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Optimal Capacity Sizing of Park-and-Ride Lots with Information-Aware Commuters

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We study capacity sizing of park-and-ride lots that offer services to commuters sensitive to congestion and parking availability information. The goal is to determine parking lot capacities that maximize the total social welfare for commuters whose parking lot choices are predicted using the multinomial logit model. We formulate the problem as a non-convex nonlinear program that involves a lower and an upper bound on each lot's capacity, and a fixed-point constraint reflecting the effects of parking information and congestion on commuters' lot choices. We show that except for at most one lot, the optimal capacity of each lot takes one of three possible values. Based on analytical results, we develop a one-variable search algorithm to solve the model. We learn from numerical results that the optimal capacity of a lot with a high intrinsic utility tends to be equal to the upper bound. By contrast, a lot with a low or moderate-sized intrinsic utility tends to attain an optimal capacity on its effective lower bound. We evaluate the performance of the optimal solution under different choice scenarios of commuters who are shared with real-time parking information. We learn that commuters are better off in an average choice scenario when both the effects of parking information and congestion are considered in the model than when either effect is ignored from the model.

Key words: Park and ride, Optimal capacity sizing, Information-aware commuters, Multinomial logit model, Equilibrium

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1. Introduction

Park-and-ride has become an indispensable part of life for many people who commute between their homes in suburban areas and workplaces in central business districts (CBDs). A park-and-ride lot (or simply, a lot) is a key facility in a park-and-ride system where commuters can park their cars and get access to public transit. Park-and-ride lots are designed, maintained, and managed by state or local departments of transportation (DOTs) to serve communities in the satellite cities of a core urban area. For example, there are 134 park-and-ride lots with around 22,453 spaces scattered in the satellite cities surrounding the Seattle CBD (WSDOT, 2022). Table 1 shows example park-and-ride lots, their capacities, and average daily utilization in 2021 (PSRC, 2022).

Park and ride in the United States can date back to the 1930s (Noel, 1988) and received great attention in the 1970s when the DOTs in different urban areas attempted to introduce an economical travel mode as an alternative to driving private cars all the way to office, in response to growing

Table 1 The spaces and utilization of example park-and-ride lots in the Greater Seattle Area in 2021.

Park & Ride Name	Address	Capacity	Utilization
Arlington P&R	SR 9 & SR 530	25	36%
Smokey Point Community Church	SmokeyPt Blvd/77th St NE	50	24%
All Saints Lutheran Church	27225 Military Rd S	75	46%

oil prices brought about by the Oil Crisis (Spillar, 1997). As green mobility attracts the growing attention of policymakers, recent years have witnessed an increasing interest in park and ride that is believed to reduce traffic jams and emissions (Holgúin-Veras et al., 2012; Stieffenhofer et al., 2016).

This work investigates the *optimal capacity sizing problem* for a park-and-ride system, in which a DOT plans to design or re-design a set of park-and-ride lots in a specific service area, e.g., a satellite city, and needs to determine the optimal capacity of each lot. The capacities of all the lots form a vector called a *capacity plan*. The ridership demand model that forecasts the *demand* at a lot plays a central role in the problem. The demand is interchangeably used with the *traffic flow* routed through the lot in this paper and measured as the number of commuter vehicles parked at the lot during a period of time (e.g., peak hours on weekday mornings) in practice. The number of park-and-ride commuters per parked vehicle is often estimated according to the experience of practitioners and historical data. For instance, the number is estimated to be 1.4 in Bullard and Christiansen (1983). Without loss of generality (w.l.o.g.), we assume that one parked vehicle is occupied by one commuter.

A ridership demand model is expected to possess several basic features. First, the demands at all lots within the same service area are interdependent, since they serve the same group of commuters. Second, the demands are influenced by the DOT's capacity sizing decision. To elaborate, the capacities of the lots will affect the parking utilization that will be shared with commuters (cf. Table 1), and therefore affect the attractiveness of each lot to the commuters who are aware of (and sensitive to) the information. As a result, the capacity sizing decision may affect the likelihood of commuters choosing to visit a particular lot, thus affecting the demands. Third, the congestion effect should be captured by the demand model. The congestion effect refers to the phenomenon that the utility of a service or facility erodes as it serves a higher demand. Given a capacity sizing decision, the utility of a lot perceived by commuters, which represents the preference of commuters toward the lot, drops as more commuters choose to use the lot due to the congestion effect and increasing parking utilization. Observe that the park-and-ride demands depend on the utilities of the lots, which are, in turn, affected by the demands. This relation may lead to the well-known phenomenon called *traffic equilibrium*, which has been observed in queueing systems (Maglaras et al., 2018), seaport operation systems (Wang and Meng, 2019), and transportation systems (Cascetta, 2009). Discrete choice models, such as the multinomial logit (MNL) model, nested logit (NL) model, and mixed logit (ML) model (Train, 2009; Ben-Akiva and Lerman, 1985), have been broadly used to model the choice behavior of travelers facing a set of transportation modes such as riding public transit and driving a private car (Naumov et al., 2020; Holgúin-Veras et al., 2012).

We will model the ridership demand at a lot using an MNL-based commuter choice model. The commuter choice model forecasts the probability of commuters choosing a lot based on the utilities of all the lots, thus reflecting the interdependence among the demands at all the lots. A lot's utility is first affected by a variety of physical attributes such as the lot's economic attraction, accessibility, and the transit fare. This part of the utility is called the *intrinsic utility*. Meanwhile, the utility is formulated to depend on the demand and (average) parking utilization to account for the effects of congestion and parking information on commuter choice. The commuter choice model developed in this work can be viewed as an extended version of the site-level demand model which has been applied to forecast the demands for individual park-and-ride lots (Abdus-Samad and Grecco, 1972; Bullard and Christiansen, 1983; Spillar, 1997; Niles and Pogodzinski, 2021). Using the MNL-based commuter choice model, we represent the demands at all lots at equilibrium (given a capacity sizing decision) as a solution to a (vector) fixed-point equation.

Next, we explain the objective value and various constraints in the problem. As explained in Litman (2019), one of the primary goals of public transit is to achieve social welfare benefits. The *total social welfare* can be interpreted as the overall satisfaction received by all park-and-ride commuters, where the satisfaction of a commuter can be measured by the utility of the park-and-ride lot that the commuter chooses to use. As a policymaker, the DOT intends to maximize the total social welfare when sizing a set of park-and-ride lots. Meanwhile, the demand realized at each lot cannot exceed the lot's capacity, which is referred to as a *capacity constraint*. Also, the demands at all lots need to satisfy the fixed-point equation, which is referred to as the *fixed-point constraint*. Besides, as explained in Bullard and Christiansen (1983), there are often constraints on both the minimum and maximum lot sizes due to design and operational features of park-and-ride services. For example, a lot should be designed with a minimum size to ensure that it can at least accommodate the demand arriving between two consecutive buses during the morning commute, which we refer to as a *lower-bound constraint*. Meanwhile, there is always an upper bound on the capacity due to limited space and budget, which is referred to as an *upper-bound constraint*. Both the lower- and upper-bound constraints are called bound constraints.

1.1. The Objective and Contributions

This section describes our objective and contributions. Since we now live in the era of smartphones, it is easier than ever for commuters to access parking availability information through mobile apps or websites. However, the effect of parking information on the travel mode choice of commuters has not been well addressed in the literature on park-and-ride capacity sizing.

This work is aimed at bridging the gap by studying optimal capacity sizing of park-and-ride lots with consideration of information-aware commuters. The problem will be formulated as an optimization model, in which we intend to determine an optimal capacity plan for a set of park-and-ride lots in a specific service area to maximize the total social welfare received by commuters while honoring the capacity constraints, lower- and lower-bound constraints, and the fixed-point constraint.

The main objective of this work is threefold: (i) to characterize the optimal capacities of the lots, (ii) to develop effective algorithms to determine optimal capacities by leveraging the characterization results, and (iii) to provide managerial insights into the properties of optimal capacities and the value of modeling the effects of parking information and congestion in park-and-ride capacity sizing.

This work makes the following *theoretical* contributions:

(a) We formulate the optimal capacity sizing problem as a non-convex nonlinear optimization model with capacity and bound constraints as well as the fixed-point constraint reflecting the effect of parking information on commuter choice. The problem is challenging to solve due to these constraints. To resolve the challenge, we transform the model into an equivalent flow-based model, in which the optimal traffic flow rather than capacity is determined for each lot and the fixed-point constraint is dropped. To the best of our knowledge, this work is the first to model the effect of parking information in park-and-ride capacity sizing, which extends the existing literature by adding a novel model and opens doors for future studies on park-and-ride with information-aware commuters.

(b) We show that the flow-based model is equivalent to a univariate optimization model with only one decision variable (i.e., the total traffic flow through all lots). The univariate model is built upon a subproblem in which the capacity, bound, and fixed-point constraints are reformulated as constraints over intervals. The subproblem is tractable to analyze due to its simple form of constraints, which paves the way for obtaining analytical results for the capacity sizing problem.

(c) We provide analytical results that characterize the structure of the optimal solution to the subproblem. Based on the structural results, we characterize the capacity plan that solves the optimal capacity sizing problem.

(d) We develop a one-variable search algorithm to determine a near-optimal capacity plan by leveraging the characterization results. The algorithm involves an efficient solution procedure to determine the traffic flow pattern at equilibrium under any given capacity plan. The procedure is not only useful in sizing park-and-ride lots, but also provides traffic engineers with a method to forecast equilibrium traffic flows under the MNL model.

We provide numerical results by solving the optimal capacity sizing problem for the park-and-ride lots in the city of Bellevue in Washington State. The numerical analysis serves two purposes: to obtain insights into park-and-ride lot design in a satellite city such as Bellevue and to investigate the value of modeling the effects of congestion and parking information in sizing park-and-ride lots.

We obtain the following *managerial insights* from the theoretical and numerical results:

(1) The optimal capacity sizing model may be infeasible, which signals the need to adopt smaller lower bounds and/or larger upper bounds on the lots' capacities. When the problem is feasible, except for at most one lot whose optimal capacity is strictly between the lower and upper bounds, the optimal capacity of each lot is equal to one of three values: the lower bound, the upper bound, and the traffic flow through the lot.

(2) We learn from numerical results that a lot with a high intrinsic utility tends to have its optimal capacity equal to the upper bound. A lot with a low intrinsic utility and a small lower bound tends to have the optimal capacity equal to the optimal traffic flow through the lot, leading to full utilization of the lot. However, when the lower bound is not small, the optimal capacity tends to be equal to the lower bound. We also observe that the parking availability at a lot with full utilization may not be improved by allowing for a smaller lower bound and/or a larger upper bound on the lot's capacity.

(3) When a lot has a high initial utilization rate, its utilization seems to be non-increasing as commuters become more sensitive to parking availability. However, if a lot attains low utilization with commuters weakly sensitive to parking availability, the utilization tends to be non-decreasing with possible jumps to lower levels as commuters care more about parking availability.

(4) Opening a subset of the potential park-and-ride lots may yield a higher total social welfare than utilizing all of the available lots.

1.2. The Outline of the Work

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 formally defines the optimal capacity sizing problem. In Section 4, we formulate the problem as a non-convex nonlinear program and transform it into an equivalent flow-based model. We then pass on to Section 5 to reformulate the flow-based model as a one-variable optimization model. Section 6 presents a search algorithm to solve the problem. Numerical studies are provided in Section 7 and conclusions are drawn in Section 8. Supplement materials and all proofs are provided in the e-companion.

2. Literature Review

There are two streams of research related to our work: park-and-ride capacity sizing and multiproduct price optimization under customer choice, which are discussed in Sections 2.1 and 2.2, respectively.

2.1. Capacity Sizing for Park-and-Ride Lots

This work follows the line of research on capacity sizing for park-and-ride lots. Early studies, such as Abdus-Samad and Grecco (1972) and Spillar (1997), were mostly performed in collaboration with local DOTs and focused on developing *site-level demand models*, in which regression analysis is applied to forecast the demand at an individual park-and-ride lot based on characteristics such as the location and accessibility. These studies treat demand forecasting and capacity sizing as two separate decision-making processes: Demand is estimated first and a capacity size is then chosen to serve the anticipated demand. This two-step method suffers from two drawbacks. First, the method results in an independent-demand model (Talluri and van Ryzin, 2004): The demand at a lot is not influenced by capacity sizing decisions, which is not necessarily true in practice. Second, the demand is estimated for a single lot and does not depend on the demands at all other lots, which may be unrealistic when multiple lots serve the same area and are thus regarded as substitutable or interchangeable by commuters. Nordstrom and Christiansen (1981) and Bullard and Christiansen (1983) applied site-level models to estimate park-and-ride demands in Texas based on nonlinear regression that uses parking capacity as one of the predictor variables. They use a presumed capacity size for demand forecasting, which may, however, result in a forecasted demand violating the capacity constraint. In a recent study, Niles and Pogodzinski (2021) developed ridership demand models for individual park-and-ride lots in San José, Seattle, and Los Angeles using regression.

Region-level demand models have also been employed by researchers, such as Hendricks and Outwater (1998), Spillar (1997), and Holgúin-Veras et al. (2012), to forecast park-and-ride demands. In these models, the flows routed through different paths/travel modes between each pair of origin and destination zones over a study area are estimated and the park-and-ride demands are then calculated based on the flows. Nonetheless, site-level demand models have continued to receive attention from practitioners for two reasons. First, a site-level demand model is often more effective in providing accurate site-specific forecasts, while a region-level model tends to overestimate park-and-ride demands, particularly when the travel characteristics incorporated into the model do not correspond to the realized design and construction conditions of the park-and-ride lots (Spillar, 1997). Second, a site-level model often requires a smaller amount of data to estimate than a region-level model, thus particularly favored by practitioners with limited data and computational resources. The commuter choice model developed in this work can be thought of as an improved site-level demand model, by which the forecasted park-and-ride demands are interdependent across all the lots in the same service area and dependent on capacity sizing decisions.

The literature on addressing demand forecasting and capacity sizing for park-and-ride lots under a unified optimization model is limited. For instance, García and Marín (2002) formulated a joint capacity sizing and pricing problem for park-and-ride lots as a bi-level programming model, which involves a nested logit model to describe the travel mode choice of travelers. Song et al. (2017) solved the optimal location problem for park-and-ride lots by determining the optimal capacities and frequencies of transit services at the lots. The problem was formulated as a mathematical program with equilibrium constraints under an MNL model. Liu et al. (2018) studied optimal capacity sizing for the park-and-ride facilities in a suburban area and formulated the problem as a mathematical program with equilibrium constraints, where a crossed-nested logit model is used to describe the choice behavior of travelers. Henry et al. (2022) investigated the park-and-ride facility location problem for an on-demand transportation system and used an MNL model to forecast commuters' choice. Rezaei et al. (2022) studied the optimal park-and-ride facility location for Nashville, Tennessee using a mixed-integer linear program integrated with an MNL-based demand model. Readers are referred to Haque et al. (2021) for more literature on park-and-ride.

Our work sets itself apart from these studies in two major aspects. First, these studies do not address the impact of parking availability information on commuter choice. Second, their studies are focused on developing numerical solutions to find a (local or global) optimal solution, while our work aims to obtain analytical results that characterize the global optimal solution and gain insights from the numerical results.

2.2. Multiproduct Price Optimization under Customer Choice

This work is also related to the literature on multiproduct price optimization under customer choice, particularly on the methodological side. For the sake of comparison, loosely speaking, a park-and-ride lot in this work corresponds to a product in multiproduct pricing, the traffic flow routed through a lot resembles the sales of a product, and the capacity of a lot, like the price, serves as a decision lever to adjust demands.

In this literature, one of the inspiring results is that the objective value function, i.e., the total profit of the products, under the MNL model is generally not concave with respect to the price vector, but it turns out to be concave with respect to the market share vector (there exists a one-to-one correspondence between the price and market share vectors) (Song and Xie, 2007; Li and Huh, 2011). As a result, it has been shown that the optimal markup prices for different products are equal when the price sensitivity parameters are identical for the products under the MNL model (Hopp and Xu, 2005; Li and Huh, 2011).

Efforts have recently been invested in understanding whether or not the same or similar results hold under other choice models. To mention only a few, Gallego and Wang (2014) studied price optimization and competition for multiple products under the NL model with heterogeneous price sensitivity parameters within and across the nests. The authors maintained that the objective value function with respect to the market share vector is not necessarily concave under the NL model. Rayfield et al. (2015) developed approximation methods to solve the multiproduct price and assortment optimization problems under the NL model. A unique feature considered in this work is that there are lower and upper bounds on the prices. Zhang et al. (2018) studied multiproduct price optimization with convex constraints on purchase probabilities under the generalized extreme value models (which include the MNL and NL models as special cases). The authors showed that the optimal markup price is constant across products when the problem is unconstrained and price sensitivities are homogeneous. They also showed that the market-share-based transformation with constraints is, in fact, a convex program. More examples include Dong et al. (2019) based on Markov choice models and Li (2020) with diffusion-choice models. However, these works do not involve the fixed-point constraint, which is critical in the park-and-ride capacity sizing problem considered in our work.

Du et al. (2016) are the first to study multiproduct price optimization under the network effect that a product becomes more appealing to consumers as it attracts a larger market share. The problem was formulated as a nonlinear program with a vector fixed-point equation constraint based on the MNL model. The authors highlighted that the objective value function in the sales-based (i.e. market-share-based) transformation is not necessarily convex or concave. The authors obtained an interesting result that the optimal prices can take at most two distinct values even when the intrinsic utilities, network sensitivities, and price sensitivities are identical for different products. In their subsequent work, Du et al. (2018) solved the worst-case pricing problem for a single product under the network effects. Nosrat et al. (2021) studied optimal pricing for a single product with multiple customer segments and the network effects under a mixture of MNL models. Our work distinguishes itself from this series of works by the following features. First, these studies do not consider bound constraints on sales or prices, while we consider that the traffic flow (corresponding to the sales) at each lot is limited by its capacity and the capacity (corresponding to the price) is bounded from below and above. Second, the utility function of a product is separable in the sales and price, while the utility of a park-and-ride lot is non-separable in the traffic flow and capacity. Third, the utility of a park-and-ride lot is non-increasing in the traffic flow due to the effects of congestion and parking information, while the utility of a product is non-decreasing in the sales due to the network effects in their studies. Fourth, the objective is to maximize the total profit of a firm in their works, while it is to maximize the total social welfare of commuters in our work. Due to all these different features, which are ultimately attributed to the fact that we consider a different application in transportation, the results and proof techniques in these works do not easily carry through. For instance, the first-order optimality condition that is one of the key proof techniques in these and other existing studies on optimal pricing needs to be revisited with capacity and bound constraints. Therefore, new analysis techniques are needed to tackle the problem considered in our work.

3. Problem Description

This section describes the optimal capacity sizing problem for park-and-ride lots with information-aware commuters. More specifically, Section 3.1 introduces the notation adopted in this work. In Section 3.2, we develop the MNL-based commuter choice model with the effects of congestion and parking information. In Section 3.3, we discuss the properties of the traffic flows at equilibrium. We formulate the total social welfare in Section 3.4.

3.1. Notation

We use $\mathbb{R}_+ := [0,\infty)$ to denote the set of non-negative real numbers and $\mathbb{R}_{++} := (0,\infty)$ to represent the set of positive real numbers. Let $[n]$ denote the integer set $[n] := \{1,2,\dots,n\}$ for any $n \in \mathbb{N}$. For any $\mathbf{x} \in \mathbb{R}^n$, where $n \in \mathbb{N}$, let $\text{diag}(\mathbf{x})$ denote the $n \times n$ diagonal matrix generated from $\mathbf{x}$ such that x_i is the i-th diagonal entry for each $i \in [n]$ and all other entries are zeros. Let $\mathbf{e} \in \mathbb{R}^J$ denote a J-dimensional vector with all entries equal to one. All vectors are column vectors unless otherwise specified. However, for the economical use of space, we write $\mathbf{x} = (x_1, x_2, \dots, x_n)$ inside of the text to describe entries of vector $\mathbf{x} \in \mathbb{R}^n$. In addition, we conform to the convention that $1/\infty = 0$, $\log(0) = -\infty$, and $a + \infty = \infty$ for all $a \in \mathbb{R}$ and that any set $\{m, m+1, \dots, n\}$ is empty if $m > n$ for any $m, n \in \mathbb{Z}$. Let $\mathbf{1}\{A\}$ denote the indicator function for any event A and $\|\cdot\|_1$ denote the 1-norm. Define $x \vee y := \max\{x,y\}$ and $x \wedge y := \min\{x,y\}$ for all $x, y \in \mathbb{R}$. Let $|A|$ denote the cardinality of any discrete set A.

3.2. Commuter Choice Model with Congestion and Parking Information

In this section, we formulate the commuter choice model with congestion and parking effects.

Consider a population of commuters who travel on workdays from their homes within a service area (e.g., a satellite city) to their work sites in an urban CBD. There exists a set $[J]$ of park-and-ride lots available to the commuters in the service area, where $J \in \mathbb{N}$. For each $j \in [J]$, let $q_j \in \mathbb{R}_+$ denote the (average) traffic flow attracted by j during a given time period, e.g., the morning commute between 7–9 am on workdays. Define $\mathbf{q} := (q_j, j \in [J])$ as the vector of flows for all the lots, which is referred to as the *flow pattern*. Let $C_j \in [0,\infty]$ denote the capacity of the lot j and define $\mathbf{C} := (C_1, C_2, \dots, C_J)$ as the vector of capacities, which is referred to as the *capacity plan*. Let $\ell_j \in \mathbb{R}_{++}$ denote the lower bound on the capacity C_j and $u_j \in \mathbb{R}_{++}$ denote the upper bound on C_j, where $\ell_j < u_j$. The lower bound represents the minimum design capacity to meet operations constraints, such as the minimum number of commuters to be served during the time headway between two consecutive transit services and the upper bound is due to constraints such as space and budget limits.

Let $U_j(q_j, C_j)$ denote the utility of any $j \in [J]$ perceived by commuters, which is expressed as a random variable consisting of two components as follows:

$$U_j(q_j, C_j) = \nu_j(q_j, C_j) + \epsilon_j, \tag{1}$$

where $\nu_j(q_j, C_j)$ is the systematic utility and ϵ_j is the random error reflecting the variation in the preferences of commuters toward j. The systematic utility is a function of q_j and C_j defined as

$$\nu_j(q_j, C_j) = b_j - \beta q_j^{\theta} + \varphi\left(1 - q_j/C_j\right). \tag{2}$$

The first term $b_j \in \mathbb{R}$ represents the intrinsic utility of the park-and-ride lot j, which measures the attractiveness of j due to characteristics such as accessibility, location, reputation, and transit fares. The second term is similar in form to the widely used BPR function in urban transportation (Sheffi, 1985), which reflects the congestion effect. As in the BPR function, $\theta \in \mathbb{R}_+$ is a parameter to be estimated from data and $\beta \in \mathbb{R}_+$ is the congestion sensitivity parameter. The third term captures the effect of parking information, in which q_j/C_j is the utilization of the lot j shared with commuters such as those in Table 1, $(1 - q_j/C_j)$ is the corresponding parking availability at j, and $\varphi \in \mathbb{R}_+$ is the parking sensitivity parameter, which is assumed to be nonnegative to reflect that commuters appreciate a lot better if it has more availability. Define $\boldsymbol{\nu}(\mathbf{q}, \mathbf{C}) := (\nu_j(q_j, C_j), j \in [J])$ as the vector of utilities for all the park-and-ride lots.

Let $Q \in \mathbb{R}_{++}$ denote the total travel demand originating from the service area during the given time period. Then, we have $Q = \sum_{j=0}^{J} q_j$, where $q_0 \in [0, Q]$ denotes the number of commuters choosing not to park and ride, such as those driving private cars to office. Let $j = 0$ denote the no-park-and-ride alternative representing the choice of commuters not to park and ride at any lot in $[J]$. Let ν_0 and ϵ_0 denote the systematic utility and random error of the no-park-and-ride alternative, respectively.

Define $p_j : \mathbb{R}^J \mapsto [0, 1]$ as a choice function mapping from $\boldsymbol{\nu}(\mathbf{q}, \mathbf{C})$ to the probability of commuters choosing the park-and-ride lot j from the set $[J]$. We assume that each commuter is a utility maximizer who will choose the alternative with the largest utility among from $[J]$. We also assume that the random errors $\{\epsilon_j\}_{j=0}^{J}$ are independent and identically distributed Gumbel random variables with a location parameter 0 and scale parameter 1. Then, according to the random utility theory (Ben-Akiva and Lerman, 1985), we have

$$p_j(\boldsymbol{\nu}(\mathbf{q}, \mathbf{C})) := \frac{\exp(\nu_j(q_j, C_j))}{\exp(\nu_0) + \sum_{k=1}^{J} \exp(\nu_k(q_k, C_k))} = \frac{\exp(\nu_j(q_j, C_j))}{1 + \sum_{k=1}^{J} \exp(\nu_k(q_k, C_k))}, \tag{3}$$

where ν_0 is normalized to zero. It follows from the traffic equilibrium principle (Cascetta, 2009) that the traffic flow through j at equilibrium satisfies,

$$q_j = Q p_j(\boldsymbol{\nu}(\mathbf{q}, \mathbf{C})) = Q \frac{\exp\left(b_j - \beta q_j^{\theta} + \varphi\left(1 - \frac{q_j}{C_j}\right)\right)}{1 + \sum_{k=1}^{J} \exp\left(b_k - \beta q_k^{\theta} + \varphi\left(1 - \frac{q_k}{C_k}\right)\right)}.$$

Note that we can assume, w.l.o.g., that $Q = 1$; otherwise, by setting $\beta' := \beta Q^{\theta}$ and $q'_k := q_k/Q$ and $C'_k := C_k/Q$ for all $k \in [J]$, the above equation is equivalent to $q'_j = p_j(\boldsymbol{\nu}(\mathbf{q}', \mathbf{C}'))$ with β replaced by β', where $\mathbf{q}' := (q'_j, j \in [J])$ and $\mathbf{C}' := (C'_j, j \in [J])$. Thus, it follows that $\mathbf{q}$ satisfies the following vector fixed-point equation,

$$q_j = p_j(\boldsymbol{\nu}(\mathbf{q}, \mathbf{C})) = \frac{\exp\left(b_j - \beta q_j^{\theta} + \varphi\left(1 - \frac{q_j}{C_j}\right)\right)}{1 + \sum_{k=1}^{J} \exp\left(b_k - \beta q_k^{\theta} + \varphi\left(1 - \frac{q_k}{C_k}\right)\right)}, \quad \forall j \in [J]. \tag{4}$$

Note that $\mathbf{q} \in [0,1]^J$. Given any capacity plan $\mathbf{C} \in (0,\infty]^J$, define a choice operator $\mathbf{F}(\cdot,\mathbf{C}) : [0,1]^J \mapsto [0,1]^J$ such that $\mathbf{F}(\mathbf{q},\mathbf{C}) := (p_j(\boldsymbol{\nu}(\mathbf{q},\mathbf{C})), j \in [J])$ for any $\mathbf{q} \in [0,1]^J$. Then, (4) can be written in a vector form:

$$\mathbf{q} = \mathbf{F}(\mathbf{q},\mathbf{C}). \tag{5}$$

3.3. Properties of the Equilibrium Flow Pattern

In this section, we discuss the properties of the solution to the fixed-point equation (5), which will be useful in later sections.

