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On weak/Strong Attractor for a 3-D Structural-Acoustic Interaction with Kirchhoff–Boussinesq Elastic Wall Subject to Restricted Boundary Dissipation

Irena Lasiecka[1,2] · José H. Rodrigues[3]

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Abstract
Existence of global attractors for a structural-acoustic system, subject to restricted boundary dissipation, is considered. Dynamics of the acoustic environment is given by a linear 3-D wave equation subject to locally distributed boundary dissipation, while the dynamics on the (flat) structural wall is given by a 2D-Kirchhoff-Boussinesq plate equation, subject to linear dissipation and supercritical nonlinear restoring forces. It is shown that the trajectories of the dynamical system defined on finite energy phase space are attracted asymptotically to a global attractor. The main challenges of the problem are related to: (i) superlinearity of the elastic energy of the structural component, (ii) Boussinesq effects of internal forces potentially leading to a finite time blowing up solutions, (iii) partially-restricted boundary dissipation placed on the interface only. The resulting system lacks dissipativity along with the suitable compactness properties, both corner stones of PDE dynamical system theories [1]. To contend with the difficulties, a new hybrid approach based on a suitable adaptation of the so called "energy methods" [2, 3] and compensated compactness [4] has been developed. The geometry of the acoustic chamber plays a critical role.

Keywords Structure-acoustic dynamical system · Kirchhoff-Boussinesq plate and interface · Boundary-interface dissipation · Global attractors

Mathematics Subject Classification 35B41 · 35L05 · 74K20 · 37L30 · 74F99

✉ Irena Lasiecka
lasiecka@memphis.edu

José H. Rodrigues
jose-rodrigues@oist.jp

1 Department of Mathematical Sciences, The University of Memphis, Memphis, TN, USA

2 IBS, Polish Academy of Sciences, Warsaw, Poland

3 Okinawa Institute of Science and Technology, Okinawa, Japan

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1 Introduction

We consider an acoustic-structure interaction which comprises of acoustic waves propagating in a three dimensional domain (an acoustic chamber Ω) and interacting with the elastic vibrations on a part of the boundary $\partial\Omega$. The acoustic chamber is surrounded by the elastic and hard walls. The elastic wall is modeled by a nonlinear plate equation (Kirchhoff-Boussinesq) defined on a two dimensional manifold $\Gamma_0 \subset \partial\Omega$. The remaining walls of the acoustic chamber, denoted by Γ_1, are hard walls. The interaction between two media occurs on the interface between the chamber and the structure. Acoustic waves propagating in a 3D bounded medium can be forced by an external source affecting the chamber and the elastic wall provides a feedback response to the medium in a form of vibrations. The goal is to study the long time behavior of such interaction.

Structural acoustic models have acquired a considerable attention in the literature. Physical aspects of the modeling and related technological applications can be traced back to [5] and more recently [6]. Control theoretic and PDE analysis of the coupled system can be found [7–14] and references therein. Such models have a wide variety of applications which include: noise reduction in the acoustic chambers, pressure reduction in a cockpit of a helicopter, sound control in a musical chamber. Therefore, understanding and having an ability to control the induced oscillations became so important. Good control of these interactions on the common interface (structural wall), which is the main carrier of propagation of oscillations, is fundamental to the phenomena described above. In the particular model under consideration, the nonlinearity governing the elastic wall due to the restorative forces is *supercritical*, with respect to the topology induced by the natural energy function. By supercritical, we mean the property that the nonlinear term can not be controlled by the topology of the phase space. On the other hand, this structure results from physical modeling of restorative forces which involves p-Laplacian—intrinsically not bounded on $H^2(\Gamma_0) \to L_2(\Gamma_0)$, unless $p = 2$. In addition, the Boussinesq effects, when left uncontrolled, may lead to finite time blow up of solutions [15, 16]. Thus, controlling the interaction of these effects with the overall behavior of the structure is of paramount interest. Of particular relevance are boundary control actions which are easily accessible through the external manipulations. On the other hand, mathematical aspects of control theory with restricted geometric support are challenging [11, 13, 17, 18], particularly, from the point of view of controllability and stabilization. The latter have been studied in the literature of the last two decades with main results pertinent to *linear models and dissipative interactions*. The system under consideration in this work, motivated by the applications, involves *supercritical* nonlinearity and lacks the *dissipativity*. The aim put forward is to study long time behavior in the situations when stabilization may not occur (the lack of dissipativity, multiple equilibria, potential blow ups of energy even in a finite time). This leads naturally, to a study of attractors-invariant sets which capture dynamics asymptotically in time. The goal is to show that subject certain conditions, a feedback control located *only* on the elastic wall, leads to an existence of global attractor.This is achieved under: (1) the geometric assumption of convexity imposed on the non-dissipated part of the boundary Γ_1 -Assumption (G), along with the requirement that support of the boundary acoustic dissipation contains the interactive part of the boundary Γ_0 (point 2 in Theorem 1); (2) non-explosion condition (i) in Theorem 2 imposed on the internal source f of the plate; and (3): sufficiently large damping affecting the plate, as relative to the strength of Boussinesq source-condition (ii) in Theorem 2 .

To our best knowledge, this is the first time such study is conducted within the context of structural acoustic interaction with structural *supercriticality* and *boundary restricted* feedback control. See [10, 18] for semilinear dissipative systems with internal/locally distributed damping. The problem, as stated, brings aboard number of challenging mathematical aspects within the realm of PDE dynamical systems of hyperbolic type. To appreciate this, we just list features such as the lack of dissipativity and supercriticality along with the associated loss of compactness—the corner stones of dynamical systems theories. In fact, solutions of a single structural component may blow up in a finite time. This is prevented by applying feedback control on the interface. Our final goal is to show that the overall dynamics is captured asymptotically by an attracting set. This will be accomplished by developing new tools in the theory of attractors which routed in compensated compactness are also capable of handling non-cooperative interactions.

1.1 The Model

Let Ω be a bounded and open subset of $\mathbb{R}^3$ with a smooth boundary $\Gamma = \partial\Omega$, which consists of two connected regions Γ_0 and Γ_1, with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0 is flat. We consider the following acoustic-structural system, written in the variables $\{z, w\}$, which correspond to the dynamics in the medium Ω and to the structural wall Γ_0.

Acoustic Medium. The acoustic dynamics in the variables (z, z_t) is given by the wave equation with boundary dissipation on Γ_0:

$$\begin{aligned} z_{tt} - c^2\Delta z + f_1 = 0, \quad \text{in } Q \equiv \Omega \times (0,\infty); \\ \partial_\nu z + l(x)z_t = \begin{cases} -l_0 z, & \text{on } \Sigma_1 \equiv \Gamma_1 \times (0,\infty); \\ w_t, & \text{on } \Sigma_0 \equiv \Gamma_0 \times (0,\infty); \end{cases} \\ z|_{t=0} = z_0, \quad z_t|_{t=0} = z_1, \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where ν stands for the outward unit vector to the boundary Γ, $c > 0$ and $l_0 \geq 0$ with the support on Γ_1, while $l(x)$ is a non-negative function over the boundary Γ and f_1 is a function defined on Ω representing a possible source of noise inside the acoustic medium which may be caused by the effects of a noise affecting the acoustic environment (for example coming from the rigid walls).

Structural Elastic Wall. The dynamics of the elastic wall Γ_0 is given by the following Kirchhoff-Boussinesq plate equation with internal forces: restorative of supercritical nature [related to p-Laplacian] and internal force due to a buckling.

$$\begin{aligned} w_{tt} + \Delta^2 w + k w_t = \operatorname{div}\{|\nabla w|^2 \nabla w\} + \sigma\Delta\{w^2\} - \rho z_t|_{\Gamma_0} - f(w), \quad \text{in } \Sigma_0; \\ w = 0, \ \nabla w = 0, \quad \text{on } \partial\Gamma_0 \times (0,\infty); \\ w|_{t=0} = w_0, \quad w_t|_{t=0} = w_1, \quad \text{in } \Gamma_0, \end{aligned} \tag{1.2}$$

where $k, \rho, \sigma \geq 0$ and f is a smooth function with $f(w)$ acting on Γ_0.

Remark 1 As a side note, one may notice that the plate model in (1.2) also arises within the context of asymptotic analysis of Midlin-Timoshenko system [19–22] when taking the limit with respect to the transverse shears and rotational inertia of filaments -[19].

Remark 2 Note that both wave (1.1) and plate (1.2) models do not account for any structural damping (added terms such as Δz_t or Δw_t) [23] and references within. These terms are known to introduce regularizing [parabolic like] effects on the dynamics, thus lending a very different analysis.

Remark 3 The model under consideration is equipped with clamped boundary conditions assumed for the plate equation (1.2). *Free* boundary conditions could also be considered, but these lead to substantial technicalities even in the case of the wellposedness theory [24]. In studying attractors, the analysis of free boundary conditions [25], within the context of structural acoustics, still needs some technical developments.

Remark 4 **Connection with the dynamic output feedback**. The system comprising of (1.1) and (1.2) can be also viewed as a *dynamic output feedback control system* -see [26] for a linear variant. To see this, one writes

$$\begin{aligned} & z_{tt} - c^2 \Delta z + f_1 = 0, \quad \text{in } Q \equiv \Omega \times (0, \infty); \\ & \partial_\nu z + l(x) z_t = \begin{cases} -l_0 z, & \text{on } \Sigma_1 \equiv \Gamma_1 \times (0, \infty); \\ u, & \text{on } \Sigma_0 \equiv \Gamma_0 \times (0, \infty); \end{cases} \\ & z|_{t=0} = z_0, \quad z_t|_{t=0} = z_1, \quad \text{in } \Omega, \end{aligned} \tag{1.3}$$

$$\begin{aligned} & w_{tt} + \Delta^2 w + k w_t = \text{div}\{|\nabla w|^2 \nabla w\} + \sigma \Delta\{w^2\} - y - f(w), \quad \text{in } \Sigma_0; \\ & w = 0, \ \nabla w = 0, \quad \text{on } \partial\Gamma_0 \times (0, \infty); \\ & w|_{t=0} = w_0, \quad w_t|_{t=0} = w_1, \quad \text{in } \Gamma_0, \end{aligned} \tag{1.4}$$

where the control u and output y are given by

$$u = w_t, \ y = \rho z_t|_{\Gamma_0} \tag{1.5}$$

It is known that for the *linear* unperturbed structure ($f_1 = 0$),the boundary control-output system with boundary control and boundary output feedback given in (1.5) is asymptotically stable whenever $l > 0$ on some subportion of the boundary [13]. This follows from the appropriate Unique Continuation properties known for hyperbolic dynamics [27]. The exponential stability [of the linear problem] is also known, but under much stronger condition: $l(x) \geq l_1 > 0$ with a support on a *full boundary* Γ or with *interior localised damping* [18] where technically demanding Carleman's estimates were used.. As a corollary of the present work, the exponential stability result will be shown to hold with the support of $l(x)$ reduced to the elastic wall Γ_0 only. This is possible owing to special geometric constructs of the flow multipliers. Such result -see Remark 9 -is of independent interest in the output boundary feedback control theory of structural acoustic systems [26].

Remark 5 The case when Γ_0 is curved can be treated by using Riemanian metric which reduces the model to variable coefficients case. Under suitable restrictions imposed on the curvature one could extend the treatment of the present work to a general non-flat framework by using Riemanian calculus developed for linear models [14, 14, 28] and most recently [29].

Although each component (acoustic and structural), when decoupled, have been treated in the literature, the interface problem presents new phenomenological peculiarities, leading to new effects emerging for the overall system. Just to give a glimpse: it is known that the unforced wave equation alone can be stabilized with the boundary damping, provided suitable geometric conditions are imposed [17, 30]. However, the interaction on an elastic wall Γ_0 introduces unbounded perturbation induced by the plate with supercritical nonlinearity which creates oscillations of the acoustic pressure. This provides a feedback in a form of the acoustic pressure acting on the wall. The ultimate end behavior may be complicated and certainly chaotic.

For all these reasons, the acoustic-structural models provide a plethora of interesting problems due to their interaction and propagation effects taking place on the part of the boundary-an interface (see [12, 13, 18, 31]). In the case of the model under consideration, the features of particular interest, challenge and novelty are the following: **(1)** dissipation acts only on the interface Γ_0 with hard walls [Neumann-Robin boundary conditions] on the remaining part of the boundary Γ; **(2)** restoring force on the structure is supercritical $div[|\nabla w|^2 \nabla w]$ - outside of the phase space; **(3)** Bousinesq force Δw^2 may lead to a finite time blow up. The latter leads to a non-dissipative structure of the dynamics-a challenge in the theory of dynamical systems [1].

It is known that the behavior of boundary solutions of hyperbolic-like dynamical systems is challenging and often requires estimates based on microlocal analysis which exhibits peculiar behavior of the boundary traces, see for instance [31–33]. Indeed, under the assumption that the boundary damping is active over the structural wall Γ_0 *only*, "hidden regularity" of the wave traces will be needed in order to for exhibit an existence of global attractors. In fact, this configuration asks for a more delicate asymptotic analysis of boundary terms where previous constructs based on flow multipliers are no longer applicable [34–38].

The main objective in this paper is to provide an analysis of long-time dynamics of energy solutions corresponding to full interaction (1.1)–(1.2). More precisely, in spite of the presence of a supercritical elastic source and of the "Boussinesq" forcing term, the existence of global attractors will be established (see Sect. 1.4) for the corresponding flow subjected to *restricted* boundary dissipation which is *localised only on the interface* Γ_0. To our best knowledge, this is a very first treatment of attractors for nonlinear structural acoustic models with the dissipation taking place only on the interface. More on this will be said when discussing the literature.

1.2 Notation

Denote $\mathcal{H} \equiv H_z \times H_w$ the *finite energy space*, where $H_z \equiv H^1(\Omega) \times L_2(\Omega)$ and $H_w \equiv H_0^2(\Gamma_0) \times L_2(\Gamma_0)$. Here H^s is the L^2-based Sobolev space or order $s \geq 0$. In addition, for $X = \Omega, \Gamma, \Gamma_0, \Gamma_1$, we denote $(\cdot, \cdot)_X$ the inner product in $L^2(X)$. Furthermore, $\mathcal{H}$ is a Hilbert space equipped with norm

$$\|\{z_0, z_1, w_0, w_1\}\|_{\mathcal{H}}^2 = \|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H^2(\Gamma_0)}^2 + \|w_1\|_{L^2(\Gamma_0)}^2,$$

for every $\{z_0, z_1, w_0, w_1\} \in \mathcal{H}$.

