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Dynamics of extensible beams with nonlinear non-compact energy-level damping

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Abstract

This work is motivated by experimental studies (NASA Langley Research Center) of nonlinear damping mechanisms present in flight structures. It has been observed that the structures exhibit significant nonlinear damping effects which are functions of the energy of the system. The present work is devoted to the study of long-time dynamics to a class of extensible beams/plates featuring nonlocal nonlinear energy damping of hyperbolic nature. Such models arise frequently in aeroelasticity when modeling flight structures, see NASA-AirForce reports (Balakrishnan in A theory of nonlinear damping in flexible structures. Stabilization of flexible structures, 1988; Balakrishnan and Taylor in Proceedings Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, 1989). The main mathematical challenge in this context is twofold: (1) nonlinear and potentially degenerate energy damping coefficient, (2) energy damping at a critical level where the usual compactness arguments (critical to the theory of attractors) do not apply. Our investigation sheds some light on a long-time behavior of such class of problems, providing new results in the area of existence of attractors and their properties within this hyperbolic-like framework. This should be contrasted with widely studied parabolic-like models involving structural damping which is known to be poorly understood. The goal is achieved by developing new methodology which allows to circumvent the difficulties related to the lack of compactness and non-locality of the nonlinear damping. The results are achieved

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through a rigorous analysis that reveals an interplay between extensibility, non-locality, and nonlinear energy damping of critical exponent.

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Contents

1 Introduction

Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We consider vibrations of a nonlinear plate model described by the displacement $u(x, t)$ and its velocity $u_t(x, t)$. With these variables $(u, u_t) := U$, we associate a standard energy function in the variable U as

$$\mathcal{E}(U) \equiv \lambda||\Delta u||^2 + ||u_t||^2,$$

where $\|\cdot\|$ stands for the norm in $L^2(\Omega)$ and λ is a structural positive constant appearing in the model under consideration, which is subjected to nonlinear effects affecting both: the displacement u and the velocity u_t. The nonlinear forcing (internal or external) is modeled by a function $f(u)$ while the oscillations and the damping is subjected to a nonlocal law depending on the instantenous energy itself with the dissipation of the form

$$\mathcal{E}(U)u_t.$$

This is one of the models of current interest where the dissipation rate modulates with the strength of the energy $\mathcal{E}(U)$. Examples of applications are abundant, for instance in aeroelasticity, see [2, 3] and references therein. On the mathematical side, the difficulties are due to potential degeneracy of the damping and superlinearity of the damping above critical exponents. More detail on this matter will be said later. In short, the type of models of interest to this study can be written more generally as:

$$u_{tt} + \lambda\Delta^2 u - \mu\Delta u + \gamma H(\mathcal{E}(U) + \epsilon I)u_t + f(u) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

where $H(s)$ is a convex, $C^1(\mathbb{R}^+)$ function, $H(0) = 0$, $\lambda, \gamma, \mu > 0$, $\epsilon \geq 0$, $f(u)$ is a nonlinear source of critical exponent, h is a external force. A canonical example for $H(s)$ is a power law $H(s) = s^q$, $q \geq 1$, $s \geq 0$. So, we will consider this more specific configuration

$$u_{tt} + \lambda\Delta^2 u - \mu\Delta u + \gamma(\mathcal{E}(U) + \epsilon I)^q u_t + f(u) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+. \tag{1.2}$$

Associated with the model are the boundary conditions imposed on to the displacement $u = u(x,t)$. We will consider either clamped (C) or hinged (H) boundary conditions given, respectively, by

$$\begin{aligned} &\text{(C)} \quad u|_{\Gamma\times\mathbb{R}^+} = \frac{\partial u}{\partial \nu}|_{\Gamma\times\mathbb{R}^+} = 0, \\ &\text{(H)} \quad u|_{\Gamma\times\mathbb{R}^+} = \Delta u|_{\Gamma\times\mathbb{R}^+} = 0, \end{aligned} \tag{1.3}$$

where ν is the outward normal to Γ, and initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.4}$$

The analysis of the dynamics for (1.2)–(1.4) will be considered within the following functional framework.

$$\mathcal{H} := \begin{cases} H_0^2(\Omega) \times L^2(\Omega) & \text{for (1.3)-(C)}, \\ \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times L^2(\Omega) & \text{for (1.3)-(H)}. \end{cases}$$

Goal. The main objective in this work is to study the long-time dynamics described by (1.2)–(1.4) including asymptotic analysis with respect to the change of dynamics when $\epsilon \to 0$. More precisely, denoting by $\{S_\epsilon(t)\}_{\epsilon\geq 0}$ the C_0-semigroup of global solutions to (1.2)–(1.4) defined on $\mathcal{H}$ via the relation

$$S_\epsilon(t)(u_0, u_1) = (u^\epsilon(t), u_t^\epsilon(t)), \quad t \geq 0,$$

it will be shown that the semigroup $\{S_\epsilon(t)\}_{\epsilon\geq 0}$ has a family compact global attractors $\mathfrak{A}_{\epsilon\geq 0}$ in the phase space $\mathcal{H}$ with polynomial attraction rates for $\epsilon = 0$ and exponential for $\epsilon > 0$. Moreover, if $\epsilon > 0$ we prove that the family of global attractors $\mathfrak{A}_{\epsilon>0}$ for (1.2)–(1.4) is both finite-dimensional and smooth. Finally, we show that the family $\{\mathfrak{A}_\epsilon\}_{\epsilon \geq 0}$ is upper semicontinuous at $\epsilon = 0$, which means that the Hausdorff semidistance

$$\text{dist}_{\mathcal{H}}(\mathfrak{A}_\epsilon, \mathfrak{A}_0) \equiv \sup_{\psi \in \mathfrak{A}_\epsilon} \inf_{\phi \in \mathfrak{A}_0} ||\psi - \phi||_{\mathcal{H}}$$

tends to 0 as $\epsilon \to 0^+$.

Mathematical challenge. The main challenge in achieving our goals stems from two factors: (1) nonlinear and potentially degenerate damping, (2) nonlinearities are of critical level, where the usual compactness based arguments do not apply. Indeed, Proposition 4.6 is the main backbone estimate which provides a control of decay rates to the attracting set, accounting for potential degeneracy of the damping. The latter is handled via special nonlinear multiplier involving the energy itself, which is first given in this work. Te above with a compensated compactness argument allows to establish an existence of a compact global attractor whose measure of non-compactness is uniform in $\epsilon \geq 0$. It is the latter aspect which requires delicate analysis, being exactly the open problem addressed in [10]. In the case when $\epsilon > 0$, one also shows additional smoothness and finite dimensionality of the said attractor. This is obtained by first establishing the so called quasi-stability inequality. Here the challenge is due to critical level of nonlinear terms. Cancellation techniques such as used for full von Karman equation (also critical) prove successful, due to good structural properties even at critical level and also generalize cases of “subcritical" nonlinear terms in the damping coefficient as considered in [10] as well as cases of non-degenerate damping coefficients as addressed in [12–14, 18, 23, 24].

Main contribution in the context of prior literature. Since the model is of current and significant interest, it has been considered in the past literature extensively with a considerable volume of papers published on the topic, see e.g. [10, 15, 23, 24] and references therein. It is also important to mention that there are several other inspiring works that are pioneering in the stability of hyperbolic models with nonlocal non-degenerate weak damping [4, 19] and later problems with nonlocal degenerate damping coefficient [5, 6]. The precise comparison with prior literature will be left to Remark 3.4. However, we shortly give the details of the results obtained. In fact, the most distinct and challenging aspects of the problem are related to: (1) “hyperbolic like" nature of the dynamics and (2) energy level of the nonlinear damping which may degenerate. Most of the past literature dealt with “parabolic" versions of the model, where strong “structural" damping introduced has led to the enhanced regularity of the solutions, which is typical in parabolic dynamics, e.g. [23] and references within. Or else, the nonlinearity in the damping is below the critical energy level, e.g. [10, 24] and references within. In the latter, it may be that the damping depends on $||\Delta^\alpha u||$ where $\alpha < 1$ or the kinetic part is subcritical as in the Kirchhoff model (with rotational inertia) and so $||u_t||$ is below the critical level of $||\nabla u_t||$. The treatment of both of these aspects, while motivated by real applications, require new methodology introduced in the present work. In short, the main contribution of this work is that regularized effects of the damping are not accounted for, and the energy damping is in its *critical form*, which required the introduction of new nonlinear energy multipliers techniques. Also, we recall that this is the form of damping coefficients arising in modeling of physical phenomena [3].

Structure of the paper. We begin with the well-posedness of the dynamics mostly to set up the stage for studying the underlying dynamical system. Here the arguments are rather standard based on nonlinear semigroup theory (Sect. 2). Then, we proceed with statements of the main results of the present work, namely, Theorems 3.2, 3.3, and 3.4, dealing with existence of global attractors, qualitative properties, and upper-semicontinuity (Sect. 3). In the next step (Sect. 4) we give the proofs of the main results. More specifically, it will be shown that the dynamics is ultimately dissipative, with an absorbing ball independent on the parameter ϵ. The subsequent result shows that for every $\epsilon > 0$, the dynamics is quasi-stable. This means that the corresponding attractors $\mathfrak{A}_\epsilon$ are both smooth and finite dimensional. However, quasi-stability property is *not*—as expected—uniform in $\epsilon \geq 0$ due to high order damping and its degeneracy at critical level. Last, the latter aspect leads to a question of singular perturbation analysis: what happens when $\epsilon \to 0$? So, it will be proved that the measure of noncompactness of global attractors is uniform in ϵ and that global attractor also exists for the limiting case $\epsilon = 0$. Based on these estimates, upper-semicontinuity of the attractors, when $\epsilon \to 0$ will be established. We conclude by providing a concise review of relevant results wihin dynamical systems theory, in order to make the paper as self-contained as possible (Appendix A).

2 Generation of the dynamical system $(\mathcal{H}, S_\epsilon(t))$

Functional spaces and assumptions. We start by setting $W_0 = L^2(\Omega)$, $W_1 = H_0^1(\Omega)$, and

$$W_2 = \begin{cases} H_0^2(\Omega) & \text{for (1.3)-(C),} \\ H^2(\Omega) \cap H_0^1(\Omega) & \text{for (1.3)-(H).} \end{cases}$$

For $m = 3, 4$, we consider

$$W_m = \begin{cases} H^m(\Omega) \cap H_0^2(\Omega) & \text{for (1.3)-(C),} \\ \{u \in H^m(\Omega) \cap H_0^1(\Omega);\ \Delta u \in H_0^1(\Omega)\} & \text{for (1.3)-(H).} \end{cases}$$

Here the notation $(\cdot,\ \cdot)$ stands for L^2-inner product and $\|\cdot\|_p$ denotes L^p-norm. For simplicity, for $p = 2$ we denote $\|\cdot\|_2 = \|\cdot\|$. Thus, $\|\nabla \cdot\|$ and $\|\Delta \cdot\|$ represent the norms in W_1 and W_2, respectively. We also rewrite the phase space

$$\mathcal{H} = W_2 \times W_0, \quad ||(u, v)||_{\mathcal{H}}^2 = \lambda \|\Delta u\|^2 + \|v\|^2, \quad (u, v) \in \mathcal{H}.$$

Denoting by $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator Δ^2 with clamped or hinged boundary condition, one has

$$\lambda_1 \|u\|^2 \leq \|\Delta u\|^2, \quad \lambda_1^{1/2} \|\nabla u\|^2 \leq \|\Delta u\|^2, \quad \forall\, u \in W_2.$$

The energy $E(t)=E(u(t),u_t(t))$ related to problem (1.2)–(1.4) is given by

$$E(t)=\frac{1}{2}\Big[\,||(u(t),u_t(t))||^2_{\mathcal{H}}+\mu\|\nabla u(t)\|^2\,\Big]+\int_\Omega\big[\,\widehat{f}(u(t))-hu(t)\,\big]\,dx, \quad (2.1)$$

where $\widehat{f}(u)=\int_0^u f(\tau)d\tau$.

We impose the following standard assumptions on the nonlinear source term $f(u)$ which is of “critical" Sobolev’s exponent..

Assumption 2.1 Let $f:\mathbb{R}\to\mathbb{R}$ be a C^2-function with $f(0)=0$ and satisfying

$$|f''(u)|\;\le\;C_{f''}(1+|u|^{\rho-1}),\quad u\in\mathbb{R}, \quad (2.2)$$

$$-C_f-\frac{c_f}{2}|u|^2\le\widehat{f}(u)\le f(u)u+\frac{c_f}{2}|u|^2,\quad u\in\mathbb{R}, \quad (2.3)$$

for some constants $C_f, C_{f''}>0$, $c_f\in[0,\lambda_1\lambda)$ and growth exponent $\rho\le\frac{4}{n-4}$ for $n\ge5$.

From inequality (2.2) we also have

$$|f'(u)|\;\le\;C_{f'}(1+|u|^{\rho}),\quad u\in\mathbb{R}, \quad (2.4)$$

for some $C_{f'}>0$.

Well-posedness. Setting $U=(u,v)$ with $v=u_t$, we then rewrite the original problem (1.2)–(1.4) as the following equivalent first-order Cauchy problem

$$\begin{cases} U_t=AU+B(U), & t>0,\\ U(0)=(u_0,u_1):=U_0, \end{cases} \quad (2.5)$$

where $A:D(A)\subset\mathcal{H}\to\mathcal{H}$ is a linear operator defined by

$$AU=\begin{pmatrix} v\\ -\lambda\Delta^2 u\end{pmatrix}^{\perp},\quad U\in D(A)=\left\{U\in\mathcal{H}\,\middle|\,\begin{matrix} v\in W_2,\\ \lambda\Delta^2u\in W_0\end{matrix}\right\}, \quad (2.6)$$

and $B:\mathcal{H}\to\mathcal{H}$ is the nonlinear operator

$$B(U)=\begin{pmatrix} 0\\ \mu\Delta u-\gamma\mathcal{E}_\epsilon(U)^q v-f(u)+h\end{pmatrix}^{\perp},\quad U=(u,v)\in\mathcal{H}, \quad (2.7)$$

where we define the linear (perturbed) energy as

$$\mathcal{E}_\epsilon(U):=\mathcal{E}(U)+\epsilon I=||U||^2_{\mathcal{H}}+\epsilon I. \quad (2.8)$$

Thus, the Hadamard well-posedness result for (2.5), and consequently for the system (1.2)–(1.4), is a consequence of the next results.

Theorem 2.1 *Let us consider the cases: $q > 0$ if $\epsilon > 0$ or else $q \geq \frac{1}{2}$ if $\epsilon = 0$. Under the* Assumption 2.1 *with $h \in W_0$, it holds the following statements:*

(i) *If $U_0 \in \mathcal{H}$, then there exists $T_{\max} > 0$ such that problem* (2.5) *has a unique mild solution $U \in C([0, T_{\max}), \mathcal{H})$, which is given by*

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}B(U(s))\,ds, \quad t \in [0, T_{\max}). \tag{2.9}$$

(ii) *If $U_0 \in D(A)$, then the above mild solution U is the regular one.*
(iii) *In both cases, we have that $T_{\max} = +\infty$.*

By the virtue of Theorem 2.1 the solution of (1.2)–(1.4) will generate a family of dynamical systems on the phase space $\mathcal{H}$. Indeed, for each $\epsilon \geq 0$ the evolution operator $S_\epsilon(t) : \mathcal{H} \to \mathcal{H}$ given by the formula

$$S_\epsilon(t)U_0 = S_\epsilon(t)(u_0, u_1) = (u(t), u_t(t)), \quad t \geq 0,$$

where $U = (u, u_t)$ is the unique mild solution of the system (1.2)–(1.4), defines a nonlinear C_0-semigroup. In addition, an important property is the “robustness" with respect to the continuous dependence of initial data as given next. Consequently, the pair $(\mathcal{H}, S_\epsilon(t))$ generates a dynamical system.