Remark 1 shows the existence and uniqueness of the solution to (5) for any given capacity plan.

Remark 1. Consider any $\mathbf{C} \in (0,\infty]^J$. Since $\mathbf{F}(\cdot,\mathbf{C})$ is a continuous function defined in the set $[0,1]^J$, $[0,1]^J$ is a nonempty compact and convex set, and the range $\mathbf{F}([0,1]^J,\mathbf{C}) \subset [0,1]^J$, it follows from the Brouwer's fixed-point theorem in Section A.3.1 of Cascetta (2009) that there exists a solution to (5). Moreover, since ν_j is strictly decreasing in q_j for each $j \in [J]$, it follows from Cascetta (2009, pp. 311–312) that the solution to (5) is unique.

Define $\mathcal{Q} := \{\mathbf{q} \in [0,1]^J : q_j > 0 \ \forall\ j \in [J],\ \|\mathbf{q}\|_1 < 1\}$. Remark 2 provides an alternative form of the fixed-point equation and shows that any equilibrium flow pattern is in $\mathcal{Q}$.

Remark 2. The fixed-point equation (4) (and (5)) is equivalent to: $\log(q_j) = \log(1 - \|\mathbf{q}\|_1) + b_j - \beta q_j^\theta + \varphi(1 - q_j/C_j)$ for all $j \in [J]$. For any $\mathbf{C} \in (0,\infty]^J$, if $\mathbf{q} = \mathbf{F}(\mathbf{q},\mathbf{C})$, then $\mathbf{q} \in \mathcal{Q}$.

Proposition 1 characterizes the monotonicity of the total traffic flow served by all the lots with respect to $\mathbf{C}$ at equilibrium.

Proposition 1. *Given capacity plans $\mathbf{C}^1, \mathbf{C}^2 \in (0,\infty]^J$, let $\mathbf{q}^i := (q_j^i, j \in [J])$ be the flow pattern such that $\mathbf{q}^i = \mathbf{F}(\mathbf{q}^i,\mathbf{C}^i)$ for $i \in \{1,2\}$. If $\mathbf{C}^1 \le \mathbf{C}^2$ (component-wisely), then $\|\mathbf{q}^1\|_1 \le \|\mathbf{q}^2\|_1$.*

Proposition 1 states that if we increase or maintain the same capacity for each lot, then the total number of commuters choosing to park and ride will not decrease at equilibrium.

3.4. The Total Social Welfare

In this section, we formulate the total social welfare and define the optimal capacity sizing problem.

Let $V(\mathbf{q},\mathbf{C})$ denote the *total social welfare*, which is the sum of the systematic utilities received by all commuters, namely,

$$V(\mathbf{q},\mathbf{C}) := \sum_{j=1}^{J} q_j \nu_j(q_j, C_j) = \sum_{j=1}^{J} q_j \left[b_j - \beta q_j^\theta + \varphi\left(1 - \frac{q_j}{C_j}\right)\right]. \tag{6}$$

We hereby note that $V(\mathbf{q},\mathbf{C})$ can be explained as the expected utility that an average commuter would receive upon her arrival at a lot, which is not the same as the $\mathbb{E}[\max_{j \in [J]\cup\{0\}} U_j(q_j, C_j)]$ known as the *satisfaction* of commuters (Sheffi, 1985). The satisfaction can be thought of as the expected *perceived utility* of a commuter upon departure from home.

When the system reaches equilibrium under a given capacity plan $\mathbf{C}$, it follows from Remark 2 that $b_j - \beta q_j^\theta + \varphi(1 - q_j/C_j) = \log(q_j) - \log(1 - \|\mathbf{q}\|_1)$ for each $j \in [J]$. Substituting it into (6) gives that the total social welfare can also be re-written as a function $h : \mathcal{Q} \mapsto \mathbb{R}$:

$$h(\mathbf{q}) := \sum_{j=1}^{J} q_j \left[\log(q_j) - \log(1 - \|\mathbf{q}\|_1)\right] = V(\mathbf{q}, \mathbf{C}). \tag{7}$$

Let $W : [-e^{-1}, \infty) \mapsto [-1, \infty)$ denote the principal branch of the Lambert-W function (Stewart, 2005). Proposition 2 shows that h is strictly convex over $\mathcal{Q}$.

Proposition 2. *h is strictly convex over $\mathcal{Q}$ and attains a unique minimum over $\mathcal{Q}$ at $\mathbf{q}^\dagger = (q^\dagger, q^\dagger, \ldots, q^\dagger)$, where $q^\dagger := W(J/e)/[J(W(J/e)+1)]$. Furthermore, $h(\mathbf{q}^\dagger) = -W(J/e)$.*

We close this section by formally stating the optimal capacity sizing problem for park-and-ride lots: The DOT needs to determine the capacity plan that maximizes the total social welfare subject to the capacity and bound constraints as well as the fixed-point constraint (5).

4. The Models

In this section, we formulate the optimal capacity sizing problem as optimization models.

The optimal capacity sizing problem can be formulated as the following *basic model*:

$$(\textsc{Basic}) \quad \sup_{\mathbf{q}, \mathbf{C}} V(\mathbf{q}, \mathbf{C}) \tag{8}$$

$$\text{s.t.} \quad \log(q_j) = \log(1 - \|\mathbf{q}\|_1) + b_j - \beta q_j^\theta + \varphi\left(1 - \frac{q_j}{C_j}\right), \quad \forall j \in [J], \tag{9}$$

$$C_j \in [\ell_j, u_j], \quad \forall j \in [J], \tag{10}$$

$$q_j \le C_j, \quad \forall j \in [J], \tag{11}$$

$$\mathbf{q} \in \mathcal{Q}. \tag{12}$$

where the objective function value (8) is the total social welfare to be maximized, Constraints (9) ensure that a feasible flow pattern satisfies the fixed-point equation (5), Constraints (10) set the upper and lower bounds on the capacity of each lot, Constraints (11) ensure that the traffic flow through each lot does not exceed the lot's capacity, and Constraint (12) is the natural constraint. Throughout the remaining paper, let $(\mathbf{q}^*, \mathbf{C}^*)$ denote an optimal solution to the model (Basic), where $\mathbf{C}^* := (C_j^*, j \in [J])$ is the optimal capacity plan and $\mathbf{q}^* := (q_j^*, j \in [J])$ is the equilibrium flow pattern under $\mathbf{C}^*$.

Next, we reformulate the model (Basic) as an optimization model that has $q_1, q_2, \ldots, q_J$ rather than $C_1, C_2, \ldots, C_J$ as decision variables. Define a function

$$\xi_j(\mathbf{q}) := \frac{\varphi q_j}{y_j(\mathbf{q})}, \tag{13}$$

for all $\mathbf{q} \in \mathcal{Q}$, where $y_j(\mathbf{q}) := b_j + \varphi - \beta q_j^\theta + \log(1 - \|\mathbf{q}\|_1) - \log(q_j)$ for all $j \in [J]$. Also define a vector-valued function $\boldsymbol{\xi} := (\xi_1, \xi_2, \ldots \xi_J)$ such that $\boldsymbol{\xi}(\mathbf{q}) := (\xi_j(\mathbf{q}), j \in [J])$ for all $\mathbf{q} \in \mathcal{Q}$.

Let $\mathcal{D} := \{\mathbf{q} \in \mathcal{Q} : \boldsymbol{\xi}(\mathbf{q}) \in \mathbb{R}^J_{++}\}$ denote the domain of $\boldsymbol{\xi}$ mapping into positive vectors. Some discussions on the function $\boldsymbol{\xi}(\mathbf{q})$ are provided in Remark 3 and Lemma 1.

Remark 3. For every $\mathbf{q} \in \mathcal{D}$, $(\mathbf{q}, \mathbf{C})$ satisfies $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$ if and only if $\mathbf{C} = \boldsymbol{\xi}(\mathbf{q})$. Thus, $\mathbf{C}^* = \boldsymbol{\xi}(\mathbf{q}^*)$.

Lemma 1 claims that $\boldsymbol{\xi} : \mathcal{D} :\mapsto \mathbb{R}^J_{++}$ is bijective.

Lemma 1. *$\boldsymbol{\xi}$ is a continuous bijection in $\mathcal{D}$ and $\boldsymbol{\xi}^{-1} : \mathbb{R}^J_{++} \mapsto \mathcal{D}$ exists.*

It follows from $h(\mathbf{q})$ defined in (7) and (9) that the model (Basic) is equivalent to the following *flow-based model*, where the primary decision variables are in $\mathbf{q} = (q_1, q_2, \ldots, q_J)$:

$$
\begin{aligned}
(\text{Flow}) \quad & \max_{\mathbf{q}} \ h(\mathbf{q}) \\
& \text{s.t. } \xi_j(\mathbf{q}) \geq q_j, \quad \forall\, j \in [J], && (14)\\
& \qquad \xi_j(\mathbf{q}) \geq \ell_j, \quad \forall\, j \in [J], && (15)\\
& \qquad \xi_j(\mathbf{q}) \leq u_j, \quad \forall\, j \in [J], && (16)\\
& \qquad (12),
\end{aligned}
$$

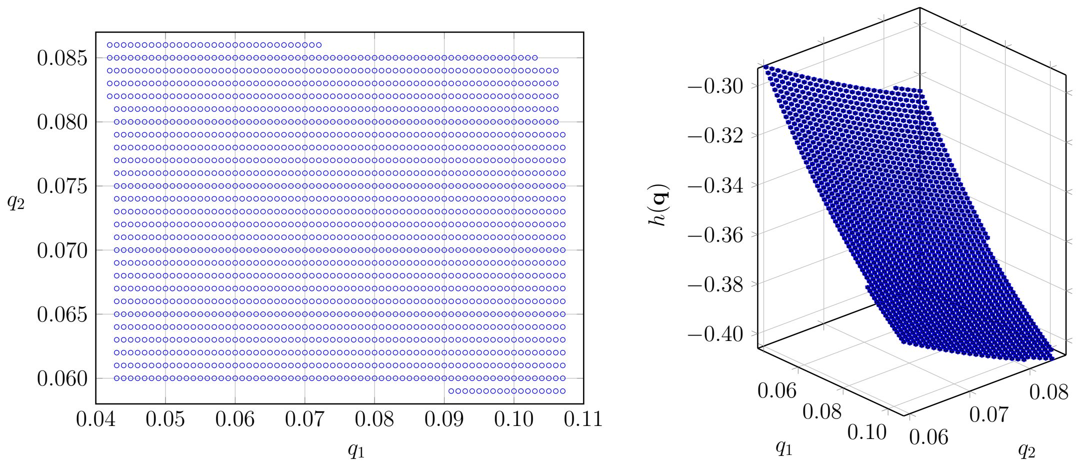
where $\xi_j(\mathbf{q})$ is defined in (13).

Let $\mathcal{S} := \{\mathbf{q} \in \mathcal{Q} : \boldsymbol{\xi}(\mathbf{q}) \in \prod_{j=1}^J [\ell_j, u_j], \boldsymbol{\xi}(\mathbf{q}) \geq \mathbf{q}\}$ denote the feasible set of the model (Flow). Since $\mathcal{Q}$ is open, it is not obvious that $\mathcal{S}$ is open or closed. We need to examine the compactness of $\mathcal{S}$ to conclude whether or not an optimal solution can be attained in $\mathcal{S}$. Proposition 3 guarantees the existence of an optimal solution in $\mathcal{S}$ when the model (Flow) is feasible, which justifies the use of "max" instead of "sup" in the model and is shown based on Lemma 1.

Proposition 3. *$\mathcal{S}$ is a compact set. If $\mathcal{S} \neq \varnothing$, then there exists an optimal solution in $\mathcal{S}$ that solves the model (Flow).*

Figure 1 illustrates the feasible set and the objective function $h(\mathbf{q})$ in the model (Flow) when $J = 2$, $\beta = 18.929$, $\theta = 0.923$, $\varphi = 3.301$, $b_1 = -2.551$, $b_2 = -3.281$, $\ell_1 = 0.05$, $u_1 = 0.95$, $\ell_2 = 0.15$, and $u_2 = 0.85$. A circle in Figure 1(a) denotes a feasible flow pattern $\mathbf{q} = (q_1, q_2)$ and a cube in Figure 1(b) represents $h(\mathbf{q})$ at a feasible $\mathbf{q}$.

As illustrated in Figure 1, the feasible set in the model (Flow) is not necessarily convex, which makes the model challenging to solve or analyze. The model is further complicated by the following features. First, the objective function $h(\mathbf{q})$ is strictly convex (cf. Proposition 2), and therefore, we are confronted with a non-convex program. Second, the unconstrained first-order optimality condition, which is one of the key analysis techniques used in the existing literature, may not hold due to the capacity and bound constraints. Third, the model may be infeasible due to these constraints. Thus, it requires a better understanding of the conditions under which the model is feasible.



(a) The feasible set. (b) The objective function.

Figure 1 The illustration of the feasible set and objective function in the model (FLOW).

5. Characterizing the Optimal Capacity Plan

In this section, we characterize the structure of the optimal capacity plan. First, we build in Section 5.1 a univariate model that is equivalent to the model (FLOW) but more amenable to analysis. Then, in Section 5.2, we characterize the optimal solution to the subproblem embedded in the univariate model, based on which we obtain characterizations of the optimal capacity plan.

5.1. An Equivalent Univariate Model

The model (FLOW) is challenging to analyze due to its non-convexity and constraints. To resolve the challenge, we build in this section a univariate model and show its equivalence to the model (FLOW).

Let $z := \|\mathbf{q}\|_1$ denote the total traffic flow attracted by all lots, where q_j is the traffic flow through j for any $j \in [J]$. Note that $z = \sum_{k=1}^{J} \exp(\nu_k(q_k, C_k))/(1+\sum_{k=1}^{J} \exp(\nu_k(q_k, C_k)))$ according to the fixed-point equation (4). By a change of variables, we will transform the model (FLOW) with $q_1, q_2, \ldots, q_J$ as decision variables into a univariate model with z as the only decision variable.

Reformulating Constraints (14)–(16). First, we reformulate Constraints (14)–(16) as a function of variable z. It can be observed from (13)–(16) that given $z = \|\mathbf{q}\|_1$, the feasible region of the traffic flow q_j for any lot $j \in [J]$ is independent of the traffic flows for all other lots. Hence, these constraints can be represented as the intersection of individual feasible regions of $q_1, q_2, \ldots, q_J$.

For $j \in [J]$, let $\mathcal{Q}_j(z)$ denote the feasible region of q_j given the total traffic flow $z \in (0,1)$. To obtain an explicit expression of $\mathcal{Q}_j(z)$, define a function $\zeta_{j,i}(\cdot, z) : [0,1] \mapsto [-\infty, \infty)$ as:

$$\zeta_{j,i}(x,z) := \beta x^{\theta} + \log(x) + \varphi\left(\mathbf{1}\{i=1\} + x\mathbf{1}\{i=2\}/\ell_j + x\mathbf{1}\{i=3\}/u_j\right) - \left(\log(1-z) + b_j + \varphi\right) \quad (17)$$

for $i \in \{1,2,3\}$.

The functions $\zeta_{j,i}(x,z)$ for $i \in \{1,2,3\}$ are used to reformulate the constraints $\xi_j(\mathbf{q}) \geq q_j$, $\xi_j(\mathbf{q}) \geq \ell_j$ and $\xi_j(\mathbf{q}) \leq u_j$ in (14)–(16) as $\zeta_{j,1}(q_j,z) \geq 0$, $\zeta_{j,2}(q_j,z) \geq 0$, and $\zeta_{j,3}(q_j,z) \leq 0$, respectively, when

z is the given total traffic flow. To explain, given z, it follows from (13) that $\xi_j(\mathbf{q}) \geq q_j$ leads to $\varphi q_j/y_j(\mathbf{q}) \geq q_j$, which is equivalent to $\varphi - y_j(\mathbf{q}) = \beta q_j^\theta + \log(q_j) - (\log(1-z) + b_j) = \zeta_{j,1}(q_j, z) \geq 0$ since $q_j > 0$ due to $\mathbf{q} \in \mathcal{Q}$. Similarly, $\xi_j(\mathbf{q}) \geq \ell_j$ and $\xi_j(\mathbf{q}) \leq u_j$ can be substituted by $\zeta_{j,2}(q_j, z) \geq 0$ and $\zeta_{j,3}(q_j, z) \leq 0$, respectively.

Remark 4 provides a description of $\mathcal{Q}_j(z)$ based on the explanations above.

REMARK 4. For any $z \in (0,1)$ and $j \in [J]$, the feasible region $\mathcal{Q}_j(z)$ of q_j specified in (14)–(16) can be represented by

$$\mathcal{Q}_j(z) = \{q_j \in [0,1] : \zeta_{j,1}(q_j, z) \geq 0, \zeta_{j,2}(q_j, z) \geq 0, \zeta_{j,3}(q_j, z) \leq 0\}.$$

The Univariate Model. The univariate model is represented as:

$$(\text{UNIVAR}) \quad \max_{z\in(0,1)} \pi(z) := \widehat{h}(z) - z\log(1-z),$$

where z is the only decision variable and $\widehat{h}(z)$ is the optimal objective value of the subproblem:

$$\begin{aligned}(\text{SUB}) \quad \widehat{h}(z) := \max_{\mathbf{q}} \; & \sum_{j=1}^{J} q_j \log(q_j) \\ \text{s.t.} \; & q_j \in \mathcal{Q}_j(z), \quad \forall j \in [J], \\ & \sum_{j=1}^{J} q_j = z.\end{aligned}$$

Proposition 4 formally establishes the equivalence between the models (FLOW) and (UNIVAR).

PROPOSITION 4. *Consider any optimal flow pattern $\mathbf{q}^*$ that solves the model (FLOW). Define $z^* := \|\mathbf{q}^*\|_1$. Then, z^* is optimal to the model (UNIVAR) and $\mathbf{q}^*$ is optimal to the model (SUB) with parameter $z = z^*$. Consider any $\widehat{z} \in (0,1)$ optimal to the model (UNIVAR) and $\widehat{\mathbf{q}}$ optimal to the model (SUB) with parameter $z = \widehat{z}$. Then, $\widehat{\mathbf{q}}$ is optimal to the model (FLOW).*

Proposition 4 have several implications. First, it implies that the model (UNIVAR) is feasible if and only if the model (FLOW) is feasible. Second, when both the models are feasible, one can retrieve an optimal solution to the model (FLOW) from that to the model (UNIVAR). Third, the structure of the optimal flow pattern that solves the model (FLOW) can be understood by studying the optimal solution to the model (SUB).

5.2. The Subproblem

In this section, we characterize the optimal solution to the model (SUB).

Feasibility. First, we discuss the feasibility of the subproblem to gain insights into the feasibility of the model (UNIVAR), which informs of the feasibility of the model (FLOW) as indicated by Proposition 4. The feasibility conditions will also be used to develop solution algorithms in Section 6.

Corollary 1 provides a complete characterization of $\mathcal{Q}_j(z)$ (i.e., the feasible region of q_j) for any $j \in J$ under any given z, which directly follows from Proposition EC.1.

COROLLARY 1. *Consider any $z \in (0,1)$ and $j \in [J]$. The following holds:*

(1) If $\log(1-z)+b_j+\varphi > \beta+\varphi\min\{1,1/\ell_j\}$, then $\mathcal{Q}_j(z)=\varnothing$.

*(2) If $\log(1-z)+b_j+\varphi \le \beta+\varphi\min\{1,1/\ell_j\}$, there exists a unique $x_i^* \in (0,1]$ such that $\zeta_{j,i}(x_i^*,z)=0$ for $i\in\{1,2\}$. Define the **lower-bound flow** $q_j^L(z) := \max\{x_1^*, x_2^*\}$.*

*(i) If $\log(1-z)+b_j+\varphi \le \beta+\varphi/u_j$, there exists a unique $x_3^*\in(0,1]$ such that $\zeta_{j,3}(x_3^*,z)=0$. If $x_3^*>u_j$, $\mathcal{Q}_j(z)=\varnothing$; otherwise, define the **upper-bound flow** $q_j^H(z):=x_3^*$, and we have $q_j^H(z)\ge q_j^L(z)$ and $\mathcal{Q}_j(z)=[q_j^L(z),q_j^H(z)]$*

(ii) If $\log(1-z)+b_j+\varphi > \beta+\varphi/u_j$, define $q_j^H(z):=1$ and we have $\mathcal{Q}_j(z)=[q_j^L(z),q_j^H(z)]$.

Corollary 1 provides conditions for $\mathcal{Q}_j(z)=\varnothing$ and represents $\mathcal{Q}_j(z)$ as a simple interval when it is nonempty. If the condition in (1) holds, either $\zeta_{j,1}(q_j,z)\ge 0$ or $\zeta_{j,2}(q_j,z)\ge 0$ fails to hold, resulting in $\mathcal{Q}_j(z)=\varnothing$. If the condition in (2)(i) holds but $x_3^*>u_j$, either $\zeta_{j,1}(q_j,z)\ge 0$ or $\zeta_{j,3}(q_j,z)\le 0$ will be violated, causing $\mathcal{Q}_j(z)=\varnothing$. Otherwise, there exist a lower-bound flow $q_j^L(z)$ and an upper-bound traffic flow $q_j^H(z)$ such that $\mathcal{Q}_j(z)$ can be represented as a nonempty interval $[q_j^L(z),q_j^H(z)]$.

Corollary 1 can be used to determine the feasibility of the model (FLOW): If $\mathcal{Q}_j(z)=\varnothing$ for some j, then the model (SUB) is infeasible for the given z. Moreover, if the model (SUB) is infeasible for all $z\in(0,1)$, both the models (UNIVAR) are (FLOW) are infeasible. Also, representing $\mathcal{Q}_j(z)$ as intervals allows for obtaining characterization results in Theorem 1. Corollary 1 will also be applied to reduce the search space of z in Algorithm 2.

Characterizations of the Optimal Solution to the Subproblem. Next, we characterize in Theorem 1 the optimal solution to the model (SUB) using Corollary 1.

THEOREM 1. *Consider any $z\in(0,1)$. Assume that $\log(1-z)+b_j+\varphi\le\beta+\varphi\min\{1,1/\ell_j\}$ for all $j\in[J]$ and that if there exists $j\in[J]$ and $x^*\in(0,1]$ such that $\zeta_{j,3}(x^*,z)=0$, it holds that $x^*\le u_j$. Let $q_j^L(z)\in(0,1]$ and $q_j^H(z)\in(0,1]$ be as in Corollary 1. Then, the model (SUB) is equivalent to*

$$\max_{\mathbf{q}} \sum_{j=1}^{J} q_j\log(q_j) \tag{18}$$

$$\text{s.t. } q_j\in\left[q_j^L(z),q_j^H(z)\right],\quad \forall j\in[J], \tag{19}$$

$$\sum_{j=1}^{J} q_j = z. \tag{20}$$

Let $\mathbf{q}^*(z):=(q_1^*(z),q_2^*(z),\ldots,q_J^*(z))$ denote an optimal solution to the model (18)–(20). Then, there exists at most one lot $k\in[J]$ such that $\mathbf{q}^*(z)$ is given by,

$$q_j^*(z)\begin{cases}\in(q_j^L(z),q_j^H(z)) & \text{if } j=k,\\ =q_j^L(z) & \text{if } j\ne k \text{ and } q_j^H(z)=1,\\ \in\{q_j^L(z),q_j^H(z)\} & \text{if } j\ne k \text{ and } q_j^H(z)<1,\end{cases} \tag{21}$$

for all $j\in[J]$, where $q_j^L(z)<1$ for all $j\in[J]$.

Theorem 1 indicates that when the model (SUB) is feasible for the given z, there exists at most one lot k such that $q_k^*(z)$ is strictly between the lower- and upper-bound flows, and for any other lot $j \neq k$, $q_j^*(z)$ is equal to either the lower- or upper-bound flow.

For each $j \in [J]$, define the effective lower bound $\ell_j^{\text{eff}} := \max\{\ell_j, q_j^*\}$. Theorem 2 characterizes the optimal capacity plan that solves the model (BASIC), which is obtained based on Remark 3, Lemma EC.3, Corollary 1, Theorem 1, and Proposition 4.

