})$, and let $y_h^* \in Y_h$ be defined by

$$y_h^* = R_h\bar{y} - \epsilon_h \Pi_h \phi. \tag{5.3}$$

First we show that y_h^* belongs to K_h for sufficiently small h. Indeed, we have $\psi - \bar{y} \ge \delta > 0$ on $\Omega \setminus G(\mathscr{A})$ and hence

$$y_h^* \le R_h\bar{y} = \bar{y} + (R_h\bar{y} - \bar{y}) \le \psi - \delta + (R_h\bar{y} - \bar{y}) \quad \text{on } \Omega \setminus G(\mathscr{A}),$$

which, in view of (3.41), implies

$$y_h^*(p) \le \psi(p) \quad \text{for any vertex } p \text{ of } \mathfrak{T}_h \text{ that belongs to } \Omega \setminus G(\mathscr{A})$$

provided that h is sufficiently small. On the other hand, for any vertex p of $\mathfrak{T}_h$ that belongs to $G(\mathscr{A})$, we have, by (5.2) and (5.3),

$$y_h^*(p) = (R_h\bar{y})(p) - \epsilon_h = \bar{y}(p) + (R_h\bar{y} - \bar{y})(p) - \epsilon_h \le \bar{y}(p) \le \psi(p).$$

Next we estimate the two terms that appear on the left-hand side of (5.1). According to Lemma 3.3, (3.18), (3.19), (5.2) and (5.3), we have

$$\|\bar{y} - y_h^*\|_{L_2(\Omega)} \le \|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \epsilon_h\|\Pi_h\phi\|_{L_2(\Omega)} \le C\epsilon_h, \tag{5.4}$$

$$\|\Delta_h(\bar{y} - y_h^*)\|_{L_2(\Omega)} = \epsilon_h\|\Delta_h(\Pi_h\phi)\|_{L_2(\Omega)} \le C\epsilon_h, \tag{5.5}$$

and therefore, in view of (4.1) and (5.2),

$$\|\bar{y} - y_h^*\|_h \le C\big((1 + |\ln h|)h^2 + h^{2\tau}\big). \tag{5.6}$$

For the second term, we find, by (3.24), (3.27), (3.28) and (3.35),

$$\begin{aligned}
\|I_h(\bar{y} - \mathfrak{C}_h y_h^*)\|_{L_\infty(\mathscr{A})} &\le \max_{T \in \mathfrak{T}_h(G(\mathscr{A}))} \|\bar{y} - \mathfrak{C}_h y_h^*\|_{L_\infty(T)} \\
&\le \max_{T \in \mathfrak{T}_h(G(\mathscr{A}))} \big(\|\bar{y} - \Pi_h\bar{y}\|_{L_\infty(T)} + \|\mathfrak{C}_h(\Pi_h\bar{y} - R_h\bar{y})\|_{L_\infty(T)} + \epsilon_h\|\Pi_h\phi\|_{L_\infty(T)}\big) \\
&\lesssim \max_{T \in \mathfrak{T}_h^*(G(\mathscr{A}))} \big(\|\bar{y} - \Pi_h\bar{y}\|_{L_\infty(T)} + \|\Pi_h\bar{y} - R_h\bar{y}\|_{L_\infty(T)}\big) + \epsilon_h \\
&\lesssim \max_{T \in \mathfrak{T}_h^*(G(\mathscr{A}))} \big(\|\bar{y} - \Pi_h\bar{y}\|_{L_\infty(T)} + \|\bar{y} - R_h\bar{y}\|_{L_\infty(T)}\big) + \epsilon_h,
\end{aligned} \tag{5.7}$$

and hence, in view of (3.38), (3.40) and (5.2),

$$\|I_h(\bar{y} - \mathfrak{C}_h y_h^*)\|_{L_\infty(\mathscr{A})}^{\frac{1}{2}} \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^\tau\big). \tag{5.8}$$

Estimate (5.1) follows from (5.6)–(5.8). □

We can now establish several concrete error estimates.

Theorem 5.2. *Let $\bar{y}_h \in K_h$ be the solution of* (2.12)–(2.13) *and $\bar{u}_h = -\Delta_h \bar{y}_h$. There exists a positive constant C independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}\big). \tag{5.9}$$

Proof. It follows from (4.1), Theorem 4.2 and Lemma 5.1 that

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}\big), \tag{5.10}$$

which together with (2.11) (where $g = 0 = g_h$) and (3.37) yields

$$\begin{aligned} \|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} &\le \|\Delta\bar{y} - \Delta_h\bar{y}\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \\ &= \|\Delta\bar{y} - Q_h(\Delta\bar{y})\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \\ &\le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}\big). \end{aligned} \tag{5.11}$$

□

The following lemmas will enable us to establish estimates for $\bar{y} - \bar{y}_h$ in other norms.

Lemma 5.3. *There exists a positive constant C independent of h such that*

$$|\!|\!| y_h |\!|\!|_h \le C\|\Delta_h y_h\|_{L_2(\Omega)} \quad \textit{for all } y_h \in Y_h.$$

Proof. In view of (3.3), (3.4), (3.16), (3.30), we have

$$|\!|\!| y_h |\!|\!|_h^2 \lesssim a_h(y_h, y_h) = a_h(E_h y_h, y_h) \le |\!|\!| E_h y_h |\!|\!|_h |\!|\!| y_h |\!|\!|_h$$

and hence

$$|\!|\!| y_h |\!|\!|_h \lesssim |\!|\!| E_h y_h |\!|\!|_h \lesssim \|\Delta(E_h y_h)\|_{L_2(\Omega)} = \|\Delta_h y_h\|_{L_2(\Omega)}$$

by (1.7), Lemma 3.1 and (3.29). □

Lemma 5.4. *There exists a positive constant C independent of h such that*

$$\|y_h\|_{L_\infty(\Omega)} \le C\|\Delta_h y_h\|_{L_2(\Omega)} \quad \textit{for all } y_h \in Y_h. \tag{5.12}$$

Proof. It follows from (3.9), (3.12), (3.29) and (3.31) that

$$\begin{aligned} \|y_h - E_h y_h\|_{L_\infty(\Omega)} &\le \|y_h - \Pi_h(E_h y_h)\|_{L_\infty(\Omega)} + \|\Pi_h(E_h y_h) - E_h y_h\|_{L_\infty(\Omega)} \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} |\!|\!| y_h - \Pi_h(E_h y_h) |\!|\!|_h + h^{\tau}\|\Delta(E_h y_h)\|_{L_2(\Omega)} \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} \big(|\!|\!| y_h - E_h y_h |\!|\!|_h + |\!|\!| E_h y_h - \Pi_h(E_h y_h) |\!|\!|_h\big) + h^{\tau}\|\Delta(E_h y_h)\|_{L_2(\Omega)} \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} h^{\tau}\|\Delta(E_h y_h)\|_{L_2(\Omega)} = (1 + |\ln h|)^{\frac{1}{2}} h^{\tau}\|\Delta_h y_h\|_{L_2(\Omega)}. \end{aligned} \tag{5.13}$$