Using this notation, we define the *total energy* functional $\mathcal{E}(t) \equiv \mathcal{E}(\mathbf{u}(t))$ associated to a solution $\mathbf{u} = \{z, z_t, w, w_t\}$ of (1.1)–(1.2) by

$$\mathcal{E}(t) \equiv E(t) + \sigma \int_{\Gamma_0} w|\nabla w|^2 + \int_{\Gamma_0} \tilde{f}(w) + \int_{\Omega} f_1 z \tag{1.6}$$

where $\tilde{f}$ denotes anti-derivative of f and $E(t) \equiv E(\mathbf{u}(t)) = E_z(t) + E_w(t)$ with

$$E_z(t) = \frac{1}{2} \int_{\Omega} \left[|z_t|^2 + |\nabla z|^2\right] d\Omega + \frac{c^2 l_0}{2} \int_{\Gamma_1} |z|_{\Gamma_1}|^2 d\Gamma_1, \tag{1.7}$$

$$E_w(t) = \frac{1}{2} \int_{\Gamma_0} \left[|w_t|^2 + |\Delta w|^2 + \frac{1}{2}|\nabla w|^4\right] d\Gamma_0, \tag{1.8}$$

stand for the corresponding positive energy in the medium and structural wall. Formally, the energy functional satisfies the following energy balance equation

$$\mathcal{E}(t) + \int_s^t D(\tau) d\tau = \mathcal{E}(s) + \sigma \int_s^t \int_{\Gamma_0} w_t |\nabla w|^2, \quad s \leq t, \tag{1.9}$$

$$\text{where } D(t) \equiv D(\mathbf{u}(t)) = k \int_{\Gamma_0} |w_t|^2 + c^2 \int_{\Gamma} l(x)|z_t|_{\Gamma}|^2 \tag{1.10}$$

stands for the "damping" term. The above identity [written for the time being only formally] shows that the total energy $\mathcal{E}(t)$ is not necessarily dissipative ($\sigma > 0$).

1.3 Well-posedness of the Dynamical System

We say that a pair of functions $\{z, w\}$ is a *weak solution* on the interval $[0, T]$, for $T > 0$, if $\{z, z_t, w, w_t\} \in L_\infty(0, T; \mathcal{H})$ and the following properties are satisfied:

- (i) the map $t \in [0, T] \mapsto \{z(t), z_t(t), w(t), w_t(t)\} \in \mathcal{H}$ is weakly continuous and, in addition, $l^{1/2} z_t|_\Gamma \in L_2(0, T; L_2(\Gamma))$;
- (ii) $\{z(0), z_t(0), w(0), w_t(0)\} = \{z_0, z_1, w_0, w_1\}$;
- (iii) the pair (z, w) is a distributional (in time) solution of the following equation

$$\begin{aligned} 0 = \frac{d}{dt} &\left[(z_t, \phi)_\Omega + (w_t, \psi)_{\Gamma_0} + \rho(z|_{\Gamma_0}, \psi)_{\Gamma_0} \right] \\ &+ c^2(\nabla z, \nabla \phi)_\Omega + c^2 l_0(z|_{\Gamma_1}, \phi|_{\Gamma_1})_{\Gamma_1} + (\Delta w, \Delta \psi)_{\Gamma_0} \\ &+ c^2(l^{\frac{1}{2}} z_t|_\Gamma, l^{\frac{1}{2}} \phi|_\Gamma)_\Gamma + k(w_t, \psi)_{\Gamma_0} - c^2(w_t, \phi|_{\Gamma_0})_{\Gamma_0} + (f_1, \phi)_\Omega + G(w, \psi), \end{aligned} \tag{1.11}$$

for all $(\phi, \psi) \in H^1(\Omega) \times H_0^2(\Gamma_0)$, where

$$G(u, v) \equiv \int_{\Gamma_0} \left[|\nabla u|^2 \nabla u \cdot \nabla v + \sigma \nabla\{u^2\} \cdot \nabla v + f(u)v \right] d\Gamma_0, \tag{1.12}$$

for every $u, v \in H_0^2(\Gamma_0)$. Furthermore, we say that a weak solution $\{z, w\}$ on the interval $[0, T]$ is *strong* (or classical) if $\{z, z_t, z_{tt}\} \in C(0, T; H^{2-\theta}(\Omega) \times H_z)$, $0 < \theta < 1$, and $\{w, w_t, w_{tt}\} \in C(0, T; H^4(\Gamma_0) \times H_w)$.

Remark 6 The reason why $0 < \theta < 1$ in the definition of strong solution is due to the fact that Robin boundary conditions with a potential jump in $l(x)$ on Γ leads to slightly compromised regularity of the displacement z [39]. If one assumes that $l(x)$ is continuous across the interface, the incremental loss of differentiability does not occur.

Existence, uniqueness and continuous dependence of of solutions of (1.1)–(1.2) has been shown in [24]. The corresponding result is formulated below.

Theorem 1 (Existence-Uniqueness, Robustness and Regularity) *With respect to the dynamics of system* (1.1)–(1.2)*, the following holds:*

1. *Existence of a strongly continuous semigroup in a weak topology of the phase space $\mathcal{H}$. Let $f_1 \in L_2(\Omega)$, and $f \in C^1(\mathbb{R})$ satisfies the following condition:*

$$\tilde{f}(s) = \int_0^s f(\xi)d\xi \geq -\delta s^4 - \beta, \quad \textit{for some } \delta \geq 0 \textit{ and } \beta \in \mathbb{R}. \tag{1.13}$$

For every initial data $\mathbf{u}_0 \equiv \{z_0, z_1, w_0, w_1\} \in \mathcal{H}$, there exists a unique weak solution $\{z, w\}$ in the class $\mathbf{u} \equiv \{z, z_t, w, w_t\} \in C([0, T]; \mathcal{H})$, for any $T > 0$. Furthermore, the solutions generate a strongly continuous semigroup S_t with respect to weak topology in $\mathcal{H}$, given by the formula

$$S_t \mathbf{u}_0 \equiv \mathbf{u}(t), \quad \textit{for every } \mathbf{u}_0 \in \mathcal{H}. \tag{1.14}$$

This is to say that continuous dependence of solutions on initial data is with respect to weak topology of $\mathcal{H}$. The corresponding solutions satisfy the following energy inequality

$$\mathcal{E}(t) + \int_s^t D(\tau)d\tau \leq \mathcal{E}(s) + \sigma \int_s^t (w_t, |\nabla w|^2)_{\Gamma_0} d\tau, \quad \text{for } s \leq t. \tag{1.15}$$

2. *Energy identity and strong continuity with respect to the initial data. In the case when $\Gamma_0 \subset supp\{l\}$, the weak solution $\{z, w\}$ must satisfy (1.15) where "$\leq$" is replaced by equality "$=$". In addition, the nonlinear semigroup in part 1 becomes continuous with respect to the strong topology of $\mathcal{H}$.*
3. *Regularity. Assume that Ω is sufficiently smooth. Weak solutions in reference above become strong provided that $\mathbf{u}_0 \in H^{2-\theta}(\Omega) \times H^1(\Omega) \times H^4(\Gamma_0) \times H^2(\Gamma_0)$, $0 < \theta < 1$ satisfy the requisite compatibility conditions on the boundary.*

$$\partial_\nu z_0 + l(x) z_1 = \begin{cases} -l_0 z_0 & on\ \Gamma_1; \\ w_1 & on\ \Gamma_0; \end{cases}$$

$$w_0 = w_1 = 0 \text{ and } \nabla w_0 = \nabla w_1 = 0 \text{ on } \partial\Gamma_0.$$

Remark 7 The growth condition (1.13) yields to a lower estimate for total energy functional $\mathcal{E}$ (see [21]), namely there are constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$c_1 E(\mathbf{u}) - c_2 \leq \mathcal{E}(\mathbf{u}), \quad \text{for every } \mathbf{u} \in \mathcal{H}. \tag{1.16}$$

This estimate plays an important role in the analysis of system (1.1)–(1.2).

1.4 Main Results

Our main goal is to establish an existence of global attractor for the dynamical system $\{\mathcal{H}, S_t\}$. We begin by recalling needed definitions and properties [1, 40–42] and references therein. A bounded closed subset $\mathfrak{A} \subset \mathcal{H}$ is said to be a *global attractor* for S_t if the following conditions hold:

(1) $\mathfrak{A}$ is an invariant set; that is, $S_t\mathfrak{A} = \mathfrak{A}$ for $t \geq 0$.
(2) $\mathfrak{A}$ is uniformly attracting; that is, for all bounded set $D \subset \mathcal{H}$, we have

$$\lim_{t\to\infty} d_{\mathcal{H}}\{S_t D \mid \mathfrak{A}\} = 0;$$

where $d_{\mathcal{H}}\{A \mid B\} = \sup_{x\in A} d_{\mathcal{H}}(x, B)$ is a Hausdorff semidistance.

In what follows we shall assume the following geometric assumption:

(G) Let Γ_1 be star-shaped and convex. This is to say there exists $x_0 \in R^n$ such that $(x - x_0) \cdot \nu(x) \leq 0$ for $x \in \Gamma_1$, Γ_1 is convex and $\nu(x)$ is external normal direction to Γ_1 at the point x.

Our result is stated below:

Theorem 2 *Existence of Global Attractor Assume that (G) holds and $\Gamma_0 \subset supp\{l\}$. In addition, assume that*

(i) $f \in C^1(\mathbb{R})$ satisfies the non-explosion condition condition

$$\liminf_{|s|\to\infty} \frac{f(s)}{s|s|^2} \geq 0 \tag{1.17}$$

(ii) The following relation holds

$$\sigma^2 < \frac{k}{4}\min\{1, k\}. \tag{1.18}$$

Then, the evolution operator S_t possess a weak compact global attractor on $\mathcal{H}$. If, in addition, $\sigma = 0$ the said attractor is strong on $\mathcal{H}$.

Remark 8 One could assume much more general structure of the source function f_1 to be given as a Nemytski's operator by a function $f_1(z, x)$ with appropriate growth (typically cubic) conditions [4, 10, 35]. However, in order to focus the attention on structural nonlinearity-which is the main difficulty and novelty in the problem, we keep the exposition simple by restricting the range of functions interpreted as a noise present in an acoustic cabin.

Remark 9 Assuming, in addition, that $f_1 = 0, \sigma = 0$ and condition (1.17) holds for all $s \in R$, all weak solutions converge exponentially to zero. This follows from the proof of the first part in Theorem 2 after accounting for these additional hypotheses and using the technique as in [17]

1.5 Methodology and Discussion Within the Context of the Literature

Over the last decade or so,there has been a resurgence of interest in the theory of structural acoustic interactions. Stimulated by physical/engineering applications, the new acoustic models lend themselves to the study by analytical and computational methods. The area has witnessed a broad range of new mathematical results and developments [23, 43].

Questions such as well-posedness of solutions, optimal control, stabilization and long time behavior have been the topics of prime interest with many results already in place [7–9, 12, 13, 18, 28, 31, 32, 34, 44, 45]. However, the models of past analyses did not account for either *super-criticality* of the structural sources, or destabilizing effects of the nonlinear forces [46–51]. These are the game changers within the context of structural acoustic interaction particularly with the dissipation reduced to the interface Γ_0 only. *Super criticality* of the structural forces, dictated by physical considerations within the framework of *coupled structural interaction*, is the focal point. Within this framework, one should mention [18] where attractors for structural acoustic interactions with nonlinear dissipation-acting in Ω and on Γ_0 subject to nonlinear forcing were considered. This is a rather comprehensive structural acoustic model, which however does not address super-criticality and the lack of dissipativity on the elastic wall.

This prompts a natural question: *what are the challenges in proving the main result which states that boundary dissipation on an elastic wall alone leads to an overall stable nonlinear dynamics with an attraction property?* Since the structural interaction consists of hybrid coupling between wave and plate dynamics, the results pertaining to these isolated dynamics are of obvious relevance. Wave dynamics with *critical nonlinearities* in 3-D has been considered in the past and a mechanism for handling the lack of compactness is in place—see [4, 52] and references within. For quintic waves, with however a full interior linear damping, Strichratz estimates [3] provide an effective tool. In the case of geometrically restricted damping, such as localised [18, 53]—or boundary damping—potentially nonlinear [54], Carleman;s estimates have been successfully employed. They allow to propagate the dissipation from geometrically restricted area to the full domain. The developments in the area of the waves led to several results obtained for structural acoustic models with *non-compact nonlinearities*, full interior nonlinear damping placed on the wave *and* plate [10]. Geometrically restricted interior damping has been considered by exploiting Carleman's estimates developed in [53] and used later in [18]. In all these works the overall system under consideration is a gradient [dissipative] system. It is well known that the two main corner stones of the theory of attractors are: *dissipativity* of the system and *compactification* of the nonlinear effects. Dissipativity is used as an ingredient to assert existence of an absorbing ball. Compactness is needed in order to establish strong convergence of trajectories to a bounded invariant set. Neither of these properties are present in the model under consideration. Dissipativity is lost due to the Boussinesq force $\sigma > 0$, so the system does not enjoy gradient property. The compactness of trajectories, or even local Lipschitz continuity, in finite energy space is out of reach due to supercriticality of restoring forces.

Methodology In order to show that the overall system is absorbing, in the case of of dissipation restricted to the boundary or interface, it is necessary to "propagate" the dissipation from the interface into the interior of the domain. This point becomes problematic when dealing with wave dynamics. The wave dynamics, alone, is subject to Neumann's boundary data which are not fully dissipated (the presence of the hard walls). As such, the boundary data do not satisfy Lopatinski condition [55]. It is well known that in the absence of Lopatinski condition there is a loss of 1/3 (one third) of derivative [33] (with respect to energy norm) which provides for a major predicament when studying long time dynamics of waves with the so called "uncontrolled" Neumann boundary. To handle this part, very special flow multipliers [30] are applied which propagate the damping from Γ_0 to Ω. The above construct allows to propagate the damping modulo lower order terms (terms below the topology of the phase space). The latter are absorbed by special unique continuation type argument [35] which allows to establish *ultimate dissipativity*. This leads to a construction of *weak* attractor in the *finite energy space*—result stated in the first part of Theorem 2. The issue of improving *weak* to *strong* is haltered by super-criticality in plate dynamics. The restorative force modeled by p-Laplacian is not bounded as on the phase space. Super-criticality has been handled in the literature [2, 3, 56] by using the so called **Lyapunov energy identity**. And this method is also successful when dealing with a single plate model [21]. However, the presence of the boundary absorbing component in the wave dynamics, and the resulting lack of time reversibility for the entire structure, prevents from obtaining the Lyapunov energy *identity* for the overall system (note that Energy *identity* is valid for the overall system see Theorem 1—due to the cancelations occurring on the interface. But not the Lyapunov Energy identity where such cancellations do not occur). This leads to a necessity of first asserting strong convergence of the acoustic energy, without any reliance on the said "cancellations". The latter requires all together different methodology- compensated compactness method based on the estimates of the differences of solutions [18, 44, 53, 54]. However, this again, meets with the challenge due to the super-criticality of the energy restorative term modeled by p-Laplacian—thus creating a vicious circle effect.