THEOREM 2 **(Characterization of the Optimal Capacity Plan)**. *Either the model (BASIC) is infeasible or there exists at most one lot $k \in [J]$ such that $C_k^* \in (\ell_k^{\text{eff}}, u_k)$ and $C_j^* \in \{\ell_j^{\text{eff}}, u_j\}$ for all $j \in [J]$ and $j \neq k$.*

Theorem 2 implies that when the optimal capacity sizing model (BASIC) is feasible, except for at most one lot k whose optimal capacity is strictly between the lower bound ℓ_j and upper bound u_j, the optimal capacity of each lot $j \neq k$ is equal to one of the following three values: ℓ_j, u_j, and the optimal traffic flow q_j^*. The theorem also suggests that under an optimal capacity plan, one of the three constraints (14)–(16) is binding for each $j \neq k$ in the model (FLOW).

6. Algorithms

In this section, we develop a one-variable search algorithm to determine an optimal capacity plan. More specifically, first, we refine the feasible range of z in the model (UNIVAR). Then, in Section 6.1, we propose an algorithm to determine the unique equilibrium flow pattern under any capacity plan, by which we can determine the refined feasible range of z. In Section 6.2, we present the one-variable search algorithm.

Let $\mathbf{q}^L := (q_j^L, j \in [J])$ denote the equilibrium flow pattern under the *all-lower-bound capacity plan* $\mathbf{C}^L := (\ell_1, \ell_2, \ldots, \ell_J)$ and $\mathbf{q}^H := (q_j^H, j \in [J])$ denote the equilibrium flow pattern under the *all-upper-bound capacity plan* $\mathbf{C}^H := (u_1, u_2, \ldots, u_J)$. Proposition 5 provides a refined range of z that consists of all the feasible points of z to the model (UNIVAR).

PROPOSITION 5. *If $z \in (0,1)$ is feasible to the model (UNIVAR), then $z \in [\|\mathbf{q}^L\|_1, \|\mathbf{q}^H\|_1]$.*

Proposition 5 indicates that the feasible region $(0,1)$ of z can be replaced by $[\|\mathbf{q}^L\|_1, \|\mathbf{q}^H\|_1]$ without excluding any feasible solutions in the model (UNIVAR). The refined range will later be used in Algorithm 2 to speed up the solution procedure.

6.1. Solving the Equilibrium Flow Pattern for a Given Capacity Plan

We need to determine $\mathbf{q}^i$ under $\mathbf{C}^i$ for $i \in \{L, H\}$ to obtain the refined range of z. To that end, in this section, we present in Algorithm 1 a solution procedure to solve the unique equilibrium flow pattern $\mathbf{q}$ that satisfies $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$ under any given capacity plan $\mathbf{C} \in (0, \infty]^J$.

Algorithm 1 is developed based on Proposition EC.2 and the classical bisection search Algorithm EC.1. The key idea is to treat the total traffic flow z under $\mathbf{C}$ as a variable and show that

Algorithm 1 SOLVING THE EQUILIBRIUM FLOWS UNDER A CAPACITY PLAN

Input: J, f_j in (EC.17) for $j \in [J]$, $\delta_1, \delta_2 > 0$, $\text{ub} \leftarrow 1$, $\text{lb} \leftarrow z^0$, $z \leftarrow (\text{lb} + \text{ub})/2$, and $\mathbf{C}$.

Output: $\mathbf{q}$ such that $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$.

Find $q_j \in [0,1]$ such that $f_j(q_j) = \log(1-z)$ using Algorithm EC.1 for each $j \in [J]$;
Set $\Gamma(z) \leftarrow \sum_{j=1}^{J} q_j - z$;
while $|\Gamma(z)| > \delta_1$ and $\text{ub} - \text{lb} > \delta_2$ **do**
    Set $\text{lb} \leftarrow z$ if $\Gamma(z) > 0$; set $\text{ub} \leftarrow z$ otherwise;
    Set $z \leftarrow (\text{lb} + \text{ub})/2$;
    Find $q_j \in [0,1]$ such that $f_j(q_j) = \log(1-z)$ using Algorithm EC.1 for each $j \in [J]$;
    Set $\Gamma(z) \leftarrow \sum_{j=1}^{J} q_j - z$;
end while

the equilibrium z that satisfies the fixed-point equation $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$ is the unique root of a strictly decreasing and continuous function (cf. Proposition EC.2). Then, the equilibrium z can be determined using the bisection search method (see Algorithm EC.1) and the equilibrium flow pattern can be obtained using Proposition EC.2. While Algorithm 1 will be used to determine $\mathbf{q}^L$ and $\mathbf{q}^H$ in Algorithm 2, it is useful in its own right in forecasting traffic flows under an MNL choice model.

6.2. A One-Variable Search Algorithm

In this section, we present in Algorithm 2 a solution procedure to determine an optimal capacity plan by solving the model (UNIVAR) based on Corollary 1 and Theorem 1.

Discretizing z. In Algorithm 2, we discretize the range $[\|\mathbf{q}^L\|_1, \|\mathbf{q}^H\|_1]$ with a step size of $\Delta z > 0$. For each discretized value of z, we will calculate the objective value $\pi(z)$ in the model (UNIVAR). Then, an optimal solution z^* can be obtained by comparing $\pi(z)$ at all discretized values (cf. Step 6).

Generating candidate optimal solutions to the model (Sub). For each discretized z, we need to solve the model (SUB) to obtain $\widehat{h}(z)$, by which we can compute $\pi(z) = \widehat{h}(z) - z\log(1-z)$.

We can determine an optimal solution $\mathbf{q}^*(z)$ to the model (SUB) (if feasible) using Theorem 1, which suggests that there exists at most one lot k such that $q_k^*(z)$ is between $q_k^L(z)$ and $q_k^H(z)$ and $q_j^*(z)$ is equal to either $q_j^L(z)$ or $q_j^H(z)$ for any lot $j \neq k$.

For each such k, we can generate a candidate $\mathbf{q}^*(z)$ as follows. Assign $q_j^L(z)$ or $q_j^H(z)$ to $q_j^*(z)$ for all $j \neq k$ and calculate $q_k^*(z) = z - \sum_{j \neq k} q_j^*(z)$. If $q_k^*(z) \in [q_k^L(z), q_k^H(z)]$, the $\mathbf{q}^*(z)$ so generated is a valid candidate optimal solution; it is non-valid and will be dropped otherwise. To facilitate the exposition of the generation of candidate $\mathbf{q}^*(z)$ in Algorithm 2, let $\widehat{\mathcal{Q}}^k(z)$ denote the set of (valid and non-valid) candidate $\mathbf{q}^*(z)$ generated for any $k \in [J]$, which is represented by

$$\widehat{\mathcal{Q}}^k(z) := \left\{ \mathbf{q} \in \mathbb{R}^J : q_j \begin{cases} = q_j^L(z) & \text{if } q_j^H(z) = 1, \\ \in \left\{ q_j^L(z), q_j^H(z) \right\} & \text{if } q_j^H(z) < 1, \end{cases}, q_k = z - \sum_{j \neq k} q_j, \ \forall j \in [J] \text{ and } j \neq k \right\}. \quad (22)$$

Algorithm 2 A ONE-VARIABLE SEARCH METHOD

Input: φ, β, θ, J, $(b_j, j \in [J])$, $(\ell_j, j \in [J])$, $(u_j, j \in [J])$, $\mathbf{C}^L$, $\mathbf{C}^H$, Δz, $\mathcal{Z} \leftarrow \varnothing$.

Output: $\mathbf{C}^*$.

Step 1. Obtain $\mathbf{q}^i$ by solving $\mathbf{q}^i = \mathbf{F}(\mathbf{q}^i, \mathbf{C}^i)$ for $i \in \{L, H\}$ using Algorithm 1. Set $z \leftarrow \|\mathbf{q}^L\|_1$.

Step 2. If $z > \|\mathbf{q}^H\|$, go to Step 6; execute the following otherwise. Set $\widetilde{\mathcal{Q}} \leftarrow \varnothing$, $k \leftarrow 1$, and $n \leftarrow 1$. If there exists $j \in [J]$ such that $\log(1-z) + b_j + \varphi > \beta + \varphi \min\{1, 1/\ell_j\}$, then set $z \rightarrow z + \Delta z$ and go to Step 2; go to Step 3 otherwise.

Step 3. For each $j \in [J]$, solve $x_k^* \in (0,1]$ such that $\zeta_{j,k}(x_k^*, z) = 0$ for $k \in \{1, 2\}$ using Algorithm EC.1 and set $q_j^L(z) \leftarrow \max\{x_1^*, x_2^*\}$.

Step 4. For each $j \in [J]$, if $\log(1-z) + b_j + \varphi > \beta + \varphi/u_j$, set $q_j^H(z) \leftarrow 1$; Otherwise, solve $x_3^* \in (0,1]$ such that $\zeta_{j,3}(x_3^*, z) = 0$ using Algorithm EC.1, set $q_j^H(z) \leftarrow x_3^*$ if $x_3^* \le u_j$, and set $z \rightarrow z + \Delta z$ and go to Step 2 if $x_3^* > u_j$.

Step 5. Set $\mathbf{q} = (q_j, j \in [J]) \leftarrow$The n-th element in $\widehat{\mathcal{Q}}^k(z)$ as defined in (22). If $q_k \in [q_k^L(z), q_k^H(z)]$, set $\widetilde{\mathcal{Q}} \leftarrow \widetilde{\mathcal{Q}} \cup \{\mathbf{q}\}$. Then, execute the following:

(a) If $n < |\widehat{\mathcal{Q}}^k(z)|$, set $n \leftarrow n+1$ and go to Step 5;

(b) If $k < J$ and $n = |\widehat{\mathcal{Q}}^k(z)|$, set $k \leftarrow k+1$, $n \leftarrow 1$, and go to Step 5;

(c) If $k = J$ and $n = |\widehat{\mathcal{Q}}^k(z)|$, find $\mathbf{q}^*(z) := (q_j^*(z), j \in [J]) \in \arg\max_{\mathbf{q} \in \widetilde{\mathcal{Q}}} \sum_{j=1}^J q_j \log(q_j)$, set $\widehat{h}(z) = \sum_{j=1}^J q_j^*(z) \log(q_j^*(z))$, calculate $\pi(z) = \widehat{h}(z) - z\log(1-z)$, set $\mathcal{Z} \leftarrow \mathcal{Z} \cup \{z\}$, set $z \rightarrow z + \Delta z$, and go to Step 2;

Step 6. Find $z^* \in \arg\max_{z \in \mathcal{Z}} \pi(z)$. Set $\mathbf{C}^* \leftarrow \boldsymbol{\xi}(\mathbf{q}^*(z^*))$, where $\boldsymbol{\xi}$ is defined in (13).

Then, as in Step 5, we can obtain $\widehat{h}(z)$ by comparing the objective values in the model (SUB) at all valid candidate $\mathbf{q}^*(z)$ in $\widehat{\mathcal{Q}}^k(z)$ for all $k \in [J]$. Note $|\widehat{\mathcal{Q}}^k(z)| \le 2^{J-1}$.

More discussions on $\widehat{\mathcal{Q}}^k(z)$. For each discretized z, we find a flow pattern $\mathbf{q}^*(z)$ that satisfies (21) for all $j \neq k$ and is thus a candidate optimal solution to the model (SUB). It further requires that the total traffic flow $\|\mathbf{q}^*(z)\|_1$ is equal to z. The set $\widehat{\mathcal{Q}}^k(z)$ consists of all such $\mathbf{q}^*(z)$'s.

Feasibility check and expedition. Algorithm 2 involves procedures skipping infeasible values of z in Step 2 and Step 4, by which the solution procedure is expedited. The feasibility of each discretized z is checked using the conditions provided in Corollary 1. More specifically, if $\mathcal{Q}_j(z) = [q_j^L(z), q_j^H(z)] = \varnothing$ for some j at z, then such z is infeasible. Note that $q_j^L(z)$ and $q_j^H(z)$ can be obtained as in Corollary 1 using Algorithm EC.1.

7. Numerical Experiments

In this section, we provide numerical results to gain insights into the optimal capacity design for park-and-ride lots in a satellite city and to investigate the value of modeling the congestion and parking effects in optimal sizing of park-and-ride lots. In Section 7.1, we describe the data related to the park-and-ride lots in Bellevue of Washington State (WA). In Section 7.2, we apply our developed models and algorithms to solve the optimal capacities of Bellevue's park-and-ride lots. Section 7.3 provides sensitivity analysis. In Section 7.4, we evaluate the performance of the optimal capacity plan under different choice behavior of commuters.

The model (Univar) is solved using Algorithm 2 with $\Delta z = 10^{-7}$ and the numerical results are obtained using Matlab R2022a on a desktop computer with Intel i7 CPU@3.20Ghz and 16GB RAM.

7.1. Bellevue's Park-and-Ride Lots and Data

In this section, we study the park-and-ride lots in the city of Bellevue in Washington State (WA), which is a satellite city of Seattle.

According to King County Metro (2022a), there are $J = 7$ park-and-ride lots accessible to commuters in Bellevue. Figure 2 shows the names and locations of the lots on the map of Bellevue. Table EC.1 shows their addresses and current capacities.

We will (re)design the capacities of these lots using our developed models and algorithms. The objective of this section is twofold: (i) to demonstrate that our models and algorithms can be applied to real-world cases, where the DOT needs to design a set of new park-and-ride lots or re-design existing lots to accommodate the growing demand for parking electric vehicles, and (ii) to obtain insights into the optimal capacity design of park-and-ride lots in a satellite city such as Bellevue.

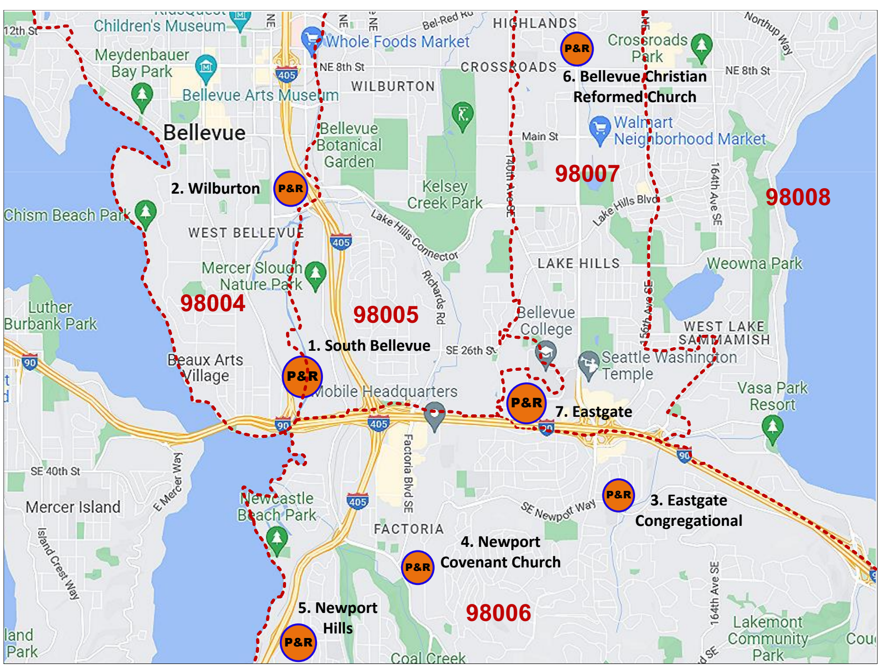


Figure 2 The names and locations of the park-and-ride lots in Bellevue, WA (King County Metro, 2022a).

Determining catchment areas. We geographically partition Bellevue into seven *catchment areas* such that each catchment area contains a single park-and-ride lot, the entire Bellevue is the union of all the catchment areas, and the catchment areas of any two different lots do not overlap. The catchment area of any lot is an area where commuters would choose to visit that lot (rather than others) if they base their decisions solely on congestion-free travel time. Let $H_j \in \mathbb{R}_+$ denote the number of households in the catchment area of any lot $j \in [J]$, which is provided in Table EC.2. The catchment areas and H_j's are determined in Section EC.5.2.

Table EC.3 shows bi-directional time distances between different pairs (i, j) of lots in Bellevue, where $i, j \in [J]$, which are collected around 3 am on October 8, 2022 (Saturday) using Google Map (2022). Since congestion seldom occurs around 3 am on a Saturday morning, these data can be used to approximate the congestion-free travel time from the catchment area of a lot $i \in [J]$ to that of a lot $j \in [J]$, which is denoted by $\tau_{i,j} \in \mathbb{R}_+$.

Next, we estimate the values of *attributes* such as the economic value, number of bus routes, bus frequency, and congestion-free access time, which will be used to estimate the intrinsic utilities of Bellevue's park-and-ride lots.

Economic value. Let $\widehat{r}_{j,1} \in \mathbb{R}_+$ denote the economic value of any lot $j \in [J]$, which is represented by the median house/condo value for j as shown in Table EC.2. The median house/condo value for each $j \in [J]$ is the median value of the houses/condominiums in the area with the same zip code as j, which reflects the economic value of j's location and is obtained from City-Data (2016).

The number of bus routes. There is a bus stop next to each lot, through which commuters access transit services. Let $\widehat{r}_{j,2} \in \mathbb{R}_+$ denote the number of bus routes traversing (the bus stop next to) lot j, which is provided in Table EC.2 and obtained from King County Metro (2022a).

Bus frequency. Let $\widehat{r}_{j,3} \in \mathbb{R}_+$ denote the bus frequency that measures the frequency of bus arrivals at lot j, which is estimated as follows. For each bus route passing by j, we calculate the average inter-arrival time (in minutes) of the buses arriving at j between 7 am and 9 am on workdays using King County Metro (2022b) and Sound Transit (2022). Next, the average bus headway for j as shown in Table EC.2 is calculated as the mean of the average inter-arrival times for all the bus routes passing along j. Then, $\widehat{r}_{j,3}$ can be computed as the reciprocal of the average bus headway for j.

Congestion-free access time. Let $\widehat{r}_{j,4} \in \mathbb{R}_+$ denote the congestion-free access time to lot j as shown in Table EC.2, which is the average travel time to j per trip (or vehicle) under no congestion. Let η denote the average number of park-and-ride trips per household on a workday morning. Hence, $H_j\eta$ approximates the number of commuters' trips generated from the catchment area of j. Then, $\widehat{r}_{j,4}$ can be estimated by,

$$\widehat{r}_{j,4} = \frac{\sum_{i\in[J]} H_i \eta \tau_{i,j}}{\sum_{i\in[J]} H_i \eta} = \frac{\sum_{i\in[J]} H_i \tau_{i,j}}{\sum_{i\in[J]} H_i},$$

where we assume that for the commuters generated from the catchment area of any lot, the congestion-free travel time to that lot is zero.

Scaling. We will conduct sensitivity analysis in Section 7.3 to investigate and compare the impacts of the sensitivities of commuters to different attributes on optimal capacity sizing. Since the data in Table EC.2 are of different scales, it often requires scaling the data to facilitate the comparison (and make optimization used to estimate the choice model well-conditioned) (Lantz, 2013). To that end, we choose the South Bellevue (i.e., $j = 1$) as the reference lot. Then, we obtain $r_{j,k} := \widehat{r}_{j,k}/\widehat{r}_{1,k}$ as the normalized attribute value for all $j \in [J]$ and $k \in \{1,2,3,4\}$, which is provided in Table EC.4.

Then, for each $j \in [J]$, the intrinsic utility b_j can be estimated by,

$$b_j = \alpha_1 r_{j,1} + \alpha_2 r_{j,2} + \alpha_3 r_{j,3} - \alpha_4 r_{j,4}, \tag{23}$$

where $\alpha_k \in \mathbb{R}_+$ is the sensitivity parameter for $k \in \{1,2,3,4\}$.

7.2. Optimal Capacities and Flows under Different Lower- and Upper-Bound Capacities

In this section, we solve the optimal capacity sizing problem for Bellevue's park-and-ride lots.

We set $\alpha_k = 2.5$ for all $k \in \{1,2,3,4\}$, $\beta = 2.5$, $\theta = 0.5$, and $\varphi = 2.5$ throughout this section, where the parameters are arbitrarily chosen and we will explore the influence of the variations in these parameters on optimal solutions in Section 7.3.

We choose South Bellevue (i.e., $j = 1$) as the reference lot. First, we select values of ℓ_1 and u_1. Then, for any lot $j \in \{2,3,\ldots,7\}$, ℓ_j (u_j) is computed as ℓ_1 (u_1) multiplied by the ratio of the current capacity of lot j (cf. Table EC.1) to that of South Bellevue, by which the relative sizes of the lower and upper bounds of all the lots are consistent with the relative sizes of their current capacities.

We obtain numerical results for a variety of cases specified by different values of ℓ_1 and u_1 (as well as different lower and upper bounds for $j \in \{2,3,\ldots,7\}$ determined as previously mentioned). Define $\rho_j^* := q_j^*/C_j^*$ as the optimal utilization at lot $j \in [J]$. Tables 2 and 3 show the intrinsic utilities, optimal capacities, traffic flows, and utilizations under different cases of ℓ_1 and u_1. Table EC.5 shows the same content as Tables 2–3 except that different ℓ_1 and u_1 are considered to provide results under a broader range of lower and upper bounds.

We observe the following from Tables 2, 3, and EC.5.

(1) *A lot with a high intrinsic utility tends to attain the optimal capacity on the upper bound. If a lot has a low/moderate-sized intrinsic utility and a small lower bound, its optimal capacity tends to be equal to the optimal traffic flow through the lot, yielding high utilization at the lot. However, a lot with a low/moderate-sized intrinsic utility and a lower bound that is not small tends to attain the optimal capacity on the lower bound.* For example, Lot 6 attains its optimal capacity on the lower bound when ℓ_1 exceeds a threshold value (e.g., when $\ell_1 > 0.05$). Since $b_6 = -0.4637 < 0$, lot 6 is potentially less attractive to commuters than other lots and the no-park-and-ride alternative. Therefore, it tends to be optimal to set the optimal capacity $C_6^* = \ell_6$

Table 2 The optimal capacities, traffic flows, and utilizations under $\ell_1 \in \{0.05, 0.25, 0.5, 0.7\}$ and $u_1 \in \{0.75, 0.85\}$.

j	b_j	$\ell_1 = 0.05,\ u_1 = 0.75$					$\ell_1 = 0.05,\ u_1 = 0.85$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.0500	0.7500	0.7500	0.2612	34.82%	0.0500	0.8500	0.8500	0.2538	29.85%
2	2.4119	0.0062	0.0930	0.0930	0.0376	40.48%	0.0062	0.1054	0.1054	0.0367	34.85%
3	1.6794	0.0007	0.0100	0.0055	0.0055	100.00%	0.0007	0.0113	0.0048	0.0048	100.00%
4	2.5824	0.0025	0.0375	0.0375	0.0251	67.02%	0.0025	0.0425	0.0425	0.0254	59.70%
5	1.3456	0.0092	0.1375	0.1369	0.0248	18.13%	0.0092	0.1558	0.1558	0.0231	14.80%
6	-0.4637	0.0007	0.0100	0.0007	0.0007	100.00%	0.0007	0.0113	0.0007	0.0007	97.72%
7	7.7539	0.0538	0.8070	0.8070	0.6437	79.77%	0.0538	0.9146	0.9146	0.6546	71.57%
j	b_j	$\ell_1 = 0.25,\ u_1 = 0.75$					$\ell_1 = 0.25,\ u_1 = 0.85$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.2500	0.7500	0.7500	0.2607	34.76%	0.2500	0.8500	0.8500	0.2534	29.81%
2	2.4119	0.0310	0.0930	0.0930	0.0376	40.40%	0.0310	0.1054	0.1054	0.0367	34.79%
3	1.6794	0.0033	0.0100	0.0056	0.0056	99.18%	0.0033	0.0113	0.0047	0.0047	100.00%
4	2.5824	0.0125	0.0375	0.0375	0.0251	66.91%	0.0125	0.0425	0.0425	0.0253	59.63%
5	1.3456	0.0458	0.1375	0.1375	0.0248	18.03%	0.0458	0.1558	0.1552	0.0230	14.81%
6	-0.4637	0.0033	0.0100	0.0033	0.0020	58.61%	0.0033	0.0113	0.0033	0.0018	54.70%
7	7.7539	0.2690	0.8070	0.8070	0.6430	79.68%	0.2690	0.9146	0.9146	0.6540	71.51%
j	b_j	$\ell_1 = 0.5,\ u_1 = 0.75$					$\ell_1 = 0.5,\ u_1 = 0.85$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.5000	0.7500	0.7500	0.2602	34.69%	0.5000	0.8500	0.8500	0.2527	29.73%
2	2.4119	0.0620	0.0930	0.0930	0.0375	40.30%	0.0620	0.1054	0.1054	0.0366	34.69%
3	1.6794	0.0067	0.0100	0.0069	0.0064	93.06%	0.0067	0.0113	0.0067	0.0060	89.17%
4	2.5824	0.0250	0.0375	0.0375	0.0250	66.79%	0.0250	0.0425	0.0425	0.0253	59.48%
5	1.3456	0.0917	0.1375	0.1375	0.0247	17.97%	0.0917	0.1558	0.1558	0.0229	14.70%
6	-0.4637	0.0067	0.0100	0.0067	0.0028	42.56%	0.0067	0.0113	0.0067	0.0026	39.20%
7	7.7539	0.5380	0.8070	0.8070	0.6422	79.57%	0.5380	0.9146	0.9146	0.6529	71.39%
j	b_j	$\ell_1 = 0.7,\ u_1 = 0.75$					$\ell_1 = 0.7,\ u_1 = 0.85$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.7000	0.7500	0.7500	0.2595	34.60%	0.7000	0.8500	0.8500	0.2521	29.66%
2	2.4119	0.0868	0.0930	0.0930	0.0374	40.19%	0.0868	0.1054	0.1054	0.0365	34.59%
3	1.6794	0.0093	0.0100	0.0095	0.0079	83.50%	0.0093	0.0113	0.0093	0.0074	79.47%
4	2.5824	0.0350	0.0375	0.0375	0.0250	66.64%	0.0350	0.0425	0.0425	0.0252	59.36%
5	1.3456	0.1283	0.1375	0.1375	0.0246	17.90%	0.1283	0.1558	0.1554	0.0228	14.68%
6	-0.4637	0.0093	0.0100	0.0093	0.0033	35.57%	0.0093	0.0113	0.0093	0.0030	32.55%
7	7.7539	0.7532	0.8070	0.8070	0.6411	79.44%	0.7532	0.9146	0.9146	0.6519	71.28%

to accommodate the limited demand it attracts, provided that ℓ_6 is not smaller than the demand (which happens when ℓ_1 is not small). However, when ℓ_6 is smaller than the demand (which happens when ℓ_1 is small, e.g., when $\ell_1 = 0.05$), it is optimal to set $C_6^* = q_6^*$ to accommodate the demand, resulting in the utilization of 100% of the lot.

