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A heat–structure interaction model with (formal) ‘square-root’ damping: analyticity and uniform stability

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Abstract In Part I, the present paper studies a homogeneous, uncontrolled 2D or 3D heat–structure interaction model, where the structure is modeled by an elastic system with (formally) ‘square-root’ damping, and where the two components are subject to high-level coupled conditions at the interface between the two media. Physically, the model occupies a doughnut-like domain: the heat (fluid) occupies the exterior domain, while the elastic structure occupies an interior subdomain. The novelty over past literature is the (formal) ‘square root’ damping of the structure versus either no damping at all or else Kelvin–Voigt (viscoelastic) damping. It is shown that such homogeneous (uncontrolled) model generates a strongly continuous contraction semigroup on a natural energy space, which moreover is analytic and uniformly stable. Next, the paper provides a characterization of the domain of a fractional power related to the generator. This result is then used to study, in Part II, the corresponding non-homogeneous model subject to control action at the interface between the two media and provide for it an optimal regularity result. The choice of the heat component over the (linearized) Navier–Stokes fluid component is only a preliminary step for initial simplicity. The fluid-model introduces serious conceptual and technical difficulties. How to overcome them has been accomplished in past literature [Avalos, G., Triggiani, R.: *AMS Contemp. Math. Fluids Waves* 440, p. 15-55 (2007); Triggiani, R.: *Mathematical Theory of Evolutionary Fluid-Flow Structure Interaction*. p. 53-172. vol. 48 (2018)] and will guide a subsequent publication.

Keywords Heat-structure interaction · Analyticity · Stability

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1 Introduction and statement of main results

Part I: The homogeneous (uncontrolled) problem

1.1 The homogeneous coupled PDE model

Throughout the paper, $\Omega_f \subseteq \mathbb{R}^d$, $d = 2$ or 3, will denote the bounded domain on which the fluid component of the coupled PDE system evolves. Its boundary will be denoted here as $\partial\Omega_f = \Gamma_s \cup \Gamma_f$, $\Gamma_s \cap \Gamma_f = \emptyset$, with each boundary piece being sufficiently smooth. Moreover, the geometry Ω_s, immersed within Ω_f, will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct fluid and elastic dynamics occurs across boundary interface $\Gamma_s = \partial\Omega_s$; see Fig. 1. In addition, the unit normal vector $\nu(x)$ will be directed away from Ω_f, and so *toward* Ω_s. (This specification of the direction of ν will influence the computations to be done below.)

On this geometry in Fig 1, we thus consider the following fluid (heat)–structure PDE model in solution variables $u = [u_1(t,x), u_2(t,x), \dots, u_d(t,x)]$ (the heat component here replacing the usual velocity field), and $w = [w_1(t,x), w_2(t,x), \dots, w_d(t,x)]$ (the structural displacement field):

$$\text{(PDE)} \begin{cases} u_t - \Delta u = 0 & \text{in } (0,T) \times \Omega_f \equiv Q_f; \quad (1.1\text{a}) \\ w_{tt} + \Delta^2 w - \Delta w_t = 0 & \text{in } (0,T) \times \Omega_s \equiv Q_s; \quad (1.1\text{b}) \end{cases}$$

$$\text{(BC)} \begin{cases} w\big|_{\Gamma_s} = 0, \quad \Delta w\big|_{\Gamma_s} = \dfrac{\partial u}{\partial \nu}\bigg|_{\Gamma_s} & \text{on } (0,T) \times \Gamma_s \equiv \Sigma_s; \quad (1.1\text{c}) \\ u\big|_{\Gamma_f} = 0; \quad u\big|_{\Gamma_s} = \dfrac{\partial w_t}{\partial \nu}\bigg|_{\Gamma_s} & \text{on } (0,T) \times \Gamma_s \equiv \Sigma_s. \quad (1.1\text{d}) \end{cases}$$

$$\text{(IC)} \ [w(0,\cdot), w_t(0,\cdot), u(0,\cdot)] = [w_0, w_1, u_0] \in \mathbf{H}. \quad (1.1\text{e})$$

The space $\mathbf{H}$ of well-posedness for $\{w, w_t, u\}$ is taken to be the finite energy space

$$\mathbf{H} = \mathcal{D}(A_{D,s}) \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f), \quad (1.2\text{a})$$

where, on the basis of the B.C. $w\big|_{\Gamma_s} = 0$, $A_{D,s} : \mathbf{L}^2(\Omega_s) \supset \mathcal{D}(A_{D,s}) \to \mathbf{L}^2(\Omega_s)$ is defined by the positive self-adjoint operator

$$A_{D,s} f = -\Delta f, \quad \mathcal{D}(A_{D,s}) = \mathbf{H}^2(\Omega_s) \cap \mathbf{H}^1_0(\Omega_s). \quad (1.2\text{b})$$

$\mathbf{H}$ is a Hilbert space with the following norm-inducing inner product, where $(f,g)_\Omega = \int_\Omega f\overline{g}\,d\Omega$:

$$\left(\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{h} \end{bmatrix} \right)_{\mathbf{H}} = (\Delta v_1, \Delta \hat{v}_1)_{\Omega_s} + (v_2, \hat{v}_2)_{\Omega_s} + (h, \hat{h})_{\Omega_f}. \quad (1.3)$$

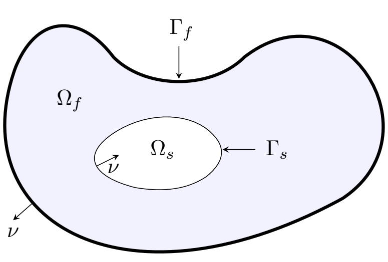

Fig. 1 The heat–structure interaction

Abstract model of Problem (1.1). The operator $\mathcal{A}$. We rewrite problem (1.1a)–(1.1e) as a first-order equation

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} w \\ w_t \\ h \end{bmatrix} = \mathcal{A}\begin{bmatrix} w \\ w_t \\ h \end{bmatrix}, \tag{1.4}$$

where we have introduced the operator $\mathcal{A} : \mathbf{H} \supset \mathcal{D}(\mathcal{A}) \to \mathbf{H}$:

$$\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -\Delta(\Delta v_1 - v_2) \\ \Delta h \end{bmatrix} \subset \mathbf{H} \tag{1.5}$$

for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$. A description of $\mathcal{D}(\mathcal{A})$ is as follows:

(i) $v_1, v_2 \in \mathcal{D}(A_{D,s}) \equiv \mathbf{H}^2(\Omega_s) \cap \mathbf{H}^1_0(\Omega_s)$, so that

$$v_1\big|_{\Gamma_s} = 0, \qquad v_2\big|_{\Gamma_s} = 0, \qquad \frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s). \tag{1.6}$$

$$\begin{cases} \Delta(\Delta v_1 - v_2) \in \mathbf{L}^2(\Omega_s) & (1.7a) \\ \left[\Delta v_1 - v_2\right]_{\Gamma_s} = \Delta v_1\big|_{\Gamma_s} = \dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s); & (1.7b) \end{cases}$$

whose solution then satisfies $\Delta v_1 - v_2 = F \in \mathbf{L}^2(\Omega_s)$ via elliptic theory, so that via (1.6)

$$\begin{cases} \Delta v_1 = v_2 + F \in \mathbf{L}^2(\Omega_s), \quad \text{or} \quad -A_{D,s} v_1 = v_2 + F \in \mathbf{L}^2(\Omega_s) \\ v_1\big|_{\Gamma_s} = 0, \end{cases}$$

and $v_1 = -A^{-1}_{D,s}[v_2 + F] \in \mathcal{D}(A_{D,s})$, confirming the statement (1.6) above.

(ii)

$$\begin{cases} \Delta h \in \mathbf{L}^2(\Omega_f) & (1.8a) \\ h\big|_{\Gamma_f} = 0, \quad h\big|_{\Gamma_s} = \dfrac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s). & (1.8b) \end{cases}$$

The h-problem then implies via elliptic theory

$$h \in \mathbf{H}^1(\Omega_f) \quad \text{and} \quad \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \text{ [33, (1.9)]} \tag{1.8c}$$

as anticipated in (1.7b). The implication on $\dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s}$ is reviewed in Appendix B.

In conclusion

$$\mathcal{D}(\mathcal{A}) = \Bigg\{[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f) \subset \mathbf{H}, \text{ so that } v_1\bigg|_{\Gamma_s} = 0, \quad v_2\bigg|_{\Gamma_s} = 0 : \tag{1.8d}$$
$$\begin{cases} \Delta h \in \mathbf{L}^2(\Omega_f) \\ h\bigg|_{\Gamma_f} = 0, \quad h\bigg|_{\Gamma_s} = \dfrac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s), \end{cases} \quad \text{hence } \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$$
$$\begin{cases} \Delta(\Delta v_1 - v_2) \in \mathbf{L}^2(\Omega_s), \quad \text{that is} \quad \Delta^2 v_1 \in \mathbf{L}^2(\Omega_s) \\ \Delta v_1\bigg|_{\Gamma_s} = \dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \end{cases} \Bigg\}.$$

In conclusion, for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, we have: $v_2 \in \mathcal{D}(A_{D,s})$ is compact in its component space $\mathbf{L}^2(\Omega_s)$, $h \in \mathbf{H}^1(\Omega_f)$ is compact in its component space $\mathbf{L}^2(\Omega_f)$, but v_1 is only in its component space $\mathcal{D}(A_{D,s})$. This indicates that $\mathcal{A}^{-1}\mathbf{H}$ does not produce smoothing and thus that $\mathcal{A}$ does not have compact resolvent.

Literature The present paper belongs to the general research area of fluid–structure interaction models, where a fluid is coupled with an elastic structure, thus with parabolic/hyperbolic coupling, which in fact takes place at the interface between the two domains. Both configurations have been considered: the case of static interface (a model considered physically acceptable in the case of small rapid oscillations) as well as the mathematically more challenging case of a moving interface, which leads to a quasilinear system. Both linear and non-linear (full Navier–Stokes equations) models have been studied. An early reference where such topic is mentioned (but not analyzed) is Lions [34]. A more specific linear model with static interface was later proposed and studied in [18]. It was probably this paper that triggered interest in this research area. While we cannot be exhaustive, we quote [3–9] for a rather complete, comprehensive, optimal analysis of the linear model introduced in [18] with no (or only frictional) damping and static interface, an extended account of which is given in the lengthy article [45]. Paper [3] introduces a trick or technique on how to eliminate the pressure, as the classical Leray projection approach is no longer valid due to the coupling at the interface. We quote also [30], [36] for corresponding non-linear models; and [22–24] for models with moving interface. The first work that introduces the (physically relevant) case of a wave structure with a (strong) Kelvin–Voigt (viscoelastic) damping is [33]. As a preliminary step, such equation is coupled with the heat equation rather than a genuine fluid. The subsequent case where the heat component is replaced by a (linearized) Navier–Stokes component is taken up and studied in [37] by employing the technique of [3,45] to eliminate the pressure. Subsequent works involved plate-like models for the structure, initially coupled with a heat component [46,47], and next replacing the heat equation with a fluid equation (the linearized Navier–Stokes equations [38], using [3]). A heat-wave optimal stability result is in [2].

The present work is the first study in the literature where the elastic structure is endowed with a (formal) 'square root' damping: The 'plate-type' structure has the double Laplacian acting on the displacement of the structure and the Laplacian (with the correct minus sign) acting on the velocity of the structure. While the original models [3–9,25–28] with no damping display a parabolic (fluid)–hyperbolic (elastic structure) coupling and ultimately lead to a strongly continuous contraction semigroup on the natural energy space describing the overall coupled system, the cases mentioned above [33,37,38] where the structure is endowed with viscoelastic Kelvin–Voigt damping ultimately generate an s.c. contraction semigroup which moreover is analytic on its natural finite energy space. The present case with (formal) 'square root' damping also produces analyticity of the natural contraction semigroup. All this is in line with the abstract theory in Hilbert space on the wave equation with various degrees of strong damping, studied in [14–17], as discussed in the Orientation on analyticity below Theorem 1.3.

Subsequent work will replace the heat equation by the natural fluid equation by employing the technique of [3,45] to eliminate the pressure.

Since the final evolution in the present paper is described by an analytic contraction semigroup, it is then of interest to seek to characterize the domains of fractional power related to its generator, to study the optimal regularity of the control-solution operator, when the control function acts at the interface between the two domains. This is done here in Part II, for brevity only for the fractional power $\theta = \frac{1}{2}$.

Models involving fluid–structure interaction in blood flows, where now the fluid domain is internal to the structure domain, were also studied, e.g., [11,12]. Papers where the domain of fractional powers in the parabolic/parabolic fluid–structure interaction models with Kelvin–Voigt damping include [32,42–44]. Higher regularity for a parabolic-hyperbolic fluid-structure system is given in [1].