To resolve this vicious circle, the idea is to exploit the validity of Lyapunov energy “equality” only for the elastic structure perturbed by boundary-interface acoustic pressure.The latter is controlled by the hyperbolic "hidden trace" regularity. This will lead to strong convergence of plate energy by energetic arguments inspired by [2, 56], see also [3] for the quintic wave equations. As pointed out above, the energy method described above can not be applied to the full interaction due to the presence of boundary pressure and the resulting irreversibility of wave forward dynamics. The wave component *does not cooperate* with the structure due to the boundary interaction. On the other hand, the asymptotic smoothness for waves with a boundary damping [35, 53] is typically shown by using compensated compactness method [4] which requires estimates on the *difference* of two solutions. However, this aspect of the method does not cooperate with supercriticality of the plate dynamics. This is in contrast to

[18] where structural acoustic model with partial *interior* damping imposed on the wave and *up to critical* nonlinear plate effects were considered. The latter are modeled by Nemytski's operator bounded on the phase space. In this latter configuration, which leads to a gradient system, compensated compactness method is applied to the difference of two trajectories of the *full interaction* by exploiting natural cancellations in the model. This is no longer an option in the presence of supercritical terms in the plate dynamics. The above conundrum is resolved by discovering additional mechanisms of compensation when accounting for the full interaction and analyzing separately the two parts of the dynamics which *do not cooperate* on the *boundary interface*. The final part of the argument amounts to exploiting decays of non-cooperative interface part of the dynamics with the convergence of both structural and acoustic elements resulting from the so called *hidden boundary regularity* and compensated compactness. As a consequence, the designed hybrid method depends on the following features: energy *equality* for the overall system, the support of l must contain Γ_0 and the *linearity* of the dissipation kw_t imposed on the interface Γ_0.

One should also note that the supercriticality of plate dynamics raises the fundamental issue of wellposedness (uniqueness and continuous dependence) of weak solutions. This preliminary step, and a starting point of the present analysis, has been resolved in [24] by exploiting "hidden" trace regularity secured due to a strategic location of boundary dissipation along with logarithmic control of Sobolev's embeddings-see also [21] where a single Kirchhofff-Bousinesq plate was considered.

1.6 Possible Extensions and Open Problems

While there are several aspects of the model which can be generalized without a substantial new effort, there are some others which, at the present stage, do not appear to have a solution in sight. We shall elaborate on these. The method proposed in the present work can be easily adjusted to a more general wave model where sources up to critical exponenti.e; locally Lipschitz $H^s(\Omega) \to L_2(\Omega)$, $s \geq 1$, with appropriate dissipativity assumptions are added. Moreover, the boundary dissipation on the wave component can be taken in a more general nonlinear form such as $l(x)g(z_t)$ where the function $g(s)$ is continuous, monotone increasing with appropriate growth restrictions and $g(0) = 0$. This is in line with the results available for single waves [4] and also for structural acoustic models in [18]. However, the restriction to linear dissipation on the plate component is intrinsic to the (energy) method, see [2, 3]. Similarily, the results related to the properties of the attractor such as smoothness or finite dimensionality are not within the reach at this moment, in contrast to [18]. The main reason is that the analysis of smoothness requires estimates on the differences of two solutions—where super criticality is again an obstacle. Notable difference are quintic wave equations [3] with an *interior linear damping* where one can resort to Strichratz estimates and backward regularity of the trajectories on the attractor. Here, again, boundary damping and the lack of reversibility of the wave dynamics is the predicament.

2 Existence of Global Weak Attractor

This section is devoted to the proof of the first statement in Theorem 2. First, we note that, as a consequence of the assumption $\Gamma_0 \subset \text{supp}\{l\}$, we are dealing with a dynamical system $\{\mathcal{H}, S_t\}$ which is strongly continuous with respect to the finite energy of $\mathcal{H}$ -see Theorem

1. Moreover, the *energy identity* (see (1.15) with "=" instead of "≤") has been previously established [24].

2.1 Ultimate Dissipativity

The first step, in fact highly challenging in the present context in the proof of Theorem 2, is to show that the system $\{\mathcal{H}, S_t\}$ is *ultimately dissipative* [57, 58]. It should be noted that the system does not possess a *gradient structure*, so the validity of the absorbing property does not follow from the usual arguments in dynamical system theory based on asymptotic compactness [4]. In the present case, existence of a bounded *absorbing ball* needs to be established independently.

This is accomplished in Proposition 1 below.

Proposition 1 *Under the assumptions of Theorem 2, there exists an absorbing set $\mathcal{B} \subset \mathcal{H}$ for the system (1.1)–(1.2), i.e. for all $R_0 > 0$ there exists $t_0 = t_0(R_0)$ such that, for every initial data $\mathbf{u}_0 \in \mathcal{H}$ satisfying $\|\mathbf{u}_0\|_{\mathcal{H}} \leq R_0$ we have*

$$S_t\mathbf{u}_0 \in \mathcal{B}, \quad \text{for } t \geq t_0. \tag{2.1}$$

The proof is technical and it proceeds through several lemmas.

The key ingredient is a careful reconstruction of the energy from the dissipation modulo some constant which however *does not depend on the solution.* This reconstructions of energy is given by the following Lemma.

Lemma 1 *Under the assumptions of Proposition 1, let $\{z, w\}$ be a solution of system (1.1)–(1.2). Then, for $T > 0$ sufficiently large, the corresponding energy functional $\mathcal{E}$ satisfies*

$$T\mathcal{E}(T) \leq C_T \left\{ \int_0^T D(t)dt + \underset{[0,T]}{\text{l.o.t.}}\{z\} \right\} + K_T, \tag{2.2}$$

where $C_T > 0$ and $K_T \in \mathbb{R}$ are constants that do not depend on the initial data, and

$$\underset{[0,T]}{\text{l.o.t.}}\{z\} \equiv \sup_{t\in[0,T]} \|z(t)\|^2_{L^2(\Omega)} + \sup_{t\in[0,T]} \|z(t)\|^2_{L_2(\Gamma)} \tag{2.3}$$

The proof of the estimate (2.2) requires a reconstruction of acoustic and elastic energies governed by the dynamics (1.1) and (1.2).

Remark 10 From this moment on, we are going to use the symbol "$\lesssim$" to express an inequality in which the constants that do not depend on fixed instants of time or the energy are omitted.

2.1.1 Reconstruction of the Elastic Energy

Lemma 2 (Plate Energy Reconstruction) *Let $T > 0$. If $\{z, w\}$ is a strong solution of (1.1)–(1.2), then there exist constants $C_1 > 0$ and $K_{1,T} \in \mathbb{R}$ such that the following estimate holds:*

$$\int_0^T \left[E_w(t) + \int_{\Gamma_0} f(w(t))w(t) \right] dt \leq C_1 \left\{ E_w(0) + E_w(T) + \int_0^T D(t)dt \right\} + K_{1,T} \tag{2.4}$$

Proof We are working with smooth solutions first. Multiplying the plate equation (1.2) by w and integrating by parts on $\Sigma_0^T \equiv \Gamma_0 \times (0,T)$, we obtain:

$$\begin{aligned}
&\int_{\Sigma_0^T}\big[|w_t|^2+|\Delta w|^2+|\nabla w|^4\big]+\int_{\Sigma_0^T} f(w)w \\
&= -\beta_0^T+\frac{2}{k}\int_0^T k\|w_t\|_{\Gamma_0}^2-\rho\int_{\Sigma_0^T} z_t w-2\sigma\int_{\Sigma_0^T} w|\nabla w|^2 \qquad (2.5)\\
&\text{with}\quad \beta_0^T \equiv \left[(w_t,w)_{\Gamma_0}+\frac{k}{2}\|w\|_{L^2(\Gamma_0)}^2\right]\Bigg|_0^T .
\end{aligned}$$

The terms in the right-hand-side (RHS) of (2.5) are handled next using Hölder inequality and Poincaré-type inequality as follows:

$$|\beta_0^T| \le E_w(T)+E_w(0); \qquad (2.6)$$

$$\left|\rho\int_{\Sigma_0^T} z_t w\right| \le \frac{\rho^2}{2}\int_{\Sigma_0^T}|z_t|^2+\frac{1}{2}\int_{\Sigma_0^T}|w|^2 \lesssim \int_0^T\|l^{1/2}z_t\|_{L^2(\Gamma)}^2+\varepsilon\int_{\Sigma_0^T}|\nabla w|^4+C_{T,\varepsilon}; \qquad (2.7)$$

$$\left|2\sigma\int_{\Sigma_0^T} w|\nabla w|^2\right| \le \varepsilon\int_{\Sigma_0^T}|\nabla w|^4+\frac{\sigma^2}{\varepsilon}\int_{\Sigma_0^T}|w|^2 \lesssim \varepsilon\int_{\Sigma_0^T}|\nabla w|^4+C_{T,\varepsilon}, \qquad (2.8)$$

for any $\varepsilon > 0$, where $C_{T,\varepsilon}$ is a positive constant.

Returning to (2.5) and using (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
&\int_{\Sigma_0^T}\big[|w_t|^2+|\Delta w|^2+(1-\varepsilon)|\nabla w|^4\big] \\
&+\int_{\Sigma_0^T} f(w)w \lesssim E_w(0)+E_w(T)+\int_0^T D(t)dt+C_{T,\varepsilon}, \qquad (2.9)
\end{aligned}$$

for every $\varepsilon > 0$. Choosing $\varepsilon > 0$ small enough, inequality (2.4) thus follow from (2.9) by setting the constants C_1 and $K_{1,T}$ properly. □

2.1.2 Reconstruction of the Acoustic Energy

This part is way more challenging and it requires careful consideration of the geometry. The dissipation is localised on Γ_0 and one needs to reconstruct the energy propagating in tangential directions to Γ_1. We recall that the wave equation (1.1) considers Neumann boundary conditions with the boundary damping locally distributed on Γ_0 only. This leads to an *uncontrolled portion* Γ_1 *of the boundary* that remains under Neumann boundary conditions. It is known that in such cases standard "boundary" multipliers do not work [30]. In view of this, a special multiplier is considered which accounts for the geometry and it is constructed using a very specific vector field acting on Γ. Such approach was introduced in [30] and leads to the presence of *tangential derivatives* which provide for an additional technical difficulty -see [34].

Under the geometrical assumption (G) imposed on Γ_1, a key point is to construct a vector field $h \in C^1(\mathbb{R}^3)$ such that

$$h\cdot\nu = 0 \text{ on } \Gamma_1 \text{ and } J_h x\cdot x \ge c_0|x|^2 \text{ for every } x\in\mathbb{R}^3, \qquad (2.10)$$

for some $c_0 > 0$, where J_h denotes the Jacobian of h. As argued in [30], (Appendix A.4 p.301), the above can be done by using the appropriate convexity arguments validated by the fact that *uncontrolled* part of the boundary Γ_1 is *convex* and $(x - x_0) \cdot \nu(x) \leq 0$ on Γ_1. These two conditions are critical for the construction of the needed vector field h -see Lemma II.I, Appendix II [59].

Lemma 3 *Let $T > 0$. If $\{z, w\}$ is a strong solution of (1.1)–(1.2), and the Assumptions of Theorem 2 hold, then there are positive constants C_2, $C_{2,T}$ and $K_{2,T}$ such that the following energy estimate holds*

$$\int_0^T E_z(t)dt \leq C_2 \left\{ E_z(0) + E_z(T) + \int_0^T D(t) \right\} + C_{2,T} \underset{[0,T]}{\text{l.o.t.}}\{z\} + K_{2,T}, \tag{2.11}$$

for every $\varepsilon > 0$, where $\underset{[0,T]}{\text{l.o.t.}}\{z\}$ is given in (2.3).

Proof We begin with the classical step: multiplying the wave Eq. (1.1) by $h \cdot \nabla z$, integrating by parts in $Q_\alpha^{T-\alpha} \equiv \Omega \times (\alpha, T - \alpha)$ and using the divergence theorem, one obtains

$$\begin{aligned} \int_{Q_\alpha^{T-\alpha}} c^2 J_h \nabla z \cdot \nabla z = &- (z_t, h \cdot \nabla z)_\Omega \Big|_\alpha^{T-\alpha} - \frac{1}{2} \int_{Q_\alpha^{T-\alpha}} \left[|z_t|^2 - c^2 |\nabla z|^2 \right] \operatorname{div}\{h\} \\ &+ \frac{1}{2} \int_{\Sigma_\alpha^{T-\alpha}} \left[|z_t|^2 - c^2 |\nabla z|^2 \right] (h \cdot \nu) + c^2 \int_{\Sigma_\alpha^{T-\alpha}} (h \cdot \nabla z) \partial_\nu z - \int_{Q_\alpha^{T-\alpha}} f_1 h \cdot \nabla z, \end{aligned} \tag{2.12}$$

where $\Sigma_\alpha^{T-\alpha} \equiv \Gamma \times (\alpha, T - \alpha)$. Now, we multiply (1.1) by $z\operatorname{div}\{h\}$ and integrate by parts in $Q_\alpha^{T-\alpha}$ in order to obtain

$$\begin{aligned} \frac{1}{2} \int_{Q_\alpha^{T-\alpha}} \left[|z_t|^2 - c^2 |\nabla z|^2 \right] \operatorname{div}\{h\} = &\ (z_t, (1/2) z \operatorname{div}\{h\})_\Omega \Big|_\alpha^{T-\alpha} - c^2 \int_{\Sigma_\alpha^{T-\alpha}} (1/2) z \operatorname{div}\{h\} \partial_\nu z \\ &+ \frac{c^2}{2} \int_{Q_\alpha^{T-\alpha}} z (\nabla z \cdot \nabla \operatorname{div}\{h\}) + \frac{1}{2} \int_{Q_\alpha^{T-\alpha}} f_1 z \operatorname{div}\{h\} \\ \leq &\ (z_t, (1/2) z \operatorname{div}\{h\})_\Omega \Big|_\alpha^{T-\alpha} - c^2 \int_{\Sigma_\alpha^{T-\alpha}} (1/2) z \operatorname{div}\{h\} \partial_\nu z \\ &+ \varepsilon \int_{Q_\alpha^{T-\alpha}} c^2 |\nabla z|^2 + C_\varepsilon \int_{Q_\alpha^{T-\alpha}} |z|^2 + \frac{1}{2} \int_{Q_\alpha^{T-\alpha}} f_1 z \operatorname{div}\{h\}, \end{aligned} \tag{2.13}$$

for every $\varepsilon > 0$, where C_ε is a positive constant.

