ON ROOT FRAMES IN $\mathbb{R}^d$

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ABSTRACT. *A root frame* in $\mathbb{R}^d$ is a finite frame whose vectors form a root system. In this note we establish some elementary properties of this class of frames and prove that root frames constitute a subclass of scalable frames. In addition, we show that root frames are examples of a larger class of frames called *eigenframes*.

1. INTRODUCTION AND PRELIMINARIES

The goal of this note is to introduce and investigate the properties of a class of finite frames associated to root systems in $\mathbb{R}^d$. We begin with the following definition

Definition 1.1. *A finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is a root system provided that*

$$\sigma_\alpha(\beta) \in R, \quad \forall \alpha, \beta \in R,$$

where σ_α is the reflection through the hyperplane orthogonal to α–$\{x \in \mathbb{R}^d : \langle x, \alpha \rangle = 0\}$– and defined by

$$\sigma_\alpha(x) := x - 2 \langle x, \alpha \rangle \alpha / \|\alpha\|^2, \quad x \in \mathbb{R}^d.$$

In addition, R is said to be an indecomposable root system if it contains an element that is not orthogonal to any element of R.

Since R is a finite subset of $\mathbb{R}^d$, there exists $\beta \in \mathbb{R}^d$ such that $\langle \alpha, \beta \rangle \neq 0$ for all $\alpha \in R$. Therefore, the set

$$R_+ := R_{+,\beta} = \{\alpha \in R, \langle \alpha, \beta \rangle > 0\},$$

is called a positive subsystem of R. Note that $\#R = 2\#R_{+,\beta}$ for all β. We refer to [9, 10, 12, 13], for more details on root systems and to [15, 16] for some of their applications in the theory of orthogonal polynomials.

The class of finite frames we will investigate can now be defined as follows.

Definition 1.2. *Let $R = \{\alpha_j\}_{j=1}^{2N}$ be a root system on $\mathbb{R}^d$. The collection $\Phi_R = \{\alpha\}_{\alpha \in R_+}$ is called a root frame in $\mathbb{R}^d$ if R_+ spans $\mathbb{R}^d$.*

Remark 1.3. (a) *While root systems have been the subject of many investigations, to the best of our knowledge, their spanning properties has received less attention. Nonetheless, we point out that a root frame in $\mathbb{R}^d$ as defined above is the same as a rank d root system defined in* [10], *and as an effective root system* [12, Proposition 4.1.2].

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(b) *The terminology indecomposable root system used in Definition 1.1 (in* [10, Section 6.2]*) seems not universal and seems synonym to the notion of a regular root system.*

(c) *Given a root system R, we let $W := W_R = \langle \sigma_\alpha \rangle$ be the (finite) group generated by the reflections σ_α where $\alpha \in R$. W is called the Weyl (or Coxeter) group associated to R, and is a subgroup of the orthogonal group $O(d)$,* [13, Section 1.2].

(d) *If Φ is a root frame for $\mathbb{R}^d$, then, $W_R\Phi = \Phi$. That is W_R is a group of isometries leaving Φ invariant. We note that the symmetry group of (tight) frames was investigated in* [18].

(e) *A parameter function associated to the root system R is any function $k : R \to \mathbb{C}$ which is W-invariant. Given such a positive parameter function k and a root system $R = \{\alpha_j\}_{j=1}^{2N}$, we could consider a root frame of the form $\Phi_{R,k} = \{\sqrt{k(\alpha)}\alpha\}_{\alpha \in R_+}$. As we will show in the sequel, this can be be viewed as scaling the length of each of the frame vector α and can be understood in the context of scalable frames* [7, 14].

In the rest of the note, we establish some properties of root frames including their classification. In addition, we prove that all root frames are scalable. Finally, we show that root frames constitute a sub-class of a family of frames we called eigenframes which can be viewed themselves as examples of fusion frames.

2. Root frames and their frame operators in $\mathbb{R}^d$

In this section we prove some basic properties of root frames focusing on their frame operators whose spectral properties seem quite unique compared to other frames.

2.1. **Elementary properties of root frames.** Let Φ_R be a root frame associated to the root system R. The frame operator associated to Φ_R will be denoted by S_R and is given by

$$S_R := \sum_{\alpha \in R_+} \alpha\alpha^T = \sum_{\alpha \in R_+} \langle \cdot, \alpha \rangle \, \alpha. \tag{1}$$

When the root system R is fixed, we will denote the corresponding frame operator by S when there is no confusion.