Theorem 2.2 *Let Assumptions of* Theorem 2.1 *be in force. Then, for any two (strong or mild) solutions $U^1(t) = S_\epsilon(t)U_0^1$ and $U^2(t) = S_\epsilon(t)U_0^2$ of problem* (1.2)–(1.4) *corresponding to initial data $U^1(0) = U_0^1 = (u_0^1, u_1^1)$, $U^2(0) = U_0^2 = (u_0^2, u_1^2)$, respectively, there exists a positive constant $C = C(||U_0^1||_{\mathcal{H}}, ||U_0^2||_{\mathcal{H}})$, independent on $\epsilon \geq 0$, such that*

$$||S_\epsilon(t)U_0^1 - S_\epsilon(t)U_0^2||_{\mathcal{H}} \leq Ce^{Ct}||U_0^1 - U_0^2||_{\mathcal{H}}, \quad t \in [0, T]. \tag{2.10}$$

The proofs of Theorems 2.1 and 2.2 are relegated to Sect. 4.

3 Main results

In this section we state the main results of this work, which are established in Theorems 3.2, 3.3, and 3.4. In what follows, It will be shown that for every $\epsilon > 0$ the dynamical systems $(\mathcal{H}, S_\epsilon(t))$ has a compact global attractor $\mathfrak{A}_\epsilon$, which moreover is smooth and finite-dimensional. The above property does not hold uniformly in $\epsilon \geq 0$, hence not for $\epsilon = 0$. However, it is shown that all the dynamics S_ϵ, $\epsilon \geq 0$, have a common absorbing ball. This implies, by standard methods, the existence of *weak* attractors $\mathfrak{A}_\epsilon$ with uniformity in $\epsilon \geq 0$. Under additional restrictions on the parameter q, it will be also shown that the dynamical systems $(\mathcal{H}, S_\epsilon(t))$ admit global compact attractors, whose measure of non-compactness for $\mathfrak{A}_\epsilon$ is independent of $\epsilon \geq 0$. This is, of course, a weaker condition than uniform additional regularity of the attractors $\mathfrak{A}_\epsilon$ independently of $\epsilon \geq 0$ but, however, it is enough to conclude strong convergence of the perturbed semigroups, hence upper-semicontinuity of the attractors at the threshold parameter $\epsilon = 0$. Moreover, since under the restriction $\epsilon > 0$ the attractor $\{\mathfrak{A}_\epsilon\}_{\epsilon>0}$ is both finite-dimensional and smooth, then the upper-semicontinuity at $\epsilon = 0$ ensures the attractor $\mathfrak{A}_0$ can not *"explode"*.

3.1 Global attractors for $\epsilon \geq 0$

For each $\epsilon \geq 0$, let $\mathcal{N}_\epsilon$ be the set of *stationary points* of $(\mathcal{H}, S_\epsilon(t))$ and $\mathrm{M}^u(\mathcal{N}_\epsilon)$ be the *unstable manifold* emanating from $\mathcal{N}_\epsilon$. For readers' convenience, all definitions on long time behavior of dynamical systems are recalled in the Appendix A. The main results of this work are formulated below.

Theorem 3.1 (Uniform Ultimate Dissipativity) *Under the assumptions of* Theorem 2.1*, for any bounded set $B \subset \mathcal{H}$, $U_0 \in B$, there exist $R > 0$ and $t_B > 0$, both independent of $\epsilon \geq 0$, such that*

$$||S_\epsilon(t)U_0||_{\mathcal{H}} \leq R, \quad \forall t > t_B. \tag{3.1}$$

Remark 3.1 Note that the size of the absorbing ball is uniform in $\epsilon \geq 0$. *Weak attractors* have a measure of non-compactness uniform in ϵ in the *weak* topology.

Theorem 3.2 (Case $\epsilon > 0$) *Let Assumptions of Theorem 2.1 be valid with $\epsilon > 0$. Then:*

- **(i)** **(Finite-dimensionality)** *the associate dynamical system $(\mathcal{H}, S_\epsilon(t))_{\epsilon>0}$ of problem (1.2)–(1.4) has a compact global attractor $\mathfrak{A}_\epsilon = \mathrm{M}^u(\mathcal{N}_\epsilon)$ in $\mathcal{H}$ with finite fractal dimension $dim_f^{\mathcal{H}}\mathfrak{A}_\epsilon$;*
- **(ii)** **(Regularity)** *any full trajectory $\Upsilon = \{U(t) = (u(t), u_t(t));\ t \in \mathbb{R}\}$ from attractor $\mathfrak{A}_\epsilon$ enjoys the following regularity properties,*

$$u_t \in L^\infty(\mathbb{R}; W_2) \cap C(\mathbb{R}, W_0), \quad u_{tt} \in L^\infty(\mathbb{R}; W_0). \tag{3.2}$$

Moreover, there exists $R_\epsilon > 0$ such that

$$\|u_{tt}(t)\|^2 + \|\Delta u_t(t)\|^2 \leq R_\epsilon^2, \quad t \in \mathbb{R}, \tag{3.3}$$

where R_ϵ depends on the parameter $\epsilon > 0$, on the seminorm $\eta_{W_2}(u) \equiv ||u||^2_{\rho+2}$, and on a finite intrinsic structural constant (to be denoted as c_∞).

Remark 3.2 Since the damping depends nonlinearly on the full energy and it is degenerate when $\epsilon = 0$, it is expected that the global attractor may not be always compact in the weak phase space with an arbitrary choice of $H(s)$. Indeed, a counterexample can be found in [7, Chapt. 5] for certain choices of the function $H(s)$. See also [10, Sect. 7].

Remark 3.3 Note that the results of Theorem 3.2 hold for each $\epsilon > 0$ with the estimates potentially blowing up when $\epsilon \to 0$. This is expected, due to intrinsic nonlinearity of the damping with a positive derivative for low frequencies. So, the dimension of each attractor $dim_f^{\mathcal{H}}\mathfrak{A}_\epsilon$ depends on $\epsilon > 0$. The result dealing with the asymptotic analysis of the attractors when $\epsilon \to 0$ is given below under the additional restrictions on the parameter q.

Theorem 3.3 (Case $\epsilon \geq 0$) *Let us consider the assumptions of Theorem 2.1 with $\epsilon \geq 0$ and $1 \leq 2^{q-1}q < 2$. Then, we have:*

- **(i)** **(Global attractor)** *the associate dynamical system $(\mathcal{H}, S_\epsilon(t))$ of problem (1.2)–(1.4) has a compact global attractor $\mathfrak{A}_\epsilon$ in $\mathcal{H}$ whose measure of non-compactness is uniform in $\epsilon \geq 0$. In particular, the dynamical system $(\mathcal{H}, S_0(t))$ has a global compact attractor $\mathfrak{A}_0$.*
- **(ii)** **(Characterization)** *the global attractor $\mathfrak{A}_0$ is precisely the unstable manifold $\mathfrak{A}_0 = \mathrm{M}^u(\mathcal{N}_0)$ emanating from the set of stationary solution $\mathcal{N}_0$. In addition, $\mathfrak{A}_0$ consists of full trajectories $\Upsilon = \{S_0(U_0) = U(t) : t \in \mathbb{R}\}$ such that*

$$\lim_{t\to-\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}_0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}_0) = 0.$$

Remark 3.4 (Comparison with prior literature) As we have already mentioned, there is a considerable volume of recent papers related to long time behavior of the general model described in the introduction. We shall focus only on the most relevant ones. The n-dimensional version of the model proposed by Balakrishinan–Taylor ([3], Section 4, Eq. (4.2)) in a bounded domain $\Omega \subset \mathbb{R}^n$ with clamped boundary condition and perturbed by a source term $f(u)$ was considered by Jorge Silva et al. in [15]:

$$u_{tt} - \mu\Delta u + \Delta^2 u - \gamma\left[\,\|\Delta u\|^2 + \|u_t\|^2\,\right]^q \Delta u_t + f(u) = 0. \tag{3.4}$$

Assuming that $q \geq 1$, the authors proved the existence and uniqueness of a global regular solution, polynomial stability and a non-exponential decay prospect. Recently, Sun and Yang [23, 24] consider n-dimensional models of extensible beams (without rotational forces and with rotational forces) with an energy-like damping. In [23] a parabolic version (strong damping) with non-degenerate case was considered, that is, they considered the damping like $M(\|\Delta u\|^2 + \|u_t\|^2)\Delta u_t$ with a strong hypothesis $M(s) > 0$ for $s \in \mathbb{R}^+$. For that model, existence of smooth attractors with additional properties has been established. In [24] the authors focus on asymptotic behavior of attractors where models under consideration are also subjected to non-degenerate damping coefficient $M > 0$. In addition, in the “hyperbolic" case, the nonlinear damping is of subcritical nature with respect to rotational inertia energy and, consequently, the authors prove the existence of strong global and exponential attractors. More recently, [10] considers a class of models associated with (3.4) its non-homogeneous version under the presence of degenerate and non-degenerate damping. More specifically, the energy models class was treated with the following dissipation

$$\gamma\left[\,\|(-\Delta)^\alpha u\|^2 + \|u_t\|^2\,\right]^q u_t, \quad \alpha \in [0, 1].$$

The authors prove the existence of a compact global attractor for the restriction of $\alpha \in [0, 1)$, which again is of subcritical level of the energy. This result is in line with the counterexample in [7]. Hence, the compact global attractor in Theorem 3.3 greatly improves the result achieved in [10]. In view of the above, the results presented in this work deal with “hyperbolic" dynamics, critical level of the energy damping (H possibly degenerating and $\alpha = 1$). In the degenerate case, existence and upper-semicontinuity (with respect to “degeneracy") of attractors is shown for a restricted range of q. The proofs of these results require new methodology and approaches, which will be explained in the process of the proofs.

3.2 Upper-semiconinuity of global attractors at $\epsilon = 0$

As already mentioned, the family of attractors $\{\mathfrak{A}_\epsilon\}_{\epsilon \geq 0}$ is upper semiconituous at $\epsilon = 0$. Since the semicontinuity is analyzed at $\epsilon = 0$, then without loss of generality, let us fix any $\epsilon_0 > 0$ and take $\epsilon \in [0, \epsilon_0)$. The third main result dealing with upper-semicontinuity of global attractor $\mathfrak{A}_\epsilon$ at $\epsilon = 0$ can be stated as follows.

Theorem 3.4 (Upper-semicontinuity) *Let the assumptions of* Theorem 3.3 *be in force. Then, the family of the attractors* $\{\mathfrak{A}_\epsilon\}_{\epsilon \geq 0}$ *to problem* (1.2)–(1.4) *is upper-semi-continuous at* $\epsilon = 0$*, namely,*

$$dist_{\mathcal{H}}\left(\mathfrak{A}_\epsilon, \mathfrak{A}_0\right) \equiv \sup_{\psi \in \mathfrak{A}_\epsilon} \inf_{\phi \in \mathfrak{A}_0} ||\psi - \phi||_{\mathcal{H}} \to 0 \quad as \quad \epsilon \to 0^+. \tag{3.5}$$

The proofs of Theorems 3.1, 3.2, 3.3, and 3.4, are relegated to Sect. 4 after presentation of several technical results. To conclude the final statements in the stated theorems, we evoke several abstract results on dynamical systems recalled in Appendix A.

4 Proofs of the main results

This section is devoted to the proofs of Theorems 2.1, 2.2, 3.1, 3.2, 3.3, and 3.4.

4.1 Proof of Theorem 2.1

This proof is rather standard and relies on application of semigroup theory. We provide it for sake of completeness. In *Step 1* we show that the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ given in (2.6) is a infinitesimal generator of a C_0- semigroup of contractions on $\mathcal{H}$. This is proved by showing that A is dissipative and maximal, and application of the Lumer–Phillips Theorem [20, Theorem 1.4.3]. In *Step 2* we show that the operator $B : \mathcal{H} \to \mathcal{H}$ is locally Lipschtz. *Step 1* and *Step 2* guarantee the existence of local solution [20, Theorem 6.1.4]. The existence of global solution is asserted in *Step 3*.
Step 1. The operator A defined in (2.6) is the infinitesimal generator of a C_0-semigroup in $\mathcal{H}$. Indeed, we take arbitrary element $U \in D(A)$. Then

$$\langle AU(t), U(t)\rangle_{\mathcal{H}} = \lambda\left(\Delta v, \Delta u\right) + \left(-\lambda \Delta^2 u, v\right) = 0,$$

which proves dissipativity for A. To show that A is maximal we need to prove that $R(I-A)=\mathcal{H}$, where $R(I-A)$ is the range of $I-A$. Indeed, let $U^*=(u^*,v^*)\in\mathcal{H}$, and consider the equation $(I-A)U=U^*$ which, written in components, reads

$$\begin{aligned} u-v&=u^*,\\ v+\lambda\Delta^2 u&=v^*. \end{aligned} \tag{4.1}$$

Substituting $u=v+u^*$ in the second equations of (4.1), we obtain

$$v+\lambda\Delta^2 v=v^*-\lambda\Delta^2 u^* =: w^*\in W_2'. \tag{4.2}$$

Since the corresponding weak formulation is

$$a(v,w)=\int_\Omega w^* w\,dx,\quad \forall w\in W_2,$$

where

$$a(v,w):=\int_\Omega [\,vw+\lambda\Delta v\Delta w\,]\,dx,$$

by the Lax-Milgram Theorem we can conclude that problem (4.2) admits a unique solution $v\in W_2$. Then we deduce from the second equation of (4.1) that $\lambda\Delta^2 u=v^*-v\in W_0$. This implies that $R(I-A)=\mathcal{H}$. Therefore, A is maximal monotone and due to Lummer-Phillips Theorem A is a infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.

