Computers and Mathematics with Applications 155 (2024) 80–90

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

C^0 interior penalty methods for an elliptic distributed optimal control problem with general tracking and pointwise state constraints ☆

Susanne C. Brenner [a,*], SeongHee Jeong [b,1], Li-yeng Sung [a], Zhiyu Tan [c,2]

[a] Department of Mathematics, Center for Computation & Technology, Louisiana State University, Baton Rouge, LA 70803, USA
[b] Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
[c] Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA

ARTICLE INFO

Keywords:
Elliptic distributed optimal control problems
General tracking function
C^0 interior penalty methods

ABSTRACT

We consider C^0 interior penalty methods for a linear-quadratic elliptic distributed optimal control problem with pointwise state constraints in two spatial dimensions, where the cost function tracks the state at points, curves and regions of the domain. Error estimates and numerical results that illustrate the performance of the methods are presented.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygon, β be a positive constant, $\mathscr{P} = \{p_1, \ldots, p_J\}$ be a finite set of points in Ω, and $\mathscr{C} = \bigcup_{\ell=1}^{L} \mathscr{C}_\ell \subset \Omega$ be the union of the curves $\mathscr{C}_1, \ldots, \mathscr{C}_L$, where each curve is parametrized by a Lipschitz continuous function defined on $[0,1]$. The weight functions w_0, w_1 and w_2 are bounded nonnegative Borel measurable functions defined on $\mathscr{P}$, $\mathscr{C}$ and Ω respectively. The desired/observed states y_0, y_1 and y_2 are Borel measurable functions defined on $\mathscr{P}$, $\mathscr{C}$ and Ω such that

$$\sum_{j=1}^{J} y_0(x_j)^2 w_0(x_j) + \sum_{\ell=1}^{L} \int_{\mathscr{C}_\ell} y_1^2 w_1\, ds + \int_\Omega y_2^2 w_2\, dx < \infty. \tag{1.1}$$

The optimal control problem is to find

$$(\bar{y}, \bar{u}) = \operatorname*{argmin}_{(y,u)\in\mathbb{K}} \frac{1}{2}\big[G(y) + \beta \|u\|_{L_2(\Omega)}^2\big], \tag{1.2}$$

where

$$G(y) = \sum_{j=1}^{J} (y(p_j) - y_0(p_j))^2 w_0(p_j) + \sum_{\ell=1}^{L} \int_{\mathscr{C}_\ell} (y - y_1)^2 w_1\, ds + \int_\Omega (y - y_2)^2 w_2\, dx, \tag{1.3}$$

$\mathbb{K} \subset H_0^1(\Omega) \times L_2(\Omega)$, and $(y,u) \in H_0^1(\Omega) \times L_2(\Omega)$ belongs to $\mathbb{K}$ if and only if

$$\int_\Omega \nabla y \cdot \nabla z\, dx = \int_\Omega uz\, dx \qquad \forall z \in H_0^1(\Omega), \tag{1.4}$$

$$\psi_- \le y \le \psi_+ \qquad \text{a.e. in } \Omega. \tag{1.5}$$

We assume that the functions $\psi_\pm$ satisfy

$$\psi_\pm \in W^{3,q}(\Omega) \quad \text{for} \quad q > 2, \tag{1.6}$$

$$\psi_- < \psi_+ \quad \text{on} \quad \bar{\Omega}, \tag{1.7}$$

$$\psi_- < 0 < \psi_+ \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

Here and below we will follow the standard notation for differential operators, function spaces and norms that can be found for example in [36,1,18].

Remark 1.1. Since Ω is convex, the partial differential equation constraint (1.4) implies through elliptic regularity (cf. [43,37,51]) that

☆ This work was supported in part by the National Science Foundation under Grant No. DMS-19-13035 and Grant No. DMS-22-08404.
* Corresponding author.
E-mail addresses: brenner@math.lsu.edu (S.C. Brenner), sjeong5@lsu.edu, sj20bk@fsu.edu (S. Jeong), sung@math.lsu.edu (L.-y. Sung), ztan@cct.lsu.edu, zhiyutan@xmu.edu.cn (Z. Tan).
[1] Current address: Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA.
[2] Current address: School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Fujian, 361005, China.

https://doi.org/10.1016/j.camwa.2023.11.032
Received 25 February 2023; Received in revised form 16 November 2023; Accepted 24 November 2023
Available online 11 December 2023

$y \in H^2(\Omega)$ if $(y, u) \in \mathbb{K}$. Therefore the function G is well-defined by the Sobolev embedding $H^2(\Omega) \subset C(\bar{\Omega})$ (cf. [1]).

Remark 1.2. In the case where $w_0 = w_1 = 0$ and $w_2 = 1$, the function G reduces to a standard tracking function (cf. [45,55]), and the optimal control problem with pointwise state constraints was introduced in [31]. In the case where $w_1 = w_2 = 0$, it becomes a point-tracking function (cf. [28,33,27,6,4,5,3]).

Remark 1.3. In general the last term on the right-hand side of (1.3) defines tracking on the region that is the support of w_2.

Remark 1.4. The optimal control problem defined by (1.2)–(1.5) can be interpreted as a heat conduction problem (cf. [55]), where y is the temperature, u is the heat source, y_0 (resp., y_1 and y_2) is the desired temperature at the points (resp., curves and regions), and $\psi_\pm$ are the constraints on the temperature in Ω. The weights w_0, w_1 and w_2 allow preferences for the desired temperature.

In the case where y_0, y_1 and y_2 satisfy the constraints in (1.5) and the points, curves and regions are disjoint, the optimal control problem can also be interpreted as a least-squares data fitting problem for a mathematical model that connects the input u to the output y through (1.4). In this interpretation y_0, (resp., y_1 and y_2) is the observed output at the points (resp., curves and regions), the constraints in (1.5) provide *a priori* modeling information, and the weights w_0, w_1 and w_2 allow preferences in the data fitting.

Remark 1.5. By introducing the Radon measure ν on $\bar{\Omega}$ defined by

$$\int_\Omega f\,d\nu = \sum_{j=1}^{J} f(p_j)w_0(p_j) + \sum_{\ell=1}^{L} \int_{\mathscr{C}_\ell} f w_1\,ds + \int_\Omega f w_2\,dx, \tag{1.9}$$

we can write

$$G(y) = \int_\Omega (y - y_d)^2\,d\nu = \|y - y_d\|^2_{L_2(\Omega;\nu)}, \tag{1.10}$$

where

$$y_d = \begin{cases} y_0 & \text{on } \mathscr{P} \\ y_1 & \text{on } \mathscr{C} \setminus \mathscr{P} \\ y_2 & \text{on } \Omega \setminus (\mathscr{C} \cup \mathscr{P}) \end{cases}, \tag{1.11}$$

and the condition (1.1) becomes $\|y_d\|^2_{L_2(\Omega;\nu)} < \infty$.

The optimal control papers with point-tracking mentioned in Remark 1.2 are concerned with control constraints. For optimal control problems with the standard tracking function and pointwise state constraints, the traditional approach (cf. [49,52,45,34,32,53]) is based on reducing the optimal control problem to a problem that only involves the control. Here we adopt the opposite approach where the optimal control problem is reduced to a problem that only involves the state, which can be reformulated as a fourth order variational inequality.

This reformulation was discussed in [54], and the first numerical scheme based on this idea appeared in [50], where the analysis was carried out under the ad hoc assumptions from [7] on the free boundary. These assumptions were later removed by the new convergence analysis in [20], where the regularity results in [41,42,29] for fourth order elliptic variational inequalities were exploited. Various finite element methods based on this new approach have appeared in [24,11,16,12,25,13,17,22].

Comparing with the traditional approach, a distinct feature of the new approach is that the convergence of the state can also be established in the L_∞ norm. Another important feature is that the discrete problems are quadratic programs with simple box constraints where the system matrices are available and consequently they can be solved efficiently by the primal-dual active set algorithm in [8,44]. These features are also shared by the C^0 interior penalty methods in the current paper.

The rest of the paper is organized as follows. We discuss the reformulation of the continuous problem in Section 2 and present the C^0 interior penalty methods in Section 3. The convergence analysis is carried out in Section 4, followed by numerical results in Section 5. We end the paper with some concluding remarks in Section 6. The appendix A contains the construction of an exact solution for a problem with point-tracking.

Throughout the paper we will use C (with or without subscripts) to denote a generic positive constant that is independent of the mesh size.