(2) *In general, a lot with a large (small) intrinsic utility tends to attract a high (low) traffic flow. However, this is not always true: A lot with a high intrinsic utility and high parking utilization may attract fewer commuters than that with a low intrinsic utility and low utilization.* For

Table 3 The optimal capacities, traffic flows, and utilizations under $\ell_1 \in \{0.05, 0.25, 0.5, 0.7\}$ and $u_1 \in \{0.95, 1\}$.

j	b_j	$\ell_1 = 0.05,\ u_1 = 0.95$					$\ell_1 = 0.05,\ u_1 = 1$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.0500	0.9500	0.9500	0.2525	26.58%	0.0500	1.0000	1.0000	0.2493	24.93%
2	2.4119	0.0062	0.1178	0.1178	0.0368	31.20%	0.0062	0.1240	0.1240	0.0363	29.30%
3	1.6794	0.0007	0.0127	0.0043	0.0043	100.00%	0.0007	0.0133	0.0041	0.0041	100.00%
4	2.5824	0.0025	0.0475	0.0473	0.0260	54.92%	0.0025	0.0500	0.0500	0.0261	52.23%
5	1.3456	0.0092	0.1742	0.0092	0.0064	69.71%	0.0092	0.1833	0.0093	0.0063	67.78%
6	-0.4637	0.0007	0.0127	0.0007	0.0006	95.19%	0.0007	0.0133	0.0007	0.0006	93.48%
7	7.7539	0.0538	1.0222	1.0222	0.6725	65.79%	0.0538	1.0760	1.0760	0.6763	62.86%

j	b_j	$\ell_1 = 0.25,\ u_1 = 0.95$					$\ell_1 = 0.25,\ u_1 = 1$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.2500	0.9500	0.9500	0.2470	26.00%	0.2500	1.0000	1.0000	0.2441	24.41%
2	2.4119	0.0310	0.1178	0.1178	0.0359	30.44%	0.0310	0.1240	0.1240	0.0355	28.61%
3	1.6794	0.0033	0.0127	0.0041	0.0041	100.00%	0.0033	0.0133	0.0039	0.0039	100.00%
4	2.5824	0.0125	0.0475	0.0475	0.0255	53.77%	0.0125	0.0500	0.0500	0.0256	51.25%
5	1.3456	0.0458	0.1742	0.1712	0.0215	12.55%	0.0458	0.1833	0.1822	0.0209	11.48%
6	-0.4637	0.0033	0.0127	0.0033	0.0017	51.65%	0.0033	0.0133	0.0033	0.0017	50.35%
7	7.7539	0.2690	1.0222	1.0222	0.6633	64.89%	0.2690	1.0760	1.0760	0.6675	62.03%

j	b_j	$\ell_1 = 0.5,\ u_1 = 0.95$					$\ell_1 = 0.5,\ u_1 = 1$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.5000	0.9500	0.9500	0.2463	25.93%	0.5000	1.0000	1.0000	0.2433	24.33%
2	2.4119	0.0620	0.1178	0.1178	0.0357	30.34%	0.0620	0.1240	0.1240	0.0353	28.51%
3	1.6794	0.0067	0.0127	0.0067	0.0057	85.72%	0.0067	0.0133	0.0067	0.0056	84.15%
4	2.5824	0.0250	0.0475	0.0475	0.0255	53.64%	0.0250	0.0500	0.0500	0.0256	51.11%
5	1.3456	0.0917	0.1742	0.1652	0.0212	12.86%	0.0917	0.1833	0.1787	0.0207	11.60%
6	-0.4637	0.0067	0.0127	0.0067	0.0024	36.62%	0.0067	0.0133	0.0067	0.0024	35.53%
7	7.7539	0.5380	1.0222	1.0222	0.6622	64.78%	0.5380	1.0760	1.0760	0.6662	61.91%

j	b_j	$\ell_1 = 0.7,\ u_1 = 0.95$					$\ell_1 = 0.7,\ u_1 = 1$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.7000	0.9500	0.9500	0.2457	25.86%	0.7000	1.0000	1.0000	0.2428	24.28%
2	2.4119	0.0868	0.1178	0.1178	0.0356	30.26%	0.0868	0.1240	0.1240	0.0353	28.44%
3	1.6794	0.0093	0.0127	0.0093	0.0071	75.99%	0.0093	0.0133	0.0093	0.0070	74.52%
4	2.5824	0.0350	0.0475	0.0475	0.0254	53.52%	0.0350	0.0500	0.0500	0.0255	51.02%
5	1.3456	0.1283	0.1742	0.1675	0.0212	12.67%	0.1283	0.1833	0.1726	0.0205	11.89%
6	-0.4637	0.0093	0.0127	0.0093	0.0028	30.24%	0.0093	0.0133	0.0093	0.0027	29.28%
7	7.7539	0.7532	1.0222	1.0222	0.6612	64.68%	0.7532	1.0760	1.0760	0.6653	61.83%

example, if commuters base their decisions only on intrinsic utilities, $q_2^* \leq q_4^*$ holds under all cases since $b_2 < b_4$. When commuters are additionally information-aware, we have $q_2^* = 0.0376 > q_4^* = 0.0251$ under the case of $\ell_1 = 0.25$ and $u_1 = 0.75$, since lot 2 has higher parking availability than lot 4. Thus, under an optimal capacity plan, it tends to reallocate information-aware commuters from high-utilization lots to low-utilization lots. This finding highlights the impact of parking information on capacity sizing with information-aware commuters.

(3) *The parking availability at a high-utilization lot may not be improved by allowing for a smaller lower or a larger upper bound for the lot.* For example, although lot 3 has high utilization, the

utilization may not drop and commuters may not be better off if the DOT allows for a different lower or upper bound for the lot. When $\ell_1 = 0.5, u_1 = 0.75$, C_3^* is strictly between ℓ_3 and u_3, indicating non-binding lower- and upper-bound constraints. Hence, the utilization will not drop even when ℓ_3 decreases or u_3 increases.

(4) *It is possibly both inevitable and beneficial to deploy lots around the peripheral area of a satellite city, and a peripheral lot may gain full or nearly full utilization.* Examples include lots 3 & 6 in Bellevue. On the one hand, practically, these lots are needed to provide nearby households with points of access to public transit, as the transit center is usually distant from the peripheral area (e.g., Bellevue Transit Center is located in the downtown). Analytically, even though a peripheral lot is initially used by very few commuters, it will gain more visits when the system reaches equilibrium due to the effects of congestion and parking information. *Meanwhile, it seems reasonable not to set large upper bounds for the lots in the peripheral area.* After all, a peripheral lot attracts a limited number of commuters due to its long access time and limited bus services near the lot, the optimal capacity tends to be equal to the effective lower bound.

(5) *It seems desirable to have a large-sized lot close to the downtown area as long as the budget and space allow. However, the lot may not gain a good utilization rate, possibly due to the long bus waiting time caused by congestion.* An example is lot 2 which is close to downtown Bellevue. As Table EC.2 shows, lot 2 has the longest average bus headway, i.e., the average inter-arrival time between two consecutive buses, which jeopardizes its attractiveness to morning commuters who are eager to arrive at their office as early as possible, and thus, results in a low utilization rate.

Optimal capacities with unused lots. We have thus far considered that all the seven candidate lots are opened with positive lower bounds on their capacities. Our model and algorithms are also applicable when the DOT is allowed to use only a subset of the lots. Let $\bar{J} \in \{1, 2, \ldots, 7\}$ denote the number of used lots. Figure EC.1 shows the optimal total social welfare as a function of $\bar{J}$ under different upper bounds (indicated by different u_1) but no lower bounds. We observe that: (i) Opening only one lot may be unable to accommodate the need of all commuters, particularly when the upper bound is not large enough, e.g., when $u_1 \in \{0.75, 0.85\}$; (ii) Opting to open only a portion of the lots may lead to a higher overall social welfare compared to utilizing all available lots. For example, the maximum total social welfare when $\bar{J} = 3$ is greater than that when $\bar{J} = 7$.

7.3. Sensitivity Analysis

In this section, we investigate how the changes in the sensitivity parameters α_k for $k \in \{1, 2, 3, 4\}$, β, and φ affect optimal capacities.

We choose $\ell_1 = 0.25$ and $u_1 = 0.75$. Then, the lower and upper bounds for all lots are shown in Table 2. Figures EC.2–EC.7 show the optimal capacity and traffic flow of each Bellevue's park-and-ride lot as functions of α_1, α_2, α_3, α_4, β, and φ, respectively, and Figure 3 shows the optimal utilization of each lot as a function of φ. In each figure, the optimal capacities, traffic flows, or utilizations are obtained under different values of a particular sensitivity parameter, while the values of all other sensitivity parameters are fixed and as specified in Section 7.2.

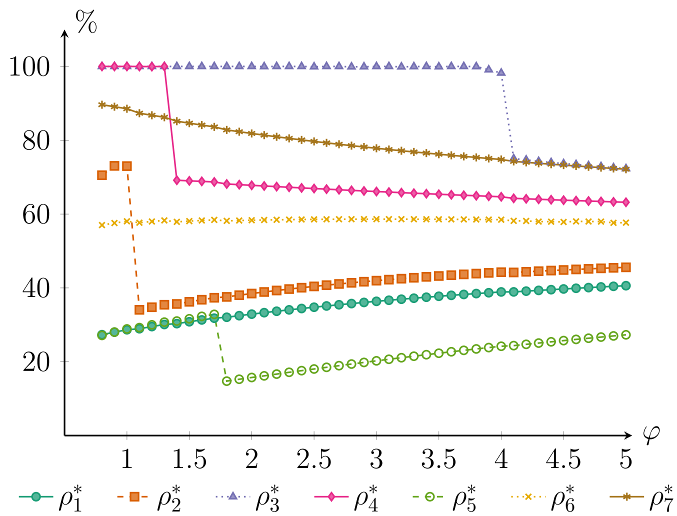


Figure 3 The optimal utilization ρ_j^* as a function of φ for all $j \in \{1, 2, \ldots, 7\}$.

We make the following observations from Figures EC.2–EC.7 and 3.

(1) *As commuters become more sensitive to a particular attribute, such as economic value, the number of bus routes, bus frequency, or congestion-free access time, the optimal traffic flows through the lots with large values of the attribute tends to be increasing, while those through the lots with small values of the attribute appear to be non-increasing.* For example, as α_2 increases, the optimal traffic flow through lot 7 is increasing, while that through any other lot is decreasing. With a larger α_2, commuters are more sensitive to the number of bus routes passing by a lot. Since there are 14 bus routes serving lot 7, which exceeds the numbers of bus routes passing through all other lots by at least nine, the lot 7 is likely to attract commuters from other lots as α_2 increases.

(2) *As commuters become more sensitive to congestion, it tends to reallocate commuters from lots with high traffic flows to those with low traffic flows under the optimal capacity plan.* For example, lot 7 has the largest traffic flow among all lots when commuters are congestion-insensitive, i.e., when $\beta = 0$. As β increases, the optimal traffic flow through lot 7 decreases, while that through any other lot increases.

(3) *As commuters become more sensitive to parking information, the utilization of a lot with high initial utilization tends to be non-increasing, while that of a lot with low initial utilization tends to be non-decreasing with possible jumps to lower values.* For example, the former result applies to lots 3, 4, and 7, while the latter result applies to lots 1, 2, 5, and 6.

(4) *For any lot, the optimal capacity seems to be non-decreasing in the optimal traffic flow.* This is likely due to the capacity constraint $q_j \leq C_j$ for all $j \in [J]$ in the model (Basic).

7.4. The Performance of the Optimal Capacity Plan Under Real-Time Parking Information

In this section, we use simulation to evaluate the performance of the optimal capacity plan for Bellevue's park-and-ride lots when commuters are provided with (and sensitive to) *real-time parking information*. To that end, we first introduce three capacity plans to be compared as follows:

$\mathbf{C}^*$: The DOT considers that commuters are information-, congestion-, and intrinsic-utility-aware, solves the model (Basic), and obtains the optimal capacity plan $\mathbf{C}^*$.

$\mathbf{C}^1$: (**The congestion effect is ignored**) The DOT considers that commuters are information- and intrinsic-utility-aware but congestion-unaware, which is equivalent to treating β as zero, solves the model (Basic) (with $\beta = 0$), and obtains the optimal solution as $\mathbf{C}^1$.

$\mathbf{C}^2$: (**The parking effect is ignored**) The DOT considers that commuters are congestion- and intrinsic-utility-aware but information-unaware, which is equivalent to presuming $\varphi = 0$. Then, the DOT solves the fixed-point equation (5) (with $\varphi = 0$) using Algorithm 1 and obtains the equilibrium flow pattern $\widetilde{\mathbf{q}} = (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_J)$. The DOT sets $\widetilde{C}_j := \min\{\max\{\ell_j, \widetilde{q}_j\}, u_j\}$ for all $j \in [J]$ and obtains the capacity plan $\mathbf{C}^2 := (\widetilde{C}_j, j \in [J])$.

Then, in Section EC.5.1, we define nine choice scenarios, describe the departure process of commuters, and build commuter choice models under real-time parking information and different choice scenarios. To evaluate the performance of the optimal capacity plan $\mathbf{C}^*$, we will compare the expected total social welfare generated by $\mathbf{C}^*$, $\mathbf{C}^1$, and $\mathbf{C}^2$ under the nine choice scenarios.

Next, we generate the departures and choices of commuters using simulation, and then calculate the expected total social welfare (i.e., $\Pi^m(\mathbf{C})$ in (EC.21)) using sample average approximations.

We set $T = 7,200$ seconds to account for commuters departing within 7–9 am on Mondays and $\kappa = 1$. Hence, $Q = \lfloor 7200\kappa \rfloor$. The values of $r_{j,k}$'s are as in Table EC.4 and those of α_k for $k \in \{1,2,3,4\}$, θ, β, and φ are as specified in Section 7.2. Similar to Section 7.2, the comparison results are obtained under different lower and upper bounds.

For each choice scenario $m \in \{1,2,\ldots,9\}$ and capacity plan $\mathbf{C} \in \{\mathbf{C}^*, \mathbf{C}^1, \mathbf{C}^2\}$, let $\bar{\Pi}^m(\mathbf{C})$ denote the sample average approximation of $\Pi^m(\mathbf{C})$ computed using $1,000$ sample paths of the joint departure and choice processes of commuters. Define the relative gap

$$\text{Gap}_i^m := \frac{\bar{\Pi}^m(\mathbf{C}^*) - \bar{\Pi}^m(\mathbf{C}^i)}{|\bar{\Pi}^m(\mathbf{C}^*)|} \times 100,$$

for each m and the *total relative gap* $\text{TG}_i := \sum_{m=1}^{9} \text{Gap}_i^m$, where $i \in \{1,2\}$. The total relative gap TG_i measures the performance of $\mathbf{C}^*$ versus $\mathbf{C}^i$ under an average choice scenario of commuters.

Table 4 $\bar{\Pi}^m(\mathbf{C})$, Gap_i^m, and TG_i for $i \in \{1,2\}$, $m \in \{1,2,\ldots,9\}$, and $\mathbf{C} \in \{\mathbf{C}^*, \mathbf{C}^1, \mathbf{C}^2\}$ when $\ell_1 \in \{0.05, 0.25, 0.5, 0.7\}$ and $u_1 \in \{0.75, 0.85\}$.

m	$\ell_1=0.05,\ u_1=0.75$					$\ell_1=0.05,\ u_1=0.85$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	61307	61662	58281	-0.58	4.94	62377	62770	58304	-0.63	6.53
2	61334	61671	58284	-0.55	4.97	62414	62799	58310	-0.62	6.57
3	55796	55643	52974	0.27	5.06	62680	62509	52975	0.27	15.48
4	46738	53124	49486	-13.66	-5.88	46743	54067	49453	-15.67	-5.80
5	9756	7241	8275	25.78	15.19	9802	7222	8304	26.32	15.28
6	55152	54882	52390	0.49	5.01	62087	61795	52391	0.47	15.62
7	55669	55528	52860	0.25	5.05	62568	62413	52861	0.25	15.51
8	9741	7215	8270	25.93	15.10	9786	7196	8299	26.47	15.19
9	21652	16119	16605	25.56	23.31	22263	16133	16634	27.53	25.29
TG_i	–	–	–	63.49	72.75	–	–	–	64.39	109.67

m	$\ell_1=0.25,\ u_1=0.75$					$\ell_1=0.25,\ u_1=0.85$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	61296	61425	59492	-0.21	2.94	62367	62500	59489	-0.21	4.61
2	61325	61450	59485	-0.20	3.00	62404	62536	59482	-0.21	4.68
3	55797	55765	53055	0.06	4.91	62681	62656	53056	0.04	15.36
4	46626	51293	49021	-10.01	-5.14	46631	52061	48991	-11.64	-5.06
5	9771	8545	8941	12.55	8.50	9817	8551	8970	12.90	8.63
6	55157	55098	52444	0.11	4.92	62093	62037	52445	0.09	15.54
7	55670	55640	52935	0.05	4.91	62569	62549	52935	0.03	15.40
8	9756	8519	8931	12.68	8.46	9802	8524	8959	13.03	8.59
9	21670	17753	17840	18.08	17.67	22279	17794	17869	20.13	19.80
TG_i	–	–	–	33.11	50.17	–	–	–	34.16	87.55

m	$\ell_1=0.5,\ u_1=0.75$					$\ell_1=0.5,\ u_1=0.85$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	61286	61322	60349	-0.06	1.53	62352	62396	60345	-0.07	3.22
2	61314	61353	60340	-0.06	1.59	62390	62435	60336	-0.07	3.29
3	55799	55786	53126	0.02	4.79	62684	62677	53126	0.01	15.25
4	46458	48797	48002	-5.04	-3.32	46442	49407	47967	-6.39	-3.28
5	9816	9510	9562	3.12	2.59	9879	9518	9590	3.65	2.92
6	55162	55147	52487	0.03	4.85	62101	62085	52488	0.02	15.48
7	55671	55659	53001	0.02	4.80	62571	62568	53001	0.00	15.29
8	9800	9478	9549	3.29	2.56	9863	9487	9578	3.81	2.89
9	21711	19854	19808	8.55	8.77	22337	19892	19836	10.95	11.19
TG_i	–	–	–	9.87	28.16	–	–	–	11.91	66.25

m	$\ell_1=0.7,\ u_1=0.75$					$\ell_1=0.7,\ u_1=0.85$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	61273	61279	60599	-0.01	1.10	62340	62353	60595	-0.02	2.80
2	61302	61312	60588	-0.02	1.16	62378	62394	60584	-0.03	2.87
3	55800	55792	53149	0.02	4.75	62686	62683	53149	0.01	15.21
4	46271	46694	46437	-0.91	-0.36	46255	47179	46398	-2.00	-0.31
5	9886	9884	9871	0.02	0.15	9949	9896	9900	0.53	0.50
6	55167	55159	52505	0.01	4.82	62106	62098	52506	0.01	15.46
7	55673	55665	53022	0.01	4.76	62573	62574	53023	0.00	15.26
8	9871	9851	9855	0.20	0.16	9934	9863	9883	0.71	0.51
9	21783	21489	21446	1.35	1.55	22407	21575	21474	3.72	4.16
TG_i	–	–	–	0.67	18.09	–	–	–	2.93	56.46

Table 4 shows $\bar{\Pi}^m(\mathbf{C})$, Gap_i^m, and TG_i for $i \in \{1,2\}$, $m \in \{1,2,\ldots,9\}$, and $\mathbf{C} \in \{\mathbf{C}^*, \mathbf{C}^1, \mathbf{C}^2\}$ when $\ell_1 \in \{0.05, 0.25, 0.5, 0.7\}$ and $u_1 \in \{0.75, 0.85\}$. Table EC.6 shows the same content as Table 4 except for $u_1 \in \{0.95, 1\}$.

The following observations are obtained from Tables 4 and EC.6:

(1) *The optimal capacity plan performs the best under an average choice scenario of commuters* compared with $\mathbf{C}^1$ and $\mathbf{C}^2$. This is justified by $\text{TG}_i > 0$ for $i \in \{1,2\}$.

(2) *The information-aware commuters may be better off if commuters are treated as congestion-unaware regardless of whether or not they are congestion-aware in reality.* Since $\text{Gap}_1^m < 0$ for $m \in \{1,2,4\}$, $\mathbf{C}^1$ outperforms $\mathbf{C}^*$ under these scenarios where commuters are information-aware but may be congestion-aware or not.

(3) *The optimal capacity plan may perform the best even when commuters are information-unaware.* This is justified by $\text{Gap}_i^m > 0$ for $i \in \{1,2\}$ and $m \in \{5,6,7,8,9\}$ where commuters are information-unaware.

(4) *The two capacity plans $\mathbf{C}^*$ and $\mathbf{C}^1$ perform closely with a negligible gap ($\leq 1\%$) when commuters base their decisions on attributes such as economic value, the number of buses, and bus frequency.* This observation is due to $|\text{Gap}_i^m| < 1\%$ for $m \in \{1,2,3,6,7\}$, where commuters are sensitive to the economic value, number of buses, and bus frequency for a lot.

(5) *The optimal capacity plan outperforms $\mathbf{C}^2$ under all but one choice scenarios.*

8. Conclusions

This work is concerned with optimal sizing of park-and-ride lots from the perspective of a public transit policymaker that attempts to determine the optimal capacities of a set of lots. The problem is formulated as a non-convex nonlinear optimization model with a fixed-point equation constraint reflecting the effects of congestion and parking information on commuter choice. A novel feature of the model is to involve information-aware commuters who make lot selections based on the published parking availability information.

We transform the model into a univariate optimization model with a single decision variable and a subproblem involving only interval constraints, allowing for tractable analysis to obtain insights into the structure of the optimal capacities. We show that except for at most one lot whose optimal capacity is strictly between the lower and upper bounds on the capacity, the optimal capacity of each lot is equal to one of the three values: the lower bound, the upper bound, and the optimal traffic flow through the lot. We develop a one-variable search algorithm to solve the model by leveraging the structural results.

We learn from the numerical results that the optimal capacity of a lot with a low or moderate-sized intrinsic utility and a small lower bound tends to be equal to its optimal traffic flow. However, when the lower bound is not small, the optimal capacity tends to equal its lower bound. The optimal capacity of a lot with a high intrinsic utility tends to be equal to the upper bound. We observe that if a lot has high utilization when commuters are weakly sensitive to parking information, as commuters become more sensitive, the utilization is non-increasing. If a lot has low utilization when customers are weakly sensitive to parking information, its utilization is non-decreasing, with possible jumps to lower levels as commuters care more about parking availability. We also learn that under real-time parking information, the optimal capacity plan obtained when both congestion and parking effects are considered in the model outperforms that obtained when either effect is ignored from the model.

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E-Companion

EC.1. Supplementary Results, Supporting Lemmas, and Proofs for Section 3

Proof of Remark 2. Taking logarithm on both sides of (4), we have

$$\log(q_j) = \log(1 - \|\mathbf{q}\|_1) + b_j - \beta q_j^\theta + \varphi(1 - q_j/C_j), \quad \forall j \in [J]. \tag{EC.1}$$

If $q_j = 0$ for some j, it follows from (EC.1) that $\|\mathbf{q}\|_1 = 1$, which with (EC.2) further indicates $q_j = 0$ for all j. Thus, we have $\|\mathbf{q}\|_1 = 0$, contradicting $\|\mathbf{q}\|_1 = 1$. Thus, $q_j > 0$ for all j. If $\|\mathbf{q}\|_1 = 1$, it again follows from (EC.2) that $q_j = 0$ for all j. Thus, we have $\|\mathbf{q}\|_1 = 0$, contradicting $\|\mathbf{q}\|_1 = 1$. Thus, $\mathbf{q} \in \mathcal{Q}$. Q.E.D.

Proof of Proposition 1. It follows from Remark 2 that

$$\log(1 - \|\mathbf{q}^i\|_1) = \beta(q_j^i)^\theta + \frac{\varphi}{C_j^i} q_j^i + \log(q_j^i) - b_j - \varphi, \quad \forall j \in [J], \ i \in \{1, 2\}. \tag{EC.2}$$

Assume by contradiction that $\|\mathbf{q}^1\|_1 > \|\mathbf{q}^2\|_1$, which yields $\log(1 - \|\mathbf{q}^1\|_1) < \log(1 - \|\mathbf{q}^2\|_1)$. It follows from (EC.2) that

$$\log(q_j^1) + \beta(q_j^1)^\theta + \varphi\frac{q_j^1}{C_j^1} < \log(q_j^2) + \beta(q_j^2)^\theta + \varphi\frac{q_j^2}{C_j^2}, \quad \forall\ j \in [J], \tag{EC.3}$$

which implies $q_j^1 < q_j^2$ for all j. To see this, suppose that $q_j^1 \geq q_j^2$ for some j. Since $\mathbf{C}^1 \leq \mathbf{C}^2$,

$$\log(q_j^1) + \beta(q_j^1)^\theta + \varphi\frac{q_j^1}{C_j^1} \geq \log(q_j^2) + \beta(q_j^2)^\theta + \varphi\frac{q_j^2}{C_j^1} \geq \log(q_j^2) + \beta(q_j^2)^\theta + \varphi\frac{q_j^2}{C_j^2},$$

contradicting (EC.3). Since $q_j^1 < q_j^2$ for all j, it follows that $\|\mathbf{q}^1\|_1 < \|\mathbf{q}^2\|_1$, contradicting $\|\mathbf{q}^1\|_1 > \|\mathbf{q}^2\|_1$. This proves the desired result. Q.E.D.