Combining (2.12), (2.13) and choosing $\varepsilon = \frac{c_0}{2}$, we obtain

$$\begin{aligned} \int_{Q_\alpha^{T-\alpha}} c^2 |\nabla z|^2 \leq &- \frac{2}{c_0} (z_t, h \cdot \nabla z - (1/2) z \operatorname{div}\{h\})_\Omega \Big|_\alpha^{T-\alpha} + \frac{1}{c_0} \int_{\Sigma_\alpha^{T-\alpha}} \left[|z_t|^2 - c^2 |\nabla z|^2 \right] (h \cdot \nu) \\ &+ \frac{2c^2}{c_0} \int_{\Sigma_\alpha^{T-\alpha}} \partial_\nu z \left[h \cdot \nabla z - (1/2) z \operatorname{div}\{h\} \right] - \frac{2}{c_0} \int_{Q_\alpha^{T-\alpha}} f_1 [h \cdot \nabla z + (1/2) z \operatorname{div}\{h\}] \\ &+ \frac{2T}{c_0} C_{h,\varepsilon} \sup_{t \in [0,T]} \|z(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{2.14}$$

The kinetic energy is reconstructed multiplying (1.1) by z and integrating by parts in $Q_\alpha^{T-\alpha}$, namely

$$\frac{1}{2}\int_{Q_\alpha^{T-\alpha}}|z_t|^2-\frac{1}{2}\int_{Q_\alpha^{T-\alpha}}c^2|\nabla z|^2=(z_t,(1/2)z)_\Omega\big|_\alpha^{T-\alpha}-\frac{c^2}{2}\int_{\Sigma_\alpha^{T-\alpha}}z\partial_\nu z-\frac{1}{2}\int_{Q_\alpha^{T-\alpha}}f_1 z. \tag{2.15}$$

Adding (2.14) and (2.15) we conclude

$$\frac{1}{2}\int_{Q_\alpha^{T-\alpha}}\left[|z_t|^2+c^2|\nabla z|^2\right]\le\gamma_\alpha^T+B\Sigma_\alpha^T+SQ_\alpha^T+\frac{2T}{c_0}C_{h,\varepsilon}\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)} \tag{2.16}$$

where the following notation was introduced

$$\text{[pointwise energy terms]}\quad \gamma_\alpha^T\equiv-\frac{1}{c_0}\left(z_t,2h\cdot\nabla z-z\left[\operatorname{div}\{h\}+\frac{c_0}{2}\right]\right)_\Omega\Big|_\alpha^{T-\alpha}; \tag{2.17}$$

$$\text{[boundary terms]}\quad B\Sigma_\alpha^T\equiv\frac{1}{c_0}\int_{\Sigma_\alpha^{T-\alpha}}\left[|z_t|^2-c^2|\nabla z|^2\right](h\cdot\nu)+\frac{c^2}{c_0}\int_{\Sigma_\alpha^{T-\alpha}}\partial_\nu z\left(2h\cdot\nabla z-z\left[\operatorname{div}\{h\}+\frac{c_0}{2}\right]\right). \tag{2.18}$$

$$\text{[internal source terms]}\quad SQ_\alpha^T\equiv-\frac{2}{c_0}\int_{Q_\alpha^{T-\alpha}}f_1\left[h\cdot\nabla z+z(1/2)(\operatorname{div}\{h\}+c_0/2)\right]. \tag{2.19}$$

Estimating γ_α^T. Pointwise energy terms (2.17) are handled by performing straightforward calculations

$$\left|\gamma_\alpha^T\right|\lesssim E_z(\alpha)+E_z(T-\alpha)+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}. \tag{2.20}$$

Estimating $B\Sigma_\alpha^T$. For the boundary terms, we first use property $h\cdot\nu=0$ over Γ_1 (see assumption (2.10)) and the decomposition $\nabla z=\partial_\nu z\,\nu+\nabla_\tau z$, where τ denotes tangential direction, which yields the following estimate for the first integral in (2.18)

$$\left|\frac{1}{c_0}\int_{\Sigma_\alpha^{T-\alpha}}\left[|z_t|^2-c^2|\nabla z|^2\right](h\cdot\nu)\right|\le\frac{1}{c_0}\max_{x\in\overline{\Gamma_0}}|h(x)|\left\{\int_\alpha^{T-\alpha}\int_{\Gamma_0}\left[|z_t|^2+c^2|\partial_\nu z|^2+c^2|\nabla_\tau z|^2\right]\right]$$
$$\lesssim\int_0^T D+\int_\alpha^{T-\alpha}\int_{\Gamma_0}|\nabla_\tau z|^2, \tag{2.21}$$

where the boundary conditions of (1.1) were used. The remaining integral in the RHS of (2.18) is rewritten in the form

$$\int_{\Sigma_\alpha^{T-\alpha}}\partial_\nu z\left(2h\cdot\nabla z-z\left[\operatorname{div}\{h\}+\frac{c_0}{2}\right]\right)=2\underbrace{\int_{\Sigma_\alpha^{T-\alpha}}\partial_\nu z(h\cdot\nabla z)}_{I_1}-\underbrace{\int_{\Sigma_\alpha^{T-\alpha}}z\partial_\nu z\left[\operatorname{div}\{h\}+\frac{c_0}{2}\right]}_{I_2}$$
$$\equiv 2I_1-I_2. \tag{2.22}$$

Estimate for I_2. Using trace theorem and the boundary conditions in (1.1), we have

$$|I_2|\le\frac{1}{2}\left(\max_{x\in\overline{\Gamma}}\left|\frac{c_0}{2}+\operatorname{div}\{h(x)\}\right|\right)\int_0^T\left[\|\partial_\nu z\|^2_{L^2(\Gamma)}+\|z\|^2_{L^2(\Gamma)}\right]$$
$$\lesssim\int_0^T D+T\sup_{t\in[0,T]}\|z(t)\|^2_{H^{1/2}(\Omega)}. \tag{2.23}$$

Estimate for I_1. Using the decomposition $\nabla z = \partial_\nu z\ \nu + \nabla_\tau z$ on Γ, we have

$$I_1 = \int_{\Sigma_\alpha^{T-\alpha}} \partial_\nu z(h\cdot[\partial_\nu z\ \nu + \nabla_\tau z]) = \underbrace{\int_{\Sigma_\alpha^{T-\alpha}} |\partial_\nu z|^2(h\cdot\nu)}_{I_{11}} + \underbrace{\int_{\Sigma_\alpha^{T-\alpha}} \partial_\nu z(h\cdot\nabla_\tau z)}_{I_{12}}. \quad (2.24)$$

Now, using assumption (2.10) on h it follows that $\operatorname{supp}\{h\cdot\nu\}\cap\Gamma\subset\Gamma_0$, from which we compute the first integral in the RHS of (2.24) as follows

$$\begin{aligned} I_{11} \equiv \int_{\Sigma_\alpha^{T-\alpha}} |\partial_\nu z|^2(h\cdot\nu) &= \int_\alpha^{T-\alpha}\int_{\operatorname{supp}\{h\cdot\nu\}\cap\Gamma} |\partial_\nu z|^2(h\cdot\nu) \\ &\le \max_{x\in\overline{\Gamma_0}} |h(x)| \int_0^T\int_{\Gamma_0} |\partial_\nu z|^2 \\ &\lesssim \int_0^T D(t)dt + \int_0^T |l_0 z|^2_{\Gamma_1}. \end{aligned} \quad (2.25)$$

The second integral in the RHS of (2.24) is handled using the boundary conditions in (1.1)

$$\begin{aligned} I_{12} &\equiv \int_{\Sigma_\alpha^{T-\alpha}} \partial_\nu z(h\cdot\nabla_\tau z) \\ &= \int_{\Sigma_\alpha^{T-\alpha}} l(x)z_t(h\cdot\nabla_\tau z) + \int_\alpha^{T-\alpha}\int_{\Gamma_0} w_t(h\cdot\nabla_\tau z) - l_0\int_\alpha^{T-\alpha}\int_{\Gamma_1} z(h\cdot\nabla_\tau z) \\ &\lesssim \int_0^T D(t)dt + \int_\alpha^{T-\alpha}\left[\int_\Gamma l(x)|\nabla_\tau z|^2 + \int_{\Gamma_0}|\nabla_\tau z|^2\right] + T\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma_1)}. \end{aligned} \quad (2.26)$$

Returning to (2.24) and using (2.25) and (2.26) we obtain

$$|I_1| \lesssim \int_0^T D(t)dt \int_\alpha^{T-\alpha}\left[\int_\Gamma l(x)|\nabla_\tau z|^2 + \int_{\Gamma_0}|\nabla_\tau z|^2\right] + T\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma_1)}. \quad (2.27)$$

Estimates (2.23) and (2.27) are used to improve (2.22) and to obtain

$$\begin{aligned} &\left|\int_{\Sigma_\alpha^{T-\alpha}} \partial_\nu z\left(2h\cdot\nabla z - z\left[\operatorname{div}\{h\} + \frac{c_0}{2}\right]\right)\right| \\ &\lesssim \int_0^T D(t)dt + \int_\alpha^{T-\alpha}\left[\int_\Gamma l(x)|\nabla_\tau z|^2 + \int_{\Gamma_0}|\nabla_\tau z|^2\right] + T\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma_1)}. \end{aligned} \quad (2.28)$$

Combining (2.21) and (2.28), we obtain

$$|B\Sigma_\alpha^T| \lesssim \int_0^T D(t)dt + \int_\alpha^{T-\alpha}\left[\int_\Gamma l(x)|\nabla_\tau z|^2 + \int_{\Gamma_0}|\nabla_\tau z|^2\right] + T\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma_1)}. \quad (2.29)$$

Tangential derivatives are estimated using Propsotion 5 in [54] and the boundary conditions in (1.1). This is the crucial element in the proof. Indeed, for $0<\eta<1/2$, there exists $C_\eta>0$ such that

$$\begin{aligned} &\int_\alpha^{T-\alpha}\int_{\operatorname{supp}\{h\cdot\nu\}\cap\Gamma}|\nabla_\tau z|^2 + \int_\alpha^{T-\alpha}\int_\Gamma l(x)|\nabla_\tau z|^2 \\ &\le C_\eta\left\{\int_0^T\int_{\Gamma_0}\left[|\partial_\nu z|^2 + |z_t|^2\right] + \int_0^T\int_\Gamma l(x)\left[|\partial_\nu z|^2 + |z_t|^2\right] + \|z\|^2_{H^{\eta+\frac{1}{2}}(Q_0^T)}\right\} \end{aligned}$$

$$\lesssim C_\eta \left\{ \int_0^T D(t)dt + \|z\|^2_{L^2(\Sigma_0^T)} + \|z\|^2_{H^{\eta+\frac{1}{2}}(Q_0^T)} \right\}. \tag{2.30}$$

Note that "almost lower order terms" in the (2.30) can be handled as follows

$$\|z\|^2_{H^{\eta+\frac{1}{2}}(Q_0^T)} \le C_\varepsilon \|z\|^2_{L^2(Q_0^T)} + \varepsilon \int_0^T E_z, \tag{2.31}$$

for any $\varepsilon > 0$. Combining (2.29), (2.30) and (2.31) and using the Trace Theorem, we have

$$|B\Sigma_\alpha^T| \lesssim \varepsilon \int_0^T E_z + \int_0^T D(t)dt + T \left[\sup_{t\in[0,T]} \|z(t)\|^2_{L^2(\Omega)} + \sup_{t\in[0,T]} \|z(t)\|^2_{L_2(\Gamma)} \right]. \tag{2.32}$$

Estimating SQ_α^T.

$$\left| \frac{1}{2} \int_{Q_\alpha^{T-\alpha}} f_1 z[\mathrm{div}\{h\} + c_0/2] \right| \lesssim C_T \left[1 + \sup_{[0,T]} \|z(t)\|^2_{L^2(\Omega)} \right]. \tag{2.33}$$

The remaining integral in (2.19) is estimated using the divergence theorem as follows

$$\begin{aligned} -\int_{Q_\alpha^{T-\alpha}} f_1(h\cdot\nabla z) &= -\int_{Q_\alpha^{T-\alpha}} \mathrm{div}\{f_1 z h\} + \int_\Omega f_1 z \mathrm{div}\{h\} \\ &= -\int_{\Sigma_\alpha^{T-\alpha}} f_1 z(h\cdot\nu) + \int_{Q_\alpha^{T-\alpha}} f_1 z\mathrm{div}\{h\}, \end{aligned} \tag{2.34}$$

and consequently

$$\left| -\frac{2}{c_0} \int_{Q_\alpha^{T-\alpha}} f_1(h\cdot\nabla z) \right| \lesssim \int_{\Sigma_\alpha^{T-\alpha}} |f_1 z| + \int_{Q_\alpha^{T-\alpha}} |f_1 z|. \tag{2.35}$$

Moreover

$$\int_{Q_\alpha^{T-\alpha}} |f_1 z| \le C_{f_1} \int_{Q_\alpha^{T-\alpha}} \left[|z|^2\right] \lesssim C_T \left[1 + \sup_{[0,T]} \|z(t)\|^2_{L^2(\Omega)} \right]. \tag{2.36}$$

Analogously, the boundary integral in the RHS of (2.35) is estimated as follows

$$\begin{aligned} \int_{\Sigma_\alpha^{T-\alpha}} |f_1 z| &\le C_{f_1} \int_\alpha^{T-\alpha} \int_\Gamma \left[|z| + |z|^2\right] \\ &\lesssim C_T \left[1 + \sup_{[0,T]} \|z(t)\|^2_{L_2(\Gamma_1)} \right]. \end{aligned} \tag{2.37}$$

Combining (2.33) and (2.35), and using estimates (2.36) and (2.37), we conclude

$$|SQ_\alpha^T| \le C_T \left[\sup_{[0,T]} \|z(t)\|^2_{L^2(\Omega)} \right] + K_2(T), \tag{2.38}$$

for some positive constants C_T and $K_2(T)$. Finally, we return to (2.16) and use estimates (2.20), (2.32) and (2.38) in order to obtain

$$(1-\varepsilon) \int_\alpha^{T-\alpha} E_z(t)dt \le E_z(\alpha) + E_z(T-\alpha) + \int_0^T D(t)dt$$

$$+C_{T,\varepsilon}\left\{1+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}+\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma_1)}\right\}, \tag{2.39}$$

for some positive constant $C_{T,\varepsilon}$ and for any $\varepsilon > 0$.

In order to handle the pointwise energy terms in the RHS of (2.39), we multiply the wave equation (1.1) by z_t and integrate by parts in $Q_s^t = \Omega \times (s,t)$ for $s \le t$ in order to obtain

$$E_z(t) + c^2\int_s^t\int_\Gamma l(x)|z_t|^2 = E_z(s) + c^2\int_s^t\int_{\Gamma_0} z_t w_t - \int_s^t\int_\Omega f_1 z_t. \tag{2.40}$$

Applying (2.40) for $\{s,t\} = \{0,\alpha\}$ and $\{s,t\} = \{T-\alpha, T\}$, we obtain

$$\begin{aligned}&E_z(\alpha) + E_z(T-\alpha)\\&\quad\lesssim E_z(0) + E_z(T) + \int_0^T D(t)dt + C_T\left[1+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}\right].\end{aligned} \tag{2.41}$$

Combining (2.39) and (2.41), and choosing $\varepsilon > 0$ sufficiently small, we conclude

$$\begin{aligned}&\int_\alpha^{T-\alpha} E_z(t)dt \lesssim E_z(0) + E_z(T) + \int_0^T D(t)dt\\&\quad + C_T\left\{1+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}+\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma)}\right\},\end{aligned} \tag{2.42}$$

for some positive constants C and C_T.