2. The continuous problem

As mentioned in Remark 1.1, the constraint (1.4) implies that $y \in H^2(\Omega) \cap H^1_0(\Omega) \subset C(\bar{\Omega})$. In view of (1.10) and the relation $u = -\Delta y$ from (1.4), the optimal control problem defined by (1.2)–(1.5) is equivalent to the following minimization problem:

$$\text{Find} \quad \bar{y} = \operatorname*{argmin}_{y \in K} \frac{1}{2}\Big[\beta\|\Delta y\|^2_{L_2(\Omega)} + \|y - y_d\|^2_{L_2(\Omega;\nu)}\Big], \tag{2.1}$$

where

$$K = \{y \in H^2(\Omega) \cap H^1_0(\Omega) :\ \psi_- \le y \le \psi_+ \ \text{ in } \Omega\}. \tag{2.2}$$

It follows from the standard theory of calculus of variations (cf. [48,39]) that the minimization problem defined by (2.1) and (2.2) has a unique solution $\bar{y} \in K$ characterized by the fourth order variational inequality

$$\beta \int_\Omega (\Delta\bar{y})(\Delta y - \Delta\bar{y})\,dx + \int_\Omega (\bar{y} - y_d)(y - \bar{y})\,d\nu \ge 0 \qquad \forall\, y \in K,$$

which in turn is equivalent to the following generalized Karush-Kuhn-Tucker conditions (cf. [47, Chapter 1, Theorem 1.6]):

$$\beta \int_\Omega (\Delta\bar{y})(\Delta z)dx + \int_\Omega (\bar{y} - y_d)z\,d\nu = \int_\Omega z\,d\mu \qquad \forall\, z \in H^2(\Omega) \cap H^1_0(\Omega), \tag{2.3}$$

where μ is a regular Borel measure, such that

$$\mu \ge 0 \quad \text{if} \quad \bar{y} = \psi_-, \tag{2.4}$$

$$\mu \le 0 \quad \text{if} \quad \bar{y} = \psi_+, \tag{2.5}$$

$$\mu = 0 \quad \text{otherwise.} \tag{2.6}$$

Remark 2.1. It follows from (2.4)–(2.6) that the support of μ is the union of the active sets $\mathscr{A}_\pm$ defined by

$$\mathscr{A}_\pm = \{x \in \Omega :\ \bar{y}(x) = \psi_\pm(x)\}. \tag{2.7}$$

Note that $\mathscr{A}_\pm$ are compact subsets of Ω by the assumption (1.8). Therefore μ is a bounded measure.

We can also rewrite (2.3) as

$$\mathcal{A}(\bar{y}, z) - \int_\Omega y_d z\,d\nu = \int_\Omega z\,d\mu, \tag{2.8}$$

where

$$\mathcal{A}(y, z) = \beta a(y, z) + \int_\Omega yz\,d\nu \tag{2.9}$$

and (cf. [43, Section 4.3])

$$a(y, z) = \int_\Omega (\Delta y)(\Delta z)\,dx = \int_\Omega D^2 y : D^2 z\,dx \qquad \forall\, y, z \in H^2(\Omega) \cap H^1_0(\Omega).$$

(2.10)

Here D^2y is the Hessian matrix of y and $D^2y : D^2z$ is the Frobenius inner product of D^2y and D^2z.

2.1. Regularity of $\bar{y}$

Let the adjoint state $\bar{p} \in L_2(\Omega)$ be defined by

$$\int_\Omega \bar{p}(-\Delta z)\,dx = \int_\Omega (\bar{y} - y_d) z\,d\nu - \int_\Omega z\,d\mu \tag{2.11}$$

for all $z \in H^2(\Omega) \cap H^1_0(\Omega)$. Then we have (cf. [30, Theorem 1])

$$\bar{p} \in W^{1,s}_0(\Omega) \qquad \forall\, s < 2. \tag{2.12}$$

Remark 2.2. Since $\mathscr{C} \cup \mathscr{P}$ and the support of μ are compact subsets of Ω, the adjoint state $\bar{p}$ belongs to H^2 in a neighborhood of $\partial\Omega$ and vanishes on $\partial\Omega$.

It follows from (2.3) and (2.11) that

$$\bar{u} = -\Delta \bar{y} = -(1/\beta)\bar{p} \in W^{1,s}_0(\Omega) \qquad \forall\, s < 2, \tag{2.13}$$

and, in view of Remark 2.2, $\bar{u}$ belongs to H^2 in a neighborhood of $\partial\Omega$ and $\bar{u} = 0$ on $\partial\Omega$.

According to (2.13), the solution $\bar{y} \in K \subset H^2(\Omega) \cap H^1_0(\Omega)$ of (2.1) satisfies

$$\Delta \bar{y} = \bar{p}/\beta \text{ in } \Omega \text{ and } \bar{y} = 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

It then follows from (2.12), (2.14) and interior elliptic regularity (cf. [2, Section 14] and [46, Lemma 17.1.1]) that

$$\bar{y} \in W^{3,s}_{loc}(\Omega) \qquad \forall\, s < 2. \tag{2.15}$$

Moreover we can conclude from Remark 2.2, (2.14), (2.15) and the elliptic regularity theory for polygonal domains (cf. [43,37]) that globally

$$\bar{y} \in H^{2+\alpha}(\Omega) \tag{2.16}$$

for some $\alpha \in (0,1)$, where the index of elliptic regularity α is determined by the angles at the corners of Ω.

Remark 2.3. In the absence of point-tracking (the case where $w_0 = 0$ in (1.3)), the index α in (2.16) can equal 1 by the regularity result for fourth order variational inequalities in [41]. In this paper we always assume that point-tracking is present and therefore $\alpha < 1$ because of (2.15).

2.2. Regularity of μ

In view of (2.7) and the assumption (1.7), we have

$$\mathscr{A}_- \cap \mathscr{A}_+ = \emptyset. \tag{2.17}$$

The Hahn decomposition of μ is given by

$$\mu = \mu_- + \mu_+ \tag{2.18}$$

where μ_- is the finite nonnegative Borel measure (cf. (2.5)) defined by

$$\mu_-(B) = \mu(B \cap \mathscr{A}_-) \qquad \text{for all Borel subsets } B \text{ of } \Omega, \tag{2.19}$$

and μ_+ is the finite nonpositive Borel measure (cf. (2.4)) defined by

$$\mu_+(B) = \mu(B \cap \mathscr{A}_+) \qquad \text{for all Borel subsets } B \text{ of } \Omega. \tag{2.20}$$

Since the support of μ_+ (resp., μ_-) is a subset of the active set $\mathscr{A}_+$ (resp., $\mathscr{A}_-$), we have

$$\int_\Omega (\bar{y} - \psi_+)\,d\mu_+ = 0 \quad \text{and} \quad \int_\Omega (\bar{y} - \psi_-)\,d\mu_- = 0. \tag{2.21}$$

Let $z \in H^2(\Omega) \cap H^1_0(\Omega)$ be arbitrary. Because of (2.17) we can construct $\phi \in C^\infty_c(\Omega)$ such that $\phi = 1$ in a neighborhood of $\mathscr{A}_+$ and $\phi = 0$ in a neighborhood of $\mathscr{A}_-$. It follows from (1.9), (1.11), (2.3), (2.15), (2.18) and integration by parts that

$$\begin{aligned}
\int_\Omega z\,d\mu_+ &= \int_\Omega \phi z\,d\mu \\
&= \beta \int_\Omega (\Delta\bar{y})[\Delta(\phi z)]\,dx + \int_\Omega (\bar{y} - y_d)(\phi z)\,d\nu \\
&= \beta \int_\Omega \nabla(\Delta\bar{y}) \cdot \nabla(\phi z)\,dx + \sum_{j=1}^{J} w_0(p_j)[\bar{y}(p_j) - y_0(p_j)]\phi(p_j) z(p_j) \\
&\quad + \sum_{\ell=1}^{L} \int_{\mathscr{C}_\ell} w_1(\bar{y} - y_1)(\phi z)\,ds + \int_\Omega w_2(\bar{y} - y_2)(\phi z)\,dx,
\end{aligned}$$

and hence, by (2.15), the trace theorem and the Sobolev embedding $H^{1+\epsilon}(\Omega) \subset C(\bar{\Omega})$, we have

$$\Big|\int_\Omega z\,d\mu_+\Big| \le C_\epsilon \|z\|_{H^{1+\epsilon}(\Omega)} \qquad \forall\, z \in H^2(\Omega) \cap H^1_0(\Omega) \tag{2.22}$$

and for any $\epsilon > 0$.

Given any $z \in H^{1+\epsilon}(\Omega)$, we can construct a sequence $z_n \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $z_n\phi$ converges to $z\phi$ in $H^{1+\epsilon}(\Omega)$ as $n \to \infty$. Hence we can extend the definition of the integral $\int_\Omega z\,d\mu_+$ to $z \in H^{1+\epsilon}(\Omega)$ such that the estimate in (2.22) remains valid. In other words, we have

$$\Big|\int_\Omega z\,d\mu_+\Big| \le C_\epsilon \|z\|_{H^{1+\epsilon}(\Omega)} \qquad \forall\, z \in H^{1+\epsilon}(\Omega), \tag{2.23}$$

which holds in particular if $z \in H^1_0(\Omega)$ is a finite element function.

Similarly, the estimate (2.23) also holds if μ_+ is replaced by μ_-.

3. C^0 interior penalty methods

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω. We will denote the set of the vertices of $\mathcal{T}_h$ by $\mathcal{V}_h$ and the set of the edges of $\mathcal{T}_h$ interior to Ω by $\mathcal{E}^i_h$. The diameter of $T \in \mathcal{T}_h$ is denoted by h_T and the mesh parameter h equals $\max_{T\in\mathcal{T}_h} h_T$. The nodal interpolation operator for the P_1 finite element space associated with $\mathcal{T}_h$ is denoted by I_h.