1.2 Main result

A main result of the present paper is that the operator $\mathcal{A}$ in (1.5)–(1.8) is the generator of an s.c. contraction semigroup $e^{\mathcal{A}t}$ on the energy space $\mathbf{H}$ in (1.2a) which moreover is analytic (holomorphic) as well as uniformly stable. A similar result holds also for the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ on $\mathbf{H}$ to be defined in Sect. 1.3.

Theorem 1.1 (i) *The operator $\mathcal{A}$ in (1.5)–(1.8) is dissipative on the space* $\mathbf{H}$ *in (1.2a), topologized by (1.3): for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$ described in (1.8d), we have*

$$Re\left(\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}} = -\|\nabla v_2\|^2 - \|\nabla h\|^2 \le 0, \quad [v_1, v_2, h] \in \mathcal{D}(\mathcal{A}). \tag{1.9}$$

(ii) *The origin belongs to the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A} : 0 \in \rho(\mathcal{A})$. Thus, $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H})$. The explicit expression of $\mathcal{A}^{-1}$ is given below in (1.25). Then, there exists a $r_0 > 0$, such that $\mathcal{S}_{r_0}(0) \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, where $\mathcal{S}_{r_0}(0)$ is the open disk centered at the origin and of radius r_0.*

(iii) *Thus, the operator $\mathcal{A}$ is maximal dissipative on* $\mathbf{H}$*, and so, it generates a s.c. contraction semigroup $e^{\mathcal{A}t}$ on* $\mathbf{H}$

$$\begin{bmatrix} \omega_0 \\ w_1 \\ u_0 \end{bmatrix} \in \mathbf{H} \to e^{\mathcal{A}t}\begin{bmatrix} \omega_0 \\ w_1 \\ u_0 \end{bmatrix} \in C([0, T]; \mathbf{H}). \tag{1.10}$$

Proof (i) Let $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$ as described in (1.8d). Via (1.5), we compute, recalling the topology (1.3) and using at first Green's second theorem on Ω_s where $v_2\big|_{\Gamma_s} = 0$ by (1.6), and next Green's first theorem on Ω_f for h, where $h\big|_{\Gamma_f} = 0$ by (1.8b). By (1.5), we obtain (since the unit normal ν is outward to Ω_f on both Γ_f and Γ_s)

$$\left(\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}} = \left(\begin{bmatrix} v_2 \\ -\Delta(\Delta v_1 - v_2) \\ \Delta h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}} \tag{1.11}$$

$$= (\Delta v_2, \Delta v_1) - (\Delta(\Delta v_1 - v_2), v_2) + (\Delta h, h) \tag{1.12}$$

$$= (\Delta v_2, \Delta v_1) - \left\{(\Delta v_1 - v_2, \Delta v_2) - \cancel{\left(\frac{\partial(\Delta v_1 - v_2)}{\partial \nu}, v_2\right)_{\Gamma_s}}\right.$$
$$\left. + \left(\Delta v_1, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s} + \cancel{\left(-v_2, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}}\right\} + \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} - \|\nabla h\|^2 \tag{1.13}$$

$$= (\Delta v_2, \Delta v_1) - \left\{(\Delta v_1, \Delta v_2) + \|\nabla v_2\|^2 - \cancel{\left(v_2, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}}\right.$$
$$\left. + \left(\Delta v_1, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}\right\} + \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} - \|\nabla h\|^2 \tag{1.14}$$

$$= (\Delta v_2, \Delta v_1) - (\Delta v_1, \Delta v_2) - \|\nabla v_2\|^2 - \|\nabla h\|^2$$
$$- \cancel{\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}} + \cancel{\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}}. \tag{1.15}$$

Moreover, to go from (1.13) to (1.14), we have used Green's first theorem on $(v_2, \Delta v_2)$ with $v_2\Big|_{\Gamma_s} = 0$ by (1.6). Finally, to go from (1.14) to (1.15), we have recalled the BC $\Delta v_1 = \frac{\partial h}{\partial \nu}$ on Γ_s in (1.7b) and $h = \frac{\partial v_2}{\partial \nu}$ on Γ_s in (1.8b). Taking now the Re part in (1.11)–(1.15) and using $\text{Re}(z - \bar{z}) = 0$, $z = (\Delta v_2, \Delta v_1)$, we obtain (1.9).

(ii) By way of illustration, we first show that $0 \notin \sigma_p(\mathcal{A})$, the point spectrum of $\mathcal{A}$. Let $\mathcal{A}\mathbf{x} = 0$, $\mathbf{x} = [v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, and conclude that $\mathbf{x} = 0$. In fact, by (1.5), $\mathcal{A}\mathbf{x} = 0$ implies $v_2 \equiv 0$ in Ω_s; hence, $h\big|_{\Gamma_s} = \left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s} = 0$ by (1.8b). This, along with $\Delta h \equiv 0$ in Ω_f by (1.5) and $h\big|_{\Gamma_f} = 0$ by (1.8b), implies $h \equiv 0$ in Ω_f. Then,

$\Delta v_1\big|_{\Gamma_s} = \dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} = 0$ by (1.7b). This together with $\Delta(\Delta v_1) \equiv 0$ in Ω_s by (1.5) implies $\Delta v_1 \equiv 0$ in Ω_s which along with $v_1\big|_{\Gamma_s} = 0$ by (1.6) yields $v_1 \equiv 0$ in Ω_s. Hence, $\mathbf{x} = 0$ as desired. A similar argument yields $0 \notin \sigma_p(\mathcal{A}^*)$, with $\mathcal{A}^*$ defined below in (1.26). This then implies [41, p 282] that $0 \notin \sigma_r(\mathcal{A})$, the residual spectrum of $\mathcal{A}$. We now show the full statement that $0 \in \rho(\mathcal{A})$, or $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H})$. Let $[v_1^*, v_2^*, h^*] \in \mathbf{H}$. Recalling (1.6), we seek to solve

$$\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -\Delta(\Delta v_1 - v_2) \\ \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \tag{1.16}$$

uniquely for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, with inverse $\mathcal{A}^{-1}$ being bounded in $\mathcal{L}(\mathbf{H})$. By (1.16), (1.5)–(1.8), we first obtain

$$v_2 = v_1^* \in \mathcal{D}(A_{D,s}), \quad \begin{cases} \Delta h = h^* \in \mathbf{L}^2(\Omega_f) & \text{(1.17a)} \\ h\big|_{\Gamma_f} = 0,\ h\big|_{\Gamma_s} = \dfrac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} = \dfrac{\partial v_1^*}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s), & \text{(1.17b)} \end{cases}$$

so that $h \in \mathbf{H}^1(\Omega_f)$ and $\dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$ [33, (1.9)], and Appendix B, see (B.4). Thus, the unique solution of problem (1.17) in terms of the data is

$$h = -A_{D,f}^{-1} h^* + \widetilde{D}_f \left(\frac{\partial v_1^*}{\partial \nu}\bigg|_{\Gamma_s} \right) \in \mathbf{H}^1(\Omega_f), \tag{1.18}$$

where the positive self-adjoint operator $A_{D,f}$ and the Dirichlet map $\widetilde{D}_f$ on Ω_f are

$$A_{D,f}\phi = -\Delta\phi, \qquad \mathcal{D}(A_{D,f}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f) \tag{1.19a}$$

$$\widetilde{D}_f g = q \iff \left\{ \Delta q = 0 \text{ in } \Omega_f, \quad q\big|_{\Gamma_f} = 0, \quad q\big|_{\Gamma_s} = g \right\} \tag{1.19b}$$

$$\widetilde{D}_f : \mathbf{H}^r(\partial\Omega_f) \to \mathbf{H}^{r+\frac{1}{2}}(\Omega_f) \text{ continuously for any } r \in \mathbb{R} \quad [35] \tag{1.19c}$$

$$\widetilde{D}_f : \mathbf{L}^2(\partial\Omega_f) \to \mathbf{H}^{\frac{1}{2}}(\Omega_f) \subset \mathbf{H}^{\frac{1}{2}-2\epsilon}(\Omega_f) \equiv \mathcal{D}(A_{D,f}^{\frac{1}{4}-\epsilon}) \quad [31\text{, p. 181], [42]} \tag{1.19d}$$

continuously. A next implication of (1.16) is by (1.5) and (1.7b)

$$\begin{cases} \Delta(\Delta v_1 - v_2) = -v_2^* \in \mathbf{L}^2(\Omega_s) \\ \Delta v_1\big|_{\Gamma_s} = \dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \end{cases}$$

or by $v_2 = v_1^*$, $v_2\big|_{\Gamma_s} = v_1^*\big|_{\Gamma_s} = 0$ via (1.16), (1.6)

$$\begin{cases} \Delta(\Delta v_1 - v_1^*) = -v_2^* \in \mathbf{L}^2(\Omega_s) & \text{(1.20a)} \\ \left[\Delta v_1 - v_1^*\right]_{\Gamma_s} = \dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) & \text{(1.20b)} \end{cases}$$

$$\begin{cases} \Delta v_1 - v_1^* = A_{D,s}^{-1} v_2^* + D_s \left(\dfrac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \right) & \text{(1.21a)} \\ v_1\big|_{\Gamma_s} = 0 & \text{(1.21b)} \end{cases}$$

with h given by (1.18) in terms of $\{h^*, v_1^*\}$. Here, the Dirichlet map D_s on Ω_s is defined by

$$D_s g = f \iff \left\{\Delta f = 0 \text{ in } \Omega_s;\; f\big|_{\Gamma_s} = g\right\} \tag{1.22a}$$

$$D_s : \mathbf{H}^r(\Gamma_s) \to \mathbf{H}^{r+\frac{1}{2}}(\Omega_s) \text{ continuously for any } r \in \mathbb{R} \quad [35] \tag{1.22b}$$

$$D_s : \mathbf{L}^2(\Gamma_s) \to \mathbf{H}^{\frac{1}{2}}(\Omega_s) \subset \mathbf{H}^{\frac{1}{2}-\epsilon}(\Omega_s) \equiv \mathcal{D}(A_{D,f}^{\frac{1}{4}-\epsilon}) \quad [31, \text{p. } 181], [42]. \tag{1.22c}$$

Hence, recalling $A_{D,s}$ in (1.2b), we obtain from (1.21), (1.18)

$$v_1 = -A_{D,s}^{-1}\left\{v_1^* + A_{D,s}^{-1}v_2^* + D_s\frac{\partial}{\partial\nu}\left[-A_{D,f}^{-1}h^* + \widetilde{D}_f\left(\frac{\partial v_1^*}{\partial\nu}\bigg|_{\Gamma_s}\right)\right]\right\}. \tag{1.23}$$

Thus, putting together (1.16), (1.17), (1.18), and (1.23), we can write explicitly the operator $\mathcal{A}^{-1}$ as

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \mathcal{A}^{-1}\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}$$

$$= \begin{bmatrix} -A_{D,s}^{-1}\left\{v_1^* + D_s\frac{\partial}{\partial\nu}\widetilde{D}_f\left(\frac{\partial v_1^*}{\partial\nu}\bigg|_{\Gamma_s}\right)\right\} - A_{D,s}^{-2}v_2^* + A_{D,s}^{-1}D_s\frac{\partial}{\partial\nu}(A_{D,f}^{-1}h^*) \\ v_1^* \\ -A_{D,f}^{-1}h^* + \widetilde{D}_f\left(\frac{\partial v_1^*}{\partial\nu}\bigg|_{\Gamma_s}\right) \end{bmatrix} \tag{1.24}$$

$$\mathcal{A}^{-1}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -A_{D,s}^{-1}\left[I + D_s\frac{\partial}{\partial\nu}\widetilde{D}_f\frac{\partial}{\partial\nu}(\cdot)\right] & -A_{D,s}^{-2} & A_{D,s}^{-1}D_s\frac{\partial}{\partial\nu}A_{D,f}^{-1} \\ I & 0 & 0 \\ \widetilde{D}_f\left(\frac{\partial(\cdot)}{\partial\nu}\bigg|_{\Gamma_s}\right) & 0 & -A_{D,f}^{-1} \end{bmatrix}\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}. \tag{1.25}$$

□

1.3 The adjoint operator $\mathcal{A}^*$ on $\mathbf{H}$ and its maximal dissipativity

Theorem 1.2 (i) *The adjoint $\mathcal{A}^*$ of the operator $\mathcal{A}$ in (1.5) with respect to the space $\mathbf{H}$ in (1.2a) is given by*

$$\mathcal{A}^*\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} -v_2^* \\ \Delta(\Delta v_1^* + v_2^*) \\ \Delta h^* \end{bmatrix} \subset \mathbf{H} \tag{1.26}$$

for $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}^*)$. A description of $\mathcal{D}(\mathcal{A}^*)$ is as follows:

(ii) *$v_1^*, v_2^* \in \mathcal{D}(A_{D,s}) = \mathbf{H}^2(\Omega_s) \cap \mathbf{H}^1(\Omega_s)$, so that*

$$v_1^*\big|_{\Gamma_s} = 0, \quad v_2^*\big|_{\Gamma_s} = 0, \quad \frac{\partial v_2^*}{\partial\nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) \tag{1.27}$$

$$\begin{cases} \Delta(\Delta v_1^* + v_2^*) \in \mathbf{L}^2(\Omega_s) & (1.28\text{a}) \\ [\Delta v_1^* + v_2^*]_{\Gamma_s} = \Delta v_1^*\big|_{\Gamma_s} = -\dfrac{\partial h^*}{\partial\nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s), & (1.28\text{b}) \end{cases}$$

$$\begin{cases} \Delta h^* \in \mathbf{L}^2(\Omega_f) & \text{(1.29a)} \\ h^*\big|_{\Gamma_f} = 0, \quad h^*\big|_{\Gamma_s} = \dfrac{\partial v_2^*}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s) & \text{(1.29b)} \end{cases}$$

so that the h^*-problem implies

$$h^* \in \mathbf{H}^1(\Omega_f) \quad \textit{and} \quad \frac{\partial h^*}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \quad \textit{[33, (1.9)], Appendix B, see (B.4).} \tag{1.30}$$

(iii) *The operator $\mathcal{A}^*$ in (1.26) is dissipative*

$$Re\left(\mathcal{A}^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}\right)_{\mathbf{H}} = -\|\nabla v_2^*\|^2 - \|\nabla h^*\|^2 \leq 0, \quad [v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}^*). \tag{1.31}$$

(iv) *Since both $\mathcal{A}$ and $\mathcal{A}^*$ (densely defined, closed) are dissipative, then they are both maximal dissipative [39, Cor. 4.4, p. 15], and so, they generate s.c. semigroups $e^{\mathcal{A}t}$ and $e^{\mathcal{A}^*t}$ of contraction on $\mathbf{H}$, $t \geq 0$.*

The proof of Theorem 1.2 is given in Appendix A.