Completing the energy integral. To complete the energy integral in the LHS of (2.42), we use the energy identity (2.40) for $\{s,t\} = \{0,t\}$ from which we obtain

$$\begin{aligned}&\left[\int_0^\alpha + \int_{T-\alpha}^T\right] E_z\\&\quad\le 2\alpha E_z(0) + c^2\left[\int_0^\alpha + \int_{T-\alpha}^T\right]\int_0^t\int_{\Gamma_0} z_t w_t + \left[\int_0^\alpha + \int_{T-\alpha}^T\right]\int_0^t\int_\Omega f_1 z_t\\&\quad\lesssim E_z(0) + \int_0^T D(t)dt + C_T\left[1+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}\right],\end{aligned} \tag{2.43}$$

for some positive constant C_T. Adding (2.42) and (2.43) we have

$$\int_0^T E_z(t)dt \lesssim E_z(0) + E_z(T) + \int_0^T D(t)dt + C_T\left[1+\sup_{t\in[0,T]}\|z(t)\|^2_{L^2(\Omega)}+\sup_{t\in[0,T]}\|z(t)\|^2_{L_2(\Gamma)}\right]. \tag{2.44}$$

Inequality (2.11) thus follow from (2.44) by setting the constants C_2, $C_{2,T}$ and $K_{2,T}$ properly. □

2.1.3 Proof of Lemma 1

We are now in position to establish the energy estimate (2.2). Due to the argument based on density, it is enough to consider $\{z, w\}$ a strong solution of (1.1)–(1.2). Combining the results in Lemmas 2 and 3, we obtain

$$\int_0^T\left[E + \int_{\Gamma_0} f(w)w\right] \lesssim E(0) + E(T) + \int_0^T D(t)dt + C_T \underset{[0,T]}{\text{l.o.t.}}\{z\} + K_T. \tag{2.45}$$

On the other hand, note that assumption (1.17) on f implies $F(s) \leq f(s)s + C_f$, for every $s \in \mathbb{R}$ and for some constant $C_f \in \mathbb{R}_+$. Consequently, we have the following

$$\begin{aligned}\mathcal{E} - \left[E + \int_{\Gamma_0} f(w)w\right] &\leq \sigma \int_{\Gamma_0} w|\nabla w|^2 + C_f|\Gamma_0| + \int_\Omega |f_1 z| \\ &\lesssim \varepsilon \int_{\Gamma_0} |\nabla w|^4 + C_\varepsilon + C_{f_1} \sup_{t\in[0,T]} \|z(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{2.46}$$

for every $\varepsilon > 0$. Estimate (2.46) combined with the lower estimate (1.16) and the energy identity (1.15) yields

$$\begin{aligned}T\mathcal{E}(T) + \int_0^T E &\leq (1 + 1/c_1) \int_0^T \mathcal{E} + \sigma \int_0^T \int_t^T \int_{\Gamma_0} w_t|\nabla w|^2 + \frac{Tc_2}{c_1} \\ &\lesssim \int_0^T \left[E + \int_{\Gamma_0} f(w)w\right] + \varepsilon \int_0^T \int_{\Gamma_0} |\nabla w|^4 + C_{T,\varepsilon}\left[1 + \int_0^T D(t)dt\right].\end{aligned} \tag{2.47}$$

Combining (2.45) and (2.47) we obtain

$$T\mathcal{E}(T) + \int_0^T E(t)dt \lesssim E(0) + E(T) + \varepsilon \int_0^T \int_{\Gamma_0} |\nabla w|^4 + C_{T,\varepsilon}\left[1 + \int_0^T D(t)dt + \underset{[0,T]}{\text{l.o.t.}}\{z\}\right]. \tag{2.48}$$

The pointwise energy terms in the RHS of (2.48) are handled using the lower estimate (1.16) as follows

$$E(0) + E(T) \leq \frac{1}{c_1}\left[\mathcal{E}(0) + \mathcal{E}(T) + 2c_2\right], \tag{2.49}$$

while the initial energy is replaced using the energy identity (1.9) as follows

$$\mathcal{E}(0) = \mathcal{E}(T) + \int_0^T D(t)dt - \sigma \int_0^T \int_{\Gamma_0} w_t|\nabla w|^2, \tag{2.50}$$

for every $\varepsilon > 0$. Combining (2.48), (2.49) and (2.50), we have

$$\begin{aligned}&(T - C)\mathcal{E}(T) + \int_0^T \left[E_z(t) + \frac{1}{2}\left\{\|w_t\|^2_{L^2(\Gamma_0)} + \|\Delta w\|^2_{L^2(\Omega)} + (1/2 - \varepsilon)\|\nabla w\|^4_{L^4(\Gamma_0)}\right\}\right] \\ &\leq C_T\left[1 + \int_0^T D(t)dt + \underset{[0,T]}{\text{l.o.t.}}\{z\}\right],\end{aligned} \tag{2.51}$$

for some positive constants C and $C_T > 0$. Finally, for $T > 0$ large and $\varepsilon > 0$ small enough, inequality (2.2) thus follow from (2.51) by rescaling the constants C_T and K_T. This concludes the proof of Lemma 1.

Estimate (2.2) in Lemma 1 provides a bound for the energy in terms of damping and "lower order terms", the latter depending only on wave solutions. Therefore, the next step in the proof of Proposition 1 is to "absorb" these lower order terms using the damping functional.

2.1.4 Stationary Solutions

Next step is to absorb the *lower order terms* in the estimate (2.2) by using the damping. To achieve this, an analysis of stationary states is needed. This is accomplished below for strong solutions first, the result can thus be extended to weak solutions via an approximation argument. We begin by observing that if $\mathbf{u} = S_t \mathbf{u}_0$ is a stationary solution of (1.1)–(1.2) then $\mathbf{u} = (z, 0, w, 0)$, where z and w satisfy the following stationary problems

$$\begin{aligned} -c^2 \Delta z + f_1 = 0, \quad & \text{in } \Omega, \\ \partial_\nu z = \begin{cases} -l_0 z, & \text{on } \Gamma_1, \\ 0, & \text{on } \Gamma_0; \end{cases} \end{aligned} \tag{2.52}$$

$$\begin{aligned} \Delta^2 w = \operatorname{div}\{|\nabla w|^2 \nabla w\} + \sigma \Delta\{w^2\} - f(w), \quad & \text{in } \Gamma_0, \\ w = 0, \ \nabla w = 0, \quad & \text{on } \partial\Gamma_0. \end{aligned} \tag{2.53}$$

It follows from assumption (1.17) on f_1 and the boundary conditions in (2.52) that

$$0 = -c^2 \int_\Omega z \Delta z + \int_\Omega f_1(z) z \quad \text{iff} \quad C_{f_1,\varepsilon} \geq (1-\varepsilon)\|z\|^2_{H^1(\Omega)}, \tag{2.54}$$

for every $\varepsilon > 0$ and for some positive constant $C_{f_1,\varepsilon}$. Choosing ε small enough, we conclude that the stationary solution z is bounded.

For the plate solution w, we multiply (2.53) by w and integrating by parts in Γ_0, we have

$$\int_{\Gamma_0} \left[|\Delta w|^2 + |\nabla w|^4\right] d\Gamma_0 = -2\sigma \int_{\Gamma_0} w |\nabla w|^2 d\Gamma_0 - \int_{\Gamma_0} w f(w) d\Gamma_0. \tag{2.55}$$

The first integral in the RHS of (2.55) is estimated using Poincaré's-type inequality in $W_0^{1,4}(\Gamma_0)$ as follows

$$\begin{aligned} \left| -2\sigma \int_{\Gamma_0} w |\nabla w|^2 \right| &\leq \frac{\sigma^2}{\varepsilon} \int_{\Gamma_0} |w|^2 + \varepsilon \int_{\Gamma_0} |\nabla w|^2 \\ &\leq 2\varepsilon \int_{\Gamma_0} |\nabla w|^4 + \frac{\sigma^4 |\Gamma_0|}{4\varepsilon^3}, \end{aligned} \tag{2.56}$$

for every $\varepsilon > 0$. The second integral in the RHS of (2.55) is evaluated using assumption (1.17) on f, which implies $f(s)s \geq -\varepsilon s^4 - C_\varepsilon$ for any $\varepsilon > 0$, for some $C_\varepsilon \in \mathbb{R}$ and for every $s \in \mathbb{R}$. Using this lower estimate and Poincaré-type inequality in $W_0^{1,4}(\Gamma_0)$, we obtain

$$-\int_{\Gamma_0} f(w) w \leq \varepsilon \int_{\Gamma_0} |\nabla w|^4 + C_\varepsilon |\Gamma_0|, \tag{2.57}$$

for every $\varepsilon > 0$. Finally, combining (2.55), (2.56) and (2.57), we have

$$\|\Delta w\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0} |\nabla w|^4 \leq 3\varepsilon \int_{\Gamma_0} |\nabla w|^4 + |\Gamma_0| \left(\frac{\sigma^4}{4\varepsilon^3} + C_\varepsilon \right). \tag{2.58}$$

Choosing $\varepsilon > 0$ properly, inequality (2.58) above shows that every stationary solution of the plate equation (2.53) is bounded in $H^2(\Gamma_0)$. Since z is bounded, we conclude that the stationary solution $\mathbf{u} = (z, 0, w, 0)$ of (2.52)–(2.53) is bounded in $\mathcal{H}$. Therefore, if we denote by $\mathcal{N}$ the set of stationary solution of (1.1)–(1.2), then there exists $K_\mathcal{N} > 0$ such that

$$\|\mathbf{u}\|_\mathcal{H} \leq K_\mathcal{N}, \quad \text{for every } \mathbf{u} \in \mathcal{N}. \tag{2.59}$$

The above analysis of stationary solutions will be used in the next step in order to absorb lower order terms that appeared in the previous step.

2.1.5 Absorption of Lower Order Terms by the Damping and the Stationary Set

Lemma 4 *Let $T > 0$ and assume that $\mathbf{u} = \{z, z_t, w, w_t\} \in C(0, T; \mathcal{H})$ is a solution of* (1.1)–(1.2). *There exist constants $C = C(T) > 0$ and $K_1 = K_1(T)$, both independent on $E(0)$. such that*

$$\underset{[0,T]}{\text{l.o.t.}}\{z\} \leq C \int_0^T D(t)dt + K_1, \tag{2.60}$$

provided that $T > 0$ is large enough.

Proof The proof relies on a suitable compact/uniqueness argument. By "suitable" we mean that the presence of non-dissipative terms needs to be accounted for by a size of stationary set. Argument along these lines was used for the first time in [35]. The main idea is that the damping forces solutions to remain in a bounded set determined by the stationary solutions. Assume that (2.60) is not true, i.e. for any choice of constants $C > 0$ and $K_1 > 0$, there exist a solution $\mathbf{u} = \{z, z_t, w, w_t\}$ of (1.1)–(1.2) such that

$$\underset{[0,T]}{\text{l.o.t.}}\{z\} > C \int_0^T D + K_1.$$

Let $K_1 > K_{\mathcal{N}}$ be fixed and set $C = n \in \mathbb{N}$. Therefore, we obtain a sequence of solutions $\mathbf{u}^n = \{z^n, z_t^n, w^n, w_t^n\}$ such that

$$\underset{[0,T]}{\text{l.o.t.}}\{z^n\} > n \int_0^T D_n(t)dt + K_1, \quad \text{for every } n \in \mathbb{N}, \tag{2.61}$$

where $D_n = k\|w_t^n\|^2_{L^2(\Gamma_0)} + c^2\|l^{1/2}z_t^n\|^2_{L^2(\Gamma)}$. Since $K_1 > 0$, inequality (2.61) implies

$$\underset{[0,T]}{\text{l.o.t.}}\{z^n\} > K_1, \quad \text{for every } n \in \mathbb{N}; \tag{2.62}$$

$$\lim_{n\to\infty} \frac{\underset{[0,T]}{\text{l.o.t.}}\{z^n\}}{\int_0^T D_n(t)dt} = \infty, \quad \text{as } n \to \infty. \tag{2.63}$$

Case I. $\left\{\underset{[0,t]}{\text{l.o.t.}}\{z^n\}\right\}_{n\in\mathbb{N}}$ **is uniformly bounded.**

In this case, the limit (2.63) implies that $\int_0^T D_n \to 0$ as $n \to \infty$ and consequently

$$l^{1/2} z_t^n \to 0 \quad \text{in } L^2((0,T) \times \Gamma) \text{ as } n \to \infty; \tag{2.64}$$

$$w_t^n \to 0 \quad \text{in } L^2((0,T) \times \Gamma_0) \text{ as } n \to \infty. \tag{2.65}$$

In addition, inequality (2.2) in Lemma 1 and relation (1.16) imply that $\{\mathbf{u}^n\}$ is bounded in $L^\infty(0, T; \mathcal{H})$. Hence, there exists a subsequence, also denoted by $\{\mathbf{u}^n\}$, such that

$$\mathbf{u}^n \to \mathbf{u} \equiv \{z, z_t, w, w_t\} \text{ weakly star in } L^\infty(0, T; \mathcal{H}). \tag{2.66}$$

Using the compact embedding $H^{1-\eta}(\Omega) \subset H^1(\Omega)$ and $H^{2-\eta}(\Gamma_0) \subset H^2(\Gamma_0)$, for every $0 < \eta$ sufficiently small, as well as Aubin's-Simmons Lemma (see [60]) we conclude

$$\{z^n, w^n\} \to \{z, w\} \text{ in } C(0, T; H^{1-\eta}(\Omega) \times H^{2-\eta}(\Gamma_0)). \tag{2.67}$$

In particular, we have

$$\underset{[0,T]}{\text{l.o.t.}}\{z^n\} \to \underset{[0,T]}{\text{l.o.t.}}\{z\}, \quad \text{as } n \to \infty. \tag{2.68}$$

Case Ia. $\mathbf{u} = 0$. In this case, we pass to the limit into (2.62) and use (2.68) in order to conclude $0 \geq K_1$, which is a contradiction.
Case Ib. $\mathbf{u} \neq 0$. In this case, convergences (2.64)–(2.67) allow us to pass to the limit on the variational problem

$$\begin{aligned} 0 = &\frac{d}{dt}\left[(z_t^n, \phi)_\Omega + (w_t^n, \psi)_{\Gamma_0} + \rho(z^n|_{\Gamma_0}, \psi)_{\Gamma_0}\right] \\ &+ c^2(\nabla z^n, \nabla\phi)_\Omega + c^2 l_0(z^n|_{\Gamma_1}, \phi|_{\Gamma_1})_{\Gamma_1} + (\Delta w^n, \Delta\psi)_{\Gamma_0} \\ &+ c^2(l^{\frac{1}{2}} z_t^n|_\Gamma, l^{\frac{1}{2}}\phi|_\Gamma)_\Gamma + k(w_t^n, \psi)_{\Gamma_0} - c^2(w_t^n, \phi|_{\Gamma_0})_{\Gamma_0} + G(w^n, \psi), \end{aligned}$$

for every $\phi \in H^1(\Omega)$ and $\psi \in H_0^2(\Gamma_0)$, where

$$G(w^n, \psi) \equiv -\int_{\Gamma_0}\left[|\nabla w^n|^2 \nabla w^n \cdot \nabla\psi + \sigma\nabla\{w^{n2}\} \cdot \nabla\psi + f(w^n)\psi\right] d\Gamma_0,$$

stands for the nonlinear terms.