We next establish a number of properties of root frames. The first such result shows that the associated frame operator is independent of the choice of the positive root. Consequently, we will also assume that for a root frame $\Phi_R = \{\alpha\}_{\alpha \in R_+}$, the positive roots system R_+ is fixed.

Proposition 2.1. *Let R be a root system and R_{+,β_1}, R_{+,β_2} two associated positive root systems. Denote the corresponding frame operators by S_{β_1} and S_{β_2}. Then $S_{\beta_1} = S_{\beta_2}$.*

Proof. We start by observing that

$$S_{\beta_1} = \sum_{\alpha \in R_{+,\beta_1}} \alpha\alpha^T = \sum_{\alpha \in R_{+,\beta_1} \cap R_{+,\beta_2}} \alpha\alpha^T + \sum_{\alpha \in R_{+,\beta_1} \setminus R_{+,\beta_2}} \alpha\alpha^T.$$

Now for any $\alpha \in R$ we must have $\alpha \in R_{+,\beta_i}$ or $-\alpha \in R_{+,\beta_i}$ for $i = 1, 2$. It follows that for $\alpha \in R_{+,\beta_1} \setminus R_{+,\beta_2}$ then $-\alpha \in R_{+,\beta_2} \setminus R_{+,\beta_1}$. It follows that the map $\alpha \to -\alpha$ is a bijection from $R_{+,\beta_1} \setminus R_{+,\beta_2}$ onto $R_{+,\beta_2} \setminus R_{+,\beta_1}$. Consequently,

$$\sum_{\alpha \in R_{+,\beta_1} \setminus R_{+,\beta_2}} \alpha\alpha^T = \sum_{-\alpha \in R_{+,\beta_2} \setminus R_{+,\beta_1}} \alpha\alpha^T$$

from which the result follows. □

Our first main result deals with the spectral decomposition of the frame operator associated with a root frame. In particular, we show that each of its vectors is an eigenvector of the corresponding frame operator. To the best of our knowledge, except for tight frames, root frames seem to be the only class of frames with this property. We note that in general, the frames whose vectors are eigenvectors of the frame operator, are precisely the critical points of the frame potential [4, 17].

Theorem 2.2. *Suppose that $\Phi_R = \{\alpha\}_{\alpha \in R_+}$ is a root frame for $\mathbb{R}^d$ with frame operator S. Then each frame vector $\alpha \in R_+$ is an eigenvector for S with eigenvalue λ_α given by*

$$\begin{cases} S\alpha = \lambda_\alpha \alpha \\ \lambda_\alpha := \frac{\langle S\alpha, \alpha\rangle}{\|\alpha\|^2} = \frac{1}{\|\alpha\|^2} \sum_{\beta \in R_+} \langle \alpha, \beta\rangle^2 . \end{cases}$$

Consequently, the spectrum of S is $\{\lambda_\alpha,\ \alpha \in R_+\}$ and R_+ contains a basis of $\mathbb{R}^d$ consisting of eigenvectors of S.

Proof. We first observe that the frame operator can be written as

$$S = \frac{1}{2} \sum_{\beta \in R} \beta\beta^T.$$

In addition, S commutes with the action of W_R. That is for each $x \in \mathbb{R}^d$ and each $\alpha \in R$ we have

$$\sigma_\alpha(Sx) = \tfrac{1}{2} \sum_{\beta \in R} \langle x, \beta\rangle\, \sigma_\alpha(\beta) = \tfrac{1}{2} \sum_{\beta \in R} \langle \sigma_\alpha x, \sigma_\alpha \beta\rangle\, \sigma_\alpha(\beta) = \tfrac{1}{2} \sum_{\beta' \in R} \langle \sigma_\alpha x, \beta'\rangle\, \beta' = S(\sigma_\alpha x).$$

It follows that for $\alpha \in R_+$

$$\alpha\alpha^T(S) = S(\alpha\alpha^T) \iff \alpha(S\alpha)^T = \langle S\alpha, \alpha\rangle = (S\alpha)\alpha^T.$$

Because S is symmetric we get $S\alpha = \frac{\langle S\alpha, \alpha\rangle}{\|\alpha\|^2}\alpha = \lambda_\alpha \alpha$.

□

The following is a simple consequence of Theorem 2.2.