Before stating the next two main results, we recall the following.

Classical Poincare inequality on Ω_f and Ω_s. First, if $\phi \in \mathbf{H}^1(\Omega_f)$ with ϕ vanishing on a portion of $\partial\Omega_f = \Gamma_f \cup \Gamma_s$, and if likewise $\psi \in \mathbf{H}^1(\Omega_s)$ with ψ vanishing on a portion of $\partial\Omega_s = \Gamma_s$, then Poincare inequality on Ω_f and Ω_s says that there is a positive constant $c_{p,f}$ depending only on Ω_f, and there is a positive constant $c_{p,s}$ depending only on Ω_s, such that

$$c_{p,f}\|\phi\|^2 \leq \|\nabla\phi\|^2; \qquad c_{p,s}\|\psi\|^2 \leq \|\nabla\psi\|^2. \tag{1.32}$$

Setting $c_p = \min\{c_{p,f}, c_{p,s}\} > 0$ and recalling that for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$ we have $v_1\Big|_{\Gamma_s} = v_2\Big|_{\Gamma_s} = 0$ by (1.6) and $h\Big|_{\Gamma_f} = 0$ by (1.8b), we then obtain

$$c_p[\|v_2\|^2 + \|h\|^2] \leq \|\nabla v_2\|^2 + \|\nabla h\|^2. \tag{1.33}$$

1.4 Uniform bound on $R(\alpha + i\omega, \mathcal{A})$ near the imaginary axis, and uniform stability of the semigroup $e^{\mathcal{A}t}$ on $\mathbf{H}$

Let $\alpha \leq 0$, $\omega \in \mathbb{R}$. Let $[v_1^*, v_2^*, h^*] \in \mathbf{H}$. Via (1.5), we seek to solve

$$((\alpha + i\omega)I - \mathcal{A}) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} (\alpha + i\omega)v_1 - v_2 \\ (\alpha + i\omega)v_2 + \Delta(\Delta v_1 - v_2) \\ (\alpha + i\omega)h - \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \tag{1.34}$$

in terms of $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, with a uniform bound for the resolvent $((\alpha + i\omega)I - \mathcal{A})^{-1}$ on a vertical strip, as in (1.36) below

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(\alpha + i\omega, \mathcal{A}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}. \tag{1.35}$$

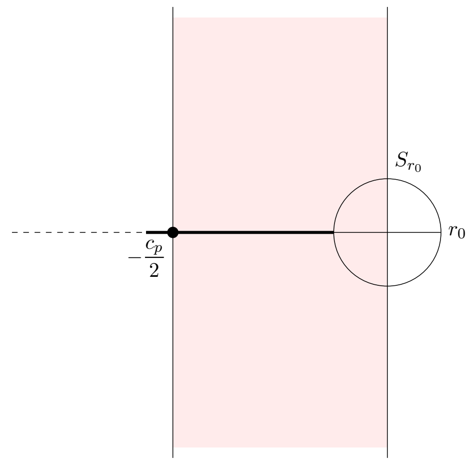

Fig. 2 The shaded area is in $\rho(\mathcal{A})$ by Theorem 1.3

Theorem 1.3 *Let c_p be the constant in (1.33) and let $\omega_0 > 0$ arbitrary. There exists a positive constant k_{c_P,ω_0} depending on c_p and ω_0, such that*

$$\|R(\alpha + i\omega, \mathcal{A})\|_{\mathcal{L}(\mathbf{H})} \leq k_{c_P,\omega_0}, \quad -\frac{c_p}{2} < \alpha \leq 0, |\omega| \geq \omega_0 > 0. \tag{1.36}$$

Thus, the specialization of (1.36) to the imaginary axis, i.e., $\alpha = 0$, combined with the property that $0 \in \rho(\mathcal{A})$, as in Theorem 1.1 (ii) yields

$$\|R(i\omega, \mathcal{A})\|_{\mathcal{L}(\mathbf{H})} \leq constant, \quad \forall\, \omega \in \mathbb{R}. \tag{1.37}$$

Thus, in view of [40], the s.c. contraction semigroup $e^{\mathcal{A}t}$ on $\mathbf{H}$ asserted by Theorem 1.1 (iii) is exponentially stable: there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| e^{\mathcal{A}t} \right\|_{\mathcal{L}(\mathbf{H})} \leq Me^{-\delta t}, \quad t \geq 0. \tag{1.38}$$

After Theorem 1.4, we can take $\delta = r_0 > 0$ (Fig. 2).

The proof is given in Sect. 2.

Orientation on analyticity A main result of the present paper is Theorem 1.4 that claims that the s.c. semigroup of contraction $e^{\mathcal{A}t}$ asserted by Theorem 1.1 (iii) is, in fact, analytic on the space $\mathbf{H}$ in (1.2a), and uniformly stable by Theorem 1.3.

Analyticity per se is not totally surprising in view of the following motivating considerations.

A motivating result: Analyticity The following is *a very special case of a much more general result* (noted below) for which we refer to [14,15], (see also [31, Appendix 3B of Chapter 3, pp. 285–296], [16,17]). These references solve and improve upon the conjectures posted in [13]. Let A be a positive, self-adjoint operator on the Hilbert space Y. On it, consider the following abstract equation with square-root damping:

$$\ddot{x} + Ax + A^{\frac{1}{2}}\dot{x} = 0; \text{ or } \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \mathbb{A}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}; \tag{1.39}$$

$$\mathbb{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -A^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -A^{\frac{1}{2}}(A^{\frac{1}{2}}x_1 + x_2) \end{bmatrix}; \tag{1.40a}$$

$$\mathcal{D}(\mathbb{A}) = \left\{[x_1, x_2] \in E \equiv \mathcal{D}(A^{\frac{1}{2}}) \times Y : \ x_2 \in \mathcal{D}(A^{\frac{1}{2}}), \ A^{\frac{1}{2}}x_1 + x_2 \in \mathcal{D}(A^{\frac{1}{2}})\right\}. \tag{1.40b}$$

The operator $\mathbb{A}$ is dissipative and with domain (1.40b) is closed and generates a s.c. contraction semigroup $e^{\mathbb{A}t}$ on the finite energy space $E \equiv \mathcal{D}(A^{\frac{1}{2}}) \times Y$, which moreover is analytic on E. This is a special case of the conjecture that was advanced in [13], and which was proved in [14] and generalized in [15]. Thus, the second-order dynamic (1.39) with square-root damping is parabolic-like. Indeed, [14,15], (see also [31, Appendix 3B of Chapter 3, pp. 285–296]) show the following more general result, and more useful in application to mixed PDEs-problems: that analyticity holds true if in Eq. (1.39) the damping term $A^{\frac{1}{2}}\dot{x}$ is replaced by $B\dot{x}$, where B is another positive self-adjoint operator (which needs not commute with A) which is comparable with A^{α}, $\frac{1}{2} \le \alpha \le 1$, in the sense of inner product: $\rho_1 A^{\alpha} \le B \le \rho_2 A^{\alpha}$, $0 < \rho_1 < \rho_2 < +\infty$.

The above abstract result for Eq. (1.40a) where the damping operator $A^{\frac{1}{2}}$ is the square root of the free dynamic operator A, may suggest or make one surmise, that Eq. (1.1b) is "parabolic", as the damping $(-\Delta)$ is the "formal square root" of the free dynamic operator Δ^2. As a consequence, the homogeneous problem (1.1a)–(1.1e) is then the coupling of "two parabolic problems", and hence, the operator $\mathcal{A}$ in (1.5) generates an analytic semigroup $e^{\mathcal{A}t}$ on the finite energy space $\mathbf{H}$ in (1.2a). Of course, the above considerations are purely indicative and qualitatively suggestive, as unlike the above operator A, the Laplacian Δ in (1.1b) is a differential operator with coupled, high-level, non-homogeneous interface boundary conditions which constitute the crux of the matter to be resolved before making the assertion of analyticity of problem (1.1a)–(1.1e). At any rate, analyticity cannot follow by a perturbation argument; see Appendix C.

Theorem 1.3 says in particular that $i\mathbb{R} \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. It raises the question, which is answered affirmatively in the following main result, and which improves upon Theorem 1.3 for $\alpha = 0$.

Theorem 1.4 *The following resolvent estimate holds true:*

$$\|R(i\omega, \mathcal{A})\|_{\mathcal{L}(\mathbf{H})} \le \frac{c}{\omega_0}, \quad \text{for all } \omega \in \mathbb{R},\ |\omega| \ge \text{some arbitrarily small } \omega_0 > 0, \tag{1.41}$$

equivalently, via (1.35) with $\alpha = 0$ recalling the $\mathbf{H}$-topology (1.3):

$$\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2 \le \frac{c}{\omega_0}\left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right]$$
$$\text{for all } \omega \in \mathbb{R},\ |\omega| \ge \text{some arbitrarily small } \omega_0 > 0. \tag{1.42}$$

Hence, the s.c. semigroup of contraction $e^{\mathcal{A}t}$ of Theorem 1.1 (iii) is analytic (holomorphic) on the energy space $\mathbf{H}$ in (1.2a) [31, Thm 3E.3, p. 334]. Accordingly [31, p. 335], there exists a constant $M > 0$, such that

$$\|R(\lambda, \mathcal{A})\|_{\mathcal{L}(\mathbf{H})} \le \frac{M}{|\lambda|}, \quad \lambda \ne 0,\ \forall\, \lambda \in \Sigma_{\theta_1}, \tag{1.43}$$

$$\Sigma_{\theta_1} = \left\{\lambda \in \mathbb{C} : |arg\lambda| \le \frac{\pi}{2} + \theta_1\right\}, \tag{1.44}$$

where one may take the angle θ_1, $0 < \theta_1 < \frac{\pi}{2}$, such that $\tan(\frac{\pi}{2} - \theta_1) = \frac{c}{\rho}$, with c the constant in (1.41), for an arbitrary fixed constant $0 < \rho < 1$. See Fig. 3.

The proof of Theorem 1.4 is given in Sect. 3.

Theorem 1.5 *For the maximal dissipative operator $\mathcal{A}$ with $0 \in \rho(\mathcal{A})$, we have*

$$\mathcal{D}((-\mathcal{A})^{\frac{1}{2}}) = [\mathcal{D}(\mathcal{A}), \mathbf{H}]_{\frac{1}{2}} \qquad (Intermediate\ space[35, p.\ 10]), \tag{1.45a}$$

where

$$\mathcal{D}((-\mathcal{A})^{\frac{1}{2}}) = \Bigg\{[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f),\ \text{so that}$$
$$v_1\Big|_{\Gamma_s} = 0,\quad v_2\Big|_{\Gamma_s} = 0 : h\Big|_{\Gamma_f} = 0,\quad h\Big|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s),$$
$$\Delta v_1\Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s),\quad \Delta v_1 - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_s)\Bigg\}. \tag{1.45b}$$

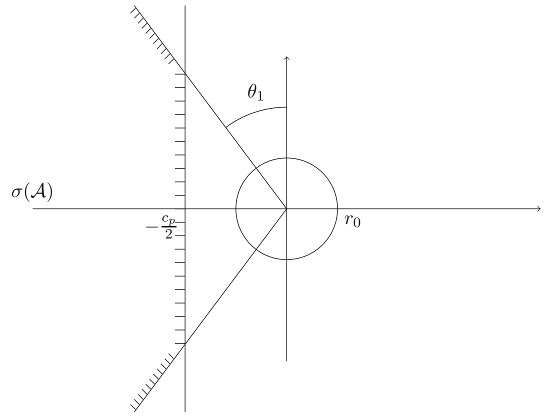

Fig. 3 By Theorem 1.3 and 1.4, the spectrum $\sigma(\mathcal{A})$, $\omega \neq 0$, of $\mathcal{A}$ is contained in the dotted area

In the case of $\mathcal{D}(((-\mathcal{A})^*)^{\frac{1}{2}})$, *the B.C.* $\Delta v_1\Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}$ *is replaced by the B.C.* $\Delta v_1\Big|_{\Gamma_s} = -\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}$, *in line with (1.28b).*

The proof of Theorem 1.5 is given in Sect. 5.

