

Research Article

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Luenberger compensator theory for heat-Kelvin-Voigt-damped-structure interaction models with interface/boundary feedback controls

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Abstract: An optimal, complete, continuous theory of the Luenberger dynamic compensator (or state estimator or state observer) is obtained for the recently studied class of heat-structure interaction partial differential equation (PDE) models, with structure subject to high Kelvin-Voigt damping, and feedback control exercised either at the interface between the two media or else at the external boundary of the physical domain in three different settings. It is a first, full investigation that opens the door to numerous and far reaching subsequent work. They will include physically relevant *fluid*-structure models, with wave- or plate-structures, possibly without Kelvin-Voigt damping, as explicitly noted in the text, all the way to achieving the ultimate discrete numerical theory, so critical in applications. While the general setting is functional analytic, delicate PDE-energy estimates dictate how to define the interface/boundary feedback control in each of the three cases.

Keywords: Luenberger compensator theory, heat-structure interaction, Kelvin-Voigt damping

MSC 2020: 93C20, 35Q93, 37C50

1 Introduction

The present article was conceived in response to the intents of the special issue of Open Mathematics. In fact, it provides a new, optimal, rather complete and comprehensive theory on a control theory topic of long standing relevance to applications, with the focus on a recently introduced, by necessity special, class of coupled partial differential equation (PDE) models: heat-structure interaction (HSI) models with high, Kelvin-Voigt damping for the wave-structure, subject to feedback control exercised on the boundary.

Homogeneous (uncontrolled) model. Throughout, $\Omega_f \subseteq \mathbb{R}^n$, $n = 2$, or 3 , will denote the bounded domain on which the heat component of the coupled PDE system evolves. Its boundary will be denoted here as $\partial\Omega_f = \Gamma_s \cup \Gamma_f$, $\Gamma_s \cap \Gamma_f = \emptyset$, with each boundary piece being sufficiently smooth. Moreover, the geometry Ω_s , immersed within Ω_f , will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct heat (fluid) and elastic dynamics occurs across boundary interface $\Gamma_s = \partial\Omega_s$; see Figure 1. In addition, the unit normal vector $\nu(x)$ will be directed away from Ω_f , and so *toward* Ω_s . (This specification of the direction of ν will influence the computations to be done in the following text.)

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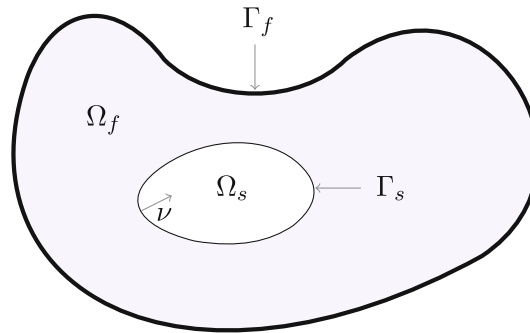


Figure 1: The fluid-structure interaction.

On this geometry in Figure 1, we thus consider the following heat-structure PDE model in solution variables $u = [u_1(t, x), u_2(t, x), \dots, u_n(t, x)]$ (the heat component here replacing the usual velocity field as a first step in this new investigation), and $w = [w_1(t, x), w_2(t, x), \dots, w_n(t, x)]$ (the structural displacement field) with boundary conditions (BC) and initial conditions (IC):

$$(PDE) \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (1.1a)$$

$$(1.1b)$$

$$(BC) \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \\ u = w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (1.1c)$$

$$(1.1d)$$

$$\left[\frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} \right] \quad \text{on } (0, T] \times \Gamma_s; \quad (1.1e)$$

$$(IC) [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}_b. \quad (1.1f)$$

The constant b in (1.1b) will take up either the value $b = 0$, or else the value $b = 1$, as explained below. Accordingly, the space of well-posedness is taken to be the finite energy space

$$\mathbf{H}_b = \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f), & b = 0; \\ \mathbf{H}^1(\Omega_s) \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f), & b = 1, \end{cases} \quad (1.2a)$$

$$(1.2b)$$

for the variable $[w, w_t, u]$. (We are using the common notation $\mathbf{H}^s \equiv [H^s]^n$.) $\mathbf{H}_b$ is a Hilbert space with the following norm inducing inner product, where $(f, g)_\Omega \equiv \int_\Omega f \bar{g} d\Omega$:

$$\left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \right)_{\mathbf{H}_b} = \begin{cases} (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, & b = 0; \\ (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, & b = 1. \end{cases} \quad (1.2c)$$

In (1.2a), the space $\mathbf{H}^1(\Omega_s) \setminus \mathbb{R} = \mathbf{H}^1(\Omega_s) \setminus \text{const}$ is endowed with the gradient norm. Relevant properties of this model, obtained in [1], will be reviewed in Section 1.3.

In particular, it was shown in [1] and reproduced in Section 1.3 that homogeneous problems (1.1a)–(1.1f) can be rewritten as the abstract model

$$y' = \mathcal{A}y, \quad y = [w_1, w_2, u] \quad (1.3)$$

with $\mathcal{A} = \mathcal{A}_b$ the generator of a s.c. contraction semigroup $e^{\mathcal{A}t}$, which is uniformly stable and analytic on $\mathbf{H}_b$.

Three controlled systems. We consider three cases of boundary/interface control g , as applied to systems (1.1a)–(1.1f).

- CASE 1 (Section 2): the control g acts on the matching of the stresses condition (2.1e) at the interface between the two media, thus as a Neumann control as follows:

$$\frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} + g \quad \text{on } (0, T] \times \Gamma_s. \quad (1.1eN)$$

See the entire system (2.1a)–(2.1f) in Section 2.

- CASE 2 (Section 4): the control g acts this time as a Dirichlet control on the matching of the “velocity” condition (1.1d) also at the interface between the two media as follows:

$$u = w_t + g \quad \text{on } (0, T] \times \Gamma_s. \quad (1.1dD)$$

See the entire system (4.1a)–(4.1f) in Section 4.

- CASE 3 (Section 5): the control g acts, still as a Dirichlet control, but this time as a boundary control on the external boundary of the heat domain in (1.1c) as follows:

$$u|_{\Gamma_f} = g \quad \text{on } (0, T] \times \Gamma_f. \quad (1.1cD)$$

See the entire system (5.1a)–(5.1f) in Section 5.

In each case, the control g introduces a control operator $\mathcal{B}$, highly unbounded. Thus, the resulting model is now

$$y' = \mathcal{A}y + \mathcal{B}g \quad (1.3\text{-controlled})$$

with the highly unbounded control operator $\mathcal{B}$ depending on the three cases, see equation (2.2) for CASE 1; equation (4.2) for CASE 2; and equation (5.13) for CASE 3. The selection of the operators entering into the development of the Luenberger theory, as described in Section 1.2.2, depends on the three cases.

Objective of the present article, as a template for future work. The initial objective in the present work is to offer a “continuous theory” of the long-standing, highly relevant topic corresponding to the Luenberger compensator program. The deliberate goal is to have this first contribution serve as a basis for further development of investigations in various directions. Among them, we cite:

- replacing the heat component with a fluid component, thus accounting for the pressure variable while keeping the Kelvin-Voigt structure (model as in [2] (wave) or as in [3] (plate)). The new technique inspired by the boundary control theory [4], which is required for the extension from the heat component to the fluid component is described in Appendix A, with a focus on the present uncontrolled model, following [2].
- replacing the present Kelvin-Voigt damped-**wave** with a Kelvin-Voigt damped-**plate** in modeling the structure. This in turn opens up a variety of different, physically relevant, coupling conditions at the interface between the two media (models as in [3] and [5]);
- obtain a corresponding “discrete theory” or (rigorous) numerical analysis theory, as done in past genuine PDE models of different types in [4, pp. 495–504], [6–8] and also [9, 10], to name just a few references. This is a very challenging and technical direction of investigation, and yet highly important in engineering applications);
- analysis of the same models as in (i) and (ii), and this time, however, with no Kelvin-Voigt damping, such as in [11–14] (wave), where the original stability properties are different. Here, in contrast with the heat-structure case, $\lambda = 0$ is a simple eigenvalue of the uncontrolled model. Heat-viscoelastic plates are studied in [15,16];
- further extension of both the continuous and the discrete analysis to nonlinear models, with static interface [17–24], and even with moving interface [25,26].

1.1 Historical orientation on Luenberger’s compensator theory

The Luenberger theory of “observers” was introduced for lumped (finite-dimensional) linear systems in 1971 [27], and it was met with great success [28, p. 48]. It subsequently stimulated investigations for PDE problems with boundary controls/boundary observations (infinite-dimensional systems with “badly unbounded” control and observation operators) of both parabolic and hyperbolic types [6–8], [4, pp. 495–504]. It consists, in its first phase, of a *continuous theory*, followed next by a rigorous numerical implementation, as in the aforementioned references. At the level of numerical implementation, it was in a sense rediscovered with the more recent topic of “data assimilation” that shares the same philosophy as the Luenberger discrete theory. Recent references include [29–32]. A more detailed description of these topics is given below.

Step 1. The continuous theory. Here, in a purely informal manner, we shall provide the special setting that we shall select in our application of the continuous Luenberger's dynamic compensator theory to heat-structure models. For a preliminary conceptual understanding, we may regard the operators below as being all finite-dimensional, in line with Luenberger's original contribution [27]. Its standard representation is as follows:

$$\begin{cases} \dot{y} = Ay + Bg, & g = Fz = \text{control}, & y(0) = y_0, \\ \dot{z} = (A + BF - KC)z + K(Cy), & z(0) = z_0. \end{cases} \quad (1.4a)$$

$$(1.4b)$$

The basic idea behind is that the full state y is inaccessible, unknown, beyond any measurement, as is often the case in applications. What we have instead at our disposal is the **partial observation** (Cy) , where C is the known observation operator. Examples abound: (i) the actual state within a furnace or (ii) the true distribution of “noise” within an acoustic chamber are not exactly accessible, and only some information from the boundary may be available in each case. Thus, the (compensator) z -equation (1.4b) is fed, or determined by, only the available partial observation (Cy) . Subtracting (1.4b) from (1.4a) with $Bg = BFz$, we obtain after a cancellation of the term BFz :

$$\begin{cases} \frac{d}{dt}[y(t) - z(t)] = (A - KC)[y(t) - z(t)], \\ [y(t) - z(t)] = e^{(A-KC)t}[y_0 - z_0], & t \geq 0. \end{cases} \quad (1.5a)$$

$$(1.5b)$$

One next assumes the detectability condition for the pair $\{A, C\}$: there exist K and k such that $\|e^{(A-KC)t}\| \leq Me^{-kt}$, $k > 0$. Thus, from (1.5b), we finally obtain

$$\|y(t) - z(t)\| = \|e^{(A-KC)t}[y_0 - z_0]\| \leq Me^{-kt}\|y_0 - z_0\|, \quad t \geq 0 \quad (1.6)$$

and the *dynamic compensator* $z(t)$, which is fed only by the known *partial observation* (Cy) of the inaccessible state y , asymptotically approximates such state $y(t)$, at an exponential rate. This is the key of Luenberger's theory in the lumped case where the state of the system is a finite dimensional vector. Nontrivial extensions were subsequently introduced and studied in the case of distributed parameter systems modeled by partial differential equations with boundary control/boundary observation [4, pp. 495–504], [6–8].

Step 2. The numerical theory. Particularly in the case of PDEs dynamics, it is critically important to provide a (finite element) approximation theory of dynamic compensators of Luenberger's type *for partially observed systems*. The aforementioned PDE references include also the discrete/numerical Luenberger theory based on finite element method. The analysis is very technical.

Connections with data assimilation. In recent years, a numerical procedure called “data assimilation” has been introduced, particularly with emphasis on nonlinear dissipative PDE dynamics with finite degrees of freedom, which in spirit is closely related in terms of goals to the discrete Luenberger's compensator theory. In common with the Luenberger's theory, in the presence of inadequate knowledge of the original system, a suitable data assimilation algorithm is introduced to force its corresponding solution to approach the original solution at an exponential rate in time. This is done by having access “to data from measurements of the system collected at much coarser spatial grid than the desired resolution of the forecast” [30]. As expected, the efficiency of data assimilation relies also on the finite dimensionality of the proposed algorithm. Inspiration comes from a rigorous result on the 2D Navier-Stokes equations (NSE) given in [33], where it is proved that if a number of Fourier modes of two different solutions of the NSE have the same asymptotic behavior as t goes to infinity, then the remaining infinite number of modes also have the same asymptotic behavior.

It seems unfortunate that data assimilation theory introduced in 2014 has not apparently been aware of the large body of works in Luenberger theory, which was introduced in 1971, to include PDE parabolic and hyperbolic problems as in [4, pp. 495–504], [6–8]. Luenberger's theory in these references emphasizes control/observation on the boundary, unlike the literature of data assimilation. The original Luenberger theory was for linear models, but it was later introduced for nonlinear models, which are the core of data assimilation. Moreover, data assimilation is passive in the sense that there is no control action. Mutual awareness and knowledge of the two communities' research effort may well benefit both. In this spirit, being this an article on

the Luenberger theory in systems of coupled PDEs with control/observation at “the boundary,” we are pleased to provide specific references in data assimilation.

The first main data assimilation was done in [29] in 2014. This gives the AOT algorithm. It describes the interpolation operators (nodes, modes, and volume averages) and uses the “nudging” algorithm, which is essentially interior control for the data assimilated problem. The technique utilizes the existence of finitely many determining functional to capture the essential asymptotic dynamics of the system.

The initial article that does data assimilation for 3D NSE without assuming any regularity of the solution is [34]. All previous works critically utilized the regularity of the 2D NSE to show asymptotic convergence of the data assimilated solution to the reference solution. In the absence of global regularity in the 3D case, the previous article [34] achieved the exact same result for the 3D NSE by imposing conditions on the observed (model) data. Next, we quote paper [30], which provides results on the Boussinesq system (and also, therefore, for 3D NSE). While [34] does data assimilation for the 3D NSE with the assumption that the reference solution is obtained via the Galerkin procedure, the article [30] makes no such assumptions and does data assimilation for a general (Leray-Hopf) weak solution, which obeys the energy identity corresponding to the system. Also, the article only uses velocity measurements to perform data assimilation.

Luenberger problems for three interface/boundary feedback controlled models: system (1.1a)–(1.1f) subject to control action g as in (1.1eN) (CASE 1); or (1.1dD) (CASE 2); or (1.1cD) (CASE 3), in the final form $y' = \mathcal{A}y + \mathcal{B}g$ as in equation (1.3-controlled). The goal of the present article is to investigate the Luenberger’s dynamic compensator theory (continuous version) as applied to a class of fluid-structure interaction models, in the particular setting where the structure is subject to visco-elastic (Kelvin-Voigt) damping, as in (1.1a)–(1.1f). How to handle the corresponding *fluid-wave* model is described in Appendix A. This focuses on the new tricky technique that is required in the present homogeneous case (1.1a)–(1.1f) following [2]. In summary, in the present article, we consider a fluid (heat)-structure interaction model with high Kelvin-Voigt damping under three different scenarios: (1) in Part I, with Neumann control g at the interface Γ_s as in (1.1eN); (2) in Part II, with Dirichlet control g at the interface Γ_s as in (1.1dD); (3) in Part III, with Dirichlet control g at the external boundary Γ_f as in (1.1cD).

1.2 Orientation on the contributions of the present article. Conceptual description of the mathematical setting and ultimate results

To ease the reading of this article, we find it most appropriate to provide a focused, synthetic orientation regarding both the mathematical setting of the article and its ultimate, sought-after Luenberger-type results.

1.2.1 Uncontrolled, homogeneous model

The uncontrolled model is a coupled heat-structure interaction, where the structure is modeled by a wave with strong Kelvin-Voigt damping, which interacts with a heat component through the interface between the two media, see the linear, coupled PDE system (1.1a)–(1.1f) and Figure 1. The state of the system is the triple $y = \{w, w_t, u\}$, displacement, velocity of the elastic structure, and temperature. A comprehensive study of this model was carried out in [1]. Selected results to be used in the present article are reviewed in Section 1.3. The uncontrolled coupled system is described by an operator $\mathcal{A}$, which is the generator of a s.c. contraction semigroup $e^{\mathcal{A}t}$ on a natural finite energy functional setting. From the purpose of the Luenberger theory to be here investigated, its main feature is that such semigroup is **uniformly (exponentially) stable**, **Theorem 1.3(ii)**. This is due to the Kelvin-Voigt damping. An additional property of such semigroup is that it is analytic in its natural setting, also Theorem 1.3(ii). This analyticity property – also due to the Kelvin-Voigt damping – adds a positive feature to the uncontrolled dynamics. One then seeks, successfully, to retain it and propagate it to the corresponding Luenberger feedback problem, that is, the dynamics of the observer variable z , expressed in feedback-form with respect to the partial observation (Cy) of the original unknown state y . But analyticity is not critical for the key Luenberger’s goal **to recover asymptotically the originally unknown full state y by using the observer z** .

1.2.2 Controlled systems $y' = \mathcal{A}y + \mathcal{B}g$

As already noted, we consider three cases of boundary/interface control g :

- Case 1 (Section 2): The control g acts on the matching of the stresses condition (1.1f) at the interface between the two media (Neumann control). See the entire system (1.1a)–(1.1f) in Section 2.
 - Case 2 (Section 4): The control g acts this time as a Dirichlet control on the matching of the “velocity” condition (3.1d) also occurring at the interface between the two media. See the entire system (3.1a)–(3.1f) in Section 4.
 - Case 3 (Section 5): The control g acts, still as a Dirichlet control, but this time as a boundary control on the external boundary of the heat domain as in (4.1c). See the entire system (4.1a)–(4.1f) in Section 5.
- CASE 1. Here, the following analysis selects the feedback form $g = Fz$ for the Neumann boundary control g at the interface, by taking the Luenberger operators as follows: $F = -\mathcal{B}^*$, $C = \mathcal{B}^*$, $K = \mathcal{B}$. This way, the observer equation becomes: $z' = (\mathcal{A} - 2\mathcal{B}\mathcal{B}^*)z + \mathcal{B}(\mathcal{B}^*y)$, with $\mathcal{B}^*$ the observation operator of the entire unknown state y . Ultimately, the difference $[y - z]$ between unknown state y and known observation z satisfies the equation $\frac{d[y - z]}{dt} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*)[y - z]$, where the feedback generator $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ in equation (2.15a) for CASE 1 is deliberately selected to preserve the property of dissipativity of the original free dynamic operator $\mathcal{A}$. The **ultimate goal** of the analysis is then to establish that such feedback generator is uniformly (exponentially) stable on its natural setting. This is Theorem 3.1, equation (3.14) in CASE 1. This result is established by PDE methods in Section 3.1.3. This way, the known observation variable z , based only on partial knowledge of the unknown state y thorough the unbounded observation/trace operator $\mathcal{B}^*$, see equation (3.8), approaches the unknown full state y asymptotically at exponential speed, the key of the Luenberger theory.
 - CASE 2. Here, the following analysis selects the feedback form $g = Fz$ for the Dirichlet boundary control g at the interface, by a different choice from CASE 1: in fact, in CASE 2 one takes: $F = \mathcal{B}^*$, $C = \mathcal{B}^*$, $K = \mathcal{B}$, so that the observation equation now becomes: $z' = \mathcal{A}z + \mathcal{B}(\mathcal{B}^*y)$, with trace operator $\mathcal{B}^*$, see (4.7). This way the difference $[y - z]$ between unknown state y and known observation z , satisfies the same-looking equation as in CASE 1: $\frac{d[y - z]}{dt} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*)[y - z]$, with dissipative feedback generator. Again, the **ultimate goal** is to establish that such new feedback generator is uniformly (exponentially) stable in its natural setting. This is Theorem 4.1, equation (4.24). This result is again obtained by PDE methods in Section 4.2.4.
 - CASE 3. Here, the following analysis selects the feedback form $g = Fz$ for the Dirichlet boundary control g at the external boundary, which *notationally* is like CASE 2: $F = \mathcal{B}^*$, $C = \mathcal{B}^*$, $K = \mathcal{B}$, with a different operator $\mathcal{B}$ of course, and hence, again observation equation $z' = \mathcal{A}z + \mathcal{B}(\mathcal{B}^*y)$, and finally the same desired form of $[y - z]$: $\frac{d[y - z]}{dt} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*)[y - z]$. In this CASE 3, the **ultimate goal** is again to show that the new feedback operator $(\mathcal{A} - \mathcal{B}\mathcal{B}^*)$ is uniformly stable. This is Theorem 5.1, equation (5.32), whose proof is given in Section 5.2.4. The operator $\mathcal{B}^*$ is again a trace operator (cf. equation (5.15)).
 - Insight on the choice $F = -\mathcal{B}^*$ versus $F = \mathcal{B}^*$ in the various cases. For CASE 2, this insight is given in (4.15), and in CASE 3, this insight is given in equation (5.23). In short, this is a purely PDE problem related to the operator $\mathcal{B}^*$ for the purpose to achieve the feedback generator still dissipative. Thus, while the setting of the analysis is functional analytic, the key technical parts are based on PDE estimates.
 - Finally, to ease the reading, each case is dealt individually. In other words, one may read CASE 3 without knowledge of Cases 1 or 2.

1.3 Review of homogeneous heat-structure interaction model with Kelvin-Voigt damping: $b = 0$, $b = 1$ [1]

We return to the homogeneous problem (1.1a)–(1.1f), which for easy reading, we reproduce here:

$$\begin{aligned} \text{(PDE)} \quad & \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \end{aligned} \quad \begin{aligned} (1.1a) \\ (1.1b) \end{aligned}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \\ u = w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad \begin{aligned} (1.1c) \\ (1.1d) \end{aligned}$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (1.1e)$$

$$\text{(IC)} \quad [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}_b. \quad (1.1f)$$

The constant b in (1.1b) will take up either the value $b = 0$, or else the value $b = 1$, as explained later. Accordingly, the space of well-posedness is taken to be the finite energy space

$$\mathbf{H}_b = \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f), & b = 0; \end{cases} \quad (1.2a)$$

$$\mathbf{H}_b = \begin{cases} \mathbf{H}^1(\Omega_s) \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f), & b = 1, \end{cases} \quad (1.2b)$$

for the variable $[w, w_t, u]$. (We are using the common notation $\mathbf{H}^s \equiv [H^s]^n$.) $\mathbf{H}_b$ is a Hilbert space with the following norm inducing inner product, where $(f, g)_\Omega \equiv \int_\Omega f \bar{g} d\Omega$:

$$\left[\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \right]_{\mathbf{H}_b} = \begin{cases} (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, & b = 0; \\ (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, & b = 1. \end{cases} \quad (1.2c)$$

In (1.2a), the space $\mathbf{H}^1(\Omega_s) \setminus \mathbb{R} = \mathbf{H}^1(\Omega_s) \setminus \text{const}$ is endowed with the gradient norm.

Abstract model of the homogeneous PDE problem (1.1a)–(1.1f). The operator $\mathcal{A}_b$ and its adjoint $\mathcal{A}_b^*$, $b = 0, 1$. **Basic results** [1]. The abstract version of the homogeneous PDE model (1.1a)–(1.1f) is given as a first-order equation by

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \quad (1.7)$$

where the operator $\mathcal{A}_b : \mathbf{H}_b \supset \mathcal{D}(\mathcal{A}_b) \rightarrow \mathbf{H}_b$ is given by

$$\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}. \quad (1.8)$$

A description of $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_b)$ is as follows:

$$\begin{aligned} \text{(i)} \quad & v_1, v_2 \in \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} \text{ for } b = 0; \quad v_1, v_2 \in \mathbf{H}^1(\Omega_s) \text{ for } b = 1; \\ & \text{so that } v_2|_{\Gamma_s} = h|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \text{ in both cases;} \\ & \Delta(v_1 + v_2) \in \mathbf{L}^2(\Omega_s); \end{aligned} \quad (1.9a)$$

$$\begin{aligned} \text{(ii)} \quad & h \in \mathbf{H}^1(\Omega_f), \quad \Delta h \in \mathbf{L}^2(\Omega_f), \quad h|_{\Gamma_f} \equiv 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s); \\ & \left. \frac{\partial h}{\partial \nu} \right|_{\Gamma_s} = \left. \frac{\partial(v_1 + v_2)}{\partial \nu} \right|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \end{aligned} \quad (1.9b)$$

Remark 1.1. The aforementioned description of $\mathcal{D}(\mathcal{A}_b)$ in (1.9a)–(1.9b) shows that the point $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_b)$ enjoys a smoothing of regularity by one Sobolev unit – from $\mathbf{L}^2(\cdot)$ to $\mathbf{H}^1(\cdot)$ – but only of the coordinates v_2 and h , with respect to the original finite energy state space $\mathbf{H}_b$ in (1.2a). In contrast, the first coordinate v_1 experiences no smoothing: it is in $\mathbf{H}^1(\Omega_s)$, the first coordinate component of the space $\mathbf{H}_b$. This amounts to the fact that $\mathcal{A}$ has *noncompact resolvent* $R(\lambda, \mathcal{A})$ on $\mathbf{H}_b$. Consistently, it was shown in [1, Proposition 2.4] that the point $\lambda = -1$ belongs to the continuous spectrum of $\mathcal{A}_b$: $-1 \in \sigma_c(\mathcal{A}_b)$.