Moreover, we have, by (1.7), (3.29) and the Sobolev inequality,

$$\|E_h y_h\|_{L_\infty(\Omega)} \lesssim \|E_h y_h\|_{H^{1+\alpha}(\Omega)} \lesssim \|\Delta_h(E_h y_h)\|_{L_2(\Omega)} = \|\Delta_h y_h\|_{L_2(\Omega)}. \tag{5.14}$$

Estimate (5.12) follows from (5.13) and (5.14). □

Theorem 5.5. *There exists a positive constant C independent of h such that*

$$|\!|\!| \bar{y} - \bar{y}_h |\!|\!|_h \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}\big), \tag{5.15}$$

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \le C\big((1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}\big). \tag{5.16}$$

Proof. In view of (3.19), (5.10) and Lemma 5.3, we have

$$|\!|\!| R_h\bar{y} - \bar{y}_h |\!|\!|_h \lesssim \|\Delta_h(R_h\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \lesssim (1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau},$$

and therefore

$$|\!|\!| \bar{y} - \bar{y}_h |\!|\!|_h \le |\!|\!| \bar{y} - R_h\bar{y} |\!|\!|_h + |\!|\!| R_h\bar{y} - \bar{y}_h |\!|\!|_h \lesssim (1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau}$$

by (3.17).

According to (3.19), (5.10) and Lemma 5.4, we have

$$\|R_h\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \lesssim \|\Delta_h(R_h\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \lesssim (1 + |\ln h|)^{\frac{1}{2}} h + h^{\tau},$$

which together with (3.41) and the triangle inequality implies (5.16). □

Improved Error Estimates

We can improve estimate (5.1) under additional regularity assumptions. For the discussion below, we assume (4.13), (4.14) and strengthen (4.15) to

$$\bar{y} \in H^4(\Omega \setminus \mathscr{A}). \tag{5.17}$$

Combining $\bar{y} \in H^3_{\mathrm{loc}}(\Omega)$ with (4.13), (4.14) and (5.17), we see that $\bar{y}$ belongs to $C^2(\bar{\Omega})$ and

$$\bar{y} \in H^{\frac{7}{2}-\epsilon}(\Omega) \quad \text{for all } \epsilon > 0. \tag{5.18}$$

For simplicity, we assume $\tau = 1$ in (3.13) and $k = 2$ in the discussion below.

It follows from (5.18) that

$$|\!|\!|\bar{y} - \Pi_h \bar{y}|\!|\!|_h \le Ch^2 |\bar{y}|_{H^3(\Omega)}, \tag{5.19}$$

and estimates (3.17) and (3.18) can then be improved to

$$\|\bar{y} - R_h \bar{y}\|_{L_2(\Omega)} + h|\!|\!|\bar{y} - R_h \bar{y}|\!|\!|_h \le Ch^3 |\bar{y}|_{H^3(\Omega)}. \tag{5.20}$$

Assumption (4.13) and the Sobolev inequality imply that $\psi \in W^{3,p}(\Omega)$ for any $p < \infty$ and hence

$$\|\psi - \Pi_T \psi\|_{L_\infty(T)} \le C_\epsilon h_T^{3-\epsilon} \quad \text{for all } T \in \mathfrak{T}_h,\ \epsilon > 0 \tag{5.21}$$

by the Bramble–Hilbert lemma (cf. [11, 39]) and scaling. We can combine estimate (4.18) with (5.21) to obtain

$$\|\bar{y} - \Pi_h \bar{y}\|_{L_\infty(G(\mathscr{A}))} \le \|(\bar{y} - \psi) - \Pi_h(\bar{y} - \psi)\|_{L_\infty(G(\mathscr{A}))} + \|\psi - \Pi_h \psi\|_{L_\infty(G(\mathscr{A}))} \le C_\epsilon h^{3-\epsilon} \tag{5.22}$$

provided that $G(\mathscr{A})$ satisfies (4.20). Similarly, the estimate

$$\|\bar{y} - \Pi_h \bar{y}\|_{W_h^{1,\infty}(G(\mathscr{A}))} \le C_\epsilon h^{2-\epsilon} \tag{5.23}$$

follows from (3.22), (4.18) and (5.21). Combining (3.21) and (5.20)–(5.23), we have the estimate

$$\epsilon_h = \|R_h \bar{y} - \bar{y}\|_{L_\infty(G(\mathscr{A}))} \le C_\epsilon h^{3-\epsilon} \tag{5.24}$$

that improves (5.2). Consequently, estimate (5.8) is improved to

$$\|I_h(\bar{y} - \mathfrak{C}_h y_h^*)\|^{\frac{1}{2}}_{L_\infty(\mathscr{A})} \le C_3 h^{\frac{3}{2}-\epsilon} \tag{5.25}$$

by (5.7), (5.22) and (5.24). Putting (5.4), (5.5), (5.24) and (5.25) together, we obtain the estimate

$$\|\bar{y} - y_h^*\|_h + \|I_h(\bar{y} - \mathfrak{C}_h y_h^*)\|^{\frac{1}{2}}_{L_\infty(\mathscr{A})} \le C_\epsilon h^{\frac{3}{2}-\epsilon} \tag{5.26}$$

that improves (5.1). Hence we have

$$\|\bar{y} - \bar{y}_h\|_h \le C_\epsilon h^{\frac{3}{2}-\epsilon}$$

by (4.24) and (5.26), and estimate (5.10) becomes

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \le C_\epsilon h^{\frac{3}{2}-\epsilon},$$

and thus, in view of (5.18), estimate (5.11) becomes

$$\begin{aligned}\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} &\le \|\Delta\bar{y} - \Delta_h \bar{y}\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \\ &\le \|\Delta\bar{y} - Q_h(\Delta\bar{y})\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \le C_\epsilon h^{\frac{3}{2}-\epsilon}.\end{aligned}$$

Therefore, we have the estimate

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} \le C_\epsilon h^{\frac{3}{2}-\epsilon} \tag{5.27}$$

that improves (5.9). Similarly estimates (5.15) and (5.16) can be improved to

$$|\!|\!|\bar{y} - \bar{y}_h|\!|\!|_h \le C_\epsilon h^{\frac{3}{2}-\epsilon} \quad \text{and} \quad \|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \le C_\epsilon h^{\frac{3}{2}-\epsilon}$$

through (3.42), (5.19) and (5.20).

6 Numerical Results

We have performed numerical experiments on the SIP method for $k = 1$ and $k = 2$, where the penalty parameter $\sigma = 3k(k+1)$ (cf. [41]). The mesh parameter for the j-th level refinement is $h_j = 8/2^j$. The discrete problems are solved by a primal-dual active set algorithm (cf. [7, 9, 48, 50]).

