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An Adaptive Nonlinear Least-Squares Finite Element Method for a Pucci Equation in Two Dimensions

Susanne C. Brenner[1,*], Li-Yeng Sung[1] and Zhiyu Tan [2]

[1] *Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA*
[2] *School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Fujian, 361005, China*

Abstract. We present an adaptive nonlinear least-squares finite element method for a two dimensional Pucci equation. The efficiency of the method is demonstrated by a numerical experiment.

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1. Introduction

The Pucci equation is a fully nonlinear second order elliptic partial differential equation that first appeared in the study of linear uniformly elliptic equations in nondivergence form (cf. [13, 35, 36]) and has found applications in optimal designs (cf. [14]) and population models (cf. [12, 37]).

Let Ω be a bounded convex polygon in $\mathbb{R}^2$. We consider in this paper the following Dirichlet boundary value problem for a Pucci equation:

$$\begin{aligned} \alpha\lambda_{\max}(D^2u) + \lambda_{\min}(D^2u) &= \psi \quad \text{in } \Omega, \\ u &= \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\alpha > 1$, $\lambda_{\max}(D^2u)$ (resp., $\lambda_{\min}(D^2u)$) is the maximum (resp., minimum) eigenvalue of D^2u (the Hessian of u), $\psi \in L^2(\Omega)$ and $\phi \in H^2(\Omega)$.

Remark 1.1. Throughout this paper we will follow the standard notation for differential operators, functions spaces and norms that can be found for example in [1, 7, 22].

*Corresponding author. *Email addresses:* `brenner@math.lsu.edu` (Susanne C. Brenner), `sung@math.lsu.edu` (Li-Yeng Sung), `zhiyutan@xmu.edu.cn` (Zhiyu Tan)

The numerical treatment of Pucci's equation began in [14, 19], followed by the work in [30]. The finite element methods in these papers were tested extensively but without convergence analysis. Finite difference methods for the viscosity solutions of the Pucci equation were investigated in [23, 34], where the convergence was established in the framework of [2] without convergence rate, and a second order consistent finite difference method was considered in [4].

Motivated by our work on the Monge-Ampère equation in [10], a nonlinear least-squares method was presented in [9] for the strong solutions of (1.1), where convergence with convergence rates was established. Our goal in this paper is to present an adaptive version of this nonlinear least-squares method and demonstrate its effectiveness through a numerical experiment.

The rest of the paper is organized as follows. We introduce the nonlinear least-squares method in Section 2 and briefly recall the theoretical results from [9]. The numerical result for the adaptive version is presented in Section 3. We end with some concluding remarks in Section 4.

2. A Nonlinear Least-Squares Finite Element Method

Let $S_{2\times 2}$ be the space of real 2×2 symmetric matrices and $P(M)$ be the Pucci operator defined on $S_{2\times 2}$ given by

$$P(M) = \alpha\lambda_{\max}(M) + \lambda_{\min}(M) \tag{2.1}$$

for a constant $\alpha > 1$. We can then write the boundary value problem (1.1) as

$$\begin{aligned} P(D^2u) &= \psi \quad \text{in } \Omega, \\ u &= \phi \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

A unique strong solution $u \in H^2(\Omega)$ of (2.2) was established in [9] by using the uniform ellipticity of $P(D)$, the Miranda-Talenti inequality

$$\|D^2v\|_{L^2(\Omega)} \le \|\Delta v\|_{L^2(\Omega)} \qquad \forall\, v \in H^2(\Omega) \cap H^1_0(\Omega),$$

that holds on convex domains (cf. [24, 32, 38]) and the theory of Companato on near operators (cf. [15, 31]).

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h, $V_h \subset H^1(\Omega)$ be the cubic Lagrange finite element space (cf. [7, 18]) associated with $\mathcal{T}_h$, and Π_h be the nodal interpolation operator from $C(\bar{\Omega})$ to V_h.