Theorem 1.1. (The adjoint $\mathcal{A}_b^*$ of $\mathcal{A}_b$, $b = 0$ or 1 [1, Appendix A]). The $\mathbf{H}_b$ -adjoint of the operator $\mathcal{A}_b$ defined in (1.7)–(1.9) is given by

$$\mathcal{A}_b^* \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ -\Delta + bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} -\tilde{v}_2 \\ \Delta(\tilde{v}_2 - \tilde{v}_1) + b\tilde{v}_1 \\ \Delta\tilde{h} \end{bmatrix}. \quad (1.10a)$$

The PDE version of

$$\frac{d}{dt} \begin{bmatrix} v \\ w \\ u \end{bmatrix} = \mathcal{A}_b^* \begin{bmatrix} v \\ w \\ u \end{bmatrix} \quad (1.10b)$$

is given by

$$(PDE) \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (1.10c)$$

$$(1.10d)$$

$$(BC) \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \\ u = -w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (1.10e)$$

$$(1.10f)$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial \nu} = -\frac{\partial u}{\partial \nu} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (1.10g)$$

compared with equation (1.1a)–(1.1e). The domain $\mathcal{D}(\mathcal{A}_b^*)$ of the operator $\mathcal{A}_b^*$ in (1.10a) is described as follows (compared with $\mathcal{D}(\mathcal{A}_b)$ in (1.9a)–(1.9b)): $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \in \mathcal{D}(\mathcal{A}_b^*)$ means:

(i)

$$\begin{aligned} \tilde{v}_1, \tilde{v}_2 &\in \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} \text{ for } b = 0; \quad \tilde{v}_1, \tilde{v}_2 \in \mathbf{H}^1(\Omega_s) \text{ for } b = 1; \\ \text{so that } \tilde{v}_2|_{\Gamma_s} &= \tilde{h}|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \text{ in both cases;} \\ \Delta(\tilde{v}_2 - \tilde{v}_1) &\in \mathbf{L}^2(\Omega_s); \end{aligned} \quad (1.11a)$$

(ii)

$$\begin{aligned} \tilde{h} &\in \mathbf{H}^1(\Omega_f), \quad \Delta\tilde{h} \in \mathbf{L}^2(\Omega_f), \quad \tilde{h}|_{\Gamma_f} \equiv 0, \quad \tilde{h}|_{\Gamma_s} = \tilde{v}_2|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s); \\ \frac{\partial\tilde{h}}{\partial \nu} \Big|_{\Gamma_s} &= \frac{\partial(\tilde{v}_2 - \tilde{v}_1)}{\partial \nu} \Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \end{aligned} \quad (1.11b)$$

Actually, [1, Appendix A] gives the detailed proof for $b = 0$. For $b = 1$, in [1, equation (A.4)] one adds the term (w_2, v_1) for the full H^1 -inner product but also the term $-b(w_1, w_2)$ with $b = 1$.

Theorem 1.2. (Generation by $\mathcal{A}_b$ and $\mathcal{A}_b^*$, $b = 0, b = 1$ [1, Theorem 1.2])

(i) The operator $\mathcal{A}_b$ defined by (1.8), (1.9) and its adjoint $\mathcal{A}_b^*$ given by (1.10a) and (1.11) are dissipative: For $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A}_b)$, and $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}_b^*)$, we have

$$\operatorname{Re} \left(\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_b} = -\|\nabla v_2\|_{\Omega_s}^2 - \|\nabla h\|_{\Omega_f}^2 \leq 0, \quad (1.12)$$

$$\operatorname{Re} \left(\mathcal{A}_b^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}_b} = -\|\nabla v_2^*\|_{\Omega_s}^2 - \|\nabla h^*\|_{\Omega_f}^2 \leq 0, \quad (1.13)$$

in the $\mathbf{L}^2(\cdot)$ norms of Ω_s and Ω_f .

(ii) Thus, $\mathcal{A}_b$ and $\mathcal{A}_b^*$ are maximal dissipative on $\mathbf{H}_b$. Then [35] gives that $\mathcal{A}_b$ generates a s.c. (C_0) -contraction semigroup $e^{\mathcal{A}_b t}$ on $\mathbf{H}_b$, which gives the unique solution of problems (1.1a)–(1.1f):

$$\begin{bmatrix} w_0 \\ w_1 \\ u_0 \end{bmatrix} \in \mathbf{H}_b \Rightarrow \begin{bmatrix} w(t) \\ w_t(t) \\ u(t) \end{bmatrix} \equiv e^{\mathcal{A}_b t} \begin{bmatrix} w_0 \\ w_1 \\ u_0 \end{bmatrix} \in C([0, T]; \mathbf{H}_b). \quad (1.14)$$

The same generation results hold also for $\mathcal{A}_b^*$ on $\mathbf{H}_b$, with $e^{\mathcal{A}_b^* t}$ solving system (1.10c)–(1.10g).

Again, [1, Section 2] gives the proof only for $b = 0$. For $b = 1$, in [1, equation (2.2a)], one adds the terms (v_2, v_1) for the full H^1 -norm and the term $-b(v_1, v_2)$, $b = 1$, leading now to a new version of such equation (2.2a) in [1] given by $-\|\nabla v_2\|^2 - \|\nabla h\|^2 + 2\operatorname{Im}(\nabla v_2, \nabla v_1) + 2\operatorname{Im}(v_2, v_1)$. Thus, taking the real part of the aforementioned expression, one obtains (1.12) for $b = 0$ and $b = 1$.

Theorem 1.3. *With reference to the operator $\mathcal{A}_b$ in (1.7)–(1.9) and its adjoint $\mathcal{A}_b^*$ in (1.10a) and (1.11), both defined on $\mathbf{H}_b$, $b = 0, 1$, we have*

(i)

$$0 \in \rho(\mathcal{A}_b), \quad 0 \in \rho(\mathcal{A}_b^*), \quad \rho(\cdot) = \text{resolvent set} \quad (1.15)$$

with explicit expression of $\mathcal{A}_b^{-1}$ given in [1, Lemma 2.2].

(ii) [1, Theorem 1.4] *The contraction semigroups $e^{\mathcal{A}_b t}$ and $e^{\mathcal{A}_b^* t}$ generated by Theorem 1.2 are analytic and uniformly stable on $\mathbf{H}_b$, $b = 0, 1$; there exist constants $M \geq 1$, $\delta > 0$, such that*

$$\|e^{\mathcal{A}_b t}\|_{\mathcal{L}(\mathbf{H}_b)} + \|e^{\mathcal{A}_b^* t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq M e^{-\delta t}, \quad t \geq 0, \quad b = 0, 1. \quad (1.16)$$

Remark 1.2.

- (1) Section 1.3 (a subset of [1]) shows that the natural functional setting for problems (1.1a)–(1.1f) is: the energy space $\mathbf{H}_{b=0}$ in (1.2a) for $b = 0$; and the energy space $\mathbf{H}_{b=1}$ in (1.2b) for $b = 1$. In each such case, $b = 0$ and $b = 1$, the free dynamic operator is maximal dissipative, it defines a corresponding expression for the adjoint $\mathcal{A}_b^*$ and the resulting contraction semigroups $e^{\mathcal{A}_b t}$ and $e^{\mathcal{A}_b^* t}$ are analytic and uniformly stable. Analyticity in Theorem 1.3(ii) above is consistent with abstract results [36–38], in view of the Kelvin-Voigt damping.
- (2) If, however, one insists in considering problems (1.1a)–(1.1f) with $b = 0$ in the energy space $\mathbf{H}_{b=1}$ with full H^1 -norm for the position variable, then stability is lost: more precisely, one can readily prove or verify that:

$$\begin{cases} \lambda = 0 \text{ is a simple eigenvalue of the free dynamics operator } \mathcal{A}_{b=0} \text{ (with } b = 0) \\ \text{with corresponding eigenvector } e = [1, 0, 0] \in \mathcal{D}(\mathcal{A}_{b=0}) \subset \mathbf{H}_{b=1}. \end{cases} \quad (1.17)$$

In fact, setting equal to zero equation (1.8) with $b = 0$ implies $v_2 \equiv 0$; hence, $h \equiv 0$ from $\Delta h = 0$, $h|_{\Gamma_f} = 0$, $h|_{\Gamma_s} = v_2|_{\Gamma_s} = 0$. This yields $\Delta v_1 = 0$ in Ω_s , $\frac{\partial v_1}{\partial \nu}\Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\Big|_{\Gamma_s} = 0$ by (1.9b), whose normalized solution is $v_1 = 1$ in $\mathbf{H}^1(\Omega_s)$ (while it would be $v_1 = 0$ in $\mathbf{H}^1(\Omega_s) \setminus \mathbb{R}$).

- (3) In this case, we may view the problem with $b = 1$ on $\mathbf{H}_{b=1}$ as having “stabilized” (and regularized) the same

problem with $b = 0$ on $\mathbf{H}_{b=1}$: $\mathcal{A}_{b=1} = \mathcal{A}_{b=0} + \mathcal{S}$, with stabilizing operator $\mathcal{S} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ -v_1 \\ 0 \end{bmatrix}$.

2 CASE 1. Heat-structure interaction with Kelvin-Voigt damping: Neumann control g at the interface Γ_s

The present article begins with this section. In the present CASE 1, we consider problem (1.1a)–(1.1f) subject this time to control g acting in the Neumann interface condition (1.1e); that is,

$$\begin{aligned} \text{(PDE)} \quad & \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \end{aligned} \quad \begin{aligned} (2.1a) \\ (2.1b) \end{aligned}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \\ u = w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad \begin{aligned} (2.1c) \\ (2.1d) \end{aligned}$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} + g & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (2.1e)$$

$$\text{(IC)} \quad [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}_b, \quad (2.1f)$$

with Neumann boundary control g acting at the interface Γ_s . The constant b in (1.1b) will take up either the value $b = 0$, or else the value $b = 1$, as explained earlier. Accordingly, the space of well-posedness is taken to be the finite energy space defined in (1.2a), or (1.2b).

2.1 Abstract model on $\mathbf{H}_b$, $b = 0, 1$, of the nonhomogeneous PDE model (2.1a)–(2.1f) with Neumann control g acting at the interface Γ_s

This topic was duly treated in [39], at least for $b = 0$. This will be reviewed below and complemented by the case $b = 1$. In either case, the abstract version of the nonhomogeneous PDE model (2.1a)–(2.1f) is given by

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathcal{B}_N^{(b)} g, \quad (2.2)$$

with the full dynamic operator $\mathcal{A}_b$ given by (1.8) and (1.9). The definition of the Neumann control $\mathcal{B}_N^{(b)}$ depends on the two cases $b = 0$ and $b = 1$, on the respective space $\mathbf{H}_b$. We shall provide a unified treatment covering the two cases $b = 0$ and $b = 1$, which will recover the case $b = 0$ in [39].

We first define the positive self-adjoint operator $A_{N,s}^{(b)}$ by

$$-A_{N,s}^{(b)} \varphi = (\Delta - bI)\varphi, \quad \mathcal{D}(A_{N,s}^{(b)}) = \begin{cases} \varphi \in \mathbf{H}^2(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0 \\ \varphi \in \mathbf{H}^2(\Omega_s) & \text{for } b = 1 \end{cases} : \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\Gamma_s} = 0, \quad (2.3)$$

so that $A_{N,s}^{(b)}$ is defined on the first component space of the state space $\mathbf{H}_b$, $b = 0$, or $b = 1$. Next, following an established procedure [40], [4, Chapter 3], we define the following Neumann map $N_s^{(b)}$ on the structure domain Ω_s by

$$\psi = N_s^{(b)} \mu \iff \begin{cases} (\Delta - bI)\psi = 0 & \text{in } \Omega_s, \\ \frac{\partial \psi}{\partial \nu} \Big|_{\Gamma_s} = \mu, \end{cases} \quad (2.4)$$

with regularity [40–43]

$$\begin{cases} N_s^{(b)} : \mathbf{H}^r(\Gamma_s) \rightarrow \begin{cases} \mathbf{H}^{\frac{3}{2}+r}(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^{\frac{3}{2}+r}(\Omega_s) & \text{for } b = 1, \end{cases} & r \text{ real,} \\ N_s^{(b)} : \mathbf{L}^2(\Gamma_s) \rightarrow \begin{cases} \mathbf{H}^{\frac{3}{2}}(\Omega_s) \setminus \mathbb{R} \subset \mathbf{H}^{\frac{3}{2}-2\varepsilon}(\Omega_s) \setminus \mathbb{R} = \mathcal{D}\left[(A_{N,s}^{(b)})^{\frac{3}{4}-\varepsilon}\right] & \text{for } b = 0, \\ \mathbf{H}^{\frac{3}{2}}(\Omega_s) \subset \mathbf{H}^{\frac{3}{2}-2\varepsilon}(\Omega_s) = \mathcal{D}\left[(A_{N,s}^{(b)})^{\frac{3}{4}-\varepsilon}\right] & \text{for } b = 1, \end{cases} & \text{continuously} \\ \text{or } (A_{N,s}^{(b)})^{\frac{3}{4}-\varepsilon} N_s^{(b)} \in \mathcal{L}(\mathbf{L}^2(\Gamma_s); \mathbf{L}^2(\Omega_s)), & b = 0 \text{ or } b = 1. \quad [30, \text{p. 195}] \end{cases} \quad (2.5)$$

Next we return to (2.1b) and rewrite it via (2.4) and (2.1e) [40], [4, Chapter 3]

$$w_{tt} = \Delta(w + w_t) - bw = (\Delta - bI) \left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} + g \Big|_{\Gamma_s} \right) \right] + b(w + w_t) - bw. \quad (2.6)$$

By (2.1), (2.1e), the term $\left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} + g \Big|_{\Gamma_s} \right) \right]$ satisfies the zero Neumann B.C. of $A_{N,s}^{(b)}$ in (2.3), so we can rewrite (2.6) as follows:

$$w_{tt} = -A_{N,s}^{(b)} \left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} + g \Big|_{\Gamma_s} \right) \right] + b(w + w_t) - bw \in \mathbf{L}^2(\Omega_s), \quad (2.7)$$

$$w_{tt} = -\tilde{A}_{N,s}^{(b)}(w + w_t) + \tilde{A}_{N,s}^{(b)} N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right) + b(w + w_t) - bw + \tilde{A}_{N,s}^{(b)} N_s^{(b)} g \in [\mathcal{D}(A_{N,s}^{(b)})]', \quad (2.8)$$

where now $\tilde{A}_{N,s}^{(b)}$ is the isomorphic extension $\mathbf{L}^2(\Omega_s) \rightarrow [\mathcal{D}(A_{N,s}^{(b)})]' = \text{dual of } \mathcal{D}(A_{N,s}^{(b)}) \text{ with respect to } \mathbf{L}^2(\Omega_s)$, of the original operator $A_{N,s}^{(b)}$ in (2.3). Next, we pass to the “fluid” domain Ω_f . We let $-A_{D,f}$ be the negative, self-adjoint operator on $\mathbf{L}^2(\Omega_f)$ by

$$-A_{D,f} \varphi = \Delta \varphi, \quad \mathcal{D}(A_{D,f}) \equiv \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f), \quad (2.9)$$

while $D_{f,s}$ is the Dirichlet map from Γ_s to Ω_f defined by

$$\varphi = D_{f,s} \chi \iff \begin{cases} \Delta \varphi = 0 & \text{in } \Omega_f; \\ \varphi|_{\Gamma_f} = 0, & \varphi|_{\Gamma_s} = \chi. \end{cases} \quad (2.10a)$$

$$(2.10b)$$

The following regularity holds true for $D_{f,s}$: for any r ,

$$D_{f,s} : \mathbf{H}^r(\partial\Omega_f) \rightarrow \mathbf{H}^{r+\frac{1}{2}}(\Omega_f), \quad [33] \quad (2.11a)$$

$$D_{f,s} : \mathbf{L}^2(\partial\Omega_f) \rightarrow \mathbf{H}^{\frac{1}{2}}(\Omega_f) \subset \mathbf{H}^{\frac{1}{2}-2\varepsilon}(\Omega_f) \equiv \mathcal{D}(A_{D,f}^{\frac{1}{4}-\varepsilon}); \quad [31, \text{p. 181}], [48] \quad (2.11b)$$

continuously. Then, as usual [1], we rewrite the u -problem in (2.1a) via (2.10) as follows:

$$\begin{cases} u_t = \Delta(u - D_{f,s}(w_t|_{\Gamma_s})) & \text{in } (0, T] \times \Omega_f; \\ [u - D_{f,s}(w_t|_{\Gamma_s})]_{\Gamma_s} = 0 & \text{in } (0, T] \times \Gamma_s; \\ [u - D_{f,s}(w_t|_{\Gamma_s})]_{\Gamma_f} = 0 & \text{in } (0, T] \times \Gamma_f; \end{cases} \quad (2.12a)$$

$$(2.12b)$$

$$(2.12c)$$

$$u_t = -A_{D,f}(u - D_{f,s}(w_t|_{\Gamma_s})) \in \mathbf{L}^2(\Omega_f); \quad u_t = -\tilde{A}_{D,f}(u + \tilde{A}_{D,f} D_{f,s}(w_t|_{\Gamma_s})) \in [\mathcal{D}(A_{D,f})]', \quad (2.13)$$

where $\tilde{A}_{D,f}$ is its isomorphic extension $\mathbf{L}^2(\Omega_f) \rightarrow [\mathcal{D}(A_{D,f})]' = \text{dual of } \mathcal{D}(A_{D,f}) \text{ with respect to } \mathbf{L}^2(\Omega_f) \text{ as a pivot space}$. Henceforth, as in [4, Chapter 3] we drop the $\sim$ for the extensions $\tilde{A}_{N,s}^{(b)}$ and $\tilde{A}_{D,f}$ for notational easiness, as no misunderstanding is likely to arise.

By combining (2.8) and (2.13), we obtain

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A_{N,s}^{(b)} & -A_{N,s}^{(b)} + bI & A_{N,s}^{(b)} N_s^{(b)} \left(\frac{\partial}{\partial \nu} \cdot \right) \\ 0 & A_{D,f} D_{f,s}(\cdot|_{\Gamma_s}) & -A_{D,f} \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)} g \\ 0 \end{bmatrix}. \quad (2.14)$$

The operator in (2.14) acting on $[w, w_t, u](g \equiv 0)$ is the same operator $\mathcal{A}_b$ in (1.8)–(1.9), except that in (2.14) the relevant BCs (1.9) are included in the operator entries.

Thus, in conclusion, the abstract model for the nonhomogeneous PDE model (2.1a)–(2.1f) is given again by (2.2), where now

$$\mathcal{B}_N^{(b)} g = \begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)} g \\ 0 \end{bmatrix}, \quad (2.15)$$

$$\mathcal{B}_N^{(b)} : \text{continuous } \mathbf{L}^2(\Gamma_s) \rightarrow \left[\begin{array}{c} \otimes \\ \mathcal{D}[(A_{N,s}^{(b)})^{\frac{1}{4}+\varepsilon}]' \\ \otimes \end{array} \right], \quad (2.16)$$

recalling (2.5). The adjoint operator $\mathcal{B}_N^{(b)*}$ (which, in fact, does not depend on b) is given by

$$\mathcal{B}_N^{(b)*} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_2|_{\Gamma_s}, x_2 \in \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s); \quad \mathcal{B}_N^{(b)*} : \text{continuous} \left[\begin{array}{c} \otimes \\ \mathcal{D}[(A_{N,s}^{(b)})^{\frac{1}{4}+\varepsilon}] \\ \otimes \end{array} \right] \rightarrow \mathbf{L}^2(\Gamma_s), \quad (2.17)$$

with $x_2 \in \mathcal{D}[(A_{N,s}^{(b)})^{\frac{1}{4}+\varepsilon}] = \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s)$, in the following sense. For $g \in \mathbf{L}^2(\Gamma_s)$ and $\{x_1, x_2, x_3\} \in [\otimes, \mathcal{D}[(A_{N,s}^{(b)})^{\frac{1}{4}+\varepsilon}], \otimes]$, we compute as a duality pairing via (1.2a)–(1.2b) and (2.15):

$$\left(\mathcal{B}_N^{(b)} g, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}_b} = \left(\begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)} g \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}_b} = (A_{N,s}^{(b)} N_s^{(b)} g, x_2)_{\mathbf{L}^2(\Omega_s)} = (g, N_s^{(b)*} A_{N,s}^{(b)} x_2)_{\mathbf{L}^2(\Gamma_s)} \quad (2.18)$$

$$= (g, -x_2|_{\Gamma_s})_{\mathbf{L}^2(\Gamma_s)} = \left(g, \mathcal{B}_N^{(b)*} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{L}^2(\Gamma_s)}, \quad (2.19)$$

where we shall establish that

$$N_s^{(b)*} A_{N,s}^{(b)} x_2 = -x_2|_{\Gamma_s} \quad \text{for } x_2 \in \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s) \quad (2.20)$$

for the operator $A_{N,s}^{(b)}$ as in (2.3) and $N_s^{(b)}$ as in (2.4).

Proof of (2.20). [see also [4, pp. 195–196].] Take initially $f \in \mathcal{D}(A_{N,s}^{(b)})$ in (2.3), so that $\frac{\partial f}{\partial \nu} \Big|_{\Gamma_s} = 0$, and $g \in \mathbf{L}^2(\Gamma_s)$. We compute by means of the second Green's theorem, where we recall that on Γ_s , the normal ν is inward. We obtain by (2.3) and (2.4):

$$-(A_{N,s}^{(b)} f, N_s^{(b)} g)_{\Omega_s} = ((\Delta - bI)f, N_s^{(b)} g)_{\Omega_s} \quad (2.21)$$

$$= (f, (\Delta - bI)N_s^{(b)} g)_{\Omega_s} - \left(\frac{\partial f}{\partial \nu} \Big|_{\Gamma_s}, N_s^{(b)} g \right)_{\Gamma_s} + \left(f \Big|_{\Gamma_s}, \frac{\partial}{\partial \nu} (N_s^{(b)} g)_{\Gamma_s} \right)_{\Gamma_s} \quad (2.22)$$

$$= (f|_{\Gamma_s}, g)_{\Gamma_s}. \quad (2.23)$$

In the aforementioned computation we have used (2.4) and $\frac{\partial f}{\partial \nu} \Big|_{\Gamma_s} = 0$, thus accounting for the two vanishing terms. For the last equality, we have invoked the BC in (2.4). In conclusion from (2.23)

$$(N_s^{(b)*} A_{N,s}^{(b)} f, g)_{\mathbf{L}^2(\Gamma_s)} = -(f|_{\Gamma_s}, g)_{\mathbf{L}^2(\Gamma_s)}, \quad \text{for any } g \in \mathbf{L}^2(\Gamma_s) \quad (2.24)$$

initially for $f \in \mathcal{D}(A_{N,s}^{(b)})$. Thus, (2.20) follows for $x_2 \in \mathcal{D}(A_{N,s}^{(b)})$. Moreover, (2.24) can be extended to all $f \in \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s)$ [40], [4, Chapter 3]. $\square$

In conclusion:

Theorem 2.1. *Let $b = 0$ or 1 . Then the abstract model of the nonhomogeneous PDE problem (2.1a)–(2.1f) on the respective energy space $\mathbf{H}_b$ in (1.2a)–(1.2b) is given by*

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathcal{B}_N^{(b)} g, \quad \mathcal{B}_N^{(b)} = \begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)} \\ 0 \end{bmatrix} \quad (2.25)$$

with $\mathcal{A}_b$ is given by (1.8), (1.9), or alternatively by (2.14), $\mathcal{B}_N^{(b)}$ is given by (2.15), and for $x_2 \in \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s) = \mathcal{D}\left[(A_{N,s}^{(b)})^{\frac{1}{4}+\varepsilon}\right]$

$$\mathcal{B}_N^{(b)*} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_2|_{\Gamma_s}; \quad N_s^{(b)*} A_{N,s}^{(b)} x_2 = -x_2|_{\Gamma_s}, \quad (2.26)$$

in both case, $b = 0$ and $b = 1$, see (2.17), (2.20).

Henceforth, in light of Theorem 2.1, we shall omit the qualifying parameter “ b ” for CASE 1, as the model (2.25) and (2.26) applies to both cases $b = 0$ and $b = 1$.

3 The Luenberger’s compensator model for the heat-structure interaction model (1.1a)–(1.1f) with Neumann boundary control g at the interface Γ_s , $b = 0$, and $b = 1$

3.1 Special selection of the data

With reference to the representation (1.4) in Step 1 of the Orientation in Section 1.1, we take in our present case

$$\begin{cases} A \text{ “exponentially stable”}: \|e^{At}\| \leq ce^{-\delta t}, \quad \delta > 0, \quad t \geq 0, \\ F = -B^*, \quad C = B^*, \quad K = B. \end{cases} \quad (3.1a)$$

$$(3.1b)$$

Thus, the special setting becomes, in this case,

$$[\text{partial observation of the state } y] = Cy = B^*y, \quad \text{control } g = Fz = -B^*z, \quad (3.2)$$

leading to the Luenberger’s dynamics

$$\dot{y} = Ay - BB^*z, \quad (3.3a)$$

$$\dot{z} = (A - 2BB^*)z + B(B^*y), \quad (3.3b)$$

and hence,

$$\frac{d}{dt}[y - z] = (A - BB^*)[y - z]; \quad [y(t) - z(t)] = e^{(A - BB^*)t}[y_0 - z_0]. \quad (3.4)$$

As noted in Section 1, Step 2, it is the PDE argument to be carried out in Section 3.1.3 in the present case of Neumann control on the interface Γ_s that will determine that the infinite dimensional version of $e^{(A - BB^*)t}$ is exponentially stable, as desired, as well as analytic.

Insight. How did we decide that $F = -B^*$ in (3.1b); that is, that the preassigned control $g = Fz$ is given by $g = -B^*z$ or $F = -B^*$? We first notice that, regardless of the choice of F , the Luenberger scheme in (1.4)–(1.6), yields that $(A - KC)$ is the resulting sought-after operator in characterizing the quantity $[y - z]$ of interest. As A is, in our case, dissipative and we surely seek to retain dissipativity, then we choose $KC = BB^*$, or $K = B, C = B^*$. Then, the (dissipative) operator $(A - BB^*)$ is the key operator to analyze for the purpose of concluding that the semigroup $e^{(A - BB^*)t}$ is (analytic as well as) uniformly stable. Thus, at this stage, with F yet not committed and $K = B, C = B^*$ committed, the z -equation becomes $\dot{z} = [A - BF - BB^*]z + B(B^*y)$. It is natural to test either $F = B^*$ or else $F = -B^*$, whichever choice may yield the desired properties for the

feedback operator $A_F = A - BB^*$. What is the right sign? Thus, passing from the historical scheme (1.4a)–(1.4b) to our present PDE problem (2.1a)–(2.1f), the corresponding operator $\mathcal{B}_N^*$ is critical in imposing the boundary conditions for the feedback operator $\mathcal{A}_{F,N}^{(b)} = \mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*$ in our CASE 1. In our present CASE 1, if we choose $F \rightarrow -\mathcal{B}_N^*$ and so $g = -\mathcal{B}_N^* \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = z_2|_{\Gamma_s}$ by (2.26), this then implies the boundary condition $\frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} + v_2|_{\Gamma_s}$ for $\{v_1, v_2, h\}$ in $\mathcal{D}(A_{F,N})$ as in (3.17b) below, via (1.9b). With this B.C., the argument in (3.34) and (3.35) leads to the trace term $-i\omega \|h|_{\Gamma_s}\|^2$ in (3.37a), and hence to the critical term $i\omega[\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2]$ in (3.38), with the correct “minus” sign “–” for the argument of Theorem 3.4, in particular estimate (3.32), to succeed. Therefore, in view of the interface condition $\frac{\partial(w + w_t)}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} + g$ in (2.1e), the B.C. $\frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} + v_2|_{\Gamma_s}$ confirms that $g = v_2|_{\Gamma_s}$, or $g = F \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -\mathcal{B}_N^* \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = w_t|_{\Gamma_s}$. Hence, the choice $F \rightarrow -\mathcal{B}_N^*$, as in (2.1b) is the correct one in our present CASE 1.