Let $V_h \subset H^1_0(\Omega)$ be the P_k $(k \ge 2)$ Lagrange finite element space associated with $\mathcal{T}_h$ (cf. [36,18]). The discrete problem is to find

$$\bar{y}_h = \underset{y_h \in K_h}{\operatorname{argmin}}\ \frac{1}{2}\Big[\beta a_h(y_h, y_h) + \|y_h - y_d\|^2_{L_2(\Omega;\nu)}\Big], \tag{3.1}$$

where

$$K_h = \{y_h \in V_h : I_h\psi_- \le I_h y_h \le I_h\psi_+\}, \tag{3.2}$$

$$\begin{aligned}
a_h(y_h, z_h) = &\sum_{T\in\mathcal{T}_h} \int_T D^2 y_h : D^2 z_h\,dx + \sum_{e\in\mathcal{E}^i_h} \sigma h_e^{-1} \int_e [\![\partial y_h/\partial n]\!][\![\partial z_h/\partial n]\!]\,ds \\
&+ \sum_{e\in\mathcal{E}^i_h} \int_e \Big[\{\!\{\partial^2 y_h/\partial n^2\}\!\}[\![\partial z_h/\partial n]\!] + \{\!\{\partial^2 z_h/\partial n^2\}\!\}[\![\partial y_h/\partial n]\!]\Big] ds,
\end{aligned} \tag{3.3}$$

h_e is the diameter of the edge e, and $\sigma > 0$ is a penalty parameter.

The bilinear form $a_h(\cdot,\cdot)$ defined by (3.3) is the C^0 interior penalty bilinear form (cf. [40,19,10]) that approximates the bilinear form $a(\cdot,\cdot)$ defined in (2.10). Given a unit normal n of an edge $e \in \mathcal{E}^i_h$, the jump $[\![\partial v/\partial n]\!]$ and the mean $\{\!\{\partial^2 v/\partial n^2\}\!\}$ across e are defined by

$$[\![\partial v/\partial n]\!] = \frac{\partial v_+}{\partial n} - \frac{\partial v_-}{\partial n} \quad \text{and} \quad \{\!\{\partial^2 v/\partial n^2\}\!\} = \frac{1}{2}\Big(\frac{\partial^2 v_+}{\partial n^2} + \frac{\partial^2 v_-}{\partial n^2}\Big),$$

where $v_\pm$ is the restriction of v to $T_\pm$, the two triangles that share e as a common edge, and the vector n points from T_- to T_+. Note that these definitions are independent of the choice of n, and they are well-defined on any function v that is piecewise H^2 with respect to $\mathcal{T}_h$.

We can describe the properties of $a_h(\cdot,\cdot)$ in terms of the norm $|\cdot|_{H^2(\Omega,\mathcal{T}_h)}$ defined by

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}. \tag{3.4}$$

The derivations of the following estimates can be found in [19,10]: there exist positive constants $C_\dagger$ and $C_\ddagger$ such that

$$a_h(y_h, z_h) \le C_\dagger |y_h|_{H^2(\Omega,\mathcal{T}_h)} |z|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, y_h, z_h \in V_h, \tag{3.5}$$

$$a_h(y_h, y_h) \ge C_\ddagger |y_h|^2_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, y_h \in V_h, \tag{3.6}$$

provided σ is sufficiently large (which is assumed to be the case from here on).

Remark 3.1. It follows from the Poincaré-Friedrichs inequality for piecewise H^2 function in [26] that

$$\|v\|_{L_2(\Omega)} \le C|v|_{H^2(\Omega,\mathcal{T}_h)} \tag{3.7}$$

for all $v \in H_0^1(\Omega)$ that is piecewise H^2 with respect to $\mathcal{T}_h$, where the positive constant C only depends on the shape regularity of $\mathcal{T}_h$.

The analysis of the C^0 interior penalty method defined by (3.1)–(3.3) relies on the regularity of $\bar{y}$ and μ presented in Section 2 and two linear operators Π_h and E_h.

3.1. The operator Π_h

The operator $\Pi_h : H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow V_h$ is the Lagrange nodal interpolation operator. It follows from (2.2) and (3.2) that

$$\Pi_h \text{ maps } K \text{ into } K_h. \tag{3.8}$$

In particular, K_h is nonempty and hence the minimization problem (3.1) has a unique solution $\bar{y}_h \in K_h$ characterized by the discrete variational inequality

$$\beta a_h(\bar{y}_h, y_h - \bar{y}_h) + \int_\Omega (\bar{y}_h - y_d)(y_h - \bar{y}_h)d\nu \ge 0 \qquad \forall\, y_h \in K_h,$$

which can be written as

$$\mathcal{A}_h(\bar{y}_h, y_h - \bar{y}_h) - \int_\Omega y_d(y_h - \bar{y}_h)d\nu \ge 0 \qquad \forall\, y_h \in K_h, \tag{3.9}$$

where the bilinear form

$$\mathcal{A}_h(y_h, z_h) = \beta a_h(y_h, z_h) + \int_\Omega y_h z_h d\nu \tag{3.10}$$

approximates the bilinear form $\mathcal{A}(\cdot,\cdot)$ in (2.9).

The following error estimates for $\Pi_h\bar{y}$, which are based on the Bramble-Hilbert lemma (cf. [9,38]) and the regularity estimates (2.15)–(2.16), can be found in [19,23].

We have

$$\|\bar{y} - \Pi_h\bar{y}\|_{L_2(\Omega)} \le Ch^{2+\tau}, \tag{3.11}$$

$$|\bar{y} - \Pi_h\bar{y}|_{H^1(\Omega)} \le Ch^{1+\tau}, \tag{3.12}$$

$$\|\bar{y} - \Pi_h\bar{y}\|_{L_\infty(\Omega)} \le Ch^{1+\tau}, \tag{3.13}$$

$$\Big(\sum_{T\in\mathcal{T}_h} |\bar{y} - \Pi_h\bar{y}|^2_{H^2(T)}\Big)^{\frac{1}{2}} \le Ch^\tau, \tag{3.14}$$

where

$$\tau = \begin{cases} \alpha & \text{if } \mathcal{T}_h \text{ is quasi-uniform} \\ 1-\epsilon & \text{if } \mathcal{T}_h \text{ is graded around the corners of } \Omega \text{ where the interior angles are } > (\pi/2) \end{cases} \tag{3.15}$$

Here α is the index of elliptic regularity in (2.16) and ϵ can be any positive number.

Remark 3.2. Details of the graded mesh can be found in [24, Section 2.1]. Since the singularities around the corners of Ω (with respect to H^3 regularity) are resolved by the graded meshes, the interpolation error estimates for $\Pi_h\bar{y}$ are determined by the interior regularity (2.15) which leads to $\tau = 1-\epsilon$, and the constant C in (3.11)–(3.14) increases to ∞ as ϵ decreases to 0. This dependence on ϵ also holds for the constants in the estimates in the rest of the paper where $\tau = 1-\epsilon$.

Remark 3.3. In the case where Ω is a square, the optimal control $\bar{y}$ is H^3 near $\partial\Omega$ and hence its overall regularity is determined by the $H^{3-\epsilon}_{loc}(\Omega)$ interior regularity. Therefore for such a domain the estimates in (3.11)–(3.14) (and the results in Section 4) hold for $\tau = 1-\epsilon$ on quasi-uniform meshes.

There are two simple consequences of (3.11)–(3.14). We have, by (3.12), (3.14) and the trace inequality with scaling,

$$\Big(\sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|(\partial\bar{y}/\partial n) - (\partial(\Pi_h\bar{y})/\partial n)\|^2_{L_2(e)}\Big)^{\frac{1}{2}} \le Ch^\tau. \tag{3.16}$$

Moreover it follows from (1.9) and (3.11)–(3.13) that

$$\|\bar{y} - \Pi_h\bar{y}\|_{L_2(\Omega;\nu)} \le Ch^{1+\tau}. \tag{3.17}$$

3.2. The operator E_h

Let $W_h \subset H^2(\Omega)\cap H_0^1(\Omega)$ be the Hsieh-Clough-Tocher finite element space (cf. [35]) associated with $\mathcal{T}_h$. One can define an operator $E_h : V_h \longrightarrow W_h$ by averaging (cf. [14,10]) such that

$$(E_h y_h)(p) = y_h(p) \qquad \forall\, p \in \mathcal{V}_h \tag{3.18}$$

and

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|y_h - E_h y_h\|^2_{L_2(T)} + h_T^{-2}|y_h - E_h y_h|^2_{H^1(T)} + |E_h y_h|^2_{H^2(T)}\Big) \le C|y_h|^2_{H^2(\Omega,\mathcal{T}_h)}, \tag{3.19}$$

where the positive constant C depends only on the shape regularity of $\mathcal{T}_h$.

It follows from (1.9), (3.19), the trace theorem and a discrete Sobolev inequality (cf. [18, Section 4.9]) that

$$\|y_h - E_h y_h\|_{L_2(\Omega;\nu)} \le Ch(1+|\ln h|)^{\frac{1}{2}}|y_h|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, y_h \in V_h. \tag{3.20}$$

Remark 3.4. There is no need to invoke the discrete Sobolev inequality if the tracking points $p_1,\ldots,p_J$ belong to $\mathcal{V}_h$. In this case the term $(1+|\ln h|)^{\frac{1}{2}}$ in (3.20) can be removed. This observation is important for the analogous optimal control problem in three spatial dimensions.

The derivations of the following estimates that connect Π_h and E_h, which rely on the regularity of $\bar{y}$ in (2.15)–(2.16) and the Bramble-Hilbert lemma, can be found in [19,23,20]. We have

$$\|\bar{y} - E_h\Pi_h\bar{y}\|_{L_2(\Omega)} \le Ch^{2+\tau}, \tag{3.21}$$

$$|\bar{y} - E_h\Pi_h\bar{y}|_{H^1(\Omega)} \le Ch^{1+\tau}, \tag{3.22}$$

$$\|\bar{y} - E_h\Pi_h\bar{y}\|_{L_\infty(\Omega)} \le Ch^{1+\tau}, \tag{3.23}$$

$$|\bar{y} - E_h\Pi_h\bar{y}|_{H^2(\Omega)} \le Ch^{\tau}, \tag{3.24}$$

and

$$a_h(\Pi_h\bar{y}, y_h) - a(\bar{y}, E_h y_h) \le Ch^{\tau}|y_h|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, y_h \in V_h, \tag{3.25}$$

where τ is given by (3.15).