Step 2. The operator $B:\mathcal{H}\to\mathcal{H}$ given in (2.6) is locally Lipschitz. Indeed, let us take $R>0$ and $U^1=(u^1,u_t^1),\ U^2=(u^2,u_t^2)$ such that $||U^1||_{\mathcal{H}},\ ||U^2||_{\mathcal{H}}\le R$. Denoting $w=u^1-u^2$, we have

$$\begin{aligned} ||B(U^1)-B(U^2)||_{\mathcal{H}}=\|\,\mu\Delta w+\gamma\Big[\,\mathcal{E}_\epsilon(U^1)^q u_t^1-\mathcal{E}_\epsilon(U^2)^q u_t^2\,\Big]\\ -\Big[f(u^1)-f(u^2)\Big]\,\|. \end{aligned} \tag{4.3}$$

Now, let us estimate the terms on the right hand side of the above equality. First we have

$$\mu\int_\Omega \Delta w\varphi\,dx\le\mu\|\Delta w\|\|\varphi\|\le\frac{\mu}{\lambda^{\frac{1}{2}}}||U^1-U^2||_{\mathcal{H}}\|\varphi\|,\quad\forall\varphi\in W_0.$$

Note that we can rewrite

$$\begin{aligned} &\gamma\int_\Omega\Big[\,\mathcal{E}_\epsilon(U^1)^q u_t^1-\mathcal{E}_\epsilon(U^2)^q u_t^2\,\Big]\varphi\,dx\\ &\quad=\gamma\mathcal{E}_\epsilon(U^1)^q\int_\Omega w_t\varphi\,dx+\gamma\Big[\,\mathcal{E}_\epsilon(U^1)^q-\mathcal{E}_\epsilon(U^2)^q\,\Big]\int_\Omega u_t^2\varphi\,dx. \end{aligned}$$

It is easy to see that

$$\gamma\mathcal{E}_\epsilon(U^1)^q\int_\Omega w_t\varphi\,dx \le \gamma\left[\,||U||^2_{\mathcal{H}}+\epsilon\,\right]^q\|w_t\|\|\varphi\|$$
$$\le \gamma\left(R^2+\epsilon\right)^q||U^1-U^2||_{\mathcal{H}}\|\varphi\|,\quad \forall\varphi\in W_0.$$

Next, to estimate the term $\gamma\left[\,\mathcal{E}_\epsilon(U^1)^q-\mathcal{E}_\epsilon(U^2)^q\,\right]\int_\Omega u_t^2\varphi\,dx$ we will separate the cases $\epsilon>0$ and $\epsilon=0$.

Case $\epsilon>0$ **and** $q>0$: Note that, from Mean Value Theorem, we have

$$\mathcal{E}_\epsilon(U^1)^q-\mathcal{E}_\epsilon(U^2)^q = q\left[\,\theta||U^1||^2_{\mathcal{H}}+(1-\theta)||U^2||^2_{\mathcal{H}}+\epsilon\,\right]^{q-1}\left[\,||U^1||^2_{\mathcal{H}}-||U^2||^2_{\mathcal{H}}\,\right]$$
$$\le \begin{cases} q\epsilon^{q-1}\left|\,||U^1||^2_{\mathcal{H}}-||U^2||^2_{\mathcal{H}}\,\right| & \text{for}\quad 0<q<1,\\ q\left[\,R^2+\epsilon\,\right]^{q-1}\left|\,||U^1||^2_{\mathcal{H}}-||U^2||^2_{\mathcal{H}}\,\right| & \text{for}\quad q\ge 1.\end{cases}$$

Hence, taking $C=q\max\{\epsilon,\left[\,R^2+\epsilon\,\right]\}^{q-1}$, we have

$$\gamma\left[\,\mathcal{E}_\epsilon(U^1)^q-\mathcal{E}_\epsilon(U^2)^q\,\right]\int_\Omega u_t^2\varphi\,dx \le C\left[\,||U^1||_{\mathcal{H}}+||U^2||_{\mathcal{H}}\,\right]\left|\,||U^1||_{\mathcal{H}}-||U^2||_{\mathcal{H}}\,\right|\|u_t^1\|\|\varphi\|$$
$$\le 2CR^2||U^1-U^2||_{\mathcal{H}}\|\varphi\|,\quad \forall\varphi\in W_0.$$

Case $\epsilon=0$ **and** $q\ge\frac{1}{2}$: Also from Mean Value Theorem there exists $\theta\in(0,1)$ such that

$$\gamma\left[\,\mathcal{E}_\epsilon(U^1)^q-\mathcal{E}_\epsilon(U^2)^q\,\right]\int_\Omega u_t^2\varphi\,dx=\gamma\left[\,||U^1||^{2q}_{\mathcal{H}}-||U^2||^{2q}_{\mathcal{H}}\,\right]\int_\Omega u_t^2\varphi\,dx$$
$$=\ 2\gamma q\left[\,\theta||U^1||_{\mathcal{H}}+(1-\theta)||U^1||_{\mathcal{H}}\,\right]^{2q-1}\left[\,||U^1||_{\mathcal{H}}-||U^2||_{\mathcal{H}}\,\right]\int_\Omega u_t^2\varphi\,dx$$
$$\le\ 2\gamma q\,[\,R\,]^{2q}\,||U^1-U^2||_{\mathcal{H}}\|\varphi\|,\quad \forall\varphi\in W_0.$$

Finally, from Mean Value Theorem, Assumption (2.4), Hölder's inequality with $\frac{\rho}{2(\rho+1)}+\frac{1}{2(\rho+1)}+\frac{1}{2}=1$, and embedding $W_2\hookrightarrow L^{2(\rho+1)}(\Omega)$, we get

$$\int_\Omega\left(f(u^1)-f(u^2)\right)\varphi dx \le C_{f'}\int_\Omega\left[\,1+\left(|u^1|+|u^2|\right)^\rho\,\right]|w||\varphi|\,dx$$
$$\le 2^{\rho+1}C_{f'}\left[\,|\Omega|+\|u^1\|^{2(\rho+1)}_{2(\rho+1)}+\|u^2\|^{2(\rho+1)}_{2(\rho+1)}\,\right]^{\frac{\rho}{2(\rho+1)}}\|w\|_{2(\rho+1)}\|\varphi\|$$
$$\le C_R||U^1-U^2||_{\mathcal{H}}\|\varphi\|,\quad \forall\varphi\in W_0.$$

Thus, returning to (4.3), there exists a constant $C_R>0$ such that

$$||B(U^1)-B(U^2)||_{\mathcal{H}}\le C_R||U^1-U^2||_{\mathcal{H}}. \tag{4.4}$$

This completes the proof of **(i)** and **(ii)**.

Step 3. It remains to check that both mild and regular solutions are globally defined, that is, $T_{\max} = +\infty$. Indeed, in order we define

$$\omega := 1 - \frac{c_f}{\lambda\lambda_1} > 0.$$

Next, multiplying the problem (1.2) by u_t and integrating over Ω we get

$$\frac{d}{dt}E(U(t)) = -\gamma\mathcal{E}_\epsilon(U(t))^q\|u_t(t)\|^2 \leq 0, \tag{4.5}$$

where we remember that E is defined in (2.1) and $\mathcal{E}_\epsilon$ in (2.8). Equation (4.5) implies

$$E(U(t)) \leq E(U(0)), \quad \forall t \in [0, T_{\max}). \tag{4.6}$$

Now, from assumption (2.3) and using that $W_2 \hookrightarrow W_0$, we have

$$\int_\Omega \widehat{f}(u)dx \geq -\frac{c_f}{2}\|u(t)\|^2 - C_f|\Omega| \geq -\frac{c_f}{2\lambda_1}\|\Delta u(t)\|^2 - C_f|\Omega|. \tag{4.7}$$

On the other hand, from Hölder and Young inequalities, and using again that $W_2 \hookrightarrow W_0$, we have

$$\int_\Omega hudx \leq \|h\|\|u(t)\| \leq \|h\|\frac{1}{\lambda_1^{1/2}}\|\Delta u(t)\| \leq \frac{1}{\omega\lambda\lambda_1}\|h\|^2 + \frac{\omega\lambda}{4}\|\Delta u(t)\|^2. \tag{4.8}$$

Then, from definition of $E(U(t))$ and inequalities (4.7) and (4.8), we obtain that

$$E(U(t)) \geq \frac{1}{2}\|u_t(t)\|^2 + \frac{\omega\lambda}{4}\|\Delta u(t)\|^2 - C_f|\Omega| - \frac{1}{\omega\lambda\lambda_1}\|h\|^2 \geq \frac{\omega}{4}||U(t)||^2_{\mathcal{H}} - \omega_0, \tag{4.9}$$

where $\omega_0 = \left[C_f|\Omega| + \frac{1}{\omega\lambda\lambda_1}\|h\|_2^2\right]$. Hence, from (4.9) and (4.6), we obtain

$$||U(t)||^2_{\mathcal{H}} \leq \frac{4}{\omega}E(U_0) + \frac{4\omega_0}{\omega}, \quad \forall t \in [0, T_{\max}). \tag{4.10}$$

Estimate (4.10) implies that any (mild or strong) solution is globally bounded in time. Therefore, from Pazy [20, Theorem 1.4] we conclude that $T_{\max} = +\infty$. Which proves **(iii)** and completes the proof of Theorem 2.1. □

4.2 Proof of Theorem 2.2

From (2.9) we have

$$S_\epsilon(t)U_0^1 - S_\epsilon(t)U_0^2 = e^{At}\left[U_0^1 - U_0^2\right] + \int_0^t e^{A(t-s)}\left[B(U^1(s)) - B(U^2(s))\right]ds. \tag{4.11}$$

Then, from (4.11) and (4.4) there exists a constant $C = C(||U_0^1||_{\mathcal{H}}, ||U_0^2||_{\mathcal{H}}) > 0$ such that

$$||S_\epsilon(t)U_0^1 - S_\epsilon(t)U_0^2||_{\mathcal{H}} \le ||U_0^1 - U_0^2||_{\mathcal{H}} + C\int_0^t ||S_\epsilon(s)U_0^1 - S_\epsilon(s)U_0^2||_{\mathcal{H}}ds.$$

Applying Gronwall's lemma we obtain (2.10), which completes the proof of Theorem 2.2.

4.3 Proof of Theorem 3.1

In what follows we shall perform several calculations which require adequate degree of smoothness of solutions. Since solutions can be smooth if we take sufficiently smooth initial data, the usual density argument allows to obtain the needed estimates for solutions in the phase space. This fact will be used without further mention.

4.3.1 Gradient system

We begin with asserting that the dynamical systems $(\mathcal{H}, S_\epsilon(t))$ is gradient for each $\epsilon \ge 0$. While gradient property along with an asymptotic smoothness (to be established later) imply an existence of an absorbing ball, the needed piece of information is an uniformity of the size of the absorbing balls in terms of the paraneter $\epsilon \ge 0$. To accomplish the latter, we shall construct the absorbing balls explicitely.

Proposition 4.1 *Assume that the assumptions of* Theorem 2.1 *hold. Then,* $\Phi : \mathcal{H} \to \mathbb{R}$ *given by*

$$\Phi(U) := \frac{1}{2}||U||_{\mathcal{H}}^2 + \frac{\mu}{2}\|\nabla u\|^2 + \int_\Omega \widehat{f}(u(x))\,dx - \int_\Omega h(x)u(x)\,dx$$

is a strict Lyapunov functional for the dynamical system $(\mathcal{H}, S_\epsilon(t))$*. Consequently,* $(\mathcal{H}, S_\epsilon(t))$ *is a gradient dynamical system.*

Proof Let us define $\Phi := E$. From (4.5) one sees that the mapping

$$t \mapsto E(U(t)) = \Phi(S_\epsilon(t)U_0)$$

is non-increasing for every $U_0 := (u_0, u_1) \in \mathcal{H}$. Multiplying the problem (1.2) by u_t and integrating over $\Omega \times [0, t]$ one gets

$$\Phi(S_\epsilon(t)U_0) + \gamma \int_0^t \mathcal{E}_\epsilon(U(\tau))^q \|u_t(\tau)\|^2 \, d\tau = \Phi(U_0), \quad t > 0, \tag{4.12}$$

for every $U_0 \in \mathcal{H}$. From (4.12), we easily conclude that

$$\Phi(S_\epsilon(t)U_0) = \Phi(U_0) \quad \Rightarrow \quad U_0 \in \mathcal{N}_\epsilon, \quad t > 0,$$

where $\mathcal{N}_\epsilon$ is the set of stationary points (with the size independent on ϵ) of the dynamical system $(\mathcal{H}, S_\epsilon(t))$. Since we know that

$$U_0 \in \mathcal{N}_\epsilon \quad \Leftrightarrow \quad S_\epsilon(t)(U_0) = U_0, \quad t > 0,$$

then Φ is a strict Lyapunov functional for the dynamical system $(\mathcal{H}, S_\epsilon(t))$. □

4.3.2 Ultimate dissipativity–Completion of Theorem 3.1

Now we show that the dynamic system $(\mathcal{H}, S_\epsilon(t))$ with $\epsilon \geq 0$ is (ultimately) dissipative, that is, that the C_0-semigroup $\{S_\epsilon(t)\}_{\epsilon \geq 0}$ has a bounded absorbing set $\mathcal{B}$ with the size independent on $\epsilon \geq 0$.

Proposition 4.2 *Let us consider a bounded set $B \subset \mathcal{H}$ and take the solution $S_\epsilon(t)(U_0) = U(t)$ of problem* (1.2)–(1.4) *with $U_0 = (u_0, u_1) \in B$. Under the hypotheses of* Theorem 2.1, *there exist positive constants $c_{0,B}$, $c_{1,B,\epsilon}$ ($c_{0,B}$ independent on ϵ), and $\theta_\epsilon > 0$ such that*

$$\widetilde{E}(U(t)) \leq \begin{cases} \left[c_{0,B}(t-1)^+ + \widetilde{E}(U_0)^{-q} \right]^{-\frac{1}{q}} + 8\omega_0 & \text{if} \quad \epsilon \geq 0, \\ c_{1,B,\epsilon} \widetilde{E}(U_0) e^{-\theta_\epsilon t} + 4\omega_0, & \text{if} \quad \epsilon > 0. \end{cases} \tag{4.13}$$

where $\widetilde{E}$ is a perturbed energy given by

$$\widetilde{E}(U(t)) = E(U(t)) + \omega_0, \quad \text{with} \quad \omega_0 := \frac{1}{\omega\lambda\lambda_1}\|h\|^2 + C_f|\Omega|, \tag{4.14}$$

where $\omega \equiv 1 - \frac{c_f}{\lambda\lambda_1} > 0$.