2 Proof of Theorem 1.3

Proof Let $\alpha \leq 0$, $\omega \in \mathbb{R}$. Let $[v_1^*, v_2^*, h^*] \in \mathbf{H}$ in (1.2a). We return to (1.34), and we seek to solve

$$((\alpha + i\omega)I - \mathcal{A})\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} (\alpha + i\omega)v_1 - v_2 \\ (\alpha + i\omega)v_2 + \Delta(\Delta v_1 - v_2) \\ (\alpha + i\omega)h - \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \tag{2.1a}$$

in terms of $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, with a uniform bound for the resolvent $((\alpha + i\omega)I - \mathcal{A})^{-1}$ on a vertical strip, as in (1.36): $-\frac{c_p}{2} < \alpha \leq 0,\ |\omega| > \omega_0 > 0$

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(\alpha + i\omega, \mathcal{A})\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}. \tag{2.1b}$$

Proving (1.36) is equivalent to showing, by the topology of $\mathbf{H}$ in (1.3), that there is $k_{c_P,\omega_0} > 0$, such that for the solution of (2.1b), we have

$$\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2 \leq k_{c_P,\omega_0}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big], \quad -\frac{c_p}{2} < \alpha \leq 0, |\omega| \geq \omega_0 > 0, \tag{2.2}$$

ω_0 arbitrarily small. This estimate will be established below.

Step 1: Inner product the third row of (2.1a) with h and integrate by parts (Green's first theorem) to get

$$(\alpha + i\omega)\|h\|^2 - \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} + \|\nabla h\|^2 = (h^*, h), \tag{2.3}$$

as $h\Big|_{\Gamma_f} = 0$ by (1.8b). Similarly, inner product the second row of (2.1a) with v_2 and integrate by parts to get

$$(\alpha + i\omega)\|v_2\|^2 + (\alpha - i\omega)\|\Delta v_1\|^2 + \|\nabla v_2\|^2 + \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} = (v_2^*, v_2) + (\Delta v_1, \Delta v_1^*). \tag{2.4}$$

In fact, recall from (1.12)–(1.14) that

$$(\Delta(\Delta v_1 - v_2), v_2) = (\Delta v_1, \Delta v_2) + \|\nabla v_2\|^2 + \left(\Delta v_1, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}$$
$$\text{(by (2.1a))} \qquad = (\alpha - i\omega)\|\Delta v_1\|^2 - (\Delta v_1, \Delta v_1^*) + \|\nabla v_2\|^2 + \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}$$

invoking the B.C. (1.7b) and (1.8b). The above identity then allows us to obtain (2.4).

Summing up (2.3) and (2.4) yields after a cancelation of the boundary term $\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}$

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha\left[\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right] + i\omega\left[\|v_2\|^2 + \|h\|^2 - \|\Delta v_1\|^2\right]$$
$$= (\Delta v_1, \Delta v_1^*) + (v_2^*, v_2) + (h^*, h). \tag{2.5}$$

Step 2: We take the Real part and the Imaginary part of (2.5), thus obtaining

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha\left[\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right] = \text{Re}\left\{(\Delta v_1, \Delta v_1^*) + (v_2^*, v_2) + (h^*, h)\right\} \tag{2.6}$$

and

$$\omega\left[\|v_2\|^2 + \|h\|^2 - \|\Delta v_1\|^2\right] = \text{Im}\left\{(\Delta v_1, \Delta v_1^*) + (v_2^*, v_2) + (h^*, h)\right\}. \tag{2.7}$$

Remark 2.1 Equation (2.6) would of course follow at once by the dissipativity identity (1.9) applied to Eq. (2.1a) rewritten after inner product with $\mathbf{x}$ as

$$(\mathcal{A}\mathbf{x}, \mathbf{x})_{\mathbf{H}} = (\alpha + i\omega)\|\mathbf{x}\|^2_{\mathbf{H}} - (\mathbf{x}^*, \mathbf{x})_{\mathbf{H}}, \quad \mathbf{x} = [v_1, v_2, h], \quad \mathbf{x}^* = [v_1^*, v_2^*, h^*]. \tag{2.8}$$

Taking the Real part of (2.8) and using (1.9) yield (2.6).

Step 3: We divide (2.7) by ω for $\omega \neq 0$ and obtain

$$\|\Delta v_1\|^2 = \|v_2\|^2 + \|h\|^2 - \text{Im}\left\{(\Delta v_1, \frac{\Delta v_1^*}{\omega}) + (\frac{v_2^*}{\omega}, v_2) + (\frac{h^*}{\omega}, h)\right\} \tag{2.9}$$

from which we obtain for $\epsilon > 0$ small

$$(1-\epsilon)\|\Delta v_1\|^2 \le (1+\epsilon)\left[\|v_2\|^2 + \|h\|^2\right] + \frac{\widetilde{C}_\epsilon}{|\omega|}\left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right] \tag{2.10}$$

or

$$\|\Delta v_1\|^2 \le \frac{1+\epsilon}{1-\epsilon}\left[\|v_2\|^2 + \|h\|^2\right] + \left(\frac{C_\epsilon}{|\omega|}\right)\left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right], \quad \omega \neq 0. \tag{2.11}$$

Step 4: We rewrite the Real part in (2.6) as

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha\left[\|v_2\|^2 + \|h\|^2\right] = |\alpha|\|\Delta v_1\|^2 + \text{Re}\left\{(\Delta v_1, \Delta v_1^*) + (v_2^*, v_2) + (h^*, h)\right\}. \tag{2.12}$$

Using the Poincare inequalities as in (1.33) on the LHS of (2.12), we then obtain

$$(c_p + \alpha)\left[\|v_2\|^2 + \|h\|^2\right] \le (|\alpha| + \epsilon)\|\Delta v_1\|^2 + \epsilon\left[\|v_2\|^2 + \|h\|^2\right] + C_\epsilon\left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right] \tag{2.13}$$

or

$$(c_p + \alpha - \epsilon)\left[\|v_2\|^2 + \|h\|^2\right] \le (|\alpha| + \epsilon)\|\Delta v_1\|^2 + C_\epsilon\left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right]. \tag{2.14}$$

We next assume $c_p + \alpha > 0$, and we select $\epsilon > 0$, so that $c_p + \alpha - \epsilon > 0$. Then, (2.14) yields

$$\|v_2\|^2 + \|h\|^2 \le \frac{|\alpha| + \epsilon}{c_p + \alpha - \epsilon}\|\Delta v_1\|^2$$

$$+\frac{C_\epsilon}{c_p+\alpha-\epsilon}\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right], \quad -c_p<\alpha\leq 0. \tag{2.15}$$

Step 5: We now use estimate (2.15) on the RHS of estimate (2.11) to obtain for $|\omega|\geq$ some $\omega_0>0$

$$\|\Delta v_1\|^2 \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{|\alpha|+\epsilon}{c_p+\alpha-\epsilon}\right)\|\Delta v_1\|^2$$

$$+\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{C_\epsilon}{c_p+\alpha-\epsilon}+\frac{C_\epsilon}{\omega_0}\right]\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right] \tag{2.16}$$

or

$$\left[1-\left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{|\alpha|+\epsilon}{c_p+\alpha-\epsilon}\right)\right]\|\Delta v_1\|^2$$

$$\leq\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{C_\epsilon}{c_p+\alpha-\epsilon}+\frac{C_\epsilon}{\omega_0}\right]\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right] \tag{2.17}$$

still with $c_p+\alpha-\epsilon>0$ and $|\omega|\geq\omega_0>0$. We henceforth impose the further restriction

$$\frac{|\alpha|}{c_p+\alpha}=\frac{|\alpha|}{c_p-|\alpha|}<1, \quad \text{or} \quad |\alpha|<\frac{c_p}{2}, \quad \text{or} \quad -\frac{c_p}{2}<\alpha\leq 0. \tag{2.18}$$

Thus, we can select $\epsilon_1>0$ arbitrarily small and restrict further $\epsilon>0$, so that

$$\begin{cases}\dfrac{|\alpha|+\epsilon}{c_p+\alpha-\epsilon}=1-\epsilon_1, \text{ and hence} & \text{(2.19a)}\\ \left(\dfrac{1+\epsilon}{1-\epsilon}\right)\left(\dfrac{|\alpha|+\epsilon}{c_p+\alpha-\epsilon}\right)=\left(1+\dfrac{2\epsilon}{1-\epsilon}\right)(1-\epsilon_1)=(1-\epsilon_1^2)\equiv k_{\epsilon,\epsilon_1}<1 & \text{(2.19b)}\end{cases}$$

by selecting $\dfrac{2\epsilon}{1-\epsilon}=\epsilon_1$. Thus, (2.19b) used in (2.17) yields

$$\|\Delta v_1\|^2\leq\frac{1}{K_{\epsilon,\epsilon_1}}\left[\left(1+\frac{2\epsilon}{1-\epsilon}\right)\frac{C_\epsilon}{c_p+\alpha-\epsilon}+\frac{C_\epsilon}{\omega_0}\right]\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right],$$

$$|\omega|\geq\omega_0>0, \; -\frac{c_p}{2}<\alpha\leq 0 \tag{2.20}$$

with $K_{\epsilon,\epsilon_1}=1-k_{\epsilon,\epsilon_1}$, $0<K_{\epsilon,\epsilon_1}<1$, recalling the restriction on α in (2.18); or

$$\|\Delta v_1\|^2\leq C_{\epsilon,\epsilon_1,c_p,\omega_0}\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right], \qquad |\omega|\geq\omega_0>0, \; -\frac{c_p}{2}<\alpha\leq 0. \tag{2.21}$$

Step 6: We use estimate (2.21) on the RHS of estimate (2.15) and obtain

$$\|v_2\|^2+\|h\|^2\leq\widetilde{C}_{\epsilon,\epsilon_1,c_p,\omega_0}\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right], \qquad |\omega|\geq\omega_0>0, \; -\frac{c_p}{2}<\alpha\leq 0. \tag{2.22}$$

Step 7: Combining (2.21) and (2.22), we obtain the final desired estimate

$$\|\Delta v_1\|^2+\|v_2\|^2+\|h\|^2\leq \text{const}_{\epsilon,\epsilon_1,c_p,\omega_0}\left[\|\Delta v_1^*\|^2+\|v_2^*\|^2+\|h^*\|^2\right],$$

$$|\omega|\geq\omega_0>0, \; -\frac{c_p}{2}<\alpha\leq 0 \tag{2.23}$$

with ω_0 arbitrarily small. Then, (2.23) proves (2.2) and estimates (1.36). Theorem 1.3 is established. □

3 Proof of the analyticity Theorem 1.4

With reference to (2.1a) specialized of $\alpha=0$

$$(i\omega I-\mathcal{A})\begin{bmatrix}v_1\\v_2\\h\end{bmatrix}=\begin{bmatrix}i\omega v_1-v_2\\ i\omega v_2+\Delta(\Delta v_1-v_2)\\ i\omega h-\Delta h\end{bmatrix}=\begin{bmatrix}v_1^*\\v_2^*\\h^*\end{bmatrix}\in\mathbf{H} \tag{3.1a}$$

for $[v_1^*, v_2^*, h^*] \in \mathbf{H}$ and $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, or

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(i\omega, \mathcal{A}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \tag{3.1b}$$

we shall establish the bound required in (1.42)

$$\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2 \leq \frac{c}{\omega_0}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big], \quad \text{for all } \omega \in \mathbb{R},\ |\omega| \geq \text{some } \omega_0 > 0, \tag{3.2}$$

ω_0 arbitrarily small. We shall proceed by specializing the proof of Theorem 1.3 in Sect. 2 to the case $\alpha = 0$.