Since the maps $w \mapsto |\nabla w|^2\nabla w$, $w \mapsto |\nabla w|^2$ and $w \mapsto f(w)$ are continuous from $H^{2-\eta}(\Gamma_0)$ into $L^2(\Gamma_0)$, for $\eta > 0$ small enough, we can pass with the limit on $G(w^n, \psi)$. Also, from the convergence (2.63) we conclude that $w_t = 0$. Therefore, w is a stationary solution of (1.2), while z is a (distributional) solution of

$$\begin{cases} z_{tt} - c^2\Delta z + f_1 = 0, \text{ in } \Omega \times (0, T); \\ \partial_\nu z = \begin{cases} -l_0 z, & \text{on } \Gamma_1 \times (0, T), \\ 0, & \text{on } \Gamma_0 \times (0, T). \end{cases} \end{cases}$$

Henceforth, since $\Gamma_0 \subset \operatorname{supp}\{l\}$, it follows that $v = z_t$ is a (distributional) solution of

$$\begin{cases} v_{tt} - c^2\Delta v = 0, \text{ in } \Omega \times (0, T); \\ v = 0, \text{ on } \Gamma_0 \times (0, T), \\ \partial_\nu v = \begin{cases} -l_0 v, & \text{on } \Gamma_1 \times (0, T), \\ 0, & \text{on } \Gamma_0 \times (0, T). \end{cases} \end{cases} \tag{2.69}$$

which by applying uniqueness continuation result due to Ruiz (see [27]) for the above equation (2.69) with potential and overdetermined on Γ_0, we conclude that $v = 0$ and, consequently, $z_t = 0$ in $\Omega \times (0, T)$. Therefore, z is a stationary solution of (1.1), hence $\mathbf{u} = (z, 0, w, 0)$ is a stationary solution of system (1.1)–(1.2), i.e. $\mathbf{u} \in \mathcal{N}$ and, consequently, $\|\mathbf{u}\|_{\mathcal{H}} \leq K_{\mathcal{N}}$ (see (2.59)). On the other hand, passing to the limit into (2.62), and accounting for (2.68), we have $\|\mathbf{u}\|_{\mathcal{H}} \geq K_1 > K_{\mathcal{N}}$ which provides a contradiction.

Case 2. $\left\{\underset{[0,T]}{\text{l.o.t.}}\{z^n\}\right\}_{n\in\mathbb{N}}$ **not bounded.**

This case is ruled out by using the following rescaling argument. Let us define $\lambda_n^2 \equiv \underset{[0,T]}{\text{l.o.t.}}\{z^n\}$, for every $n \in \mathbb{N}$. Hence, setting $\hat{\mathbf{u}}^n = \mathbf{u}^n/\lambda_n$, it follows that $\hat{z}^n = z^n/\lambda_n$ and $\hat{w}^n = w^n/\lambda_n$ are solutions of (1.1)–(1.2) verifying

$$\underset{[0,T]}{\text{l.o.t.}}\{\hat{z}^n\} = 1, \quad \text{for every } n \in \mathbb{N}. \tag{2.70}$$

In addition, inequality (2.61) implies

$$1 = \frac{\underset{[0,T]}{\text{l.o.t.}}\{z^n\}}{\lambda_n^2} > n\int_0^T \hat{D}_n + \frac{K_1}{\lambda_n^2}, \tag{2.71}$$

where $\hat{D}_n \equiv D_n/\lambda_n^2$, for every $n \in \mathbb{N}$. Inequality (2.71) implies $\lim_{n\to\infty}\int_0^T \hat{D}_n = 0$. Consequently, applying inequality (2.2) in Lemma 1 to the solutions $\{z^n, w^n\}$, dividing by appropriate powers λ_n, we conclude that

$$\frac{t}{\lambda_n^2}\mathcal{E}(\mathbf{u}^n(t)) + \frac{1}{\lambda_n^2}\int_0^t E_n \le C_T\left\{\int_0^T \hat{D}_n + \underset{[0,T]}{\text{l.o.t.}}\{\hat{z}^n\} + \frac{K_T}{\lambda_n^2}\right\}, t \le T. \tag{2.72}$$

where $\hat{E}_n = E(\hat{\mathbf{u}}^n)$. Inequality (2.72) above provides a uniform bound for the total energy at $t = T$ and the energy integral. On the other hand, applying the energy identity (1.9) to the solution $\{z^n, w^n\}$, using the lower bound (1.16) and dividing by λ_n^2, we have

$$\|\hat{\mathbf{u}}^n(t)\|_{\mathcal{H}}^2 \le \frac{1}{c_0}\left[\frac{1}{\lambda_n^2}\mathcal{E}(\mathbf{u}^n(T)) + \left(1+\frac{1}{k}\right)\int_0^T \hat{D}_n + \frac{\sigma^2}{\lambda_n^2}\int_0^T\int_{\Gamma_0}\frac{1}{4}|\nabla w^n|^4 + c_1\right]. \tag{2.73}$$

Since the RHS of (2.73) is bounded [recall the definition of nonlinear energy $E_w(t)$ as controlling L_4 norm ∇w^n and (2.72)] it follows that $\{\hat{\mathbf{u}}^n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0, T; \mathcal{H})$. Therefore, reducing to a subsequence if necessary, there exists $\hat{\mathbf{u}} = \{\hat{z}, \hat{z}_t, \hat{w}, \hat{w}_t\}$ such that

$$\hat{\mathbf{u}}^n \to \hat{\mathbf{u}} \quad \text{weakly star in } L^\infty(0, T; \mathcal{H}). \tag{2.74}$$

However, since $\int_0^T \hat{D}_n$ goes to 0 as n goes to infinity, we conclude

$$\hat{w}_t^n \to 0 \quad \text{in } L^2((0, T) \times \Gamma_0), \tag{2.75}$$

$$l^{1/2}\hat{z}_t^n \to 0 \quad \text{in } L^2((0, T) \times \Gamma). \tag{2.76}$$

Note that convergences (2.74), (2.75) and (2.76) imply that $\hat{w}_t = 0$ and $\hat{z}_t|_{\Gamma_0} = 0$. In addition, $\hat{z}$ is a (distributional) solution of

$$\begin{cases} \hat{z}_{tt} - c^2\Delta\hat{z} = 0 & \text{in } \Omega \times (0, T); \\ \hat{z}_t = 0 & \text{in } \Gamma_0 \times (0, T); \\ \partial_\nu\hat{z} = \begin{cases} -l_0\hat{z} & \text{on } \Gamma_1 \times (0, T); \\ 0 & \text{on } \Gamma_0 \times (0, T). \end{cases} \end{cases}$$

Proceeding as in the previous case, after using unique continuation for the wave solution $\hat{z}$, we conclude that $\hat{z} = 0$ and, consequently, $\underset{[0,T]}{\text{l.o.t.}}\{\hat{z}\} = 0$. On the other hand, it follows from convergence (2.74) and Aubin-Simmon's Lemma, we have $\hat{z}^n \to \hat{z}$ strongly in $C(0, T; H^{1-\eta}(\Omega))$, for $\eta > 0$ small enough and, therefore,

$$1 = \underset{[0,T]}{\text{l.o.t.}}\{\hat{z}^n\} = \underset{[0,T]}{\text{l.o.t.}}\{\hat{z}\},$$

which provides a contradiction. □

Remark 11 The idea of the proof of the contradiction argument above relies on two principles: the damping takes care of the dynamics while the constant K_1 takes care of the stationary behavior, the coercivity of $div[\nabla w]^4$ [see the definition of $E_w(t)$ and the inequality (2.72)] plays critical role.

2.1.6 Completion of the Proof of Proposition 1 and of first part of Theorem 2

The next step is in merging the estimates obtained so far for the elastic and acoustic energies. This will lead to a construction of the absorbing ball for the full dynamical system.

Proof of Proposition 1 Let $R_0 > 0$ and $\mathbf{u}_0 \in \mathcal{H}$ such that $\|\mathbf{u}_0\|_{\mathcal{H}} \leq R_0$. It follows from Lemmas 1 and 4 that the corresponding total energy functional $\mathcal{E}$ satisfies

$$\mathcal{E}(T) \leq C(T) \int_0^T D(t)dt + K(T), \tag{2.77}$$

for $T > 0$ sufficiently large with $K(T)$ *independent on a solution*.

We define $\tilde{\mathcal{E}} \equiv \mathcal{E} + \epsilon_1 (w, w_t)_{\Gamma_0}$ where $\epsilon_1 > 0$ is small enough. Proceeding as in [21], we have

$$|\tilde{\mathcal{E}} - \mathcal{E}| \leq \epsilon_1 [E_w + \|w\|^2_{L^2(\Gamma_0)}],$$

which implies, for all $\epsilon_1 \leq 1/4$,

$$\mathcal{E} \leq 2\tilde{\mathcal{E}} + M_1 \quad \text{and} \quad \tilde{\mathcal{E}} \leq 2\mathcal{E} + M_2, \tag{2.78}$$

where $M_1, M_2 \in \mathbb{R}$ are constants which depend on σ. Using (2.78), we rewrite (2.77) as follows

$$\tilde{\mathcal{E}}(T) \leq C(T) \int_0^T D(t)dt + K(T). \tag{2.79}$$

Now, using the energy identity (1.9), the plate equation (1.2) and assumption (1.18) on σ, as well as the estimate

$$\sigma \int_{\Gamma_0} |\nabla w|^2 w_t dx \leq \frac{k}{4} \|w_t\|^2_{L^2(\Gamma_0)} + \frac{\sigma^2}{k} \int_{\Gamma_0} |\nabla w|^4 dx,$$

we obtain

$$\frac{d}{dt}\tilde{\mathcal{E}} \leq -\left(\frac{k}{2} - \epsilon_1\right) \|w_t\|^2_{L^2(\Gamma_0)} - (c^2 - \gamma\rho)\|l^{1/2} z_t\|^2_{L^2(\Gamma)}$$
$$- \epsilon_1 \left\{ \|\Delta w\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0} \left[\left(1 - \delta - \frac{\sigma^2}{k\epsilon_1}\right) |\nabla w|^4 - \left(\frac{\epsilon_1 \rho}{4\tilde{l}\gamma} + k\epsilon_1 + \frac{\sigma^2}{\delta}\right) |w|^2 + w f(w) \right] d\Gamma_0 \right\},$$

for any $\epsilon_1, \delta, \gamma > 0$. Choosing these parameters such that

$$\frac{k}{2} - \epsilon_1 > 0, \quad 1 - \delta - \frac{\sigma^2}{k\epsilon_1} > 0, \quad c^2(1-\gamma) > 0,$$

and exploiting assumption (1.17) on f, we obtain

$$\frac{d}{dt}\tilde{\mathcal{E}} + C_0 \left[E_w + \|w\|^2_{L^2(\Gamma_0)} + D \right] \leq C_1, \tag{2.80}$$

where $C_0, C_1 > 0$ are constants. Notice that a Poincaré-type inequality in $W_0^{1,4}(\Gamma_0)$ was used and $\zeta > 0$ must be chosen small enough, as well. Integrating (2.80) in the interval $[0, T]$, we conclude

$$\int_0^T D(t)dt \leq \frac{1}{C_0} \left[\tilde{\mathcal{E}}(0) - \tilde{\mathcal{E}}(T) + TC_1 \right]. \tag{2.81}$$

Combining estimates (2.79) and (2.81) we obtain

$$\tilde{\mathcal{E}}(T) \leq C_0(T)\left[\tilde{\mathcal{E}}(0) - \tilde{\mathcal{E}}(T) + TC_1\right] + K(T), \tag{2.82}$$

where $C_0(T) = \frac{C(T)}{C_0}$. From estimate (1.16), there exists $c > 0$ such that $\tilde{\mathcal{E}}^c(t) \equiv \tilde{\mathcal{E}}(t)+c > 0$ for every $t \geq 0$. Replacing $\tilde{\mathcal{E}}$ by $\tilde{\mathcal{E}}^c$ in (2.82), we have

$$\tilde{\mathcal{E}}^c(T) \leq \gamma\tilde{\mathcal{E}}^c(0) + \frac{\tilde{K}_T}{1+C(T)}, \tag{2.83}$$

where $0 < \gamma \equiv \frac{C(T,E(0))}{1+C(T,E(0))} < 1$ and $\tilde{K}_T = C_0(T)TC_1 + K(T) + c$.

Remark 12 The constant c above does not depend on the solution, but only on the parameter σ, the function f and the domain Γ_0. Therefore, the constant $\tilde{K}_T$ depend on T but not on the initial data. This fact is essential in the computations to follow.

Let $m \in \mathbb{N}$, re-iterating the above estimate (2.83) on the interval $[(m-1)T, mT]$ we conclude

$$\tilde{\mathcal{E}}^c(mT) \leq \gamma\tilde{\mathcal{E}}^c((m-1)T) + \frac{\tilde{K}_T}{1+C(T)} \leq \gamma^{m-1}\tilde{\mathcal{E}}^c(0) + \left(\sum_{i=0}^{m-1}\gamma^i\right)\frac{\tilde{K}_T}{1+C(T)}. \tag{2.84}$$

For $t > T$, let $m \in \mathbb{N}$ such that $t = mT + s$, with $s \in [0, T)$. Hence, using (2.80) and (2.84), we have

$$\begin{aligned}\tilde{\mathcal{E}}^c(t) &\leq \tilde{\mathcal{E}}^c(mT) + (t-mT)C_1 \\ &\leq \gamma^{m-1}\tilde{\mathcal{E}}^c(0) + \frac{1}{1-\gamma}\cdot\frac{\tilde{K}_T}{1+C(T)} + TC_1 \\ &\leq \frac{1}{\gamma}e^{-\frac{|\ln\gamma|}{T}t}\tilde{\mathcal{E}}^c(0) + K_T.\end{aligned} \tag{2.85}$$

where $K_T \equiv TC_1 + \tilde{K}_T$. Using the expressions of $\tilde{\mathcal{E}}^c$ and $\tilde{\mathcal{E}}$, we have $|\tilde{\mathcal{E}}^c(0)| \leq C_{R_0}$, for some $C_{R_0} > 0$. Consequently, inequalities (2.78) and (2.85) imply

$$\mathcal{E}(t) \leq 2\left[\frac{1}{\gamma}e^{-\frac{|\ln\gamma|}{T}t}C_{R_0} + K_T\right] + M_1.$$

Finally, choosing $t_0 = t_0(R_0) > 0$ sufficiently large and using the lower bound (1.16), we conclude

$$\|\mathbf{u}(t)\|_{\mathcal{H}} \leq E(t)^{1/2} \leq \left\{\frac{1}{c_1}\left[2(1+K_T) + M_1 + c_2\right]\right\}^{1/2} \equiv R, \quad \text{for } t \geq t_0.$$

Therefore, the conclusion will follow from the above inequality by setting $\mathcal{B} = \{\mathbf{y} \in \mathcal{H} : \|\mathbf{y}\|_{\mathcal{H}} \leq R\}$. □

Remark 13 Note that the inequality in (2.77) is not sufficient to conclude absorption property. This is due to the presence of Boussinesq nonlinear force which makes the energy relation non dissipative. To handle this issue, we have used perturbed energy method which allows to account on the restorative effects of the restorative forces.