Proof From the definition of $\widetilde{E}$ it follows directly from (4.9) that

$$\widetilde{E}(U(t)) \geq \frac{\omega}{4}||U(t)||^2_{\mathcal{H}}. \tag{4.15}$$

By multiplying the first equation in (1.2) by u_t and integrating over $\Omega \times [t, t+1]$, we have

$$\gamma \int_t^{t+1} \mathcal{E}_\epsilon(U(s))^q \|u_t(s)\|^2 ds = E(U(t)) - E(U(t+1)) \equiv \mathcal{Q}(t)^2. \tag{4.16}$$

Using that

$$\gamma\mathcal{E}_\epsilon(U(t))\|u_t(t)\|^2 = \gamma\left[\lambda\|\Delta u(t)\|^2 + \|u_t(t)\|^2 + \epsilon I\right]^q \|u_t(t)\|^2 \\ \geq \gamma\left[\|u_t(t)\|^{2q} + \epsilon^q\right]\|u_t(t)\|^2,$$

returning to (4.16), we have

$$\gamma\int_t^{t+1}\|u_t(s)\|^{2(q+1)}ds + \gamma\epsilon^q\int_t^{t+1}\|u_t(s)\|^2 ds \leq \mathcal{Q}(t)^2. \tag{4.17}$$

Note that, if $\epsilon > 0$ follows directly from (4.17) that

$$\int_t^{t+1}\|u_t(s)\|^2 ds \leq \frac{1}{\gamma\epsilon^q}\mathcal{Q}(t)^2.$$

On the other hand, if $\epsilon \geq 0$, from Hölder inequality with $\frac{q}{q+1} + \frac{1}{q+1} = 1$ and (4.17), we have

$$\int_t^{t+1}\|u_t(s)\|^2 ds \leq \left[\int_t^{t+1} ds\right]^{\frac{q}{q+1}}\left[\int_t^{t+1}\|u_t(s)\|^{2(q+1)}ds\right]^{\frac{1}{q+1}} \leq \frac{1}{\gamma^{\frac{1}{q+1}}}\mathcal{Q}(t)^{\frac{2}{q+1}}.$$

Thus, we obtain that

$$\int_t^{t+1}\|u_t(s)\|^2 ds \leq \mathcal{J}_\epsilon(t), \tag{4.18}$$

where,

$$\mathcal{J}_\epsilon(t) := \begin{cases} \frac{1}{\gamma^{\frac{1}{q+1}}}\mathcal{Q}(t)^{\frac{2}{q+1}}, & \text{if } \epsilon \geq 0, \\ \frac{1}{\gamma\epsilon^q}\mathcal{Q}(t)^2, & \text{if } \epsilon > 0. \end{cases} \tag{4.19}$$

From Mean Value Theorem there exist $t_1 \in [t, t+\frac{1}{4}]$, $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|^2 \leq 4\mathcal{J}_\epsilon(t). \tag{4.20}$$

Next, multiplying the first equation in (1.2) by u and integrating over $\Omega \times [t_1, t_2]$ we get

$$\int_{t_1}^{t_2}\int_\Omega\left[\lambda|\Delta u|^2 + \mu|\nabla u|^2 + f(u)u - hu\right]dxds = \int_{t_1}^{t_2}\|u_t(s)\|^2 ds + \sum_{i=1}^{2} I_i, \tag{4.21}$$

where

$$
\begin{aligned}
I_1 &= -\gamma \int_{t_1}^{t_2} \int_\Omega \mathcal{E}_\epsilon(U)^q u_t u dx ds, \\
I_2 &= \left[\int_\Omega u_t(s) u(s) dx \right]_{t_1}^{t_2}.
\end{aligned}
$$

From Assumption (2.3) and definition of $\widetilde{E}$, we obtain that

$$
\int_{t_1}^{t_2} \widetilde{E}(U(s)) ds \leq \omega_0 + \frac{3}{2} \int_{t_1}^{t_2} \|u_t(s)\|^2 ds + \sum_{i=1}^{2} I_i. \tag{4.22}
$$

The terms I_1 and I_2 can be estimated as follows. First, from Hölder inequality, (4.10), (4.15), (4.16), and Young inequality, we have

$$
\begin{aligned}
I_1 &\leq \gamma \int_{t_1}^{t_2} \int_\Omega \mathcal{E}_\epsilon(U(s))^{\frac{q}{2}} |u_t| \mathcal{E}_\epsilon(U(s))^{\frac{q}{2}} |u| dx ds \\
&\leq \left[\gamma \int_t^{t+1} \mathcal{E}_\epsilon(U(s))^q \|u_t(s)\|^2 ds \right]^{\frac{1}{2}} \left[\frac{\gamma}{\lambda_1} \int_t^{t+1} \mathcal{E}_\epsilon(U(s))^q \|\Delta u(s)\|^2 ds \right]^{\frac{1}{2}} \\
&\leq C_B \mathcal{Q}(t) \sup_{t \leq s \leq t+1} \widetilde{E}(U(s))^{\frac{1}{2}} \\
&\leq C_B \mathcal{Q}(t)^2 + \frac{1}{8} \sup_{t \leq s \leq t+1} \widetilde{E}(U(s)).
\end{aligned}
$$

Using immersion $W_2 \hookrightarrow W_0$, (4.15), and (4.18) we have

$$
\begin{aligned}
I_2 &\leq \frac{1}{\lambda_1^{1/2}} \sum_{i=1}^{2} \|u_t(t_i)\| \|\Delta u(t_i)\| \\
&\leq \frac{4}{\lambda_1^{1/2}} \mathcal{J}_\epsilon(t)^{\frac{1}{2}} \sup_{t \leq s \leq t+1} \widetilde{E}(U(s))^{\frac{1}{2}} \\
&\leq \frac{32}{\lambda_1} \mathcal{J}_\epsilon(t) + \frac{1}{8} \sup_{t \leq s \leq t+1} \widetilde{E}(U(s)).
\end{aligned}
$$

Substituting I_1 and I_2 into (4.22) we get

$$
\int_{t_1}^{t_2} \widetilde{E}(U(s)) ds \leq \omega_0 + \frac{1}{4} \sup_{t \leq s \leq t+1} \widetilde{E}(U(s)) + C_B \mathcal{Q}(t)^2 + \left(\frac{32}{\lambda_1} + 1 \right) \mathcal{J}_\epsilon(t). \tag{4.23}
$$

Again using the MVT, there exists $\tau \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \widetilde{E}(U(s))ds = \widetilde{E}(\tau)(t_2 - t_1) \geq \frac{1}{2}\widetilde{E}(U(t+1)). \tag{4.24}$$

On the other hand, from (4.16), we have

$$\widetilde{E}(U(t)) = \widetilde{E}(U(t+1)) + \mathcal{Q}(t)^2. \tag{4.25}$$

From (4.23), (4.24), (4.25), and using that $\widetilde{E}(U(t)) = \sup_{t\leq s\leq t+1} \widetilde{E}(U(s))$, we obtain

$$\sup_{t\leq s\leq t+1} \widetilde{E}(U(s)) \leq 2\omega_0 + \frac{1}{2}\sup_{t\leq s\leq t+1} \widetilde{E}(U(s)) + (1+2C_B)\mathcal{Q}(t)^2 + 2\left(\frac{32}{\lambda_1}+1\right)\mathcal{J}_\epsilon(t).$$

Thus, there exists a constant $K_B > 0$ such that

$$\sup_{t\leq s\leq t+1} \widetilde{E}(U(s)) \leq 4\omega_0 + K_B\mathcal{J}_\epsilon(t). \tag{4.26}$$

From (4.26), we have

$$\sup_{t\leq s\leq t+1} \widetilde{E}(U(s))^{q+1} \leq (8\omega_0)^{q+1} + K_{1,B}\left[\widetilde{E}(U(t)) - \widetilde{E}(U(t+1))\right] \quad \text{if} \quad \epsilon \geq 0,$$

with $K_{1,B} = \frac{(2K_B)^{q+1}}{\gamma}$, and

$$\sup_{t\leq s\leq t+1} \widetilde{E}(U(s)) \leq 4\omega_0 + K_{2,B}\left[\widetilde{E}(U(t)) - \widetilde{E}(U(t+1))\right] \quad \text{if} \quad \epsilon > 0,$$

with $K_{2,B} = \frac{K_B}{\gamma\epsilon^q}$. Therefore, applying Nakao's Lemma (cf. [17]), we have

$$\widetilde{E}(U(t)) \leq \begin{cases} \left[\frac{1}{K_{1,B}}(t-1)^+ + \widetilde{E}(U_0)^{-q}\right]^{-\frac{1}{q}} + 8\omega_0 & \text{if} \quad \epsilon \geq 0, \\ \widetilde{E}(U_0)\left(\frac{1+K_B}{K_B}\right)e^{-\theta t} + 4\omega_0, & \text{if} \quad \epsilon > 0. \end{cases}$$

where $(t-1)^+ = \max\{t-1, 0\}$ and $\theta = \ln\left(\frac{1+K_{2,B}}{K_{2,B}}\right) > 0$. Taking $c_{0,B} = \frac{1}{K_{1,B}}$ and $c_{1,B} = \frac{1+K_B}{K_B}$ we obtain (4.13). This completes the proof of the Proposition 4.2. □

Remark 4.1 Proposition 4.2 shows that bounded sets of $\mathcal{H}$ are attracted by polynomial rates in case $\epsilon = 0$ and by exponential rates in case $\epsilon > 0$. However the polynomial rates are uniform in $\epsilon \to 0$. This is sufficient to assert that the size of the absorbing ball is independent on $\epsilon \geq 0$.

The next corollary gives the conclusion of the proof of Theorem 3.1.

Corollary 4.3 (Ultimate dissipativity) *Under the hypotheses of Theorem 2.1, let us consider any bounded set $B \subset \mathcal{H}$. For $U_0 \in B$, there exist $R > 0$ and $t_B > 0$ (both independent on $\epsilon \geq 0$) such that*

$$||S_\epsilon(t)U_0||_{\mathcal{H}} \leq R, \quad \forall t > t_B, \ \epsilon \geq 0. \tag{4.27}$$

Proof For initial data $U_0 \in B$ we obtain from (4.13) and (4.15) that there exists $T_{B,\epsilon} > 0$ depending on $B \subset \mathcal{H}$ and ϵ such that

$$||S_\epsilon(t)U_0||^2_{\mathcal{H}} \leq \begin{cases} \frac{32\omega_0}{\omega} \text{ if } & \epsilon \geq 0, \\ \frac{16\omega_0}{\omega} \text{ if } & \epsilon > 0, \end{cases} \tag{4.28}$$

where we recall that ω_0 is set in (4.14). One notices that $T_{B,\epsilon}$ can be made independent on ϵ, by applying the polynomial rate of convergence to the absorbing ball, which is uniform for all ϵ. Thus, taking $R = \frac{32\omega_0}{\omega}$, we obtain (4.27) and the set

$$\mathcal{B} = \{U \in \mathcal{H};\ ||U||_{\mathcal{H}} \leq R\}$$

is a bounded absorbing set for $\{S_\epsilon(t)\}_{\epsilon \geq 0}$ with t_B independent on $\epsilon \geq 0$. Therefore, from Definition A.3, the dynamical system $(\mathcal{H}, S_\epsilon(t))$ is ultimately dissipative uniformly in $\epsilon \geq 0$. □

4.4 A useful identity for the difference of trajectories

In the proof of the subsequent results (Propositions 4.5 and 4.6), a useful energy identity is ensured by Lemma 4.4 when dealing with the difference of two trajectories.

Let us consider two mild solutions $U^i_\epsilon = (u^i_\epsilon, u^i_{t,\epsilon})$ of (1.2)–(1.4) with initial data (u^i_0, u^i_1), $i = 1, 2$, and $w_\epsilon = u^1_\epsilon - u^2_\epsilon$. Then, $w_\epsilon := w$ satisfies

$$\begin{cases} w_{tt} - \mu\Delta w + \lambda\Delta^2 w + \gamma \Big[\mathcal{E}_\epsilon(U^1)^q u^1_t - \mathcal{E}_\epsilon(U^2)^q u^2_t \Big] + F(w) = 0, \\ w(0) = w_0 = u^1_0 - u^2_0, \quad w_t(0) = w_1 = u^1_1 - u^2_1, \end{cases} \tag{4.29}$$

where $F(w) = f(u^1) - f(u^2)$, $\mathcal{E}_\epsilon(U^i)$, $i = 1, 2$, is set in (2.8), and

$$\begin{aligned} \gamma\mathcal{E}_\epsilon(U^1)^q u^1_t - \mathcal{E}_\epsilon(U^2)^q u^2_t = \frac{\gamma}{2}\Big[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q \Big] w_t \\ + \frac{\gamma}{2}\Big[\mathcal{E}_\epsilon(U^1)^q - \mathcal{E}_\epsilon(U^2)^q \Big][u^1_t + u^2_t]. \end{aligned}$$

The energy functional associated with (4.29) is given by

$$E_{w,\epsilon}(t) := \frac{1}{2}\Big[||(w, w_t)||^2_{\mathcal{H}} + \mu\|\nabla w\|^2 \Big] = \frac{1}{2}\Big[||W||_{\mathcal{H}} + \mu\|\nabla w\|^2 \Big]. \tag{4.30}$$

In what follows we shall use simply w, E_w-omitting ϵ.

Lemma 4.4 *Under the above notations, the following identity holds true*

$$E_w(T) - E_w(\tau) + \int_\tau^T D_\epsilon(s)ds = \int_\tau^T \left[G_\epsilon(s)ds - \int_\Omega F(w)w_t dx \right] ds,\ T > \tau \geq 0, \tag{4.31}$$

where

$$D_\epsilon(t) = \frac{\gamma}{2}\left[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q \right] \|w_t\|^2 + \frac{\gamma q}{2}\,[\,\xi_\theta\,]^{q-1} \left[\|u_t^1\|^2 - \|u_t^2\|^2 \right]^2, \tag{4.32}$$

$$G_\epsilon(t) = -\frac{\gamma q}{2}\,[\,\xi_\theta\,]^{q-1} \left[\lambda\|\Delta u^1\|^2 - \lambda\|\Delta u^2\|^2 \right] \int_\Omega [\,u_t^1 + u_t^2\,] w_t dx, \tag{4.33}$$

with $\xi_\theta = \theta\mathcal{E}_\epsilon(U^1) + (1-\theta)\mathcal{E}_\epsilon(U^2)$ *for some* $\theta \in (0, 1)$.

Proof Multiplying (4.29) by w_t, applying polarization formula, and integrating over $\Omega \times [\tau, T]$, we get

$$\begin{aligned} E_w(T) - E_w(\tau) &+ \frac{\gamma}{2}\int_\tau^T \left[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q \right] \|w_t\|^2 ds \\ &+ \frac{\gamma}{2}\int_\tau^T \left[\mathcal{E}_\epsilon(U^1)^q - \mathcal{E}_\epsilon(U^2)^q \right] \int_\Omega [\,u_t^1 + u_t^2\,] w_t dx ds \\ &= -\int_\tau^T \int_\Omega F(w) w_t dx ds. \end{aligned} \tag{4.34}$$

Denoting $\xi_\theta = \theta\mathcal{E}_\epsilon(U^1) + (1-\theta)\mathcal{E}_\epsilon(U^2)$, from Mean Value Theorem there exists $\theta \in (0, 1)$ such that

$$\begin{aligned} &\frac{\gamma}{2}\left[\mathcal{E}_\epsilon(U^1)^q - \mathcal{E}_\epsilon(U^2)^q \right] \int_\Omega [\,u_t^1 + u_t^2\,] w_t dx \\ &= \frac{\gamma q}{2}\,[\,\xi_\theta\,]^{q-1} \left[\|u_t^1\|^2 - \|u_t^2\|^2 + \lambda\|\Delta u^1\|^2 - \lambda\|\Delta u^2\|^2 \right] \int_\Omega [\,u_t^1 + u_t^2\,] w_t dx \\ &= \frac{\gamma q}{2}\,[\,\xi_\theta\,]^{q-1} \left[\|u_t^1\|^2 - \|u_t^2\|^2 \right]^2 \\ &\quad + \frac{\gamma q}{2}\,[\,\xi_\theta\,]^{q-1} \left[\lambda\|\Delta u^1\|^2 - \lambda\|\Delta u^2\|^2 \right] \int_\Omega [\,u_t^1 + u_t^2\,] w_t dx. \end{aligned}$$

Therefore, returning to (4.34) we obtain the identity (4.31). □

4.5 Quasi-stability property: case $\epsilon > 0$

In this subsection we show that under the restriction $\epsilon > 0$ the dynamical system $(\mathcal{H}, S_\epsilon(t))$ associated with problem (1.2)–(1.4) is quasi-stable and then will possess a compact global attractor $\{\mathfrak{A}_\epsilon\}_{\epsilon>0}$ that is both finite-dimensional and smooth-see the Appendix..