Remark 3.5. The estimates (3.21)–(3.24) indicate that $E_h\Pi_h$ behaves like a quasi-interpolation operator. The estimate (3.25) indicates that E_h and Π_h are approximate adjoint operators with respect to the bilinear forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$.

4. Convergence analysis

We will carry out the convergence analysis in terms of the mesh-dependent norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \beta|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \|v\|^2_{L_2(\Omega;\nu)}. \tag{4.1}$$

In view of (3.4)–(3.6), (3.10) and (4.1), there exist positive constants $C_\sharp$ and $C_\flat$ such that

$$\mathcal{A}_h(y_h, z_h) \le C_\sharp\|y_h\|_h\|z_h\|_h \qquad \forall\, y_h, z_h \in V_h, \tag{4.2}$$

$$\mathcal{A}_h(y_h, y_h) \ge C_\flat\|y_h\|_h^2 \qquad \forall\, y_h \in V_h. \tag{4.3}$$

Moreover (3.14)–(3.17) and (4.1) imply

$$\|\bar{y} - \Pi_h\bar{y}\|_h \le Ch^{\tau}. \tag{4.4}$$

It follows from (3.8), (3.9), (4.3) and (4.4) that

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_h^2 &\le 2\|\bar{y} - \Pi_h\bar{y}\|_h^2 + 2\|\Pi_h\bar{y} - \bar{y}_h\|_h^2\\ &\le C_1h^{2\tau} + C_2\mathcal{A}_h(\Pi_h\bar{y} - \bar{y}_h, \Pi_h\bar{y} - \bar{y}_h)\\ &\le C_1h^{2\tau} + C_2\Big[\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - \int_\Omega y_d(\Pi_h\bar{y} - \bar{y}_h)d\nu\Big].\end{aligned} \tag{4.5}$$

The key is to bound the second term on the right-hand side of (4.5). We will show that

$$\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - \int_\Omega y_d(\Pi_h\bar{y} - \bar{y}_h)d\nu \le C\big(h^{2\tau} + h^{\tau}\|\Pi_h\bar{y} - \bar{y}_h\|_h\big), \tag{4.6}$$

and then the convergence analysis is completed as follows.

Theorem 4.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_h \le Ch^{\tau}, \tag{4.7}$$

where τ is given by (3.15).

Proof. We have, by (4.4)–(4.6),

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_h^2 &\le C\big[h^{2\tau} + h^{\tau}\big(\|\Pi_h\bar{y} - \bar{y}\|_h + \|\bar{y} - \bar{y}_h\|_h\big)\big]\\ &\le C\big[h^{2\tau} + h^{\tau}\|\bar{y} - \bar{y}_h\|_h\big],\end{aligned}$$

which implies (4.7) through the inequality of arithmetic and geometric means. □

We will establish (4.6) by reducing it to an estimate at the continuous level.

4.1. Reducing (4.6) to the continuous level

We begin with an analog of (3.25).

Lemma 4.2. *There exists a positive constant C independent of h such that*

$$\mathcal{A}_h(\Pi_h\bar{y}, y_h) - \mathcal{A}(\bar{y}, E_h y_h) \le Ch^{\tau}\|y_h\|_h \qquad \forall\, y_h \in V_h. \tag{4.8}$$

Proof. According to (2.9) and (3.10), we have

$$\mathcal{A}_h(\Pi_h\bar{y}, y_h) - \mathcal{A}(\bar{y}, E_h y_h) = \beta\big[a_h(\Pi_h\bar{y}, y_h) - a(\bar{y}, E_h y_h)\big] + \int_\Omega(\Pi_h\bar{y} - \bar{y})y_h\,d\nu - \int_\Omega\bar{y}(y_h - E_h y_h)d\nu,$$

which together with (3.7), (3.15), (3.17), (3.20), (3.25) and (4.1) implies (4.8). □

It follows from (4.2) and Lemma 4.2 that

$$\begin{aligned}\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) &= \mathcal{A}_h(\bar{y}, \Pi_h\bar{y} - \bar{y}_h) + \mathcal{A}_h(\Pi_h\bar{y} - \bar{y}, \Pi_h\bar{y} - \bar{y}_h)\\ &\le \mathcal{A}\big(\bar{y}, E_h(\Pi_h\bar{y} - \bar{y}_h)\big) + Ch^{\tau}\|\Pi_h\bar{y} - \bar{y}_h\|_h.\end{aligned} \tag{4.9}$$

Furthermore, the estimate (3.20) implies

$$-\int_\Omega y_d(\Pi_h\bar{y} - \bar{y}_h)d\nu \le -\int_\Omega y_d E_h(\Pi_h\bar{y} - \bar{y}_h)d\nu + Ch^{\tau}\|\Pi_h\bar{y} - \bar{y}_h\|_h. \tag{4.10}$$

Putting (4.9) and (4.10) together, we arrive at

$$\begin{aligned}&\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - \int_\Omega y_d(\Pi_h\bar{y} - \bar{y}_h)d\nu\\ &\le \Big[\mathcal{A}\big(\bar{y}, E_h(\Pi_h\bar{y} - \bar{y}_h)\big) - \int_\Omega y_d E_h(\Pi_h\bar{y} - \bar{y}_h)d\nu\Big] + Ch^{\tau}\|\Pi_h\bar{y} - \bar{y}_h\|_h.\end{aligned} \tag{4.11}$$

Comparing (4.6) and (4.11), we see that the proof of (4.6) has been reduced to an estimate for the first term on the right-hand side of (4.11), which does not involve the discrete bilinear form $\mathcal{A}_h(\cdot,\cdot)$. Therefore we can use (2.8) and (2.18) to write

$$\begin{aligned}&\mathcal{A}\big(\bar{y}, E_h(\Pi_h\bar{y} - \bar{y}_h)\big) - \int_\Omega y_d E_h(\Pi_h\bar{y} - \bar{y}_h)d\nu\\ &= \int_\Omega E_h(\Pi_h\bar{y} - \bar{y}_h)d\mu_+ + \int_\Omega E_h(\Pi_h\bar{y} - \bar{y}_h)d\mu_-.\end{aligned} \tag{4.12}$$

4.2. Completing the proof of (4.6)

We will focus on the first integral on the right-hand side of (4.12). The other integral can be bounded analogously.

In view of (2.21), we can write

$$\begin{aligned}\int_\Omega E_h(\Pi_h\bar{y} - \bar{y}_h)d\mu_+ &= \int_\Omega(E_h\Pi_h\bar{y} - \bar{y})d\mu_+ + \int_\Omega(\psi_+ - I_h\psi_+)d\mu_+\\ &\quad + \int_\Omega(I_h\psi_+ - I_hE_h\bar{y}_h)d\mu_+\\ &\quad + \int_\Omega(I_hE_h\bar{y}_h - E_h\bar{y}_h)d\mu_+.\end{aligned} \tag{4.13}$$

For the first term on the right-hand side of (4.13), we have

$$\int_\Omega(E_h\Pi_h\bar{y} - \bar{y})d\mu_+ \le |\mu_+(\mathscr{A}_+)|\,\|E_h\Pi_h\bar{y} - \bar{y}\|_{L_\infty(\Omega)} \le Ch^{1+\tau} \tag{4.14}$$

by (3.23), and we can use a standard interpolation error estimate for I_h (cf. [36,18]) to bound the second term by

$$\int_\Omega (\psi_+ - I_h\psi_+)d\mu_+ \le |\mu_+(\mathscr{A}_+)|\,\|\psi_+ - I_h\psi_+\|_{L_\infty(\Omega)} \le Ch^2 \tag{4.15}$$

because ψ_+ belongs to $W^{2,\infty}(\Omega)$ by the assumption (1.6) and the Sobolev embedding $W^{1,q}(\Omega) \subset C(\bar\Omega)$ for $q > 2$.

Since μ_+ is nonpositive, the third term satisfies

$$\int_\Omega (I_h\psi_+ - I_hE_h\bar y_h)d\mu_+ = \int_\Omega (I_h\psi_+ - I_h\bar y_h)d\mu_+ \le 0 \tag{4.16}$$

by (3.2) and (3.18).

We can split the last term on the right-hand side of (4.13) as

$$\int_\Omega (I_hE_h\bar y_h - E_h\bar y_h)d\mu_+ \tag{4.17}$$
$$= \int_\Omega \big[I_h(E_h\bar y_h - \bar y) - (E_h\bar y_h - \bar y)\big]d\mu_+ + \int_\Omega (I_h\bar y - \bar y)d\mu_+,$$

and we have

$$\int_\Omega (I_h\bar y - \bar y)d\mu_+ \le |\mu_+(\mathscr{A}_+)|\,\|I_h\bar y - \bar y\|_{L_\infty(\mathscr{A}_+)} \le C_\epsilon h^{2-\epsilon} \tag{4.18}$$

by (2.15) and a standard interpolation error estimate for I_h.