We report the errors of the optimal state in $\|\cdot\|_{L_2(\Omega)}$ and $|\!|\!|\cdot|\!|\!|_h$, and the errors of the optimal control in $\|\cdot\|_{L_2(\Omega)}$. We also report the approximation of $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$ by $\|\Pi_h \bar{y} - \bar{y}_h\|_\infty$, which is the ∞-norm of the finite element coefficient vector representing $\Pi_h \bar{y} - \bar{y}_h$.

The approximate solution $\bar{y}_{h_j}$ and the operator Π_{h_j} are denoted by $\bar{y}_j$ and Π_j, respectively, in all the tables.

Remark 6.1. We have also tested the SIP method based on the $\mathbb{P}_3$ finite element. The results are similar to the ones for the $\mathbb{P}_2$ element for the examples in Section 6.1 and Section 6.2.

6.1 Square Example

Let $\Omega = (-4, 4)^2$. We consider the optimal control problem (1.1) with $\psi(x) = |x|^2 - 1$, $\beta = 1$, $g = 0$ and

$$y_d(x) = \begin{cases} \Delta^2 \bar{y}(x) + \bar{y}(x) & \text{if } |x| > 1, \\ \Delta^2 \bar{y}(x) + \bar{y}(x) + 2 & \text{if } |x| \le 1, \end{cases}$$

where the optimal state $\bar{y}$ is given by

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \le 1, \\ v(|x|) + [1 - \phi(|x|)] w(x) & \text{if } 1 \le |x| \le 3, \\ w(x) & \text{if } |x| \ge 3, \end{cases}$$

with

$$\begin{aligned} v(r) &= (r^2 - 1)\Big(1 - \frac{r-1}{2}\Big)^4 + \frac{1}{4}(r-1)^2(r-3)^4, \\ \phi(r) &= \Big[1 + 4\Big(\frac{r-1}{2}\Big) + 10\Big(\frac{r-1}{2}\Big)^2 + 20\Big(\frac{r-1}{2}\Big)^3\Big]\Big(1 - \frac{r-1}{2}\Big)^4, \\ w(x) &= 2\sin\Big(\frac{\pi}{8}(x_1 + 4)\Big)\sin\Big(\frac{\pi}{8}(x_2 + 4)\Big). \end{aligned}$$

This example from [26] is designed such that $\mathscr{A}$ is the disc defined by $|x| \le 1$, and $\bar{y}$ and $\bar{u}$ have zero traces on the boundary. The numerical results for $k = 1$ and $k = 2$ are reported in Table 1 and Table 2, respectively.

Note that the additional regularity assumptions (4.13), (4.14) and (5.17) are satisfied for this example. In view of (5.18), we have

$$\bar{u} = -\Delta \bar{y} \in H^{\frac{3}{2}-\epsilon}(\Omega) \quad \text{for all } \epsilon > 0, \tag{6.1}$$

j	$\|\bar{y} - \bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\!\|\!\|\bar{y} - \bar{y}_j\|\!\|\!\|_h$	Order	$\|\Pi_j \bar{y} - \bar{y}_j\|_\infty$	Order	$\|\bar{u} - \bar{u}_j\|_{L_2(\Omega)}$	Order
0	4.5810×10^{1}	—	5.3417×10^{1}	—	9.2024×10^{0}	—	1.8989×10^{1}	–
1	3.5379×10^{1}	0.37	3.6566×10^{1}	0.55	9.5624×10^{0}	−0.06	4.5579×10^{1}	−1.26
2	3.0432×10^{0}	3.54	4.0927×10^{0}	3.16	1.1012×10^{0}	3.12	7.5447×10^{0}	2.59
3	1.3713×10^{0}	1.15	2.7151×10^{0}	0.59	4.1099×10^{-1}	1.42	6.1330×10^{0}	0.30
4	1.9044×10^{-1}	2.85	1.1673×10^{0}	1.22	8.0634×10^{-2}	2.35	2.9217×10^{0}	1.07
5	4.8280×10^{-2}	1.98	5.5406×10^{-1}	1.08	2.5790×10^{-2}	1.64	1.0700×10^{0}	1.45
6	1.7648×10^{-2}	1.45	2.7205×10^{-1}	1.03	7.9764×10^{-3}	1.69	3.6576×10^{-1}	1.55
7	1.2534×10^{-2}	0.49	1.3587×10^{-1}	1.00	2.9755×10^{-3}	1.42	1.6622×10^{-1}	1.14
8	2.4529×10^{-3}	2.35	6.7554×10^{-2}	1.01	6.5540×10^{-4}	2.18	5.8690×10^{-2}	1.50
9	5.1696×10^{-4}	2.25	3.3747×10^{-2}	1.00	1.3571×10^{-4}	2.27	2.0999×10^{-2}	1.48

Table 1: Example on square ($k = 1$)

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\|\bar{y}-\bar{y}_j\|\|\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	3.7032×10^{0}	—	1.4658×10^{1}	—	1.1510×10^{0}	—	9.1462×10^{0}	–
1	8.3726×10^{0}	−1.18	7.3305×10^{0}	1.00	1.7694×10^{0}	−0.62	9.2196×10^{0}	−0.01
2	3.2350×10^{0}	1.37	5.3145×10^{0}	0.46	1.3622×10^{0}	0.38	1.4794×10^{1}	−0.68
3	8.3648×10^{-1}	1.95	1.5522×10^{0}	1.78	2.6935×10^{-1}	2.34	6.0855×10^{0}	1.28
4	1.2604×10^{-1}	2.73	3.3384×10^{-1}	2.22	5.7337×10^{-2}	2.23	2.1357×10^{0}	1.51
5	2.8159×10^{-2}	2.16	7.5093×10^{-2}	2.15	6.8294×10^{-3}	3.07	8.1279×10^{-1}	1.39
6	7.2141×10^{-3}	1.96	1.8852×10^{-2}	1.99	2.2932×10^{-3}	1.57	2.5827×10^{-1}	1.65
7	1.2074×10^{-3}	2.58	4.1633×10^{-3}	2.18	4.1299×10^{-4}	2.47	9.9289×10^{-2}	1.38
8	4.1976×10^{-4}	1.52	1.1245×10^{-3}	1.89	1.3026×10^{-4}	1.66	3.3727×10^{-2}	1.56

Table 2: Example on square ($k = 2$)

and also the Hölder regularity

$$\bar{y} \in C^{2,\frac{1}{2}-\epsilon}(\bar{\Omega}) \quad \text{for all } \epsilon > 0 \tag{6.2}$$

by the Sobolev Embedding Theorem (cf. [60, Section 2.8]).