Proposition 1 ensure the existence of absorbing ball for the dynamical system $\{\mathcal{H}, S_t\}$. In order to complete the proof of the existence of weak global attractor, the first part of Theorem

2 it suffices to appeal to general results in dynamical systems. We have weakly continuous flow in $\mathcal{H}$ with an absorbing ball which is *weakly* compact. Theorem 1, Chapter 2 in Sect 2 [41] [see also sect 2.3.2 in [61] provides the desired conclusion.

In order to establish the second part of the Theorem, the task is more challenging and more work is required. Indeed, one needs to establish *asymptotic smoothness* of the flow. This will be done under the assumption that $\sigma = 0$. The above condition implies the finiteness of dissipative integrals $\int_0^\infty D(s) \le C(E(0))$ a property of critical use in the proof.

3 Global Strong Attractor

In order to prove an existence of strong attractor, one needs to establish asymptotic smoothness in *strong* topology of the phase space. This is done below.

3.1 Asymptotic Smoothness in Strong Topology-Proof of the Second Part of Theorem 2

In order to complete the proof of Theorem 2, one needs to establish asymptotic smoothness. in strong topology of the phase space. We recall that (see [61], Definition 2.2.1, page 52) an evolution operator S_t is *asymptotically smooth* if: for every bounded set $B \subset \mathcal{H}$ such that $S_t B \subset B$ for $t > 0$, there exists a compact set $K \subset \overline{B}$ such that $S_t B$ converges uniformly to K in the sense that

$$\lim_{t\to\infty} d_{\mathcal{H}}\{S_t B \mid K\} = 0,$$

A sufficient condition to establish the asymptotic smoothness of $(\mathcal{H}, S_t)$ is given by the Ladyzhenskaya condition (see Proposition 7.1.6 in [42], page 340): for every bounded sequence $\{\mathbf{u}_n\} \subset \mathcal{H}$ and every $t_n \to \infty$, the sequence $\{S_{t_n}\mathbf{u}_n\}$ is relatively compact in $\mathcal{H}$.

The asymptotic smoothness is formulated in the following Proposition.

Proposition 2 *Under the assumptions of Theorem 2, the dynamical system $\{\mathcal{H}, S_t\}$ is asymptotically smooth.*

In order to prove Proposition 2, first we establish the asymptotic smoothness for elastic-plate solutions. This is quite involved due to the following two reasons: (1) the presence of supercritical internal nonlinearity, (ii) the fact that the two parts of the interaction *wave and plate* do not cooperate through the cancellations of the boundary terms.Thus the boundary terms [intrinsically unbounded] need to be dealt with. The first predicament is dealt with by adopting the method of J. Ball's. The second one, by adapting the technology of "dissipation integral" [4] which forces the assumption $\sigma = 0$ but still allows for supercriticality. The proof is carried out through a sequence of Lemmas.

3.1.1 Strong Convergence of the Elastic Component

Due to the supercriticality of nonlinear terms in the plate equation, we shall adapt the so called "energy" method [2] which is intrinsically based on the following two features: [i.] energy identity for the Lyapunov's function and [ii.] linear damping. To proceed, we introduce the following notation: for $\mathbf{u}_0 \in \mathcal{H}$, let $\mathbf{u}(t) = S_t\mathbf{u}_0$ be the corresponding solution of (1.1)–(1.2) given by $\mathbf{u} = \{z, z_t, w, w_t\}$. It is convenient to denote $\mathbf{u} = \{\mathbf{z}, \mathbf{w}\}$, where $\mathbf{z} = \{z, z_t\}$ and

$\mathbf{w} = \{w, w_t\}$ correspond to the wave and plate components, respectively. With this notation, the asymptotic compactness of plate solutions is proven next.

Lemma 5 *Let $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_j \to \infty$ as $j \to \infty$. For every sequence $\{\mathbf{u}^j\}$ of solutions of (1.1)–(1.2), with $\{\mathbf{u}^j(0)\} \subset \mathcal{H}$ bounded, the sequence $\{\mathbf{w}^j(t_j)\} \subset H_w$ has a convergent subsequence.*

Proof Let $\epsilon > 0$ to be chosen later. For a single solution $\mathbf{u} = \{\mathbf{z}, \mathbf{w}\}$ of (1.1)–(1.2), define

$$\mathcal{I}(\mathbf{w}) = \mathcal{E}_w + \epsilon(w, w_t)_{\Gamma_0}$$
$$\text{where} \quad \mathcal{E}_w \equiv E_w + \sigma \int_{\Gamma_0} w|\nabla w|^2 + \int_{\Gamma_0} F(w).$$

Proceeding as in [21], we have the identity valid for all weak solutions. [Recall that this argument requires geometric assumption $\Gamma_0 \subset$ *supp l*.]

$$\frac{d}{dt}\mathcal{I}(\mathbf{w}) = -(k-\epsilon)\|w_t\|^2_{\Gamma_0} + \sigma \int_{\Gamma_0} w_t|\nabla w|^2 - \epsilon a(w,w) - \epsilon \int_{\Gamma_0} |\nabla w|^4 - 2\epsilon\sigma \int_{\Gamma_0} w|\nabla w|^2$$
$$- \epsilon \int_{\Gamma_0} wf(w) - \epsilon k(w, w_t)_{\Gamma_0} - \rho \int_{\Gamma_0} w_t z_t - \epsilon\rho \int_{\Gamma_0} w z_t. \tag{3.1}$$

Choosing $\epsilon = k/2$, we rewrite (3.1) as

$$\frac{d}{dt}\mathcal{I}(\mathbf{w}) + k\mathcal{I}(\mathbf{w}) = \Phi(\mathbf{u}), \tag{3.2}$$

where

$$\Phi(\mathbf{u}) \equiv \sigma \int_{\Gamma_0} w_t|\nabla w|^2 - \frac{k}{4}\int_{\Gamma_0} |\nabla w|^4 - \frac{k}{2}\int_{\Gamma_0} wf(w)$$
$$+k\int_{\Gamma_0} F(w) - \frac{k}{2}\rho \int_{\Gamma_0} w z_t - \rho \int_{\Gamma_0} w_t z_t. \tag{3.3}$$

Integrating in time (3.2) we obtain:

$$\mathcal{I}(\mathbf{w}(t)) = e^{-kt}\mathcal{I}(\mathbf{w}(0)) + \int_0^t e^{-k(t-s)}\Phi(\mathbf{u}(s))ds, \quad \text{for } t > 0. \tag{3.4}$$

Remark 14 Note that Φ consists of subcritical terms except for the last two integrals. However, for the noncompact terms in (3.4) we have

$$\left| -\frac{k\rho}{2}\int_{\Gamma_0} w z_t - \rho \int_{\Gamma_0} w_t z_t \right| \le C_{k,\rho,l}\left[D(\mathbf{u}) + C_\sigma \|w\|^2_{L^2(\Gamma_0)} \right], \tag{3.5}$$

for some constant $C_{k,\rho,l} > 0$. In addition, since $k >> \sigma^2$, it follows from the energy identity (1.9) and the dissipativity proved in Proposition 1 that

$$\int_0^T D(\mathbf{u})dt \le C_R + C\sigma \int_0^T \int_{\Gamma_0} |\nabla w|^4, t \ge 0. \tag{3.6}$$

Inequalities (3.5) and (3.6) suggest the decomposition $\Phi = \Phi^1 + \Phi^2$ where

$$\Phi^1(\mathbf{u}) \equiv \sigma \int_{\Gamma_0} w_t|\nabla w|^2 - \frac{k}{4}\int_{\Gamma_0} |\nabla w|^4 - \frac{k}{2}\int_{\Gamma_0} wf(w) + k\int_{\Gamma_0} F(w) \tag{3.7}$$
$$\Phi^2(\mathbf{u}) \equiv -\frac{k\rho}{2}\int_{\Gamma_0} w z_t - \rho \int_{\Gamma_0} w_t z_t. \tag{3.8}$$

Let $\{\mathbf{u}^j\}$ be a sequence of solutions of (1.1)–(1.2) such that $\{\mathbf{u}^j(0)\} \subset \mathcal{H}$ is bounded, and let $\{t_j\} \subset \mathbb{R}_+$ such that $t_j \to \infty$. It follows from the ultimate dissipativity-, proved in the previous section (see Proposition 1), that $\{\mathbf{u}^j(t)\}$ is uniformly bounded in $\mathcal{H}$. Restricting to a subsequence if necessary, there is $\xi \in \mathcal{H}$ such that $\mathbf{u}^j(t_j) \rightharpoonup \xi$, where the symbol "$\rightharpoonup$" denotes weak convergence. For the same reason, we have $\mathbf{u}^j(t_j - T) \rightharpoonup \xi_T$, for some $\xi_T \in \mathcal{H}$. Defining $\overline{\mathbf{u}}^j(t) = \mathbf{u}^j(t_j + t - T)$, for $t \geq 0$, we conclude that $\{\overline{\mathbf{u}}^j\}$ is a sequence of solutions of (1.1)–(1.2) such that $\overline{\mathbf{u}}^j(0) = \mathbf{u}^j(t_j - T) \rightharpoonup \xi_T$. If $\overline{\mathbf{u}}$ is the corresponding solution satisfying $\overline{\mathbf{u}}(0) = \xi_T$, it follows from the fact that S_t is a strongly continuous semigroup, that $\overline{\mathbf{u}}^j(t) \rightharpoonup \overline{\mathbf{u}}(t)$ uniformly over compact sets. In particular, we have $\mathbf{u}^j(t_j) = \overline{\mathbf{u}}^j(T) \rightharpoonup \overline{\mathbf{u}}(T)$ and, consequently, $\overline{\mathbf{u}}(T) = \xi$. Writing in two coordinates, [wave, plate] we have by the uniqueness of the limit: $\mathbf{u}^j = \{\mathbf{z}^j, \mathbf{w}^j\}$, $\overline{\mathbf{u}}^j = \{\overline{\mathbf{z}}^j, \overline{\mathbf{w}}^j\}$, $\overline{\mathbf{u}} = \{\overline{\mathbf{z}}, \overline{\mathbf{w}}\}$, $\xi = \{\xi^1, \xi^2\}$ and $\xi_T = \{\xi_T^1, \xi_T^2\}$, from which, in particular,

$$\mathbf{w}^j(t_j) \rightharpoonup \overline{\mathbf{w}}(T) = \xi^2 \quad \text{and} \quad \mathbf{w}^j(t_j - T) \rightharpoonup \overline{\mathbf{w}}(0) = \xi_T^2.$$

Applying (3.4) for $\overline{\mathbf{w}}^j$, we have

$$\mathcal{I}(\mathbf{w}^j(t_j)) = e^{-kT}\mathcal{I}(\mathbf{w}^j(t_j - T)) + \int_0^T e^{-k(T-s)}\Phi(\overline{\mathbf{u}}^j(s))ds. \tag{3.9}$$

Note that Φ^1 is subcritical with respect to the strong topology of $\mathcal{H}$ and, therefore, compact. In this case, we have $\Phi^1(\overline{\mathbf{u}}^j(s)) \to \Phi^1(\overline{\mathbf{u}}(s))$ on $[0, T]$. Therefore,

$$\limsup_{t_j \to \infty} \int_0^T e^{-k(T-s)}\Phi^1(\overline{\mathbf{u}}^j(s))ds = \int_0^T e^{-k(T-s)}\Phi^1(\overline{\mathbf{u}}(s))ds. \tag{3.10}$$

By compactness properties of Φ_1 and (3.9)

$$\begin{aligned}
&\lim_{t_j\to\infty} \int_0^T e^{-k(T-s)}\Phi^1(\overline{\mathbf{u}}^j(s))ds + \int_0^T e^{-k(T-s)}\Phi^2(\overline{\mathbf{u}}^j(s))ds \\
&= \int_0^T e^{-k(T-s)}\Phi^1(\overline{\mathbf{u}}(s))ds + \lim_{t_j\to\infty} \int_0^T e^{-k(T-s)}\Phi^2(\overline{\mathbf{u}}^j(s))ds \\
&= \mathcal{I}(\overline{\mathbf{w}}(T)) - e^{-kT}\mathcal{I}(\overline{\mathbf{w}}(0)) + \lim_{t_j\to\infty} \int_0^T e^{-k(T-s)}[\Phi^2(\overline{\mathbf{u}}^j(s)) - \Phi^2(\overline{\mathbf{u}}(s))]ds \\
&= \mathcal{I}(\xi^2) - e^{-kT}\mathcal{I}(\overline{\mathbf{w}}(0)) + \lim_{t_j\to\infty} \int_0^T e^{-k(T-s)}[\Phi^2(\overline{\mathbf{u}}^j(s)) - \Phi^2(\bar{\mathbf{u}}(s))]ds.
\end{aligned} \tag{3.11}$$

Since $|\mathcal{E}_w(t)| \leq C(E_w(t))$, for every $t \geq 0$, it follows from the ultimate dissipativity once more, that $\{\mathcal{I}(\mathbf{w}^j(t_j - T))\}$ is bounded. Therefore, taking the $\limsup_{t_j\to\infty}$ of identity (3.9), accounting for the limit above yields:

$$\begin{aligned}
\limsup_{t_j\to\infty} \mathcal{I}(\mathbf{w}^j(t_j)) &\leq Ce^{-kT} + \mathcal{I}(\xi^2) - e^{-kT}\mathcal{I}(\overline{\mathbf{w}}(0)) \\
&\quad + \limsup_{t_j} \int_0^T e^{-k(T-s)}[\Phi^2(\bar{\mathbf{u}}^j(s) - \Phi^2(\bar{\mathbf{u}}(s)]ds.
\end{aligned} \tag{3.12}$$

We shall show that the last term goes to zero when $T \to \infty$. It is only here where we use $\sigma = 0$. Indeed, $\int_0^\infty \Phi^2(\mathbf{u}^j) \leq \infty$ by the virtue of $k > 0$ and the fiiniteness of the dissipation integral $[\sigma = 0]$

$$\int_0^\infty |(z_t^j, w_t^j)|dt \leq \int_0^\infty ||(z_t^j||_{\Gamma_0}^2 + ||w_t^j||_{\Gamma_0}^2 dt < \infty;$$

uniformly in t_j. Moreover, for every $\epsilon > 0$, there exists $J = J(\epsilon)$ such that

$$\limsup_{t_j} \int_0^T e^{-k(T-s)}\Phi^2(\bar{\mathbf{u}}^j(s)ds \leq \int_0^T e^{-k(T-s)}[||\bar{w}_t^J(s)||^2 + ||\bar{z}_t^J(s)||^2]ds + \epsilon$$

$$\int_0^T e^{-k(T-s)}\Phi^2(\bar{\mathbf{u}}(s)ds \le \int_0^T e^{-k(T-s)}[||\bar{w}_t(s)||^2 + ||\bar{z}_t(s)||^2]ds. \tag{3.13}$$

Moreover, for every index j we have

$$\lim_{T\to\infty}\int_0^T e^{-k(T-s)}[||\bar{w}_t^j(s)||^2 + ||\bar{z}_t^j(s)||^2] = 0 \tag{3.14}$$

The proof of (3.14) amounts to splitting the splitting the interval $[0, T] = [0, N] \cup [N, T]$. Uniform [in j] $L_1(0,\infty)$ bounds $C(B)$ for $||z_t^j||^2_{\Gamma_0} + ||w_t^j||^2_{\Gamma_0}$ allow to select N large enough to obtain the integral on $[0, N]$ small. Then, since $k > 0$ exponential decay allows to make the integral on $[0, N]$ small.