Finally, it follows from (2.23), (3.19), (3.24) and a standard interpolation error estimate for I_h that

$$\begin{aligned}\int_\Omega &\big[I_h(E_h\bar y_h - \bar y) - (E_h\bar y_h - \bar y)\big]d\mu_+\\ &\le C|I_h(E_h\bar y_h - \bar y) - (E_h\bar y_h - \bar y)|_{H^{1+\epsilon}(\Omega)}\\ &\le Ch^{1-\epsilon}|E_h\bar y_h - \bar y|_{H^2(\Omega)}\\ &\le Ch^{1-\epsilon}\big(|E_h(\bar y_h - \Pi_h\bar y)|_{H^2(\Omega)} + |E_h\Pi_h\bar y - \bar y|_{H^2(\Omega)}\big)\\ &\le Ch^{1-\epsilon}\big(\|\bar y_h - \Pi_h\bar y\|_h + h^\tau\big).\end{aligned} \tag{4.19}$$

Combining (4.13)–(4.19), we have

$$\int_\Omega E_h(\Pi_h\bar y - \bar y_h)d\mu_+ \le C\big(h^{1+\tau} + h^2 + h^{2-\epsilon} + h^{1+\tau-\epsilon} + h^{1-\epsilon}\|\Pi_h\bar y - \bar y_h\|_h\big)$$

and hence, in view of (3.15),

$$\int_\Omega E_h(\Pi_h\bar y - \bar y_h)d\mu_+ \le C\big(h^{2\tau} + h^\tau\|\Pi_h\bar y - \bar y_h\|_h\big). \tag{4.20}$$

Similarly, we have

$$\int_\Omega E_h(\Pi_h\bar y - \bar y_h)d\mu_- \le C\big(h^{2\tau} + h^\tau\|\Pi_h\bar y - \bar y_h\|_h\big). \tag{4.21}$$

The estimate (4.6) follows from (4.11), (4.12), (4.20) and (4.21).

4.3. Other convergence results

We can approximate the optimal control $\bar u$ by $\bar u_h = -\Delta_h\bar y_h$, where Δ_h is the piecewise defined Laplacian with respect to $\mathcal{T}_h$. The following result is a direct consequence of (4.1) and Theorem 4.1.

Corollary 4.3. *There exists a positive constant C independent of h such that*

$$\|\bar u - \bar u_h\|_{L_2(\Omega)} \le Ch^\tau.$$

We can also establish error estimates for $\bar y - \bar y_h$ in lower order norms.

Corollary 4.4. *There exists a positive constant C independent of h such that*

$$\|\bar y - \bar y_h\|_{L_2(\Omega)} + |\bar y - \bar y_h|_{H^1(\Omega)} + \|\bar y - \bar y_h\|_{L_\infty(\Omega)} \le Ch^\tau.$$

Proof. The estimates for $\|\bar y - \bar y_h\|_{L_2(\Omega)}$ and $\|\bar y - \bar y_h\|_{H^1(\Omega)}$ follow immediately from (4.1), Theorem 4.1 and the Poincaré-Friedrichs inequality for piecewise H^2 functions in [26].

Similarly the estimate for $\|\bar y - \bar y_h\|_{L_\infty(\Omega)}$ follows from (4.1), Theorem 4.1 and the Sobolev inequality for piecewise H^2 functions in [15, Appendix]. □

Remark 4.5. Numerical results in Section 5 indicate that the estimates in Corollary 4.4 are not sharp.

5. Numerical results

In this section we report the results for two numerical examples on square domains where the computations were carried out on uniform meshes. Consequently τ equals $1-\epsilon$ for any $\epsilon > 0$ in Theorem 4.1, Corollary 4.3 and Corollary 4.4 (cf. Remark 3.3). The first example only involves point tracking and region tracking, and the exact solution is constructed through the procedure in Appendix A. The second example involves all three types of tracking functions, where the exact solution is not available.

The discrete problems are solved by the PDAS algorithm in [8,44].

Example 5.1. We follow the procedure in Appendix A to obtain an exact solution for (2.1)–(2.2) on $\Omega = (-4,4)^2$, where $w_1 = 0$, $w_2 = 1$, $\beta = 1$, $\psi_- = -\infty$ and

$$\psi_+(x) = |x|^2 - 1.$$

The active set $\mathscr{A}$ is the disk $\{x : |x| \le 1\}$ and we use the function v in [13, Example 7.1].

We track the state at the following points that do not belong to $\mathcal{V}_h$:

$$p_1 = (-2.49, 2.51),\ p_2 = (2.51, -2.51),\ p_3 = (2.49, 2.51),$$
$$p_4 = (-2.51, 2.49),$$

with the weights $w_0(p_1) = w_0(p_2) = w_0(p_3) = w_0(p_4) = 100$.

The function $\Phi \in C_c^3(\Omega)$ that satisfies (A.5) is given by $\Phi(x) = \zeta(x_1)\zeta(x_2)$, where

$$\zeta(t) = \begin{cases} 0 & \text{if } d_2 \le t,\\ \eta(\frac{t-c_2}{d_2-c_2}) & \text{if } c_2 \le t \le d_2,\\ 1 & \text{if } b_2 \le t \le c_2,\\ \eta(\frac{b_2-t}{b_2-a_2}) & \text{if } a_2 \le t \le b_2,\\ 0 & \text{if } d_1 \le t \le a_2,\\ \eta(\frac{t-c_1}{d_1-c_1}) & \text{if } c_1 \le t \le d_1,\\ 1 & \text{if } b_1 \le t \le c_1,\\ \eta(\frac{b_1-t}{b_1-a_1}) & \text{if } a_1 \le t \le b_1,\\ 0 & \text{if } t \le a_1,\end{cases}$$

with

$$a_1 = -7/2,\quad b_1 = -3,\quad c_1 = -2,\quad d_1 = -3/2,$$
$$a_2 = 3/2,\quad b_2 = 2,\quad c_2 = 3,\quad d_2 = 7/2,$$

and

$$\eta(t) = (1 + 4t + 10t^2 + 20t^3)(1-t)^4.$$

The exact solution $\bar y$ is then defined by (A.6), where $\theta = 0.1$. The targets $y_0(p_j)$ for $1 \le j \le 4$ are given by (A.8) and the target y_2 is given by (A.9).

Table 5.1
Relative errors of the state versus mesh size h and orders of convergence (P_2 element) for Example 5.1.

h	e^r_h	order	$e^r_{1,h}$	order	$e^r_{0,h}$	order	$e^r_{\infty,h}$	order
2^{-1}	9.6413e-1	–	5.4460e-1	–	2.0670e-1	–	3.4617e-1	–
2^{-2}	6.1450e-1	0.65	2.5521e-1	1.09	8.5745e-2	1.27	1.4600e-1	1.25
2^{-3}	5.1126e-1	0.27	1.6734e-1	0.61	4.6488e-2	0.88	1.4090e-1	0.05
2^{-4}	2.6839e-1	0.93	7.1839e-2	1.22	1.7819e-2	1.38	5.7271e-2	1.30
2^{-5}	1.3523e-1	0.99	2.8121e-2	1.35	5.8638e-3	1.60	2.1555e-2	1.41
2^{-6}	6.4002e-2	1.08	9.1345e-3	1.62	1.6972e-3	1.79	6.7055e-3	1.68

Table 5.2
Relative errors of the control versus mesh size h and orders of convergence (P_2 element) for Example 5.1.

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)}/\|\bar{u}\|_{L^2(\Omega)}$	9.8501e-1	6.2449e-1	5.2082e-1	2.7151e-1	1.3591e-1	6.3854e-2
Order	–	0.66	0.26	0.94	1.00	1.09

Table 5.3
Relative errors of the state versus mesh size h and orders of convergence (P_3 element) for Example 5.1.

h	e^r_h	order	$e^r_{1,h}$	order	$e^r_{0,h}$	order	$e^r_{\infty,h}$	order
2^{1}	2.8861e0	–	5.6489e0	–	4.5143e0	–	3.4806e0	–
2^{0}	1.7906e0	0.69	3.7298e0	0.60	3.6158e0	0.32	3.2830e0	0.08
2^{-1}	8.4475e-1	1.08	3.8210e-1	3.29	1.3201e-1	4.78	2.0883e-1	3.97
2^{-2}	4.7260e-1	0.84	9.9812e-2	1.94	1.7288e-2	2.93	2.6302e-2	2.99
2^{-3}	1.6540e-1	1.51	2.2119e-2	2.17	3.1072e-3	2.48	1.1167e-2	1.24
2^{-4}	7.2488e-2	1.19	4.9158e-3	2.17	1.2571e-3	1.31	2.7623e-3	2.02

Table 5.4
Relative errors of the control versus mesh size h and orders of convergence (P_3 element) for Example 5.1.

h	2^{1}	2^{0}	2^{-1}	2^{-2}	2^{-3}	2^{-4}
$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)}/\|\bar{u}\|_{L^2(\Omega)}$	2.8807e0	1.7714e0	8.6255e-1	4.8265e-1	1.6892e-1	7.4030e-2
Order	–	0.70	1.04	0.84	1.51	1.19

We discretize (2.1)–(2.2) by the P_2 (resp., P_3) Lagrange finite element on uniform meshes with penalty parameter $\sigma = 100$ (resp., 1000). The numerical results are presented in Tables 5.1–5.4, where the relative errors of the state are defined by

$$e^r_h = \|\bar{y}-\bar{y}_h\|_h/\|\bar{y}\|_{H^2(\Omega)}, \qquad e^r_{1,h} = \|\bar{y}-\bar{y}_h\|_{H^1(\Omega)}/\|\bar{y}\|_{H^1(\Omega)},$$
$$e^r_{0,h} = \|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}/\|\bar{y}\|_{L^2(\Omega)}, \qquad e^r_{\infty,h} = \max_{p\in\mathcal{V}_h} |\bar{y}(p)-\bar{y}_h(p)|/\|\bar{y}\|_{L^\infty(\Omega)}.$$

The convergence for the state in the $\|\cdot\|_h$ norm (resp., the convergence of the control in the L_2 norm) agrees with Theorem 4.1 (resp., Corollary 4.3) where $\tau = 1-\epsilon$. The convergence of the state in the other norms is better than the convergence predicted by Corollary 4.4.