3.1.1 The counterpart of $\dot{y} = Ay - BB^*z$ in (2.3a) for the heat structure interaction (2.1a)–(2.1f), with $F \rightarrow \mathcal{B}_N^*$

Accordingly, for $z = [z_1, z_2, z_3]$, the Luenberger’s compensator variable, in line with (3.2), we select the Neumann control g in (2.1e) in the form

$$g = -\mathcal{B}_N^* z = z_2|_{\Gamma_s}, \quad z_2 \in \mathcal{D}\left(A_{N,s}^{\frac{1}{4}+\varepsilon}\right) \equiv \mathbf{H}^{\frac{1}{2}+2\varepsilon}(\Omega_s), \quad (3.5)$$

where we have critically invoked the trace result (2.26) of Theorem 2.1 in both cases $b = 0$ and $b = 1$ (we are omitting the superscript “ b ”). With $y = [w, w_t, u]$, the PDE version of (3.3a) corresponding to the abstract feedback problem ($b = 0, b = 1$):

$$\dot{y} = \mathcal{A}_b y - \mathcal{B}_N \mathcal{B}_N^* z, \quad \text{or} \quad \frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} w_t \\ \Delta(w + w_t) - bw \\ \Delta u \end{bmatrix} + \mathcal{B}_N(z_2|_{\Gamma_s}) \quad (3.6)$$

from problem (2.25), $\mathcal{A}_b$ as in (1.8), (1.9), alternatively in (2.14), with g as in (3.5), is

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (3.7a)$$

$$\begin{cases} w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (3.7b)$$

$$\begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \quad u = w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (3.7c)$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} + z_2 & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (3.7d)$$

3.1.2 The counterpart of the dynamic compensator equation $\dot{z} = (A - 2BB^*)z + B(B^*y)$ in (2.3b) for the heat-structure interaction (2.1a)–(2.1f)

With partial observation as in (3.2) according to (2.26)

$$\mathcal{B}_N^{(b)*} y = \mathcal{B}_N^{(b)*} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -w_t|_{\Gamma_s} = \text{partial observation of state } y = \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \quad (3.8)$$

the compensator equation (3.3b) in $z = [z_1, z_2, z_3]$ becomes

$$\dot{z} = \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = (\mathcal{A}_b - 2\mathcal{B}_N^{(b)}\mathcal{B}_N^{(b)*})z + \mathcal{B}_N^{(b)}(-w_t|_{\Gamma_s}) \quad (3.9)$$

or, via (1.8) for $\mathcal{A}_b$, (2.15) = (2.25) for $\mathcal{B}_N^{(b)}$, (2.26) for $\mathcal{B}_N^{(b)*}$

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = \begin{bmatrix} z_2 \\ \Delta(z_1 + z_2) - bz_1 \\ \Delta z_3 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ A_{N,s}^{(b)}N_s^{(b)}(-z_2|_{\Gamma_s}) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A_{N,s}^{(b)}N_s^{(b)}(-w_t|_{\Gamma_s}) \\ 0 \end{bmatrix}, \quad (3.10)$$

whereby $z_{1t} = z_2$, $z_{1tt} = z_{2t}$, and thus (3.10) is re-written as follows:

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_{1t} \\ z_3 \end{bmatrix} = \begin{bmatrix} z_{1t} \\ \Delta(z_1 + z_{1t}) - bz_1 \\ \Delta z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ A_{N,s}^{(b)}N_s^{(b)}[2z_2|_{\Gamma_s} - w_t|_{\Gamma_s}] \\ 0 \end{bmatrix}. \quad (3.11)$$

The PDE version of the abstract z -model (3.11) with partial observation $-w_t|_{\Gamma_s}$ in the Neumann condition at the interface Γ_s (see (3.8)) is

$$\begin{cases} z_{3t} - \Delta z_3 = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (3.12a)$$

$$\begin{cases} z_{1tt} - \Delta z_1 - \Delta z_{1t} + bz_1 = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (3.12b)$$

$$\begin{cases} z_3|_{\Gamma_f} = 0 \text{ on } (0, T] \times \Gamma_f; \quad z_3 = z_{1t} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (3.12c)$$

$$\begin{cases} \frac{\partial(z_1 + z_{1t})}{\partial \nu} = \frac{\partial z_3}{\partial \nu} + [2z_{1t} - w_t] & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (3.12d)$$

3.1.3 The dynamics $\dot{d} = (\mathcal{A}_b - \mathcal{B}_N^{(b)}\mathcal{B}_N^{(b)*})d$, $d(t) = y(t) - z(t)$, corresponding to (3.4): analyticity, $b = 0$ and $b = 1$

We are omitting the superscript “ b ” on $\mathcal{B}_N$, $\mathcal{B}_N^*$. The main result of the present section is as follows:

Theorem 3.1. *Let $b = 0, 1$. The (feedback) operator*

$$\mathcal{A}_{F,N}^{(b)} = \mathcal{A}_b - \mathcal{B}_N\mathcal{B}_N^*, \quad \mathbf{H}_b \supset \mathcal{D}(\mathcal{A}_{F,N}^{(b)}) = \{x \in \mathbf{H}_b : (I - \mathcal{A}_b^{-1}\mathcal{B}_N\mathcal{B}_N^*)x \in \mathcal{D}(\mathcal{A}_b)\} \quad (3.13)$$

is the infinitesimal generator of a s.c. contraction semigroup $e^{\mathcal{A}_{F,N}^{(b)}t}$ on $\mathbf{H}_b$, which moreover is analytic and exponentially stable on $\mathbf{H}_b$: there exist constants $C \geq 1$, $\rho > 0$, possibly depending on “ b ,” such that

$$\|e^{\mathcal{A}_{F,N}^{(b)}t}\|_{\mathcal{L}(\mathbf{H}_b)} = \|e^{(\mathcal{A}_b - \mathcal{B}_N\mathcal{B}_N^*)t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq Ce^{-\rho t}, \quad t \geq 0. \quad (3.14)$$

A more detailed description of $\mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ is given in (3.17a)–(3.17b).

The proof of Theorem 3.1 is by PDE methods, which consist of analyzing the corresponding PDE system (3.16). We proceed through a series of steps.

Step 1. Identification of the $[y - z]$ -abstract equation.

Lemma 3.2. *Let $b = 0, 1$. With $d = [d_1, d_2, d_3] \in \mathbf{H}_b$ ($d = \text{difference} = y - z$), the abstract equation*

$$\dot{d} = \mathcal{A}_{F,N}^{(b)}d = (\mathcal{A}_b - \mathcal{B}_N\mathcal{B}_N^*)d, \quad \mathcal{B}_N(\mathcal{B}_N^*d) = \begin{bmatrix} 0 \\ A_{N,s}^{(b)}N_s^{(b)}(-d_2|_{\Gamma_s}) \\ 0 \end{bmatrix} \quad (3.15a)$$

recalling (1.8) for $\mathcal{A}_b$, (2.25) and (2.26) for $\mathcal{B}_N$ and $\mathcal{B}_N^$*

$$\begin{bmatrix} d_{1t} \\ d_{2t} \\ d_{3t} \end{bmatrix} = \begin{bmatrix} d_2 \\ \Delta(d_1 + d_2) - bd_1 \\ \Delta d_3 \end{bmatrix} - \begin{bmatrix} 0 \\ A_{N,s}^{(b)}N_s^{(b)}(-d_2|_{\Gamma_s}) \\ 0 \end{bmatrix} \quad (3.15b)$$

so that $d_2 = d_{1t}, d_{2t} = d_{1tt}$ corresponds to the following PDE system, where we relabel the variable $[d_1, d_2 = d_{1t}, d_3] = [\hat{w}, \hat{w}_t, \hat{u}]$ for convenience

$$\begin{cases} \hat{u}_t - \Delta \hat{u} = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (3.16a)$$

$$\begin{cases} \hat{w}_{tt} - \Delta \hat{w} - \Delta \hat{w}_t + b\hat{w} = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (3.16b)$$

$$\begin{cases} \hat{u}|_{\Gamma_f} = 0 \text{ on } (0, T] \times \Gamma_f; \quad \hat{u} = \hat{w}_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (3.16c)$$

$$\begin{cases} \frac{\partial(\hat{w} + \hat{w}_t)}{\partial \nu} = \frac{\partial \hat{u}}{\partial \nu} + \hat{w}_t & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (3.16d)$$

Remark 3.1. We note that the term $\frac{\partial \hat{w}}{\partial \nu}$ and $\hat{w}_t$ have the same sign across the equality sign in (3.16d). This is due to the normal vector ν being inward with respect to Ω_s as shown in Figure 1 and as noted in (2.20). This is consistent with the fluid-structure interaction model in [12], [39, equation 2.16.1e, p. 128], also with ν being inward with respect to Ω_s . This model without the Kelvin-Voigt term is known in these references to be uniformly stable by PDE-techniques.

Description of $\mathcal{D}(\mathcal{A}_{F,N}^{(b)})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)}) = \mathcal{D}(\mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*)$ if and only if

(i)

$$\begin{aligned} v_1, v_2 &\in \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} \text{ for } b = 0; \quad v_1, v_2 \in \mathbf{H}^1(\Omega_s) \text{ for } b = 1; \\ \text{so that } v_2|_{\Gamma_s} &= h|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \text{ in both cases;} \\ \Delta(v_1 + v_2) &\in \mathbf{L}^2(\Omega_s); \end{aligned} \quad (3.17a)$$

(ii)

$$\begin{aligned} h &\in \mathbf{H}^1(\Omega_f), \quad \Delta h \in \mathbf{L}^2(\Omega_f), \quad h|_{\Gamma_f} \equiv 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s); \\ \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} &= \frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} - v_2|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \end{aligned} \quad (3.17b)$$

The adjoint operator $\mathcal{A}_{F,N}^{(b)*} = \mathcal{A}_b^* - \mathcal{B}_N \mathcal{B}_N^*$. For $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)*})$ (to be characterized below), we have

recalled $\mathcal{A}_b^*$ in (1.10a) and $\mathcal{B}_N^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = -v_2|_{\Gamma_s}$ in (2.26):

$$\mathcal{A}_{F,N}^{(b)*} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = (\mathcal{A}_b^* - \mathcal{B}_N \mathcal{B}_N^*) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \mathcal{B}_N(-v_2|_{\Gamma_s}) \quad (3.18)$$

$$= \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)}(-v_2|_{\Gamma_s}) \\ 0 \end{bmatrix} \quad (3.19)$$

recalling also $\mathcal{B}_N$ in (2.25) = (2.15).

Description of $\mathcal{D}(\mathcal{A}_{F,N}^{(b)*})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)*}) = \mathcal{D}(\mathcal{A}_b^* - \mathcal{B}_N \mathcal{B}_N^*)$ if and only if the same conditions for $\mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ in (3.17a)–(3.17b) apply, except that new (3.17a) is replaced by $\Delta(v_2 - v_1) \in \mathbf{L}^2(\Omega_s)$ and (3.17b) is replaced by

$$\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial(v_2 - v_1)}{\partial \nu} \Big|_{\Gamma_s} + v_2|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \quad (3.20)$$

The PDE corresponding to $\mathcal{A}_{F,N}^{(b)*}$ is given by $\frac{d}{dt} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = \mathcal{A}_{F,N}^{(b)*} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$ or

$$\begin{cases} h_t - \Delta h = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (3.21a)$$

$$\begin{cases} w_{1t} - \Delta w_1 - \Delta w_{1t} - bw_1 = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (3.21b)$$

$$\begin{cases} h|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \end{cases} \quad (3.21c)$$

$$\begin{cases} h|_{\Gamma_s} = -w_{1t}|_{\Gamma_s}; \frac{\partial(w_1 + w_{1t})}{\partial \nu} \Big|_{\Gamma_s} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} + w_{1t}|_{\Gamma_s} & \text{on } (0, T] \times \Gamma_s, \end{cases} \quad (3.21d)$$

(where $w_{1t} = -w_2$ by (3.18), top line) on $\mathbf{H}_b$, with $\{w_{10}, w_{11}, h_0\} \in \mathbf{H}_b$, recalling (1.10b)–(1.10g) for $\mathcal{A}_b^*$.

Step 2. (Analysis of the PDE problem (3.16): the operator $\mathcal{A}_{F,N}^{(b)} = \mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*$ in (3.13))

Proposition 3.3. *Let $b = 0, 1$. The operator $\mathcal{A}_{F,N}^{(b)} = \mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*$ in (3.13) and its $\mathbf{H}_b$ -adjoint $\mathcal{A}_{F,N}^{(b)*} = \mathcal{A}_b^* - \mathcal{B}_N^* \mathcal{B}_N$ are dissipative*

$$\operatorname{Re} \left((\mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_b} = -\|\nabla v_2\|_{\Omega_s}^2 - \|\nabla h\|_{\Omega_f}^2 - \|v_2\|_{\Gamma_s}^2, \quad \{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)}), \quad (3.22)$$

$$\operatorname{Re} \left((\mathcal{A}_b^* - \mathcal{B}_N^* \mathcal{B}_N) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}_b} = -\|\nabla v_2^*\|_{\Omega_s}^2 - \|\nabla h^*\|_{\Omega_f}^2 - \|v_2^*\|_{\Gamma_s}^2, \quad \{v_1^*, v_2^*, h^*\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)*}) \quad (3.23)$$

in the $\mathbf{L}^2(\cdot)$ -norms of Ω_s and Ω_f , and the $\mathbf{L}^2(\Gamma_s)$ -norm on Γ_s . Hence, both $\mathcal{A}_{F,N}^{(b)}$ and $\mathcal{A}_{F,N}^{(b)*}$ are maximal dissipative and thus generate s.c. contraction semigroups $e^{\mathcal{A}_{F,N}^{(b)}t}$ and $e^{\mathcal{A}_{F,N}^{(b)*}t}$ on $\mathbf{H}_b$ [35]. Explicitly in terms of the corresponding PDE systems, we have:

$$\begin{bmatrix} \widehat{w}(t) \\ \widehat{w}_t(t) \\ \widehat{u}(t) \end{bmatrix} = e^{\mathcal{A}_{F,N}^{(b)}t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} = e^{(\mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*)t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} \quad (3.24)$$

for the $\{\widehat{w}, \widehat{w}_t, \widehat{u}\}$ -fluid-structure interaction model given by (3.16a)–(3.16d) on $\mathbf{H}_b$: with I.C. $\{\widehat{w}_0, \widehat{w}_1, \widehat{u}_0\} \in \mathbf{H}_b$. Similarly,

$$\begin{bmatrix} w_1(t) \\ w_{1t}(t) \\ h(t) \end{bmatrix} = e^{\mathcal{A}_{F,N}^{(b)*}t} \begin{bmatrix} w_{10} \\ w_{11} \\ h_0 \end{bmatrix} = e^{(\mathcal{A}_b^* - \mathcal{B}_N^* \mathcal{B}_N)t} \begin{bmatrix} w_{10} \\ w_{11} \\ h_0 \end{bmatrix} \quad (3.25)$$

for the $\{w_1, w_{1t} = -w_2, u\}$ -fluid-structure interaction model given by (3.21a)–(3.21d).

Proof of (3.22). For $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)})$, we compute via (3.15a)–(3.15b)

$$\operatorname{Re} \left((\mathcal{A}_b - \mathcal{B}_N \mathcal{B}_N^*) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_b} = \operatorname{Re} \left(\begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_b} + \spadesuit, \quad (3.26)$$

where

$$\spadesuit = - \left(\begin{bmatrix} 0 \\ A_{N,s}^{(b)} N_s^{(b)} (-v_2|_{\Gamma_s}) \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_b} \quad (3.27)$$

$$= (v_2|_{\Gamma_s}, N_s^{(b)*} A_{N,s}^{(b)} v_2|_{\Gamma_s})_{\Gamma_s} = -\|v_2|_{\Gamma_s}\|^2 \quad (3.28)$$

recalling (2.26). On the other hand, the first term on the right-hand side (RHS) of (3.26) is equal to $[-\|\nabla v_2\|^2 - \|\nabla h\|^2]$ by (1.12), which added to (3.28) yields (3.22). Similarly for (3.23), starting from (3.19). $\square$

Step 3. This step provides the key PDE-energy estimate, $b = 0$ and $b = 1$, of the entire present section. The case $b = 1$ is more challenging.

Remark 3.2. Given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$, and $\omega \in \mathbb{R} \setminus \{0\}$, we seek to solve the equation

$$(i\omega I - \mathcal{A}_{F,N}^{(b)}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}$$

in terms of $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ uniquely. We have

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(i\omega, \mathcal{A}_{F,N}^{(b)}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix},$$

$$\mathcal{A}_{F,N}^{(b)} R(i\omega, \mathcal{A}_{F,N}^{(b)}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}.$$

Theorem 3.4. Let $b = 0, 1$. Let $\omega \in \mathbb{R}$ and

$$(i\omega I - \mathcal{A}_{F,N}^{(b)}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \in \mathbf{H}_b \quad (3.29)$$

for $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ identified in (3.17a)–(3.17b). Then:

(i) the following estimate holds true: given $\varepsilon > 0$ sufficiently small, there exists a constant $C_\varepsilon > 0$ such that:

$$\begin{aligned} & \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + \|\nabla v_2\|^2 + b\|v_2\|^2 + \|h|_{\Gamma_s}\|^2 + \|\nabla h\|^2 \\ & \leq C_\varepsilon \{\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\}, \quad \forall |\omega| \geq \varepsilon > 0. \end{aligned} \quad (3.30)$$

(ii) (Analyticity) In view of Remark 3.2, Estimate (3.30) without the term $\|h|_{\Gamma_s}\|^2 + \|\nabla h\|^2$ is equivalent to

$$\|\mathcal{A}_{F,N}^{(b)} R(i\omega, \mathcal{A}_{F,N}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq C_\varepsilon, \quad \forall |\omega| \geq \varepsilon > 0, \quad (3.31)$$

$R(i\omega, \mathcal{A}_{F,N}^{(b)}) = (i\omega I - \mathcal{A}_{F,N}^{(b)})^{-1}$, which in turn is equivalent to

$$\|R(i\omega, \mathcal{A}_{F,N}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \frac{C_\varepsilon}{|\omega|}, \quad \forall |\omega| \geq \varepsilon > 0. \quad (3.32)$$

Thus, the s.c. contraction semigroup $e^{\mathcal{A}_{F,N}^{(b)} t}$ asserted by Proposition 3.3 is analytic on $\mathbf{H}_b$ by [4, Theorem 3E.3 p. 334] and similarly for $e^{\mathcal{A}_{F,N}^{(b)*} t}$ on $\mathbf{H}_b$. Their explicit PDE version is given by (3.24) for system $\{\hat{w}(t), \hat{w}_t(t), \hat{u}(t)\}$ in (3.16a)–(3.16d) and, respectively, by (3.25) for system $\{w_1(t), w_{1t}(t), h(t)\}$ in (3.21a)–(3.21d).

Proof. (i) The proof of estimate (3.30) follows closely the technical proof of [1, Section 3] for the operator $\mathcal{A}$ except that now the argument uses the B.C. of $\mathcal{A}_{F,N}$ rather than of $\mathcal{A}$, $b = 0$ and $b = 1$. We indicate the relevant changes.

Step 1. Return to (3.29) re-written for $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ and $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$.

$$i\omega v_1 - v_2 = v_1^*, \quad (3.33a)$$

$$i\omega v_2 - [\Delta(v_1 + v_2) - bv_1] = v_2^*, \quad (3.33b)$$

$$i\omega h - \Delta h = h^*, \quad (3.33c)$$

Step 2. Take the $L^2(\Omega_f)$ -inner product of equation (3.33c) against Δh , use Green's First theorem, recall the B.C. $h|_{\Gamma_f} = 0$ in (3.17b) for $\mathcal{D}(\mathcal{A}_{F,N})$ and obtain the counterpart of [1, equation (3.10)]

$$i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \quad (3.34)$$

Similarly, we take the $L^2(\Omega_s)$ -inner product of (3.33b) against $[\Delta(v_1 + v_2) - bv_1]$, use Green's first theorem to evaluate $\int_{\Omega_s} v_2 \Delta(\bar{v}_1 + \bar{v}_2) d\Omega_s$, recalling that the normal vector ν is inward with respect to Ω_s , and obtain (see [1,

equation (3.11))

$$\begin{aligned} & -i\omega \int_{\Gamma_s} v_2 \frac{\partial(\bar{v}_1 + \bar{v}_2)}{\partial v} d\Gamma_s - i\omega(\nabla v_2, \nabla(v_1 + v_2)) - i\omega(v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 \\ & = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \end{aligned} \quad (3.35)$$

We now invoke the B.C. $h|_{\Gamma_s} = v_2|_{\Gamma_s}$ and $\frac{\partial(v_1 + v_2)}{\partial v}\Big|_{\Gamma_s} = \frac{\partial h}{\partial v}\Big|_{\Gamma_s} + v_2|_{\Gamma_s}$ for $\{v_1, v_2, h\}$ in $\mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ (see (3.17)), and we rewrite (3.35) as follows:

$$\begin{aligned} & -i\omega \int_{\Gamma_s} h \left[\frac{\partial \bar{h}}{\partial v} + \bar{h} \right] d\Gamma_s - i\omega \|\nabla v_2\|^2 - i\omega(\nabla v_2, \nabla v_1) - i\omega(v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 \\ & = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \end{aligned} \quad (3.36)$$

Summing up (3.34) and (3.36) yields after a cancellation of the boundary term $i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial v} d\Gamma_s$:

$$\begin{aligned} & -i\omega \|h|_{\Gamma_s}\|^2 - i\omega[\|\nabla v_2\|^2 + \|\nabla h\|^2] = \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega(v_2, bv_1) + i\omega(\nabla v_2, \nabla v_1) \\ & + (v_2^*, [\Delta(v_1 + v_2) - bv_1]) + (h^*, \Delta h). \end{aligned} \quad (3.37a)$$

Using, via (3.33a), the identities

$$-i\omega(\nabla v_2, \nabla v_1) = (\nabla v_2, \nabla(i\omega v_1)) = \|\nabla v_2\|^2 + (\nabla v_2, \nabla v_1^*), \quad (3.37b)$$

$$-i\omega(v_2, v_1) = (v_2, i\omega v_1) = \|v_2\|^2 + (v_2, v_1^*), \quad (3.37c)$$

we obtain from (3.37a) the final identity

$$\begin{aligned} & \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega[\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2] \\ & = \|\nabla v_2\|^2 + b\|v_2\|^2 + (\nabla v_2, \nabla v_1^*) + b(v_2, v_1^*) - (v_2^*, [\Delta(v_1 + v_2) - bv_1]) - (h^*, \Delta h). \end{aligned} \quad (3.38)$$

Step 3. We take the real part of identity (3.38), thus obtaining the new identity:

$$\begin{aligned} & \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 = \|\nabla v_2\|^2 + b\|v_2\|^2 + \operatorname{Re}(\nabla v_2, \nabla v_1^*) + b \operatorname{Re}(v_2, v_1^*) \\ & - \operatorname{Re}(v_2^*, [\Delta(v_1 + v_2) - bv_1]) - \operatorname{Re}(h^*, \Delta h) \end{aligned} \quad (3.39)$$

or

$$\begin{aligned} & (1 - \varepsilon)[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ & \leq (1 + \varepsilon)[\|\nabla v_2\|^2 + b\|v_2\|^2] + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (3.40)$$

Step 4. We now take the imaginary part of identity (3.38), thus obtaining the new identity

$$\omega\{\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2\} = \operatorname{Im}(\nabla v_2, \nabla v_1^*) + b \operatorname{Im}(v_2, v_1^*) - \operatorname{Im}\{(v_2^*, [\Delta(v_1 + v_2) - bv_1]) + (h^*, \Delta h)\}, \quad (3.41)$$

$$\begin{aligned} & |\omega\{\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2\}| \leq \frac{\varepsilon^2}{2}\|\nabla v_2\|^2 + b\frac{\varepsilon^3}{3}\|v_2\|^2 + \varepsilon^3[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ & + C_{\varepsilon^2}[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2] + C_{\varepsilon^3}[\|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (3.42)$$

or

$$\begin{aligned} & \left| |\omega| - \frac{\varepsilon^2}{2} \right| \|\nabla v_2\|^2 + |\omega|[\|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2] \leq b\frac{\varepsilon^3}{2}\|v_2\|^2 + \varepsilon^3[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ & + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (3.43)$$

Take now

$$\frac{\varepsilon^2}{2} \leq |\omega| - \frac{\varepsilon^2}{2} \Leftrightarrow \varepsilon^2 \leq |\omega|, \quad (3.44)$$

so that (3.43) yields for $|\omega|$ as in (3.44):

$$\begin{aligned} \frac{\varepsilon^2}{2} \|\nabla v_2\|^2 + \varepsilon^2 [\|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2] &\leq b \frac{\varepsilon^3}{2} \|v_2\|^2 + \varepsilon^3 [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]; \end{aligned} \quad (3.45)$$

hence, for $|\omega|$ as in (3.44)

$$\begin{aligned} \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2 &\leq b\varepsilon \|v_2\|^2 + 2\varepsilon [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (3.46)$$

Remark 3.3. In the case $b = 0$, the proof on the space $\mathbf{H}_{b=0}$, hence with the first component in $\mathbf{H}^1(\Omega_s) \setminus \mathbb{R}$ topologized by the gradient norm, proceeds as follows. Estimate (3.46) is “too good for our purposes”: we drop the terms $\|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2$ and substitute the new estimate on $\|\nabla v_2\|^2$ on the RHS of (3.40) with $b = 0$. We obtain

$$(1 - \varepsilon) [\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] \leq (1 + \varepsilon) 2\varepsilon [\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] + (1 + \varepsilon) C_\varepsilon [\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (3.47)$$

or

$$[(1 - \varepsilon) - (1 + \varepsilon) 2\varepsilon] [\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] \leq \tilde{C}_\varepsilon [\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (3.48)$$

or

$$\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \leq \hat{C}_\varepsilon [\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (3.49)$$

Estimate (3.48) coincides with estimate [1, equation (3.26)], case $b = 0$.