The following are the best possible results allowed by the regularity (5.18), (6.1) and (6.2):

$$\|\bar{y}-\bar{y}_h\|_{L_2(\Omega)} \le \begin{cases} Ch^2, & k=1,\\ Ch^3, & k=2,\end{cases} \tag{6.3}$$

$$\|\|\bar{y}-\bar{y}_h\|\|_h \le \begin{cases} Ch, & k=1,\\ Ch^2, & k=2,\end{cases} \tag{6.4}$$

$$\|\bar{y}-\bar{y}_h\|_{L_\infty(\Omega)} \le \begin{cases} Ch^2, & k=1,\\ C_\epsilon h^{\frac{5}{2}-\epsilon}, & k=2,\end{cases} \tag{6.5}$$

$$\|\bar{u}-\bar{u}_h\|_{L_2(\Omega)} \le \begin{cases} C_\epsilon h^{\frac{3}{2}-\epsilon}, & k=1,\\ C_\epsilon h^{\frac{3}{2}-\epsilon}, & k=2.\end{cases} \tag{6.6}$$

The numerical results for $\|\|\bar{y}-\bar{y}_h\|\|_h$ and $\|\bar{u}-\bar{u}_h\|_{L_2(\Omega)}$ in Table 1 and Table 2 match exactly (6.4) and (6.6). The orders of convergence for $\|\bar{y}-\bar{y}_h\|_{L_2(\Omega)}$ and $\|\Pi_h\bar{y}-\bar{y}_h\|_{L_\infty(\Omega)}$ are less stable, possibly due to the fact that the free boundary is a circle that is not captured exactly by the mesh, and that the quality of the representation of the circle by the meshes varies from refinement level to refinement level. Nevertheless, the numerical results for $\|\Pi_h\bar{y}-\bar{y}_h\|_{L_\infty(\Omega)}$ in Table 1 and in Table 2 (up to the seventh level refinement) indicate (6.5) asymptotically. The order of convergence for $\|\bar{y}-\bar{y}_h\|_{L_2(\Omega)}$ in Table 1 also agrees with (6.3). Finally, the order of convergence for $\|\bar{y}-\bar{y}_h\|_{L_2(\Omega)}$ in Table 2 appears to be at most $5/2$, which is less than the optimal order in (6.3) for $k = 2$.

Note that the $O(h)$ convergence for $\|\|\bar{y}-\bar{y}_h\|\|_h$ in Table 1 agrees with estimate (5.15) (for $\tau = 1$), and the $O(h^{\frac{3}{2}})$ convergence for $\|\bar{u}-\bar{u}_h\|_{L_2(\Omega)}$ in Table 2 agrees with the estimate in (5.27). While these two estimates are the only ones we can establish within our theoretical framework that match the numerical results exactly, we have provided a heuristic justification of the other numerical results in Appendix A.

Pictures for the optimal state, the optimal control and the active set computed by the $\mathbb{P}_1$ SIP method at level 7 are depicted in Figure 1.

6.2 L-Shaped Domain Example

We extend the previous example to the L-shaped domain $\Omega = (-8,8)^2 \setminus ([0,8]\times[-8,0])$ by shifting the active set to the center of the upper left quadrant (cf. Figure 2) and by adding the corner singular function

$$\psi_s(x) = r^{\frac{2}{3}} \sin\Big(\frac{2}{3}\theta\Big),$$

where (r,θ) are the polar coordinates of x.

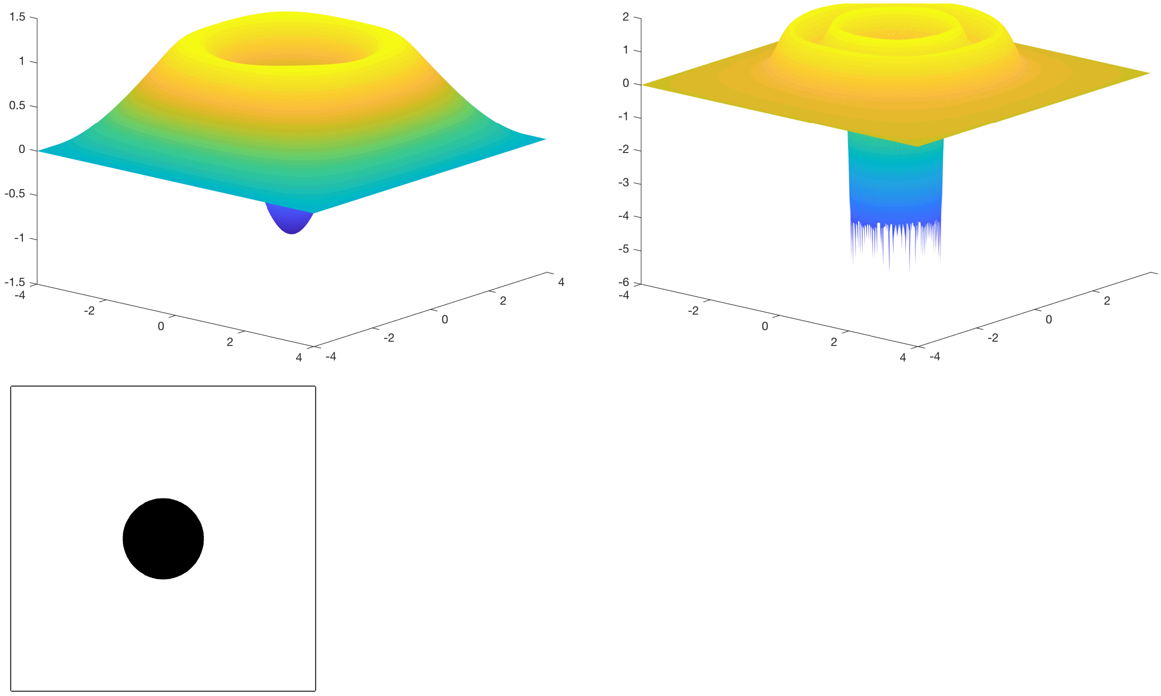

Figure 1: State, control and active set computed by a uniform mesh ($k = 1, j = 7$)

More precisely, we take $\beta = 1$, $g \in H^4(\Omega)$ such that $g = 4\psi_s$ on $\partial\Omega$, $x_* = (-4, 4)$ (the center of the upper left quadrant),

$$\psi(x) = |x - x_*|^2 - 1 + 4\psi_s(x),$$

and

$$y_d(x) = \begin{cases} \Delta^2\bar{y}(x) + \bar{y}(x) & \text{if } |x - x_*| > 1, \\ \Delta^2\bar{y}(x) + \bar{y}(x) + 2 & \text{if } |x - x_*| \le 1, \end{cases}$$

where

$$\bar{y}(x) = 4\psi_s(x) + \begin{cases} |x - x_*|^2 - 1 & \text{if } |x - x_*| \le 1, \\ v(|x - x_*|) + [1 - \phi(|x - x_*|)]w(x - x_*) & \text{if } 1 \le |x - x_*| \le 3, \\ w(x - x_*) & \text{if } |x - x_*| \ge 3, \end{cases}$$

with v, ϕ, w as in the previous example.