It is also observed that the relative errors for the P_3 finite element method at $h = 2^{-4}$ are comparable to the relative errors for the P_2 finite element method at $h = 2^{-6}$. Therefore the cubic C^0 interior penalty method is more efficient for this example.

The graphs for the optimal states, optimal controls and active sets are displayed in Fig. 5.1 and Fig. 5.2.

Example 5.2. We consider an optimal control problem on $\Omega = (0,1)^2$ that involves all three types of tracking functions. We take $\beta = 1$, $\psi_- = -\infty$ and

$$\psi_+ = 5 - [(x_1-0.5)^2 + (x_2-0.5)^2].$$

The point tracking occurs at $p_1 = (0.375, 0.625)$ and $p_2 = (0.625, 0.625)$ (cf. Fig. 5.3), and we take $y_0(p_1) = 4.8$, $y_0(p_2) = 4.6$, $w_0(p_1) = 2000$ and $w_0(p_2) = 500$.

The curve tracking takes place at the boundary of the square with vertices $(0.125, 0.125)$, $(0.875, 0.125)$, $(0.875, 0.875)$ and $(0.125, 0.875)$ (cf. Fig. 5.3), and we take $y_1 = 4.5$ and $w_1 = 700$.

For the region tracking we choose $w_2 = 1000\chi_R$, where χ_R is the characteristic function of the rectangle R with vertices $(0.375, 0.25)$, $(0.625, 0.25)$, $(0.625, 0.375)$, and $(0.375, 0.375)$ (cf. Fig. 5.3) and we take $y_2 = 4.7$.

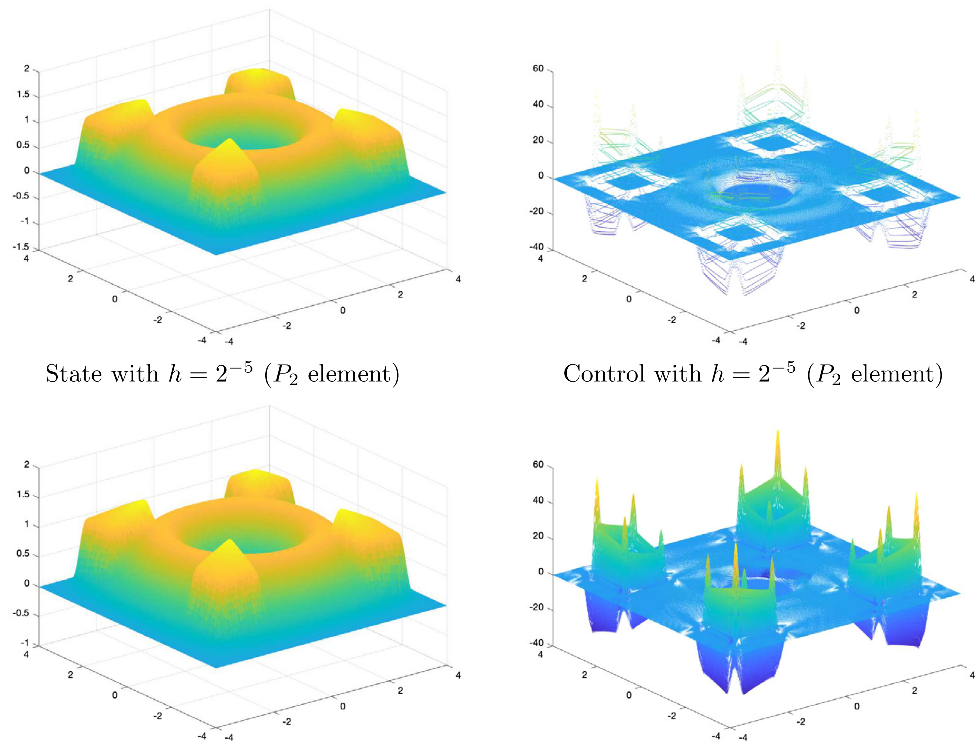
Note that for this example the pointwise state constraints are satisfied by the target functions y_0, y_1 and y_2. Therefore it can be interpreted as a least-squares data fitting problem (cf. Remark 1.4).

The optimal control problem is solved by the quadratic C^0 finite element method with $\sigma = 100$. The numerical results are reported in Table 5.5 and Table 5.6, where the relative errors of the state are defined by

$$e^r_h = \|\bar{y}_{h/2}-\bar{y}_h\|_h/\|\bar{y}_{2^{-9}}\|_h,$$
$$e^r_{1,h} = \|\bar{y}_{h/2}-\bar{y}_h\|_{H^1(\Omega)}/\|\bar{y}_{2^{-9}}\|_{H^1(\Omega)},$$
$$e^r_{0,h} = \|\bar{y}_{h/2}-\bar{y}_h\|_{L_2(\Omega)}/\|\bar{y}_{2^{-9}}\|_{L_2(\Omega)},$$
$$e^r_{\infty,h} = \max_{p\in\mathcal{V}_h} |\bar{y}_{h/2}(p)-\bar{y}_h(p)|/\max_{p\in\mathcal{V}_h} |\bar{y}_{2^{-9}}(p)|.$$

The convergence for the state in the $\|\cdot\|_h$ norm (resp., the convergence of the control in the L_2 norm) agrees with Theorem 4.1 (resp., Corollary 4.3), where $\tau = 1-\epsilon$. The convergence of the state in the other norms is better than the convergence predicted by Corollary 4.4.

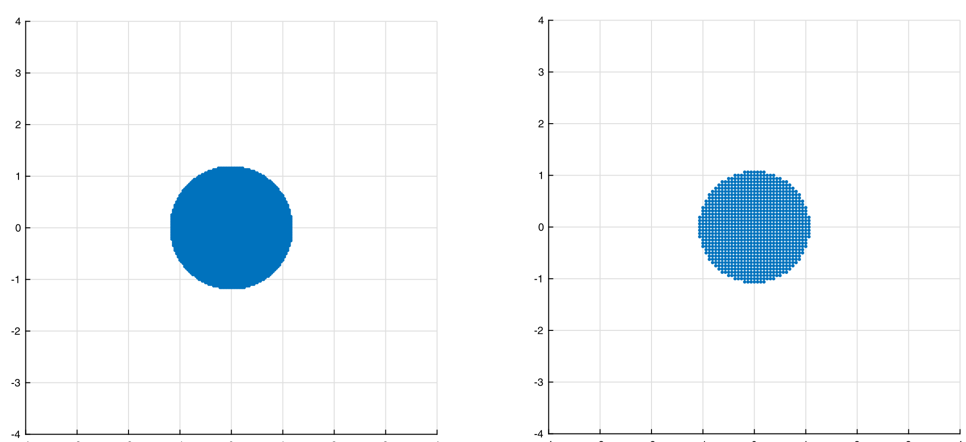
The graphs for the optimal state, optimal control and active set at $h = 2^{-7}$ are displayed in Fig. 5.4 and Fig. 5.5.



State with $h = 2^{-5}$ (P_2 element) Control with $h = 2^{-5}$ (P_2 element)

State with $h = 2^{-4}$ (P_3 element) Control with $h = 2^{-4}$ (P_3 element)

Fig. 5.1. Graphs of optimal states and optimal controls for Example 5.1.



Active set with $h = 2^{-5}$ (P_2 element) Active set with $h = 2^{-4}$ (P_3 element)

Fig. 5.2. Active sets for Example 5.1.

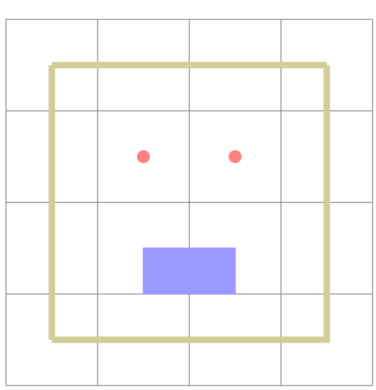


Fig. 5.3. Point, line and region tracking.

6. Concluding remarks

We have constructed C^0 interior penalty methods for a linear quadratic elliptic distributed optimal control problem with pointwise state constraints and a general cost function that involves tracking at points, curves and regions of the domain. It is based on reformulating the optimal control problem as a minimization problem that only involves the state. The results in this paper can be extended to problems with both pointwise state and pointwise control constraints by using a cubic C^0 interior penalty method (cf. [21]).