Corollary 2.3. *Suppose that $\Phi_R = \{\alpha\}_{\alpha \in R_+}$ is a root frame for $\mathbb{R}^d$ with corresponding frame operator S. Let $\{\lambda_i\}_{i=1}^r$ be the list of the distinct eigenvalues $\{\lambda_\alpha\}_{\alpha \in R_+}$ of S. For each $i = 1, \dots, r$ let*

$$\begin{cases} R_i := \{\alpha \in R_+ :\ \lambda_\alpha = \lambda_i\} \\ E_i := \operatorname{span} R_i \end{cases}$$

with $d_i = \dim E_i$.

The following statements hold.

(a) *$\{R_i\}_{i=1}^r$ is a collection of pairwise orthogonal and W_R-invariant sub-root systems.*
(b) *For all $i = 1, \dots, r$, $E_i = \ker(S - \lambda_i)$ and $\lambda_i \times d_i = \# R_{i,+}$.*

(c) *If $A \leq B$ denote the optimal frame bounds of Φ_R, then*

$$A \leq \frac{\# R_+}{d} \leq B.$$

(d) *For each $i = 1, \dots, r$, $\Phi_{R_i} := \{\alpha\}_{\alpha \in R_{i,+}}$ is a tight frame for E_i with $S_{R_{i,+}} = \lambda_i Id_{E_i}$. Furthermore, if the root-system R is regular then the root frame $\Phi_R = \{\alpha\}_{\alpha \in R_+}$ is tight.*

Proof. Part (a) is straightforward and we omit its proof.

(b) The fact that $\lambda_i \times d_i = \# R_{i,+}$ follows from taking the trace of the matrices in:

$$\lambda_i Id_{E_i} = S_{|E_i} = \sum_{\alpha \in R_{i,+}} \alpha \otimes \alpha.$$

(c) Given that A, B are the optimal frame bounds, We see from the definition that

$$Ad \leq \text{trace}(S) = \sum_{\alpha \in R_+} \|\alpha\|^2 = \# R_+ \leq Bd.$$

(d) The fact that Φ_{R_i} is a tight frame for its span E_i is trivial. It follows that the root frame Φ_R is tight if and only if R is a regular root system.

□

Another immediate consequence of the spectral properties of S is the construction of Parseval frames starting with any root system R. We recall that a frame $\{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$ is scalable if there exist $\{c_k\}_{k=1}^N \subset [0, \infty)$ such that $\{c_k \varphi_k\}_{k=1}^N$ is a tight frame for $\mathbb{R}^d$ [7, 8, 14]. In the setting of root frames, the next result shows that we can always rescale each vector in a root frame to obtain a Parseval frame.

Proposition 2.4. *The canonical dual of the root frame Φ_R is the root frame generated by the same root system R and given by*

$$\tilde{\Phi}_R := \left\{ \sqrt{\frac{1}{\lambda_\alpha}} \alpha \right\}_{\alpha \in R_+}.$$

Furthermore, $\tilde{\Phi}_R$ is a Parseval frame and

$$\frac{1}{d} \sum_{\alpha \in R_+} \frac{1}{\lambda_\alpha} = 1.$$

Consequently, the root frame Φ_R is a scalable frame.

Proof. Let S be the frame operator for the root frame Φ_R. For $x \in \mathbb{R}^d$ we have

$$Sx = \sum_{\alpha \in R_+} \langle x, \alpha \rangle \alpha.$$

Thus

$$x = \sum_{\alpha \in R_+} \langle x, \alpha \rangle S^{-1} \alpha = \sum_{\alpha \in R_+} \tfrac{1}{\lambda_\alpha} \langle x, \alpha \rangle \alpha.$$

□

Remark 2.5. (a) *If a frame* $\{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$ *is scalable, then we can find scalars* $\{c_k\}_{k=1}^N \subset [0,\infty)$ *such that* $\{c_k\varphi_k\}_{k=1}^N$ *is a Parseval frame. Consequently,* $\{\varphi_k\}_{k=1}^N$ *can be viewed as a continuous frame for* $(\mathbb{R}^d, J, \mu)$ *where* $J = \{1, 2, \ldots, N\} \subset \mathbb{N}$*, and* μ *is the (discrete) measure defined on* J *by* $\mu(k) = c_k^{1/2}$ [2, 3]*. For an applications of the solution to the frame discretization problem by Freeman and Speegle, see* [11, Corollary 1.6]*.*

(b) *Because every root frame* Φ_R *is scalable, we conclude that the ellipsoid of minimum volume (also known as the Löwner ellipsoid) that circumscribed the convex hull of* $\{\pm\alpha\}_{\alpha\in R}$ *is the unit ball* [7]*.*

(c) *An interesting question we have not been to settle is the characterization of scalable frames that are also root-frames. This reduces to proving that the group generated by the reflections corresponding to the frame vectors is finite. See Theorem 2.6 for more.*

2.2. **Classification of root frames.** In this section we classify all the frames of unit-norm vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{R}^d$ that are root frames.