Step 1: Estimate (2.17) with $\alpha = 0$ specializes to

$$\left[1 - \left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{\epsilon}{c_p - \epsilon}\right)\right]\|\Delta v_1\|^2 \leq \left[\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{C_\epsilon}{c_p - \epsilon} + \frac{C_\epsilon}{\omega_0}\right]\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big], \tag{3.3}$$

where we take $c_p - \epsilon > 0$ and $|\omega| \geq \omega_0 > 0$. However, for the coefficient term [ ] on the LHS of (3.3), we obtain for small ϵ

$$\frac{1}{2} \leq 1 - \left(\frac{1-\epsilon+2\epsilon}{1-\epsilon}\right)\left(\frac{\epsilon}{c_p-\epsilon}\right) = 1 - \frac{\epsilon}{c_p - \epsilon} - \frac{2\epsilon^2}{(1-\epsilon)(c_p-\epsilon)}, \tag{3.4}$$

so that (3.3) is rewritten as

$$\|\Delta v_1\|^2 \leq 2\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{C_\epsilon}{c_p-\epsilon} + \frac{C_\epsilon}{\omega_0}\right]\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big] \tag{3.5}$$

for all $\omega \in \mathbb{R}$, $|\omega| \geq \omega_0 > 0$. By now taking $\omega_0 > 0$ sufficiently small

$$\frac{1+\epsilon}{1-\epsilon} \cdot \frac{C_\epsilon}{c_p - \epsilon} < \frac{1}{\omega_0} \quad \text{or} \quad \omega_0 < \frac{1-\epsilon}{1+\epsilon} \cdot \frac{c_p - \epsilon}{C_\epsilon}, \tag{3.6}$$

then (3.4) becomes for such ω as in (3.6)

$$\|\Delta v_1\|^2 \leq \frac{4C_\epsilon}{\omega_0}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big], \quad \text{for all } |\omega| \geq \omega_0 > 0, \tag{3.7}$$

which is the desired bound for the term $\|\Delta v_1\|^2$ in (3.2).

Step 2: Substitute estimate (3.7) in estimate (2.15) for $\alpha = 0$, thus obtaining

$$\|v_2\|^2 + \|h\|^2 \leq \left\{\left(\frac{\epsilon}{c_p-\epsilon}\right)\frac{4C_\epsilon}{\omega_0} + \frac{C_\epsilon}{c_p-\epsilon}\right\}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big] \tag{3.8}$$

$$(\text{by } (3.6)) \qquad \leq \left\{\left(\frac{\epsilon}{c_p-\epsilon}\right)\frac{4C_\epsilon}{\omega_0} + C_\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{1}{\omega_0}\right\}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big] \tag{3.9}$$

again taking $|\omega| \geq \omega_0 > 0$, where we have used $\frac{C_\epsilon}{c_p - \epsilon} < \left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{1}{\omega_0}$ by (3.6). Hence, (3.9) ultimately yields

$$\|v_2\|^2 + \|h\|^2 \leq \frac{C_{\epsilon,c_p}}{\omega_0}\Big[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\Big] \tag{3.10}$$

for all $\omega \in \mathbb{R}$, $|\omega| \geq \omega_0 > 0$ arbitrarily small. Estimate (3.10) is the desired estimate for the terms $\|v_2\|^2 + \|h\|^2$ in (3.2). Summing up (3.7) and (3.10), we obtain (3.2) and Theorem 1.4 is proved.

4 Explicit expression of $\mathcal{A}$ in (1.5) incorporating B.C.

Equation (1.5) shows the action of the operator $\mathcal{A}$, whose domain $\mathcal{D}(\mathcal{A})$ is then described by (1.8d). In the present section, we incorporate the B.C. directly in the expression of $\mathcal{A}$. This is given in (4.19).

Step 1: Start from the u-equation (1.1a) supplemented by the BC in (1.1d). Recall the Dirichlet map on Ω_f from Γ_s, introduced in (1.19b)

$$\phi = \widetilde{D}_f g \iff \{\Delta\phi = 0 \text{ in } \Omega_f : \quad \phi\Big|_{\Gamma_f} = 0; \quad \phi\Big|_{\Gamma_s} = g\}. \tag{4.1}$$

Then, rewrite (1.1a) as

$$\begin{cases} u_t = \Delta\left(u - \widetilde{D}_f\left(\dfrac{\partial w_t}{\partial \nu}\Big|_{\Gamma_s}\right)\right), \\ \left[u - \widetilde{D}_f\left(\dfrac{\partial w_t}{\partial \nu}\Big|_{\Gamma_s}\right)\right]_{\partial\Omega_f} = 0 = \begin{cases} u\Big|_{\Gamma_f} - 0 = 0 & on \Gamma_f, \\ u - \dfrac{\partial w_t}{\partial \nu} = 0 & on \Gamma_s, \end{cases} \quad \text{by } (1.1d) & (4.2a) \\ u_t = -A_{D,f}\left[u - \widetilde{D}_f\left(\dfrac{\partial w_t}{\partial \nu}\Big|_{\Gamma_s}\right)\right] \in \mathbf{L}^2(\Omega_f), & (4.2b) \end{cases}$$

where $-A_{D,f}$ is the Dirichlet–Laplacian on Ω_f (strictly negative self-adjoint) introduced in (1.19a)

$$-A_{D,f}\psi = \Delta\psi, \quad \mathcal{D}(A_{D,f}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f). \tag{4.3}$$

Ultimately, extending $A_{D,f}$ from $\mathbf{L}^2(\Omega_f) \supset \mathcal{D}(A_{D,f}) \to \mathbf{L}^2(\Omega_f)$ to $\mathbf{L}^2(\Omega_f) \to [\mathcal{D}(A_{D,f})]'$ by isomorphism, while keeping the same notation, we obtain from (4.2b)

$$u_t = -A_{D,f}u + A_{D,f}\widetilde{D}_f\left(\frac{\partial w_t}{\partial \nu}\Big|_{\Gamma_s}\right) \in [\mathcal{D}(A_{D,f})]'. \tag{4.4}$$

Step 2: Next, we consider the w-equation (1.1b) supplemented by the BC in (1.1c). Introduce the Green map G_2 defined by [31, p. 212]

$$y = G_2 v \iff \{\Delta^2 y = 0 \text{ in } \Omega_s, \quad y\Big|_{\Gamma_s} = 0, \quad \Delta y\Big|_{\Gamma_s} = v\} \tag{4.5}$$

as well as recall the Dirichlet map on Ω_s introduced in (1.22a)

$$x = D_s v \iff \{\Delta x = 0 \text{ in } \Omega_s, \quad x\Big|_{\Gamma_s} = v\}. \tag{4.6}$$

We have [31, Eq. (3.6.6), p. 212]

$$G_2 = -\mathbb{A}^{-\frac{1}{2}} D_s, \tag{4.7}$$

where we have defined the positive, self-adjoint operator $\mathbb{A}$ by

$$\mathbb{A}\phi = \Delta^2\phi, \quad \mathcal{D}(\mathbb{A}) = \{\phi \in \mathbf{H}^4(\Omega_s) : \phi\Big|_{\Gamma_s} = \Delta\phi\Big|_{\Gamma_s} = 0\}, \tag{4.8}$$

so that [31, p. 205]

$$\mathbb{A} = A_{D,s}^2, \quad \text{or} \quad \mathbb{A}^{\frac{1}{2}} = A_{D,s}, \tag{4.9}$$

where $A_{D,s}$ is the Dirichlet–Laplacian on Ω_s (strictly positive self-adjoint)

$$A_{D,s}\psi = -\Delta\psi, \quad \mathcal{D}(A_{D,s}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f). \tag{4.10}$$

introduced in (1.2b). Via (4.5) and (4.10), we rewrite (1.1b) as

$$\begin{cases} w_{tt} + \Delta^2 \left(w - G_2 \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) \right) - \Delta w_t = 0 & \text{(4.11a)} \\ \left[w - G_2 \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) \right]_{\Gamma_s} = 0 & \text{(4.11b)} \\ \left[\Delta \left(w - G_2 \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) \right) \right]_{\Gamma_s} = \left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} - \left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} = 0 & \text{(4.11c)} \end{cases}$$

by recalling $\left. w \right|_{\Gamma_s} = 0$ in (4.11b) and $\left. \Delta w \right|_{\Gamma_s} = \left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} = 0$ in (4.11c) via (1.1c). Hence, recalling (4.8) and (4.10), we see that (4.11) is rewritten abstractly as

$$w_{tt} = -\mathbb{A} \left(w - G_2 \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) \right) - A_{D,s} w_t \in \mathbf{L}^2(\Omega_s) \tag{4.12}$$

$$w_{tt} = -\mathbb{A} w + \mathbb{A} G_2 \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) - A_{D,s} w_t \in [\mathcal{D}(\mathbb{A})]' \tag{4.13}$$

the original $\mathbb{A} : \mathbf{L}^2(\Omega_s) \supset \mathcal{D}(\mathbb{A}) \to \mathbf{L}^2(\Omega_s)$ being extended by isomorphism to $\mathbf{L}^2(\Omega_s) \to [\mathcal{D}(\mathbb{A}^*)]' = [\mathcal{D}(\mathbb{A})]'$, as usual, while maintaining the same notation; or by (4.7)

$$w_{tt} = -\mathbb{A} w - \mathbb{A}^{\frac{1}{2}} D_s \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) - A_{D,s} w_t \in [\mathcal{D}(\mathbb{A})]'. \tag{4.14}$$

Finally, by (4.9)

$$w_{tt} = -A_{D,s} \left[A_{D,s} w + D_s \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) + w_t \right] \in \mathbf{L}^2(\Omega_s) \tag{4.15}$$

$$w_{tt} = -A_{D,s}^2 w - A_{D,s} D_s \left(\left. \frac{\partial u}{\partial \nu} \right|_{\Gamma_s} \right) - A_{D,s} w_t \in [\mathcal{D}(\mathbb{A})]', \tag{4.16}$$

where $-A_{D,s}$ is defined in (1.2b) = (4.10) and is extended $\mathbf{L}^2(\Omega_s) \to [\mathcal{D}(A_{D,s})]'$ by isomorphic extension, with no change of notation.

Step 3: We combine (4.4) and (4.16) in a first-order system

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A_{D,s}^2 & -A_{D,s} & -A_{D,s} D_s \left(\left. \frac{\partial \cdot}{\partial \nu} \right|_{\Gamma_s} \right) \\ 0 & A_{D,f} \widetilde{D}_f \left(\left. \frac{\partial \cdot}{\partial \nu} \right|_{\Gamma_s} \right) & -A_{D,f} \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \tag{4.17}$$

$$\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A_{D,s}^2 & -A_{D,s} & -A_{D,s}D_s\left(\frac{\partial\,\cdot}{\partial\nu}\bigg|_{\Gamma_s}\right) \\ 0 & A_{D,f}\widetilde{D}_f\left(\frac{\partial\,\cdot}{\partial\nu}\bigg|_{\Gamma_s}\right) & -A_{D,f} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \tag{4.18}$$

$$\equiv \begin{bmatrix} v_2 \\ -A_{D,s}\left[A_{D,s}v_1 + v_2 + D_s\left(\frac{\partial h}{\partial\nu}\bigg|_{\Gamma_s}\right)\right] \\ A_{D,f}\left[\widetilde{D}_f\left(\frac{\partial v_2}{\partial\nu}\bigg|_{\Gamma_s}\right) - h\right] \end{bmatrix}, \quad [v_1, v_2, h] \in \mathcal{D}(\mathcal{A}). \tag{4.19}$$

The domain of $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, is described in (1.8d). In particular

$$v_1, v_2 \in \mathbf{H}^2(\Omega_s) \cap \mathbf{H}_0^1(\Omega_s); \qquad h \in \mathbf{H}^1(\Omega_f), \text{ such that} \tag{4.20}$$

$$A_{D,s}v_1 + v_2 + D_s\left(\frac{\partial h}{\partial\nu}\bigg|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,s}); \qquad h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial\nu}\bigg|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,f}). \tag{4.21}$$

5 Proof of Theorem 1.5 (after [33, Section 4])

The present proof is patterned after [33, Section 4].

Step 1: Preliminary setting.