For simplicity we have performed numerical experiments on rectangular domains with uniform meshes which is sufficient to guarantee $O(h^{1-\epsilon})$ estimates for the errors that appear in Theorem 4.1, Corollary 4.3 and Corollary 4.4. These $O(h^{1-\epsilon})$ error estimates are also valid for a general convex domain provided we use local mesh refinements

Table 5.5
Relative errors of the state versus mesh size h and orders of convergence for Example 5.2.

h	e_h^r	order	$e_{1,h}^r$	order	$e_{0,h}^r$	order	$e_{\infty,h}^r$	order
2^{-3}	9.5382e-2		9.8750e-2		6.0017e-2		6.4915e-2	
2^{-4}	4.8085e-2	0.99	4.2391e-2	1.22	2.5509e-2	1.23	3.4311e-2	0.92
2^{-5}	2.3055e-2	1.06	1.4364e-2	1.56	7.7981e-3	1.71	1.1588e-2	1.57
2^{-6}	1.1124e-2	1.05	4.6165e-3	1.64	2.1994e-3	1.83	3.6465e-3	1.67
2^{-7}	5.3350e-3	1.06	1.3428e-3	1.78	5.8472e-4	1.91	1.0256e-3	1.83
2^{-8}	2.6061e-3	1.03	3.6717e-4	1.87	1.5988e-4	1.87	2.7192e-4	1.92

Table 5.6
Relative errors of the control versus mesh size h and order of convergence for Example 5.2.

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$\|\bar{u}_{h/2}-\bar{u}_h\|_{L_2(\Omega)}/\|\bar{u}_{2^{-9}}\|_{L_2(\Omega)}$	2.70e-1	1.41e-1	7.03e-2	3.44e-2	1.65e-2	8.04e-3
order	-	0.93	1.01	1.03	1.06	1.04

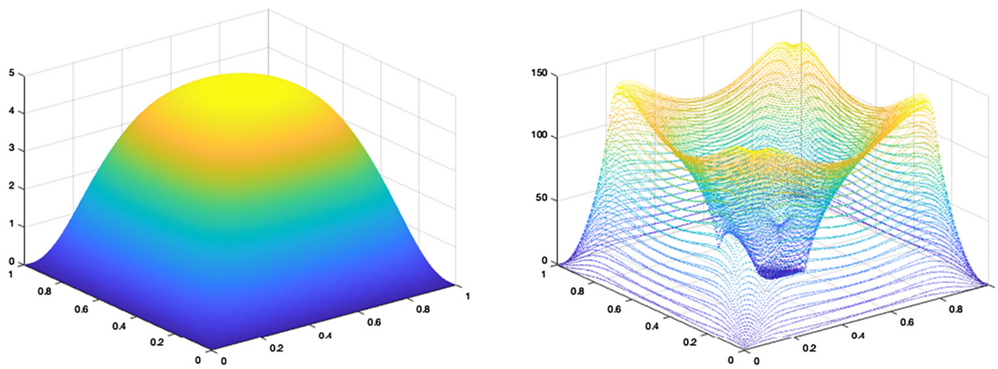


Fig. 5.4. Graphs of optimal state (left) and optimal control (right) for Example 5.2 at $h = 2^{-7}$.

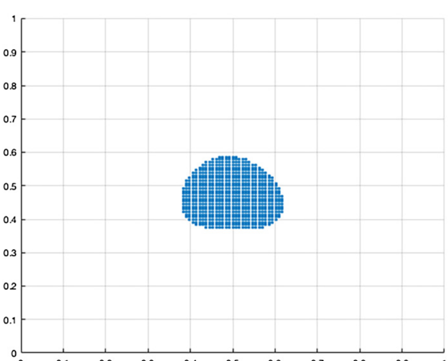


Fig. 5.5. Active set for Example 5.2 at $h = 2^{-7}$.

around the corners where the interior angles are strictly larger than $\pi/2$.

One can consider an analogous optimal control problem on three dimensional domains, where the cost function also includes tracking on surfaces inside the domain in addition to tracking on points, curves and regions. In this case the regularity estimate (2.12) for the adjoint state $\bar{p}$ becomes (cf. [30, Theorem 1])

$$\bar{p} \in W_0^{1,s}(\Omega) \qquad \forall\, s < \frac{3}{2},$$

and hence the regularity estimate (2.15) for the optimal state $\bar{y}$ becomes

$$\bar{y} \in W_{loc}^{3,s}(\Omega) \qquad \forall\, s < \frac{3}{2}.$$

Consequently the error estimates on uniform meshes are $O(h^{(1/2)-\epsilon})$ even for a rectangular parallelepiped. However, since the deterioration of the elliptic regularity is due to the existence of point tracking at known positions, one can include these points in $\mathcal{V}_h$ and use local mesh refinement around them (and around $\partial\Omega$ for a general convex Ω) to recover $O(h^{1-\epsilon})$ error estimates.

Data availability

Data will be made available on request.

Appendix A. The construction of an exact solution

We will construct an exact solution to the problem (2.1)–(2.2) where $w_1 = 0$ in (1.3) (i.e., there is no tracking on curves), $w_2 = 1$ (i.e., standard tracking for the domain), and $\psi_- = -\infty$ (i.e., there is only an upper constraint on the state).

Let $\Omega = (-4,4)\times(-4,4)$, $\psi_+(x) = |x|^2 - 1$ and $\mathscr{A} = \{x : |x| \le 1\}$. The starting point is a function $v \in H^3(\Omega)$ that is piecewise smooth with respect to $\mathscr{A}$ and has the following properties:

$$v = \psi \text{ on } \mathscr{A} \text{ and } v < \psi \text{ on } \Omega\setminus\mathscr{A}, \tag{A.1}$$

$$v = \Delta v = 0 \text{ on } \partial\Omega, \tag{A.2}$$

$$[\![\partial(\Delta v)/\partial n]\!] \ge 0 \text{ on } \partial\mathscr{A}. \tag{A.3}$$

The construction of such a function can be found in [13, Example 7.1].

Next we choose the points $p_1, \ldots, p_J \in \Omega\setminus\mathscr{A}$ and define

$$\phi_j(x) = \frac{1}{8\pi}|x-p_j|^2 \ln|x-p_j|$$

Note that

$$\phi_j(p_j) = 0, \quad \Delta\phi_j = \frac{1}{2\pi}\big(\ln|x-p_j| + 1\big) \quad \text{and} \quad \Delta^2\phi_j = \delta_{p_j}, \tag{A.4}$$

where δ_{p_j} is the Dirac point measure associate with the point p_j.

Let $\Phi \in C_c^3(\Omega)$ such that

$$\Phi = 1 \text{ near } p_1, \ldots, p_J \text{ and } \Phi = 0 \text{ in a neighborhood of } \mathscr{A}. \tag{A.5}$$

The exact solution $\bar{y}$ is given by

$$\bar{y} = v + \theta \sum_{j=1}^{J} \phi_j \Phi, \tag{A.6}$$

where θ is sufficiently small so that (cf. (A.1))

$$\bar{y} = \psi \text{ on } \mathscr{A} \text{ and } \bar{y} < \psi \text{ on } \Omega \setminus \mathscr{A}. \tag{A.7}$$

Now we choose positive weights $w_0(p_1), \ldots, w_0(p_J)$ and set the targets $y_0(p_1), \ldots, y_0(p_J)$ by

$$y_0(p_j) = \bar{y}(p_j) + \frac{\theta\beta}{w_0(p_j)} = v(p_j) + \theta\Big(\sum_{i=1}^{J} \phi_i(p_j) + \frac{\beta}{w_0(p_j)}\Big) \quad \text{for} \quad 1 \le j \le J. \tag{A.8}$$

Finally we set the target function $y_2 \in L_2(\Omega)$ by

$$y_2 = \begin{cases} Lv + \theta\beta \sum_{j=1}^{J} R(\phi_j, \Phi) + \theta \sum_{j=1}^{J} \phi_j \Phi & \text{on } \Omega \setminus \mathscr{A} \\ Lv + \gamma & \text{on } \mathscr{A} \end{cases}, \tag{A.9}$$

where L is the differential operator $\beta\Delta^2 + 1$, $R(\cdot,\cdot)$ is the bilinear form defined by

$$R(f,g) = 4\nabla(\Delta f)\cdot\nabla g + 2(\Delta f)(\Delta g) + 4D^2 f : D^2 g + 4\nabla f \cdot \nabla(\Delta g) + f(\Delta^2 g),$$

and γ is a nonnegative constant.

A key observation is that

$$\int_{\Omega} \big[\Delta(\phi_j\Phi)\big](\Delta z)dx = z(p_j) + \int_{\Omega} R(\phi_j,\Phi) z\, dx, \tag{A.10}$$

which follows from (A.4), (A.5) and integration by parts.

Combining (A.1)–(A.3), (A.6)–(A.10) and integration by parts, we find

$$\beta \int_{\Omega} (\Delta\bar{y})(\Delta z)dx + \int_{\Omega} (\bar{y} - y_2)z\, dx + \sum_{j=1}^{J} w_0(p_j)[\bar{y}(p_j) - y_0(p_j)]z(p_j) = -\gamma \int_{\mathscr{A}} z\, dx - \beta \int_{\partial\mathscr{A}} [\![\partial(\Delta v)/\partial n]\!] z\, ds \qquad \forall z \in H^2(\Omega)\cap H_0^1(\Omega), \tag{A.11}$$

which verifies that $\bar{y}$ is the exact solution of (2.1)–(2.2) (cf. (2.3), (2.5) and (2.6)).

References

[1] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second edition, Academic Press, Amsterdam, 2003.

[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Commun. Pure Appl. Math. 12 (1959) 623–727.