Next, substitute estimate (3.49) into the RHS of estimate (3.46) with $b = 0$, to obtain for $|\omega|$ as in (3.44):

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2 \leq \tilde{C}_\varepsilon [\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (3.50)$$

Summing up estimate (3.49) with estimate (3.50) finally yields for $|\omega|$ as in (3.44):

$$\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_s}\|^2 \leq C_\varepsilon [\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (3.51)$$

for all points $i\omega$, with $|\omega| \geq \frac{\varepsilon^2}{2}$ as in (3.44). Then estimate (3.51) coincides with (3.30) with $b = 0$ as desired. In view of Remark 3.2, such estimate is equivalent to

$$\|\mathcal{A}_{F,N}^{(0)} R(i\omega, \mathcal{A}_{F,N}^{(0)})\|_{\mathcal{L}(\mathbf{H}_0)} \leq C_\varepsilon, \quad \forall |\omega| \geq \varepsilon > 0. \quad (3.52)$$

The analyticity of the s.c. contraction semigroup $e^{\mathcal{A}_{F,N}^{(0)} t} = e^{(\mathcal{A}_0 - \mathcal{B}_N \mathcal{B}_N^*) t}$ on $\mathbf{H}_{b=0}$ is established and similarly for $e^{\mathcal{A}_{F,N}^{(0)*} t}$.

Step 5. We proceed now with the proof of analyticity in the case $b = 1$ on $\mathbf{H}_{b=1}$. This case is more challenging and requires the following additional result (in substitution of the Poincaré inequality, which does not hold true for v_2 on Ω_s).

Lemma 3.5. [44, p. 260] *On a sufficiently smooth bounded domain Ω in $\mathbb{R}^n$, let $\Psi \in H^1(\Omega)$. Then:*

- (a) $\int_{\Omega} |\Psi|^2 d\Omega \leq c_1 \left[\int_{\Omega} |\nabla \Psi|^2 d\Omega + \int_{\tilde{\Gamma}} |\Psi|^2 d\Gamma \right],$
- (b) $\int_{\tilde{\Gamma}} |\Psi|^2 d\Gamma \leq c_2 \int_{\Omega} [|\Psi|^2 + |\nabla \Psi|^2] d\Omega,$
- (c) hence for positive constant $0 < k_1 < k_2 < \infty$,

$$k_1 \int_{\Omega} [|\Psi|^2 + |\nabla \Psi|^2] d\Omega \leq \int_{\Omega} |\nabla \Psi|^2 d\Omega + \int_{\tilde{\Gamma}} |\Psi|^2 d\Gamma \leq k_2 \int_{\Omega} [|\Psi|^2 + |\nabla \Psi|^2] d\Omega,$$

where $\tilde{\Gamma}$ is any fixed portion of the boundary $\Gamma = \partial\Omega$ of Ω of positive measure.

We now return to inequality (3.46) with $b = 1$, where we use $h|_{\Gamma_s} = v_2|_{\Gamma_s}$ from (3.17a) on its left-hand side (LHS) and then invoke Lemma 3.5(a) for $\Phi = v_2$ to obtain

$$\frac{1}{c_1} \|v_2\|^2 + \|\nabla h\|^2 \leq \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|v_2|_{\Gamma_s}\|^2 \leq b\varepsilon \|v_2\|^2 + \boxed{\text{OK}_\varepsilon}, \quad (3.53a)$$

$$\boxed{\text{OK}_\varepsilon} = 2\varepsilon[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (3.53b)$$

hence,

$$\frac{1}{2c_1} \|\nabla v_2\|^2 + \|\nabla h\|^2 \leq \left(\frac{1}{c_1} - b\varepsilon\right) \|v_2\|^2 + \|\nabla h\|^2 \leq \boxed{\text{OK}_\varepsilon}. \quad (3.54)$$

by taking $\left(\frac{1}{c_1} - b\varepsilon\right) \geq \frac{1}{2c_1}$ (where $b = 1$), which yields $b\|v_2\|^2 \leq 2c_1 \boxed{\text{OK}_\varepsilon}$, which along with the RHS inequality in (3.53a) gives the desired estimate for $b = 1$:

$$\begin{aligned} \|\nabla v_2\|^2 + b\|v_2\|^2 + \|\nabla h\|^2 + \|v_2|_{\Gamma_s}\|^2 &\leq b\varepsilon \|v_2\|^2 + \boxed{\text{OK}_\varepsilon} + b\|v_2\|^2 \\ &= (1 + \varepsilon)b\|v_2\|^2 + \boxed{\text{OK}_\varepsilon} \leq [(1 + \varepsilon)2c_1 + 1] \boxed{\text{OK}_\varepsilon} \\ &\leq 2\varepsilon[(1 + \varepsilon)2c_1 + 1]\{\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2\} + C_\varepsilon[\|\nabla v_1^*\|^2 \\ &\quad + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (3.55)$$

The LHS in (3.55) estimates the two terms $[\|\nabla v_2\|^2 + b\|v_2\|^2]$ on the RHS in (3.40). We may now proceed as in going from (3.47) to (3.51) in Remark 3.3 for $b = 0$.

Step 6. We substitute the new estimate (3.55) on the RHS of (3.40). We obtain

$$\begin{aligned} &(1 - \varepsilon)[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\leq (1 + \varepsilon)2\varepsilon[(1 + \varepsilon)2c_1 + 1][\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (3.56)$$

or

$$\begin{aligned} &\{(1 - \varepsilon) - (1 + \varepsilon)2\varepsilon[(1 + \varepsilon)2c_1 + 1]\}[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\leq \tilde{C}_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (3.57)$$

or

$$\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 \leq \hat{C}_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (3.58)$$

Estimate (3.58) for $b = 1$ is the counterpart of estimate (3.49) for $b = 0$.

Step 7. Summing up estimates (3.58) with estimate (3.55) finally yields for $|\omega|$ as in (3.44):

$$\begin{aligned} &\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + \|\nabla v_2\|^2 + b\|v_2\|^2 + \|\nabla h\|^2 + \|v_2|_{\Gamma_s}\|^2 \\ &\leq C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (3.59)$$

for all points ω , with $|\omega| \geq \frac{\varepsilon^2}{2}$ as in (3.44). Then estimate (3.58) coincides with estimate (3.30) as desired. The analyticity of the s.c. contraction semigroup $e^{\mathcal{A}_{F,N}^{(1)}t} = e^{(\mathcal{A}_1 - \mathcal{B}_N \mathcal{B}_N^*)t}$ on $\mathbf{H}_{b=1}$ is established. Similarly for $e^{\mathcal{A}_{F,N}^{(1)*}t}$, $b = 1$. $\square$

3.1.4 Exponential stability of $e^{\mathcal{A}_{F,N}^{(b)}t}$ and $e^{\mathcal{A}_{F,N}^{(b)*}t}$ on $\mathbf{H}_b$, $b = 0, 1$

In Proposition 3.6, we shall prove, in both cases $b = 0$ and $b = 1$, that we have

$$0 \in \rho(\mathcal{A}_{F,N}^{(b)}), \quad 0 \in \rho(\mathcal{A}_{F,N}^{(b)*}), \quad (\mathcal{A}_{F,N}^{(b)})^{-1} \in \mathcal{L}(\mathbf{H}_b), \quad (\mathcal{A}_{F,N}^{(b)*})^{-1} \in \mathcal{L}(\mathbf{H}_b), \quad (3.60)$$

$\rho(\cdot)$ denoting the resolvent set, so that there exists a disk S_{r_0} centered at the origin and of suitable radius $r_0 > 0$ such that $S_{r_0} \subset \rho(\mathcal{A}_{F,N}^{(b)})$. Then, the resolvent bound (3.32) combined with $(\mathcal{A}_{F,N}^{(b)})^{-1} \in \mathcal{L}(\mathbf{H}_b)$ in (3.60) allows one to conclude that the resolvent operator is uniformly bounded on the imaginary axis $i\mathbb{R}$:

$$\|R(i\omega, \mathcal{A}_{F,N}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \text{const}, \quad \omega \in \mathbb{R}. \quad (3.61)$$

Hence, [45] the s.c. analytic semigroup $e^{\mathcal{A}_{F,N}^{(b)}t}$ is, moreover, (uniformly) exponentially bounded: There exist constants $M \geq 1$, $\delta > 0$, possibly depending on b , such that

$$\|e^{\mathcal{A}_{F,N}^{(b)}t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq Me^{-\delta t}, \quad t \geq 0, b = 0, b = 1. \quad (3.62)$$

Similarly for the adjoint $\mathcal{A}_{F,N}^{(b)*}$.

Proposition 3.6. *Statement (3.60) holds true. Hence, the exponential stability for $e^{\mathcal{A}_{F,N}^{(b)}t}$ in (3.62) holds true. More precisely, with reference to $\mathcal{A}_{F,N}^{(b)}$, we have: given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$, the unique solution $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,N}^{(b)})$ of*

$$\mathcal{A}_{F,N}^{(b)} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \quad (3.63)$$

is given explicitly by

$$v_1 = (-A_{N,s}^{(b)})^{-1}[-\Delta v_1^* + v_2^*] + N_s^{(b)} \left\{ \frac{\partial}{\partial v} [-A_{D,f}^{-1} h^* + D_{f,s}(v_1^*|_{\Gamma_s})] - \frac{\partial v_1^*}{\partial v} \Big|_{\Gamma_s} + v_1^*|_{\Gamma_s} \right\} \\ \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0; \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1. \end{cases} \quad (3.64)$$

$$v_2 = v_1^* \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad h = -A_{D,f}^{-1} h^* + D_{f,s}(v_1^*|_{\Gamma_s}) \in \mathbf{H}^1(\Omega_f). \quad (3.65)$$

where the positive self-adjoint operator $A_{D,f}$ and the Dirichlet map $D_{f,s}$ from Γ_s into Ω_f were defined in (2.9) and (2.10), and are repeated as follows:

$$-A_{D,f}\varphi = \Delta\varphi \text{ in } \Omega_f; \quad \varphi \in \mathcal{D}(A_{D,f}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f); \quad (3.66)$$

$$D_{f,s} : H^s(\partial\Omega_f) \rightarrow H^{s+\frac{1}{2}}(\Omega_f), s \in \mathbb{R} : D_{f,s}\mu = \psi \Leftrightarrow \begin{cases} \Delta\psi = 0 & \text{in } \Omega_f; \\ \psi|_{\Gamma_f} = 0, \quad \psi|_{\Gamma_s} = \mu. \end{cases} \quad (3.67a)$$

$$(3.67b)$$

While $A_{N,s}^{(b)}$ and $N_s^{(b)}$, $b = 0, 1$, are defined in (2.3) and (2.4), respectively. In the operator form, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = (\mathcal{A}_{F,N}^{(b)})^{-1} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} (-A_{N,s}^{(b)})^{-1}[-\Delta v_1^* + v_2^*] + N_s^{(b)} \left\{ \frac{\partial}{\partial v} [-A_{D,f}^{-1} h^* + D_{f,s}(v_1^*|_{\Gamma_s})] - \frac{\partial v_1^*}{\partial v} \Big|_{\Gamma_s} + v_1^*|_{\Gamma_s} \right\} \\ v_1^* \\ -A_{D,f}^{-1} h^* + D_{f,s}(v_1^*|_{\Gamma_s}) \end{bmatrix} \quad (3.68)$$

$$= \begin{bmatrix} (A_{N,s}^{(b)})^{-1}\Delta + N_s^{(b)} \frac{\partial}{\partial v} D_{f,s}(\cdot|_{\Gamma_s}) + (\cdot|_{\Gamma_s}) - \frac{\partial \cdot}{\partial v} \Big|_{\Gamma_s} & (-A_{N,s}^{(b)})^{-1} & -N_s^{(b)} \frac{\partial}{\partial v} A_{D,f}^{-1} \\ I & 0 & 0 \\ D_{f,s}(\cdot|_{\Gamma_s}) & 0 & -A_{D,f}^{-1} \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \quad (3.69)$$

$$\|[v_1, v_2, h]\|_{\mathbf{H}_b} \leq c\|[v_1^*, v_2^*, h^*]\|_{\mathbf{H}_b}. \quad (3.70)$$

Proof. Identity (3.63) yields by (3.17b)

$$v_2 = v_1^* \begin{cases} \Delta h = h^* \in \mathbf{L}(\Omega_f); \\ h|_{\Gamma_f} = 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} = v_1^*|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s), \end{cases} \quad (3.71a)$$

$$(3.71b)$$

and the h -problem in (3.71) yields the solution h in (3.65).

Moreover, (3.63), $v_2 = v_1^*$ in (3.65) and (3.17b) yield

$$\begin{cases} \Delta(v_1 + v_1^*) - bv_1 = v_2^*, & \text{or} \quad \Delta v_1 - bv_1 = -\Delta v_1^* + v_2^*; \end{cases} \quad (3.72a)$$

$$\begin{cases} \left. \frac{\partial(v_1 + v_1^*)}{\partial \nu} \right|_{\Gamma_s} = \left. \frac{\partial h}{\partial \nu} \right|_{\Gamma_s} + v_1^*|_{\Gamma_s}, & \text{or} \quad \left. \frac{\partial v_1}{\partial \nu} \right|_{\Gamma_s} = \left[-\frac{\partial v_1^*}{\partial \nu} + \frac{\partial h}{\partial \nu} + v_1^* \right]_{\Gamma_s} \end{cases} \quad (3.72b)$$

from which the expression for v_1 in (3.64) followed by invoking the operator $A_{N,s}^{(b)}$ and $N_s^{(b)}$ in (2.3) and (2.4). The proof is complete. $\square$

4 CASE 2. Heat-structure interaction with Kelvin-Voigt damping: Dirichlet control g at the interface Γ_s : abstract model

We return to the homogeneous heat-structure interaction model (1.1a)–(1.1f) with Kelvin-Voigt damping. In this CASE 2, we insert a control g in the Dirichlet interface condition (1.1d). Thus, with the same geometry (Figure 1) and notation $\{w, w_t, u\}$ as in Section 1.3 for the uncontrolled problem, in the present CASE 2, we consider the following controlled problem:

$$\begin{aligned} \text{(PDE)} \quad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \end{aligned} \quad \begin{aligned} (4.1a) \\ (4.1b) \end{aligned}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \\ u = w_t + g & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad \begin{aligned} (4.1c) \\ (4.1d) \end{aligned}$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (4.1e)$$

$$\text{(IC)} \quad [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}_b, \quad (4.1f)$$

this time with Dirichlet control g acting at the interface Γ_s . Compare against model (2.1a)–(2.1f) of CASE 1 with Neumann control g at the interface Γ_s , as in (2.1e). We shall likewise consider two cases: $b = 0$ and $b = 1$. $\mathbf{H}_b$ is the same finite energy space as in (1.2a)–(1.2b).

4.1 Abstract model on $\mathbf{H}_b$, $b = 0, 1$ of the nonhomogeneous PDE model (4.1a)–(4.1f) with Dirichlet control g acting at the interface Γ_s

This topic was duly treated in [39, Section 6]. Here, it was shown that the abstract version of the nonhomogeneous PDE model (4.1a)–(4.1f) is given by

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathcal{B}_D g, \quad (4.2)$$

where the operator $\mathcal{A}_b : \mathbf{H}_b \supset \mathcal{D}(\mathcal{A}_b) \rightarrow \mathbf{H}_b$ is of course the same as given by (1.8) and (1.9). Instead, the (boundary) control operator $\mathcal{B}_D$ is given by

$$\mathcal{B}_D g = \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,s} g \end{bmatrix}, \quad \mathcal{B}_D : \text{continuous } \mathbf{L}^2(\Gamma_s) \rightarrow \left[\begin{array}{c} \otimes \\ \otimes \\ \mathcal{D} \left(A_{D,f}^{\frac{3}{4} + \varepsilon} \right) \end{array} \right]^\gamma. \quad (4.3)$$

Here, $-A_{D,f}$ is the negative, self-adjoint operator on $\mathbf{L}^2(\Omega_f)$ defined by (2.9) = (3.66), i.e., by

$$-A_{D,f}\varphi = \Delta\varphi, \quad \mathcal{D}(A_{D,f}) \equiv \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f), \quad (4.4)$$

while $D_{f,s}$ is the Dirichlet map from Γ_s to Ω_f defined by (2.10) = (3.66), i.e., by

$$\varphi = D_{f,s}\chi \iff \begin{cases} \Delta\varphi = 0 & \text{in } \Omega_f; \\ \varphi|_{\Gamma_f} = 0, & \varphi|_{\Gamma_s} = \chi. \end{cases} \quad (4.5a)$$

$$(4.5b)$$

The following regularity holds true for $D_{f,s}$ [40,41], [4, Chapter 3]:

$$\begin{cases} D_{f,s} : \mathbf{L}^2(\Gamma_s) \rightarrow \mathcal{D}(A_{D,f}^{\frac{1}{4}-\varepsilon}) \equiv \mathbf{H}^{\frac{1}{2}-2\varepsilon}(\Omega_f), \\ \text{or } A_{D,f}^{\frac{1}{4}-\varepsilon} D_{f,s} \in \mathcal{L}(\mathbf{L}^2(\Gamma_s); \mathbf{L}^2(\Omega_f)). \end{cases} \quad (4.6a)$$

$$(4.6b)$$

Thus, with $x_3 \in \mathcal{D}(A_{D,f}^{\frac{3}{4}+\varepsilon}) \subset \mathbf{H}^{\frac{3}{2}+2\varepsilon}(\Omega_f)$, we have:

$$\mathcal{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_s}; \quad \mathcal{B}_D^* : \text{continuous} \begin{bmatrix} \otimes \\ \otimes \\ \mathcal{D}(A_{D,f}^{\frac{3}{4}+\varepsilon}) \end{bmatrix} \rightarrow \mathbf{L}^2(\Gamma_s) \quad (4.7)$$

in the following sense. For $g \in \mathbf{L}^2(\Gamma_s)$ and $\{x_1, x_2, x_3\} \in \left[\otimes, \otimes, \mathcal{D}(A_{D,f}^{\frac{3}{4}+\varepsilon}) \right]$, we compute as a duality pairing via (1.2a)–(1.2b) and (4.3):

$$\left(\mathcal{B}_D g, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}_b} = \left(\begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,s} g \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}_b} \quad (4.8)$$

$$= (A_{D,f} D_{f,s} g, x_3)_{\mathbf{L}^2(\Omega_f)} = (g, D_{f,s}^* A_{D,f} x_3)_{\mathbf{L}^2(\Gamma_s)} \quad (4.9)$$

$$= \left(g, -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_s} \right)_{\mathbf{L}^2(\Gamma_s)} = \left(g, \mathcal{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{L}^2(\Gamma_s)}, \quad (4.10)$$

where we have recalled (the normal ν is outward with respect to Ω_f)

$$D_{f,s}^* A_{D,f} x_3 = -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_s} \quad (4.11)$$

from [1, p. 181], [4, Chapter 3]. Thus (4.7) is established. The proof initially takes $x_3 \in \mathcal{D}(A_{D,f})$, and thus, $x_3|_{\partial\Omega_f} = 0$, and proceeds analogously to the path (2.21)–(2.23) via Green's second theorem to obtain (4.11) in this case. Next, we extend (4.11) to $x_3 \in \mathbf{H}^{\frac{3}{2}+2\varepsilon}(\Omega_f) \equiv \mathcal{D}(A_{D,f}^{\frac{1}{4}+\varepsilon})$.

As noted in Theorem 0.3 from [1], the operator $\mathcal{A}_b$ in (1.8) and (1.9) is boundedly invertible on $\mathbf{H}_b$: $\mathcal{A}_b^{-1} \in \mathcal{L}(\mathbf{H}_b)$ is explicitly given by [1], from which it then follows that $\mathcal{A}_b^{-1} \mathcal{B}_D \in \mathcal{L}(\mathbf{H}^{\frac{1}{2}}(\Gamma_s); \mathbf{H}_b)$.

4.2 The Luenberger's compensator model for the heat-structure interaction model (4.1a)–(4.1f) with Dirichlet control g at the interface Γ_s , $b = 0, 1$

4.2.1 Special selection of the data

For the present heat-structure interaction problem with Dirichlet control at the interface Γ_s , we shall modify the special selection made in CASE 1 of Neumann control at the interface Γ_s , on the basis of the representation (1.4) in Step 1 of the orientation in Section 1.1. In fact, in the present case, we now take, in the notation of (1.4)–(1.6):

$$\begin{cases} A \text{ "exponentially stable": } \|e^{At}\| \leq ce^{-\delta t}, \delta > 0, t \geq 0, \\ F = B^*, \quad C = B^*, \quad K = B. \end{cases} \quad (4.12a)$$

$$F = B^*, \quad C = B^*, \quad K = B. \quad (4.12b)$$

Thus, the special setting becomes, in this case,

$$[\text{partial observation of the state } y] = Cy = B^*y, \quad \text{control } g = Fz = B^*z, \quad (4.13)$$

leading to the Luenberger's dynamics

$$\begin{cases} \dot{y} = Ay + BB^*z, \\ \dot{z} = Az + B(B^*y) \end{cases} \quad (4.14a)$$

$$(4.14b)$$

(as $BF - KC = 0$ in the present case) and hence to

$$\frac{d}{dt}[y - z] = (A - BB^*)[y - z]; \quad [y(t) - z(t)] = e^{(A - BB^*)t}[y_0 - z_0]. \quad (4.15)$$

This is the setting that will be selected in the study of the Luenberger's theory below, as applied to heat (fluid)-structure interaction models with Dirichlet control g at the interface Γ_s , as in (4.1d).

Insight. How did we decide that $F = B^*$ in (4.12b); that is, that the preassigned control $g = Fz$ is given by $g = B^*z$ or $F = B^*$? We first notice that, regardless of the choice of F , the Luenberger scheme in (1.4)–(1.6), yields that $(A - KC)$ is the resulting sought-after operator in characterizing the quantity $[y - z]$ of interest. As A is, in our case, dissipative and we surely seek to retain dissipativity, then we choose $KC = BB^*$, or $K = B, C = B^*$. Then, the (dissipative) operator $(A - BB^*)$ is the key operator to analyze for the purpose of concluding that the semigroup $e^{(A - BB^*)t}$ is (analytic as well as) uniformly stable. Thus, at this stage, with F not yet committed and $K = B, C = B^*$ committed, the z -equation becomes $\dot{z} = [A - BF - BB^*]z + B(B^*y)$. It is natural to test either $F = B^*$ or else $F = -B^*$, whichever choice may yield the desired properties for the feedback operator $A_F = A - BB^*$. What is the right sign? Thus, passing from the historical scheme (1.4a)–(1.4b) to our present PDE problem (4.1a)–(4.1f), the corresponding operator B_D^* is critical in imposing the boundary conditions for the feedback operator $\mathcal{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*$ in our CASE 2. But in our present CASE 2, if we choose

$$F \rightarrow \mathcal{B}_D^* \text{ and so } g = \mathcal{B}_D^* \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = -\frac{\partial z_3}{\partial \nu} \Big|_{\Gamma_s} \text{ by (4.7)}. \text{ This then implies the boundary condition } h|_{\Gamma_s} = v_2|_{\Gamma_s} - \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s}$$

for $\{v_1, v_2, h\}$ in $\mathcal{D}(A_{F,D}^{(b)})$ as in (4.28b). With this B.C., the argument in (4.46), (4.47) leads to the trace term

$$-i\omega \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right\|^2 \text{ in (4.48), and hence to the critical term } i\omega \left[\|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right\|^2 \right] \text{ in (4.50) with the correct$$

“minus” sign “-” for the argument of Theorem 4.4, in particular estimate (4.52), to succeed. Therefore,

in view of the interface condition $u = w_t + g$ in (4.1d), the B.C. $u = w_t - \frac{\partial u}{\partial \nu} \Big|_{\Gamma_s}$ confirms that $g = -\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s}$,

or $g = F \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{B}_D^* \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s}$. Hence, the choice $F \rightarrow \mathcal{B}_D^*$ as in (4.12b) is the correct one in our CASE 2.

4.2.2 The counterpart of $\dot{y} = Ay + BB^*z$ in (4.14a) for the HSI model (4.1a)–(4.1f) with $F \rightarrow \mathcal{B}_D^*$

Accordingly, for $z = [z_1, z_2, z_3]$, the Luenberger's compensator variable, we select the Dirichlet control g in (4.1d) in the form

$$g = \mathcal{B}_D^* z = -\frac{\partial z_3}{\partial \nu} \Big|_{\Gamma_s}, \quad z_3 \in \mathcal{D} \left(A_{D,f}^{\frac{3}{4} + \varepsilon} \right) \subset \mathbf{H}^{\frac{3}{2} + 2\varepsilon}(\Omega_f), \quad (4.16)$$

i.e., with $F \rightarrow \mathcal{B}_D^*$, where we have critically invoked the trace result (4.7). With $y = [w, w_t, u]$, the PDE version of (3.3a) corresponding to the abstract feedback problem

$$\dot{y} = \mathcal{A}_b y + \mathcal{B}_D \mathcal{B}_D^* z, \quad \text{or} \quad \frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} w_t \\ \Delta(w + w_t) - bw \\ \Delta u \end{bmatrix} + \mathcal{B}_D \left(-\frac{\partial z_3}{\partial v} \Big|_{\Gamma_s} \right) \quad (4.17)$$

from problem (4.2) with operator $\mathcal{A}_b$ as in (1.8) and with g as in (4.16), is

$$u_t - \Delta u = 0 \quad \text{in } (0, T] \times \Omega_f; \quad (4.18a)$$

$$w_{tt} - \Delta w - \Delta w_t + bw = 0 \quad \text{in } (0, T] \times \Omega_s; \quad (4.18b)$$

$$u|_{\Gamma_f} = 0 \text{ on } (0, T] \times \Gamma_f; \quad u = w_t - \frac{\partial z_3}{\partial v} \quad \text{on } (0, T] \times \Gamma_s; \quad (4.18c)$$

$$\frac{\partial(w + w_t)}{\partial v} = \frac{\partial u}{\partial v} \quad \text{on } (0, T] \times \Gamma_s. \quad (4.18d)$$

4.2.3 The counterpart of the dynamic compensator equation $\dot{z} = Az + B(B^*y)$ in the special setting (4.14b) for the HSI model (4.1a)–(4.1f)

With partial observation as in (4.13), $C \rightarrow \mathcal{B}_D^*$, according to (4.7)

$$Cy \rightarrow \mathcal{B}_D^* y = \mathcal{B}_D^* \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -\frac{\partial u}{\partial v} \Big|_{\Gamma_s} = \text{partial observation of state} = \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \quad (4.19)$$

the compensator equation (4.14b) in $z = [z_1, z_2, z_3]$ becomes in our present HSI case

$$\dot{z} = \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathcal{A}_b z + \mathcal{B}_D \left(-\frac{\partial u}{\partial v} \Big|_{\Gamma_s} \right) \quad (4.20)$$

or via (1.8) on $\mathcal{A}_b$, (4.3), (4.7) on $\mathcal{B}_D$, $\mathcal{B}_D^*$

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = \begin{bmatrix} z_2 \\ \Delta(z_1 + z_2) - bz_1 \\ \Delta z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,s} \left(-\frac{\partial u}{\partial v} \Big|_{\Gamma_s} \right) \end{bmatrix}, \quad (4.21)$$

whereby $z_{1t} = z_2$, $z_{1tt} = z_{2t}$. The PDE version of the abstract z -model (4.21) with partial observation $-\frac{\partial u}{\partial v} \Big|_{\Gamma_s}$ in the Dirichlet condition at the interface Γ_s as in (4.19) is given as follows:

$$z_{3t} - \Delta z_3 = 0 \quad \text{in } (0, T] \times \Omega_f; \quad (4.22a)$$

$$z_{1tt} - \Delta z_1 - \Delta z_{1t} + bz_1 = 0 \quad \text{in } (0, T] \times \Omega_s; \quad (4.22b)$$

$$z_3|_{\Gamma_f} = 0 \text{ on } (0, T] \times \Gamma_f; \quad z_3 = z_{1t}|_{\Gamma_s} - \frac{\partial u}{\partial v} \Big|_{\Gamma_s} \quad \text{on } (0, T] \times \Gamma_s; \quad (4.22c)$$

$$\frac{\partial(z_1 + z_{1t})}{\partial v} = \frac{\partial z_3}{\partial v} \quad \text{on } (0, T] \times \Gamma_s. \quad (4.22d)$$

4.2.4 The dynamics $\dot{d} = (\mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*)d$, $d(t) = y(t) - z(t)$ corresponding to (4.15): analyticity and exponential decay, $b = 0, 1$

The main result of the present section is as follows:

Theorem 4.1. Let $b = 0, 1$. The (feedback) operator

$$\mathcal{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*, \quad \mathbf{H}_b \supset \mathcal{D}(\mathcal{A}_{F,D}) = \{x \in \mathbf{H}_b : (I - \mathcal{A}_b^{-1} \mathcal{B}_D \mathcal{B}_D^*)x \in \mathcal{D}(\mathcal{A}_b)\} \quad (4.23)$$

in the infinitesimal generator of a s.c. contraction semigroup $e^{\mathcal{A}_{F,D}^{(b)}t}$ on $\mathbf{H}_b$, which moreover is analytic and exponential stable on $\mathbf{H}_b$: there exist constants $C \geq 1, \rho > 0$ possibly depending on “ b ” such that

$$\|e^{\mathcal{A}_{F,D}^{(b)}t}\|_{\mathcal{L}(\mathbf{H}_b)} = \|e^{(\mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*)t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq Ce^{-\rho t}, \quad t \geq 0. \quad (4.24)$$

A more detailed description of $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ is given in (4.28a)–(4.28b). The proof of Theorem 4.1 is by PDE methods, which consists of analyzing the corresponding PDE system (4.27).