(i) We return to the positive self-adjoint operator $A_{D,s}$ on Ω_s in (1.2b) = (4.10)

$$A_{D,s}f = -\Delta f \in \mathbf{L}^2(\Omega_s), \quad \mathcal{D}(A_{D,s}) = \mathbf{H}^2(\Omega_s) \cap \mathbf{H}_0^1(\Omega_s) \tag{5.1}$$

and recall [20,21,29]

$$\mathcal{D}(A_{D,s}^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega_s) \equiv \{\phi \in \mathbf{H}^1(\Omega_s), \phi\Big|_{\Gamma_s} = 0\}. \tag{5.2}$$

(ii) Similarly, we return to the positive self-adjoint operator $A_{D,f}$ on Ω_f in (1.19a) = (4.3)

$$A_{D,f}q = -\Delta q \in \mathbf{L}^2(\Omega_f), \quad \mathcal{D}(A_{D,f}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f) \tag{5.3}$$

and recall

$$\mathcal{D}(A_{D,f}^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega_f) \equiv \{\phi \in \mathbf{H}^1(\Omega_f), \phi\big|_{\partial\Omega_f} = 0\}. \tag{5.4}$$

(iii) Next, recall the Dirichlet map D_s on Ω_s in (1.22a) = (4.6)

$$\psi = D_s g \iff \left\{\Delta\psi = 0 \text{ in } \Omega_s;\ \psi\big|_{\Gamma_s} = g\right\} \tag{5.5a}$$

$$D_s : \mathbf{H}^s(\Gamma_s) \to \mathbf{H}^{s+\frac{1}{2}}(\Omega_s) \text{ continuously}, \quad s \in \mathbb{R}. \tag{5.5b}$$

(iv) Finally, recall the Dirichlet map $\widetilde{D}_f$ in (1.19) = (4.1)

$$q = \widetilde{D}_f g \iff \left\{\Delta q = 0 \text{ in } \Omega_f, \quad q\big|_{\Gamma_f} = 0, \quad q\big|_{\Gamma_s} = g\right\} \tag{5.6a}$$

$$\widetilde{D}_f : \mathbf{H}^s(\partial\Omega_f) \to \mathbf{H}^{s+\frac{1}{2}}(\Omega_f) \text{ continuously}, \quad s \in \mathbb{R}. \tag{5.6b}$$

(v) We start with the pair $\{v_1 \in \mathcal{D}(A_{D,s}), v_2 \in \mathcal{D}(A_{D,s})\}$, so that $v_2\big|_{\Gamma_s} = 0$, $\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$ and $\widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_s)$ by (5.6b). Next, by imposing the 'static' version of (4.2), we shall obtain that

$$h \in \mathbf{H}^1(\Omega_f), \quad \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \quad \text{so that } D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right) \in \mathbf{L}^2(\Omega_s) \tag{5.7}$$

recalling (1.8c) (or [33, Eq. (1.9)], or Appendix B, see (B.4)) for the normal derivative. In fact, we next note that

$$\left[h - \widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s}\right)\right] \in \mathcal{D}(A_{D,f}), \ \text{as in (5.3) means via (5.3), (5.6a)} \tag{5.8}$$

$$\begin{cases} \Delta\left[h - \widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s}\right)\right] = \Delta h \in \mathbf{L}^2(\Omega_f) & (5.9a) \\ \left[h - \widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s}\right)\right]_{\Gamma_f} = h\Big|_{\Gamma_f} = 0 & (5.9b) \\ \left[h - \widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s}\right)\right]_{\Gamma_s} = h\Big|_{\Gamma_s} - \left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s} = 0, \ \text{or } h\Big|_{\Gamma_s} = \left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s), & (5.9c) \end{cases}$$

so that by elliptic theory, we obtain (5.7), as desired. Thus, the triple $[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f)$ has the same regularity as in $\mathcal{D}(\mathcal{A})$, see (1.8d).

(vi) Similarly, we note that

$$\left[(\Delta v_1 - v_2) - D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right)\right] \in \mathcal{D}(A_{D,s}), \ \text{as in (5.1) means via (5.1), (5.5a)} \tag{5.10}$$

$$\begin{cases} \Delta\left[(\Delta v_1 - v_2) - D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right)\right] = \Delta(\Delta v_1 - v_2) = F \in \mathbf{L}^2(\Omega_s) & (5.11a) \\ \left[(\Delta v_1 - v_2) - D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right)\right]_{\Gamma_s} = \Delta v_1\Big|_{\Gamma_s} - \cancel{v_2\Big|_{\Gamma_s}} - \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s} = 0, & \\ \qquad \text{or } \Delta v_1\Big|_{\Gamma_s} = \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s), \ \text{by(5.7)}. & (5.11b) \end{cases}$$

Problem (5.11) implies by (5.1) and (5.5) that

$$\Delta v_1 - v_2 = -A_{D,s}^{-1}F + D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,s}) + \mathbf{L}^2(\Omega_s) \tag{5.12}$$

by (5.5b) with $s = -\frac{1}{2}$, which along with $v_2 \in \mathcal{D}(A_{D,s})$ and $v_1\Big|_{\Gamma_s} = 0$ yields

$$v_1 = -A_{D,s}^{-1}v_2 + A_{D,s}^{-2}F - A_{D,s}^{-1}D_s\left(\left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,s})$$

confirming the original choice of v_1, v_2 in (v).

Step 2: With the triple $\{v_1 \in \mathcal{D}(A_{D,s}), v_2 \in \mathcal{D}(A_{D,s}), h \in \mathbf{H}^1(\Omega_f)\}$ as taken above in (v), we introduce the following positive self-adjoint diagonal operator $\mathfrak{A}$ acting explicitly on the variables:

$$\left\{v_1, v_2, h, \left[(\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right)\right], \left[h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\right]\right\} \tag{5.13}$$

by

$$\mathfrak{A}\begin{bmatrix} v_1 \\ v_2 \\ h \\ (\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right) \\ h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\end{bmatrix} = \begin{bmatrix} I_s & & & & \\ & I_s & & & \\ & & I_f & & \\ & & & A_{D,s} & \\ & & & & A_{D,f}\end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \\ (\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right) \\ h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\end{bmatrix}, \tag{5.14}$$

where I_s is the identity on $\mathcal{D}(A_{D,s})$, while I_f is the identity on $\mathbf{H}^1(\Omega_f)$

$$\begin{aligned}\mathcal{D}(\mathfrak{A}) = &\Big\{[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f) \subset \mathbf{H}, \text{ so that } v_1\Big|_{\Gamma_s} = 0, \quad v_2\Big|_{\Gamma_s} = 0 : \\ &\left[(\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right)\right] \in \mathcal{D}(A_{D,s}), \quad \left[h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\right] \in \mathcal{D}(A_{D,f})\Big\}\end{aligned} \tag{5.15}$$

$$\begin{aligned} = &\Big\{[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f) \subset \mathbf{H}, \text{ so that } v_1\Big|_{\Gamma_s} = 0, \quad v_2\Big|_{\Gamma_s} = 0 : \\ &(i)\ \Delta h \in \mathbf{L}^2(\Omega_f),\ h\Big|_{\Gamma_f} = 0,\ h\Big|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s), \text{ hence } \frac{\partial h}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s) \\ &(ii)\ \ \Delta(\Delta v_1 - v_2) \in \mathbf{L}^2(\Omega_s), \quad \Delta v_1\Big|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)\Big\},\end{aligned} \tag{5.16}$$

where in going from (5.15) to (5.16), we have invoked (5.8)–(5.9a)–(5.9c), (5.10)–(5.11a)–(5.11b). Thus, recalling the operator $\mathcal{A}$ in (1.5), with domain $\mathcal{D}(\mathcal{A})$ defined in (1.8d), we conclude from (5.16) that

$$\mathcal{D}(\mathfrak{A}) = \mathcal{D}(\mathcal{A}) \tag{5.17}$$

in the variables $[v_1, v_2, h]$ in $\mathbf{H}$.

Step 3: Here, we consider only the case $\theta = \frac{1}{2}$ of fractional power. For the diagonal, positive self-adjoint operator $\mathfrak{A}$ in (5.14)–(5.15), its positive square-root operator is

$$\mathfrak{A}^{\frac{1}{2}}\begin{bmatrix} v_1 \\ v_2 \\ h \\ (\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right) \\ h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\end{bmatrix} = \begin{bmatrix} I_s & & & & \\ & I_s & & & \\ & & I_f & & \\ & & & A_{D,s}^{\frac{1}{2}} & \\ & & & & A_{D,f}^{\frac{1}{2}}\end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \\ (\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right) \\ h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right)\end{bmatrix} \tag{5.18}$$

$$\mathcal{D}(\mathfrak{A}^{\frac{1}{2}}) = \Big\{[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f) \subset \mathbf{H}, \text{ so that } v_1\Big|_{\Gamma_s} = 0,\ v_2\Big|_{\Gamma_s} = 0 :$$

$$\left.\begin{aligned}&(i)\ (\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,s}^{\frac{1}{2}}) \equiv \mathbf{H}_0^1(\Omega_s),\\ &(ii)\ h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,f}^{\frac{1}{2}}) \equiv \mathbf{H}_0^1(\Omega_f)\end{aligned}\right\} \tag{5.19}$$

invoking (5.2), (5.4). Statement (5.19) includes a constraint on the regularity of the triple $[v_1, v_2, h]$, as well as a constraint on the boundary conditions. Statement (ii) imposes no new condition in its regularity statement, over $[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f)$, since then $\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$, and hence, $\widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_f)$ by (1.19d), so that $h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_f)$ automatically. However, statement (ii) imposes the following boundary condition by (5.6a):

$$\left[h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s}\right)\right]_{\Gamma_f} = h\bigg|_{\Gamma_f} - 0 = 0, \text{ or } h\bigg|_{\Gamma_f} = 0, \tag{5.20}$$

$$\left[h - \widetilde{D}_f\left(\frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s}\right)\right]_{\Gamma_s} = h\bigg|_{\Gamma_s} - \frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} = 0, \text{ or } h\bigg|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s). \tag{5.21}$$

Instead, statement (i) imposes both a regularity condition

$$\left[(\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s}\right)\right] \in \mathbf{H}^1(\Omega_s), \tag{5.22}$$

where $v_2 \in \mathbf{H}^2(\Omega_s) \subset \mathbf{H}^1(\Omega_s)$, as well as a condition at the boundary by (5.5a)

$$\left[(\Delta v_1 - v_2) - D_s\left(\frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s}\right)\right]_{\Gamma_s} = \Delta v_1\bigg|_{\Gamma_s} - \cancel{v_2\bigg|_{\Gamma_s}} - \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} = 0,$$

$$\text{thus } \Delta v_1\bigg|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s). \tag{5.23}$$

Thus, ultimately

$$\begin{aligned}\mathcal{D}(\mathfrak{A}^{\frac{1}{2}}) = \Bigg\{&[v_1, v_2, h] \in \mathcal{D}(A_{D,s}) \times \mathcal{D}(A_{D,s}) \times \mathbf{H}^1(\Omega_f) \subset \mathbf{H}, \text{ so that}\\ &v_1\bigg|_{\Gamma_s} = 0,\ v_2\bigg|_{\Gamma_s} = 0,\ h\bigg|_{\Gamma_f} = 0,\ h\bigg|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s),\\ &\Delta v_1\bigg|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s), \quad \Delta v_1 - D_s\left(\frac{\partial h}{\partial \nu}\bigg|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_s)\Bigg\}.\end{aligned} \tag{5.24}$$

Step 4: Since $\mathcal{A}$ is maximal dissipative and $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H})$ (Theorem 1.1), we have [10, Prop. 6.1, p. 171], [33, p. 5]

$$\mathcal{D}((-\mathcal{A})^{\frac{1}{2}}) = [\mathcal{D}(\mathcal{A}), \mathbf{H}]_{\frac{1}{2}}. \tag{5.25}$$

Step 5: Finally, from (5.17) and (5.25), we have for the positive self-adjoint operator $\mathfrak{A}$ in terms of the triple $[v_1, v_2, h]$

$$\mathcal{D}(\mathfrak{A}^{\frac{1}{2}}) = [\mathcal{D}(\mathfrak{A}), \mathbf{H}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}), \mathbf{H}]_{\frac{1}{2}} = \mathcal{D}((-\mathcal{A})^{\frac{1}{2}}), \tag{5.26}$$

and thus, $\mathcal{D}((-\mathcal{A})^{\frac{1}{2}})$ is given explicitly by (5.24). This proves (1.45) of Theorem 1.5 for $\mathcal{D}((-\mathcal{A})^{\frac{1}{2}})$. The proof for $\mathcal{D}((-\mathcal{A}^*)^{\frac{1}{2}})$ is similar, using now $\mathcal{D}(\mathcal{A}^*)$ in (1.26).

Part II: Dirichlet control $\mathbf{g} \in \mathbf{L^p(0, T; H^{\frac{1}{2}}(\Gamma_s))}$ at the interface Γ_s. Main results

6 Model with Dirichlet control at the interface

In the present Part II, we consider the case where the boundary control g acts in the Dirichlet B.C. of the heat variable u at the interface between the two media

$$\text{(PDE)} \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T) \times \Omega_f; \quad (6.1a) \\ w_{tt} + \Delta^2 w - \Delta w_t = 0 & \text{in } (0, T) \times \Omega_s; \quad (6.1b) \end{cases}$$

$$\text{(BC)} \begin{cases} w|_{\Gamma_s} = 0; \quad \Delta w|_{\Gamma_s} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma_s} & \text{on } (0, T) \times \Gamma_s \equiv \Sigma_s; \quad (6.1c) \\ u|_{\Gamma_f} = 0 \text{ on } (0, T) \times \Gamma_f; \quad u|_{\Gamma_s} = \left.\dfrac{\partial w_t}{\partial \nu}\right|_{\Gamma_s} + g & \text{on } (0, T) \times \Gamma_s; \quad (6.1d) \end{cases}$$

$$\text{(IC)}\ [w(0,\cdot), w_t(0,\cdot), u(0,\cdot)] = [w_0, w_1, u_0] \in \mathbf{H}. \tag{6.1e}$$

H being the finite energy state space in (1.2a).