The numerical results for $k = 1$ and $k = 2$ on uniform and graded meshes are displayed in Tables 3 and 4 and Tables 5 and 6, respectively.

Due to the singularity at the reentrant corner, we observe $O(h^{\frac{2}{3}})$ convergence for $|||\bar{y} - \bar{y}_h|||_h$ on uniform meshes in Table 3 and Table 5, which agrees with estimate (5.15) for $\tau = \frac{2}{3} - \epsilon$. For graded mesh refinement with grading parameter 0.6 for $k = 1$ and 0.3 for $k = 2$, the convergence of $|||\bar{y} - \bar{y}_h|||_h$ is improved to $O(h)$ and $O(h^2)$, respectively, as in the square domain example of Section 6.1. The performance of other approximations on graded meshes is also similar to the ones in Section 6.1.

The pictures of the optimal state, the optimal control and the active set computed by the $\mathbb{P}_1$ SIP method on the uniform mesh at level 6 are depicted in Figure 2. We have also included the picture of the active set computed on the graded mesh, which shows a better approximation.

6.3 Cube Example

We have also tested the SIP method on a three-dimensional example, which is an extension of the example in Section 6.1.

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\|\bar{y}-\bar{y}_j\|\|\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	3.9920×10^{1}	—	4.4007×10^{1}	—	9.4995×10^{0}	—	2.2870×10^{1}	–
1	3.9904×10^{1}	0.00	3.3403×10^{1}	0.40	1.0482×10^{1}	−0.14	4.0431×10^{1}	−0.82
2	3.4537×10^{0}	3.53	5.7522×10^{0}	2.54	1.4600×10^{0}	2.84	7.4429×10^{0}	2.44
3	1.4574×10^{0}	1.24	3.6802×10^{0}	0.64	9.7520×10^{-1}	0.58	6.1192×10^{0}	0.28
4	2.1943×10^{-1}	2.73	1.9689×10^{0}	0.90	6.1792×10^{-1}	0.66	2.9231×10^{0}	1.07
5	5.7923×10^{-2}	1.92	1.1369×10^{0}	0.79	3.8999×10^{-1}	0.66	1.0723×10^{0}	1.45
6	1.8490×10^{-2}	1.65	6.7892×10^{-1}	0.74	2.4632×10^{-1}	0.66	3.6672×10^{-1}	1.55
7	1.4113×10^{-2}	0.39	4.1360×10^{-1}	0.72	1.5534×10^{-1}	0.67	1.6617×10^{-1}	1.14
8	2.7271×10^{-3}	2.37	2.5482×10^{-1}	0.70	9.7860×10^{-2}	0.67	5.8672×10^{-2}	1.50

Table 3: Example on L-shaped domain for uniform meshes ($k = 1$)

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\|\bar{y}-\bar{y}_j\|\|\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	3.9920×10^{1}	—	4.4007×10^{1}	—	9.4995×10^{0}	—	2.2870×10^{1}	–
1	2.6192×10^{1}	0.61	2.8876×10^{1}	0.61	1.0147×10^{1}	−0.10	3.3432×10^{1}	−0.55
2	2.6055×10^{0}	3.33	4.4817×10^{0}	2.69	1.1320×10^{0}	3.16	7.5177×10^{0}	2.15
3	1.5013×10^{0}	0.80	2.9781×10^{0}	0.59	4.1680×10^{-1}	1.44	6.1210×10^{0}	0.30
4	2.0284×10^{-1}	2.89	1.3167×10^{0}	1.18	1.5541×10^{-1}	1.42	2.9203×10^{0}	1.07
5	5.7017×10^{-2}	1.83	6.3409×10^{-1}	1.05	7.3336×10^{-2}	1.08	1.0716×10^{0}	1.45
6	2.0081×10^{-2}	1.51	3.1756×10^{-1}	1.00	3.8830×10^{-2}	0.92	3.6665×10^{-1}	1.55
7	1.1602×10^{-2}	0.79	1.5707×10^{-1}	1.02	1.5405×10^{-2}	1.33	1.6612×10^{-1}	1.14
8	2.5718×10^{-3}	2.17	7.8324×10^{-2}	1.00	7.2773×10^{-3}	1.08	5.8662×10^{-2}	1.50

Table 4: Example on L-shaped domain for graded meshes with grading $\mu = 0.6$ ($k = 1$)

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\|\bar{y}-\bar{y}_j\|\|\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	4.2702×10^{0}	—	1.1243×10^{1}	—	2.2648×10^{0}	—	8.6780×10^{0}	–
1	8.2827×10^{0}	−0.96	7.3681×10^{0}	0.61	1.7615×10^{0}	0.36	9.1700×10^{0}	−0.08
2	3.3045×10^{0}	1.33	5.5664×10^{0}	0.40	1.3632×10^{0}	0.37	1.4789×10^{1}	−0.69
3	8.3352×10^{-1}	1.99	1.8441×10^{0}	1.59	4.3771×10^{-1}	1.64	6.0881×10^{0}	1.28
4	1.3230×10^{-1}	2.66	7.0735×10^{-1}	1.38	2.7683×10^{-1}	0.66	2.1354×10^{0}	1.51
5	3.2286×10^{-2}	2.03	3.9996×10^{-1}	0.82	1.7469×10^{-1}	0.66	8.1272×10^{-1}	1.39
6	7.3717×10^{-3}	2.13	2.4830×10^{-1}	0.69	1.1000×10^{-1}	0.67	2.5828×10^{-1}	1.65
7	1.6831×10^{-3}	2.13	1.5603×10^{-1}	0.67	6.9299×10^{-2}	0.67	9.9293×10^{-2}	1.38

Table 5: Example on L-shaped domain for uniform meshes ($k = 2$)

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\|\bar{y}-\bar{y}_j\|\|\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	4.2702×10^{0}	—	1.1243×10^{1}	—	2.2648×10^{0}	—	8.6780×10^{0}	–
1	1.2868×10^{1}	−1.59	1.7278×10^{1}	−0.62	5.9386×10^{0}	−1.39	2.4659×10^{1}	−1.51
2	3.3202×10^{0}	1.95	5.3727×10^{0}	1.69	1.4139×10^{0}	2.07	1.4917×10^{1}	0.73
3	8.3547×10^{-1}	1.99	1.5466×10^{0}	1.80	2.6935×10^{-1}	2.39	6.0817×10^{0}	1.29
4	1.2014×10^{-1}	2.80	3.2962×10^{-1}	2.23	5.7276×10^{-2}	2.23	2.1342×10^{0}	1.51
5	2.7127×10^{-2}	2.15	7.4216×10^{-2}	2.15	6.8314×10^{-3}	3.07	8.1287×10^{-1}	1.39
6	8.0864×10^{-3}	1.75	1.8834×10^{-2}	1.98	2.3745×10^{-3}	1.52	2.5826×10^{-1}	1.65
7	1.4089×10^{-3}	2.52	4.0875×10^{-3}	2.20	4.8728×10^{-4}	2.28	9.9273×10^{-2}	1.38