The nonlinear least-squares method in [9] is given by

$$u_h = \underset{v_h \in L_h}{\operatorname{argmin}} J_h(v_h), \tag{2.3}$$

where the constraint set L_h is defined by

$$L_h = \{v_h \in V_h : \; v_h = \Pi_h\phi \text{ on } \partial\Omega\}, \tag{2.4}$$

and the objective function J_h is defined by

$$J_h(v_h) = \frac{h^4}{2}\|D_h^2 v_h\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{e\in\mathscr{E}_h^i} |e|^{-1}\|[\![\partial v_h/\partial n]\!]\|_{L^2(e)}^2 + \frac{1}{2}\|P(D_h^2 v_h) - \psi\|_{L^2(\Omega)}^2. \tag{2.5}$$

Here $D_h^2 v_h$ is the piecewise defined Hessian of v_h, $\mathscr{E}_h^i$ is the set of the interior edges of $\mathscr{T}_h$, $[\![\partial v_h/\partial n]\!]$ is the jump of the normal derivative of v_h across an interior edge, and $|e|$ denotes the length of the edge e.

It follows from a Poincaré-Friedrichs inequality for piecewise H^2 functions (cf. [11]) that J_h has a global minimizer in L_h. Let $u_h \in L_h$ be a solution of the minimization problem defined by (2.3)–(2.5). It was shown in [9] that

$$\|u - u_h\|_h^2 \le C\big[J_h(v_h) + \|\phi - \phi_h\|_h^2\big] \qquad \forall\, v_h, \phi_h \in L_h, \tag{2.6}$$

where

$$\|v\|_h^2 = \|D_h^2 v\|_{L^2(\Omega)}^2 + \sum_{e\in\mathscr{E}_h^i} |e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L^2(e)}^2$$

is the standard discrete norm that appears in C^0 interior penalty methods for fourth order problems (cf. [5, 8, 21]).

The quasi-optimal error estimate (2.6) and standard interpolation error estimates imply that

$$\|u - u_h\|_h \le Ch^{\min\{s-2,2\}} \tag{2.7}$$

if the solution u of (1.1)/(2.2) belongs to $H^s(\Omega)$ for some $s > 2$. It follows from (2.7) and the Poincaré-Friedrichs and Sobolev inequalities in [6, 11] that

$$\|u - u_h\|_{L^2(\Omega)} + |u - u_h|_{H^1(\Omega)} + \|u - u_h\|_{L^\infty(\Omega)} \le Ch^{\min\{s-2,2\}}. \tag{2.8}$$

Remark 2.1. Numerical results in [9] indicate that the error estimate (2.7) is sharp, but the error estimate in (2.8) for the lower order norms is not. This can also be observed in the numerical results from Section 3.

Since the discrete problem defined by (2.3)–(2.5) is a nonlinear optimization problem, in general the outcome of an optimization algorithm is not guaranteed to be a global minimizer. However, we can monitor the convergence of the computed numerical solution according to the following estimate in [9].

Let $\tilde{u}_h \in L_h$ be an approximate solution of the discrete minimization problem. We have

$$\|u - \tilde{u}_h\|_h^2 \le C\Big[\|P(D_h^2\tilde{u}_h) - \psi\|_{L^2(\Omega)} + \Big(\sum_{e\in\mathscr{E}_h^i} |e|^{-1}\|[\![\partial \tilde{u}_h/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}} + Osc(\phi)\Big], \tag{2.9}$$

where $Osc(\phi) = \|\phi - \Pi_h\phi\|_h$ is the oscillation term (which is a higher order term if ϕ is smooth). Hence we can conclude that $\tilde{u}_h$ is converging to u if the right-hand side of (2.9) is approaching zero.

On the other hand we also have

$$\begin{aligned}\|P(D_h^2\tilde{u}_h)-\psi\|_{L^2(T)} &= \|P(D^2u)-P(D_h^2\tilde{u}_h)\|_{L^2(T)}\\ &\leq \sqrt{2}\alpha\|D_h^2(u-\tilde{u}_h)\|_{L^2(T)} \qquad \forall\, T\in\mathscr{T}_h\end{aligned}\tag{2.10}$$

by the Hoffman-Wielandt inequality (cf. [28]), and the obvious relation

$$\sum_{e\in\mathscr{E}_h^i}|e|^{-1}\|[\![\partial\tilde{u}_h/\partial n]\!]\|_{L^2(e)}^2=\sum_{e\in\mathscr{E}_h^i}|e|^{-1}\|[\![\partial(u-\tilde{u}_h)/\partial n]\!]\|_{L^2(e)}^2.\tag{2.11}$$

It follows from (2.9)–(2.11) that the residual-based error estimator defined by

$$\eta_h(\tilde{u}_h)=\|P(D_h^2\tilde{u}_h)-\psi\|_{L^2(\Omega)}+\Big(\sum_{e\in\mathscr{E}_h^i}|e|^{-1}\|[\![\partial\tilde{u}_h/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}}\tag{2.12}$$

is both reliable and locally efficient and hence it can be used for adaptive mesh refinement.