Abstract model We modify accordingly the argument in Sect. 4. With reference to the Dirichlet map $\widetilde{D}_f$ on the fluid domain introduced in (1.19b) = (4.1), the counterpart of Eqns. (4.1), (4.4) for the present problem (6.1a)–(6.1e) with control g in (6.1d) is

$$\begin{cases} u_t = \Delta \left(u - \widetilde{D}_f \left(\left.\dfrac{\partial w_t}{\partial \nu}\right|_{\Gamma_s} + g \right) \right) & \text{in } (0, T) \times \Omega_f; \quad (6.2a) \\ \left[u - \widetilde{D}_f \left(\left.\dfrac{\partial w_t}{\partial \nu}\right|_{\Gamma_s} + g \right) \right]_{\partial \Omega_f} = 0 & \text{in } (0, T) \times \partial\Omega_f; \quad (6.2b) \end{cases}$$

hence

$$\begin{aligned} u_t &= -A_{D,f} \left(u - \widetilde{D}_f \left(\left.\frac{\partial w_t}{\partial \nu}\right|_{\Gamma_s} + g \right) \right); \\ &= -A_{D,f} u + A_{D,f} D_f \left(\left.\frac{\partial w_t}{\partial \nu}\right|_{\Gamma_s} \right) + A_{D,f} \widetilde{D}_f g \in [\mathcal{D}(A_{D,f})]', \end{aligned} \tag{6.3}$$

where $-A_{D,f}$ is originally defined in (1.19a) = (4.3): $\mathcal{D}(A_{D,f}) \to \mathbf{L}^2(\Omega_f)$ and is here extended in (6.3): $\mathbf{L}^2(\Omega_f) \to [\mathcal{D}(A_{D,f})]'$ by isomorphic extension, with no change of notation. The counterpart of Eq. (4.16) now does not contain g and thus remains the same

$$w_{tt} = -A_{D,s}^2 w - A_{D,s} D_s \left(\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_s} \right) - A_{D,s} w_t \in [\mathcal{D}(\mathbb{A})]'. \tag{6.4}$$

Combining (6.3) and (6.4), we obtain

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathcal{B}_D g, \tag{6.5}$$

the non-homogeneous version of (4.17), where $\mathcal{A}$ is given by (4.18) and the boundary operator $\mathcal{B}_D$ is given via (6.3) by

$$\mathcal{B}_D g = \begin{bmatrix} 0 \\ 0 \\ A_{D,f}\widetilde{D}_f g \end{bmatrix}, \mathcal{B}_D : \text{ continuous } \mathbf{L}^2(\Gamma_s) \to \begin{bmatrix} \otimes \\ \otimes \\ \left[\mathcal{D}\left(A_{D,f}^{\frac{3}{4}+\epsilon}\right)\right]' \end{bmatrix}, \tag{6.6}$$

since (recalling (1.19d), i.e., [31, p. 181], [42])

$$\begin{cases} \widetilde{D}_f : \mathbf{L}^2(\Gamma_s) \to \mathcal{D}\left(A_{D,f}^{\frac{1}{4}-\epsilon}\right) \equiv \mathbf{H}^{\frac{1}{2}-2\epsilon}(\Omega_f), & (6.7a) \\ \text{or } A_{D,f}^{\frac{1}{4}-\epsilon}\widetilde{D}_f \in \mathcal{L}(\mathbf{L}^2(\Gamma_s); \mathbf{L}^2(\Omega_f)). & (6.7b) \end{cases}$$

Thus, with $x_3 \in \mathcal{D}(A_{D,f}^{\frac{3}{4}+\epsilon}) \subset \mathbf{H}^{\frac{3}{2}+2\epsilon}(\Omega_f)$, we have for the adjoint $\mathcal{B}_D^*$

$$\mathcal{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left.\frac{\partial x_3}{\partial \nu}\right|_{\Gamma_s}; \quad \mathcal{B}_D^* : \text{ continuous } \begin{bmatrix} \otimes \\ \otimes \\ \mathcal{D}(A_{D,f}^{\frac{3}{4}+\epsilon}) \end{bmatrix} \to \mathbf{L}^2(\Gamma_s), \tag{6.8}$$

in the following sense. For $g \in \mathbf{L}^2(\Gamma_s)$ and $\{x_1, x_2, x_3\} \in \left[\otimes, \otimes, \mathcal{D}(A_{D,f}^{\frac{3}{4}+\epsilon})\right]$, we compute as a duality pairing via (6.6):

$$\left(\mathcal{B}_D g, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{\mathbf{H}} = \left(\begin{bmatrix} 0 \\ 0 \\ A_{D,f}\widetilde{D}_f g \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{\mathbf{H}} \tag{6.9}$$

$$= (A_{D,f}\widetilde{D}_f g, x_3)_{\mathbf{L}^2(\Omega_f)} = (g, \widetilde{D}_f^* A_{D,f} x_3)_{\mathbf{L}^2(\Gamma_s)} \tag{6.10}$$

$$= \left(g, \left.\frac{\partial x_3}{\partial \nu}\right|_{\Gamma_s}\right)_{\mathbf{L}^2(\Gamma_s)} = \left(g, \mathcal{B}_D^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{\mathbf{L}^2(\Gamma_s)}, \tag{6.11}$$

where we have recalled $\widetilde{D}_f^* A_{D,f} x_3 = \left.\frac{\partial x_3}{\partial \nu}\right|_{\Gamma_s}$ from [33, p. 181]. Thus, (6.8) is established.

Theorem 6.1 *With reference to $\mathcal{A}^{-1}$ in (1.25) and $\mathcal{B}_D$ in (6.6), we have for $g \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$*

$$\mathcal{A}^{-1}\mathcal{B}_D g = \begin{bmatrix} A_{D,s}^{-1} D_s \frac{\partial}{\partial \nu} A_{D,f}^{-1} A_{D,f} \widetilde{D}_f g \\ 0 \\ -A_{D,f}^{-1} A_{D,f} \widetilde{D}_f g \end{bmatrix} = \begin{bmatrix} A_{D,s}^{-1} D_s \frac{\partial}{\partial \nu}(\widetilde{D}_f g) \\ 0 \\ -\widetilde{D}_f g \end{bmatrix} \in \begin{bmatrix} \mathcal{D}(A_{D,s}) \\ 0 \\ \mathbf{H}^1(\Omega_f) \end{bmatrix} \in \mathbf{H}. \tag{6.12}$$

[However, $\mathcal{A}^{-1}\mathcal{B}_D g \notin \mathcal{D}((-\mathcal{A})^{\frac{1}{2}})$, because recalling the requirement on the third component $h\Big|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}$ in (1.45), we use that the required B.C. $\widetilde{D}_f g\Big|_{\Gamma_s} = g = \frac{\partial(0)}{\partial \nu}\Big|_{\Gamma_s} = 0\Big|_{\Gamma_s}$, as $v_2 = 0$ in the present case, is not satisfied, unless $g = 0$.]

A-fortiori

$$\mathcal{A}^{-1}\mathcal{B}_D \in \mathcal{L}(\mathbf{H}^{\frac{1}{2}}(\Gamma_s); \mathbf{H}). \tag{6.13}$$

Proof Verification of the expression in (6.12) is immediate. Next, the regularity in $\mathbf{H}$. In fact, for $g \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$, we have continuously

$$\widetilde{D}_f g \in \mathbf{H}^1(\Omega_f); \qquad \frac{\partial(\widetilde{D}_f g)}{\partial \nu}\bigg|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s); \tag{6.14a}$$

$$D_s \frac{\partial}{\partial \nu}(\widetilde{D}_f g) \in \mathbf{L}^2(\Omega_s); \qquad A_{D,s}^{-1} D_s \frac{\partial(\widetilde{D}_f g)}{\partial \nu} \in \mathcal{D}(A_{D,s}), \tag{6.14b}$$

recalling also (1.19c) for $\widetilde{D}_f$ with $r = \frac{1}{2}$, since $\widetilde{D}_f g = \varphi$ is a harmonic function satisfying $\Delta\varphi = 0$ in Ω_f, $\varphi|_{\Gamma_f} = 0$, $\varphi|_{\Gamma_s} = g$ [see (1.19b)]. Then, $\frac{\partial}{\partial \nu}\varphi|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$, as it follows from $0 = \int_{\Omega_f} \Delta\varphi\, \psi \, d\Omega_f$ via Green's First Theorem with test function $\psi \in \mathbf{H}^1(\Omega_f)$, $\psi|_{\Gamma_f} = 0$, see Appendix B. In (6.14b), we have also used (1.22b) on D_s for $r = -\frac{1}{2}$. We notice that the regularity of the top term in (6.12) as being in $\mathcal{D}(A_{D,s})$; see (6.14a), is tight, and so is the regularity of the bottom term. □

Corollary 6.2 *With reference to problem (6.1a)–(6.1e), we have via the variation of parameters formula on (6.5)*

$$\begin{bmatrix} w(t) \\ w_t(t) \\ u(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \\ u_0 \end{bmatrix} + (\mathbf{L}_D g)(t); \tag{6.15}$$

$$(\mathbf{L}_D g)(t) = \int_0^t e^{\mathcal{A}(t-\tau)} \mathcal{B}_D g(\tau) d\tau \tag{6.16}$$

$$= \int_0^t \mathcal{A} e^{\mathcal{A}(t-\tau)} \mathcal{A}^{-1} \mathcal{B}_D g(\tau) d\tau \tag{6.17}$$

$$: L_p(0, \infty; \mathbf{H}^{\frac{1}{2}}(\Gamma_s)) \to L_p(0, \infty; \mathbf{H}), \ 1 < p < \infty. \tag{6.18}$$

Moreover

$$((-\mathcal{A})^{-\frac{1}{2}} \mathbf{L} g)(t) = -\int_0^t (-\mathcal{A})^{\frac{1}{2}} e^{\mathcal{A}(t-\tau)} \mathcal{A}^{-1} \mathcal{B}_D g(\tau) d\tau \tag{6.19}$$

$$: L_2(0, T; \mathbf{H}^{\frac{1}{2}}(\Gamma_s)) \to C([0, T]; \mathbf{H}) \tag{6.20}$$

$$(\mathbf{L} g)(\tau) : L_2(0, T; \mathbf{H}^{\frac{1}{2}}(\Gamma_s)) \to C([0, T]; \left[\mathcal{D}((-\mathcal{A}^*)^{\frac{1}{2}})\right]') \tag{6.21}$$

continuously, with $1 < p < \infty$, recalling (6.13) and invoking De Simon's result [19], [31, p. 4] in (6.18), (6.19).

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Appendix A: Proof of Theorem 1.2 for $\mathcal{A}^*$ on H

Proof of (i). For $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$ in (1.8d) and $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}^*)$ in (1.27)–(1.30), we return to (1.5), (1.3) and compute by Green's second theorem, recalling that

$$v_2^*\Big|_{\Gamma_f} = 0, \quad v_2\Big|_{\Gamma_s} = 0, \quad h^*\Big|_{\Gamma_f} = 0, \quad h\Big|_{\Gamma_f} = 0, \tag{A.1}$$

and that ν is inward to Ω_s:

$$\left(\mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}} = \left(\begin{bmatrix} v_2 \\ -\Delta(\Delta v_1 - v_2) \\ \Delta h \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}} \tag{A.2}$$

$$= (\Delta v_2, \Delta v_1^*) - (\Delta(\Delta v_1 - v_2), v_2^*) + \left(\Delta h, h^*\right) \tag{A.3}$$

$$= \left[(v_2, \Delta^2 v_1^*) - \left(\frac{\partial v_2}{\partial \nu}, \Delta v_1^*\right)_{\Gamma_s} + \left(v_2, \frac{\partial \Delta v_1^*}{\partial \nu}\right)_{\Gamma_s} \right]$$
$$- \left[(\Delta v_1 - v_2, \Delta v_2^*) - \left(\frac{\partial(\Delta v_1 - v_2)}{\partial \nu}, v_2^*\right)_{\Gamma_s} + \left(\Delta v_1, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s} - \left(v_2, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s} \right]$$
$$+ \left[(h, \Delta h^*) + \left(\frac{\partial h}{\partial \nu}, h^*\right)_{\Gamma_s} - \left(h, \frac{\partial h^*}{\partial \nu}\right)_{\Gamma_s} \right] \tag{A.4}$$

[next we recall from (1.7a), (1.8b), (1.28b), (1.29b)

$$\left.\frac{\partial v_2}{\partial \nu}\right|_{\Gamma_s} = h\Big|_{\Gamma_s}; \quad \Delta v_1^*\Big|_{\Gamma_s} = -\left.\frac{\partial h^*}{\partial \nu}\right|_{\Gamma_s}; \quad \Delta v_1\Big|_{\Gamma_s} = \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s}; \quad \left.\frac{\partial v_2^*}{\partial \nu}\right|_{\Gamma_s} = h^*\Big|_{\Gamma_s} \Bigg] \tag{A.5}$$

to obtain from (A.4)

$$= \left[(v_2, \Delta^2 v_1^*) + \left(h, \frac{\partial h^*}{\partial \nu}\right)_{\Gamma_s} \right] - \left[(\Delta v_1, \Delta v_2^*) - (v_2, \Delta v_2^*) + \left(\frac{\partial h}{\partial \nu}, h^*\right)_{\Gamma_s} \right]$$
$$+ \left[(h, \Delta h^*) + \left(\frac{\partial h}{\partial \nu}, h^*\right)_{\Gamma_s} - \left(h, \frac{\partial h^*}{\partial \nu}\right)_{\Gamma_s} \right] \tag{A.6}$$