Step 1.

Lemma 4.2. Let $b = 0, 1$. With $d = [d_1, d_2, d_3] \in \mathbf{H}_b$, the abstract equation

$$\dot{d} = \mathcal{A}_{F,D}^{(b)}d = (\mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*)d \quad (4.25)$$

or via (1.8) on $\mathcal{A}_b$, (4.3) and (4.7) on $\mathcal{B}_D, \mathcal{B}_D^*$

$$\begin{bmatrix} d_{1t} \\ d_{2t} \\ d_{3t} \end{bmatrix} = \begin{bmatrix} d_2 \\ \Delta(d_1 + d_2) - bd_1 \\ \Delta d_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,s} \left(-\frac{\partial d_3}{\partial v} \Big|_{\Gamma_s} \right) \end{bmatrix} \quad (4.26)$$

so that $d_2 = d_{1t}, d_{2t} = d_{1tt}$, corresponds to the following PDE system, where we relabel the variable $[d_1, d_2 = d_{1t}, d_3] = [\hat{w}, \hat{w}_t, \hat{u}]$ for convenience

$$\begin{cases} \hat{u}_t - \Delta \hat{u} = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (4.27a)$$

$$\begin{cases} \hat{w}_{tt} - \Delta \hat{w} - \Delta \hat{w}_t + b\hat{w} = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (4.27b)$$

$$\begin{cases} \hat{u}|_{\Gamma_f} = 0 \text{ on } (0, T] \times \Gamma_f; \quad \hat{u} = \hat{w}_t - \frac{\partial \hat{u}}{\partial \nu} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (4.27c)$$

$$\begin{cases} \frac{\partial(\hat{w} + \hat{w}_t)}{\partial \nu} = \frac{\partial \hat{u}}{\partial \nu} & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (4.27d)$$

Description of $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)}) = \mathcal{D}(\mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*)$ if any only if

(i)

$$\begin{aligned} v_1, v_2 \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad \text{so that } v_2|_{\Gamma_s} = h|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \text{ in both cases;} \\ \Delta(v_1 + v_2) \in \mathbf{L}^2(\Omega_s); \end{aligned} \quad (4.28a)$$

(ii)

$$\begin{aligned} h \in \mathbf{H}^1(\Omega_f), \quad \Delta h \in \mathbf{L}^2(\Omega_f), \quad h|_{\Gamma_f} \equiv 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} - \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s); \\ \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \end{aligned} \quad (4.28b)$$

The adjoint operator $\mathcal{A}_{F,D}^{(b)*}$ $= \mathcal{A}_b^* - \mathcal{B}_D \mathcal{B}_D^*$. For $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)*})$ (to be characterized later), we have after

recalling $\mathcal{A}_b^*$ in (1.10a) and $\mathcal{B}_D^*$ $\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s}$ from (4.7)

$$\mathcal{A}_{F,D}^{(b)*} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = (\mathcal{A}_b^* - \mathcal{B}_D \mathcal{B}_D^*) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \mathcal{B}_D \left(-\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right) \quad (4.29)$$

$$= \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,s} \left(-\frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right) \end{bmatrix} \quad (4.30)$$

recalling $\mathcal{B}_D$ in (4.3).

Description of $\mathcal{D}(\mathcal{A}_{F,D}^{(b)*})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)*}) = \mathcal{D}(\mathcal{A}_b^* - \mathcal{B}_D \mathcal{B}_D^*)$ if and only if the same conditions for $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ in (4.28a)–(4.28b) apply, except that now (4.28a) is replaced by $\Delta(v_2 - v_1) \in \mathbf{L}^2(\Omega_s)$ and (4.28b) is replaced by

$$\frac{\partial v_2 - v_1}{\partial v} \Big|_{\Gamma_s} = \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \quad (4.31)$$

The PDE version corresponding to the adjoint operator $\mathcal{A}_{F,D}^{(b)*}$ is given by $\frac{d}{dt} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = \mathcal{A}_{F,D}^{(b)*} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$ or (counterpart of (3.21a)–(3.21d))

$$\begin{cases} h_t - \Delta h = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (4.32a)$$

$$\begin{cases} w_{1tt} - \Delta w_1 - \Delta w_{1t} + bw_1 = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (4.32b)$$

$$\begin{cases} h|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f; \end{cases} \quad (4.32c)$$

$$\begin{cases} h|_{\Gamma_s} = -w_{1t}|_{\Gamma_s} - \frac{\partial h}{\partial v} \Big|_{\Gamma_s}; \quad \frac{\partial(w_1 + w_{1t})}{\partial v} \Big|_{\Gamma_s} = -\frac{\partial h}{\partial v} \Big|_{\Gamma_s} & \text{on } (0, T] \times \Gamma_s, \end{cases} \quad (4.32d)$$

(where $w_{1t} = -w_2$ by (4.29), top line) with I.C. in $\mathbf{H}_b$.

Step 2. (Analysis of the PDE problem (4.27): the operator $\mathcal{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*$ in (4.23))

Proposition 4.3. *The operator $\mathcal{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*$ in (4.23) and its $\mathbf{H}_b$ -adjoint $\mathcal{A}_{F,D}^{(b)*} = \mathcal{A}_b^* - \mathcal{B}_D \mathcal{B}_D^*$ are dissipative*

$$\operatorname{Re} \left\langle \begin{bmatrix} \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} = -\|\nabla v_2\|_{\Omega_s}^2 - \|\nabla h\|_{\Omega_f}^2 - \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|_{\Gamma_s}^2, \quad \{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)}), \quad (4.33)$$

$$\operatorname{Re} \left\langle \begin{bmatrix} \mathcal{A}_b^* - \mathcal{B}_D \mathcal{B}_D^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right\rangle_{\mathbf{H}_b} = -\|\nabla v_2^*\|_{\Omega_s}^2 - \|\nabla h^*\|_{\Omega_f}^2 - \left\| \frac{\partial h^*}{\partial v} \Big|_{\Gamma_s} \right\|_{\Gamma_s}^2, \quad \{v_1^*, v_2^*, h^*\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)*}) \quad (4.34)$$

in the $\mathbf{L}^2(\cdot)$ norms of Ω_s and Ω_f , and the $\mathbf{L}^2(\Gamma_s)$ norm on Γ_s . Hence, both $\mathcal{A}_{F,D}^{(b)}$ and $\mathcal{A}_{F,D}^{(b)*}$ are maximal dissipative and thus generate s.c. contraction semigroups $e^{\mathcal{A}_{F,D}^{(b)} t}$ and $e^{\mathcal{A}_{F,D}^{(b)*} t}$, respectively, on $\mathbf{H}_b$. Explicitly in terms of the corresponding PDE systems (4.27) we have:

$$\begin{bmatrix} \widehat{w}(t) \\ \widehat{w}_t(t) \\ \widehat{u}(t) \end{bmatrix} = e^{\mathcal{A}_{F,D}^{(b)} t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} = e^{(\mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^*) t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} \quad (4.35)$$

for the $\{\widehat{w}, \widehat{w}_t, \widehat{u}\}$ -fluid-structure interaction model given by (4.27a)–(4.27d) on $\mathbf{H}_b$; with I.C. $\{\widehat{w}_0, \widehat{w}_1, \widehat{u}_0\} \in \mathbf{H}_b$, similarly, for the adjoint $\mathcal{A}_{F,D}^{(b)*}$, corresponding to model (4.32a)–(4.32d).

Proof of (4.33). For $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D})$ in (4.28a) and (4.28b), we compute via (1.8), (4.26):

$$\operatorname{Re} \left\langle \begin{bmatrix} \mathcal{A}_b - \mathcal{B}_D \mathcal{B}_D^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} = \operatorname{Re} \left[\left\langle \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} - \boxed{1_D} \right], \quad (4.36)$$

where recalling (4.3) for $\mathcal{B}_D$ and (4.7) for $\mathcal{B}_D^*$

$$\boxed{1_D} = \left[\begin{array}{c} 0 \\ 0 \\ A_{D,f} D_{f,s} \left(-\frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right) \end{array} \right]_{\mathbf{H}_b} \left[\begin{array}{c} v_1 \\ v_2 \\ h \end{array} \right]_{\mathbf{H}_b} = \left(-\frac{\partial h}{\partial v} \Big|_{\Gamma_s}, D_{f,s}^* A_{D,f} h \right)_{L^2(\Gamma_s)} \quad (4.37)$$

$$= \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|_{L^2(\Gamma_s)}^2, \quad (4.38)$$

recalling $D_{f,s}^* A_{D,f} h = -\frac{\partial h}{\partial v} \Big|_{\Gamma_s}$ from (4.11). On the other hand, recalling (1.12)

$$\operatorname{Re} \left\langle \mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} = -\|\nabla v_2\|^2 - \|\nabla h\|^2. \quad (4.39)$$

Thus, (4.38) and (4.39), used in (4.36), yield (4.33), as desired. Similarly for (4.34) starting from (1.10a) for $\mathcal{A}_b^*$. $\square$

4.2.5 Analyticity of $e^{\mathcal{A}_{F,D}^{(b)} t}$ and $e^{\mathcal{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$, $b = 0, 1$

Remark 4.1. Given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$, and $\omega \in \mathbb{R} \setminus \{0\}$, we seek to solve the equation

$$(i\omega I - \mathcal{A}_{F,D}^{(b)}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}$$

in terms of $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ uniquely. We have (counterpart of Remark 3.2)

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(i\omega, \mathcal{A}_{F,D}^{(b)}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix},$$

$$\mathcal{A}_{F,D}^{(b)} R(i\omega, \mathcal{A}_{F,D}^{(b)}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}.$$

Theorem 4.4. Let $b = 0, 1$. Let $\omega \in \mathbb{R}$ and

$$(i\omega I - \mathcal{A}_{F,D}^{(b)}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \in \mathbf{H}_b \quad (4.40)$$

for $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ identified in (4.28a)–(4.28b). Then:

(i) the following estimate holds true: there exists a constant $C_\varepsilon > 0$ such that:

$$b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|_{L^2(\Gamma_s)}^2;$$

$$\leq C_\varepsilon \{b\|v_1^*\|^2 + \|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\}, \quad \forall |\omega| \geq \varepsilon > 0;$$

(ii) In view of Remark 4.1, estimate (4.41) is equivalent to

$$\|\mathcal{A}_{F,D}^{(b)} R(i\omega, \mathcal{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq C_\varepsilon, \quad \forall |\omega| \geq \varepsilon > 0, \quad (4.42)$$

$R(i\omega, \mathcal{A}_{F,D}^{(b)}) = (i\omega I - \mathcal{A}_{F,D}^{(b)})^{-1}$, in turn equivalent to estimate

$$\|R(i\omega, \mathcal{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \frac{C_\varepsilon}{|\omega|}, \quad \forall |\omega| \geq \varepsilon > 0. \quad (4.43)$$

Thus, the s.c. contraction semigroup $e^{\mathcal{A}_{F,D}^{(b)}t}$ asserted by Proposition 4.3 is analytic on $\mathbf{H}_b$ by [4, Theorem 3E.3 p. 334], similarly for $e^{\mathcal{A}_{F,D}^{(b)*}t}$ on $\mathbf{H}_b$. The explicit PDE version of (4.35) is system $\{\widehat{w}(t), \widehat{w}_t(t), \widehat{u}(t)\}$ in (4.27a)–(4.27d).

Proof. (i) The proof of estimate (4.41) follows closely the technical proof of estimate (3.30) for the operator $\mathcal{A}_{F,N}^{(b)}$ except that now the argument uses the B.C. of $\mathcal{A}_{F,D}^{(b)}$ rather than of $\mathcal{A}_{F,N}^{(b)}$. This, in particular, requires a different argument to handle the more challenging case $b = 1$, with full $H^1(\Omega)$ -norm on the first component space. We indicate the relevant changes.

Step 1. Return to (4.40), re-written for $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ and $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$:

$$i\omega v_1 - v_2 = v_1^*, \quad (4.44a)$$

$$i\omega v_2 - [\Delta(v_1 + v_2) - bv_1] = v_2^*, \quad (4.44b)$$

$$i\omega h - \Delta h = h^*. \quad (4.44c)$$

Step 2. Take the $L^2(\Omega_f)$ -inner product of equation (4.44c) against Δh , use Green's First theorem, recall the B.C. $h|_{\Gamma_f} = 0$ in (4.28b) for $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ and obtain the counterpart of [1, equation (3.10)] or (3.34)

$$i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \quad (4.45)$$

Similarly, we take the $L^2(\Omega_s)$ -inner product of (4.44b) against $[\Delta(v_1 + v_2) - bv_1]$, use Green's First Theorem to evaluate $\int_{\Omega_s} v_2 \Delta(\bar{v}_1 + \bar{v}_2) d\Omega_s$, recalling that the normal vector ν is inward with respect to Ω_s , and obtain as in (3.35)

$$\begin{aligned} -i\omega \int_{\Gamma_s} v_2 \frac{\partial(\bar{v}_1 + \bar{v}_2)}{\partial \nu} d\Gamma_s - i\omega(\nabla v_2, \nabla(v_1 + v_2)) - i\omega(v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 \\ = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \end{aligned} \quad (4.46)$$

We now invoke the B.C. $h|_{\Gamma_s} = v_2|_{\Gamma_s} - \frac{\partial h}{\partial \nu}|_{\Gamma_s}$ and $\frac{\partial(v_1 + v_2)}{\partial \nu}|_{\Gamma_s} = \frac{\partial h}{\partial \nu}|_{\Gamma_s}$ for $\{v_1, v_2, h\}$ in $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ (see (4.28b)), we rewrite (4.46) as follows:

$$\begin{aligned} -i\omega \int_{\Gamma_s} \left[h + \frac{\partial h}{\partial \nu} \right] \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla v_2\|^2 - i\omega(\nabla v_2, \nabla v_1) - i\omega(v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 \\ = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \end{aligned} \quad (4.47)$$

Summing up (4.45) and (4.47) yields after a cancellation of the boundary terms $i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s$:

$$\begin{aligned} -i\omega \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 - i\omega[\|\nabla v_2\|^2 + \|\nabla h\|^2] = \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega(v_2, bv_1) + i\omega(\nabla v_2, \nabla v_1) \\ + (v_2^*, [\Delta(v_1 + v_2) - bv_1]) + (h^*, \Delta h). \end{aligned} \quad (4.48)$$

By using via (4.44a), the identities

$$-i\omega(\nabla v_2, \nabla v_1) = (\nabla v_2, \nabla(i\omega v_1)) = \|\nabla v_2\|^2 + (\nabla v_2, \nabla v_1^*), \quad (4.49a)$$

$$-i\omega(v_2, v_1) = (v_2, i\omega v_1) = \|v_2\|^2 + (v_2, v_1^*), \quad (4.49b)$$

and we obtain the final identity

$$\begin{aligned} \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega \left[\|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 \right] \\ = \|\nabla v_2\|^2 + b\|v_2\|^2 + (\nabla v_2, \nabla v_1^*) + b(v_2, v_1^*) - (v_2^*, [\Delta(v_1 + v_2) - bv_1]) - (h^*, \Delta h). \end{aligned} \quad (4.50)$$

Step 3. We take the real part of identity (4.50), thus obtaining the new identity:

$$\begin{aligned} \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 &= \|\nabla v_2\|^2 + b\|v_2\|^2 + \operatorname{Re}(\nabla v_2, \nabla v_1^*) + b\operatorname{Re}(v_2, v_1^*) \\ &\quad - \operatorname{Re}(v_2^*, [\Delta(v_1 + v_2) - bv_1]) - \operatorname{Re}(h^*, \Delta h) \end{aligned} \quad (4.51)$$

or

$$\begin{aligned} (1 - \varepsilon)[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] &\leq (1 + \varepsilon)[\|\nabla v_2\|^2 + b\|v_2\|^2] \\ &\quad + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (4.52)$$

Step 4. We now take the imaginary part of identity (4.50), thus obtaining the new identity

$$\begin{aligned} \omega \left\{ \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 \right\} &= \operatorname{Im}\{(\nabla v_2, \nabla v_1^*)\} + b\operatorname{Im}\{(v_2, v_1^*)\} - \operatorname{Im}\{(v_2^*, \Delta(v_1 + v_2) - bv_1) \\ &\quad + (h^*, \Delta h)\}, \end{aligned} \quad (4.53)$$

$$\begin{aligned} |\omega| \left\{ \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 \right\} &\leq \frac{\varepsilon^2}{2} \|\nabla v_2\|^2 + b \frac{\varepsilon^3}{2} \|v_2\|^2 + \varepsilon^3 [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_{\varepsilon^2}[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2] + C_{\varepsilon^3}[\|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (4.54)$$

or

$$\begin{aligned} \left[|\omega| - \frac{\varepsilon^2}{2} \right] \|\nabla v_2\|^2 + |\omega| \left[\|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 \right] &\leq b \frac{\varepsilon^3}{2} \|v_2\|^2 + \varepsilon^3 [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (4.55)$$

Take now

$$\frac{\varepsilon^2}{2} \leq |\omega| - \frac{\varepsilon^2}{2} \Leftrightarrow \varepsilon^2 \leq |\omega| \quad (4.56)$$

so that (4.55) yields for $|\omega|$ as in (4.56):

$$\begin{aligned} \frac{\varepsilon^2}{2} \|\nabla v_2\|^2 + \varepsilon^2 \left[\|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 \right] &\leq b \frac{\varepsilon^3}{2} \|v_2\|^2 + \varepsilon^3 [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]; \end{aligned} \quad (4.57)$$

hence for $|\omega|$ as in (4.56)

$$\begin{aligned} \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 &\leq b\varepsilon \|v_2\|^2 + 2\varepsilon [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (4.58)$$

Remark 4.2. In the case $b = 0$, the proof on the space $\mathbf{H}_{b=0}$, hence via (1.2a) with the first component in $H^1(\Omega_s) \setminus \mathbb{R}$ topologized by the gradient norm, proceeds as follows. Estimate (4.58) is “too good for our purposes

at this stage”: on its LHS, we drop the terms $\left[\|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \right\|_{\Gamma_s}^2 \right]$ and substitute the new estimate on the remaining $\|\nabla v_2\|^2$ on the RHS of (4.52) with $b = 0$. We obtain

$$\begin{aligned} (1 - \varepsilon)[\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] &\leq (1 + \varepsilon)2\varepsilon[\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] \\ &\quad + (1 + \varepsilon)C_\varepsilon[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (4.59_0)$$

or

$$[(1 - \varepsilon) - (1 + \varepsilon)2\varepsilon][\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2] \leq \tilde{C}_\varepsilon[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (4.60_0)$$

or

$$\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \leq \hat{C}_\varepsilon[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (4.61_0)$$

Estimate (4.61₀) coincides with estimate [1, equation (3.26)], case $b = 0$.

Substitute estimate (4.61₀) into the RHS of estimate (4.58) with $b = 0$ to obtain for $|\omega|$ as in (4.56):

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 \leq \tilde{C}_\varepsilon[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (4.62_0)$$

Summing up estimate (4.61₀) with estimate (4.62₀) finally yields for $|\omega|$ as in (4.56):

$$\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 \leq C_\varepsilon[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (4.63_0)$$

for all points $i\omega$, with $|\omega| \geq \frac{\varepsilon^2}{2}$ as in (4.56). Then estimate (4.63₀) coincides with estimate (4.41) with $b = 0$ as desired.

Step 5. We proceed now with the proof of the case $b = 1$ on the space $\mathbf{H}_{b=1}$ via (1.2b) with $H^1(\Omega_s)$ first component. As in the Neumann control on Γ_s of CASE 1, this case is more challenging and requires the same Lemma 3.5 of CASE 1. However, its use will be tuned to the present case of Dirichlet control on Γ_s , with trace $\frac{\partial h}{\partial \nu}|_{\Gamma_s}$ characterized by (4.28b) in $\mathcal{D}(\mathcal{A}_{F,D})$:

$$\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} = v_2|_{\Gamma_s} - h|_{\Gamma_s}. \quad (4.64)$$

Thus, with $b = 1$, we return to estimate (4.58) where on its LHS we use (4.64):

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|v_2|_{\Gamma_s} - h|_{\Gamma_s}\|^2 \leq b\varepsilon\|v_2\|^2 + \boxed{\text{OK}_\varepsilon}, \quad (4.59_1)$$

where

$$\boxed{\text{OK}_\varepsilon} = 2\varepsilon[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (4.60_1)$$

For $0 \leq \varepsilon_1 \leq 1$ to be chosen below, we return to (4.59₁) and readily obtain, also after adding $\varepsilon_1 \|h|_{\Gamma_s}\|^2$ to both sides:

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \varepsilon_1 \|v_2|_{\Gamma_s} - h|_{\Gamma_s}\|^2 + \varepsilon_1 \|h|_{\Gamma_s}\|^2 \leq b\varepsilon\|v_2\|^2 + \varepsilon_1 \|h|_{\Gamma_s}\|^2 + \boxed{\text{OK}_\varepsilon}. \quad (4.61_1)$$

Next from

$$\|v_2|_{\Gamma_s}\|^2 = \|(v_2|_{\Gamma_s} - h|_{\Gamma_s}) + h|_{\Gamma_s}\|^2 \leq [\|v_2|_{\Gamma_s} - h|_{\Gamma_s}\| + \|h|_{\Gamma_s}\|]^2 \leq 2\|v_2|_{\Gamma_s} - h|_{\Gamma_s}\|^2 + 2\|h|_{\Gamma_s}\|^2, \quad (4.62_1)$$

and hence,

$$\frac{\varepsilon_1}{2} \|v_2|_{\Gamma_s}\|^2 \leq \varepsilon_1 \|v_2|_{\Gamma_s} - h|_{\Gamma_s}\|^2 + \varepsilon_1 \|h|_{\Gamma_s}\|^2. \quad (4.63_1)$$

Using (4.63₁) on the LHS of (4.61₁) yields ($\varepsilon_1 \leq 1$)

$$\frac{\varepsilon_1}{2} [\|v_2|_{\Gamma_s}\|^2 + \|\nabla v_2\|^2] + \|\nabla h\|^2 \leq b\varepsilon\|v_2\|^2 + \varepsilon_1 \|h|_{\Gamma_s}\|^2 + \boxed{\text{OK}_\varepsilon}. \quad (4.64_1)$$

Next we invoke Lemma 3.5(a) on the term $\left[\right]$ on the LHS of (3.64₁), to obtain

$$\left(\frac{\varepsilon_1}{2C_1} - b\varepsilon \right) \|v_2\|^2 + \|\nabla h\|^2 \leq \varepsilon_1 \|h|_{\Gamma_s}\|^2 + \boxed{\text{OK}_\varepsilon} \quad (4.65_1)$$

$$\leq \varepsilon_1 C_p \|\nabla h\|^2 + \boxed{\text{OK}_\varepsilon}, \quad (4.66_1)$$

where going from (4.65₁) to (4.66₁) we have invoked the Poincare inequality

$$\|h|_{\Gamma_s}\|^2 \leq C_p \|\nabla h\|^2, \quad (4.67_1)$$

which is legal since $h|_{\Gamma_f} = 0$ by (4.28) on $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$. Then (4.66₁) is rewritten as follows:

$$\left(\frac{\varepsilon_1}{2C_1} - b\varepsilon \right) \|v_2\|^2 + (1 - \varepsilon_1 C_p) \|\nabla h\|^2 \leq \boxed{\text{OK}_\varepsilon}. \quad (4.68_1)$$

Now we select $1 \geq \varepsilon_1 > 0$ ($b = 1$) so that

$$\sqrt{\varepsilon} < \frac{\varepsilon_1}{2C_1} - b\varepsilon \quad \text{or} \quad 2C_1(\sqrt{\varepsilon} + \varepsilon) < \varepsilon_1 \leq 1, \quad (4.69_1)$$

and

$$\frac{1}{2} < 1 - \varepsilon_1 C_p \quad \text{or} \quad \varepsilon_1 < \frac{1}{2C_p} \leq 1 \quad (4.70_1)$$

(we can always take $C_p \geq \frac{1}{2}$). Using (4.69₁) and (4.70₁) in (4.68₁) yields

$$\sqrt{\varepsilon} \|v_2\|^2 + \frac{1}{2} \|\nabla h\|^2 \leq 2\varepsilon [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (4.71_1)$$

recalling $\boxed{\text{OK}_\varepsilon}$ in (4.60₁), for $|\omega| \geq \varepsilon^2$ as in (4.56).