Table 6: Example on L-shaped domain for graded meshes with grading $\mu = 0.3$ ($k = 2$)

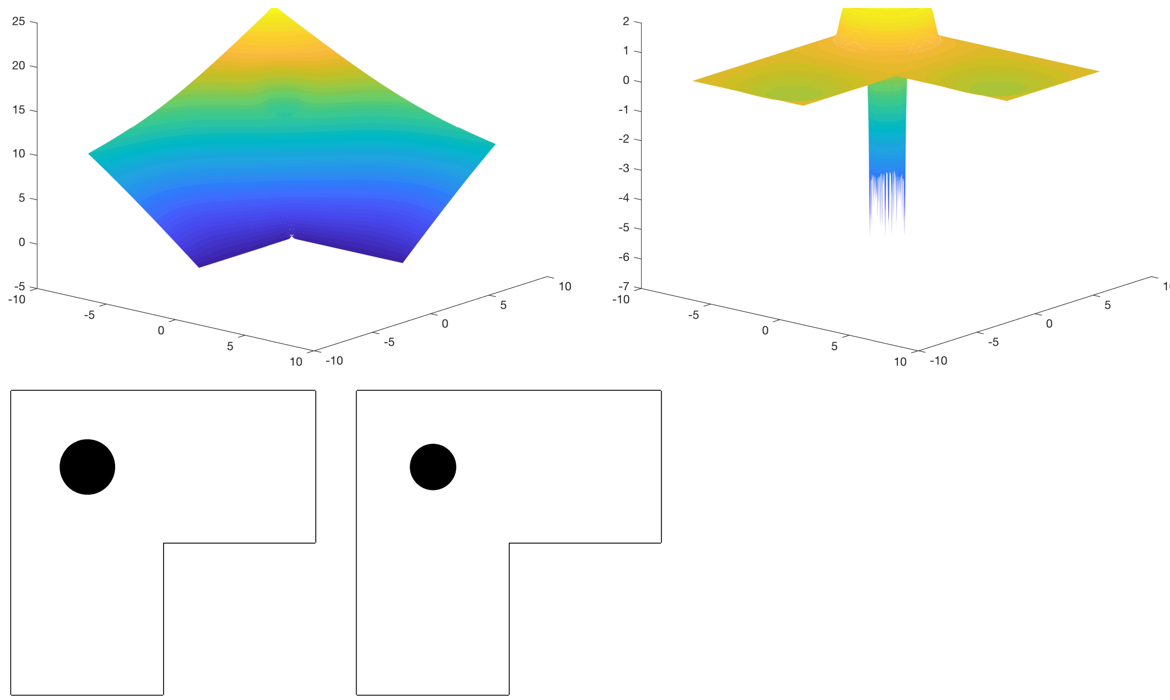

Figure 2: State, control and active set computed by a uniform mesh and the active set computed by a graded mesh for the L-shaped domain example ($k = 1, j = 6$)

Let $\Omega = (-4, 4)^3$. We consider the optimal control problem with $\psi(x) = |x|^2 - 1$, $\beta = 1$, $g = 0$ and

$$y_d(x) = \begin{cases} \Delta^2\bar{y}(x) + \bar{y}(x) & \text{if } |x| > 1, \\ \Delta^2\bar{y}(x) + \bar{y}(x) + 2 & \text{if } |x| \le 1, \end{cases}$$

where the optimal state $\bar{y}$ is given by

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \le 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{if } 1 \le |x| \le 3, \\ w(x) & \text{if } |x| \ge 3, \end{cases}$$

with radial functions v, ϕ as in the two-dimensional example, and

$$w(x) = 2\sin\Big(\frac{\pi}{8}(x_1 + 4)\Big)\sin\Big(\frac{\pi}{8}(x_2 + 4)\Big)\sin\Big(\frac{\pi}{8}(x_3 + 4)\Big).$$

The numerical results are reported in Table 7 for $k = 1$ and Table 8 for $k = 2$. The performance of the SIP method is similar to that in Section 6.1.

j	$\|\bar{y} - \bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\|\bar{y} - \bar{y}_j\|\|_h$	Order	$\|\Pi_j\bar{y} - \bar{y}_j\|_\infty$	Order	$\|\bar{u} - \bar{u}_j\|_{L_2(\Omega)}$	Order
0	1.6410×10^1	—	2.3815×10^1	—	9.7347×10^{-1}	—	2.3152×10^1	–
1	7.5940×10^0	1.11	1.4455×10^1	0.72	1.1095×10^0	−0.19	2.0836×10^1	0.15
2	2.7130×10^0	1.48	4.8940×10^0	1.56	5.5711×10^{-1}	0.99	9.4985×10^0	1.13
3	1.3542×10^0	1.00	3.3239×10^0	0.56	4.2026×10^{-1}	0.41	7.6705×10^0	0.31
4	9.5533×10^{-2}	3.83	1.5610×10^0	1.09	8.2658×10^{-2}	2.35	3.4858×10^0	1.14
5	4.5772×10^{-2}	1.06	7.7556×10^{-1}	1.01	1.8483×10^{-2}	2.16	1.3494×10^0	1.37

Table 7: Example on cube ($k = 1$)

j	$\|\bar{y}-\bar{y}_j\|_{L_2(\Omega)}$	Order	$\|\!\|\bar{y}-\bar{y}_j\|\!\|_h$	Order	$\|\Pi_j\bar{y}-\bar{y}_j\|_\infty$	Order	$\|\bar{u}-\bar{u}_j\|_{L_2(\Omega)}$	Order
0	2.6455×10^{1}	—	4.8283×10^{1}	—	2.0914×10^{0}	—	2.8380×10^{1}	–
1	3.0710×10^{0}	3.11	3.9681×10^{0}	3.60	2.7922×10^{-1}	2.90	7.0809×10^{0}	2.00
2	2.3105×10^{0}	0.41	2.9753×10^{0}	0.42	1.5391×10^{0}	−2.46	9.9390×10^{0}	−0.49
3	2.6989×10^{-1}	3.10	6.1120×10^{-1}	2.28	6.7021×10^{-2}	4.52	3.1649×10^{0}	1.65
4	1.0293×10^{-2}	4.71	1.6018×10^{-1}	1.93	2.1690×10^{-2}	1.63	9.7816×10^{-1}	1.69
5	4.2952×10^{-3}	1.26	4.5806×10^{-2}	1.81	3.9147×10^{-3}	2.47	3.8867×10^{-1}	1.33

Table 8: Example on cube ($k = 2$)

7 Concluding Remarks

We have developed a SIP method for an elliptic distributed optimal control problem with pointwise state constraints on general polygonal domains. The resulting discrete problems are quadratic programs with box constraints that can be solved efficiently by a primal-dual active set method.