Proposition 4.5 (Quasi-stabilizability estimate) *Let $\epsilon > 0$. Under the assumptions of* Theorem 2.1*, let B a bounded set in $\mathcal{H}$ and assume that $U^1(t) = (u^1(t), u^1_t(t))$ and $U^2(t) = (u^2(t), u^2_t(t))$ be two mild solutions to* (1.2)–(1.4) *with initial data $U^i_0 \equiv (u^i_0, u^i_1) \in B$, $i = 1, 2$. Then we have the following estimate*

$$||S_\epsilon(t)U^1_0 - S_\epsilon(t)U^2_0||^2_{\mathcal{H}} \leq b_\epsilon(t)||U^1_0 - U^2_0||^2_{\mathcal{H}} + c_\epsilon(t) \sup_{s\in[0,t]} \|u^1(s) - u^2(s)\|^2_{\rho+2}, \tag{4.35}$$

where $b_\epsilon(t)$ and $c_\epsilon(t)$ are nonnegative scalar functions satisfying the properties $b_\epsilon \in L^1(\mathbb{R}^+)$ with $\lim_{t\to+\infty} b_\epsilon(t) = 0$ and $c_\epsilon(t)$ is locally bounded on $[0,\infty)$. *However, these functions do depend in a critical way on $\epsilon \geq 0$.*

Proof We consider again the difference $w = u^1 - u^2$ of two mild solution u^1, u^2 of the problem (1.2)–(1.4). Then, the difference $U^1 - U^2 = (w, w_t)$ solves the problem (4.29) with $\epsilon > 0$ and the following equality holds

$$E_w(T) - E_w(\tau) + \int_\tau^T D_\epsilon(s)ds = \int_\tau^T \left[G_\epsilon(s)ds - \int_\Omega F(w)w_t dx \right] ds. \tag{4.36}$$

where $E_w(t)$, $D_\epsilon(t)$, and $G_\epsilon(t)$ are given in (4.30), (4.32), and (4.33), respectively.
Claim: For any $\delta > 0$, there exist positive constants $C_{B,\delta}$, C_B, such that

$$\begin{aligned}\left|\int_\tau^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds\right| \leq C_B \sup_{0\leq s\leq T} \|w(s)\|^2_{\rho+2} + \delta \int_\tau^T E_w(s)ds \\ + C_{B,\delta} \int_\tau^T d(s, u^1, u^2)E_w(s)ds,\end{aligned} \tag{4.37}$$

where

$$d(t, u^1, u^2) := \|u^1_t(t)\|^2 + \|u^2_t(t)\|^2.$$

Indeed, from definition of $G_\epsilon(t)$ given in (4.33) and Young inequality with any $\delta_1 > 0$, we get

$$\begin{aligned}&\int_\tau^T G_\epsilon(s)ds \\ &\leq \frac{\gamma q}{2} \int_\tau^T [\,\xi_\theta\,]^{q-1} \left[\lambda^{\frac12} \|\Delta u^1\| + \lambda^{\frac12} \|\Delta u^2\| \right] \lambda^{\frac12} \|\Delta w\| \left[\|u^1_t\| + \|u^2_t\| \right] \|w_t\| ds\end{aligned}$$

$$
\begin{aligned}
&\leq C_B \int_\tau^T \lambda^{\frac{1}{2}} \|\Delta w\| \Big[\|u_t^1\| + \|u_t^2\| \Big] \|w_t\| ds \\
&\leq \frac{\delta_1}{2} \int_\tau^T E_w(s) ds + C_{B,\delta_1} \int_\tau^T d(s, u^1, u^2) E_w(s) ds. \qquad (4.38)
\end{aligned}
$$

On the other hand, we can rewrite

$$
\begin{aligned}
\int_\Omega F(w) w_t dx &= \int_\Omega \Big[f(u^1) - f(u^2) \Big] w_t dx \\
&= \frac{d}{dt} \int_\Omega \Big[f(u^1) - f(u^2) \Big] w dx - \int_\Omega \Big[f'(u^1) u_t^1 - f'(u^2) u_t^2 \Big] w dx \\
&= \frac{d}{dt} \int_\Omega \Big[f(u^1) - f(u^2) \Big] w dx - \int_\Omega f'(u^1) w_t w dx \\
&\quad - \int_\Omega \Big[f'(u^1) - f'(u^2) \Big] u_t^2 w dx \\
&= \frac{d}{dt} \int_\Omega F(w) w dx - \frac{1}{2} \frac{d}{dt} \int_\Omega f'(u^1) w^2 dx \\
&\quad + \frac{1}{2} \int_\Omega f''(u^1) u_t^1 w^2 dx - \int_\Omega \Big[f'(u^1) - f'(u^2) \Big] u_t^2 w dx \\
&= \frac{d}{dt} Q(t) + R(t),
\end{aligned}
$$

where

$$
\begin{aligned}
Q(t) &:= \int_\Omega F(w) w dx - \frac{1}{2} \int_\Omega f'(u^1) w^2 dx, \\
R(t) &:= \frac{1}{2} \int_\Omega f''(u^1) u_t^1 w^2 dx - \int_\Omega \Big[f'(u^1) - f'(u^2) \Big] u_t^2 w dx.
\end{aligned}
$$

Thus, we have

$$
- \int_\tau^T \int_\Omega F(w(s)) w_t(s) ds = - \int_\tau^T \frac{d}{ds} Q(s) ds - \int_\tau^T R(s) ds. \qquad (4.39)
$$

Defining $\xi_\theta(t) := \theta u^1(t) + (1-\theta) u^2(t)$ and using Assumption (2.4), Hölder inequality with $\frac{\rho}{\rho+2} + \frac{2}{\rho+2} = 1$, and immersion $W_2 \hookrightarrow L^{\rho+2}(\Omega)$, we obtain

$$
\begin{aligned}
- \int_\tau^T \frac{d}{ds} Q(s) ds &= - \int_\Omega F(w(T)) w(T) dx + \int_\Omega F(w(\tau)) w(\tau) dx \\
&\quad + \frac{1}{2} \int_\Omega f'(u^1(T)) w(T)^2 dx - \frac{1}{2} \int_\Omega f'(u^1(\tau)) w(\tau)^2 dx \\
&= - \int_\Omega \int_0^1 f'(\xi_\theta(T)) d\theta\, w(t)^2 dx + \int_\Omega \int_0^1 f'(\xi_\theta(\tau)) d\theta\, w(\tau)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_\Omega f'(u^1(T))w(t)^2dx-\frac{1}{2}\int_\Omega f'(u^1(\tau))w(\tau)^2dx\\
\leq\ & C_{f'}\int_\Omega\left[1+\left(|u^1(T)|+|u^2(T)|\right)^\rho\right]|w(T)|^2dx\\
&+C_{f'}\int_\Omega\left[1+\left(|u^1(\tau)|+|u^2(\tau)|\right)^\rho\right]|w(\tau)|^2dx\\
&+\frac{C_{f'}}{2}\int_\Omega\left[1+|u^1(T)|^\rho\right]|w(T)|^2dx\\
&+\frac{C_{f'}}{2}\int_\Omega\left[1+|u^1(\tau)|^\rho\right]|w(\tau)|^2dx\\
\leq\ & C\left[1+\left(\|u^1(T)\|^\rho_{\rho+2}+\|u^2(T)\|^\rho_{\rho+2}\right)\right]\|w(T)\|^2_{\rho+2}\\
&+C\left[1+\left(\|u^1(\tau)\|^\rho_{\rho+2}+\|u^2(\tau)\|^\rho_{\rho+2}\right)\right]\|w(\tau)\|^2_{\rho+2}\\
\leq\ & C_B\sup_{0\leq s\leq T}\|w(s)\|^2_{\rho+2}.
\end{aligned}
$$

Using Assumption (2.2), Hölder inequality with $\frac{\rho-1}{2(\rho+1)}+\frac{1}{2}+\frac{1}{\rho+1}=1$, and the embedding $W_2\hookrightarrow L^{2(\rho+1)}(\Omega)$, and Young inequality with any $\delta_2>0$, we have

$$
\begin{aligned}
-\int_\tau^T R(s)ds\leq\ & \frac{1}{2}\int_\tau^T\int_\Omega|f''(u^1)|\,|u^1_t|\,|w|^2dxds\\
&+\int_\tau^T\int_\Omega\int_0^1|f''(\xi_\theta(s))|d\theta|u^2_t|\,|w|^2dxds\\
\leq\ & \frac{C_{f''}}{2}\int_\tau^t\int_\Omega\left[1+|u^1|^{\rho-1}\right]|u^1_t||w|^2dxds\\
&+C_{f''}\int_\tau^T\int_\Omega\left[1+\left(|u^1|+|u^2|\right)^{\rho-1}\right]|u^2_t||w|^2dxds\\
\leq\ & C\int_\tau^T\left[1+\|u^1\|_{2(\rho+1)}+\|u^2\|^{\rho-1}_{2(\rho+1)}\right]\left[\|u^1_t\|+\|u^2_t\|\right]\|w\|^2_{2(\rho+1)}ds\\
\leq\ & C_B\int_\tau^T\left[\|u^1_t\|+\|u^2_t\|\right]\|\Delta w\|^2ds\\
\leq\ & \frac{\delta_2}{2}\int_\tau^T E_w(s)ds+C_{B,\delta_2}\int_\tau^T d(s,u^1,u^2)E_w(s)ds.
\end{aligned}
$$

Substituting the last two inequalities in (4.39) we get that

$$
\begin{aligned}
-\int_\tau^T\int_\Omega F(w(s))w_t(s)ds\leq C_B\sup_{0\leq s\leq T}\|w(s)\|^2_{\rho+2}+\frac{\delta_2}{2}\int_\tau^T E_w(s)ds&\\
+C_{B,\delta_2}\int_\tau^T d(s,u^1,u^2)E_w(s)ds.&
\end{aligned}
\tag{4.40}
$$

Therefore, taking $\delta_1=\delta_2:=\delta>0$, we obtain the inequality (4.37) from (4.38) and (4.40).

Now, multiplying the equation in (4.29) by w and integrating over $\Omega \times [0, T]$, we get

$$\int_0^T E_w(s)ds = \sum_{i=1}^{5} I_i, \tag{4.41}$$

where

$$\begin{aligned}
I_1 &= 2\int_0^T \|w_t\|^2 ds,\\
I_2 &= -\frac{\gamma}{2}\int_0^T\int_\Omega \Big[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q\Big] w_t w dxds,\\
I_3 &= -\frac{\gamma}{2}\int_0^T\int_\Omega \Big[\mathcal{E}_\epsilon(U^1)^q - \mathcal{E}_\epsilon(U^2)^q\Big][\,u_t^1 + u_t^2\,]wdxds,\\
I_4 &= -\int_0^T\int_\Omega F(w)wdxds,\\
I_5 &= \int_\Omega [\,w(0)w_t(0) - w(T)w_t(T)\,]\,dx.
\end{aligned}$$

Let us estimate the terms $I_1, \ldots, I_5$. Firstly, from (4.32) with $\epsilon > 0$, we have

$$I_1 \le \frac{2}{\gamma\epsilon^q}\int_\tau^T D_\epsilon(s)ds.$$

It is here where the dependence on ϵ becomes critical. Next, it is easy to see that

$$\begin{aligned}
I_2 &\le C_B\int_0^T \|w_t\|\|w\|ds \le \frac{1}{6}\int_0^T E_w(s)ds + C_B\int_0^T \|w\|^2 ds,\\
I_3 &\le C_B\int_0^T E_w(s)^{\frac{1}{2}}\|w\|ds \le \frac{1}{6}\int_0^T E_w(s)ds + C_B\int_0^T \|w\|^2 ds.
\end{aligned}$$

From Assumption (2.4), Hölder inequality with $\frac{\rho}{\rho+2} + \frac{2}{\rho+2} = 1$, immersion $W_2 \hookrightarrow L^{\rho+2}(\Omega)$, we get

$$I_4 \le C_{f'}\int_0^T\int_\Omega \Big[1 + (\,|u^1| + |u^2|\,)^\rho\Big]|w|^2 dxds \le C_B\int_0^T \|w\|_{\rho+2}^2 ds.$$

Finally, using that $W_2 \hookrightarrow L^2(\Omega)$, we have

$$I_5 \le \frac{2}{\lambda_1^{\frac{1}{2}}}[\,E_w(0) + E_w(T)\,].$$

Returning to (4.41) and using that $L^{\rho+2}(\Omega) \hookrightarrow L^2(\Omega)$, we get

$$\int_0^T E_w(s)ds \le \frac{4}{\lambda_1^{\frac{1}{2}}}\left[\, E_w(0)+E_w(T)\,\right] + \frac{4}{\gamma\epsilon^q}\int_0^T D_\epsilon(s)ds + C_B\int_0^T \|w\|_{\rho+2}^2 ds. \tag{4.42}$$

On the other hand, integrating (4.36) from 0 to T, we get

$$T E_w(T) + \int_0^T\int_\tau^T D_\epsilon(s)dsd\tau = \int_0^T E_w(\tau)d\tau + \int_0^T\int_\tau^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] dsd\tau. \tag{4.43}$$

Combining (4.42) and (4.43), we obtain that

$$\begin{aligned} T E_w(T) + \int_0^T E_w(s)ds \le{}& \frac{8}{\lambda_1^{1/2}}\left[\, E_w(0)+E_w(T)\,\right] + \frac{8}{\gamma\epsilon^q}\int_0^T D_\epsilon(s)ds \\ &+ \int_0^T\int_\tau^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] dsd\tau \\ &+ T C_B \sup_{0\le t\le T} \|w(t)\|_{\rho+2}^2. \end{aligned} \tag{4.44}$$

From (4.36) with $\tau = 0$, we have

$$E_w(0) = E_w(T) + \int_0^T D_\epsilon(s)ds - \int_0^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds. \tag{4.45}$$

Substituting (4.45) in (4.44) and using without loss of generality that $\widetilde{T} = T - \frac{16}{\lambda_1^{1/2}} > 0$, we get

$$\begin{aligned} \widetilde{T} E_w(T) + \int_0^T E_w(s)ds \le{}& 8\left(\frac{1}{\lambda_1^{1/2}} + \frac{1}{\gamma\epsilon^q} \right)\int_0^T D_{\epsilon,\kappa}(s)ds \\ &+ \int_0^T\int_\tau^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] dsd\tau \\ &- \frac{8}{\lambda_1^{1/2}}\int_0^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds \\ &+ T C_B \sup_{0\le s\le T} \|w(s)\|_{\rho+2}^2. \end{aligned} \tag{4.46}$$

Again, from (4.36) with $\tau = 0$, we get

$$\int_0^T D_\epsilon(s)ds = [\, E_w(0) - E_w(T)\,] + \int_0^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds.$$

Using this equality in (4.46) we get

$$\begin{aligned}\widetilde{T} E_w(T) + \int_0^T E_w(s)ds \le C_\epsilon [\, E_w(0) - E_w(T)\,] + \int_0^T \int_\tau^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds d\tau \\ + \frac{8}{\gamma \epsilon^q} \int_0^T \left[G_\epsilon(s) - \int_\Omega F(w)w_t dx \right] ds \\ + T C_B \sup_{0 \le s \le T} \|w(s)\|^2_{\rho+2},\end{aligned} \tag{4.47}$$

where $C_\epsilon = 8\left(\frac{1}{\lambda_1^{1/2}} + \frac{1}{\gamma\epsilon^q}\right)$. From (4.37), we obtain that

$$\begin{aligned}\widetilde{T} E_w(T) + \int_0^T E_w(s)ds \le C_\epsilon [\, E_w(0) - E_w(T)\,] + \delta \int_0^T \int_\tau^T E_w(s) ds d\tau \\ + \frac{8\delta}{\gamma\epsilon^q} \int_0^T E_w(s)ds + C_{B,\delta} \int_0^T \int_\tau^t d(s, u^1, u^2) E_w(s) ds d\tau \\ + C_{B,\delta} \int_0^T d(s, u^1, u^2) E_w(s) ds + T C_B \sup_{0 \le t \le T} \|w(t)\|^2_{\rho+2}.\end{aligned} \tag{4.48}$$