We finally obtain the desired estimate also for $b = 1$ to include also $\|v_2\|^2$:

$$\|v_2\|^2 + \|\nabla h\|^2 \leq 4\sqrt{\varepsilon} [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \quad (4.72_1)$$

for $|\omega| \geq \varepsilon^2$ as in (4.56). Substituting the estimate for $\|v_2\|^2$ from (4.2.5) into the RHS of (4.58) with $b\varepsilon \leq b$ yields

$$\begin{aligned} \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|^2 &\leq 4b\varepsilon\sqrt{\varepsilon} [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (4.73_1)$$

Now, add the estimate for $\|v_2\|^2$ in (4.72₁) to (4.73₁) and obtain

$$\begin{aligned} b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|^2 &\leq 4\sqrt{\varepsilon}(1 + b\varepsilon) [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \end{aligned} \quad (4.73_1^*)$$

Estimate (4.73₁) is the counterpart of estimate (4.58) for $b = 0$.

Step 6. The rest of the proof for $b = 1$ now proceeds as in the case $b = 0$. In (4.73₁^{*}) we drop the terms

$\left[\|\nabla h\|^2 + \left\| \frac{\partial h}{\partial v} \Big|_{\Gamma_s} \right\|^2 \right]$ and substitute the resulting estimate for $[b\|v_2\|^2 + \|\nabla v_2\|^2]$ into the RHS of (4.52) and obtain

$$\begin{aligned} (1 - \varepsilon) [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] &\leq (1 + \varepsilon) 4\sqrt{\varepsilon}(1 + b\varepsilon) [\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ &\quad + (1 + \varepsilon) C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (4.74_1)$$

or, since $[(1 - \varepsilon) - (1 + \varepsilon) 4\sqrt{\varepsilon}(1 + b\varepsilon)] \geq k > 0$,

$$\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 \leq \widehat{C}_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (4.75_1)$$

which is a counterpart of (4.63₀) for $b = 0$. By substituting (4.75₁) into the RHS of (4.73₁^{*}), we finally obtain

$$b \|v_2\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 \leq \widetilde{C}_\varepsilon [\|\nabla v_1^*\|^2 + b \|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (4.76_1)$$

Finally, summing up (4.75₁) and (4.76₁) yields

$$\begin{aligned} b \|v_2\|^2 + \|\nabla v_2\|^2 + \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_s}^2 \\ \leq \overline{C}_\varepsilon [\|\nabla v_1^*\|^2 + b \|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \end{aligned} \quad (4.77_1)$$

for all $|\omega| > \varepsilon^2$, as in (4.56).

Estimate (4.77₁) is the sought-after estimate (4.41) for $b = 0$, $b = 1$. $\square$

4.2.6 Exponential stability of $e^{\mathcal{A}_{F,D}^{(b)} t}$ and $e^{\mathcal{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$, $b = 0, 1$

In Proposition 4.5, we shall prove that, in both cases $b = 0$ and $b = 1$, we have

$$0 \in \rho(\mathcal{A}_{F,D}^{(b)}), \quad 0 \in \rho(\mathcal{A}_{F,D}^{(b)*}), \quad \mathcal{A}_{F,D}^{(b)-1} \in \mathcal{L}(\mathbf{H}_b), \quad \mathcal{A}_{F,D}^{(b)*-1} \in \mathcal{L}(\mathbf{H}_b), \quad (4.78)$$

so that there exists a disk S_{r_0} centered at the origin and of suitable radius $r_0 > 0$ such that $S_{r_0} \subset \rho(\mathcal{A}_{F,D}^{(b)})$. Then, the resolvent bound (4.43) combined with $\mathcal{A}_{F,D}^{(b)-1} \in \mathcal{L}(\mathbf{H}_b)$ in (4.78) allows one to conclude that the resolvent is uniformly bounded on the imaginary axis $i\mathbb{R}$:

$$\|R(i\omega, \mathcal{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \text{const}, \quad \omega \in \mathbb{R}. \quad (4.79)$$

Hence, [45] the s.c. analytic semigroup $e^{\mathcal{A}_{F,D}^{(b)} t}$ is, moreover, (uniformly) exponentially stable: There exist constants $M \geq 1$, $\delta > 0$, possibly depending on “ b ” such that

$$\|e^{\mathcal{A}_{F,D}^{(b)} t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq M e^{-\delta t}, \quad t \geq 0. \quad (4.80)$$

It was similar for the adjoint $\mathcal{A}_{F,D}^{(b)*}$.

Proposition 4.5. *Statement (4.78) holds true. Hence, the exponential stability for $e^{\mathcal{A}_{F,D}^{(b)} t}$ in (4.80) holds true. More precisely, with reference to $\mathcal{A}_{F,D}^{(b)}$, we have: given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$, the unique solution $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ of*

$$\mathcal{A}_{F,D}^{(b)} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \quad (4.81)$$

is given explicitly by

$$v_1 = (A_{N,s}^{(b)})^{-1} [v_2^* - \Delta v_1^*] + N_s^{(b)} \left\{ \frac{\partial}{\partial \nu} [-A_{R,f}^{-1} h^* + R_{f,s}(v_1^*|_{\Gamma_s}) - v_1^*|_{\Gamma_s}] \right\} \in \mathbf{H}^1(\Omega_s) \setminus \mathbb{R}, \quad (4.82)$$

$$v_2 = v_1^* \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad h = -A_{R,f}^{-1} h^* + R_{f,s}(v_1^*|_{\Gamma_s}) \in \mathbf{H}^{\frac{3}{2}}(\Omega_f). \quad (4.83)$$

In the operator form, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \mathcal{A}_{F,D}^{(b)-1} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} (A_{N,s}^{(b)})^{-1} [\Delta v_1^* - v_2^*] + N_s^{(b)} \left\{ \frac{\partial}{\partial \nu} [-A_{R,f}^{-1} h^* + R_{f,s}(v_1^*|_{\Gamma_s}) - v_1^*|_{\Gamma_s}] \right\} \\ v_1^* \\ -A_{D,f}^{-1} h^* + R_{f,s}(v_1^*|_{\Gamma_s}) \end{bmatrix}, \quad (4.84a)$$

$$= \begin{bmatrix} (A_{N,s}^{(b)})^{-1}\Delta + N_s^{(b)} \frac{\partial}{\partial v} [-R_{f,s}(\cdot|_{\Gamma_s})] & -A_{N,s}^{-1} & -N_s^{(b)} \frac{\partial}{\partial v} A_{D,f}^{-1} \\ I & 0 & 0 \\ R_{f,s}(\cdot|_{\Gamma_s}) & 0 & -A_{D,f}^{-1} \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \quad (4.84b)$$

$$\| [v_1, v_2, h] \|_{\mathbf{H}_b} \leq c \| [v_1^*, v_2^*, h^*] \|_{\mathbf{H}_b}, \quad (4.84c)$$

where the operators $A_{N,s}^{(b)}, N_s^{(b)}$ are defined in (2.3), (2.4). Moreover, $-A_{R,f}$ is the Robin Laplacian on Ω_f :

$$-A_{R,f}\varphi = \Delta\varphi \text{ in } \Omega_f; \quad \varphi \in \mathcal{D}(A_{R,f}) = \left\{ \phi \in \mathbf{H}^2(\Omega_f) : \phi|_{\Gamma_f} = 0, \quad \left[\frac{\partial\phi}{\partial\nu} + \phi \right]_{\Gamma_s} = 0 \right\}, \quad (4.85)$$

and $R_{f,s}$ is the Robin map from Γ_s to Ω_f :

$$R_{f,s} : H^s(\partial\Omega_f) \rightarrow H^{s+\frac{3}{2}}(\Omega_f), s \in \mathbb{R} : R_{f,s}\mu = \psi \Leftrightarrow \begin{cases} \Delta\psi = 0 & \text{in } \Omega_f; \\ \psi|_{\Gamma_f} = 0, \quad \left[\frac{\partial\psi}{\partial\nu} + \psi \right]_{\Gamma_s} = \mu. \end{cases} \quad (4.86a)$$

$$(4.86b)$$

Proof. Identity (4.81) and the characterization of $\mathcal{D}(\mathcal{A}_{F,D}^{(b)})$ in (4.28a)–(4.28b) yield

$$v_2 = v_1^* \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad \begin{cases} \Delta h = h^* \in \mathbf{L}^2(\Omega_f); \\ h|_{\Gamma_f} = 0, \quad \left[\frac{\partial h}{\partial\nu} \right]_{\Gamma_s} + h|_{\Gamma_s} = v_1^*|_{\Gamma_s}, \end{cases} \quad (4.87a)$$

$$(4.87b)$$

and the h -problem in (4.87) yields the solution h in (4.83). Moreover, (4.81), $v_2 = v_1^*$ in (4.87a) and (4.87b) yield

$$\begin{cases} \Delta(v_1 + v_2) - bv_1 = v_2^*, & \text{or} & \Delta v_1 - bv_1 = -\Delta v_1^* + v_2^*, \\ \left[\frac{\partial(v_1 + v_2)}{\partial\nu} \right]_{\Gamma_s} = \left[\frac{\partial h}{\partial\nu} \right]_{\Gamma_s}, & \text{or} & \left[\frac{\partial v_1}{\partial\nu} \right]_{\Gamma_s} = -\left[\frac{\partial v_1^*}{\partial\nu} \right]_{\Gamma_s} + \left[\frac{\partial h}{\partial\nu} \right]_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \end{cases} \quad (4.88a)$$

$$(4.88b)$$

Then, the solution of problem (4.88) is given by (4.82) via (4.83), where the operators $A_{N,s}^{(b)}$ and $N_s^{(b)}$ were defined in (2.3), (2.4), and recalled as follows:

$$-A_{N,s}^{(b)}\varphi = (\Delta - bI)\varphi, \quad \mathcal{D}(A_{N,s}^{(b)}) = \left\{ \varphi \in \begin{cases} \mathbf{H}^2(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^2(\Omega_s) & \text{for } b = 1, \end{cases} : \left[\frac{\partial\varphi}{\partial\nu} \right]_{\Gamma_s} = 0 \right\}, \quad (4.89)$$

where $N_s^{(b)}$ is the Neumann map

$$\psi = N_s^{(b)}\mu \Leftrightarrow \begin{cases} (\Delta - bI)\psi = 0 & \text{in } \Omega_s; \\ \left[\frac{\partial\psi}{\partial\nu} \right]_{\Gamma_s} = \mu. \end{cases} \quad (4.90)$$

Proposition 4.5 regarding (4.79) and (4.80) is proved. $\square$

5 CASE 3: Heat-structure interaction with Kelvin-Voigt damping: Dirichlet control g at the external boundary Γ_f

We return to the homogeneous heat-structure interaction model (1.1a)–(1.1f) with Kelvin-Voigt damping. In the present CASE 3, we apply a Dirichlet control g on the external boundary Γ_f . Thus, with the same geometry (Figure 1) and notation $\{w, w_t, u\}$ as in Section 1.3 (see (1.1a)–(1.1f)), in the present CASE 3, we consider the problem

$$(PDE) \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (5.1a)$$

$$(5.1b)$$

$$(BC) \begin{cases} u|_{\Gamma_f} = g & \text{on } (0, T] \times \Gamma_f; \\ u = w_t & \text{on } (0, T] \times \Gamma_s; \\ \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (5.1c)$$

$$(5.1d)$$

$$(5.1e)$$

$$(IC) [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}, \quad (5.1f)$$

with Dirichlet boundary control g acting at the external boundary Γ_f . Compare against model (2.1a)–(2.1f) of CASE 1 with Neumann control g at the interface Γ_s , as in (2.1e), and against model (4.1a)–(4.1f) of CASE 2 with Dirichlet control g at the interface Γ_s , as in (4.1d). We shall likewise consider two cases: $b = 0$ and $b = 1$. $\mathbf{H}_b$ is the same finite energy space defined throughout in (1.2a)–(1.2b).

5.1 Abstract model on $\mathbf{H}_b$, $b = 0, 1$, of the nonhomogeneous PDE model (5.1a)–(5.1f) with Dirichlet control g acting at the external boundary Γ_f

This is the counterpart of the treatment in [39], where an interior Neumann or Dirichlet control acts at the interface Γ_s . To this end, we define two boundary $\rightarrow$ interior maps, with interior Ω_f : the map $D_{f,s}$ (introduced in (2.10) = (3.67) = (4.5)) acting from Γ_s , and the map $D_{f,f}$ acting from Γ_f :

$$\varphi = D_{f,s}\chi \Leftrightarrow \begin{cases} \Delta\varphi = 0 & \text{in } \Omega_f; \\ \varphi|_{\Gamma_f} = 0, & \varphi|_{\Gamma_s} = \chi, \end{cases} \quad \psi = D_{f,f}\mu \Leftrightarrow \begin{cases} \Delta\psi = 0 & \text{in } \Omega_f; \\ \psi|_{\Gamma_f} = \mu, & \psi|_{\Gamma_s} = 0, \end{cases} \quad (5.2a)$$

$$D_{f,s}, D_{f,f} : \mathbf{H}^r(\partial\Omega_f) \rightarrow \mathbf{H}^{r+\frac{1}{2}}(\Omega_f), \quad \text{for any } r \in \mathbb{R}, \quad (5.2b)$$

$$D_{f,s}, D_{f,f} : \mathbf{L}^2(\partial\Omega_f) \rightarrow \mathbf{H}^{\frac{1}{2}}(\Omega_f) \subset \mathbf{H}^{\frac{1}{2}-2\varepsilon}(\Omega_f) \equiv \mathcal{D}\left(A_{D,f}^{\frac{1}{4}-\varepsilon}\right), \quad (5.2c)$$

[40,41], [4, p. 181], [39], where $-A_{D,f}$ is the negative, self-adjoint operator on $\mathbf{L}^2(\Omega_f)$ defined in (2.9) = (3.66) = (4.4) and repeated here

$$-A_{D,f}\varphi = \Delta\varphi, \quad \mathcal{D}(A_{D,f}) \equiv \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f). \quad (5.3)$$

Similarly, we recall from (2.4) the Neumann map $N_s^{(b)}$ on Ω_s :

$$\psi = N_s^{(b)}\mu \Leftrightarrow \begin{cases} (\Delta - bI)\psi = 0 & \text{in } \Omega_s; \\ \frac{\partial\psi}{\partial\nu}\Big|_{\Gamma_s} = \mu, \end{cases} \quad (5.4)$$

with regularity given by (2.5), as well as the Neumann-Laplacian in Ω_s ; $-A_{N,s}^{(b)}$, the negative self-adjoint operator on $\mathbf{L}^2(\Omega_s)$, introduced in (2.3) and repeated here

$$-A_{N,s}^{(b)}\varphi = (\Delta - bI)\varphi = 0, \quad \mathcal{D}(A_{N,s}^{(b)}) = \left\{ \varphi \in \begin{cases} \mathbf{H}^2(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^2(\Omega_s) & \text{for } b = 1, \end{cases} : \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_s} = 0 \right\}. \quad (5.5)$$

To obtain the abstract model of problem (5.1), we proceed as usual [4,39]. We re-write the u -problem in (5.1) as follows:

$$\begin{cases} u_t = \Delta(u - D_{f,f}g - D_{f,s}(w_t|_{\Gamma_s})) & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (5.6a)$$

$$\begin{cases} [u - D_{f,f}g - D_{f,s}w_t]_{\partial\Omega_f} = 0 = \begin{cases} g - g - 0 & \text{on } \Gamma_f; \\ w_t - 0 - w_t & \text{on } \Gamma_s; \end{cases} & \text{in } (0, T] \times \partial\Omega_f. \end{cases} \quad (5.6b)$$

$$\begin{cases} u_t = -A_{D,f}u - D_{f,f}g - D_{f,s}(w_t |_{\Gamma_s}) \in \mathbf{L}^2(\Omega_f); \\ u_t = -\tilde{A}_{D,f}u + \tilde{A}_{D,f}D_{f,f}g + \tilde{A}_{D,f}D_{f,s}(w_t |_{\Gamma_s}) \in [\mathcal{D}(A_{D,f})]', \end{cases} \quad \begin{matrix} (5.7a) \\ (5.7b) \end{matrix}$$

where $\tilde{A}_{D,f}$ is the isomorphic extension of the operator $A_{D,f}$ in (5.3): $\mathbf{L}^2(\Omega_f) \rightarrow [\mathcal{D}(A_{D,f})]' = \text{dual of } \mathcal{D}(A_{D,f})$ with respect to $\mathbf{L}^2(\Omega_f)$ as a pivot space. Similarly, we re-write the w -problem in (5.1b) via the RHS of (5.1e) as (compare with (2.6) of CASE 1):

$$w_{tt} = \Delta(w + w_t) - bw = \Delta bI \left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right) \right] + b(w + w_t) - bw \text{ in } (0, T] \times \Omega_s. \quad (5.8)$$

The term $\left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right) \right]$ satisfies the zero Neumann B.C. of the operator $A_{N,s}^{(b)}$ in (5.5), so we can re-write (5.8) as (compare with (2.7) and (2.8) of CASE 1):

$$w_{tt} = -A_{N,s}^{(b)} \left[(w + w_t) - N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right) \right] + b(w + w_t) - bw \in \mathbf{L}^2(\Omega_s), \quad (5.9a)$$

$$w_{tt} = -\tilde{A}_{N,s}^{(b)}(w + w_t) + \tilde{A}_{N,s}^{(b)}N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right) + b(w + w_t) - bw \in [\mathcal{D}(A_{N,s}^{(b)})]', \quad (5.9b)$$

where $\tilde{A}_{N,s}^{(b)}$ is the isomorphic extension of the operator $A_{N,s}^{(b)}$ in (5.5): $\mathbf{L}^2(\Omega_s) \rightarrow [\mathcal{D}(A_{N,s}^{(b)})]' = \text{dual of } \mathcal{D}(A_{N,s}^{(b)})$ with respect to $\mathbf{L}^2(\Omega_s)$. Henceforth, as in CASE 1, we write simply $A_{N,s}^{(b)}$ to denote also the extension $\tilde{A}_{N,s}^{(b)}$, and likewise $A_{D,f}$ to denote the extension $\tilde{A}_{D,f}$. The action of $-A_{N,s}^{(b)}$ on the terms $(w + w_t)$ in (5.9a) is $\Delta bI(w + w_t)$ and (5.9a) has an extra term $b(w + w_t)$ and their combination produces a cancellation of the term $b(w + w_t)$ for $b = 1$. Thus, (5.9b) yields via (5.5)

$$w_{tt} = -A_{N,s}^{(0)}(w + w_t) - bw + \tilde{A}_{N,s}^{(b)}N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right), \quad (5.10)$$

ultimately via (5.5) with $b = 0$

$$w_{tt} = \Delta(w + w_t) - bw + \tilde{A}_{N,s}^{(b)}N_s^{(b)} \left(\frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \right). \quad (5.11)$$

Combining (5.10) for the w -problem with (5.7) for the u -problem, we obtain the corresponding first order system

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A_{N,s}^{(0)} - bI & -A_{N,s}^{(0)} & A_{N,s}^{(b)}N_s^{(b)} \left(\frac{\partial}{\partial \nu} \Big|_{\Gamma_s} \right) \\ 0 & A_{D,f}D_{f,s}(\cdot |_{\Gamma_s}) & -A_{D,f} \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A_{D,f}D_{f,f}g \end{bmatrix}. \quad (5.12)$$

The operator in (5.12) on $[w, w_t, u](g \equiv 0)$ is of course the same operator $\mathcal{A}_b$ in (1.8)–(1.9), except that in (5.12) the relevant BCs in (1.9a)–(1.9b) are included in the operator entries. Equation (5.12) can be rewritten as follows:

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathbb{B}_D g, \quad (5.13)$$

where the operator $\mathcal{A}_b : \mathbf{H}_b \supset \mathcal{D}(\mathcal{A}_b) \rightarrow \mathbf{H}_b$ is of course the same as given by (1.8) and (1.9), while the (boundary) control operator $\mathbb{B}_D$ is given by

$$\mathbb{B}_D g = \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,f} g \end{bmatrix}, \quad \mathbb{B}_D : \text{continuous } \mathbf{L}^2(\Gamma_s) \rightarrow \left[\begin{array}{c} \otimes \\ \otimes \\ \mathcal{D}\left(A_{D,f}^{\frac{3}{4}+\varepsilon}\right) \end{array} \right]'. \quad (5.14)$$

The adjoint operator $\mathbb{B}_D^*$ of $\mathbb{B}_D$ in (5.14) is given by

$$\mathbb{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_f}; \quad \mathbb{B}_D^* : \text{continuous } \left[\begin{array}{c} \otimes \\ \otimes \\ \mathcal{D}\left(A_{D,f}^{\frac{3}{4}+\varepsilon}\right) \end{array} \right] \rightarrow \mathbf{L}^2(\Gamma_s) \quad (5.15)$$

in the following sense. For $g \in \mathbf{L}^2(\Gamma_s)$ and $\{x_1, x_2, x_3\} \in \left[\otimes, \otimes, \mathcal{D}\left(A_{D,f}^{\frac{3}{4}+\varepsilon}\right) \right]$, we compute as a duality pairing via (5.14).

$$\left(\mathbb{B}_D g, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}} = \left(\begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,f} g \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{H}_b} \quad (5.16)$$

$$= (A_{D,f} D_{f,f} g, x_3)_{\mathbf{L}^2(\Omega_f)} = (g, D_{f,f}^* A_{D,f} x_3)_{\mathbf{L}^2(\Gamma_s)} \quad (5.17)$$

$$= \left(g, -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_s} \right)_{\mathbf{L}^2(\Gamma_s)} = \left(g, \mathbb{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)_{\mathbf{L}^2(\Gamma_s)}, \quad (5.18)$$

where we have recalled from [4, Chapter 3] (same technique as in obtaining (2.24), CASE 1 or (4.11), CASE 2)

$$D_{f,f}^* A_{D,f} x_3 = -\frac{\partial x_3}{\partial \nu} \Big|_{\Gamma_f}. \quad (5.19)$$

Thus, (5.15) is established. As shown in (4.11), the proof initially takes $x_3 \in \mathcal{D}(A_{D,f})$, i.e. $x_3|_{\partial\Omega_f} = 0$, and uses Green's second theorem to establish (5.19) in this case. Then (5.19) is extended to $\mathcal{D}(A_{D,f}^{\frac{3}{4}+\varepsilon}) \subset \mathbf{H}^{\frac{3}{2}+2\varepsilon}(\Omega_f)$.

5.2 The Luenberger's compensator for the heat-structure interaction model (5.1a)–(5.1f) with Dirichlet control g at the external boundary Γ_f , $b = 0, 1$

5.2.1 Special selection of the data

For the present heat-structure interaction problem with Dirichlet control at the external boundary Γ_f in the notation of (1.4)–(1.6), we take

$$\begin{cases} A \text{ "exponentially stable": } \|e^{At}\| \leq ce^{-\delta t}, \delta > 0, t \geq 0, \\ F = B^*; \quad C = B^*, \quad K = B, \end{cases} \quad (5.20a)$$

$$(5.20b)$$

as in CASE 2, and unlike CASE 1. Thus, the special setting becomes, in this case,

$$[\text{partial observation of the state } y] = Cy = B^*y, \quad \text{control } g = Fz = B^*z \quad (5.21)$$

leading to the Luenberger's dynamics

$$\begin{cases} \dot{y} = Ay + BB^*z, \\ \dot{z} = Az + B(B^*y) \end{cases} \quad (5.22a)$$

$$(5.22b)$$

(as $BF - KC = 0$ as in CASE 2, unlike CASE 1) and hence to

$$\frac{d}{dt}[y - z] = (A - BB^*)[y - z]; \quad [y(t) - z(t)] = e^{(A - BB^*)t}[y_0 - z_0]. \quad (5.23)$$

Insight. How did we decide that $F = B^*$ in (5.20b); that is, that the preassigned control $g = Fz$ is given by $g = B^*z$ or $F = B^*$? We first notice that, regardless of the choice of F , the Luenberger scheme in (1.4)–(1.6), yields that $(A - KC)$ is the resulting sought-after operator in characterizing the quantity $[y - z]$ of interest. As A is, in our case, dissipative and we surely seek to retain dissipativity, then we choose $KC = BB^*$, or $K = B, C = B^*$. Then, the (dissipative) operator $(A - BB^*)$ is the key operator to analyze for the purpose of concluding that the semigroup $e^{(A - BB^*)t}$ is (analytic as well as) uniformly stable. Thus, at this stage, with F not yet committed and $K = B, C = B^*$ committed, the z -equation becomes $\dot{z} = [A - BF - BB^*]z + B(B^*y)$. It is natural to test either $F = B^*$ or else $F = -B^*$, whichever choice may yield the desired properties for the feedback operator $A_F = A - BB^*$. What is the right sign? Thus, passing from the historical scheme (1.4a)–(1.4f) to our present PDE problem (5.1a)–(5.1f), the corresponding operator B_D^* is critical in imposing the boundary conditions for the feedback operator $\mathbb{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*$ in our CASE 3. But in our present CASE 3, if we choose

$F \rightarrow \mathbb{B}_D^*$ and so $g = \mathbb{B}_D^* \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = -\frac{\partial z_3}{\partial v} \Big|_{\Gamma_f}$ by (5.15), this then implies the boundary condition $h|_{\Gamma_f} = -\frac{\partial h}{\partial v} \Big|_{\Gamma_f}$ for

$\{v_1, v_2, h\}$ in $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ as in (5.34) below. With this B.C., the argument in (5.51)–(5.53) below leads to the trace term $-i\omega \|h|_{\Gamma_f}\|^2$ in (5.54), and hence to the critical term $i\omega [\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2]$ in (5.57), with the correct “minus” sign “−” for the argument of Theorem 5.4, in particular estimate (5.47), to succeed. Therefore, in view of the interface condition $u|_{\Gamma_f} = g$ in (5.1c) at the external boundary Γ_f , the B.C. $h|_{\Gamma_f} = -\frac{\partial h}{\partial v} \Big|_{\Gamma_f}$ confirms that $g = -\frac{\partial h}{\partial v} \Big|_{\Gamma_f}$,

or that $g = F \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathbb{B}_D^* \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -\frac{\partial h}{\partial v} \Big|_{\Gamma_f}$. Hence, the choice $F \rightarrow \mathbb{B}_D^*$ as in (5.20b) is the correct one in our CASE 3.