Corollary EC.1. *Consider capacity plans $\mathbf{C}^1, \mathbf{C}^2 \in (0, \infty]^J$. There exists some $k \in [J]$ such that $C_k^1 < C_k^2$ and $C_j^1 = C_j^2$ for all $j \in [J]$ and $j \neq k$. Let $\mathbf{q}^i := (q_j^i, j \in [J])$ denote the flow pattern satisfying $\mathbf{q}^i = \mathbf{F}(\mathbf{q}^i, \mathbf{C}^i)$ for $i \in \{1, 2\}$. Then, it holds that $\|\mathbf{q}^1\|_1 < \|\mathbf{q}^2\|_1$, $q_k^2 > q_k^1$, and $q_j^2 < q_j^1$ for all $j \in [J]$ and $j \neq k$.*

Proof. It follows from Proposition 1 and Remark 1 that $\|\mathbf{q}^1\|_1 < \|\mathbf{q}^2\|_1$. Since $C_j^1 = C_j^2$ for all $j \in [J]$ and $j \neq k$, it follows from (EC.2) and $\|\mathbf{q}^1\|_1 < \|\mathbf{q}^2\|_1$ that $q_j^1 > q_j^2$ for $j \neq k$. Since $\|\mathbf{q}^1\|_1 = q_k^1 + \sum_{j \neq k} q_j^1 < \|\mathbf{q}^2\|_1 = q_k^2 + \sum_{j \neq k} q_j^2$, implying $q_k^2 > q_k^1$. Q.E.D.

Corollary EC.1 suggests that if the capacity of a park-and-ride lot becomes larger while the capacities of all other lots do not change, then at equilibrium, more commuters will choose to use that lot but any other park-and-ride lot will attract fewer commuters.

Lemma EC.1 will be useful, which follows from Stewart (2005).

LEMMA EC.1. *The following holds:*

(1) $W(x) \geq 0$ *for all* $x \geq 0$ *and* $W(x) > 0$ *for all* $x > 0$.

(2) $W(x)$ *is a strictly monotone increasing continuously differentiable function over* $x \in \mathbb{R}_+$.

Proof of Proposition 2. First, we show that h is strictly convex over $\mathcal{Q}$. We have that

$$h(\mathbf{q}) = \sum_{j=1}^{J} q_j \log(q_j) - \log\left(1 - \sum_{k=1}^{J} q_k\right)\left(\sum_{j=1}^{J} q_j\right),$$

and for each $j \in [J]$,

$$\begin{aligned}\frac{\partial h(\mathbf{q})}{\partial q_j} &= \log(q_j) + 1 + \frac{\sum_{j=1}^{J} q_j}{1 - \sum_{k=1}^{J} q_k} - \log\left(1 - \sum_{k=1}^{J} q_k\right) \\ &= \log(q_j) + \frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}} + \log\left(\frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}}\right).\end{aligned} \tag{EC.4}$$

Thus, the Hessian of h is represented as,

$$\nabla^2 h(\mathbf{q}) = \operatorname{diag}\left\{\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_J}\right\} + \mathbf{e}\mathbf{e}^\intercal \left(\frac{1}{(1 - \mathbf{e}^\intercal \mathbf{q})^2} + \frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}}\right).$$

Consider any $\mathbf{x} \in \mathbb{R}^J$ and $\mathbf{x} \neq 0$. It follows that

$$\begin{aligned}\mathbf{x}^\intercal \nabla^2 h(\mathbf{q}) \mathbf{x} &= \sum_{j=1}^{J} \frac{x_j^2}{q_j} + \mathbf{x}^\intercal \mathbf{e}\mathbf{e}^\intercal \mathbf{x} \left(\frac{1}{(1 - \mathbf{e}^\intercal \mathbf{q})^2} + \frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}}\right) \\ &= \sum_{j=1}^{J} \frac{x_j^2}{q_j} + \left(\sum_{j=1}^{J} x_j\right)^2 \left(\frac{1}{(1 - \mathbf{e}^\intercal \mathbf{q})^2} + \frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}}\right) > 0,\end{aligned}$$

for all $\mathbf{q} \in \mathcal{Q}$, implying that $\nabla^2 h(\mathbf{q})$ is positive-definite and that h is strictly convex over $\mathcal{Q}$. Thus, h attains its unique minimum.

Second, we show that the vector $\mathbf{q}^\dagger := (q_1^\dagger, q_2^\dagger, \ldots, q_J^\dagger)$ satisfying $\nabla h(\mathbf{q}^\dagger) = 0$ is in $\mathcal{Q}$. Note that for any j, $\partial h(\mathbf{q})/\partial q_j$ is strictly increasing in q_j with q_k fixed for all $k \neq j$. Moreover, we have that

$$\lim_{q_j \to 0} \frac{\partial h(\mathbf{q})}{\partial q_j} = -\infty, \quad \lim_{\mathbf{e}^\intercal \mathbf{q} \to 1} \frac{\partial h(\mathbf{q})}{\partial q_j} = \infty.$$

Thus, any $\mathbf{q}^\dagger$ satisfying $\nabla h(\mathbf{q}^\dagger) = 0$ is in $\mathcal{Q}$ and $\mathbf{q}^\dagger$ is the unique minimizer of $h(\mathbf{q})$ according to the strict convexity of h.

Third, we show that $q_1^\dagger = q_2^\dagger = \cdots = q_J^\dagger$. It follows from (EC.4) that

$$\log(q_j^\dagger) = \log(q_k^\dagger) = -\left[\frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}^\dagger} + \log\left(\frac{1}{1 - \mathbf{e}^\intercal \mathbf{q}^\dagger}\right)\right]$$

for any j, k. Since $\log(x)$ is monotone strictly increasing, we have that $q_j^\dagger = q_k^\dagger$ and $q_1^\dagger = q_2^\dagger = \cdots q_J^\dagger$.

Define $g(q) := Jq \log[q/(1 - Jq)]$. Note that $g(q) = h(\mathbf{q})$ for all $q \in (0, 1/J)$ and $\mathbf{q} = (q, q, \ldots, q)$. Since h is strictly convex over $\mathcal{Q}$, it follows that $g(\lambda q^1 + (1 - \lambda) q^2) = h(\lambda \mathbf{q}^1 + (1 - \lambda)\mathbf{q}^2) < \lambda h(\mathbf{q}^1) +$

$(1-\lambda)\mathbf{q}^2 = \lambda g(q^1) + (1-\lambda)g(q^2)$ for all $q^1, q^2 \in (0, 1/J)$ and $\lambda \in [0,1]$, where $\mathbf{q}^i = (q^i, q^i, \ldots, q^i)$ for $i \in \{1,2\}$. Thus, g is strictly convex.

By setting $g'(\widehat{q}) = 0$, we have

$$\log \frac{\widehat{q}}{1-J\widehat{q}} + \frac{1}{1-J\widehat{q}} = 0,$$

which is equivalent to

$$\log\left(\frac{J\widehat{q}}{1-J\widehat{q}}\right) + \frac{J\widehat{q}}{1-J\widehat{q}} = \log\left(J/e\right). \tag{EC.5}$$

It follows from the definition of the Lambert-W function $W(x)$ and (EC.5) that

$$\frac{J\widehat{q}}{1-J\widehat{q}} = W\left(J/e\right),$$

implying $\widehat{q} = \frac{W(J/e)}{W(J/e)+1} \cdot \frac{1}{J}$ and

$$\log\left(W\left(J/e\right)\right) + W\left(J/e\right) = \log\left(J/e\right), \tag{EC.6}$$

which further yields that

$$\log\left(\frac{\widehat{q}}{1-J\widehat{q}}\right) = \log\left(\frac{W(J/e)}{J}\right) = \log\left(W(J/e)\right) - \log J = -W(J/e) - 1 \tag{EC.7}$$

It follows from (EC.7) that

$$\begin{aligned} g(\widehat{q}) &= J\widehat{q}\log\frac{\widehat{q}}{1-J\widehat{q}} \\ &= J\frac{W(J/e)}{W(J/e)+1}\frac{1}{J}\cdot(-W(J/e)-1) \\ &= -W(J/e). \end{aligned}$$

Note that $\widehat{q} \in (0, 1/J)$. Since g is strictly convex, $\widehat{q}$ is the unique minimizer of $g(q)$, i.e., $g(q) > g(\widehat{q})$ for all $q \in (0, 1/J)$ and $q \neq \widehat{q}$.

Define $q^\dagger := q_1^\dagger = q_2^\dagger = \cdots = q_J^\dagger$. Therefore, $\mathbf{q}^\dagger = (q^\dagger, q^\dagger, \ldots, q^\dagger)$. Since $\mathbf{q}^\dagger$ is the unique minimizer of h, we have that $h(\mathbf{q}) > h(\mathbf{q}^\dagger)$ for all $\mathbf{q} \in \mathcal{Q}$ and $\mathbf{q} \neq \mathbf{q}^\dagger$. Assume by contradiction that $\widehat{q} \neq q^\dagger$. Define $\widehat{\mathbf{q}} := (\widehat{q}, \widehat{q}, \ldots, \widehat{q})$. Thus, $\widehat{\mathbf{q}} \neq \mathbf{q}^\dagger$ and $h(\widehat{\mathbf{q}}) = g(\widehat{q}) > h(\mathbf{q}^\dagger) = g(q^\dagger)$, contradicting $\widehat{q}$ being the unique minimizer of $g(q)$. Therefore, we have $q^\dagger = \widehat{q} = \frac{W(J/e)}{W(J/e)+1} \cdot \frac{1}{J}$ and $h(\mathbf{q}^\dagger) = g(q^\dagger) = g(\widehat{q}) = -W(J/e)$. This completes the proof. Q.E.D.

EC.2. Proofs for Section 4

Proof of Lemma 1. First, $\boldsymbol{\xi}$ is continuous since $y_j(\mathbf{q})$ is continuous for all $j \in [J]$. Then, we show that $\boldsymbol{\xi}$ is surjective. For any given capacity plan $\mathbf{z} \in \mathbb{R}_{++}^J$, it follows from Remark 1 that there exists a solution $\mathbf{q} \in \mathcal{Q}$ that satisfies the fixed-point equation (5). Then, it follows from Remark 2 that $\mathbf{q}$ satisfies (9), which is equivalent to $\boldsymbol{\xi}(\mathbf{q}) = \mathbf{z}$, indicating that $\boldsymbol{\xi}$ is surjective.

Next, we show that $\boldsymbol{\xi}$ is injective. Consider any two points $\mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}^J_{++}$ such that $\mathbf{z}^1 = \mathbf{z}^2$, where $\mathbf{z}^i := (z^i_1, z^i_2, \dots, z^i_J)$. We represent $\mathbf{z} := \mathbf{z}_1 = \mathbf{z}_2$. Consider $\mathbf{q}^i = (q^i_1, q^i_2, \dots, q^i_J) \in \mathcal{Q}$ such that $\boldsymbol{\xi}(\mathbf{q}^i) = \mathbf{z}^i$ for $i \in \{1,2\}$. Thus, we have from the definition of $\boldsymbol{\xi}$ that

$$z_j = \frac{\varphi q^i_j}{y_j(\mathbf{q})} \quad \forall\, i \in \{1,2\},\ j \in [J].$$

which implies that both $\mathbf{q}^1$ and $\mathbf{q}^2$ satisfy (9) for the same $\mathbf{z}$ and satisfy the fixed-point equation (5) for the same given $\mathbf{z}$. We have from Remark 1 again that the fixed-point equation (5) has a unique solution. Thus, we have $\mathbf{q}^1 = \mathbf{q}^2$, showing that $\boldsymbol{\xi}$ is injective. Thus $\boldsymbol{\xi}$ is a continuous bijection and $\boldsymbol{\xi}^{-1}$ exists. Q.E.D.

Proof of Proposition 3. If $\mathcal{S}$ is empty, $\mathcal{S}$ is trivially compact. We consider $\mathcal{S} \neq \varnothing$ from now on.

Note that $\mathcal{Q}$ is bounded. Since $\mathcal{S}$ is a subset of $\mathcal{Q}$, $\mathcal{S}$ is also bounded. It remains to show that $\mathcal{S}$ is closed. Consider any given convergent sequence $\{\mathbf{q}^k\}_{k=1}^{\infty} \subset \mathcal{S}$ such that $\mathbf{q}^k \to \hat{\mathbf{q}} = (\hat{q}_j, j \in [J])$ as $k \to \infty$. Let $\bar{\mathcal{Q}}$ denote the closure of $\mathcal{Q}$. Since $\mathcal{S} \subset \mathcal{Q} \subset \bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}$ is a closed set, we have $\hat{\mathbf{q}} \in \bar{\mathcal{Q}}$. Since $\boldsymbol{\xi}$ is continuous according to Lemma 1, it follows that $\boldsymbol{\xi}(\hat{\mathbf{q}}) = \lim_{k\to\infty} \boldsymbol{\xi}(\mathbf{q}^k) \in \prod_{j=1}^J [\ell_j, u_j]$ and $\boldsymbol{\xi}(\hat{\mathbf{q}}) \geq \hat{\mathbf{q}}$.

Next, we show that $\hat{\mathbf{q}} \in \mathcal{Q}$. Since $\hat{\mathbf{q}} \in \bar{\mathcal{Q}}$, we consider the following two cases.

Case 1. $\sum_{k=1}^J \hat{q}_k = 1$. Since $\boldsymbol{\xi}(\hat{\mathbf{q}}) \in \prod_{j=1}^J [\ell_j, u_j]$ and $\ell_j > 0$ for all $j \in [J]$, it follows from the definition of $\boldsymbol{\xi}$ that

$$b_j + \varphi - \beta(\hat{q}_j)^\theta + \log\left(1 - \sum_{k=1}^J q_k\right) = \frac{\varphi \hat{q}_j}{\xi_j(\hat{\mathbf{q}})} + \log(\hat{q}_j), \tag{EC.8}$$

holds for all j, yielding that $\hat{q}_j = 0$ for all j. It thus follows $\sum_{k=1}^J \hat{q}_k = 0$, contradicting $\sum_{k=1}^J \hat{q}_k = 1$.

Case 2. $\sum_{k=1}^J \hat{q}_k \neq 1$ and there exists some $j \in [J]$ such that $\hat{q}_j = 0$. Fix such j. Thus, the left-hand side of (EC.8) is finite but the right-hand side of (EC.8) is equal to $-\infty$, arriving at a contradiction.

Combining $\hat{\mathbf{q}} \in \bar{\mathcal{Q}}$ with the results from the two cases gives that $\sum_{k=1}^J \hat{q}_k \neq 1$ and $\hat{\mathbf{q}} > 0$, indicating that $\hat{\mathbf{q}} \in \mathcal{Q}$. It follows from $\boldsymbol{\xi}(\hat{\mathbf{q}}) \in \prod_{j=1}^J [\ell_j, u_j]$, $\boldsymbol{\xi}(\hat{\mathbf{q}}) \geq \hat{\mathbf{q}}$, and $\hat{\mathbf{q}} \in \mathcal{Q}$ that $\hat{\mathbf{q}} \in \mathcal{S}$, implying that $\mathcal{S}$ is a closed set. Thus, $\mathcal{S}$ is compact.

Since h is continuous and $\mathcal{S}$ is a nonempty compact set, h attains its maximum and minimum in $\mathcal{S}$. Thus, there exists an optimal solution in $\mathcal{S}$ that solves the model (Flow). Q.E.D.

EC.3. Supporting Lemmas and Proofs for Section 5

Proof of Proposition 4. Due to the optimality of $\mathbf{q}^* = (q^*_j, j \in [J])$ to the model (Flow), we have $\mathbf{q}^* \in \mathcal{Q}$, which indicates $q^*_j > 0$ for all $j \in [J]$ and $z^* = \|\mathbf{q}^*\| \in (0,1)$.

First, we show that $\mathbf{q}^*$ is optimal to the model (Sub) with parameter $z = z^* \in (0,1)$. Assume by contradiction that there exists $\mathbf{q} \neq \mathbf{q}^*$ that is feasible to the model (Sub) and $\sum_{j=1}^J q_j \log(q_j) > \sum_{j=1}^J q^*_j \log(q^*_j)$. Since $\mathbf{q}$ is feasible to the model (Sub), we have $z^* = \|\mathbf{q}\|_1$. It follows from Remark 4 that $\mathbf{q}$ satisfies (14)–(16). Note $q_j > 0$ for all j; $\sum_{j=1}^J q_j \log(q_j) > \sum_{j=1}^J q_j^* \log(q_j^*)$ is violated otherwise. Since $\|\mathbf{q}\|_1 = z^* < 1$, we have $\mathbf{q} \in \mathcal{Q}$, implying $\mathbf{q}$ is feasible to the model (Flow). Therefore, we have

$$h(\mathbf{q}) = \sum_{j=1}^J q_j \log(q_j) - z^* \log(1-z^*) > \sum_{j=1}^J q_j^* \log(q_j^*) - z^* \log(1-z^*) = h(\mathbf{q}^*),$$

contradicting the optimality of $\mathbf{q}^*$ to the model (Flow). Thus, $\mathbf{q}^*$ is optimal to the model (Sub) with parameter $z = z^*$.

Next, we show that z^* is optimal to the model (Univar). Since $\mathbf{q}^*$ is optimal to the model (Sub) with parameter $z = z^*$, we have $\widehat{h}(z^*) = \sum_{j=1}^J q_j^* \log(q_j^*)$. Consider any $z \in (0,1)$ and $\mathbf{q}$ that is optimal to the model (Sub) with parameter z. Then, we have $z = \|\mathbf{q}\|_1$. It follows from Remark 4 that $\mathbf{q}$ is feasible to the model (Flow). Then, it follows that

$$\begin{aligned}\pi(z) = \widehat{h}(z) - z\log(1-z) &= \sum_{j=1}^J q_j \log(q_j) - z\log(1-z) = h(\mathbf{q}) \\ &\le h(\mathbf{q}^*) = \sum_{j=1}^J q_j^* \log(q_j^*) - z^* \log(1-z^*) \\ &= \widehat{h}(z^*) - z^* \log(1-z^*) = \pi(z^*),\end{aligned}$$

indicating that z^* is optimal to the model (Univar).

Since $\widehat{\mathbf{q}}$ is optimal to the model (Sub), we have from Remark 4 that $\widehat{\mathbf{q}}$ is feasible to the model (Flow) and $\widehat{z} = \|\widehat{\mathbf{q}}\|_1$. For any $\mathbf{q}$ feasible to the model (Flow), define $z := \|\mathbf{q}\|_1$. We then have from Remark 4 that $\mathbf{q}$ is feasible to the model (Sub) with parameter z, which implies

$$h(\mathbf{q}) = \sum_{j=1}^J q_j \log(q_j) - z\log(1-z) \le \widehat{h}(z) - z\log(1-z) \le \widehat{h}(\widehat{z}) - \widehat{z}\log(1-\widehat{z}) = h(\widehat{\mathbf{q}}),$$

implying that $\widehat{\mathbf{q}}$ is optimal to the model (Flow), where the first inequality follows from the feasibility of $\mathbf{q}$ to the model (Sub) and the second inequality follows from the optimality of $\widehat{z}$ to the model (Univar). Q.E.D.

Lemma EC.2 will be useful.

Lemma EC.2. *Consider $f(x) := \beta x^\theta + \log(x) + Ax - B$ or $f(x) := \beta x^\theta + \log(x) + A - B$, where $A \in \mathbb{R}_{++}$, $B \in \mathbb{R}$, and $x \in [0,1]$. Then, there exists $x \in (0,1]$ such that $f(x) \ge 0$ if and only if there exists a unique $x^* \in (0,1]$ such that $f(x^*) = 0$ and if and only if $\beta + A \ge B$.*

Proof. Suppose that there exists $x_0 \in (0,1]$ such that $f(x) \ge 0$. Since $f(x)$ is strictly increasing and continuous over $x \in [0,1]$ and note that $f(0) = -\infty$. Thus, there exists a unique $x^* \in (0, x_0] \subset (0,1]$ such that $f(x^*) = 0$. Suppose that there $x^* \in (0,1]$ such that $f(x^*) = 0$. Since $f(x)$ is increasing in

x, we have that $0 = f(x^*) \le f(1) = \beta + A - B$, implying $B \le \beta + A$. Suppose that $\beta + A \ge B$. Then, $f(1) = \beta + A - B \ge 0$, implying that there exists $x = 1 \in (0,1]$ such that $f(x) \ge 0$. Q.E.D.

Lemma EC.3 compares the solutions to $\zeta_{j,1}(x,z) = 0$, $\zeta_{j,2}(x,z) = 0$, and $\zeta_{j,3}(x,z) = 0$ for any $z \in (0,1)$ and $j \in [J]$, which follows from Lemma EC.2.

Lemma EC.3. *Consider any $z \in (0,1)$ and $j \in [J]$. Suppose that there exists $x_i \in (0,1]$ such that $\zeta_{j,i}(x_i, z) = 0$ for $i \in \{1,2,3\}$. Then, the following holds:*

(1) $x_2 < x_3$,

(2) $x_2 > \ell_j$ ($x_2 = \ell_j$) if and only if $x_2 < x_1$ ($x_2 = x_1$),

(3) $x_3 > u_j$ ($x_3 = u_j$) if and only if $x_3 < x_1$ ($x_3 = x_1$).

Proof. Fix z and j. Since $\zeta_{j,i}(x_i, z) = 0$ for $i \in 2,3$, it holds that

$$\beta x_2^\theta + \log(x_2) + \frac{\varphi x_2}{\ell_j} = \beta x_3^\theta + \log(x_3) + \frac{\varphi x_3}{u_j} = \log(1-z) + b_j + \varphi,$$

implying that $x_2 < x_3$ since $\ell_j < u_j$. We only show Result (2) and Result (3) can be proved by a similar argument. Drop argument z and subscript j from $\zeta_{j,i}$ for $i \in \{1,2,3\}$ for simplicity. Since $\zeta_1(x_1) = \zeta_2(x_2) = 0$, we have that

$$\beta x_1^\theta + \log(x_1) - [\beta x_2^\theta + \log(x_2)] = \frac{\varphi x_2}{\ell} - \varphi. \tag{EC.9}$$

Suppose that $x_2 > \ell$ ($x_2 = \ell$). Then, it follows from (EC.9) that

$$\beta x_1^\theta + \log(x_1) - [\beta x_2^\theta + \log(x_2)] = \frac{\varphi x_2}{\ell} - \varphi > 0 (= 0),$$

yielding $x_1 > x_2$ ($x_1 = x_2$). Now, suppose that $x_2 < x_1$ ($x_2 = x_1$). Then, it follows from (EC.9) that

$$\beta x_1^\theta + \log(x_1) - [\beta x_2^\theta + \log(x_2)] = \frac{\varphi x_2}{\ell} - \varphi > 0 (= 0),$$

yielding that $x_2 > \ell$ ($x_2 = \ell$) This completes the proof. Q.E.D.

Proposition EC.1 characterizes $\mathcal{Q}_j(z)$ based on Lemmas EC.2 and EC.3.

Proposition EC.1. *For any $z \in (0,1)$ and $j \in [J]$, the following cases hold:*

Case 1. *$u_j > \ell_j \ge 1$.*

(1) If $\log(1-z) + b_j + \varphi \in (\beta + \varphi/\ell_j, \infty)$, then $\mathcal{Q}_j(z) = \varnothing$.

(2) If $\log(1-z) + b_j + \varphi \in (\beta + \varphi/u_j, \beta + \varphi/\ell_j]$, then there exists a unique $x_i^* \in (0,1]$ such that $\zeta_{j,i}(x_i^*, z) = 0$ for $i \in \{1,2\}$, $x_1^* \le x_2^*$, and $\mathcal{Q}_j(z) = [x_2^*, 1]$.

(3) If $\log(1-z) + b_j + \varphi \in (-\infty, \beta + \varphi/u_j]$, then there exists a unique $x_i^* \in (0,1]$ such that $\zeta_{j,i}(x_i^*, z) = 0$ for $i \in \{1,2,3\}$, $x_1^* \le x_2^* < x_3^*$, and $\mathcal{Q}_j(z) = [x_2^*, x_3^*]$.

Case 2. $u_j \geq 1 > \ell_j$.

(1) If $\log(1-z)+b_j+\varphi \in (\beta+\varphi,\infty)$, *then* $\mathcal{Q}_j(z)=\varnothing$.

(2) If $\log(1-z)+b_j+\varphi \in (\beta+\varphi/u_j, \beta+\varphi]$, *then there exists a unique* $x_i^* \in (0,1]$ *such that* $\zeta_{j,i}(x_i^*,z)=0$ *for* $i\in\{1,2\}$ *and,*

(i) If $x_2^* \leq \ell_j$, *then* $x_1^* \leq x_2^*$ *and* $\mathcal{Q}_j(z)=[x_2^*,1]$.

(ii) If $x_2^* > \ell_j$, *then* $x_1^* > x_2^*$ *and* $\mathcal{Q}_j(z)=[x_1^*,1]$.