Taking δ small enough such that $1 - \delta(T + \frac{8}{\gamma\epsilon^q}) \ge \frac{1}{2}$, we obtain

$$\begin{aligned}\widetilde{T} E_w(T) + \frac{1}{2} \int_0^T E_w(s)ds \le C_\epsilon [\, E_w(0) - E_w(T)\,] \\ + T C_{B,\epsilon} \int_0^T d(s, u^1, u^2) E_w(s) ds \\ + T C_B \sup_{0 \le t \le T} \|w(t)\|^2_{\rho+2}.\end{aligned} \tag{4.49}$$

Reiterating the estimate on the intervals $[mT, (m+1)T]$ yelds

$$E_w((m+1)T) \le \eta E_w(mT) + C_B b_m, \quad m = 0, 1, 2, \dots,$$

with $0 < \eta \equiv \frac{C_\epsilon}{\widetilde{T} + C_\epsilon} < 1$ and $C_B = \frac{T(C_{B,\epsilon} + C_B)}{\widetilde{T} + C_\epsilon}$, where

$$b_m \equiv \sup_{s \in [mT, (m+1)T]} \|w(s)\|^2_{\rho+2} + \int_{mT}^{(m+1)T} d(s; u^1, u^2) E_w(s) ds.$$

This yeelds

$$E_w(t) \leq \eta^m E_w(0) + c\sum_{l=1}^{m} \eta^{m-l} b_{l-1}.$$

Since $\eta < 1$, using the same argument as in ([8], Remark 3.3) along with the definition of b_l we obtain that there exists $\omega > 0$ such that

$$E_w(t) \leq C_1 e^{-\omega t} E_w(0) + C_2 \left[\sup_{0\leq s\leq t} \|w(s)\|^2_{\rho+2} + \int_0^t e^{-\omega(t-s)} d(s, u^1, u^2) E_w(s) ds \right],$$

for all $t \geq 0$. Therefore, applying Gronwall's lemma we find

$$E_w(t) \leq \left[C_1 e^{-\omega t} E_w(0) + C_2 \sup_{0\leq s\leq t} \|w(s)\|^2_{\rho+2} \right] e^{C_2 \int_0^t d(s,u^1,u^2)ds}.$$

Using that $\frac{1}{2}||U^1 - U^2||^2_{\mathcal{H}} \leq E_w(t) \leq \frac{1}{2}(1 + \frac{\mu}{\lambda_1^{1/2}})||U^1 - U^2||^2_{\mathcal{H}}$, we have

$$||U^1 - U^2||^2_{\mathcal{H}} \leq b(t)||U_0^1 - U_0^2||^2_{\mathcal{H}} + c(t) \sup_{0\leq s\leq t} \|w(s)\|^2_{\rho+2}.$$

where

$$b_\epsilon(t) := (1 + \frac{\mu}{\lambda_1^{1/2}}) C_1 e^{-\omega t} e^{C_2 \int_0^t d(s,u^1,u^2)ds} \quad \text{and} \quad c_\epsilon(t) := 2C_2 e^{C_2 \int_0^t d(s,u^1,u^2)ds}$$

Thus, using that $d(s, u^1, u^2) = \left[\|u_t^1\|^2 + \|u_t^2\|^2 \right] \in L^1(0, t)$ we obtain for every positive $\epsilon > 0$

$$b_\epsilon(t) \in L^1(\mathbb{R}^+) \quad \text{and} \quad \lim_{t\to+\infty} b_\epsilon(t) = 0$$

and $c_\epsilon(t)$ is locally bounded on $[0, \infty)$. The proof of Proposition 4.5 is now complete. □

4.6 Asymptotically smooth: case $\epsilon \geq 0$

Before presenting Proposition 4.7, which claims that for $\epsilon \geq 0$ the dynamical system $(\mathcal{H}, S_\epsilon(t))$ is asymptotically smooth, we are going to establish the crucial Proposition 4.6, which plays a key result in our analysis since it provides a very new estimate whose proof requires a new way of dealing with the energy damping coefficient at the critical level with respect to potential energy.

Proposition 4.6 *Under the assumptions of* Theorem 2.1, *let us also take q such that $1 \leq 2^{q-1}q < 2$. Let B be a bounded set in $\mathcal{H}$ and $S_\epsilon(t)(U_0^i) = (u^i, u_t^i)$ be two mild* *solutions of problem* (1.2)–(1.4) *with initial data* $U_0^i \equiv (u_0^i, u_1^i) \in B$, $i = 1, 2$. *Then, there exist positive constants* $C, C_{B,t}$ *such that the following inequality holds*

$$E_w(t) \le \frac{C}{t^{\frac{1}{q+1}}} E_w(0)^{\frac{1}{q+1}} + C_{B,t} \sup_{0\le s\le t} \|u^1(s) - u^2(s)\|_{\rho+2}^{\frac{1}{q+1}} + \frac{1}{t}\left|\int_0^t \int_\tau^t \int_\Omega F(w) w_t dx ds d\tau\right| + \frac{C}{t^{\frac{1}{q+1}}}\left|\int_0^t \int_\Omega F(w) w_t dx ds\right|^{\frac{1}{q+1}}. \tag{4.50}$$

where E_w *is given in* (4.30) *and* $w = u^1 - u^2$. *The estimate is uniform in* $\epsilon \ge 0$ *with the constants* $C, C_{B,t}$ *independent on* $\epsilon \ge 0$.

Proof **Step 1: First energy identity**. Using the definition of G_ϵ in (4.33), Young inequality, inequalities $\lambda\|\Delta u^i\|^2 \le ||U^i||^2_{\mathcal{H}}$ and $(a+b)^r \le 2^{r-1}(a^r + b^r)$ for $r \ge 1$, we estimate

$$\begin{aligned}
\int_\tau^T G_\epsilon(s) ds &= -\frac{\gamma q}{2}\int_\tau^T [\,\xi_\theta\,]^{q-1}\Big[\lambda\|\Delta u^1\|^2 - \lambda\|\Delta u^2\|^2\Big]\Big[\|u_t^1\|^2 - \|u_t^2\|^2\Big] ds \\
&\le \frac{\gamma q}{2}\int_\tau^T [\,\xi_\theta\,]^{q-1}\Big[\|u_t^1\|^2 - \|u_t^2\|\Big]^2 ds \\
&\quad + \frac{\gamma q}{8}\int_\tau^T [\,\xi_\theta\,]^{q-1}\Big[\lambda^{\frac12}\|\Delta u^1\| + \lambda^{\frac12}\|\Delta u^2\|\Big]^2 \lambda\|\Delta w\|^2 ds \\
&\le \frac{\gamma q}{2}\int_\tau^T [\,\xi_\theta\,]^{q-1}\Big[\|u_t^1\|^2 - \|u_t^2\|\Big]^2 ds \\
&\quad + \frac{\gamma q}{4}\int_\tau^T \Big[\mathcal{E}_\epsilon(U^1) + \mathcal{E}_\epsilon(U^2)\Big]^q \lambda\|\Delta w\|^2 ds \\
&\le \frac{\gamma q}{2}\int_\tau^T [\,\xi_\theta\,]^{q-1}\Big[\|u_t^1\|^2 - \|u_t^2\|\Big]^2 ds \\
&\quad + \gamma C_q \int_\tau^T \Big[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q\Big]\lambda\|\Delta w\|^2 ds,
\end{aligned}$$

where $C_q := \frac{2^q q}{8}$. Substituting the last inequality in (4.34), noting the cancelation of the first term on the RHS of the inequality above, we obtain the **first energy inequality** with the weighted dissipation:

$$\begin{aligned}
&E_w(T) + \frac{\gamma}{2}\int_\tau^T \Big[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q\Big]\|w_t\|^2 ds \\
&\quad \le E_w(\tau) + \gamma C_q \int_\tau^T \Big[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q\Big]\lambda\|\Delta w\|^2 ds - \int_\tau^T \int_\Omega F(w) w_t dx ds.
\end{aligned} \tag{4.51}$$

Step 2: Second energy inequality. The second term on the RHS of the inequality in (4.51) needs to be absorbed. This will be done by using time weighted equipartition of the energy with a positive weight $M(t) \in C^1(\mathbb{R}^+)$ to be selected later. To proceed, we multiply equation in (4.29) by $M(t)w$ and we integrate the result from τ to T.

$$\int_\tau^T M(t)\left[\lambda\|\Delta w\|^2 + \mu\|\nabla w\|^2\right]ds = \int_\tau^T M(t)\|w_t\|^2 ds + \sum_{i=1}^5 \int_\tau^T L_i(s)ds, \tag{4.52}$$

where

$$\begin{aligned}
L_1(t) &= -M(t)\int_\Omega F(w)w dx,\\
L_2(t) &= -\frac{\gamma}{2}M(t)[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q]\int_\Omega w_t w dx,\\
L_3(t) &= -\frac{\gamma}{2}M(t)[\mathcal{E}_\epsilon(U^1)^q - \mathcal{E}_\epsilon(U^2)^q]\int_\Omega [u_t^1 + u_t^2] w dx,\\
L_4(t) &= -M'(t)\int_\Omega w_t w dx,\\
L_5(t) &= M(t)\int_\Omega w_t w dx|_\tau^T.
\end{aligned} \tag{4.53}$$

We note that all the terms defined by L_i are of lower order (compact) as long as

$$|M'(t)| + |M(t)| \leq C_B. \tag{4.54}$$

We shall apply (4.52) with

$$M(t) \equiv \gamma C_q[\mathcal{E}_\epsilon(U^1)^q + \mathcal{E}_\epsilon(U^2)^q]$$

For this, we verify regularity in (4.54). From the energy *identity* valid for each solution U one has

$$\frac{d}{dt}E(U(t) + \gamma\mathcal{E}_\epsilon(U(t))^q||u_t||^2 = 0,$$

which implies, by ultimate dissipativity, that

$$\left|\frac{d}{dt}E(U(t))\right| \leq \gamma\mathcal{E}_\epsilon(U(t))^q||u_t||^2 \leq C_B.$$

On the other hand,

$$\frac{d}{dt}E(t) = \frac{d}{dt}\mathcal{E}_\epsilon(t) - \mu\int_\Omega \Delta u u_t dx + \int_\Omega [f(u)u_t - hu_t]dx,$$

which proves the regularity in (4.54) for this specific choice of the multiplier M.

Combining (4.51) and (4.52) we obtain

$$
\begin{aligned}
&E_w(T)+\gamma\left(\frac{1}{2}-C_q\right)\int_\tau^T\left[||U^1||^{2q}_{\mathcal{H}}+||U^2||^{2q}_{\mathcal{H}}+2\epsilon^q\right]\|w_t\|^2ds\\
&\quad\le\ E_w(\tau)-\int_\Omega F(w)w_t dx+\sum_{i=1}^5\int_\tau^T L_i(s)ds,
\end{aligned}
\tag{4.55}
$$

uniformly in $\epsilon>0$. Recalling $(a+b)^{2q}\le 2^{2q-1}(a^{2q}+b^{2q})$, we have

$$
\begin{aligned}
\left[||U^1||^{2q}_{\mathcal{H}}+||U^2||^{2q}_{\mathcal{H}}+2\epsilon^q I\right]\|w_t\|^2 &\ge\left[\|u_t^1\|^{2q}+\|u_t^2\|^{2q}\ +2\epsilon^q I\right]\|w_t\|^2\\
&\ge\frac{1}{2^{2q-1}}\|w_t\|^{2q}+2\epsilon^q||w_t||^2.
\end{aligned}
$$

Substituting the last inequality in (4.55), we have

$$
\begin{aligned}
E_w(T)+\int_0^T[\sigma\|w_t\|^{2(q+1)}+2\epsilon^q||w_t||^2]ds\ &\le\ E_w(\tau)-\int_\Omega F(w)w_t dx\\
&+\sum_{i=1}^5\int_\tau^T L_i(s)ds,
\end{aligned}
\tag{4.56}
$$

where $\sigma=\frac{\gamma}{2^{2q-1}}\left(\frac{1}{2}-C_q\right)$. We also recall that all the terms L_i provide compact contribution. Thus, the energy inequality (4.55) is the desired final energy inequality driven by a force F modulo compact terms. In fact, this can be summarized as follows:

Substituting the integrals of the terms L_1, L_2, L_3, L_4 and L_5 in (4.56) and using that $L^{\rho+2}(\Omega)\hookrightarrow L^2(\Omega)$, we obtain the **second energy inequality**

$$
\begin{aligned}
&E_w(T)+\int_\tau^T[\sigma\|w_t\|^{2(q+1)}+2\epsilon^q||w_t||^2]ds\\
&\quad\le E_w(\tau)-\int_\tau^T\int_\Omega F(w)w_t dxds+C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}.
\end{aligned}
\tag{4.57}
$$

Step 3: Reconstruction of the L^1-norm for full potential and kinetic energy. This is accomplished, as usual, by standard equipartition of energy with the multiplier $M(t)=I$. A first step in reconstructing **kinetic energy** is to integrate (4.57) from 0 to T. This yields:

$$
\begin{aligned}
&TE_w(T)+\int_0^T\int_\tau^T[\sigma\|w_t\|^{2(q+1)}+2\epsilon^q||w_t||^2]dsd\tau\\
&\le\int_0^T E_w(\tau)d\tau-\int_0^T\int_\tau^T\int_\Omega F(w)w_t dxdsd\tau+C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}.
\end{aligned}
\tag{4.58}
$$

Now, multiplying the equation in (4.29) by w and integrating over $\Omega \times [0, T]$, we obtain

$$\int_0^T E_w(s)ds = 2\int_0^T \|w_t\|^2 + \sum_{i=1}^4 \int_0^T \widetilde{L}_i(s)ds, \tag{4.59}$$

where

$$\begin{aligned}
\widetilde{L}_1(t) &= -\int_\Omega F(w)wdx,\\
\widetilde{L}_2(t) &= -\frac{\gamma}{2}\left[\,||U^1||_{\mathcal{H}}^{2q} + ||U^2||_{\mathcal{H}}^{2q} \ + 2\epsilon^q,\right]\int_\Omega w_t w dx,\\
\widetilde{L}_3(t) &= -\frac{\gamma}{2}\left[\,||U^1||_{\mathcal{H}}^{2q} - ||U^2||_{\mathcal{H}}^{2q}\right]\int_\Omega \left[u_t^1 + u_t^2\right]wdx,\\
\widetilde{L}_4(t) &= -\frac{d}{dt}\int_\Omega w_t w dx.
\end{aligned}$$

Since all the terms represented by L_i are of lower order, it is easy to see that

$$\sum_{i=1}^4 \int_0^T \widetilde{L}_i(s)ds \le C_{B,T} \sup_{0\le s\le T} \|w(s)\|_{\rho+2}.$$

Hence, returning to (4.59), we get

$$\int_0^T E_w(s)ds \le 2\int_0^T \|w_t\|^2 ds + C_{B,T} \sup_{0\le s\le T} \|w(s)\|_{\rho+2}. \tag{4.60}$$