Theorem 2.6. *Let* $\Phi = \{\varphi_j\}_{j=1}^N$ *be a frame for* $\mathbb{R}^d$ *such that* $\|\varphi_j\| = 1$ *for each* j*. Suppose that* C_Φ *is the group generated by the reflections* $\{\sigma_{\varphi_j}\}_{j=1}^N$*. Let*

$$R_\Phi := \{g\varphi_j,\, g \in C_\Phi, j = 1, \ldots, N\}.$$

The set R_Φ *is a root frame if and only if the group* C_Φ *is finite. In this case, the initial frame* Φ *is contained in the root frame* R_Φ*.*

Proof. Suppose that C_Φ is a finite group. Clearly, R_Φ is a frame, since it can be written as the finite union of images of Φ under the reflections $g \in C_\Phi$. We only need to show that R_Φ is a root-system in $\mathbb{R}^d$. Indeed, let $\alpha_1 = g_1\varphi_1, \alpha_2 = g_2\varphi_2 \in R_\Phi$ where with $g_1, g_2 \in C_\Phi$ and $\varphi_1, \varphi_1 \in \Phi$. We have

$$\sigma_{\alpha_1}(\alpha_2) = \sigma_{g_1\varphi_1}(g_2\varphi_2) = g_2\sigma_{g_2^{-1}g_1\varphi_1}(\varphi_2) = h\varphi_2 \in R_\Phi$$

where $h = g_2\sigma_{g_2^{-1}g_1\varphi_1} \in C_\Phi$.

The converse is trivially proved since assuming that R_Φ is a root frame implies that it is both a frame (hence a finite set) and a root system.

□

Remark 2.7. *Recall that the spark of a frame* $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$ *is the cardinality of the smallest linearly dependent subset of* Φ [1]*. If* Φ *is a root frame, then given* $k \neq \ell$ *there must exists* $j \neq k, \ell$ *such that* $\sigma_{\varphi_k}(\varphi_\ell) = \pm\varphi_j = \varphi_\ell - 2\langle\varphi_\ell, \varphi_k\rangle\varphi_k$*. Thus* $\{\varphi_j, \varphi_k, \varphi_\ell\}$ *must be linearly dependent and hence the spark of the frame must be at most* 3*. Consequently, if a frame* $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$ *is such that every subset of three vectors is linearly independent, then the frame is not (contained in) a root frame. This is the case for any frame with spark greater or equal to* 4*.*

2.3. **Examples of root frames in** $\mathbb{R}^d$**.** In this section we give some examples of root frames.

Example 2.8. (1) *Let* $\{e_1, \ldots, e_d\}$ *be an orthonormal basis of* $\mathbb{R}^d$*. Then,*

$$R_+ = \{e_i - e_j, e_i + e_j, 1 \le i < j \le d\} \cup \{e_i, i = 1, \ldots, d\}$$

is a positive root system of rank d *called a type* B_d *root system.*

By considering its frame operator, one can show that R_+ is a tight root-frame (of d^2 vectors) for $\mathbb{R}^d$.

(2) *Let D_n be the dihedral group of order n, $n \geq 2$. It is the group of symmetries of a regular convex polygon of n vertices in the Euclidean plan $\mathbb{R}^2$. If we identify $z = x_1 + ix_2 \in \mathbb{C}$ with $z = (x_1, x_2) \in \mathbb{R}^2$ and set $w = e^{i\pi/n}$, then the rotations in D_n are the transformation $r_j : z \mapsto zw^{2j}$ and the reflections are given by $\sigma_j : z \mapsto \overline{z}w^{2j}$, $j = 0, \ldots, n-1$. It can be proved that $R_+ = \{i\omega^j, j = 0, \ldots, n-1\}$ is a positive root-system that is also a tight root-frame for $\mathbb{R}^2$. Observe that this tight root frame for $\mathbb{R}^2$ is related to the tight frames obtained by taking two rows from the $2n \times 2n$ DFT matrix.*

(3) *The symmetric group S_d operates on $\mathbb{R}^d$ by its action on the components on the canonical basis. That is, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have*

$$\sigma x = (x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \quad \forall \sigma \in S_d.$$

Thus, the transposition (ij) plays the same role as the reflection σ_{ij} defined by par $\sigma_{ij}(e_i - e_j) = -(e_i - e_j)$, where $\{e_1, .., e_d\}$ is the standard basis of $\mathbb{R}^d$.