5.2.2 The counterpart of $\dot{y} = Ay + BB^*z$ in (5.22b) for the HSI model (5.1a)–(5.1f) with $F \rightarrow \mathbb{B}_D^*$

Accordingly, for $z = [z_1, z_2, z_3]$, the Luenberger’s compensator variable, in line with (5.21) we select the Dirichlet control g in the form

$$g = \mathbb{B}_D^* z = -\frac{\partial z_3}{\partial v} \Big|_{\Gamma_f}, \quad z_3 \in \mathcal{D}\left(A_{D,f}^{\frac{3}{2}+\varepsilon}\right) \subset \mathbf{H}^{\frac{3}{2}+2\varepsilon}(\Omega_f) \quad (5.24)$$

by (5.15). With $y = [w, w_t, u]$, the PDE version of the abstract feedback problem corresponding to (5.22a)

$$\dot{y} = \mathcal{A}_b y + \mathbb{B}_D \mathbb{B}_D^* z, \quad \text{or} \quad \frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} w_t \\ \Delta(w + w_t) - bw \\ \Delta u \end{bmatrix} + \mathbb{B}_D \left(-\frac{\partial z_3}{\partial v} \Big|_{\Gamma_f} \right) \quad (5.25)$$

recalling $\mathcal{A}_b$ from (1.8) and $\mathbb{B}_D^*$ from (5.15), with g as in (5.24), is

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (5.26a)$$

$$\begin{cases} w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (5.26b)$$

$$\begin{cases} u|_{\Gamma_f} = -\frac{\partial z_3}{\partial v} \Big|_{\Gamma_f} & \text{on } (0, T] \times \Gamma_f; \quad u = w_t & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (5.26c)$$

$$\begin{cases} \frac{\partial(w + w_t)}{\partial v} = \frac{\partial u}{\partial v} & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (5.26d)$$

5.2.3 The counterpart of the dynamic compensator equation $\dot{z} = Az + B(B^*y)$ in (5.22b) for the HSI model (5.1a)–(5.1f)

With partial observation as in (5.21) according to (5.24) or (5.15)

$$g = \mathbb{B}_D^* y = \mathbb{B}_D^* \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = -\frac{\partial u}{\partial \nu} \Big|_{\Gamma_f} = \text{partial observation of state } y = \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \quad (5.27)$$

the compensator equation (5.22b) in $z = [z_1, z_2, z_3]$ becomes in our present HSI case

$$\dot{z} = \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathcal{A}_b z + \mathbb{B}_D \left(-\frac{\partial u}{\partial \nu} \Big|_{\Gamma_f} \right) \quad (5.28)$$

or via (1.8) for $\mathcal{A}_b$ and (5.14) and (5.15) for $\mathbb{B}_D, \mathbb{B}_D^*$, respectively,

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = \begin{bmatrix} z_2 \\ \Delta(z_1 + z_2) - bz_1 \\ \Delta z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,f} \left(-\frac{\partial u}{\partial \nu} \Big|_{\Gamma_f} \right) \end{bmatrix}, \quad (5.29)$$

whereby $z_{1t} = z_2, z_{1tt} = z_{2t}$. The PDE version of the abstract z -model (5.29) with partial observation $-\frac{\partial u}{\partial \nu} \Big|_{\Gamma_f}$ in the Dirichlet condition at the external boundary Γ_f as in (5.27), is via (5.24)

$$\begin{cases} z_{3t} - \Delta z_3 = 0 & \text{in } (0, T] \times \Omega_f; \end{cases} \quad (5.30a)$$

$$\begin{cases} z_{1tt} - \Delta z_1 - \Delta z_{1t} + bz_1 = 0 & \text{in } (0, T] \times \Omega_s; \end{cases} \quad (5.30b)$$

$$\begin{cases} z_3|_{\Gamma_f} = -\frac{\partial u}{\partial \nu} \Big|_{\Gamma_f} \text{ on } (0, T] \times \Gamma_f; \quad z_3 = z_{1t} & \text{on } (0, T] \times \Gamma_s; \end{cases} \quad (5.30c)$$

$$\begin{cases} \frac{\partial(z_1 + z_{1t})}{\partial \nu} = \frac{\partial z_3}{\partial \nu} & \text{on } (0, T] \times \Gamma_s. \end{cases} \quad (5.30d)$$

5.2.4 The dynamics $\dot{d} = (\mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*)d$, $d(t) = y(t) - z(t)$, corresponding to (5.23): analyticity and exponential decay, $b = 0, 1$

The main result of the present section is as follows:

Theorem 5.1. *Let $b = 0, 1$. The (feedback) operator*

$$\mathbb{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*, \quad \mathbf{H}_b \supset \mathcal{D}(\mathbb{A}_{F,D}^{(b)}) = \{x \in \mathbf{H}_b : (I + \mathcal{A}_b^{-1} \mathbb{B}_D \mathbb{B}_D^*)x \in \mathcal{D}(\mathcal{A}_b)\} \quad (5.31)$$

is the infinitesimal generator of a s.c. contraction semigroup $e^{\mathbb{A}_{F,D}^{(b)} t}$ on $\mathbf{H}_b$, which moreover is analytic and exponentially stable on $\mathbf{H}_b$; there exist constants $C \geq 1, \rho > 0$ possibly depending on “ b ” such that

$$\|e^{\mathbb{A}_{F,D}^{(b)} t}\|_{\mathcal{L}(\mathbf{H}_b)} = \|e^{(\mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*)t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq C e^{-\rho t}, \quad t \geq 0. \quad (5.32)$$

A more detailed description of $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ is given in (5.35a) and (5.35b).

The proof of Theorem 5.1 is by PDE methods, which consist of analyzing the corresponding PDE system (5.30). We proceed through a series of steps.

Step 1.

Lemma 5.2. *Let $b = 0, 1$. With $d = [d_1, d_2, d_3] \in \mathbf{H}_b$, the abstract equation*

$$\dot{d} = \mathbb{A}_{F,D}^{(b)} d = (\mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*)d \quad (5.33a)$$

or via (1.8) on $\mathcal{A}_b$, (5.14) and (5.15) on $\mathbb{B}_D$ and $\mathbb{B}_D^$, respectively*

$$\begin{bmatrix} d_{1t} \\ d_{2t} \\ d_{3t} \end{bmatrix} = \begin{bmatrix} d_2 \\ \Delta(d_1 + d_2) - bd_1 \\ \Delta d_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,f} \left(-\frac{\partial d_3}{\partial v} \Big|_{\Gamma_f} \right) \end{bmatrix}, \quad (5.33b)$$

so that $d_2 = d_{1t}, d_{2t} = d_{1tt}$ corresponds to the following PDE system, where we relabel the variable $[d_1, d_2 = d_{1t}, d_3] = [\widehat{w}, \widehat{w}_t, \widehat{u}]$ for convenience

$$\widehat{u}_t - \Delta \widehat{u} = 0 \quad \text{in } (0, T] \times \Omega_f; \quad (5.34a)$$

$$\widehat{w}_{tt} - \Delta \widehat{w} - \Delta \widehat{w}_t + b\widehat{w} = 0 \quad \text{in } (0, T] \times \Omega_s; \quad (5.34b)$$

$$\widehat{u}|_{\Gamma_f} = -\frac{\partial \widehat{u}}{\partial \nu} \Big|_{\Gamma_f} \quad \text{on } (0, T] \times \Gamma_f; \quad \widehat{u} = \widehat{w}_t \quad \text{on } (0, T] \times \Gamma_s; \quad (5.34c)$$

$$\frac{\partial(\widehat{w} + \widehat{w}_t)}{\partial \nu} = \frac{\partial \widehat{u}}{\partial \nu} \quad \text{on } (0, T] \times \Gamma_s. \quad (5.34d)$$

Description of $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)}) = \mathcal{D}(\mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*)$ if and only if

(i)

$$v_1, v_2 \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad \text{so that } v_2|_{\Gamma_s} = h|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \quad \text{in both cases;} \quad (5.35a)$$

$$\Delta(v_1 + v_2) \in \mathbf{L}^2(\Omega_s);$$

(ii)

$$h \in \mathbf{H}^1(\Omega_f), \quad \Delta h \in \mathbf{L}^2(\Omega_f), \quad h|_{\Gamma_f} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_f); \quad (5.35b)$$

$$h|_{\Gamma_s} \equiv v_2|_{\Gamma_s}, \quad \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s).$$

The adjoint operator $\mathbb{A}_{F,D}^{(b)*} = \mathcal{A}_b^* - \mathbb{B}_D \mathbb{B}_D^*$. For $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)*})$ (to be characterized below), we have after

recalling $\mathcal{A}_b^*$ in (1.10a) and $\mathbb{B}_D^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f}$ from (5.15)

$$\mathbb{A}_{F,D}^{(b)*} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = (\mathcal{A}_b^* - \mathbb{B}_D \mathbb{B}_D^*) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \mathbb{B}_D \left(-\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right) \quad (5.36)$$

$$= \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) + bv_1 \\ \Delta h \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ A_{D,f} D_{f,f} \left(-\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right) \end{bmatrix} \quad (5.37)$$

recalling $\mathbb{B}_D$ in (5.14).

Description of $\mathcal{D}(\mathbb{A}_{F,D}^{(b)*})$. We have $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)*}) = \mathcal{D}(\mathcal{A}_b^* - \mathbb{B}_D \mathbb{B}_D^*)$ if and only if the same conditions for $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ in (5.35a) and (5.35b) apply, except that now (5.35a) is replaced by $\Delta(v_2 - v_1) \in \mathbf{L}^2(\Omega_s)$ and (5.35b) is replaced by

$$\frac{\partial v_2 - v_1}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \quad (5.38)$$

The PDE version corresponding to the adjoint operator $\mathbb{A}_{F,D}^{(b)*}$ is given by

$$\frac{d}{dt} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = \mathbb{A}_{F,D}^{(b)*} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} \quad \text{or} \quad \begin{cases} h_t - \Delta h = 0 & \text{in } (0, T] \times \Omega_f; \\ w_{1tt} - \Delta w_1 - \Delta w_{1t} + bw_1 = 0 & \text{in } (0, T] \times \Omega_s; \\ h|_{\Gamma_f} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} & \text{on } (0, T] \times \Gamma_f; \\ h|_{\Gamma_s} = -w_{1t}|_{\Gamma_s}, \frac{\partial(w_1 + w_{1t})}{\partial \nu} \Big|_{\Gamma_s} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} & \text{on } (0, T] \times \Gamma_s, \end{cases} \quad \begin{matrix} (5.39a) \\ (5.39b) \\ (5.39c) \\ (5.39d) \end{matrix}$$

(where $w_{1t} = -w_2$ in (5.36), top line) with I.C. in $\mathbf{H}_b$ (counterpart of (3.21a)–(3.21d) in CASE 1, and (4.32a)–(4.32d) in CASE 2).

Step 2. (Analysis of the PDE problem (5.34): the operator $\mathbb{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*$ in (5.33a))

Proposition 5.3. *The operator $\mathbb{A}_{F,D}^{(b)} = \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*$ in (5.33) and its $\mathbf{H}_b$ -adjoint $\mathbb{A}_{F,D}^{(b)*} = \mathcal{A}_b^* - \mathbb{B}_D^* \mathbb{B}_D$ are dissipative*

$$\operatorname{Re} \left\langle \begin{pmatrix} \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^* \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}} = -\|\nabla v_2\|_{\Omega_s}^2 - \|\nabla h\|_{\Omega_f}^2 - \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|_{L^2(\Gamma_f)}^2, \quad \{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)}), \quad (5.40)$$

$$\operatorname{Re} \left\langle \begin{pmatrix} \mathcal{A}_b^* - \mathbb{B}_D^* \mathbb{B}_D \end{pmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right\rangle_{\mathbf{H}} = -\|\nabla v_2^*\|_{\Omega_s}^2 - \|\nabla h^*\|_{\Omega_f}^2 - \left\| \frac{\partial h^*}{\partial \nu} \Big|_{\Gamma_f} \right\|_{L^2(\Gamma_f)}^2, \quad \{v_1^*, v_2^*, h^*\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)*}), \quad (5.41)$$

in the $L^2(\cdot)$ -norms of Ω_s and Ω_f , and the $L^2(\Gamma_s)$ -norm on Γ_s . Hence, both $\mathbb{A}_{F,D}^{(b)}$ and $\mathbb{A}_{F,D}^{(b)*}$ are maximal dissipative and thus generate s.c. contraction semigroups $e^{\mathbb{A}_{F,D}^{(b)} t}$ and $e^{\mathbb{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$. Explicitly in terms of the corresponding PDE systems we have:

$$\begin{bmatrix} \widehat{w}(t) \\ \widehat{w}_t(t) \\ \widehat{u}(t) \end{bmatrix} = e^{\mathbb{A}_{F,D}^{(b)} t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} = e^{(\mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^*) t} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{u}_0 \end{bmatrix} \quad (5.42)$$

for the $\{\widehat{w}, \widehat{w}_t, \widehat{u}\}$ -fluid-structure interaction model given by (5.34a)–(5.34d) on $\mathbf{H}_b$: with I.C. $\{\widehat{w}_0, \widehat{w}_1, \widehat{u}_0\} \in \mathbf{H}_b$. Similarly, for the adjoint $\mathbb{A}_{F,D}^{(b)*}$, corresponding to the model (5.39a)–(5.39d).

Proof of (5.3). For $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ in (5.35a) and (5.35b), we compute:

$$\operatorname{Re} \left\langle \begin{pmatrix} \mathcal{A}_b - \mathbb{B}_D \mathbb{B}_D^* \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} = \operatorname{Re} \left\langle \mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\mathbf{H}_b} - \left\| \mathbb{B}_D^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\|_{\mathbf{H}_b}^2 \quad (5.43)$$

$$= -\|\nabla v_2\|^2 - \|\nabla h\|^2 - \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|_{L^2(\Gamma_f)}^2, \quad (5.44)$$

recalling (1.12) and (5.24). Thus, (5.40) is established. Similarly for (5.41) recalling (1.13) for $\mathcal{A}_b^*$. $\square$

5.2.5 Analyticity of $e^{\mathbb{A}_{F,D}^{(b)} t}$ and $e^{\mathbb{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$, $b = 0, 1$

Remark 5.1. A remark such as Remark 3.2 (CASE 1) or Remark 4.1 (CASE 2) applies in the present case, to justify (5.47) from (5.46).

Theorem 5.4. Let $b = 0, 1$. Let $\omega \in \mathbb{R}$ and

$$(i\omega I - \mathbb{A}_{F,D}^{(b)}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \in \mathbf{H}_b \quad (5.45)$$

for $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ identified in (5.35a) and (5.35b). Then:

(i) the following estimate holds true: there exists a constant $C_\varepsilon > 0$ such that:

$$\begin{aligned} & \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_f}^2 \\ & \leq C_\varepsilon \{ \|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \}, \quad \forall |\omega| \geq \varepsilon > 0. \end{aligned} \quad (5.46)$$

(ii) In view of Remark 5.1, estimate (5.46) without the term $\left\| \frac{\partial h}{\partial \nu} \right\|_{\Gamma_f}^2$ is equivalent to

$$\|\mathbb{A}_{F,D}^{(b)} R(i\omega, \mathbb{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq C_\varepsilon, \quad \forall |\omega| \geq \varepsilon > 0, \quad (5.47)$$

$R(i\omega, \mathbb{A}_{F,D}^{(b)}) = (i\omega I - \mathbb{A}_{F,D}^{(b)})^{-1}$, which in turn is equivalent to

$$\|R(i\omega, \mathbb{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \frac{C_\varepsilon}{|\omega|}, \quad \forall |\omega| \geq \varepsilon > 0. \quad (5.48)$$

Thus, the s.c. contraction semigroup $e^{\mathbb{A}_{F,D}^{(b)} t}$ asserted by Proposition 5.3 is analytic on $\mathbf{H}_b$ by [4, Theorem 3E.3 p. 334]. Similarly for $e^{\mathbb{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$. The explicit PDE version of (5.33) for system $\{\widehat{w}(t), \widehat{w}_t(t), \widehat{u}(t)\}$ is given by (5.34a)–(5.34d).

Proof. (i) The proof of estimate (5.46) follows closely the technical proof of estimate (3.30) for the operator $\mathcal{A}_{F,N}^{(b)}$ in CASE 1, or (4.41) for the operator $\mathcal{A}_{F,D}^{(b)}$ in CASE 2 except that now the argument uses the B.C. of $\mathbb{A}_{F,D}^{(b)}$ rather than the B.C. of $\mathcal{A}_{F,N}^{(b)}$ or the B.C. of $\mathcal{A}_{F,D}^{(b)}$. This, in particular, requires an argument different from CASE 1 and CASE 2 to handle the more challenging case $b = 1$, with full $H^1(\Omega)$ -norm on the first component space. We indicate the relevant changes:

Step 1. Return to (5.45), re-written for $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ and $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$:

$$i\omega v_1 - v_2 = v_1^*, \quad (5.49a)$$

$$i\omega v_2 - [\Delta(v_1 + v_2) - bv_1] = v_2^*, \quad (5.49b)$$

$$i\omega h - \Delta h = h^*. \quad (5.49c)$$

Step 2. Take the $\mathbf{L}^2(\Omega_f)$ -inner product of equation (5.49c) against Δh , use Green's First theorem (counterpart (4.45))

$$i\omega \int_{\Gamma_f} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_f + i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \quad (5.50)$$

Similarly, we take the $\mathbf{L}^2(\Omega_s)$ -inner product of (5.49b) against $[\Delta(v_1 + v_2) - bv_1]$, use Green's First Theorem to evaluate $\int_{\Omega_s} v_2 \Delta(\bar{v}_1 + \bar{v}_2) d\Omega_s$, recalling that the normal vector ν is inward with respect to Ω_s , and obtain (4.45), repeated here

$$-i\omega \int_{\Gamma_s} v_2 \frac{\partial(\bar{v}_1 + \bar{v}_2)}{\partial \nu} d\Gamma_s - i\omega (\nabla v_2, \nabla(v_1 + v_2)) - i\omega (v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \quad (5.51)$$

Now we invoke the B.C. in (5.35a) and (5.35b) for $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$:

$$h|_{\Gamma_f} = -\frac{\partial h}{\partial \nu} \Big|_{\Gamma_f}; \quad h|_{\Gamma_s} = v_2|_{\Gamma_s}; \quad \frac{\partial(v_1 + v_2)}{\partial \nu} \Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \quad \text{in } \mathcal{D}(\mathbb{A}_{F,D}^{(b)}) \quad (5.52)$$

and re-write (5.50) and (5.51) as follows, respectively

$$-i\omega \int_{\Gamma_f} h^2 d\Gamma_s + i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h), \quad (5.53)$$

$$\begin{aligned} & -i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s - i\omega \|\nabla v_2\|^2 + (\nabla v_2, \nabla(i\omega v_1)) - i\omega(v_2, bv_1) - \|\Delta(v_1 + v_2) - bv_1\|^2 \\ & = (v_2^*, [\Delta(v_1 + v_2) - bv_1]). \end{aligned} \quad (5.54)$$

Summing up (5.53) and (5.54) yields after a cancellation of the boundary terms $i\omega \int_{\Gamma_s} h \frac{\partial \bar{h}}{\partial \nu} d\Gamma_s$:

$$\begin{aligned} -i\omega \|h|_{\Gamma_f}\|^2 - i\omega [\|\nabla v_2\|^2 + \|\nabla h\|^2] &= \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega(v_2, bv_1) + i\omega(\nabla v_2, \nabla v_1) \\ &+ (v_2^*, [\Delta(v_1 + v_2) - bv_1]) + (h^*, \Delta h). \end{aligned} \quad (5.55)$$

We now use the identities from (5.49a)

$$-i\omega(\nabla v_2, \nabla v_1) = (\nabla v_2, \nabla(i\omega v_1)) = \|\nabla v_2\|^2 + (\nabla v_2, \nabla v_1^*), \quad (5.56a)$$

$$-i\omega(v_2, v_1) = (v_2, i\omega v_1) = \|v_2\|^2 + (v_2, v_1^*) \quad (5.56b)$$

to obtain the final identity from (5.55)

$$\begin{aligned} & \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + i\omega[\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2] \\ &= \|\nabla v_2\|^2 + b\|v_2\|^2 + (\nabla v_2, \nabla v_1^*) + b(v_2, v_1^*) - (v_2^*, [\Delta(v_1 + v_2) - bv_1]) - (h^*, \Delta h). \end{aligned} \quad (5.57)$$

Identity (5.57) is the counterpart of identity (4.50) for Dirichlet control g on the interface Γ_s , after replacing

$\left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right\|^2$ in (4.50) with $\left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|^2 = \|h|_{\Gamma_f}\|^2$ in (5.57). Identity (5.57) is also the counterpart of identity (3.38) for

Neumann control g at the Interface Γ_s , after replacing $\|h|_{\Gamma_s}\|^2$ in (3.38) with $\left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|^2 = \|h|_{\Gamma_f}\|^2$ in (5.57). $\square$

Step 3. We proceed as in Step 3 of CASE 1 or of CASE 2. We take the real part of identity (5.57) and next obtain

$$(1 - \varepsilon)[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \leq (1 + \varepsilon)[\|\nabla v_2\|^2 + b\|v_2\|^2] + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (5.58)$$

the counterpart of estimate (4.52) CASE 2, or estimate (3.40), CASE 1.

Step 4. We now take the imaginary part of identity (5.57), and proceeding as in Step 4 of CASE 1 or CASE 2, we arrive at the following estimate

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 \leq \varepsilon b\|v_2\|^2 + \boxed{\text{OK}_\varepsilon}, \quad (5.59a)$$

$$\boxed{\text{OK}_\varepsilon} = 2\varepsilon[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] + C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (5.59b)$$

Equation (5.59a) and (5.59b) are the counterpart of (3.46) (CASE 1) or (4.59₁) and (4.60₁) via (4.64) (CASE 2), the difference being that the boundary term $\|h|_{\Gamma_s}\|^2$ on the LHS of (3.46) (CASE 1), respectively, and the boundary

term $\left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right\|^2$ on the LHS of (4.59₁) (CASE 2, where $\frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} = v_2|_{\Gamma_s} - h|_{\Gamma_s}$ holds as in (4.64)), is now replaced by the boundary term $\|h|_{\Gamma_f}\|^2$ on the LHS of (5.59a).

Case $b = 0$. In the case $b = 0$, the rest of the proof is the same as in Remark 3.1 (CASE 1, with Neumann control on Γ_s) or as in Remark 4.2 (CASE 2 with Dirichlet control on Γ_s) by replacing the term $\|h|_{\Gamma_s}\|^2$ in (3.46),

respectively, by replacing the term $\left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_s} \right\|^2$ in (4.58) with the term $\|h|_{\Gamma_f}\|^2 = \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|^2$ in (5.59a). Thus, in the case $b = 0$, the sought-after estimate (5.45), and hence, (5.47) are established in the space $\mathbf{H}_{b=0}$.

Step 5. Case $b = 1$. In the present case, as in CASE 2 of Dirichlet control on Γ_s , there are additional challenges in establishing the sought-after estimate (5.46) with $b = 1$, i.e., in the space $\mathbf{H}_{b=1}$. The required argument is provided in the present step, which is the counterpart of Step 5 in Section 3 (CASE 1) or of Step 5 in Section 4 (CASE 2). In line with these two cases, it will rely on Lemma 3.5.

We return to (5.59a) and add the term $\varepsilon_1 \|v_2|_{\Gamma_s}\|^2$ to both sides, $1 \geq \varepsilon_1 > 0$ to be chosen below, where on the RHS, we use $v_2|_{\Gamma_s} = h|_{\Gamma_s}$ by (5.52).

We obtain

$$\|\nabla v_2\|^2 + \varepsilon_1 \|v_2|_{\Gamma_s}\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 \leq \varepsilon b \|v_2\|^2 + \varepsilon_1 \|h|_{\Gamma_s}\|^2 + \boxed{\text{OK}_\varepsilon}, \quad (5.60_1)$$

for $|\omega|^2 \geq \varepsilon > 0$, hence

$$\varepsilon_1 [\|\nabla v_2\|^2 + \|v_2|_{\Gamma_s}\|^2] + \frac{1}{2} \|\nabla h\|^2 + \left[\frac{1}{2} \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 \right] \leq \varepsilon b \|v_2\|^2 + \varepsilon_1 [\|\nabla h\|^2 + \|h\|^2] + \boxed{\text{OK}_\varepsilon}. \quad (5.61_1)$$

In going from (5.60₁) to (5.61₁), we have used: on the LHS, that $0 < \varepsilon_1 \leq 1$; on the RHS, that the $L^2(\Gamma_s)$ -norm of $[h|_{\Gamma_s}]$ is dominated by the $H^1(\Omega_f)$ -norm of h . Next, on the LHS of (5.61₁), we invoke Lemma 3.5(a) for both v_2 in Ω_s and h in Ω_f . We obtain

$$\frac{\varepsilon_1}{c_1} \|v_2\|^2 \leq \varepsilon_1 [\|\nabla v_2\|^2 + \|v_2|_{\Gamma_s}\|^2], \quad (5.62_1)$$

$$\frac{1}{2c_1} \|h\|^2 \leq \frac{1}{2} [\|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2]. \quad (5.63_1)$$

Invoking (5.62₁), (5.63₁), we re-write (5.61₁) as follows:

$$\left(\frac{\varepsilon_1}{c_1} - \varepsilon b \right) \|v_2\|^2 + \left(\frac{1}{2} - \varepsilon_1 \right) \|\nabla h\|^2 + \left(\frac{1}{2c_1} - \varepsilon_1 \right) \|h\|^2 \leq \boxed{\text{OK}_\varepsilon}, \quad |\omega|^2 \geq \varepsilon > 0. \quad (5.64_1)$$

We now select $1 \geq \varepsilon > 0$ ($b = 1$) as follows:

$$\sqrt{\varepsilon} < \frac{\varepsilon_1}{c_1} - \varepsilon; \quad \frac{1}{4} < \frac{1}{2} - \varepsilon_1; \quad \frac{1}{4c_1} < \frac{1}{2c_1} - \varepsilon_1; \quad (\sqrt{\varepsilon} + \varepsilon) < \varepsilon_1 \leq \min \left\{ \frac{1}{4}, \frac{1}{4c_1} \right\}, \quad (5.65_1)$$

(we can always assume $c_1 > 1$, see Lemma 3.5(a)). Using (5.65₁) on the LHS of (5.64₁) yields

$$\begin{aligned} \sqrt{\varepsilon} \|v_2\|^2 + \frac{1}{4} \|\nabla h\|^2 + \frac{1}{4c_1} \|h\|^2 &\leq 2\varepsilon [|\Delta(v_1 + v_2) - bv_1|^2 + |\Delta h|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \end{aligned} \quad (5.66_1)$$

for $|\omega| \geq \varepsilon^2$, after invoking (5.59b) for $\boxed{\text{OK}_\varepsilon}$. Finally, ($b = 1$ and $\sqrt{\varepsilon} < \frac{1}{4}$, $\sqrt{\varepsilon} < \frac{1}{4c_1}$), we obtain

$$\begin{aligned} b \|v_2\|^2 + \|\nabla h\|^2 + \|h\|^2 &\leq 2\sqrt{\varepsilon} [|\Delta(v_1 + v_2) - bv_1|^2 + |\Delta h|^2] \\ &\quad + \tilde{C}_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2] \end{aligned} \quad (5.67_1)$$

for $|\omega| \geq \varepsilon^2$ as in (4.56) (CASE 2). Estimate (5.67₁) is a counterpart of estimate (3.55) (CASE 1) or estimate (4.2.5) (CASE 2). Estimate (5.67₁) corresponds to (5.59) for $b = 0$.