(3) If $\log(1-z)+b_j+\varphi \in (-\infty, \beta+\varphi/u_j]$, *then there exists a unique* $x_i^* \in (0,1]$ *such that* $\zeta_{j,i}(x_k^*,z)=0$ *for* $i\in\{1,2,3\}$, $x_2^* < x_3^*$, *and*

(i) If $x_2^* \leq \ell_j$, *then* $x_1^* \leq x_2^*$ *and* $\mathcal{Q}_j(z)=[x_2^*,x_3^*]$.

(ii) If $x_2^* > \ell_j$, *then* $x_1^* > x_2^*$ *and* $\mathcal{Q}_j(z)=[x_1^*,x_3^*]$.

Case 3. $1 > u_j > \ell_j$.

(1) If $\log(1-z)+b_j+\varphi \in (\beta+\varphi,\infty)$, *then* $\mathcal{Q}_j(z)=\varnothing$.

(2) If $\log(1-z)+b_j+\varphi \in (-\infty,\beta+\varphi]$, *then there exists a unique* $x_i^* \in (0,1]$ *such that* $\zeta_{j,i}(x_k^*,z)=0$ *for* $i\in\{1,2,3\}$ *and* $x_2^* < x_3^*$. *Furthermore, if* $x_3^* > u_j$, $\mathcal{Q}_j(z)=\varnothing$; *otherwise,*

(i) If $x_2^* \leq \ell_j$, *then* $x_1^* \leq x_2^*$ *and* $\mathcal{Q}_j(z)=[x_2^*,x_3^*]$.

(ii) If $x_2^* > \ell_j$, *then* $x_1^* > x_2^*$ *and* $\mathcal{Q}_j(z)=[x_1^*,x_3^*]$.

Proof. We drop argument z from functions $\zeta_{j,1}$, $\zeta_{j,2}$, and $\zeta_{j,3}$, and drop subscript j for simplicity. We consider the following three cases.

Case 1. $u > \ell \geq 1$. We have that $\beta+\varphi/u < \beta+\varphi/\ell \leq \beta+\varphi$. We have the following cases:

(1) Consider $\log(1-z)+b+\varphi \in (\beta+\varphi/\ell,\infty)$. It follows from Lemma EC.2 that there exists no $x\in(0,1]$ such that $\zeta_2(x)\geq 0$, implying that $\mathcal{Q}(z)=\varnothing$.

(2) Consider $\log(1-z)+b+\varphi \in (\beta+\varphi/u,\beta+\varphi/\ell]$. Thus, we have $\log(1-z)+b+\varphi \leq \beta+\varphi/\ell \leq \beta+\varphi$. It follows from Lemma EC.2 that there exist a unique $x_i^*\in(0,1]$ that $\zeta_i(x_i^*)=0$ for $i\in\{1,2\}$. Note that ζ_i is strictly increasing for $i\in\{1,2,3\}$. Then, $\zeta_i(x)\geq 0$ if and only if $x\in[x_i^*,1]$ for $i\in\{1,2\}$. Also note that for all $x\in[0,1]$, since ζ_3 is strictly increasing, we have $\zeta_3(x)\leq\zeta_3(1)=\beta+\varphi/u-(\log(1-z)+b+\varphi)\leq 0$. Thus, $\mathcal{Q}(z)=[x_1^*,1]\cap[x_2^*,1]\cap[0,1]$.

Since $x_2^*\leq 1\leq \ell$, it follows from Lemma EC.3(2) that $x_2^*\geq x_1^*$, implying that $\mathcal{Q}(z)=[x_2^*,1]$.

(3) Consider $\log(1-z)+b+\varphi\in(-\infty,\beta+\varphi/u]$. Thus, we have $\log(1-z)+b+\varphi\leq\beta+\varphi/u < \beta+\varphi/\ell\leq\beta+\varphi$. It follows from Lemma EC.2 that there exist a unique $x_i^*\in(0,1]$ that $\zeta_i(x_i^*)=0$ for $i\in\{1,2,3\}$. Since $x_i\leq 1\leq\ell<u$ for all $i\in 1,2$, it follows from Lemma EC.3 that $x_2^*<x_3^*$, $x_2^*\geq x_1^*$, and $x_3^*\geq x_1^*$, implying that $x_1^*\leq x_2^*<x_3^*$. Due to the strict increasingness and continuity of ζ_i for all $i\in\{1,2,3\}$, we have that $\mathcal{Q}(z)=[x_1^*,1]\cap[x_2^*,1]\cap[0,x_3^*]=[x_2^*,x_3^*]$.

Case 2. $u\geq 1>\ell$. We have that $\beta+\varphi/u\leq\beta+\varphi<\beta+\varphi/\ell$. We have the following cases:

(1) Consider $\log(1-z)+b+\varphi\in(\beta+\varphi,\infty)$. It follows from Lemma EC.2 that there exists no $x\in(0,1]$ such that $\zeta_1(x)\geq 0$, implying that $\mathcal{Q}(z)=\varnothing$.

(2) Consider $\log(1-z)+b+\varphi \in (\beta+\varphi/u, \beta+\varphi]$. Thus, we have $\log(1-z)+b+\varphi \leq \beta+\varphi < \beta+\varphi/\ell$. It follows from Lemma EC.2 that there exist a unique $x_i^* \in (0,1]$ that $\zeta_i(x_i^*)=0$ for $i \in \{1,2\}$. Note that ζ_i is strictly increasing and continuous for $i \in \{1,2,3\}$. Then, $\zeta_i(x) \geq 0$ if and only if $x \in [x_i^*, 1]$ for $i \in \{1,2\}$. Also note that for all $x \in [0,1]$, since ζ_3 is strictly increasing, we have $\zeta_3(x) \leq \zeta_3(1) = \beta+\varphi/u - (\log(1-z)+b+\varphi) \leq 0$. Thus, $\mathcal{Q}(z) = [x_1^*, 1] \cap [x_2^*, 1] \cap [0,1]$. Then, it follows from Lemma EC.3(2) that,

  (i) if $x_2^* \leq \ell$, we have $x_2^* \geq x_1^*$, implying that $\mathcal{Q}(z) = [x_2^*, 1]$,

  (ii) if $x_2^* > \ell$, we have $x_2^* < x_1^*$, implying that $\mathcal{Q}(z) = [x_1^*, 1]$.

(3) Consider $\log(1-z)+b+\varphi \in (-\infty, \beta+\varphi/u]$. Thus, we have $\log(1-z)+b+\varphi \leq \beta+\varphi/u < \beta+\varphi \leq \beta+\varphi/\ell$. It follows from Lemma EC.2 that there exist a unique $x_i^* \in (0,1]$ that $\zeta_i(x_i^*)=0$ for $i \in \{1,2,3\}$. Note that ζ_i is strictly increasing and continuous for $i \in \{1,2,3\}$. Thus, we have $\mathcal{Q}(z) = [x_1^*, 1] \cap [x_2^*, 1] \cap [0, x_3^*]$. Since $x_3 \leq 1 \leq u$, it follows from Lemma EC.3 that $x_2^* < x_3^*$ and $x_3^* \geq x_1^*$. It again follows from Lemma EC.3(2) that,

  (i) if $x_2^* \leq \ell$, we have $x_2^* \geq x_1^*$, implying that $\mathcal{Q}(z) = [x_2^*, x_3^*]$,

  (ii) if $x_2^* > \ell$, we have $x_2^* < x_1^*$, implying that $\mathcal{Q}(z) = [x_1^*, x_3^*]$.

Case 3. $1 > u > \ell$. We have that $\beta+\varphi < \beta+\varphi/u < \beta+\varphi/\ell$. We have the following cases:

(1) Consider $\log(1-z)+b+\varphi \in (\beta+\varphi, \infty)$. It follows from Lemma EC.2 that there exists no $x \in (0,1]$ such that $\zeta_1(x) \geq 0$, implying that $\mathcal{Q}(z) = \varnothing$.

(2) Consider $\log(1-z)+b+\varphi \in (-\infty, \beta+\varphi]$. Thus, we have $\log(1-z)+b+\varphi \leq \beta+\varphi < \beta+\varphi/u < \beta+\varphi/\ell$. It follows from Lemma EC.2 that there exist a unique $x_i^* \in (0,1]$ that $\zeta_i(x_i^*)=0$ for $i \in \{1,2,3\}$. Note that ζ_i is strictly increasing and continuous for $i \in \{1,2,3\}$. Thus, we have $\mathcal{Q}(z) = [x_1^*, 1] \cap [x_2^*, 1] \cap [0, x_3^*]$. If $x_3^* > u$, it follows from Lemma EC.3(3) that $x_3^* < x_1^*$, which indicates that $\mathcal{Q}(z) = \varnothing$; otherwise, it again follows from Lemma EC.3 that $x_3^* \geq x_1^*$, $x_2^* < x_3^*$, and

  (i) if $x_2^* \leq \ell$, we have $x_2^* \geq x_1^*$, implying that $\mathcal{Q}(z) = [x_2^*, x_3^*]$,

  (ii) if $x_2^* > \ell$, we have $x_2^* < x_1^*$, implying that $\mathcal{Q}(z) = [x_1^*, x_3^*]$. Q.E.D.

Proof of Theorem 1. Drop z from $\mathbf{q}^*(z)$, $q_j^L(z)$, and $q_j^H(z)$ for all $j \in [J]$ for notational simplicity. It follows from Corollary 1 that model (P1) is equivalent to model (18)–(20) under the assumption that $\log(1-z)+b_j+\varphi \leq \beta+\varphi\min\{1, 1/\ell_j\}$ for all $j \in [J]$ and that whenever there exists $j \in [J]$ and $x^* \in (0,1]$ such that $\zeta_{j,3}(x^*, z) = 0$, it holds that $x^* \leq u_j$. Since Constraints (19) and (20) are all linear, there exist Lagrangian multipliers $\lambda_j \geq 0$ and $\mu_j \geq 0$ for $j \in [J]$ and $\sigma \in \mathbb{R}$ such that

$$\log\left(q_j^*\right) + 1 + \lambda_j - \mu_j + \sigma = 0, \quad \forall\, j \in [J], \tag{EC.10}$$

$$\lambda_j(q_j^* - q_j^L) = 0, \quad \forall\, j \in [J], \tag{EC.11}$$

$$\mu_j(q_j^H - q_j^*) = 0, \quad \forall\, j \in [J], \tag{EC.12}$$

$$q_j^* \geq q_j^L, \ \ \forall\, j \in [J], \tag{EC.13}$$

$$q_j^* \leq q_j^H, \ \ \forall\, j \in [J], \tag{EC.14}$$

$$\sum_{j=1}^{J} q_j^* = z. \tag{EC.15}$$

It follows from (EC.13)–(EC.14) that $q_j^* \in [q_j^L, q_j^H]$. Note that we have $q_j^L < 1$, and for any $j \in [J]$, $q_j^* = q_j^L$ if $q_j^H = 1$; otherwise, it contradicts (EC.15) since $z \in (0,1)$.

First, we consider the case that $q_j^* \in \{q_j^L, q_j^H\}$ for all $j \in [J]$. Then, the result trivially holds.

From now on, we consider that there exists $k \in [J]$ such that $q_k^* \in (q_k^L, q_k^H)$. Define

$$\Lambda := \left\{ j \in [J] : q_j^* \in (q_j^L, q_j^H) \right\},$$

and we thus have $\Lambda \neq \varnothing$. For any $j_1, j_2 \in \Lambda$, we have from (EC.11)–(EC.12) that $\lambda_{j_1} = \mu_{j_1} = \lambda_{j_2} = \mu_{j_2} = 0$. We then have from (EC.10) that $\log(q_{j_1}^*) + 1 + \sigma = \log(q_{j_2}^*) + 1 + \sigma = 0$, implying that $q_{j_1}^* = q_{j_2}^*$. Define $q^M := q_j^*$ for some $j \in \Lambda$. It follows from the definition of Λ that

$$0 < q_j^L < q^M < q_j^H \leq 1, \ \ \forall\, j \in \Lambda. \tag{EC.16}$$

Next, we show that Λ is a singleton. Assume by contradiction that there exist $j_1, j_2 \in \Lambda$ and $j_1 \neq j_2$ such that $q_{j_1}^* = q_{j_2}^* = q^M$. It follows from (EC.16) that $q_{j_1}^* = q_{j_2}^* = q^M \in (q_{j_1}^L, q_{j_1}^H) \cap (q_{j_2}^L, q_{j_2}^H)$. Define $\varepsilon := \min\{(q^M - q_{j_1}^L)/2, (q_{j_2}^H - q^M)/2\} > 0$. We thus have that $q_M - \varepsilon \in (q_{j_1}^L, q_{j_1}^H)$ and $q^M + \varepsilon \in (q_{j_2}^L, q_{j_2}^H)$. It thus follows from (EC.16) that $q^M - \varepsilon \in (0,1)$ and $q^M + \varepsilon \in (0,1)$.

Define $\hat{\mathbf{q}} := (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_J)$ such that

$$\hat{q}_j = \begin{cases} q_j^* & \text{if } j \in [J], j \notin \{j_1, j_2\}, \\ q_M - \varepsilon & \text{if } j = j_1, \\ q_M + \varepsilon & \text{if } j = j_2. \end{cases}$$

Note that $\hat{\mathbf{q}}$ satisfies (19)–(20) and $\hat{\mathbf{q}} \neq \mathbf{q}^*$. It follows from the strict convexity of function $x\log(x)$ over $x \in (0,1)$ that

$$\begin{aligned} \widehat{h}(\hat{\mathbf{q}}) &:= \sum_{j=1}^{J} \hat{q}_j \log(\hat{q}_j) \\ &= \sum_{j \notin \{j_1, j_2\}} q_j^* \log(q_j^*) + (q_M - \varepsilon)\log(q_M - \varepsilon) + (q_M + \varepsilon)\log(q_M + \varepsilon) \\ &> \sum_{j \notin \{j_1, j_2\}} q_j^* \log(q_j^*) + 2(q_M)\log(q_M) = \widehat{h}(\mathbf{q}^*), \end{aligned}$$

which contradicts the optimality of $\mathbf{q}^*$. Thus, $\Lambda = \{k\}$.

Therefore, $q_M = q_k^* \in (q_k^L, q_k^H)$. For $j \neq k$, we have $q_j^* \in \{q_j^L, q_j^H\}$ and the following two cases:

(1) If $q_j^H = 1$, we have $q_j^* \neq q_j^H(z)$; otherwise, $\mathbf{q}^*$ will violate (EC.15) and $z \in (0,1)$. Then, $q_j^* = q_j^L(z)$.

(2) If $q_j^H < 1$, we have $q_j^* \in \{q_j^L(z), q_j^H(z)\}$.

Thus, we obtain $\mathbf{q}^*$ as defined in (21). Q.E.D.

Proof of Theorem 2. Assume the model (BASIC) is feasible with optimal solution $(\mathbf{q}^*, \mathbf{C}^*)$. Thus, the model (FLOW) is also feasible with $\mathbf{q}^*$ being the optimal flow pattern. Then, $\mathbf{q}^* \in \mathcal{S}$, which implies $q_j^* > 0$, $\|\mathbf{q}^*\|_1 < 1$, and $\xi_j(\mathbf{q}^*) = \varphi q_j / y_j(\mathbf{q}^*) \geq \ell_j > 0$, implying $y_j(\mathbf{q}^*) > 0$ for all $j \in [J]$.

It follows from Proposition 4 that $\mathbf{q}^*$ is optimal to the model (SUB) with parameter $z = z^* := \|\mathbf{q}^*\|_1$. It then follows Theorem 1 that there exists at most one $k \in [J]$ such that $q_k^* \in (q_k^L(z^*), q_k^H(z^*))$, where $q_k^L(z^*) = \max\{x_1^*, x_2^*\}$ and x_i^* is the solution to $\zeta_{k,i}(x, z^*) = 0$ for $i \in \{1, 2\}$ as in Corollary 1. It then follows that $q_k^* > x_i^*$ for $i \in \{1, 2\}$. Since $\zeta_{j,i}(x, z^*)$ is strictly increasing in x, we have $\zeta_{j,i}(q_k^*, z^*) > \zeta_{j,i}(x_i^*, z^*) = 0$, which with the definition of $\zeta_{j,i}$ implies

$$\zeta_{j,i}(q_k^*, z^*) = \varphi(\mathbf{1}\{i = 1\} + x\mathbf{1}\{i = 2\}/\ell_j) - y_k(\mathbf{q}^*) > 0,$$

yielding $\xi_k(\mathbf{q}^*) = \varphi q_k^* / y_k(\mathbf{q}^*) > q_k^*$ and $\xi_k(\mathbf{q}^*) = \varphi q_k^* / y_k(\mathbf{q}^*) > \ell_k$ due to $q_k^* > 0$ and $y_k(\mathbf{q}^*) > 0$. It follows from Remark 3 that $C_k^* = \xi_k(\mathbf{q}^*) > \max\{q_k^*, \ell_k\} = \ell_k^{\text{eff}}$. By the bound constraints (10), $C_k^* \leq u_k$. Then, either $C_k^* \in (\ell_k^{\text{eff}}, u_k)$ or $C_k^* = u_k$.

Consider any $j \neq k$. It follows Theorem 1 that $q_j^* = q_j^L(z^*)$ or $q_j^* = q_j^H(z^*) < 1$, where $q_j^L(z^*) = \max\{x_1^*, x_2^*\}$, $q_j^H(z^*) = x_3^*$, and x_i^* is the solution to $\zeta_{j,i}(x, z^*) = 0$ for $i \in \{1, 2, 3\}$. First, consider $q_j^* = q_j^L(z^*)$. If $x_1^* > x_2^*$, we have $q_j^* = x_1^*$, which implies $\zeta_{j,1}(q_j^*, z^*) = \varphi - y_j(\mathbf{q}^*) = 0$. Thus, we have $C_j^* = \xi_j(\mathbf{q}^*) = \varphi q_j^* / y_j(\mathbf{q}^*) = q_j^*$. Since $x_1^* > x_2^*$, it follows from Lemma EC.3 that $x_2^* > \ell_j$, which yields $x_1^* = q_j^* = C_j^* > x_2^* > \ell_j$. If $x_1^* \leq x_2^*$, we have $q_j^* = x_2^*$, which implies $\zeta_{j,2}(q_j^*, z^*) = \varphi q_j^* / \ell_j - y_j(\mathbf{q}^*) = 0$. Thus, we have $C_j^* = \xi_j(\mathbf{q}^*) = \varphi q_j^* / y_j(\mathbf{q}^*) = \ell_j$. Since $x_1^* \leq x_2^*$, it follows from Lemma EC.3 that $x_2^* \leq \ell_j$. We thus have $C_j^* = \ell_j \geq x_2^* = q_j^*$. Summarizing the above results gives $C_j^* = \ell_j^{\text{eff}} = \max\{\ell_j, q_j^*\}$. Second, consider $q_j^* = q_j^H(z^*) < 1$, where $q_j^H(z^*)$ is the solution to $\zeta_{j,3}(x, z^*) = 0$. Thus, $\zeta_{j,3}(q_j^*, z^*) = \varphi q_j^* / u_j - y_j(\mathbf{q}^*) = 0$, which implies $C_j^* = \xi_j(\mathbf{q}^*) = u_j$. This completes the proof. Q.E.D.

EC.4. Supporting Lemmas, Algorithms, and Proofs for Section 6

Proof of Proposition 5. Consider any $z \in (0, 1)$ feasible to the model (UNIVAR). Then, there exists $\mathbf{q}$ feasible to the model (SUB) with $z = \|\mathbf{q}\|_1 < 1$. It follows from Remark 4 that $\mathbf{q}$ satisfies (14)–(16). Note $q_j > 0$ for all $j \in [J]$; $\mathcal{Q}_j(z) = \varnothing$, contradicting the feasibility of $\mathbf{q}$ to the model (SUB) otherwise. Thus, $\mathbf{q} \in \mathcal{Q}$, indicating $\mathbf{q}$ is feasible to the model (FLOW). Define $\mathbf{C} := \boldsymbol{\xi}(\mathbf{q})$. We have $\mathbf{C}^L \leq \mathbf{C} \leq \mathbf{C}^H$.

We have from Remark 3 that $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$. It follows from the definitions of $\mathbf{q}^L$ and $\mathbf{q}^H$ that $\mathbf{q}^i = \mathbf{F}(\mathbf{q}^i, \mathbf{C}^i)$ for $i \in \{L, H\}$. We then have from Proposition 1 that $\|\mathbf{q}^L\|_1 \leq \|\mathbf{q}\|_1 = z \leq \|\mathbf{q}^H\|_1$, which proves the result. Q.E.D.

For each $j \in [J]$, define $f_j : [0, 1] \mapsto [-\infty, \beta + \varphi/C_j - b_j - \varphi]$ as:

$$f_j(x) := \beta x^\theta + \log(x) + \frac{\varphi x}{C_j} - b_j - \varphi. \tag{EC.17}$$

Also, define $z^0 := \max_{j\in[J]}\left(1-e^{\beta+\varphi/C_j-b_j-\varphi}\right)^+$.

Lemmas EC.4 and EC.5 will be useful.

Lemma EC.4. *For each $j\in[J]$, $f_j(x)$ is strictly increasing and continuous over $[0,1]$.*

Lemma EC.5. *For each $j\in[J]$ and $z\in[\left(1-e^{\beta+\varphi/C_j-b_j-\varphi}\right)^+,1]$, there exists a unique $x\in[0,1]$ such that $f_j(x)=\log(1-z)$.*

Proof. Fix $j\in[J]$. Define $g(x):=f_j(x)-\log(1-z)$. It follows from Lemma EC.4 that $g(x)$ is strictly increasing and continuous over $x\in[0,1]$. Note that $\log[1-(1-e^{\beta+\varphi/\ell-b_j-\varphi})^+]=(\beta+\varphi/\ell-b_j-\varphi)^-$. Thus, it follows that $\log(1-z)\in[-\infty,(\beta+\varphi/\ell-b_j-\varphi)^-]$ and,

$$g(0)=f_j(0)-\log(1-z)=-\infty-\log(1-z)\le 0,$$
$$g(1)=f_j(1)-\log(1-z)=\beta+\varphi/\ell-b_j-\varphi-\log(1-z)\ge\beta+\varphi/\ell-b_j-\varphi-(\beta+\varphi/\ell-b_j-\varphi)^-\ge 0,$$

which with the continuity and strict increasingness of $g(y)$ implies that there exists $x\in[0,1]$ such that $g(x)=0$, equivalent to $f(x)=\log(1-z)$. Q.E.D.

Lemma EC.5 allows us to define functions related f_j. For each $j\in[J]$, define a mapping $\gamma_j:[(1-e^{\beta+\varphi/C_j-b_j-\varphi})^+,1]\mapsto[0,1]$ such that $\gamma_j(z)$ is the unique solution to $f_j(x)-\log(1-z)=0$, i.e., $f_j(\gamma_j(z))=\log(1-z)$. Then, define a function $\Gamma:[z^0,1]\mapsto\mathbb{R}$ as follows:

$$\Gamma(z):=\sum_{j=1}^{J}\gamma_j(z)-z, \tag{EC.18}$$

where recall that $z^0:=\max_{j\in[J]}\left(1-e^{\beta+\varphi/C_j-b_j-\varphi}\right)^+$.

Lemma EC.6. *The following holds:*

(1) $\Gamma(z)$ is strictly decreasing and continuous over $[z^0,1]$.

(2) $\Gamma(z^0)>0$ and $\Gamma(1)<0$

(3) There exists a unique $z^*\in(z^0,1)$ such that $\Gamma(z^*)=0$.

Proof. Choose $k\in\arg\max_{j\in[J]}\left(1-e^{\beta+\varphi/C_j-b_j-\varphi}\right)^+$. We have that

$$z^0=(1-e^{\beta+\varphi/C_k-b_k-\varphi})^+.$$

It then follows from Lemma EC.5 that γ_j is well defined over $[z^0,1]$ for all $j\in[J]$. Consider $z_1,z_2\in[z^0,1]$ such that $z_1>z_2$. It follows from the definition of γ_j that $f_j(\gamma_j(z_1))=\log(1-z_1)<\log(1-z_2)=f_j(\gamma_j(z_2))$, which with Lemma EC.4 implies that $\gamma_j(z_1)<\gamma_j(z_2)$. Thus, γ_j is strictly decreasing for all $j\in[J]$. The continuity of γ_i follows from the continuity of f_i. Thus, it follows from (EC.18) that Γ is strictly decreasing and continuous over $[z^0,1]$, which is Result (1).

Note that $\log(1-z^0) = (\beta + \varphi/C_k - b_k - \varphi)^-$. Choose any $j \in [J]$. It follows from the definition of γ_j that

$$f_j(\gamma_j(z^0)) = \log(1-z^0) = (\beta + \varphi/C_k - b_k - \varphi)^- > -\infty.$$

Since $f_j(0) = -\infty$ and strictly increasing, we have that $\gamma_j(z_0) > 0$, and more specifically,

$$\gamma_j(z^0) \begin{cases} = 1 \text{ if } \beta + \varphi/C_k - b_k - \varphi < 0, \\ > 0 \text{ if } \beta + \varphi/C_k - b_k - \varphi \geq 0, \end{cases}$$

which yields that

$$\Gamma(z^0) = \sum_{j=1}^{J} \gamma_j(z^0) - z^0 \geq \gamma_1(z^0) - (1 - e^{\beta + \varphi/C_k - b_k - \varphi})^+ > 0.$$

Also note $f_j(\gamma_j(1)) = \log(1-1) = -\infty$, $f_j(0) = -\infty$, and f_j is strictly increasing and continuous, indicating that $\gamma_j(1) = 0$. Thus, it follows that $\Gamma(1) = \sum_{j=1}^{J} \gamma_j(1) - 1 < 0$, which proves Result (2). Result (3) follows from Results (1) and (2). Q.E.D.

Proposition EC.2 asserts that the traffic flow pattern $\mathbf{q}$ that satisfies $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$ can be obtained through solving an one-dimensional function $\Gamma(z) = 0$.