By using graded meshes, we can show $O(h)$ convergence (up to a log factor) for the optimal control in the L_2 norm, and for the optimal state in the L_2 norm, energy norm and L_∞ norm under general assumptions on the data. Better convergence can also be established under additional regularity assumptions.

We have tested the SIP method based on the $\mathbb{P}_1$ and $\mathbb{P}_2$ discontinuous finite element spaces. The numerical results are better than our theoretical results. A heuristic justification of the numerical results is provided in the appendix where an optimal control problem without the pointwise state constraints, but with the correct assumption on the regularity of the data, is considered.

We expect the interior maximum norm estimate (3.21) can be extended to the other discontinuous Galerkin methods in [4] along the lines taken in [47], in which case the analysis in this paper can also be applied to the discretizations of (2.1) based on these methods.

Even though our theory is purely for two-dimensional domains, numerical results indicate that the SIP method has similar performance in three dimensions. Therefore, the discontinuous Galerkin approach may provide higher-order methods for the optimal control problem on general polyhedral domains where the discrete problems are quadratic programs with box constraints that can be solved efficiently by a primal-dual active set algorithm. A rigorous analysis of the SIP method in three dimensions will require an interior maximum norm error estimate for discontinuous Galerkin methods in three dimensions, which is still absent from the literature.

A An Optimal Control Problem Without State Constraints

In order to give a heuristic justification of the numerical results observed in Section 6.1, we consider the elliptic optimal control problem (1.1) without the state constraint (1.4) on a square Ω. The optimal control problem is equivalent to the following problem after dropping the bars over y and u (cf. [52]): find $(u,y)\in H_0^1(\Omega)\times H_0^1(\Omega)$ such that

$$\int_\Omega yv\,dx+\beta\int_\Omega \nabla u\cdot\nabla v\,dx=\int_\Omega y_d v\,dx \quad \text{for all } v\in H_0^1(\Omega), \tag{A.1}$$

$$\int_\Omega \nabla y\cdot\nabla z\,dx-\int_\Omega uz\,dx=0 \quad \text{for all } z\in H_0^1(\Omega). \tag{A.2}$$

To match the regularity of the Lagrange multiplier in (4.17), we replace $y_d\in L_2(\Omega)$ by a linear functional F that belongs to $H^{-\frac{1}{2}-\epsilon}(\Omega)$ for any $\epsilon>0$ and rewrite (A.1)–(A.2) concisely as

$$\mathcal{B}\big((u,y),(v,z)\big)=F(v) \quad \text{for all } (v,z)\in H_0^1(\Omega)\times H_0^1(\Omega), \tag{A.3}$$

where

$$\mathcal{B}((u,y),(v,z)) = \beta\int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega yv\,dx - \int_\Omega uz\,dx + \int_\Omega \nabla y\cdot\nabla z\,dx.$$

Remark A.1. Equation (2.2) is identical to (A.3) with

$$F(v) = \int_\Omega y_d v\,dx + \int_\Omega v\,d\mu,$$

where the Lagrange multiplier μ satisfies (4.17).

The bilinear form $\mathcal{B}(\cdot,\cdot)$ is clearly bounded on $H_0^1(\Omega)\times H_0^1(\Omega)$. It is also coercive because

$$\mathcal{B}((v,z),(v,z)) = \beta|v|^2_{H^1(\Omega)} + |z|^2_{H^1(\Omega)} \quad \text{for all } (y,z)\in H_0^1(\Omega)\times H_0^1(\Omega).$$

Since Ω is convex, elliptic regularity and (A.1) (with y_d replaced by F) imply (cf. [37])

$$u\in H^{\frac{3}{2}-\epsilon}(\Omega) \quad \text{for any } \epsilon>0, \tag{A.4}$$

and then, since Ω is a square and $u\in H^{\frac{3}{2}-\epsilon}(\Omega)\cap H_0^1(\Omega)$, elliptic regularity and (A.2) imply (cf. [15, 38])

$$y\in H^{\frac{7}{2}-\epsilon}(\Omega) \quad \text{for any } \epsilon>0. \tag{A.5}$$

The SIP method for (A.3) is to find $(u_h,y_h)\in Y_h\times Y_h$ such that

$$\int_\Omega y_h v\,dx + \beta a_h(u_h,v) = F(v) \quad \text{for all } v\in Y_h, \tag{A.6}$$

$$a_h(y_h,z) - \int_\Omega u_h z\,dx = 0 \qquad \text{for all } z\in Y_h. \tag{A.7}$$

Remark A.2. The discussion below can also be carried out for the SIP method defined by (A.6) and (A.7) that involves additional technicalities. For simplicity, we consider instead conforming finite element methods for (A.3).

Recall $V_h = Y_h\cap H_0^1(\Omega)$ is the $\mathbb{P}_k$ ($k=1,2$) Lagrange finite element space associated with a uniform triangulation $\mathcal{T}_h$ of Ω. The discrete problem is to find $(u_h,y_h)\in V_h\times V_h$ such that

$$\int_\Omega y_h v\,dx + \beta\int_\Omega \nabla u_h\cdot\nabla v\,dx = F(v) \quad \text{for all } v\in V_h, \tag{A.8}$$

$$\int_\Omega \nabla y_h\cdot\nabla z\,dx - \int_\Omega u_h z\,dx = 0 \qquad \text{for all } z\in V_h, \tag{A.9}$$

or equivalently,

$$\mathcal{B}((u_h,y_h),(v,z)) = F(v) \quad \text{for all } (v,z)\in V_h\times V_h.$$

Comparing (A.1)–(A.2) (where y_d is replaced by F) with (A.8)–(A.9), we have the Galerkin relations

$$\int_\Omega (y-y_h)v\,dx + \beta\int_\Omega \nabla(u-u_h)\cdot\nabla v\,dx = 0 \quad \text{for all } v\in V_h, \tag{A.10}$$

$$\int_\Omega \nabla(y-y_h)\cdot\nabla z\,dx - \int_\Omega (u-u_h)z\,dx = 0 \quad \text{for all } z\in V_h. \tag{A.11}$$

Combing the boundedness and coercivity of $\mathcal{B}(\cdot,\cdot)$, the regularity (A.4)–(A.5), the Galerkin relations (A.10)–(A.11) and standard interpolation error estimates, we obtain the error estimate

$$|u-u_h|_{H^1(\Omega)} + |y-y_h|_{H^1(\Omega)} \le C\inf_{(v,z)\in V_h\times V_h}[|u-v|_{H^1(\Omega)} + |y-z|_{H^1(\Omega)}] \le C_\epsilon h^{\frac{1}{2}-\epsilon}, \tag{A.12}$$

which then implies

$$\|u - u_h\|_{L_2(\Omega)} + \|y - y_h\|_{L_2(\Omega)} \leq C_\epsilon h^{\frac{3}{2}-\epsilon} \tag{A.13}$$

by a standard duality argument.