Letting $T \to \infty$ in inequality (3.12) above and applying (3.14) with $j = J$, we conclude that $\limsup_{t_j\to\infty} \mathcal{I}(\mathbf{w}^j(t_j)) \le \mathcal{I}(\xi^2)$. To arrive at the final conclusion of strong convergence, one needs to validate the converse inequality. Here, however, the argument is classical based on weak lower semicontinuity. We observe that $\mathcal{I}$ is (sequently) weakly lower semicontinuous, from which the convergence $\mathbf{w}^j(t_j) \rightharpoonup \xi^2$ implies $\mathcal{I}(\xi^2) \le \liminf_{t_j\to\infty} \mathcal{I}(\mathbf{w}^j(t_j))$. Therefore, we must conclude that $\mathcal{I}(\mathbf{w}^j(t_j)) \to \mathcal{I}(\xi^2)$, which implies $\|\mathbf{w}^j(t_j)\|_{H_w} \to \|\xi^2\|_{H_w}$. This convergence together with the weak convergence $\mathbf{w}^j(t_j) \rightharpoonup \xi^2$ imply the strong convergence $\mathbf{w}^j(t_j) \to \xi^2$ in H_w, as desired. □

3.1.2 Strong Convergence of the Acoustic Component

The following "compensated compactness" property will be used [4]

Lemma 6 *Let $B \subset \mathcal{H}$ be a bounded forward invariant set and $\epsilon_0 > 0$. There exists $T_0 = T_0(\epsilon_0, B)$ such that*

$$E_z(T_0) \le \epsilon_0 + \Psi_{T_0}(\mathbf{u}_0, \mathbf{u}_1); \tag{3.15}$$

for any $\mathbf{u}_0, \mathbf{u}_1 \in B$ and $\mathbf{u} = \{z, z_t, w, w_t\}$ given by $\mathbf{u}(t) = S_t\mathbf{u}_0 - S_t\mathbf{u}_1$, where Ψ_{T_0} is a functional defined on $B \times B$ such that

$$\liminf_{m\to\infty}\liminf_{n\to\infty} \Psi_{T_0}(\mathbf{y}_m, \mathbf{y}_n) = 0. \tag{3.16}$$

for every sequence $\{\mathbf{y}_n\} \subset B$.

Remark 15 We note that if the result of Lemma 6 were true for the full energy of the system, then the AS smoothness will be deduced from Proposition 2.10 in [4]. The point we make is that validity of the inequality in Lemma above for E_w component is problematic due to supercriticality of the restoring force. Indeed, the usual cancellation properties do not apply when working on a difference of two solutions.

Proof Since B is a bounded forward invariant set, we have $S_t B \subset B$. Therefore, there exists $R > 0$ such that, if $\mathbf{u}(t) = S_t\mathbf{u}_0$ for $\mathbf{u}_0 \in B$, we have $\|\mathbf{u}(t)\|_{\mathcal{H}} \le R$ for every $t \ge 0$. We recall $\mathbf{u} = \{z, z_t, w, w_t\}$ corresponds to a difference of two solutions of (1.1)–(1.2) and, consequently, $\{z, z_t\}$ correspond to the difference of two wave solutions driven by the initial data with the Neumann data given by w_t, corresponding to the speed of propagation of the difference of two solutions of the plate.

With this in mind, we write the *energy identity* for the wave equation (1.1) (see [24]). The latter is due to the fact that the support of $l(x)$ contains Γ_0.

$$E_z(t) + c^2\int_s^t\int_\Gamma l(x)|z_t|^2 = E_z(s) + c^2\int_s^t\int_{\Gamma_0} w_t z_t, \quad s \le t. \tag{3.17}$$

Remark 16 Note that the energy identity required that support $l(x)$ contains Γ_0 -see Theorem 1 and [24].

Using the above identity for $s = t$ and $t = T$, and integrating over $[0, T]$ we obtain

$$\begin{aligned} TE_z(T) + c^2 \int_0^T \int_t^T \|l^{1/2} z_t\|_\Gamma^2 &= \int_0^T E_z + c^2 \int_0^T \int_t^T \int_{\Gamma_0} w_t z_t \\ &\leq \int_0^T E_z + \frac{cT}{(k\tilde{l})^{1/2}} \int_0^T k^{1/2} \|w_t\|_{L^2(\Gamma_0)} c \|l^{1/2} z_t\|_{L^2(\Gamma)} \\ &\leq \int_0^T E_z + \frac{cT}{2(k\tilde{l})^{1/2}} \int_0^T \Big[k\|w_t\|^2_{L^2(\Gamma_0)} + c^2 \|l^{1/2} z_t\|^2_{L^2(\Gamma)} \Big]. \end{aligned} \tag{3.18}$$

The energy integral is bounded using estimate (2.11) in Lemma 3, and uniform bounds for the pointwise energy terms, from which (3.18) imply

$$TE_z(T) + \int_0^T E_z \tag{3.19}$$

$$\begin{aligned} &\leq 2C_2 \{E_z(0) + E_z(T)\} + \left(2C_2 + \frac{cT}{2(k\tilde{l})^{1/2}}\right) \int_0^T \Big[k\|w_t\|^2_{L^2(\Gamma_0)} + c^2 \|l^{1/2} z_t\|^2_{L^2(\Gamma)} \Big] \\ &\quad + 2C_{2,T} \underset{[0,T]}{\text{l.o.t.}}\{z\} \\ &\leq C_R + C(T) \left\{ \int_0^T \Big[k\|w_t\|^2_{L^2(\Gamma_0)} + c^2 \|l^{1/2} z_t\|^2_{L^2(\Gamma)} \Big] + \underset{[0,T]}{\text{l.o.t.}}\{z\} \right\}. \end{aligned} \tag{3.20}$$

where $C(T) = 2(C_2 + C_{2,T}) + cT/2(k\tilde{l})^{1/2}$ and $l.o.t\{z\}$ are given by (2.3). The damping term in the RHS of (3.19) is replaced using the *energy identity* (3.17), from which we obtain

$$E_z(T) \leq \frac{C_R}{T} + \frac{C(T)}{T} \left[E_z(0) - E_z(T) + \left(1 + \frac{c^2}{2k\tilde{l}}\right) \int_0^T k\|w_t\|^2_{\Gamma_0} + \underset{[0,T]}{\text{l.o.t.}}\{z\} \right].$$

For any $\epsilon > 0$, we choose $T = T(\epsilon, R) > C_R/\epsilon$ and write $C_T \equiv C(T)/T$, from which we conclude

$$E_z(T) \leq \frac{\epsilon}{1 + C_T} + \gamma_\epsilon E_z(0) + \gamma_\epsilon \left[c_k \int_0^T k\|w_t\|^2_{L^2(\Gamma_0)} + \underset{[0,T]}{\text{l.o.t.}}\{z\} \right], \tag{3.21}$$

where $0 < \gamma_\epsilon \equiv \frac{C_T}{1+C_T} < 1$. In writing $w = w^1 - w^2$, it is convenient to denote

$$\begin{aligned} \mathfrak{L}_{[0,T]}\{z, w\} &= c_t \Big[\int_0^T k\|w_t^1\|^2_{L^2(\Gamma_0)} - \sigma \int_0^T \int_{\Gamma_0} w_t^1 |\nabla w^1|^2 + \int_0^T k\|w_t^2\|^2_{L^2(\Gamma_0)} \\ &\quad - \sigma \int_0^T \int_{\Gamma_0} w_t^2 |\nabla w^2|^2 \Big] \\ &\quad + \sigma \int_0^T \int_{\Gamma_0} w_t^1 |\nabla w^1|^2 + \sigma \int_0^T \int_{\Gamma_0} w_t^2 |\nabla w^2|^2 + \underset{[0,T]}{\text{l.o.t.}}\{z\} \end{aligned} \tag{3.22}$$

$$E_z(mT) \leq \frac{\epsilon}{1 + C_T} + \gamma_\epsilon E_z((m-1)T) + \gamma_\epsilon \mathfrak{L}_m^T\{z, w\}, \tag{3.23}$$

where $0 < \gamma_\epsilon \equiv \frac{C_T}{1+C_T} < 1$, $m \in \mathbb{N}$ and

$$\mathfrak{L}_m^T\{z, w\} = c_k \int_{(m-1)T}^{mT} k\|w_t\|_{L^2(\Gamma_0)}^2 dt + \underset{[(m-1)T, mT]}{\text{l.o.t.}}\{z\}.$$

After iterations we have

$$E_z(mT) \le \epsilon + \gamma_\epsilon^m E_z(0) + \sum_{k=0}^{m-1} \gamma_\epsilon^{k+1} \mathfrak{L}_{m-k}^T\{z, w\}, \quad m = 1, 2, \dots \tag{3.24}$$

Finally, for $\epsilon_0 > 0$ we choose m sufficiently large such that $\epsilon + \gamma_\epsilon^m E_z(0) < \epsilon_0$. Inequality (3.16) follows from (3.24) by setting $T_0 = mT$ and

$$\Psi_{\epsilon_0, B, T_0}(\mathbf{u}_0, \mathbf{u}_1) = \sum_{k=0}^{m-1} \gamma_\epsilon^{k+1} \mathfrak{L}_{m-k}^T\{z, w\}. \tag{3.25}$$

The conclusion of Lemma 6 then follows from the fact that $\mathfrak{L}_m^T\{z, w\}$ consists of subcritical (lower order (2.3)) terms and boundedness of the kinetic energy of plate solutions, that it $\int_0^\infty \|w_t\|_{L^2(\Gamma_0)}^2 < \infty$. Indeed, the latter follows from $\sum_0^{m-1} \gamma_\epsilon^{k+1} a_{m-k} \to 0$ when $m \to \infty$ where $a_k \equiv \int_{(k-1)T}^{kT} \|w_t\|_{L^2(\Gamma_0)}^2$ with $\sum a_k < \infty$. Thus we have proved that for all initial data of the structure in B, $z(t)$ components have strong convergence when $t \to \infty$. This follows from the arguments in the proof of Proposition 2.10 [4] which exploit Kuratowski measure of noncompactness $\alpha(P_z(S_t B))$ proving that such converges to zero when $t \to \infty$. By $P_z(S_t B)$ we denote projection of the solution on z -wave component. □

3.1.3 Completion of the Proof of Proposition 2

We are now in position to prove Proposition 2, which will establish the asymptotic smoothness for the system (1.1)–(1.2). Note, that Lemmas 6 and 5 provide the asymptotic smoothness for the components of a solution $\mathbf{u} = (\mathbf{z}, \mathbf{w})$ of system (1.1)–(1.2). Althought it is expected that the asymptotic smoothness property can be extended from the components $\mathbf{z}$ and $\mathbf{w}$ to the solution $\mathbf{u}$, we accomplish this by combining the forementioned results with Ladyzhenskaya's condition (see Definition 2.2.1, page 52 in [61]).

Proof of Proposition 2. According to Proposition 7.1.6 in [42], in order to conclude the asymptotic smoothness, it is sufficient to establish Ladyzhenskaya's condition. Let $\{\mathbf{u}_j\} \subset \mathcal{H}$ bounded and $t_j \to \infty$. We must show that $\{\mathbf{u}^j(t_j)\}$ is relatively compact. Setting $\mathbf{u}_j = (\mathbf{z}_j, \mathbf{w}_j)$ and $\mathbf{u}^j = (\mathbf{z}^j, \mathbf{w}^j)$ the corresponding solutions, it follows from Lemma 6 that $\{\mathbf{w}^j(t_j)\} \subset H_w$ is relatively compact. Therefore, it remains to show that $\{\mathbf{z}^j(t_j)\} \subset H_z$ is relatively compact. Since $B \equiv \{\mathbf{u}_j\} \subset \mathcal{H}$ is bounded and the system $(\mathcal{H}, S_t)$ is dissipative, there exists $\mathcal{B} \subset \mathcal{H}$ bounded absorbing set and $t_0 = t_0(B) > 0$ such that $\mathbf{u}^j(t) \in \mathcal{B}$, for every $t \ge t_0$ and every $j \in \mathbb{N}$. Applying Lemma 6, for every $\epsilon > 0$ there exists $T = T(\epsilon, \mathcal{B})$ such that

$$E_{\mathbf{z}}(T) < \epsilon + \Psi_{\epsilon, \mathcal{B}, T}(\mathbf{u}_0, \mathbf{u}_1), \tag{3.26}$$

for every $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{B}$ and $\mathbf{z}$ the difference of corresponding wave solutions.

Let $j_0 \in \mathbb{N}$ such that $t_j \ge t_0 + T$, for every $j > j_0$. Therefore, $\mathbf{u}^j(t_j - T) \in \mathcal{B}$ for every $j > j_0$ and inequality (3.26) can be written as follows

$$\|\mathbf{z}^n(t_n) - \mathbf{z}^m(t_m)\|_{H_z} < \epsilon + \Psi_{\epsilon, \mathcal{B}, T}(\mathbf{u}^n(t_n - T), \mathbf{u}^m(t_m - T)), \tag{3.27}$$

for every $n, m > j_0$. Inequality (3.27) and property (3.16) for $\Psi_{\epsilon,\mathcal{B},T}$ imply that $\{\mathbf{z}^j(t_j)\}$ is relatively compact. This concludes the proof. □

Propositions 1 and 2 allow us to conclude that the corresponding dynamical system $(\mathcal{H}, S_t)$ associated with system (1.1)–(1.2) is dissipative and asymtotically smooth. Therefore, there exists a unique compact global attractor $\mathfrak{A}$ (see Sect. 1.4),as claimed in the second part of Theorem 2.

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Declarations

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