Combining (4.58) and (4.60), we obtain that

$$\begin{aligned}
&TE_w(T) + \int_0^T E_w(s)ds + \int_0^T\int_\tau^T [\sigma\|w_t\|^{2(q+1)} + 2\epsilon^q||w_t||^2]dsd\tau\\
&\le 4\int_0^T \|w_t\|^2 ds - \int_0^T\int_\tau^T\int_\Omega F(w)w_t dxdsd\tau + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}.
\end{aligned} \tag{4.61}$$

From Hölder inequality with $\frac{q}{q+1} + \frac{1}{q+1} = 1$, we have

$$4\int_0^T \|w_t\|^2 ds \le 4T^{\frac{q}{q+1}}\left(\int_0^T \|w_t\|^{2(q+1)}ds\right)^{\frac{1}{q+1}}.$$

Using (4.57) with $\tau = 0$, we get

$$\int_0^T [\sigma\|w_t\|^{2(q+1)} + 2\epsilon^q||w_t||^2]ds \le E_w(0) + \left|\int_0^T\int_\Omega F(w)w_t dxds\right| + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}.$$

Returning to (4.61), we obtain

$$\begin{aligned} &TE_w(T) + \int_0^T E_w(s)ds + \int_0^T\int_\tau^T [\sigma\|w_t\|^{2(q+1)} + 2\epsilon^q||w_t||^2]dsd\tau \\ &\le \frac{4T^{\frac{q}{q+1}}}{\sigma^{\frac{1}{q+1}}}\left[E_w(0) + \left|\int_0^T\int_\Omega F(w)w_t dxds\right| + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}\right]^{\frac{1}{q+1}} \\ &\quad + \left|\int_0^T\int_\tau^T\int_\Omega F(w)w_t dxdsd\tau\right| + C_{B,T}\sup_{\tau\le s\le T}\|w(s)\|_{\rho+2}. \end{aligned} \tag{4.62}$$

which is our formula for reconstructing the L^1-norm of the full energy.
Step 4: Final proof of Proposition 4.6. There is a $T > 0$ so that the energy $E_w(T)$ can be made arbitrarily small modulo compensated compactness functional involving $F(w)w_t$. Indeed, it follows from (4.62) that

$$\begin{aligned} E_w(T) \le& \frac{4}{\sigma^{\frac{1}{q+1}}T^{\frac{1}{q+1}}}\left[E_w(0) + \left|\int_0^T\int_\Omega F(w)w_t dxds\right| + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}\right]^{\frac{1}{q+1}} \\ &+ \frac{1}{T}\left|\int_0^T\int_\tau^T\int_\Omega F(w)w_t dxdsd\tau\right| + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}. \end{aligned}$$

Using that $g(s) = s^{\frac{1}{q+1}}$ is a concave function that satisfies $g(s+r) \le g(s) + g(r)$, we obtain that

$$\begin{aligned} E_w(T) \le& \frac{4}{\sigma^{\frac{1}{q+1}}T^{\frac{1}{q+1}}}E_w(0)^{\frac{1}{q+1}} + C_{B,T}\sup_{0\le s\le T}\|w(s)\|_{\rho+2}^{\frac{1}{q+1}} \\ &+ \frac{1}{T}\left|\int_0^T\int_\tau^T\int_\Omega F(w)w_t dxdsd\tau\right| + \frac{4}{\sigma^{\frac{1}{q+1}}T^{\frac{1}{q+1}}}\left|\int_0^T\int_\Omega F(w)w_t dxds\right|^{\frac{1}{q+1}}. \end{aligned}$$

Therefore, taking $C = \frac{4}{\sigma^{\frac{1}{q+1}}}$ we obtain (4.50). This proves Proposition 4.6 with *the constants independent on* $\epsilon > 0$. □

Proposition 4.7 (Asymptotic smoothness) *Let us assume the hypotheses of* Proposition 4.6. *Then, the dynamical system* $(\mathcal{H}, S_\epsilon(t))$ *is asymptotically smooth.*

Proof The proof is based on arguments set out by Chueshov and Lasiecka [8, Sect. 3.4]. From inequality (4.50), using that $\frac{1}{2}||S_\epsilon(t)(U_0^1) - S_\epsilon(t)(U_0^2)||^2_{\mathcal{H}} \le E_w(t)$ and $E_w(0) \le C_B$, we have

$$||S_\epsilon(T)(U_0^1) - S_\epsilon(T)(U_0^2)||_{\mathcal{H}}^2 \le \frac{C_B}{T^{\frac{1}{q+1}}} + C_{B,T} \sup_{0\le s\le T} \|u^1(s) - u^2(s)\|_{\rho+2}^{\frac{1}{q+1}}$$
$$+ \frac{2}{T}\left|\int_0^T \int_\tau^T \int_\Omega F(w) w_t dx ds d\tau\right|$$
$$+ \frac{2C}{T^{\frac{1}{q+1}}}\left|\int_0^T \int_\Omega F(w) w_t dx ds\right|^{\frac{1}{q+1}}.$$

Note that, for any fixed $\rho > 0$, there exists $T_B > 0$ such that $\frac{C_B}{T_B^{\frac{1}{q+1}}} < \rho$ and

$$||S_\epsilon(T)(U_0^1) - S_\epsilon(T)(U_0^2)||_{\mathcal{H}}^2 \le \rho + \Psi_{B,T,\rho}(U_0^1, U_0^2), \quad \text{for} \quad T \ge T_B, \tag{4.63}$$

where $\Psi_{B,T,\rho} : B \times B \to \mathbb{R}$ is defined by

$$\Psi_{B,T}(U_0^1, U_0^2) := C_{B,T} \sup_{0\le s\le T} \|u^1(s) - u^2(s)\|_{\rho+2}^{\frac{1}{q+1}}$$
$$+ \frac{2}{T}\left|\int_0^T \int_\tau^T \int_\Omega \Big[f(u^1(s)) - f(u^2(s))\Big][u_t^1(s) - u_t^2(s)] dx ds d\tau\right|$$
$$+ \frac{2C}{T^{\frac{1}{q+1}}}\left|\int_0^T \int_\Omega \Big[f(u^1(s)) - f(u^2(s))\Big][u_t^1(s) - u_t^2(s)] dx ds\right|^{\frac{1}{q+1}},$$

where all the constants are independent on $\epsilon > 0$. Let $U^n = (u^n, u_t^n)$ be the corresponding solution of $U_0^n = (u_0^n, u_1^n) \in B$, $n \in \mathbb{N}$. From bounds $||U^n||_{\mathcal{H}} \le C$, the corresponding solution $U^n(t) = (u^n(t), u_t^n(t))$ satisfy (on a subsequence):

$$\begin{cases} u^n \to u \text{ weakly-star in} & L^\infty\big(s, T; W_2\big), \\ u_t^n \to u_t \text{ weakly-star in} & L^\infty(s, T; W_0), \end{cases} \tag{4.64}$$

and from the Aubin-Lions compactness theorem (see e.g. Simon [22]), we also have

$$u^n \to u \quad \text{strongly in} \quad C([s, T]; W_0), \tag{4.65}$$
$$u^n \to u \quad \text{strongly in} \quad C([s, T]; L^p(\Omega)), \tag{4.66}$$

for $p < \frac{4}{n-4}$ where we use the compact embedding $W_0 \hookrightarrow L^p(\Omega)$. From compact embedding $W_2 \hookrightarrow L^{\rho+2}(\Omega)$, we have

$$\lim_{n\to\infty} \lim_{m\to\infty} \sup_{t\in[0,T]} \|u^m(s) - u^n(s)\|_{\rho+2}^{\frac{1}{q+1}} = 0. \tag{4.67}$$

On the other hand, by using Lemma 8.1 in Lions and Magenes [16] (see on p. 275 therein), (4.64) also implies that u^n is bounded in $C_s\big(s,T;W_2)\big)$, and then $u^n(t)$ is bounded in W_2 for all $t \in [s,T]$. From this and (4.65) one gets

$$u^n(t) \to u(t) \quad \text{weakly in} \quad W_2,\ s \le t \le T, \tag{4.68}$$

and due to the compact embedding theorem, we infer

$$\widehat{f}\big(u^n(t)\big) \to \widehat{f}(u(t)) \quad \text{strongly in} \quad L^1(\Omega),\ s \le t \le T. \tag{4.69}$$

where we remember that $\widehat{f}(u) = \int_0^u f(\tau)d\tau$. Also, from (4.64), assumptions on f and again (4.65), we have

$$\big(f\big(u^n\big), u_t^n\big) \to \big(f(u), u_t\big) \quad \text{strongly in} \quad L^1(s,T). \tag{4.70}$$

Now, regarding that

$$\frac{\partial}{\partial t}\int_\Omega \widehat{f}\big(u^n(x,t)\big)dx = \big(f\big(u^n(t)\big), u_t^n(t)\big),$$

we get

$$\int_s^t \big(f(u^n(\tau)), u_t^n(\tau)\big)d\tau = \int_\Omega \widehat{f}(u^n(x,t))dx - \int_\Omega \widehat{f}(u^n(s,x))dx.$$

From this identity (which also holds true for u) and from the limits (4.69)–(4.70), we finally arrive at

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T \big(f\big(u^n(t)\big) - f\big(u^m(t)\big), u_t^n(t) - u_t^m(t)\big)dt \\
&= \lim_{n\to\infty}\int_\Omega \widehat{f}\big(u^n(x,T)\big)dx + \lim_{m\to\infty}\int_\Omega \widehat{f}\big(u^m(x,T)\big)dx \\
&\quad - \lim_{n\to\infty}\int_\Omega \widehat{f}\big(u^n(x,s)\big)dx - \lim_{m\to\infty}\int_\Omega \widehat{f}\big(u^m(x,s)\big)dx \\
&\quad - \lim_{n\to\infty}\lim_{n\to\infty}\int_s^T\int_\Omega f\big(u^n(t,x)\big)u_t^m(x,t)dxdt \\
&\quad - \lim_{n\to\infty}\lim_{n\to\infty}\int_s^T\int_\Omega f\big(u^m(x,t)\big)u_t^n(x,t)dxdt \\
&= 2\int_\Omega f(u(x,T))dx - 2\int_\Omega f(u(x,s))dx \\
&\quad - 2\int_s^T \big(f(u(t)), u_t(t)\big)dxdt \\
&= 0.
\end{aligned} \tag{4.71}$$

where the convergence is independent on $\epsilon > 0$. Hence, using (4.67) and (4.71), we obtain

$$\lim_{m\to\infty}\inf \lim_{n\to\infty}\inf \Psi_{\epsilon,B,T,\rho}(U_0^n, U_0^m) = 0,$$

uniformly in $\epsilon > 0$. Therefore, from Theorem A.2 (see Appendix A) the dynamical system $(\mathcal{H}, S_\epsilon(t))$ is *asymptotically smooth* with mesure of noncompactnesss independent of $\epsilon > 0$. □

4.7 Upper-semicontinuity: case $\epsilon \to 0$

Proposition 4.8 *Let $\epsilon \in [0, \epsilon_0)$. Under the assumptions of* Theorem 3.3*, let us consider an arbitrary bounded set $B \subset \mathcal{H}$ and denote by $S_\epsilon(t)U_0 = (u^\epsilon(t), u_t^\epsilon(t))$ the solution corresponding to initial data $U_0 = (u_0, u_1) \in B$. Then, there exists a positive non-decresasing function $\mathcal{P}(t) = \mathcal{P}(t, B)$ such that*

$$||S_\epsilon(t)U_0 - S_0(t)U_0||_{\mathcal{H}} \le \mathcal{P}(t)\epsilon^2, \quad t > 0. \tag{4.72}$$

Proof For simplicity, we denote $S_0(t)U_0 = (u(t), u_t(t))$ and set $w^\epsilon = u^\epsilon - u$. Then, w^ϵ is a solution (in the mild and strong sense) of the following problem

$$\begin{cases} w_{tt}^\epsilon + \lambda\Delta^2 w^\epsilon - \mu\Delta w^\epsilon + \frac{\gamma}{2}\Pi_1 w_t^\epsilon + \frac{\gamma}{2}\Pi_2[u_t^\epsilon + u_t] + F(w^\epsilon) = 0, \\ w^\epsilon(0) = 0, \quad w_t^\epsilon(0) = 0, \end{cases} \tag{4.73}$$

where $F(w^\epsilon) = f(u^\epsilon) - f(u)$ and

$$\Pi_j^\epsilon = \mathcal{E}_\epsilon(U^\epsilon)^q + (-1)^{1-j}\mathcal{E}(U)^q, \quad j = 1, 2.$$

Taking the multiplier w_t^ϵ in (4.73), we get

$$\frac{1}{2}\frac{d}{dt}E^\epsilon(t) + \frac{\gamma}{2}\Pi_1^\epsilon(t)\|w_t^\epsilon\|^2 = I_1^\epsilon + I_2^\epsilon, \tag{4.74}$$

where

$$\begin{aligned} E^\epsilon(t) &= \|w_t^\epsilon\|^2 + \lambda\|\Delta w^\epsilon\|^2 + \mu\|\nabla w^\epsilon\|^2, \\ I_1^\epsilon &= -\int_\Omega F(w^\epsilon)w_t^\epsilon dx, \\ I_2^\epsilon &= -\frac{\gamma}{2}\Pi_2^\epsilon(t)\int_\Omega [u_t^\epsilon + u_t]w_t^\epsilon dx. \end{aligned}$$

From Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$ and embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$, we have

$$|I_1^\epsilon| \le C_B\|\Delta w^\epsilon\|\|w_t^\epsilon\| \le C_B E^\epsilon(t).$$

Using that

$$\mathcal{E}_\epsilon(U^\epsilon) - \mathcal{E}(U) = ||U^\epsilon||_{\mathcal{H}}^2 - ||U||_{\mathcal{H}}^2 + \epsilon,$$

we have

$$\begin{aligned}|I_2^\epsilon| = &\Bigg| \frac{q\gamma}{2} \int_0^1 \left[\theta\mathcal{E}_\epsilon(U^\epsilon) + (1-\theta)\mathcal{E}(U)\right]^{q-1} d\theta \left[\,||U^\epsilon||_{\mathcal{H}}^2 - ||U||_{\mathcal{H}}^2\right] \int_\Omega [u_t^\epsilon + u_t] w_t^\epsilon dx \\ &+ \epsilon\, \frac{q\gamma}{2} \int_0^1 \left[\theta\mathcal{E}_\epsilon(U^\epsilon)^q + (1-\theta)\mathcal{E}(U)\right]^{q-1} d\theta \int_\Omega [u_t^\epsilon + u_t] w_t^\epsilon dx \Bigg| \\ &\leq C_B ||U^\epsilon - U||_{\mathcal{H}} \|w_t^\epsilon\| + \epsilon\, C_B \|w_t^\epsilon\| \leq C_B E^\epsilon(t) + \frac{\epsilon^2}{2}.\end{aligned}$$

Returning to (4.74), we obtain

$$\frac{d}{dt} E^\epsilon(t) \leq C_B E^\epsilon(t) + \epsilon^2. \tag{4.75}$$

From Gronwall's lemma, we get

$$E^\epsilon(t) \leq e^{C_B t} \left[E^\epsilon(0) + \epsilon^2 t \right]. \tag{4.76}$$