3. The Adaptive Method

In the adaptive nonlinear least-squares method we use a local version of the objective function J_h in (2.5) defined by

$$\begin{aligned}\tilde{J}_h(v_h)=\sum_{T\in\mathscr{T}_h}h_T^4\|D_h^2v_h\|_{L^2(T)}^2&+\frac{1}{2}\sum_{e\in\mathscr{E}_h^i}|e|^{-1}\|[\![\partial v_h/\partial n]\!]\|_{L^2(e)}^2\\ &+\frac{1}{2}\|P(D_h^2v_h)-\psi\|_{L^2(\Omega)}^2,\end{aligned}\tag{3.1}$$

where h_T is the diameter of T, and a local version of the error estimator η_h in (2.12) given by

$$\eta_T=\|P(D_h^2u_h)-\psi_h\|_{L^2(T)}+\Big(\sum_{e\in\mathscr{E}_T}|e|^{-1}\|[\![\partial u_h/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}}\quad\forall\, T\in\mathscr{T}_h,$$

where $\mathscr{E}_T$ is the set of the three edges of T.

We adopt the Dörfler strategy (cf. [20]) to mark the elements, i.e., we define the marking set $\mathscr{M}$ by the condition that

$$\sum_{T\in\mathscr{M}}\eta_T^2\geq\theta\sum_{T\in\mathscr{T}_h}\eta_T^2$$

with $\theta=0.4$. The meshes are refined by using the newest vertex bisection (cf. [29,33]).

Below we present a numerical example of (1.1) on the unit square $(0,1)\times(0,1)$ that originates from [19]. We take $\psi=0$ and $\phi=\phi_\delta$ defined by

$$\phi_\delta=\begin{cases}1, & \text{if } 0\leq x_1\leq 1/4-\delta;\\ \cos^2[1/4(x_1-1/4+\delta)(\pi/\delta)], & \text{if } 1/4-\delta\leq x_1\leq 1/4+\delta;\\ 0, & \text{if } 1/4+\delta\leq x_1\leq 3/4-\delta;\\ \cos^2[1/4(x_1-3/4-\delta)(\pi/\delta)], & \text{if } 3/4-\delta\leq x_1\leq 3/4+\delta;\\ 1, & \text{if } 3/4+\delta\leq x_1\leq 1,\end{cases}\tag{3.2}$$

on the edge $\{x=(x_1,x_2):\ 0\le x_1\le 1,\ x_2=0\}$, and similar definitions on the other three edges.

Remark 3.1. Note that ϕ_δ is a C^1 function along $\partial\Omega$ that is piecewise C^2. Hence it is the trace of a function in $H^{3-\epsilon}(\Omega)$ for an arbitrarily small positive ϵ and therefore ϕ_δ satisfies the assumption on the boundary value in (1.1). On the other hand it also indicates that the solution u of (1.1) does not belong to $H^4(\Omega)$.

We choose $\delta=1/16$ in (3.2) and $\alpha=3$ in (1.1) for the numerical experiment. We are interested in the relative errors

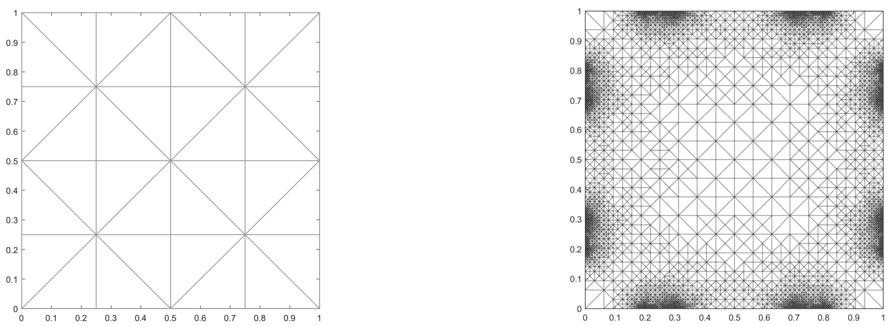
$$e^r_{2,h}=\frac{\left(\sum_{T\in\mathcal{T}_h}|u-u_h|^2_{H^2(T)}\right)^{\frac12}}{|u|_{H^2(\Omega)}},\qquad e^r_{1,h}=\frac{|u-u_h|_{H^1(\Omega)}}{|u|_{H^1(\Omega)}},$$