In the case where $k = 1$, we can improve the estimate for $|y - y_h|_{H^1(\Omega)}$ in (A.12) through relation (A.11) with $z = \Pi_h y - y_h$, which yields

$$\begin{aligned}\int_\Omega \nabla(y - y_h) \cdot \nabla(y - y_h)\, dx &+ \int_\Omega \nabla(y - y_h) \cdot \nabla(\Pi_h y - y)\, dx \\ &= \int_\Omega (u - u_h)(y - y_h)\, dx + \int_\Omega (u - u_h)(\Pi_h y - y)\, dx.\end{aligned} \tag{A.14}$$

It follows from (A.5), (A.13), (A.14) and standard interpolation error estimates that

$$|y - y_h|^2_{H^1(\Omega)} \leq C_1 h |y - y_h|_{H^1(\Omega)} + C_{2,\epsilon_1} h^{3-\epsilon_1} + C_{3,\epsilon_2} h^{\frac{5}{2}-\epsilon_2},$$

which implies (with $\epsilon_1 = 1$ and $\epsilon_2 = \frac{1}{2}$)

$$|y - y_h|_{H^1(\Omega)} \leq Ch. \tag{A.15}$$

Remark A.3. The estimate in (A.13) for $\|u - u_h\|_{L_2(\Omega)}$ and estimate (A.15) for $|y - y_h|_{H^1(\Omega)}$ match the results observed in Table 1.

In the case where $k = 2$, we need to use a duality argument that is based on the equivalence of (A.3) with the following fourth-order boundary value problem (cf. (2.2) and Remark A.1):

$$\beta \int_\Omega (\Delta y)(\Delta z)\, dx + \int_\Omega yz\, dx = F(z) \quad \text{for all } z \in H^2(\Omega) \cap H^1_0(\Omega). \tag{A.16}$$

Note that (A.3) can be interpreted as a mixed formulation of (A.16).

Let $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ be defined by

$$\beta \int_\Omega (\Delta v)(\Delta \phi)\, dx + \int_\Omega v\phi\, dx = \int_\Omega v(y - y_h)\, dx \quad \text{for all } v \in H^2(\Omega) \cap H^1_0(\Omega). \tag{A.17}$$

Since Ω is a square, we have

$$\|\phi\|_{H^{3-\epsilon}(\Omega)} \leq C_\epsilon \|y - y_h\|_{L_2(\Omega)} \tag{A.18}$$

by the elliptic regularity result in [10] for the biharmonic equation with the boundary conditions of simply supported plates.

Let $\tilde{y}_h \in H^1_0(\Omega)$ be defined by

$$\int_\Omega \nabla \tilde{y}_h \cdot \nabla z\, dx = \int_\Omega u_h z\, dx \quad \text{for all } z \in H^1_0(\Omega). \tag{A.19}$$

Then we have

$$\|\tilde{y}_h\|_{H^{3-\epsilon}(\Omega)} \leq C_\epsilon \|u_h\|_{H^1(\Omega)} \tag{A.20}$$

by the elliptic regularity for a square (cf. [37]).

In view of (A.9) and (A.19), $y_h \in V_h$ is precisely the approximation of $\tilde{y}_h$ generated by the standard $\mathbb{P}_2$ Lagrange finite element method. Consequently, we have

$$|\tilde{y}_h - y_h|_{H^1(\Omega)} \leq C_\epsilon h^{2-\epsilon} |\tilde{y}_h|_{H^{3-\epsilon}(\Omega)} \leq C_\epsilon h^{2-\epsilon}$$

by (A.20) and the fact that $\|u_h\|_{H^1(\Omega)}$ is uniformly bounded by (A.12), and then the estimate

$$\|\tilde{y}_h - y_h\|_{L_2(\Omega)} \leq Ch |\tilde{y}_h - y_h|_{H^1(\Omega)} \leq C_\epsilon h^{3-\epsilon} \tag{A.21}$$

follows from a standard duality argument.

Now we take $v = y - \tilde{y}_h$ in (A.17) and then use (A.2) and (A.19) to obtain the relation

$$\beta \int_\Omega \nabla(u - u_h)\nabla\phi\,dx + \int_\Omega (y - y_h)\phi\,dx + \int_\Omega (y_h - \tilde{y}_h)\phi\,dx = \int_\Omega (y - y_h)^2\,dx + \int_\Omega (y_h - \tilde{y}_h)(y - y_h)\,dx,$$

which implies, through (A.10),

$$\begin{aligned}\|y - y_h\|^2_{L_2(\Omega)} &= \beta \int_\Omega \nabla(u - u_h)\cdot\nabla(\phi - \Pi_h\phi)\,dx + \int_\Omega (y - y_h)(\phi - \Pi_h\phi)\,dx \\ &\quad + \int_\Omega (y_h - \tilde{y}_h)\phi\,dx - \int_\Omega (y_h - \tilde{y}_h)(y - y_h)\,dx \\ &\le \beta|u - u_h|_{H^1(\Omega)}|\phi - \Pi_h\phi|_{H^1(\Omega)} + \|y - y_h\|_{L_2(\Omega)}\|\phi - \Pi_h\phi\|_{L_2(\Omega)} \\ &\quad + \|y_h - \tilde{y}_h\|_{L_2(\Omega)}\|\phi\|_{L_2(\Omega)} + \|y_h - \tilde{y}_h\|_{L_2(\Omega)}\|y - y_h\|_{L_2(\Omega)}.\end{aligned}$$

In view of (A.13), (A.18), (A.21) and standard interpolation error estimates, we conclude that

$$\|y - y_h\|^2_{L_2(\Omega)} \le C_{1,\epsilon_1} h^{\frac{5}{2}-\epsilon_1}\|y - y_h\|_{L_2(\Omega)} + C_{2,\epsilon_2} h^{3-\epsilon_2}\|y - y_h\|_{L_2(\Omega)},$$

and hence

$$\|y - y_h\|_{L_2(\Omega)} \le C_\epsilon h^{\frac{5}{2}-\epsilon} \tag{A.22}$$

by the inequality of arithmetic and geometric means, which improves the estimate in (A.13).

Finally, we return to relation (A.14) and use (A.13), (A.22) to obtain the estimate

$$|y - y_h|^2_{H^1(\Omega)} \le C_1 h^2 |y - y_h|_{H^1(\Omega)} + C_{2,\epsilon_1} h^{4-\epsilon_1} + C_{3,\epsilon_2} h^{\frac{9}{2}-\epsilon_2},$$

and hence

$$|y - y_h|_{H^1(\Omega)} \le C_\epsilon h^{2-\epsilon}. \tag{A.23}$$

Remark A.4. The estimate in (A.13) for $\|u - u_h\|_{L_2(\Omega)}$, the estimate in (A.22) for $\|y - y_h\|_{L_2(\Omega)}$ and the estimate in (A.23) for $|y - y_h|_{H^1(\Omega)}$ match the results observed in Table 2.

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