Using that

$$w^\epsilon(0) = w_t^\epsilon(0) = 0 \quad \text{and} \quad ||S_\epsilon(t)U_0 - S_0(t)U_0||_{\mathcal{H}}^2 \leq E^\epsilon(t),$$

from (4.76), we find the following inequality

$$||S_\epsilon(t)U_0 - S_0(t)U_0||_{\mathcal{H}}^2 \leq \epsilon^2\, t e^{C_B t}, \quad t > 0$$

which proves inequality (4.72) with $\mathcal{P}(t) := te^{C_B t}$ and, therefore, completes the proof of Proposition 4.8. □

4.8 Proofs of Theorems 3.2, 3.3, 3.4 (completion)

Proof of Theorem 3.2: The conclusion of Theorem 3.2 follows by combining Proposition 4.5 with the abstract Theorem A.5 and Theorem A.6 given in Appendix A. □

Proof of Theorem 3.3-(i): Corollary 4.3 guarantees that the dynamical system $(\mathcal{H}, S_0(t))$ is dissipative and Proposition 4.7 ensures that $(\mathcal{H}, S_0(t))$ is asymptotically smooth. Here, the main issue is to trace the asymptotic behavior of measures of non-compactness. Hence the fact that the dynamical system $(\mathcal{H}, S_0(t))$ associated with the problem (1.2)–(1.4) has a compact global attractor $\{\mathfrak{A}_0\}$ is a direct application of Theorem A.1. □

Proof of Theorem 3.3-(ii): The characterization of the attractor as $\mathfrak{A}_0 = \mathrm{M}^u(\mathcal{N}_0)$ follows from the abstract Theorem A.4, after establishing gradient property for the system $(\mathcal{H}, S_0(t))$ is gradient. The latter is established in Proposition 4.1. □

Proof of Theorem 3.4: Again from Corollary 4.3, we obtain

$$\bigcup_{0 \le \epsilon < \epsilon_0} \mathfrak{A}_\epsilon \subset \mathcal{B},$$

where $\mathcal{B}$ is an absorbing ball for the semigroup $S_\epsilon(t)$ for each $\epsilon \ge 0$. Also, from Proposition 4.8 the semigroup $S_\epsilon(t) \to S_0(t)$ uniformly on bounded subsets B of $\mathcal{H}$. Therefore, the conclusion follows from Theorem A.7. □

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

A Appendix: a short review on long time behavior of evolution operators

In order to keep this work self-contained, we find convenient to recall several definitions characterizing a long time behavior of dynamic evolutions such as $S_\epsilon(t)$. This can be found in many references, including [1, 8, 9, 11, 21, 25].

A.1 Definitions

Let $(X, S(t))$ be a dynamical system, where X is a Banach space.

Definition A.1 A *global attractor* for $(X, S(t))$ is a compact set $\mathfrak{A} \subset X$ that is fully invariant and uniformly attracting, that is, $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \ge 0$ and for every bounded subset $B \subset X$

$$\mathrm{dist}_X(S(t)B, \mathfrak{A}) = \sup_{x \in S(t)B} \inf_{y \in \mathfrak{A}} d(x, y) \to 0 \quad \text{as} \quad t \to \infty.$$

Definition A.2 A bounded set $D \subset X$ is an *absorbing* set for $S(t)$ if for any bounded set $B \subset X$, there exists $t_B \geq 0$ such that

$$S(t)B \subset D, \quad \forall t \geq t_B,$$

which defines $(X, S(t))$ as a *dissipative* dynamical system.

Definition A.3 $(X, S(t))$ is said to be *(ultimate)* dissipative iff it posesses a bounded absorbing set B. If X is a Banach space, then a value $R > 0$ is said to be a *radius of dissipativity* of $(X, S(t))$ iff $B \subset \{x \in X : \|x\|_X \leq R\}$.

Definition A.4 We say that $S(t)$ is *asymptotically smooth* in X, if for any bounded positive invariant set $B \subset X$, there exists a compact set $K \subset \overline{B}$, such that

$$\text{dist}_X(S(t)B, K) = 0 \quad \text{as} \quad t \to \infty.$$

Definition A.5 The *fractal dimension* of a compact set $K \subset X$ is defined by

$$\dim K_f^X = \limsup_{\varepsilon \to 0} \frac{\ln(n(K, \varepsilon))}{\ln(1/\varepsilon)},$$

where $n(X, \varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set K.

Definition A.6 Let $\mathcal{N}$ be the set of stationary points of the dynamical system $(X, S(t))$:

$$\mathcal{N} = \{v \in X : S(t)v = v \ \text{ for all } \ t \geq 0\}.$$

We define the *unstable manifold* $\mathrm{M}^u(\mathcal{N})$ emanating from set $\mathcal{N}$ as a set of all $y \in X$ such that there exists a full trajectory $\Upsilon = \{u(t) : t \in \mathbb{R}\}$ with the properties

$$u(0) = y \quad \text{and} \quad \lim_{t \to -\infty} \text{dist}_X(u(t), \mathcal{N}) = 0.$$

Definition A.7 The dynamical system $(X, S(t))$ is said to be *gradient* if there exists a strict Lyapunov function for $(X, S(t))$ on the whole phase space X.

Definition A.8 Let X, Y be two reflexive Banach spaces with X compactly embedded in Y and set $H = X \times Y$. Consider the dynamical system $(H, S(t))$ given by an evolution operator

$$S(t)z = (u(t), u_t(t),), \qquad z = (u_0, u_1) \in H, \tag{A.1}$$

where the functions u and θ possess the regularity

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \tag{A.2}$$

Then one says that $(H, S(t))$ is *quasi-stable* on a set $B \subset H$ if there exist a compact seminorm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, such that

$$\|S(t)z^1 - S(t)z^2\|_H^2 \le a(t)\|z^1 - z^2\|_H^2, \tag{A.3}$$

and

$$\|S(t)z^1 - S(t)z^2\|_H^2 \le b(t)\|z^1 - z^2\|_H^2 + c(t) \sup_{0<s<t} \left[n_X(u^1(s) - u^2(s))\right]^2, \tag{A.4}$$

for any $z^1, z^2 \in B$. The inequality (A.4) is often called *stabilizability inequality*.

Quasi-stable systems enjoy many interesting properties that include finite dimension and smoothness, cf. [8, 9].

A.2 Abstract results

Finally, we provide several abstract theorems pertaining to long time-behavior of dynamical systems, which have been used in the process of proofs related to Sects. 3–4.

It is well known that the properties of *dissipativity* and *asymptotic smoothness* are critical for proving existence of global attractors. In fact, the following result is well-known [8, 9].

Theorem A.1 (Theorem 2.3, [8]) *Let $S(t)$ be a dissipative semigroup defined on a metric space H. Then $S(t)$ has a compact global attractor in H if and only if it is asymptotically smooth in H.*

The following result establishes a convenient criteria for asymptotic smoothness of a dynamical system.

Theorem A.2 (Theorem 7.1.11, [9]) *Let $(X, S(t))$ be a dynamical system on a complete metric space X endowed with a metric d. Assume that for any bounded positively invariant set B in X and for any $\varepsilon > 0$ there exists $T = T_{\varepsilon,B}$ such that*

$$d(S(T)y_1, S(T)y_2) < \varepsilon + \Psi_{\epsilon,B,T}(y_1, y_2), \quad y_i \in B, \tag{A.5}$$

where $\Psi_{\varepsilon,B,T}(y_1, y_2)$ is a functional defined on $B \times B$ such that

$$\liminf_{m\to\infty} \liminf_{n\to\infty} \Psi_{\varepsilon,B,T}(y_n, y_m) = 0 \tag{A.6}$$

for every sequence y_n from B. Then $(X, S(t))$ is an asymptotically smooth dynamical system.

The following result also guarantees that quasi-stable systems are also asymptotically smooth.

Proposition A.3 (Proposition 7.9.5, [9]) *Let Assumptions* (A.1) *and* (A.2) *be in force. Assume that the dynamical system* $(H, S(t))$ *is quasi-stable on every bounded forward invariant set* $\mathcal{B}$ *in* H*. The,* $(H, S(t))$ *is asymptotically smooth.*

The following result gives the characterization of the attractor for gradient systems.

Theorem A.4 (Theorem 7.5.6, [9]) *Let a dynamical system* $(X, S(t))$ *possess a compact global attractor* $\mathfrak{A}$*. Assume that there exists a strict Lyapunov function on* $\mathfrak{A}$*. Then* $\mathfrak{A} = \mathrm{M}^u(\mathcal{N})$*. Moreover, the global attractor* $\mathfrak{A}$ *consists of full trajectories* $\Upsilon = \{u(t) : t \in \mathbb{R}\}$ *with the properties*

$$\lim_{t\to-\infty} dist_X(u(t), \mathcal{N}) = 0 \quad and \quad \lim_{t\to+\infty} dist_X(u(t), \mathcal{N}) = 0.$$

The next two results show that quasi-stable systems enjoy nice properties that include both finite-dimensional and smoothness.

Theorem A.5 (Theorem 7.9.6, [9]) *Assume that the dynamical system* $(H, S(t))$ *possess a compact global attractor* $\mathfrak{A}$ *and is quasi-stable on* $\mathfrak{A}$*. Then the atractor* $\mathfrak{A}$ *of has a finite fractal dimension* $dim_f^H \mathfrak{A}$*.*

Theorem A.6 (Theorem 7.9.8, [9]) *Assume that the dynamical system* $(H, S(t))$ *possess a compact global attractor* $\mathfrak{A}$ *and is quasi-stable on* $\mathfrak{A}$*. Moreover, we assume that* (A.4) *holds with the function* $c(t)$ *possessing the property* $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) < \infty$*. Then any full trajectory* $\{(u(t); u_t(t); \theta(t)) : t \in \mathbb{R}\}$ *that belongs to the global attractor enjoys the following regularity properties,*

$$u_t \in L^\infty(\mathbb{R}; X) \cap C(\mathbb{R}; Y), \quad u_{tt} \in L^\infty(\mathbb{R}; Y), \quad \theta \in L^\infty(\mathbb{R}; Z).$$

Moreover, there exists $R > 0$ *such that*

$$|u_t(t)|_X^2 + |u_{tt}(t)|_Y^2 + |\theta_t(t)|_Z^2 \le R^2, \quad t \in \mathbb{R},$$

where R *depends on the constant* c_∞*, on the semigroup* η_X *in Definition A.8, also on the embedding properties of* X *into* Y*.*

Finally, the following abstract result deals with upper-semicontinuity of attractors, see for instance the books by Robinson [21] and Chueshov [7].

Theorem A.7 (Theorem 10.16, [21]; Proposition 2.3.30, [7]) *Assume that for each* $\epsilon \in [0, \epsilon_0)$*,* $\epsilon_0 > 0$*, the semigroup* $S_\epsilon(t)$ *have a global attractor* $\mathfrak{A}_\epsilon \subset H$ *such that:*

(i) the attractors are uniformly bounded, i.e.: there exists a bounded set $B_0 \subset H$ *such that* $\mathfrak{A}_\epsilon \subset B_0$ *for all* $\epsilon \in [0, \epsilon_0)$*;*
(ii) there exists $t_0 > 0$ *such that the semigroup* $S_\epsilon(t)x$ *converge to* $S_0(t)x$ *as* $\epsilon \to 0$ *for every* $t \ge t_0$ *uniformly with respect to* $x \in B_0$*, i.e.:*

$$\sup_{x\in B_0} |S_\epsilon(t)x - S_0(t)x| \to 0 \quad as \quad \epsilon \to 0^+.$$

Then, the Hausdorff semidistance

$$dist_H(\mathfrak{A}_\epsilon, \mathfrak{A}_0) \to 0 \quad as \quad \epsilon \to 0^+.$$

References

1. Babin, A.V., Visik, M.I.: Attractors of Evolution Equations, Nauka, Moscow, 1989. North-Holland, Amsterdam (1992). (**English translation**)
2. Balakrishnan, A.V.: A theory of nonlinear damping in flexible structures. Stabilization of flexible structures, pp. 1–12 (1988)
3. Balakrishnan, A.V., Taylor, L.W.: Distributed parameter nonlinear damping models for flight structures. In: Proceedings Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB (1989)
4. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ma, T.F.: Exponential decay of the viscoelastic Euler-Bernoulli equation with a nonlocal dissipation in general domains. Differ. Integr. Equ. **17**, 495–510 (2004)
5. Cavalcanti, M.M., Domingos Cavalcanti, V., Jorge Silva, M.A., Webler, C.M.: Exponetial stability for the wave equation with degenerate nonlocal weak damping. Israel J. Math. **210**, 189–213 (2017)
6. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Jorge Silva, M.A., Narciso, V.: Stability for extensible beams with a single degenerate nonlocal damping of Balakrishnan–Taylor type. J. Differ. Equ. **290**, 197–222 (2021)
7. Chueshov, I.: Dynamics of Quasistable Dissipative Systems. Springer, Berlin (2015)
8. Chueshov, I., Lasiecka, I.: Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. Mem. Amer. Math. Soc. 195, no. 912, Providence (2008)
9. Chueshov, I., Lasiecka, I.: Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics. Springer Monographs in Mathematics. Springer, New York (2010)
10. Gomes Tavares, E.H., Jorge Silva, M.A., Narciso, V., Vicente, A.: Dynamics of a class of extensible beams with degenerate and non-degenerate nonlocal damping. Adv. Differ. Equ. **28**(7/8), 685–752 (2023)
11. Hale, J.K.: Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs, vol. 25. American Mathematical Society, Providence (1988)
12. Jorge Silva, M.A., Narciso, V.: Long-time behavior for a plate equation with nonlocal weak damping. Differ. Integr. Equ. **27**(9–10), 931–948 (2014)
13. Jorge Silva, M.A., Narciso, V.: Attractors and their properties for a class of nonlocal extensible beams. Discrete Contin. Dyn. Syst. **35**(3), 985–1008 (2015)
14. Jorge Silva, M.A., Narciso, V.: Long-time dynamics for a class of extensible beams with nonlocal nonlinear damping. Evol. Equ. Control Theory **6**(3), 437–470 (2017)
15. Jorge Silva, M.A., Narciso, V., Vicente, A.: On a beam model related to flight structures with nonlocal energy damping. Discrete Contin. Dyn. Syst. Ser. B **24**(7), 3281–3298 (2019)
16. Lions, J.L., Magenes, E.: Nonhomogeneous Boundary Value Problems and Applications, vol. I. Springer, Berlin (1972)
17. Nakao, M.: Global attractors for wave equations with nonlinear dissipative terms. J. Differ. Equ. **227**, 204–229 (2006)
18. Narciso, V.: Attractors for a plate equation with nonlocal nonlinearities. Math. Methods Appl. Sci. **40**, 3937–3954 (2017)
19. Lange, H., Perla Menzala, G.: Rates of decay of a nonlocal beam equation. Differ. Integr. Equ. **10**, 1075–1092 (1997)
20. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, vol. 44. Springer, Berlin (1983)
21. Robinson, J.C.: Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Cambridge Texts in Applied Mathematics (2001)
22. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (4) **146**, 65–96 (1987)
23. Sun, Y., Yang, Z.: Strong attractors and their robustness for an extensible beam model with energy damping. Discrete Contin. Dyn. Syst. Ser. B **27**(6), 3101–3129 (2022)
24. Sun, Y., Yang, Z.: Attractors and their continuity for an extensible beam equation with rotational inertia and nonlocal energy damping. J. Math. Anal. Appl. **512**, 126148 (2022)

25. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences 68. Springer, New York (1988)