$$e^r_{0,h}=\frac{\|u-u_h\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}},\qquad e^r_{\infty,h}=\frac{\max_{p\in\mathcal{V}_h}|u(p)-u_h(p)|}{\|u\|_{L^\infty(\Omega)}},$$

for uniform meshes and adaptive meshes, where $\mathcal{V}_h$ is the set of all the vertices of the triangulation $\mathcal{T}_h$. Since the exact solution of this example is unknown, these relative errors are estimated by comparing the numerical solutions from two consecutive meshes.

Remark 3.2. The numerical experiments are carried out on a machine with (i) Processor: 2.1GHz 12-Core Intel Core i7; (ii) Memory: 16GB 4400 MHz DIMM. We use MATLAB (version R2023a v.9.14.0) in the computations.

The initial mesh and the adaptive mesh with 38575 degrees of freedom (dofs) are shown in Figure 1. The discrete problems are solved by a regularized version of the active set method in [25–27] and the profile of the numerical solution on the adaptive mesh is shown in Figure 2, which matches the ones in [19] and [9].



(a) The initial mesh (b) The adaptive mesh

Figure 1: The initial mesh and the final adaptive mesh.

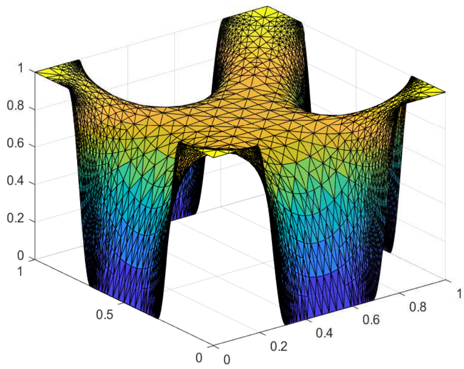


Figure 2: The numerical solution on the final adaptive mesh (38575 dofs).

The convergence histories of $e^r_{0,h}$, $e^r_{1,h}$, $e^r_{2,h}$ and $e^r_{\infty,h}$ on adaptive meshes and uniform meshes are presented in Figure 3, where N is the dimension of the finite element space associated with $\mathcal{T}_h$. It is observed that (i) the convergence in the piecewise H^2 seminorm is $O(N^{-1})$ (corresponding to $O(h^2)$ convergence for uniform meshes) and the error on the final adaptive mesh is roughly 250 times smaller than the error on a uniform mesh with a comparable number of degrees of freedom; (ii) the convergence in the L^2 norm is $O(N^{-2})$ (corresponding to $O(h^4)$ convergence for uniform meshes) and the error on the final adaptive mesh is roughly 20 times smaller than the error on a uniform mesh with a comparable number of degrees of freedom, (iii) the convergence in the H^1 and L^∞ norms are $O(N^{-3/2})$ (corresponding to $O(h^3)$ convergence for uniform meshes) and the errors on the final adaptive meshes are roughly 10 times smaller than the ones on a uniform mesh with a comparable number of degrees of freedom.

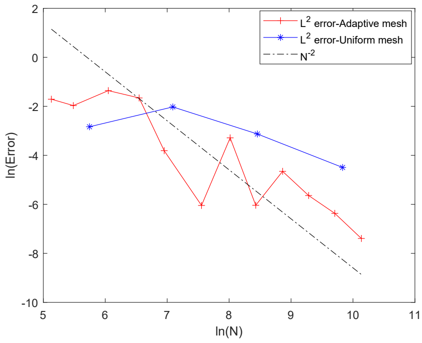
Remark 3.3. The convergence in all these norms are optimal in the sense that they agree with the interpolation errors on uniform meshes for a function in $H^4(\Omega)$, even though the exact solution u does not belong to $H^4(\Omega)$ (cf. Remark 3.1). The convergence rates in $L^2(\Omega)$, $H^1(\Omega)$ and $L^\infty(\Omega)$ are also better than the convergence predicted by (2.8).