Proposition EC.2. *Let $z^* \in (z^0, 1)$ be the unique solution to $\Gamma(z) = 0$. Define $q_j := \gamma_j(z^*)$ for all $j \in [J]$. Then, $\mathbf{q} := (q_j, j \in [J])$ satisfies (5), i.e., $\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{C})$.*

Proof. Let $z^* \in (z^0, 1)$ be the unique solution of $\Gamma(z) = 0$. Define $q_j := \gamma_j(z^*)$ for all $j \in [J]$. Then it follows from (EC.18) that $z^* = \sum_{k=1}^{J} \gamma_k(z^*) = \sum_{k=1}^{J} q_k$. It follows from the definition of γ_j that

$$f_j(\gamma_j(z^*)) = f_j(q_j) = \beta q_j^\theta + \log(q_j) + \varphi q_j/C_j - b_j - \varphi = \log(1 - z^*) = \log\left(1 - \sum_{k=1}^{J} q_k\right), \quad \forall\, j \in [J],$$

which, by Remark 2, implies that $\mathbf{q}$ satisfies (5). Q.E.D.

Algorithm EC.1 The bisection search method

Input: strictly monotone function f, $\delta_1, \delta_2 > 0$, domain $[\text{ub}, \text{ub}]$, $x^* \leftarrow (\text{lb} + \text{ub})/2$.

Output: x^* such that $f(x^*) = 0$

while $|f(x^*)| > \delta_1$ and $\text{ub} - \text{lb} > \delta_2$ **do**

  If f is decreasing, set $\text{lb} \leftarrow x^*$ if $f(x^*) > 0$; set $\text{ub} \leftarrow x^*$ otherwise;

  If f is increasing, set $\text{ub} \leftarrow x^*$ if $f(x^*) > 0$; set $\text{lb} \leftarrow x^*$ otherwise;

  Set $x^* \leftarrow (\text{lb} + \text{ub})/2$;

end while

EC.5. Supplementary Materials for Section 7

EC.5.1. The Departures and Choice Scenarios of Commuters under Real-time Parking Information

This section describes the departure process and choice models of commuters when shared with real-time parking information.

We consider that the departures of commuters (from home in Bellevue) follow a homogeneous Poisson process $\{N(t): t \geq 0\}$ with rate $\kappa > 0$, where $N(t)$ represents the number of departures till time $t \geq 0$. Denote by $t_n \geq 0$ the departure time of the n-th commuter (called customer n), where $n \in \mathbb{N}$. Let $T \in \mathbb{R}_+$ denote the time period of interest, such as 7–9 am on Monday mornings. Then, the total travel demand during the time period is expected to be $Q = \mathbb{E}[N(T)] = \lfloor \kappa T \rfloor$, where $\lfloor x \rfloor$ is the integer part of any number $x \in \mathbb{R}$.

For commuters departing at time $t \geq 0$, the travel time to any lot $j \in [J]$ can be represented by a BPR-type function $c_j(t) := r_{j,4} + \alpha'(q_j^{\text{on}}(t))^{\theta}$, where $r_{j,4}$ is the *congestion-free access time* to j as defined in Section 7.1, $q_j^{\text{on}}(t)$ is the on-road traffic flow representing the fraction of the total expected demand Q on travel to but having not arrived at j at t, and $\alpha' \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$ are parameters to be estimated from data (Sheffi, 1985). We refer to $c_j(t)$ as the *congestion-dependent access time* to j at time t.

The real-time parking utilization (or availability) of each lot at any time $t \geq 0$ is announced to commuters. Let $q_j^{\text{at}}(t)$ denote the at-park traffic at t, which denotes the fraction of Q parked at j at time t. Hence, the parking utilization of j at time t is $q_j^{\text{at}}(t)/C_j$, where C_j is the capacity of j (which is scaled by Q; see discussions on parameter scaling in Section 3.2).

Upon departure, each commuter n chooses an alternative from $[J] \cup \{0\}$, where as in Section 3.2, we use 0 to denote the no-park-and-ride alternative. However, the underlying choice model that describes the *true* choice behavior of commuters is unknown to the DOT or analysts, and hence, may not be consistent with the choice model used for modeling the optimal capacity sizing problem by the DOT. We consider the following nine choice scenarios with each imitating a particular type of *true* choice behavior of commuters:

1. Commuters are information-, congestion-, and intrinsic-utility-aware.
2. Commuters are information- and intrinsic-utility-aware but congestion-unaware.
3. Commuters are information-unaware but congestion- and intrinsic-utility-aware.
4. Commuters are information-aware but congestion- and intrinsic-utility-unaware.
5. Commuters base their decisions only on congestion-free access time.
6. Commuters base their decisions only on the utility due to economic value, the number of bus routes, and bus frequency.

7. Commuters base their decisions only on intrinsic utilities.
8. Commuters base their decisions only on congestion-dependent access time.
9. Each commuter chooses alternatives in $[J]\cup\{0\}$ with equal probabilities (flip-a-coin).

Again, the true choice behavior of commuters is unknown. These choice scenarios describe various types of commuters' choice behavior that are considered to be true from the perspective of an analyst.

For any $j\in[J]$, define $r_j:=\alpha_1 r_{j,1}+\alpha_2 r_{j,2}+\alpha_3 r_{j,3}$ as the utility of j due to its economic value, number of bus routes, and bus frequency. Let $\nu_j^m(t)$ denote the systematic utility of j *perceived* by commuters at any time $t\geq 0$ when scenario $m\in\{1,2,\dots,9\}$ is true, which is represented by,

$$\begin{aligned}\nu_j^m(t):=&r_j\mathbf{1}\{m\in\{1,2,3,6,7\}\}-\alpha_4 r_{j,4}\mathbf{1}\{m\in\{1,2,3,5,7,8\}\}-\beta(q_j^{\text{on}}(t))^\theta\mathbf{1}\{m\in\{1,3,8\}\}\\&+\varphi\left(1-\frac{q_j^{\text{at}}(t)}{C_j}\right)\mathbf{1}\{m\in\{1,2,4\}\},\end{aligned}$$

where α_4 is the time sensitivity parameter as in (23), $\beta(q_j^{\text{on}}(t))^\theta$ reflects the congestion effect, $\varphi(1-q_j^{\text{at}}(t)/C_j)$ captures the parking effect, and $\beta=\alpha_4\alpha'$ and φ are the congestion and parking sensitivity parameters, respectively, as defined in (2).

Let $j_n\in[J]\cup\{0\}$ denote the alternative chosen by customer n. The probability of commuter n choosing $j_n=j\in[J]\cup\{0\}$ upon departure at time t_n under scenario $m\in\{1,2,\dots,9\}$ can be represented by,

$$\mathbb{P}^m(j_n=j):=\begin{cases}\frac{\exp\left(\nu_j^m(t_n)\right)\mathbf{1}\left\{q_j^{\text{at}}(t_n)<C_j\right\}}{1+\sum_{k=1}^J\exp\left(\nu_k^m(t_n)\right)\mathbf{1}\left\{q_k^{\text{at}}(t_n)<C_k\right\}} & \text{for } m\in\{1,2,4\},\\ \frac{\exp\left(\nu_j^m(t_n)\right)}{1+\sum_{k=1}^J\exp\left(\nu_k^m(t_n)\right)} & \text{for } m\in\{3,5,6,7,8\},\\ \frac{1}{J+1} & \text{for } m=9,\end{cases}\tag{EC.19}$$

and $\mathbb{P}^m(j_n=0)=1-\sum_{j\in[J]}\mathbb{P}^m(j_n=j)$.

The systematic utility of $j\in[J]$ *received* by commuters departing at t is represented by,

$$\nu_j(t):=\left(b_j-\beta(q_j^{\text{on}}(t))^\theta+\varphi\left(1-\frac{q_j^{\text{at}}(t)}{C_j}\right)\right)\mathbf{1}\left\{q_j^{\text{at}}(t+c_j(t))<C_j\right\},\tag{EC.20}$$

where $b_j=r_j-\alpha_4 r_{j,4}$ is the intrinsic utility as defined in (23), $t+c_j(t)$ is the arrival time at j of customers with departure time t, and we assume that the commuters who choose to visit a parking lot but find it full upon arrival are lost and receive zero utilities.

Let $\Pi^m(\mathbf{C})$ denote the expected total social welfare under any capacity plan $\mathbf{C}\in\{\mathbf{C}^*,\mathbf{C}^1,\mathbf{C}^2\}$, choice scenario $m\in\{1,2,\dots,9\}$, and real-time parking information, which is represented by

$$\Pi^m(\mathbf{C})=\mathbb{E}\left[\sum_{n=1}^{N(T)}\sum_{j=1}^J\mathbb{P}^m(j_n=j)\nu_j(t_n)\right].\tag{EC.21}$$

EC.5.2. Supplementary Data and Materials for Section 7.1

The catchment areas are determined based on zip codes. As shown in Figure 2, Bellevue is mainly constituted by five zip codes: 98004, 98005, 98006, 98007 and 98008. Based on the zip codes of all the lots in Table EC.1, the catchment area of South Bellevue or Wilburton can approximately be estimated as half the total area of 98004 and 98005. The catchment area of of Bellevue Christian Reformed Church or Eastgate is considered to be half the total area of 98007 and 98008. The area of 98006 is evenly split into the catchment areas of Newport Hills, Newport Covenant Church, and Eastgate Congregational.

Now, we explain how to estimate H_j. According to U.S. Census Bureau (2020), the numbers of households in 98004, 98005, 98006, 98007, and 98008 are 17,460, 8,590, 13,030, 11,578, and 9,125, respectively. In the same spirit as catchment area determination, the number of households in the neighborhood of South Bellevue or Wilburton is estimated as $(17,460 + 8,590)/2 = 13025$. The number of households in the neighborhood of Bellevue Christian Reformed Church or Eastgate is approximated as $(11,578 + 9,125)/2 \approx 10,352$. The number of households in the neighborhood of Newport Hills, Newport Covenant Church, or Eastgate Congregational is equal to $13,030/3 \approx 4,343$.

Table EC.1 Current capacities of Bellevue's park-and-ride lots.

ID	Name	Address	Capacity
1	**South Bellevue P&R**	2700 Bellevue Wy SE, 98004	1,500
2	**Wilburton P&R**	720 114th Ave SE, 98004	186
3	**Eastgate Congregational**	15318 SE Newport Way, 98006	20
4	**Newport Covenant Church**	12800 SE Coal Creek Pkwy, 98006	75
5	**Newport Hills P&R**	5115 113th Pl SE, 98006	275
6	**Bellevue Christian Reformed Church**	1221 148th Ave NE, 98007	20
7	**Eastgate P&R**	14200 SE Eastgate Way, 98007	1,614

Table EC.2 The data used for sizing Bellevue's park-and-ride lots.

j	Median House/ Condo Value ($)	Number of Bus Routes	Average Bus Headway (min)	Number of Households	Congestion-free Access Time (min)
1	961,846	5	21.04	13,025	4.26
2	961,846	3	43.22	13,025	4.78
3	754,626	2	26.71	4,343	5.54
4	754,626	2	18.83	4,343	5.40
5	754,626	2	27.86	4,343	5.97
6	486,806	1	30.50	10,352	6.73
7	486,806	14	23.40	10,352	4.70

Table EC.3 Congestion-free time distances (min) between Bellevue's park-and-ride lots.

i \ j	1	2	3	4	5	6	7
1	0	4	6	5	5	9	5
2	5	0	7	6	6	8	6
3	6	7	0	5	7	7	4
4	4	5	5	0	4	9	4
5	5	6	6	3	0	9	7
6	8	8	7	9	9	0	7
7	4	7	4	5	7	7	0

Table EC.4 Normalized attribute values for calculating intrinsic utilities.

j	$r_{j,1}$	$r_{j,2}$	$r_{j,3}$	$r_{j,4}$
1	1.0000	1.0000	1.0000	1.0000
2	1.0000	0.6000	0.4868	1.1220
3	0.7846	0.4000	0.7877	1.3005
4	0.7846	0.4000	1.1174	1.2690
5	0.7846	0.4000	0.7552	1.4015
6	0.5061	0.2000	0.6898	1.5814
7	0.5061	2.8000	0.8991	1.1037

EC.5.3. Supplementary Results for Section 7.2

Table EC.5 The optimal capacities, traffic flows, and utilizations under $\ell_1 \in \{0.05, 0.15, 0.35, 0.55\}$ and $u_1 \in \{0.6, 0.7\}$.

j	b_j	$\ell_1 = 0.05,\ u_1 = 0.6$					$\ell_1 = 0.05,\ u_1 = 0.7$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.0500	0.6000	0.6000	0.2743	45.72%	0.0500	0.7000	0.7000	0.2652	37.89%
2	2.4119	0.0062	0.0744	0.0744	0.0392	52.64%	0.0062	0.0868	0.0868	0.0381	43.93%
3	1.6794	0.0007	0.0080	0.0078	0.0077	99.17%	0.0007	0.0093	0.0061	0.0061	99.65%
4	2.5824	0.0025	0.0300	0.0300	0.0246	82.10%	0.0025	0.0350	0.0350	0.0250	71.38%
5	1.3456	0.0092	0.1100	0.1100	0.0284	25.80%	0.0092	0.1283	0.1283	0.0259	20.18%
6	-0.4637	0.0007	0.0080	0.0010	0.0010	100.00%	0.0007	0.0093	0.0008	0.0008	100.00%
7	7.7539	0.0538	0.6456	0.6456	0.6230	96.50%	0.0538	0.7532	0.7532	0.6375	84.63%
j	b_j	$\ell_1 = 0.15,\ u_1 = 0.6$					$\ell_1 = 0.15,\ u_1 = 0.7$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.1500	0.6000	0.6000	0.2741	45.69%	0.1500	0.7000	0.7000	0.2650	37.86%
2	2.4119	0.0186	0.0744	0.0744	0.0391	52.60%	0.0186	0.0868	0.0868	0.0381	43.89%
3	1.6794	0.0020	0.0080	0.0076	0.0076	99.91%	0.0020	0.0093	0.0060	0.0060	99.99%
4	2.5824	0.0075	0.0300	0.0300	0.0246	82.06%	0.0075	0.0350	0.0350	0.0250	71.33%
5	1.3456	0.0275	0.1100	0.1100	0.0284	25.78%	0.0275	0.1283	0.1283	0.0259	20.16%
6	-0.4637	0.0020	0.0080	0.0020	0.0016	80.82%	0.0020	0.0093	0.0020	0.0015	74.29%
7	7.7539	0.1614	0.6456	0.6456	0.6228	96.46%	0.1614	0.7532	0.7532	0.6371	84.59%
j	b_j	$\ell_1 = 0.35,\ u_1 = 0.6$					$\ell_1 = 0.35,\ u_1 = 0.7$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.3500	0.6000	0.6000	0.2737	45.62%	0.3500	0.7000	0.7000	0.2647	37.81%
2	2.4119	0.0434	0.0744	0.0744	0.0391	52.52%	0.0434	0.0868	0.0868	0.0380	43.82%
3	1.6794	0.0047	0.0080	0.0076	0.0076	99.75%	0.0047	0.0093	0.0062	0.0062	99.04%
4	2.5824	0.0175	0.0300	0.0300	0.0246	81.96%	0.0175	0.0350	0.0350	0.0249	71.24%
5	1.3456	0.0642	0.1100	0.1100	0.0283	25.72%	0.0642	0.1283	0.1283	0.0258	20.11%
6	-0.4637	0.0047	0.0080	0.0047	0.0027	58.49%	0.0047	0.0093	0.0047	0.0025	52.80%
7	7.7539	0.3766	0.6456	0.6456	0.6222	96.38%	0.3766	0.7532	0.7532	0.6366	84.52%
j	b_j	$\ell_1 = 0.55,\ u_1 = 0.6$					$\ell_1 = 0.55,\ u_1 = 0.7$				
		ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*	ℓ_j	u_j	C_j^*	q_j^*	ρ_j^*
1	5.0000	0.5500	0.6000	0.6000	0.2735	45.58%	0.5500	0.7000	0.7000	0.2641	37.74%
2	2.4119	0.0682	0.0744	0.0744	0.0390	52.47%	0.0682	0.0868	0.0868	0.0380	43.73%
3	1.6794	0.0073	0.0080	0.0076	0.0076	99.69%	0.0073	0.0093	0.0075	0.0070	93.27%
4	2.5824	0.0275	0.0300	0.0300	0.0246	81.89%	0.0275	0.0350	0.0350	0.0249	71.12%
5	1.3456	0.1008	0.1100	0.1100	0.0283	25.69%	0.1008	0.1283	0.1283	0.0257	20.05%
6	-0.4637	0.0073	0.0080	0.0073	0.0035	47.74%	0.0073	0.0093	0.0073	0.0031	42.54%
7	7.7539	0.5918	0.6456	0.6456	0.6218	96.32%	0.5918	0.7532	0.7532	0.6358	84.41%

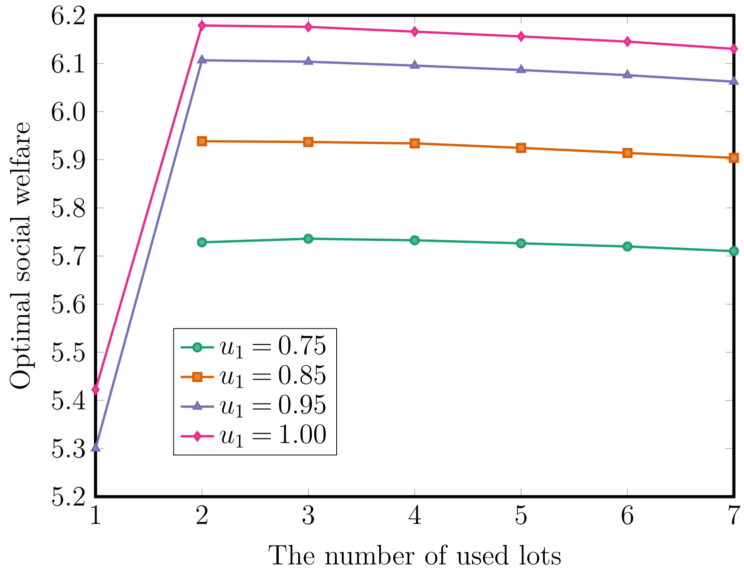


Figure EC.1 The optimal total social welfare as a function of the number $\bar{J}$ of used park-and-ride lots, where $\bar{J} \in \{1, 2, \ldots, 7\}$, under different u_1 when there are no lower-bound constraints. There are no feasible solutions when $\bar{J} = 1$ for $u_1 \in \{0.75, 0.85\}$.

EC.5.4. Supplementary Results for Section 7.3

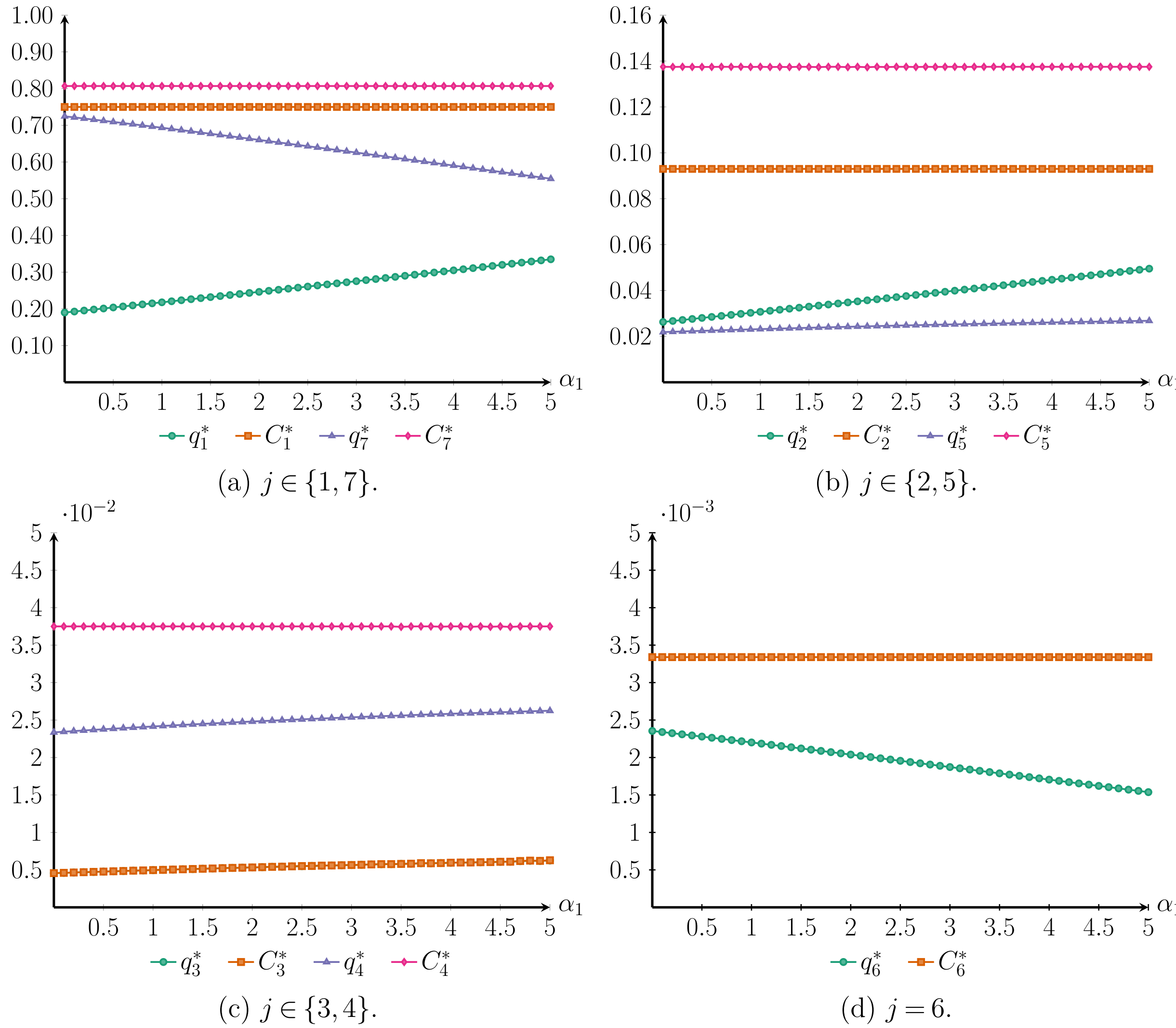


Figure EC.2 The optimal capacity C_j^* and traffic flow q_j^* as a function of α_1 for all $j \in \{1, 2, \ldots, 7\}$.

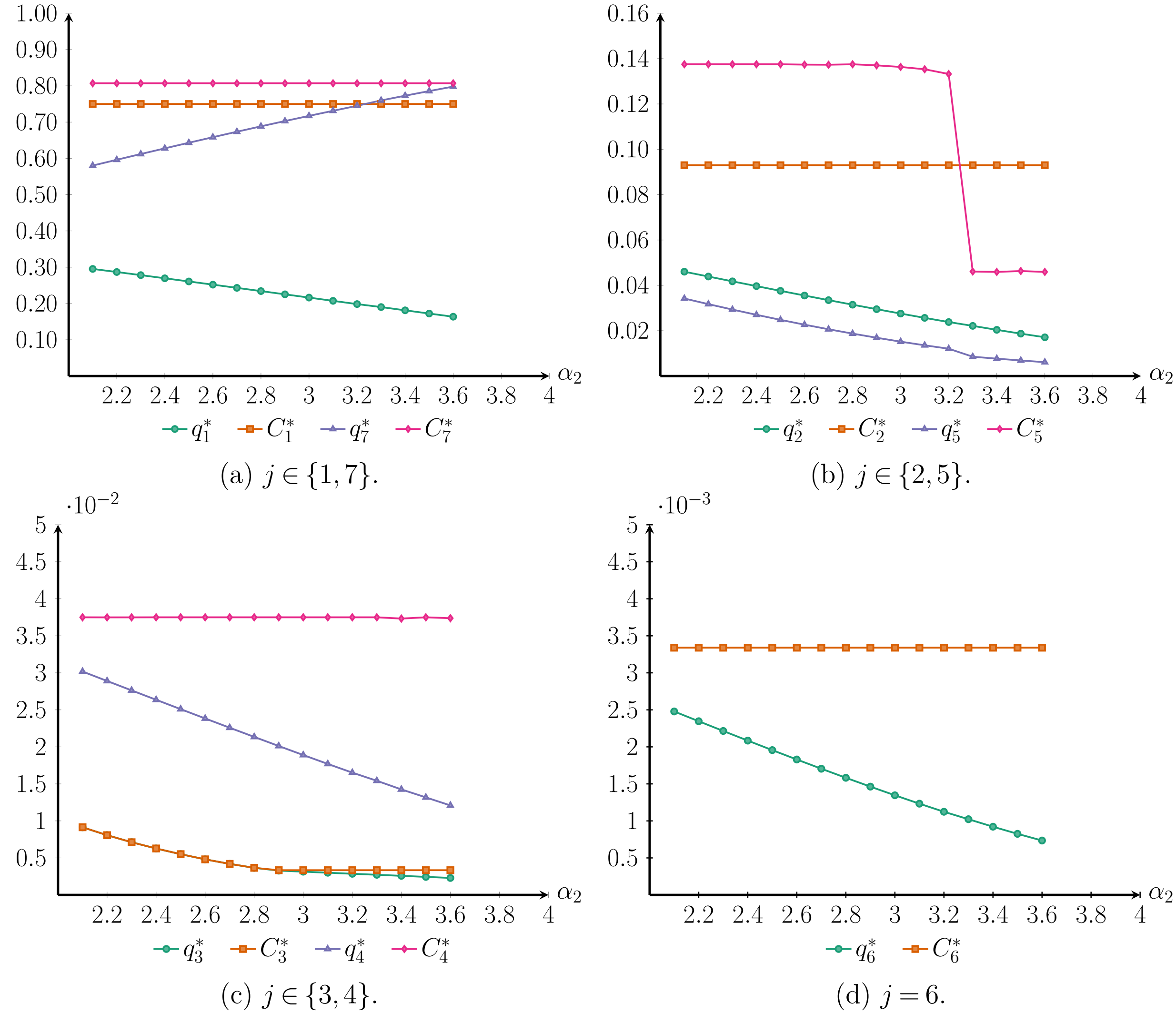


Figure EC.3 The optimal capacity C_j^* and traffic flow q_j^* as a function of α_2 for all $j \in \{1, 2, \ldots, 7\}$.