A positive sub root-system associated to S_d is given by

$$R_+ = \{e_i - e_j, 1 \leq i < j \leq d\}.$$

One can show that 0 is an eigenvalue of the frame operator S of R_+. Consequently, R_+ is an example of a positive root system that is not a root frame.

3. Eigenframes

We end this note with an extension of the notion of root frames. From Corollary 2.3, it follows that any root frame Φ_R can be written as $\Phi_R = \cup_{i=1}^r \Phi_{R_i}$ where R_{Φ_i} is a tight frame for its span, and where $\{R_{\Phi_i}\}_{i=1}^r$ are mutually orthogonal. Consequently, a root frame is an example of fusion frame [5, 6]. In fact, we can introduce a class of frames for $\mathbb{R}^d$ (or $\mathbb{C}^d$) of which the root frames are examples, and which is a subclass of fusion frames.

Definition 3.1. *A frame $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{C}^d$ is called an* eigenframe (EF), *provided that for each k, the vector φ_k is an eigenvector of its frame operator $S := S_\Phi = \sum_{k=1}^N \varphi_k \varphi_k^*$.*

Assume that $\Phi = \{\varphi_k\}_{k=1}^N$ is an EF for $\mathbb{C}^d$. Let $\{\lambda_k\}_{k=1}^r$ denote the set of distinct eigenvalues of S with

$$S\varphi_k = \lambda_k \varphi_k, \qquad \forall k = 1, \ldots, N.$$

Define

$$E_k := \{x \in \mathbb{R}^d,\ Sx = \lambda_k x\}, \tag{2}$$

and let P_k denotes the orthogonal projection onto E_k. It is easy to see that for all n we have

$$P_n = \frac{1}{\lambda_n} \sum_{\varphi_k \in E_{\lambda_n}} \varphi_k \varphi_k^*, \tag{3}$$

where $E_{\lambda_n} = \Phi \cap E_n$.

It is straightforward to establish the following result which should be compared to Corollary 2.3.

Proposition 3.2. *Let $\Phi = \{\varphi_k\}_{k=1}^N$ be an EF for $\mathbb{C}^d$. The following statements hold.*

(a) *For each $n = 1, 2, \dots, r$*

$$\lambda_n d_n = \sum_{\varphi_k \in E_{\lambda_n}} \|\varphi_k\|^2 ,$$

where $d_n = \dim E_n$.

(b) *For all $u \in E_{\lambda_n} \cap \Phi$ we have*

$$\lambda_n \geq c_\Phi(u) \|u\|^2 ,$$

where $c_\Phi(u)$ is the number of times u appears in Φ. In addition, equality holds in this inequality if and only if $\langle u, \varphi_k \rangle = 0$ for all $u \neq \varphi_k \in \Phi$.

The following result is an extension of Theorem 2.2 characterizing EFs through their frame operator. We omit its simple proof.

Theorem 3.3. *Let $\Phi = \{\varphi_k\}_{k=1}^N$ be a frame in $\mathbb{C}^d$. Then the following statements are equivalent*

(a) *Φ is an EF,*
(b) *S commutes with σ_{φ_k} for all $k = 1, \dots, N$,*
(c) *S commutes with $\varphi_k \varphi_k^* = \langle \cdot, \varphi_k \rangle \varphi_k$ for all $k = 1, \dots, N$.*
(d) *There exist mutually orthogonal subspaces $W_1, \dots, W_r$ of $\mathbb{C}^d$ such that $\mathbb{C}^d = \oplus_{i=1}^r W_i$ with $W_i = Span(\Phi_i)$ where Φ_i is a tight frame in W_i and $\Phi = \cup_{i=1}^r \Phi_i$.*

Remark 3.4. (a) *It is easy to extend Proposition 2.4 and to prove that any eigenframe Φ is scalable. Indeed, if $\Phi = \{\varphi_k\}_{k=1}^N$ is an EF, we have $S\varphi_k = \lambda_k \varphi_k$. It follows that*

$$Id_{\mathbb{C}^d} = \sum_{k=1}^N \frac{1}{\lambda_k} \varphi_k \varphi_k^T ,$$

which shows that $\{\frac{1}{\sqrt{\lambda_k}} \varphi_k\}_{k=1}^N$ is Parseval frame.

(b) *If $\Phi = \{\varphi_k\}_{k=1}^N$ is an EF, then its Gram matrix is block diagonal.*

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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