The convergence history of the error estimator is depicted in Figure 4, where we can observe an asymptotic $O(N^{-1})$ convergence that agrees with the convergence history of the error in the H^2 seminorm.

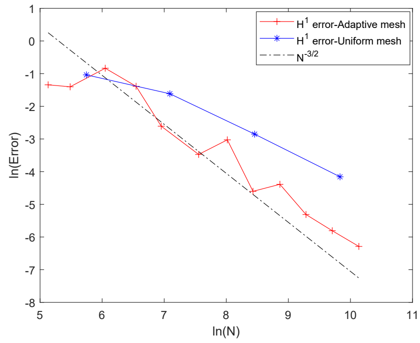
Finally a comparison of the CPU time on adaptive meshes and uniform meshes is provided in Figure 5, where it is observed that eventually the computation on the adaptive meshes is roughly four times faster than the computation on uniform meshes.

4. Conclusions

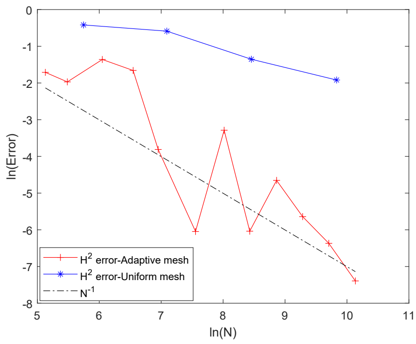
We have demonstrated numerically the superior performance of an adaptive version of the nonlinear least-squares method in [9] for the Pucci equation. Even though the convergence analysis of the adaptive nonlinear least-squares method is outside the scope of the



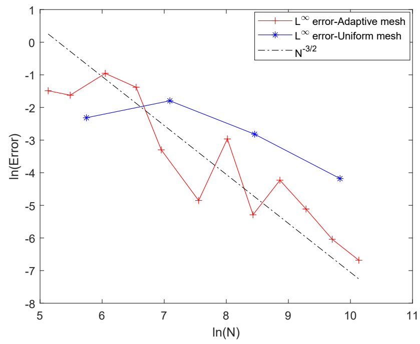
(a) L^2 error



(b) H^1 error



(c) H^2 error



(d) L^∞ error

Figure 3: Comparison of various errors on adaptive meshes and uniform meshes.

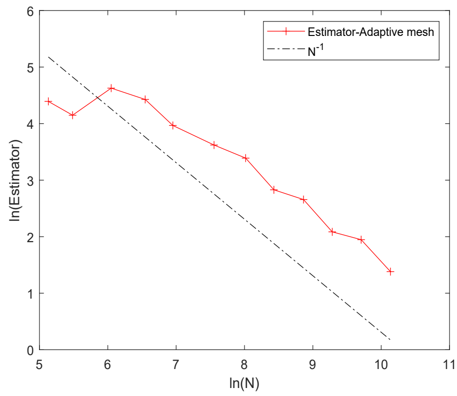


Figure 4: Convergence of the error estimator.

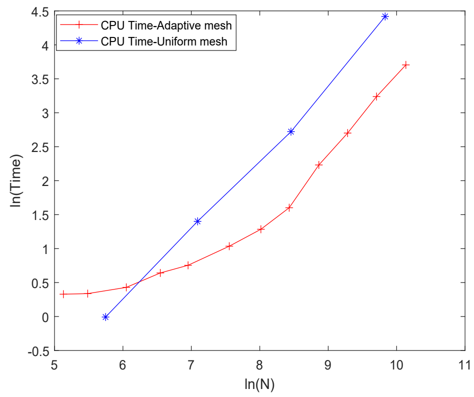


Figure 5: Comparison of the CPU time for computations using adaptive meshes and uniform meshes.

theory for adaptive methods for partial differential equations (cf. [3, 16, 17]), it is encouraging to see that the reliability and local efficiency of the error estimator can still lead to optimal convergence on adaptive meshes. We expect that adaptive nonlinear least-squares methods can also be developed for other nonlinear elliptic partial differential equations and they will play a useful computational role, especially in three dimensions.

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