Substituting the estimate for $b\|v_2\|^2$ from (5.67₁) into the RHS of (5.59a) yields

$$\begin{aligned} \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 &\leq \{\varepsilon 2\sqrt{\varepsilon} [|\Delta(v_1 + v_2) - bv_1|^2 + |\Delta h|^2] \\ &\quad + \tilde{C}_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]\} + \boxed{\text{OK}_\varepsilon}. \end{aligned} \quad (5.67_1^*)$$

Next add to (5.67₁^{*}) the estimate for $b\|v_2\|^2$ in (5.67₁) to obtain

$$\begin{aligned} b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 &\leq [2\sqrt{\varepsilon}(1 + \varepsilon) + 2\varepsilon] [|\Delta(v_1 + v_2) - bv_1|^2 + |\Delta h|^2] \\ &\quad + C_\varepsilon [\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \end{aligned} \quad (5.68_1)$$

recalling $\boxed{\text{OK}_\varepsilon}$ from (5.59b). Estimate (5.68₁) is the counterpart of estimate (5.59) for $b = 0$.

Step 6. The rest of the proof for $b = 1$ now proceeds as in the case $b = 0$. In (5.68₁), we drop the terms $[|\nabla h|^2 + \|h|_{\Gamma_f}\|^2]$ and substitute the resulting estimate for $[b\|v_2\|^2 + \|\nabla v_2\|^2]$ into the RHS of (5.58) and obtain

$$(1 - \varepsilon)[\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \leq (1 + \varepsilon)[2\sqrt{\varepsilon}(1 + \varepsilon) + 2\varepsilon][\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2] \\ + (1 + \varepsilon)C_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (5.69_1)$$

or, since $\{(1 - \varepsilon) - (1 + \varepsilon)[2\sqrt{\varepsilon}(1 + \varepsilon) + 2\varepsilon]\} \geq k > 0$, we obtain

$$\|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 \leq \widehat{C}_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (5.70_1)$$

which is the counterpart of (3.58) in CASE 1, and (4.75₁) in CASE 2. By substituting (5.70₁) into the RHS of (5.68₁), we finally obtain

$$b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 \leq \widetilde{C}_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2]. \quad (5.71_1)$$

Summing up (5.70₁) and (5.71₁) yields

$$b\|v_2\|^2 + \|\nabla v_2\|^2 + \|\Delta(v_1 + v_2) - bv_1\|^2 + \|\Delta h\|^2 + \|\nabla h\|^2 + \|h|_{\Gamma_f}\|^2 \\ \leq \widetilde{C}_\varepsilon[\|\nabla v_1^*\|^2 + b\|v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2], \quad (5.72_1)$$

which is the sought-after estimate (5.15), $b = 0$, $b = 1$, $\|h|_{\Gamma_f}\|^2 = \left\| \frac{\partial h}{\partial \nu} \Big|_{\Gamma_f} \right\|^2$.

5.2.6 Exponential stability of $e^{\mathbb{A}_{F,D}^{(b)} t}$ and $e^{\mathbb{A}_{F,D}^{(b)*} t}$ on $\mathbf{H}_b$, $b = 0, 1$

In Proposition 5.5, we shall prove that, in both cases $b = 0$ and $b = 1$, we have

$$0 \in \rho(\mathbb{A}_{F,D}^{(b)}), \quad 0 \in \rho(\mathbb{A}_{F,D}^{(b)*}), \quad \mathbb{A}_{F,D}^{(b)-1} \in \mathcal{L}(\mathbf{H}_b), \quad \mathbb{A}_{F,D}^{(b)*-1} \in \mathcal{L}(\mathbf{H}_b), \quad (5.73)$$

so that there exists a disk S_{r_0} centered at the origin and of suitable radius $r_0 > 0$ such that $S_{r_0} \subset \rho(\mathbb{A}_{F,D}^{(b)})$. Then, the resolvent bound (5.48) combined with $\mathbb{A}_{F,D}^{(b)-1} \in \mathcal{L}(\mathbf{H}_b)$ in (5.73) allows one to conclude that the resolvent is uniformly bounded on the imaginary axis $i\mathbb{R}$:

$$\|R(i\omega, \mathbb{A}_{F,D}^{(b)})\|_{\mathcal{L}(\mathbf{H}_b)} \leq \text{const}. \quad (5.74)$$

Hence, [45] the s.c. analytic semigroup $e^{\mathbb{A}_{F,D}^{(b)} t}$ is, moreover, (uniformly) exponentially bounded: There exist constants $M \geq 1$, $\delta > 0$, possibly depending on “ b ” such that

$$\|e^{\mathbb{A}_{F,D}^{(b)} t}\|_{\mathcal{L}(\mathbf{H}_b)} \leq Me^{-\delta t}, \quad t \geq 0. \quad (5.75)$$

It is similar for the adjoint $\mathbb{A}_{F,D}^{(b)*}$.

Proposition 5.5. *Statement (5.73) holds true. Hence, the exponential stability for $e^{\mathbb{A}_{F,D}^{(b)} t}$ in (5.75) holds true. More precisely, with reference to $\mathbb{A}_{F,D}^{(b)}$, we have: given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_b$, the unique solution $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ of*

$$\mathbb{A}_{F,D}^{(b)} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \quad (5.76)$$

is given explicitly by

$$v_1 = (-A_{N,s}^{(b)})^{-1}(-\Delta v_1^* + v_2^*) + N_s^{(b)} \left\{ \frac{\partial}{\partial \nu} [-A_{R,f}^{-1} h^* + \widetilde{D}_{f,s}(v_1^*|_{\Gamma_s})] - \frac{\partial v_1^*}{\partial \nu} \Big|_{\Gamma_s} \right\} \in \mathbf{H}^1(\Omega_s), \quad (5.77a)$$

$$v_2 = v_1^* \in \begin{cases} \mathbf{H}^1(\Omega_s) \setminus \mathbb{R} & \text{for } b = 0, \\ \mathbf{H}^1(\Omega_s) & \text{for } b = 1, \end{cases} \quad h = -A_{R,f}^{-1} h^* + \widetilde{D}_{f,s}(v_1^*|_{\Gamma_s}) \in \mathbf{H}^{\frac{3}{2}}(\Omega_f). \quad (5.77b)$$

Here, $A_{N,s}^{(b)}$ and $N_s^{(b)}$ are defined in (5.5) and (5.4). Moreover, the operator $-A_{R,f}$ is defined in (4.85). A new operator is the Dirichlet map $\widetilde{D}_{f,s}$ defined by

$$\widetilde{D}_{f,s}\mu = \psi \Leftrightarrow \begin{cases} \Delta\psi = 0 & \text{in } \Omega_f; \\ \psi|_{\Gamma_s} = \mu, \quad \left[\frac{\partial\psi}{\partial\nu} + \psi \right]_{\Gamma_f} = 0. \end{cases} \quad (5.78a)$$

$$\quad (5.78b)$$

In operator form, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \mathbb{A}_{F,D}^{(b)-1} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} (-A_{N,s}^{(b)})^{-1}(-\Delta v_1^* + v_2^*) + N_s^{(b)} \left\{ \frac{\partial}{\partial\nu} [-A_{R,f}^{-1} h^* + \widetilde{D}_{f,s}(v_1^*|_{\Gamma_s})] - \frac{\partial v_1^*}{\partial\nu} \Big|_{\Gamma_s} \right\} \\ v_1^* \\ -A_{R,f}^{-1} h^* + \widetilde{D}_{f,s}(v_1^*|_{\Gamma_s}) \end{bmatrix} \quad (5.79a)$$

$$= \begin{bmatrix} (-A_{N,s}^{(b)})^{-1}(-\Delta \cdot) + N_s^{(b)} \frac{\partial}{\partial\nu} \cdot \Big|_{\Gamma_s} & (-A_{N,s}^{(b)})^{-1} & -N_s^{(b)} \frac{\partial}{\partial\nu} [-A_{R,f}^{-1}] \\ 0 & I & 0 \\ \widetilde{D}_{f,s}(\cdot|_{\Gamma_s}) & 0 & -A_{R,f}^{-1} \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \quad (5.79b)$$

$$\|[v_1, v_2, h]\|_{\mathbb{H}} \leq c\|[v_1^*, v_2^*, h^*]\|_{\mathbb{H}}, \quad (5.79c)$$

where the operators $A_{N,s}^{(b)}$, $A_{R,f}$, $D_{f,f}$, and $N_s^{(b)}$, are defined in the following proof.

Proof. Identity (5.76) and the characterization of $\mathcal{D}(\mathbb{A}_{F,D}^{(b)})$ in (5.35a) and (5.35b) yield

$$\Delta h = h^* \in L^2(\Omega_f); \quad (5.80a)$$

$$v_2 = v_1^* \left\{ \left[\frac{\partial h}{\partial\nu} + h \right]_{\Gamma_f} = 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} = v_1^*|_{\Gamma_s}, \right. \quad (5.80b)$$

and the h -problem in (5.80) yields the solution h in (5.77b), invoking $A_{R,f}$ from (4.85) and $\widetilde{D}_{f,s}$ from (5.78). Moreover, (5.76), $v_2 = v_1^*$ in (5.80) and (5.35b) yield

$$\Delta(v_1 + v_2) - bv_1 = v_2^*, \quad \text{or} \quad \Delta v_1 - bv_1 = -\Delta v_1^* + v_2^*; \quad (5.81a)$$

$$\left\{ \frac{\partial(v_1 + v_2)}{\partial\nu} \Big|_{\Gamma_s} = \frac{\partial h}{\partial\nu} \Big|_{\Gamma_s}, \quad \text{or} \quad \frac{\partial v_1}{\partial\nu} \Big|_{\Gamma_s} = -\frac{\partial v_1^*}{\partial\nu} \Big|_{\Gamma_s} + \frac{\partial h}{\partial\nu} \Big|_{\Gamma_s}. \right. \quad (5.81b)$$

Then, the solution of problem (5.81) is given by (5.77a) via (5.77b). $\square$

Remark 5.2. A recent contribution of a heat-plate interaction with the plate subject to a (formal) “square root” damping in [46].

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Appendix A

The genuine fluid-structure interaction model with Kelvin-Voigt (viscoelastic) damping [2,12,14]

In this article, we have deliberately chosen to consider the simpler heat-viscoelastic structure model, as a first step of an entirely new investigation regarding the corresponding Luenberger theory. To be sure, replacing the heat equation with a fluid equation introduces conceptual and technical difficulties. These however have already been dealt with and ultimately resolved in prior work by one co-author, for a structure model originally without viscoelastic damping [12], and next with viscoelastic damping [2]. A first serious obstacle is faced at the very outset: because of the coupled nonhomogeneous boundary conditions involving the linearized Navier-Stokes equations, it is not possible to use the classical, by now standard idea of N-S problems with no-slip boundary conditions to eliminate the pressure by applying the Leray projector on the equation from $L^2(\Omega)$ onto the classical space $\{f \in (L^2(\Omega))^d : \operatorname{div} f = 0 \text{ in } \Omega_f; f \cdot \nu = 0 \text{ on } \partial\Omega_f\}$ [47, p. 7]. Accordingly, [12] introduces an entirely new idea that is inspired by boundary control theory. This is explained below in the context of the problem under present consideration.

We thus consider the following fluid-plate PDE model in solution variables

$$u = [u_1(t, x), u_2(t, x), \dots, u_d(t, x)] \quad (\text{the velocity field})$$

and

$$w = [w_1(t, x), w_2(t, x), \dots, w_d(t, x)] \quad (\text{the structural displacement field}),$$

while the scalar-valued p denotes the pressure:

$$\begin{aligned} \text{(PDE)} \quad \begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } (0, T] \times \Omega_f \equiv Q_f; \\ \operatorname{div} u = 0 & \text{in } Q_f; \\ w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T] \times \Omega_s \equiv Q_s; \end{cases} \end{aligned} \quad \begin{aligned} & \text{(A.1a)} \\ & \text{(A.1b)} \\ & \text{(A.1c)} \end{aligned}$$

$$\begin{aligned} \text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0, T] \times \Gamma_f \equiv \Sigma_f; \\ u = w_t & \text{on } (0, T] \times \Gamma_s \equiv \Sigma_s; \\ \frac{\partial u}{\partial \nu} - \frac{\partial(w + w_t)}{\partial \nu} = pv & \text{on } \Sigma_s; \end{cases} \end{aligned} \quad \begin{aligned} & \text{(A.1d)} \\ & \text{(A.1e)} \\ & \text{(A.1f)} \end{aligned}$$

$$\text{(IC)} \quad [u(0, \cdot), w(0, \cdot), w_t(0, \cdot)] = [u_0, w_0, w_1] \quad \text{on } \Omega. \quad \text{(A.1g)}$$

The constant b in (A.1c) will take up either the value $b = 0$ or else the value $b = 1$, as in the article. Accordingly, the space of well-posedness is taken to be the finite energy space:

$$\mathcal{H}_b \begin{cases} (H^1(\Omega_s)/\mathbb{R})^d \times (L^2(\Omega_s))^d \times \tilde{H}_f, & b = 0; \\ (H^1(\Omega_s))^d \times (L^2(\Omega_s))^d \times \tilde{H}_f, & b = 1, \end{cases} \quad \begin{aligned} & \text{(A.2a)} \\ & \text{(A.2b)} \end{aligned}$$

for the variable $[u, w, w_t]$, where

$$\tilde{H}_f = \{f \in (L^2(\Omega_f))^d : \operatorname{div} f \equiv 0 \text{ in } \Omega_f; f \cdot \nu \equiv 0 \text{ on } \Gamma_f\}. \quad \text{(A.3)}$$

The norm-including inner product on $\mathcal{H}_b$ is given in (1.2a)–(1.2b).

Abstract model for the free dynamics (A.1a)–(A.1g)

The previous article [12] (as well as paper [48], where the d -dimensional wave equation (A.1c) is replaced by the system of dynamic elasticity) eliminated the pressure by a completely different strategy. Following the idea of [49–52] (see also [4]), [12,48] identify a suitable elliptic problem for the pressure p , to be solved for p in terms of u , w and w_t .

Elimination of p , by expressing p in terms of u , w , and w_t . A key idea of [11,12,48] is that the pressure $p(t, x)$ solves the following elliptic problem on Ω_f in x , for each t :

$$\begin{cases} \Delta p \equiv 0 & \text{in } (0, T] \times \Omega_f \equiv Q_f; \end{cases} \quad (\text{A.4a})$$

$$\begin{cases} p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial(w + w_t)}{\partial \nu} \cdot \nu & \text{on } (0, T] \times \Gamma_s \equiv \Sigma_s; \end{cases} \quad (\text{A.4b})$$

$$\begin{cases} \frac{\partial p}{\partial \nu} = \Delta u \cdot \nu & \text{on } (0, T] \times \Gamma_f \equiv \Sigma_f. \end{cases} \quad (\text{A.4c})$$

In fact, (A.4a) is obtained by taking the divergence div across equation (A.1a), and using $\text{div } u_t \equiv 0$ in Q_f by (A.1b), as well as $\text{div } \Delta u = \Delta \text{div } u \equiv 0$ in Q_f . Next, the B.C. (A.4b) on Γ_s is obtained by taking the inner product of equation (A.1e) with ν . Finally, the B.C. (A.4c) on Γ_f is obtained by taking the inner product of equation (A.1a) restricted on Γ_f , with ν , using $u|_{\Gamma_f} \equiv 0$ by (A.1d), so that on Γ_f : $\nabla p \cdot \nu = \frac{\partial p}{\partial \nu}|_{\Gamma_f}$. This then results in (A.4c).

Explicit solution of problem (A) for p . We set

$$p = p_1 + p_2 \quad \text{in } Q_f, \quad (\text{A.5})$$

where p_1 and p_2 solve the following problems:

$$\begin{cases} \Delta p_1 \equiv 0 & \text{in } Q_f; \\ p_1 \equiv \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial(w + w_t)}{\partial \nu} \cdot \nu & \text{on } \Sigma_s; \\ \frac{\partial p_1}{\partial \nu} \Big|_{\Sigma_f} \equiv 0 & \text{on } \Sigma_f; \end{cases} \quad \begin{cases} \Delta p_2 \equiv 0 & \text{in } Q_f; \\ p_2 \equiv 0 & \text{on } \Sigma_s; \\ \frac{\partial p_2}{\partial \nu} \Big|_{\Sigma_f} \equiv \Delta u \cdot \nu & \text{on } \Sigma_f. \end{cases} \quad (\text{A.6a})$$

Accordingly, define the following “Dirichlet” and “Neumann” maps D_s and N_f :

$$h \equiv D_s g \Leftrightarrow \begin{cases} \Delta h \equiv 0 & \text{in } Q_f; \\ h = g & \text{on } \Gamma_s; \\ \frac{\partial h}{\partial \nu} \equiv 0 & \text{on } \Gamma_f; \end{cases} \quad \psi \equiv N_f \mu \Leftrightarrow \begin{cases} \Delta \psi \equiv 0 & \text{in } \Gamma_s; \\ \psi \equiv 0 & \text{on } \Gamma_s; \\ \frac{\partial \psi}{\partial \nu} \equiv \mu & \text{on } \Gamma_f. \end{cases} \quad (\text{A.7a})$$

Elliptic theory gives that D_s and N_f are well defined and possess the following regularity [43]:

$$D_s : \text{continuous } H^r(\Gamma_s) \rightarrow H^{r+\frac{1}{2}}(\Omega_f), \quad r \in \mathbb{R}, \quad (\text{A.8a})$$

$$N_f : \text{continuous } H^r(\Gamma_f) \rightarrow H^{r+\frac{3}{2}}(\Omega_f), \quad r \in \mathbb{R}. \quad (\text{A.8b})$$

Accordingly, in view of problems (A.7), we write the solutions p_1 and p_2 in (A.6), finally p in (A.5), as follows:

$$p_1 = D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial(w + w_t)}{\partial \nu} \cdot \nu \right) \Big|_{\Sigma_s} \right]; \quad p_2 = N_f[(\Delta u \cdot \nu)_{\Sigma_f}] \quad \text{in } Q_f, \quad (\text{A.9})$$

$$p = p_1 + p_2 = \Pi_1(w + w_t) + \Pi_2(u) \quad (\text{A.10a})$$

$$= D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial(w + w_t)}{\partial \nu} \cdot \nu \right) \Big|_{\Sigma_s} \right] + N_f[(\Delta u \cdot \nu)_{\Sigma_f}] \quad \text{in } Q_f, \quad (\text{A.10b})$$

where

$$\Pi_1(w + w_t) = -D_s \left[\left(\frac{\partial(w + w_t)}{\partial \nu} \cdot \nu \right) \Big|_{\Sigma_s} \right], \quad (\text{A.11a})$$

$$\Pi_2(u) = D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu \right) \Big|_{\Sigma_s} \right] + N_f[(\Delta u \cdot \nu)_{\Sigma_f}] \quad \text{in } Q_f, \quad (\text{A.11b})$$

hence via (A.10a) and (A.10b):

$$\nabla p = -G_1(w + w_t) - G_2(u) = \nabla \Pi_1(w + w_t) + \nabla \Pi_2(u) \quad (\text{A.12a})$$

$$= \nabla \left\{ D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial(w + w_t)}{\partial \nu} \cdot \nu \right) \right]_{\Sigma_s} \right\} + \nabla(N_f[(\Delta u \cdot \nu)_{\Sigma_f}]) \quad \text{in } Q_f; \quad (\text{A.12b})$$

where

$$G_1(w + w_t) = -\nabla \Pi_1(w + w_t) = \nabla \left\{ D_s \left[\left(\frac{\partial(w + w_t)}{\partial \nu} \cdot \nu \right) \right]_{\Sigma_s} \right\} \quad \text{in } Q_f, \quad (\text{A.13})$$

$$G_2(w) = -\nabla \Pi_2 u = \nabla \left\{ D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu \right) \right]_{\Sigma_s} \right\} + N_f[(\Delta u \cdot \nu)_{\Sigma_f}] \quad \text{in } Q_f, \quad (\text{A.14})$$

The linear maps G_1 and G_2 in (A.12)–(A.14) are introduced mostly for notational convenience. Equations (A.10a), (A.10b), and (A.12) have managed to eliminate the pressure p , and, more pertinently, its gradient ∇p , by expressing them in terms of the three key variables: the fluid velocity field u and the wave solution $\{w, w_t\}$. By using (A.12a), we accordingly rewrite the original model (A.1a)–(A.1g) as follows:

$$\begin{aligned} \text{(PDE)} \quad \begin{cases} u_t = \Delta u + G_1(w + w_t) + G_2 u & \text{in } Q_f; \\ \operatorname{div} u = 0 & \text{in } Q_f; \\ w_{tt} = \Delta w + \Delta w_t - bw & \text{in } Q_s; \end{cases} \end{aligned} \quad \begin{aligned} (\text{A.15a}) \\ (\text{A.15b}) \\ (\text{A.15c}) \end{aligned}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } \Sigma_f; \\ u = w_t & \text{on } \Sigma_s; \end{cases} \quad \begin{aligned} (\text{A.15d}) \\ (\text{A.15e}) \end{aligned}$$

$$\text{(IC)} \quad [u(0, \cdot), w(0, \cdot), w_t(0, \cdot)] = [u_0, w_0, w_1] \quad \text{on } \Omega, \quad (\text{A.15f})$$

only in terms of u , w , and w_t , where the pressure p has been eliminated, as desired.

Abstract model of system (A.15). The abstract model of system (A.15) is given by

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ G_1 & G_2 & \Delta + G_2 \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} \quad (\text{A.16a})$$

$$= \begin{bmatrix} w_t \\ \Delta(w + w_t) - bw \\ \Delta u + G_1(w + w_t) + G_2 u \end{bmatrix}, \quad (\text{A.16b})$$

$$[w(0), w_t(0), u(0)] = [w_0, w_1, u_0] \in \mathcal{H}_b, \quad (\text{A.16c})$$

where the matrix form for $\mathcal{A}$ on the L.H.S of (A.16a) is formal and means the action described in (A.16b).

The operator $\mathcal{A}_b$. Recalling (A.13) and (A.14) prompts the introduction of the operator

$$\mathcal{A} \equiv \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & -\rho \Delta^2 & 0 \\ G_1 & G_2 & \Delta + G_3 \end{bmatrix} \quad (\text{A.17a})$$

$$= \begin{bmatrix} 0 & I & 0 \\ \Delta^2 & \rho \Delta^2 & 0 \\ \nabla \{D_s[(\Delta \cdot) \cdot \nu]_{\Sigma_s}\} & a_{23} & a_{33} \end{bmatrix}, \quad (\text{A.17b})$$

$$a_{33} = \Delta - \nabla \left\{ D_s \left[\left(\frac{\partial \cdot}{\partial \nu} \cdot \nu \right) \right]_{\Sigma_s} \right\} + N_f[(\Delta \cdot) \cdot \nu]_{\Sigma_f}, \quad (\text{A.17c})$$

$$\mathcal{H}_b \supset \mathcal{D}(\mathcal{A}_b) \rightarrow \mathcal{H}_b. \quad (\text{A.17d})$$

The finite energy space $\mathcal{H}_b$ of well-posedness for problems (A.1a)–(A.1g), or its abstract version (A.16)–(A.17) is defined in (A). The domain $\mathcal{D}(\mathcal{A}_b)$ of $\mathcal{A}_b$ will be identified below. To this end, we find it convenient to introduce a function π , whose indicated regularity was ascertained in [12].

The scalar harmonic function π . Henceforth, with reference to (A.17b), for $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A})$, we introduce the harmonic function $\pi = \pi(v_1, v_2, f)$:

$$\pi \equiv D_s \left[\left(\frac{\partial f}{\partial \nu} \cdot \nu \right) \right]_{\Gamma_s} + N_f[(\Delta f \cdot \nu)_{\Gamma_f}] - D_s \left[\left(\frac{\partial(v_1 + v_2)}{\partial \nu} \cdot \nu \right) \right]_{\Gamma_s} \in L_2(\Omega_f) \quad (\text{A.18})$$

(compare with (A.10b) for the dynamic problem). According to the definition of the Dirichlet map D_s and Neumann map N_f given in (A.7a)–(A.7c), $\pi = \pi(v_1, v_2, f)$ in (A.18) can be equivalently given as the solution of the following elliptic problem (compare with (A.4a)–(A.4c) for the dynamic problem):

$$\begin{cases} \Delta \pi \equiv 0 & \text{in } \Omega_f; \end{cases} \quad (\text{A.19a})$$

$$\begin{cases} \pi = \frac{\partial f}{\partial \nu} \cdot \nu - \frac{\partial(v_1 + v_2)}{\partial \nu} \cdot \nu \in H^{-\frac{1}{2}}(\Gamma_s) & \text{on } \Gamma_s; \end{cases} \quad (\text{A.19b})$$

$$\begin{cases} \frac{\partial \pi}{\partial \nu} = \Delta f \cdot \nu \in H^{-\frac{3}{2}}(\Gamma_f) & \text{on } \Gamma_f. \end{cases} \quad (\text{A.19c})$$

It then follows from $\mathcal{A}_b$ in (A.17b) and (A.18) via the function π defined in (A.18) that

$$\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta f - \nabla \pi \end{bmatrix} \equiv \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \in \mathcal{H}_b, \quad \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{D}(\mathcal{A}_b). \quad (\text{A.20})$$