[3] A. Allendes, F. Fuica, E. Otárola, Error estimates for a pointwise tracking optimal control problem of a semilinear elliptic equation, SIAM J. Control Optim. 60 (2022) 1763–1790.

[4] A. Allendes, E. Otárola, R. Rankin, A.J. Salgado, Adaptive finite element methods for an optimal control problem involving Dirac measures, Numer. Math. 137 (2017) 159–197.

[5] H. Antil, E. Otárola, A.J. Salgado, Some applications of weighted norm inequalities to the error analysis of PDE-constrained optimization problems, IMA J. Numer. Anal. 38 (2018) 852–883.

[6] N. Behringer, D. Meidner, B. Vexler, Finite element error estimates for optimal control problems with pointwise tracking, Pure Appl. Funct. Anal. 4 (2019) 177–204.

[7] M. Bergounioux, K. Kunisch, Augmented Lagrangian techniques for elliptic state constrained optimal control problems, SIAM J. Control Optim. 35 (1997) 1524–1543.

[8] M. Bergounioux, K. Kunisch, Primal-dual strategy for state-constrained optimal control problems, Comput. Optim. Appl. 22 (2002) 193–224.

[9] J.H. Bramble, S.R. Hilbert, Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation, SIAM J. Numer. Anal. 7 (1970) 113–124.

[10] S.C. Brenner, C^0 interior penalty methods, in: J. Blowey, M. Jensen (Eds.), Frontiers in Numerical Analysis, Durham 2010, in: Lecture Notes in Computational Science and Engineering, vol. 85, Springer-Verlag, Berlin-Heidelberg, 2012, pp. 79–147.

[11] S.C. Brenner, C.B. Davis, L.-Y. Sung, A partition of unity method for a class of fourth order elliptic variational inequalities, Comput. Methods Appl. Mech. Eng. 276 (2014) 612–626.

[12] S.C. Brenner, J. Gedicke, L.-Y. Sung, C^0 interior penalty methods for an elliptic distributed optimal control problem on nonconvex polygonal domains with pointwise state constraints, SIAM J. Numer. Anal. 56 (2018) 1758–1785.

[13] S.C. Brenner, J. Gedicke, L.-Y. Sung, P_1 finite element methods for an elliptic optimal control problem with pointwise state constraints, IMA J. Numer. Anal. 40 (2020) 1–28.

[14] S.C. Brenner, T. Gudi, L.-Y. Sung, An a posteriori error estimator for a quadratic C^0 interior penalty method for the biharmonic problem, IMA J. Numer. Anal. 30 (2010) 777–798.

[15] S.C. Brenner, M. Neilan, A. Reiser, L.-Y. Sung, A C^0 interior penalty method for a von Kármán plate, Numer. Math. 135 (2017) 803–832.

[16] S.C. Brenner, M. Oh, S. Pollock, K. Porwal, M. Schedensack, N. Sharma, A C^0 interior penalty method for elliptic distributed optimal control problems in three dimensions with pointwise state constraints, in: S.C. Brenner (Ed.), Topics in Numerical Partial Differential Equations and Scientific Computing, in: The IMA Volumes in Mathematics and Its Applications, vol. 160, Springer, Cham-Heidelberg-New York-Dordrecht-London, 2016, pp. 1–22.

[17] S.C. Brenner, M. Oh, L.-Y. Sung, P_1 finite element methods for an elliptic state-constrained distributed optimal control problem with Neumann boundary conditions, Results Appl. Math. 8 (2020) 100090.

[18] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third edition, Springer-Verlag, New York, 2008.

[19] S.C. Brenner, L.-Y. Sung, C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, J. Sci. Comput. 22/23 (2005) 83–118.

[20] S.C. Brenner, L.-Y. Sung, A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints, SIAM J. Control Optim. 55 (2017) 2289–2304.

[21] S.C. Brenner, L.-Y. Sung, Z. Tan, A cubic C^0 interior penalty method for elliptic distributed optimal control problems with pointwise state and control constraints, Results Appl. Math. 7 (2020) 100119.

[22] S.C. Brenner, L.-Y. Sung, Z. Tan, A C^1 virtual element method for an elliptic distributed optimal control problem with pointwise state constraints, Math. Models Methods Appl. Sci. 31 (2021) 2887–2906.

[23] S.C. Brenner, L.-Y. Sung, H. Zhang, Y. Zhang, A quadratic C^0 interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates, SIAM J. Numer. Anal. 50 (2012) 3329–3350.

[24] S.C. Brenner, L.-Y. Sung, Y. Zhang, A quadratic C^0 interior penalty method for an elliptic optimal control problem with state constraints, in: O. Karakashian, X. Feng, Y. Xing (Eds.), Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, in: The IMA Volumes in Mathematics and Its Applications, vol. 157, Springer, Cham-Heidelberg-New York-Dordrecht-London, 2013, pp. 97–132, 2012 John H. Barrett Memorial Lectures.

[25] S.C. Brenner, L-Y. Sung, Y. Zhang, C^0 interior penalty methods for an elliptic state-constrained optimal control problem with Neumann boundary condition, J. Comput. Appl. Math. 350 (2019) 212–232.

[26] S.C. Brenner, K. Wang, J. Zhao, Poincaré-Friedrichs inequalities for piecewise H^2 functions, Numer. Funct. Anal. Optim. 25 (2004) 463–478.

[27] C. Brett, A. Dedner, C. Elliott, Optimal control of elliptic PDEs at points, IMA J. Numer. Anal. 36 (2016) 1015–1050.

[28] C. Brett, C.M. Elliott, M. Hintermüller, C. Löbhard, Mesh adaptivity in optimal control of elliptic variational inequalities with point-tracking of the state, Interfaces Free Bound. 17 (2015) 21–53.

[29] L.A. Caffarelli, A. Friedman, The obstacle problem for the biharmonic operator, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 4 (6) (1979) 151–184.

[30] E. Casas, L^2 estimates for the finite element method for the Dirichlet problem with singular data, Numer. Math. 47 (1985) 627–632.

[31] E. Casas, Control of an elliptic problem with pointwise state constraints, SIAM J. Control Optim. 24 (1986) 1309–1318.

[32] E. Casas, M. Mateos, B. Vexler, New regularity results and improved error estimates for optimal control problems with state constraints, ESAIM Control Optim. Calc. Var. 20 (2014) 803–822.

[33] L. Chang, W. Gong, N. Yan, Numerical analysis for the approximation of optimal control problems with pointwise observations, Math. Methods Appl. Sci. 38 (2015) 4502–4520.

[34] S. Cherednichenko, A. Rösch, Error estimates for the discretization of elliptic control problems with pointwise control and state constraints, Comput. Optim. Appl. 44 (2009) 27–55.

[35] P.G. Ciarlet, Sur l'élément de Clough et Tocher, RAIRO. Anal. Numér. 8 (1974) 19–27.
[36] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[37] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, Lecture Notes in Mathematics, vol. 1341, Springer-Verlag, Berlin-Heidelberg, 1988.
[38] T. Dupont, R. Scott, Polynomial approximation of functions in Sobolev spaces, Math. Comput. 34 (1980) 441–463.
[39] I. Ekeland, R. Témam, Convex Analysis and Variational Problems, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999.
[40] G. Engel, K. Garikipati, T.J.R. Hughes, M.G. Larson, L. Mazzei, R.L. Taylor, Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, Comput. Methods Appl. Mech. Eng. 191 (2002) 3669–3750.
[41] J. Frehse, Zum Differenzierbarkeitsproblem bei Variationsungleichungen höherer Ordnung, Abh. Math. Semin. Univ. Hamb. 36 (1971) 140–149.
[42] J. Frehse, On the regularity of the solution of the biharmonic variational inequality, Manuscr. Math. 9 (1973) 91–103.
[43] P. Grisvard, Elliptic Problems in Non Smooth Domains, Pitman, Boston, 1985.
[44] M. Hintermüller, K. Ito, K. Kunisch, The primal-dual active set strategy as a semismooth Newton method, SIAM J. Optim. 13 (2003) 865–888.
[45] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, Optimization with PDE Constraints, Springer, New York, 2009.
[46] L. Hörmander, The Analysis of Linear Partial Differential Operators. III, Springer-Verlag, Berlin, 1985.
[47] K. Ito, K. Kunisch, Lagrange Multiplier Approach to Variational Problems and Applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008.
[48] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Society for Industrial and Applied Mathematics, Philadelphia, 2000.
[49] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.
[50] W. Liu, W. Gong, N. Yan, A new finite element approximation of a state-constrained optimal control problem, J. Comput. Math. 27 (2009) 97–114.
[51] V. Maz'ya, J. Rossmann, Elliptic Equations in Polyhedral Domains, American Mathematical Society, Providence, RI, 2010.
[52] C. Meyer, Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints, Control Cybern. 37 (2008) 51–83.
[53] I. Neitzel, J. Pfefferer, A. Rösch, Finite element discretization of state-constrained elliptic optimal control problems with semilinear state equation, SIAM J. Control Optim. 53 (2015) 874–904.
[54] M. Pierre, J. Sokołowski, Differentiability of projection and applications, in: Control of Partial Differential Equations and Applications, Laredo, 1994, in: Lecture Notes in Pure and Appl. Math., vol. 174, Dekker, New York, 1996, pp. 231–240.
[55] F. Tröltzsch, Optimal Control of Partial Differential Equations, American Mathematical Society, Providence, RI, 2010.