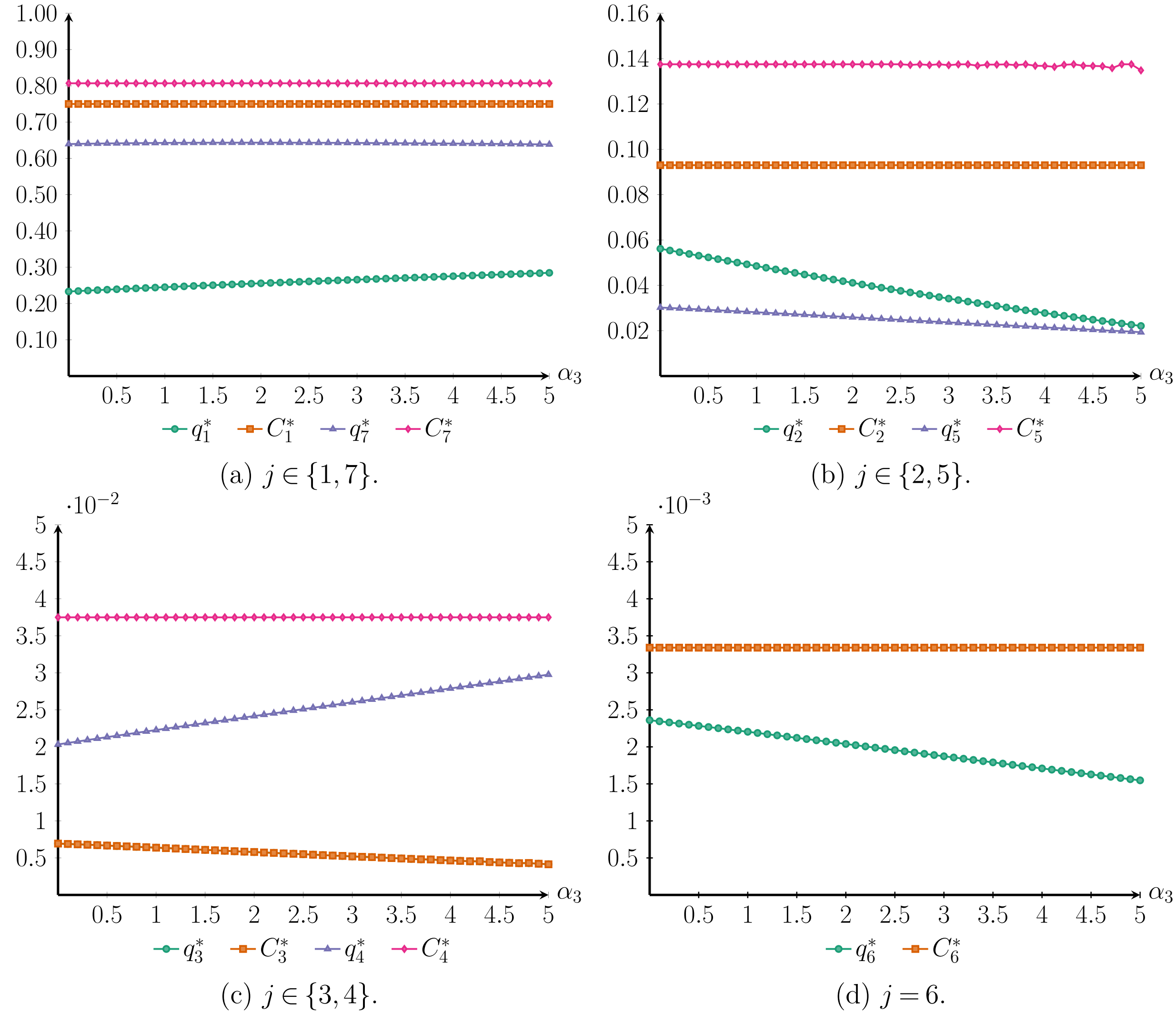


Figure EC.4 The optimal capacity C_j^* and traffic flow q_j^* as a function of α_3 for all $j \in \{1, 2, \ldots, 7\}$.

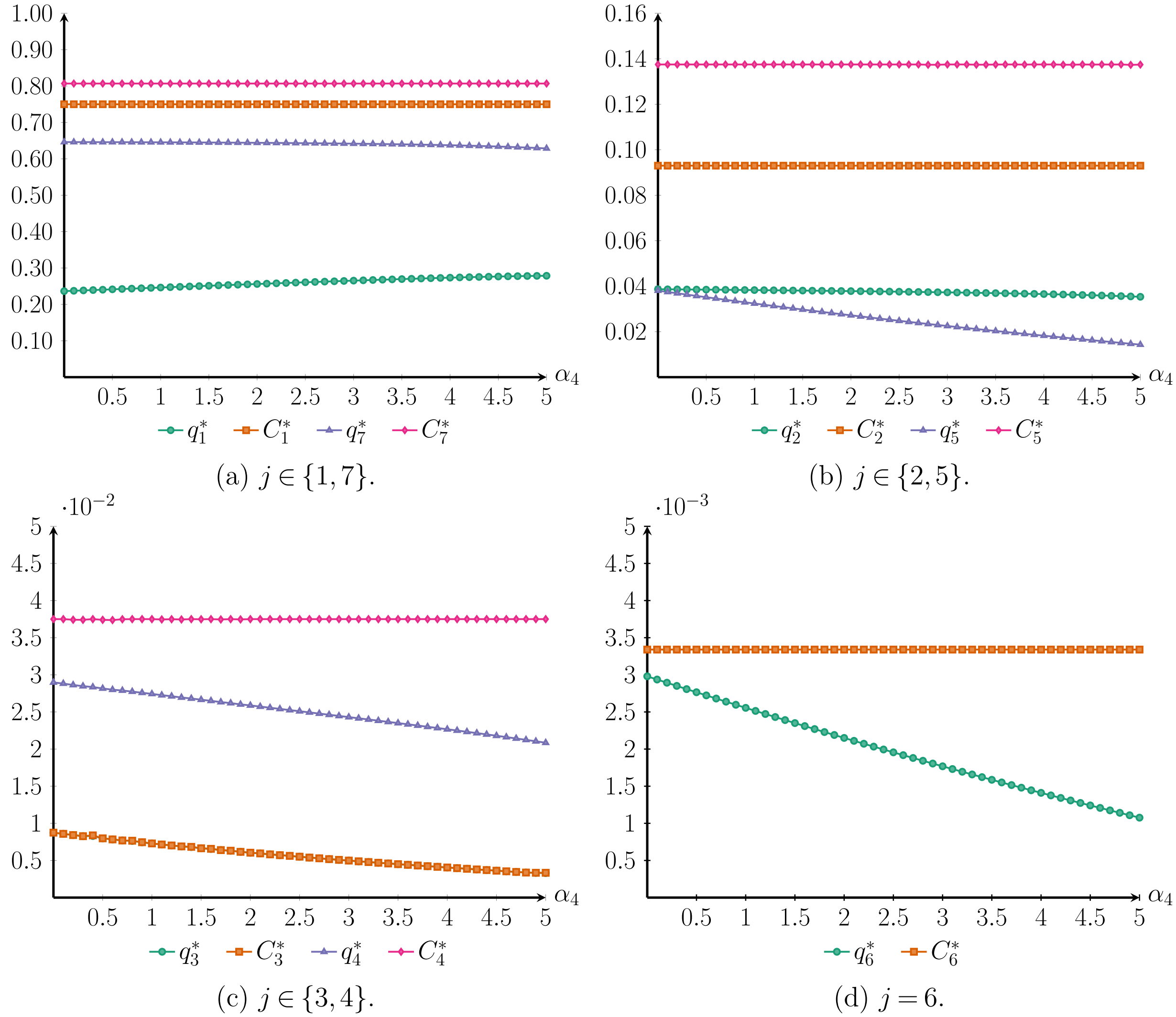
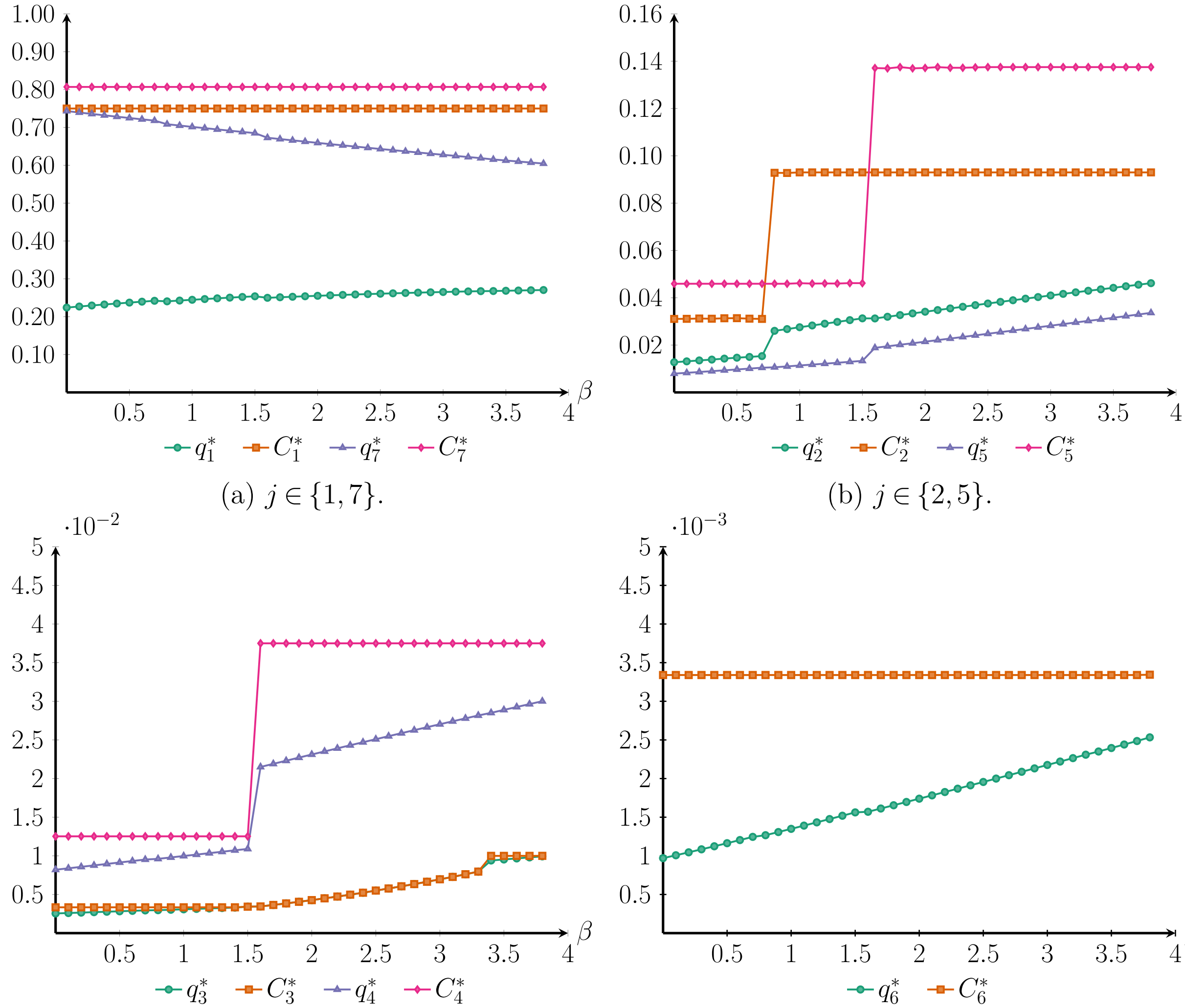


Figure EC.5 The optimal capacity C_j^* and traffic flow q_j^* as a function of α_4 for all $j \in \{1,2,\ldots,7\}$.



(a) $j \in \{1,7\}$.

(b) $j \in \{2,5\}$.

(c) $j \in \{3,4\}$.

(d) $j = 6$.

Figure EC.6 The optimal capacity C_j^* and traffic flow q_j^* as a function of β for all $j \in \{1,2,\ldots,7\}$.

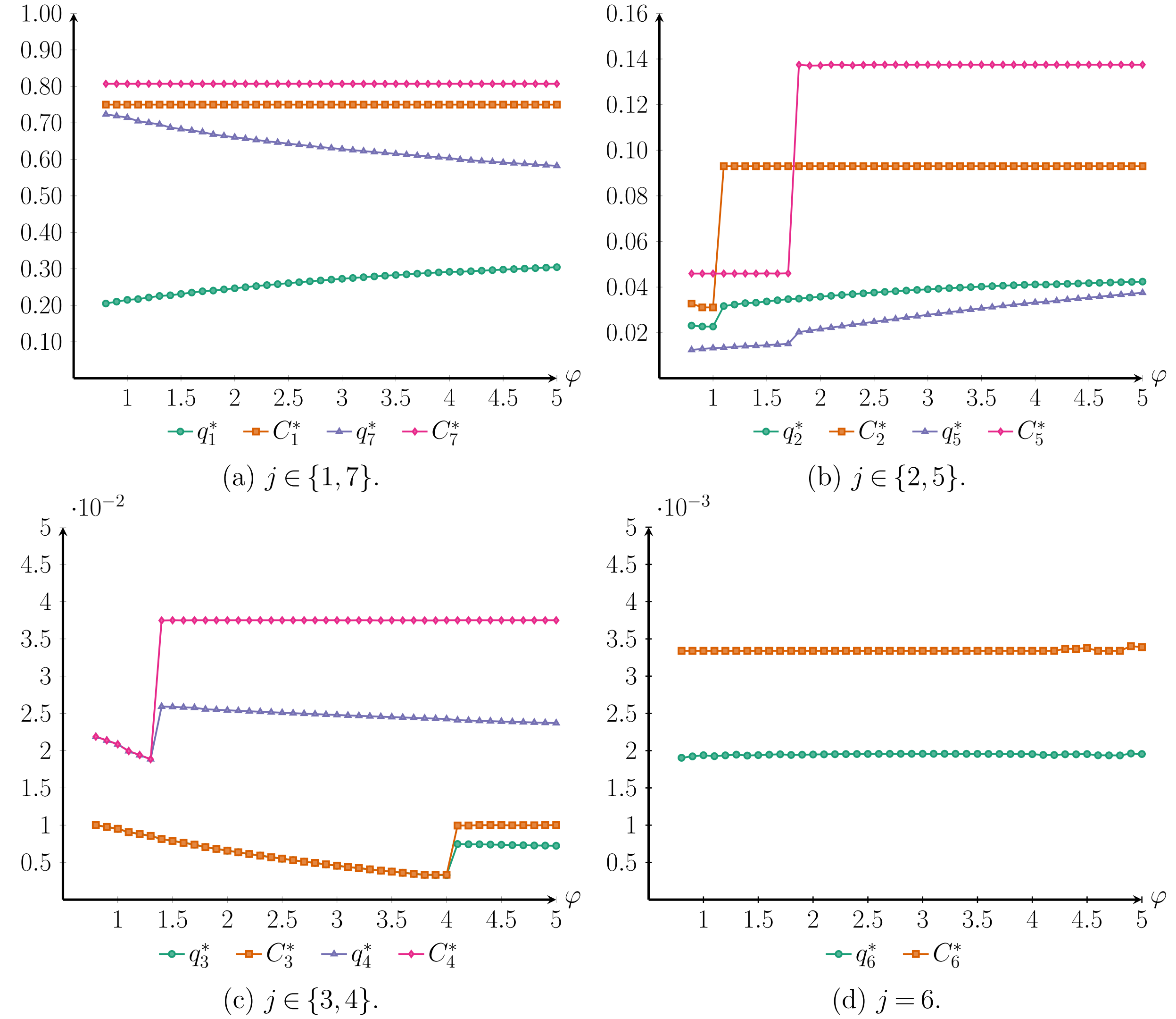


Figure EC.7 The optimal capacity C_j^* and traffic flow q_j^* as a function of φ for all $j \in \{1, 2, \ldots, 7\}$.

EC.5.5. Supplementary Results for Section 7.4

Table EC.6 $\bar{\Pi}^m(\mathbf{C})$, Gap_i^m, and TG_i for $i \in \{1,2\}$, $m \in \{1,2,\dots,9\}$, and $\mathbf{C} \in \{\mathbf{C}^*, \mathbf{C}^1, \mathbf{C}^2\}$ when $\ell_1 \in \{0.05, 0.25, 0.5, 0.7\}$ and $u_1 \in \{0.95, 1\}$.

m	$\ell_1=0.05,\ u_1=0.95$					$\ell_1=0.05,\ u_1=1$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	63283	63636	58318	-0.56	7.85	63662	64015	58318	-0.55	8.39
2	63324	63669	58314	-0.55	7.91	63705	64051	58314	-0.54	8.46
3	64154	63988	52975	0.26	17.43	64531	64361	52975	0.26	17.91
4	50992	54793	49450	-7.45	3.02	50998	55102	49450	-8.05	3.04
5	9426	7217	8319	23.44	11.75	9439	7215	8319	23.56	11.87
6	64079	63759	52391	0.50	18.24	64463	64139	52391	0.50	18.73
7	64182	64004	52861	0.28	17.64	64561	64379	52861	0.28	18.12
8	9404	7190	8314	23.54	11.60	9417	7188	8314	23.67	11.72
9	20376	16156	16648	20.71	18.30	20594	16165	16648	21.51	19.16
TG_i	–	–	–	60.17	113.74	–	–	–	60.64	117.40

m	$\ell_1=0.25,\ u_1=0.95$					$\ell_1=0.25,\ u_1=1$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	63243	63369	59488	-0.20	5.94	63623	63748	59488	-0.20	6.50
2	63285	63413	59481	-0.20	6.01	63668	63793	59481	-0.20	6.58
3	64157	64149	53056	0.01	17.30	64534	64527	53056	0.01	17.79
4	46590	52673	48976	-13.06	-5.12	46612	52924	48976	-13.54	-5.07
5	9855	8555	8984	13.19	8.84	9872	8556	8984	13.33	9.00
6	64093	64017	52446	0.12	18.17	64477	64400	52446	0.12	18.66
7	64184	64155	52936	0.04	17.53	64563	64535	52936	0.04	18.01
8	9839	8529	8973	13.32	8.80	9856	8529	8973	13.46	8.96
9	22860	17824	17882	22.03	21.77	23116	17836	17882	22.84	22.64
TG_i	–	–	–	35.25	99.24	–	–	–	35.86	103.07

m	$\ell_1=0.5,\ u_1=0.95$					$\ell_1=0.5,\ u_1=1$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	63224	63271	60344	-0.07	4.56	63604	63652	60344	-0.08	5.13
2	63268	63319	60335	-0.08	4.64	63649	63701	60335	-0.08	5.21
3	64161	64170	53127	-0.01	17.20	64539	64548	53127	-0.01	17.68
4	46394	49882	47948	-7.52	-3.35	46407	50043	47948	-7.84	-3.32
5	9927	9527	9604	4.02	3.25	9949	9535	9604	4.16	3.46
6	64105	64064	52489	0.06	18.12	64491	64449	52489	0.06	18.61
7	64188	64174	53002	0.02	17.43	64568	64554	53002	0.02	17.91
8	9911	9496	9592	4.19	3.22	9933	9504	9592	4.32	3.44
9	22929	19932	19850	13.07	13.43	23191	19989	19850	13.81	14.40
TG_i	–	–	–	13.68	78.50	–	–	–	14.36	82.52

m	$\ell_1=0.75,\ u_1=0.95$					$\ell_1=0.75,\ u_1=1$				
	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m	$\bar{\Pi}^m(\mathbf{C}^*)$	$\bar{\Pi}^m(\mathbf{C}^1)$	$\bar{\Pi}^m(\mathbf{C}^2)$	Gap_1^m	Gap_2^m
1	63213	63232	60594	-0.03	4.14	63593	63615	60594	-0.03	4.72
2	63257	63281	60583	-0.04	4.23	63639	63666	60583	-0.04	4.80
3	64163	64176	53149	-0.02	17.16	64541	64554	53149	-0.02	17.65
4	46212	47671	46378	-3.16	-0.36	46218	47857	46378	-3.55	-0.35
5	9999	9899	9914	1.00	0.85	10020	9903	9914	1.17	1.06
6	64110	64077	52507	0.05	18.10	64496	64462	52507	0.05	18.59
7	64190	64180	53023	0.02	17.40	64569	64559	53023	0.02	17.88
8	9983	9866	9897	1.17	0.86	10004	9870	9897	1.34	1.07
9	22999	21565	21488	6.24	6.57	23268	21594	21488	7.20	7.65
TG_i	–	–	–	5.23	68.95	–	–	–	6.14	73.07

EC.6. Optimal Capacity Sizing with Flow Interactions

In this section, we provide models for optimal capacity sizing of park-and-ride lots with the consideration of explicit interactions between the traffic flows through different lots.

The congestion experienced by a commuter choosing a particular lot $j \in [J]$ does not only depend on the traffic flow q_j but may also depend on the traffic flows $(q_k, k \in [J], k \neq k)$ through all other lots since all commuters originating from the same service area share the same road network. This phenomenon is referred to as *flow interactions*.

To explicitly model flow interactions, the systematic utility of any lot $j \in [J]$ in (2) can be updated as,

$$\nu_j(\mathbf{q}, C_j) = b_j - \beta q_j^\theta - \omega \sum_{k \neq j} q_k + \varphi \left(1 - q_j / C_j\right), \tag{EC.22}$$

where $\omega \in \mathbb{R}_+$ represents the sensitivity of commuters to the congestion in the entire road network except for their own paths. We also update the vector of utilities as $\boldsymbol{\nu}(\mathbf{q}, \mathbf{C}) := (\nu_j(\mathbf{q}, C_j), j \in [J])$ and the fixed-point equation (4) as

$$q_j = p_j(\boldsymbol{\nu}(\mathbf{q}, \mathbf{C})) = \frac{\exp\left(b_j - \beta q_j^\theta - \omega \sum_{k \neq j} q_k + \varphi\left(1 - \frac{q_j}{C_j}\right)\right)}{1 + \sum_{k=1}^J \exp\left(b_k - \beta q_k^\theta - \omega \sum_{k' \neq k} q_{k'} + \varphi\left(1 - \frac{q_k}{C_k}\right)\right)}, \quad \forall j \in [J]. \tag{EC.23}$$

The uniqueness and existence of the solution to (EC.23) follow from Remark 1.

The total social welfare is now defined as,

$$V(\mathbf{q}, \mathbf{C}) := \sum_{j=1}^J q_j \nu_j(q_j, C_j) = \sum_{j=1}^J q_j \left[b_j - \beta q_j^\theta - \omega \sum_{k \neq j} q_k + \varphi\left(1 - \frac{q_j}{C_j}\right)\right].$$

Therefore, at equilibrium, the total social welfare can also be represented as,

$$h(\mathbf{q}) = V(\mathbf{q}, \mathbf{C}) = \sum_{j=1}^J q_j \left[\log(q_j) - \log\left(1 - \|\mathbf{q}\|_1\right)\right].$$

The $y_j(\mathbf{q})$ function in (13) is modified as

$$y_j(\mathbf{q}) := b_j + \varphi - \beta q_j^\theta - \omega(\|\mathbf{q}\|_1 - q_j) + \log\left(1 - \|\mathbf{q}\|_1\right) - \log(q_j) \tag{EC.24}$$

for each $j \in [J]$ and we still define

$$\xi_j(\mathbf{q}) := \frac{\varphi q_j}{y_j(\mathbf{q})}.$$

The optimal capacity sizing with flow interactions can thus be formulated as the following optimization model,

$$\text{(Basic-2)} \quad \sup_{\mathbf{q}, \mathbf{C}} V(\mathbf{q}, \mathbf{C})$$
$$\text{s.t.} \quad \log(q_j) = \log\left(1 - \|\mathbf{q}\|_1\right) + b_j - \beta q_j^\theta - \omega(\|\mathbf{q}\|_1 - q_j) + \varphi\left(1 - \frac{q_j}{C_j}\right), \quad \forall j \in [J],$$

(10)–(12),

which is equivalent to the model (FLOW) with $y_j(\mathbf{q})$'s in $\xi_j(\mathbf{q})$ updated as (EC.24).

Next, we transform the model (FLOW) with flow interactions to a univariate optimization model. We redefine the $\zeta_{j,i}(x,z)$ in (17) as

$$\zeta_{j,i}(x,z) := \beta x^{\theta} - \omega x + \log(x) + \varphi\left(\mathbf{1}\{i=1\} + x\mathbf{1}\{i=2\}/\ell_j + x\mathbf{1}\{i=3\}/u_j\right) - \left(\log(1-z) + b_j + \varphi - \omega z\right) \tag{EC.25}$$

for $i \in \{1,2,3\}$, $j \in [J]$ and any given $z \in (0,1)$. Note that

$$\mathcal{Q}_j(z) = \{q_j \in [0,1] : \zeta_{j,1}(q_j,z) \geq 0, \zeta_{j,2}(q_j,z) \geq 0, \zeta_{j,3}(q_j,z) \leq 0\}$$

as defined in Remark 4. Then, the univariate optimization model with flow interactions is the same as the model (UNIVAR) with $\zeta_{j,i}(x,z)$'s in $\mathcal{Q}_j(z)$ updated as (EC.25).

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