(thus we see that the four boundary terms in (A.6) cancel out pairwise)

$$= (v_2, \Delta^2 v_1^*) - (\Delta v_1, \Delta v_2^*) + (v_2, \Delta v_2^*) + (h, \Delta h^*) \tag{A.7}$$

$$= \left(\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}}$$
$$= \left(\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} -v_2^* \\ \Delta(\Delta v_1^* + v_2^*) \\ \Delta h^* \end{bmatrix} \right)_{\mathbf{H}} = \left(\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \mathcal{A}^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}}. \tag{A.8}$$

Thus, (A.8) proves part (i). Part (ii) was used in the proof. □

Proof of (iii). We proceed as in the proof of Theorem 1.1 (i), using that $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}^*)$, as in (1.27)–(1.29), and (1.26), in particular $v_2^*\big|_{\Gamma_s} = 0$

$$\left(\mathcal{A}^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}} = \left(\begin{bmatrix} -v_2^* \\ \Delta(\Delta v_1^* + v_2^*) \\ \Delta h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_{\mathbf{H}} \tag{A.9}$$
$$= -(\Delta v_2^*, \Delta v_1^*) + (\Delta(\Delta v_1^* + v_2^*), v_2^*) + (\Delta h^*, h^*) \tag{A.10}$$
$$= -(\Delta v_2^*, \Delta v_1^*) + \left[(\Delta v_1^* + v_2^*, \Delta v_2^*) - \left(\frac{\partial(\Delta v_1^* + v_2^*)}{\partial \nu}, v_2^*\right)_{\Gamma_s} \right.$$

$$+\left(\Delta v_1^*, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s} + \cancel{\left(v_2^*, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s}}\Bigg] + \left[\left(\frac{\partial h^*}{\partial \nu}, h^*\right)_{\Gamma_s} - \left\|\nabla h^*\right\|^2\right] \tag{A.11}$$

$$\text{(by } (A.5)) \quad = -(\Delta v_2^*, \Delta v_1^*) + \left[(\Delta v_1^*, \Delta v_2^*) + (v_2^*, \Delta v_2^*) + \cancel{\left(\Delta v_1^*, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s}}\right]$$

$$+\left[-\cancel{\left(\Delta v_1^*, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s}} - \left\|\nabla h^*\right\|^2\right] \tag{A.12}$$

$$\text{(by } (A.1)) \quad = -(\Delta v_2^*, \Delta v_1^*) + (\Delta v_1^*, \Delta v_2^*) - \cancel{\left(v_2^*, \frac{\partial v_2^*}{\partial \nu}\right)_{\Gamma_s}} - \left\|\nabla v_2^*\right\|^2 - \left\|\nabla h^*\right\|^2. \tag{A.13}$$

Taking now the real part in (A.13) and using $\text{Re}\{-z+\bar{z}\} = 0$, we obtain for $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A}^*)$:

$$\text{Re}\left(\mathcal{A}^*\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}\right)_{\mathbf{H}} = -\left\|\nabla v_2^*\right\|^2 - \left\|\nabla h^*\right\|^2, \tag{A.14}$$

and part (iii) is established. □

Appendix B: Spectral theory

In this appendix, we consider the eigenvalue problem for the operator $\mathcal{A}$ in (1.5) with potential eigenvalue $\alpha+i\omega$, $\alpha < 0$, $\omega \in \mathbb{R}$, and corresponding eigenvector $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -\Delta(\Delta v_1 - v_2) \\ \Delta h \end{bmatrix} = (\alpha + i\omega)\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \tag{B.1a}$$

$$v_2 = (\alpha + i\omega)v_1, \quad \Delta(\Delta v_1 - v_2) = -(\alpha + i\omega)v_2, \quad \Delta h = (\alpha + i\omega)h. \tag{B.1b}$$

Proposition B.1 *With reference to (B.1), we have: there is a constant $k > 0$, such that*

$$k|\omega|\|h\|^2 \le \|\nabla h\|^2, \tag{B.2}$$

with constant k identified in the proof below.

Proof Consider the full problem of the h-variable in (B.1a), recalling (1.8b)

$$\begin{cases} \Delta h = (\alpha + i\omega)h, & \text{(B.3a)} \\ h\Big|_{\Gamma_f} = 0, \quad h\Big|_{\Gamma_s} = \dfrac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s). & \text{(B.3b)} \end{cases}$$

We want to establish that

$$\left\|\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \le c\|h\|_{\mathbf{H}^1(\Omega_f)}. \tag{B.4}$$

To this end, it suffices to take $\{\alpha, \omega\} = \{0, 0\}$, so that $h = \widetilde{D}_f\left(\dfrac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s}\right) \in \mathbf{H}^1(\Omega_f)$ with $\widetilde{D}_f$ in (1.19b)–(1.19c).

Next, the first Green theorem on Eq. (B.3a) (with $(\alpha + i\omega) = 0$) with a test function $\phi \in \mathbf{H}^1(\Omega_f)$, $\phi\Big|_{\Gamma_f} = 0$ yields

$$\int_{\Omega_f} \Delta h \bar{\phi}\, d\Omega_f = \int_{\Gamma_s} \frac{\partial h}{\partial \nu}\bar{\phi}\, d\Gamma_s - \int_{\Omega_f} \nabla h \cdot \nabla\bar{\phi}\, d\Omega_f = 0. \tag{B.5}$$

This makes the boundary integral on Γ_s well defined with $\bar{\phi} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$. By surjectivity of the trace [35], as $\bar{\phi}$ runs over all of $\mathbf{H}^1(\Omega_f)$, $\phi\Big|_{\Gamma_f} = 0$, then $\bar{\phi}\Big|_{\Gamma_s}$ runs over all of $\mathbf{H}^{\frac{1}{2}}(\Gamma_s)$ and this then gives $\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$, bounded by $h \in \mathbf{H}^1(\Omega_f)$ as in (B.4) via the bilinear form (B.5).

Next, from the third h-relation in (B.1), via Greens' first theorem, recalling that $h\Big|_{\Gamma_f} = 0$ by (1.8b), we obtain as ν is outward

$$(\Delta h, h) = \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} - \|\nabla h\|^2 = (\alpha + i\omega)\|h\|^2. \tag{B.6}$$

The imaginary part of (B.6) is

$$\omega\|h\|^2 = \operatorname{Im}\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}. \tag{B.7}$$

Invoking (B.4), we estimate via (B.7), as a duality pairing

$$|\omega|\|h\|^2 \le \left|\left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}, h\big|_{\Gamma_s}\right)_{\Gamma_s}\right| \le C\left\|\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \left\|h\big|_{\Gamma_s}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_s)} \tag{B.8}$$

$$\text{(by (B.4))} \qquad \le C\|h\|_{\mathbf{H}^1(\Omega_s)}\|h\|_{\mathbf{H}^1(\Omega_s)} = C\left[\|\nabla h\|^2 + \|h\|^2\right] \tag{B.9}$$

$$\text{(by (1.32))} \qquad \le C\|\nabla h\|^2 + \frac{C}{c_{p,f}}\|\nabla h\|^2 = \frac{1}{k}\|\nabla h\|^2 \tag{B.10}$$

recalling the Poincare inequality for h in (1.32), as $h\big|_{\Gamma_f} = 0$. Hence

$$k|\omega|\|h\|^2 \le \|\nabla h\|^2, \quad k = \frac{c_{p,f}}{C(1 + c_{p,f})}. \tag{B.11}$$

□

Theorem B.2 *Let the Poincare constants $c_p > 0$ and $c_{p,s} > 0$ be defined in (1.33) and (1.32), respectively. Then, with reference to the eigenvalue problem (B.1) for $\mathcal{A}$, we have the following:*

(i) there is no eigenvalue of $\mathcal{A}$ in the vertical strip $-\frac{c_p}{2} < \alpha \le 0$, $\omega \ne 0$;
(ii) there is no eigenvalue of $\mathcal{A}$ in the two upper and lower sets $-\frac{c_{p,s}}{2} < \alpha < 0$ and $|\omega| > \frac{2|\alpha|}{k}$ of Fig. 4;
(iii) [complementing (i) and (ii)] On the other hand, let $(\alpha + i\omega)$, $\alpha < 0$, be an eigenvalue of $\mathcal{A}$, then $\alpha \le -\frac{c_p}{2}$, for $\omega \ne 0$.

Proof (i) **Step 1:** We return to Eq. (2.5) for problem (B.1) = problem (2.1a) with $\{v_1^* = 0, v_2^* = 0, h^* = 0\}$. We obtain

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha\left\{\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right\} + i\omega\left\{\|v_2\|^2 + \|h\|^2 - \|\Delta v_1\|^2\right\} = 0. \tag{B.12}$$

Step 2: We next take the Real part of (B.12), as well as, for $\omega \ne 0$, the imaginary part. We obtain

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha\left\{\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right\} = 0 \tag{B.13}$$

$$\|v_2\|^2 + \|h\|^2 = \|\Delta v_1\|^2, \quad \omega \ne 0. \tag{B.14}$$

Substituting (B.14) in (B.13) yields

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + 2\alpha\left[\|v_2\|^2 + \|h\|^2\right] = 0, \quad \omega \ne 0. \tag{B.15}$$

Step 3: To show (i), let us at first invoke inequality (1.33) in (B.15). We obtain

$$(c_p + 2\alpha)\left[\|v_2\|^2 + \|h\|^2\right] \leq 0, \quad \omega \neq 0. \tag{B.16}$$

Thus, if $c_p + 2\alpha > 0$, as assumed, then (B.16) implies $v_2 \equiv 0$ in Ω_s, and hence, $v_1 \equiv 0$ in Ω_s by (B.1b) with $\omega \neq 0$, and $h \equiv 0$ in Ω_f. Thus, there are no eigenvalues of $\mathcal{A}$ in the vertical strip $-\frac{c_p}{2} < \alpha \leq 0, \omega \neq 0$. Part (i) is proved.

(ii) Next, we return to (B.15): on the v_2-variable, we invoke the Poincare constant $c_{p,s}$ in (1.32), while for the h-variable, we invoke inequality (B.2). We obtain

$$[c_{p,s} + 2\alpha]\|v_2\|^2 + [k|\omega| + 2\alpha]\|h\|^2 \leq 0, \quad \omega \neq 0. \tag{B.17}$$

Thus, if $c_{p,s} + 2\alpha > 0$ and $k|\omega| + 2\alpha > 0$, i.e., if

$$\alpha > -\frac{c_{p,s}}{2} \quad \text{and} \quad |\omega| > \frac{2|\alpha|}{k}, \tag{B.18}$$

then (B.14) implies $v_2 \equiv 0$; thus, $v_1 \equiv 0$ by (B.1b) and $h \equiv 0$ as desired. This shows (ii).

Remark B.1 Return to Eq. (2.4) for problem (B.1) = problem (2.1a) with $\{v_1^* = 0, v_2^* = 0, h^* = 0\}$ to obtain

$$(\alpha + i\omega)\|v_2\|^2 + (\alpha - i\omega)\|\Delta v_1\|^2 + \|\nabla v_2\|^2 + \left(\frac{\partial h}{\partial \nu}\Big|_{\Gamma_s}, h\big|_{\Gamma_s}\right)_{\Gamma_s} = 0. \tag{B.19}$$

Take the imaginary part of (B.19)

$$\omega\left[\|\Delta v_1\|^2 - \|v_2\|^2\right] = \operatorname{Im}\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} \tag{B.20}$$

$$\text{(by (B.10))} \qquad |\omega|\left|\|\Delta v_1\|^2 - \|v_2\|^2\right| \leq \frac{1}{k}\|\nabla h\|^2, \tag{B.21}$$

recalling the computations in (B.8)–(B.10). Invoking now (B.14) on the LHS of (B.21), we re-obtain Proposition B.1

$$k|\omega|\|h\|^2 \leq \|\nabla h\|^2. \tag{B.22}$$

(iii) Now, let $(\alpha + i\omega)$, $\alpha < 0$ be an eigenvalue of $\mathcal{A}$, with normalized eigenvector $[v_1, v_2, h]$, $\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2 = 1$. Then, by (B.13), (B.14), we obtain for $\omega \neq 0$

$$\|\nabla v_2\|^2 + \|\nabla h\|^2 + \alpha = 0, \quad \|\Delta v_1\|^2 = \|v_2\|^2 + \|h\|^2 = \frac{1}{2}, \quad \omega \neq 0. \tag{B.23}$$

Invoking inequality (1.33) on the LHS of (B.23), we obtain, also via the RHS of (B.23)

$$c_p\left[\|v_2\|^2 + \|h\|^2\right] + \alpha \leq 0, \quad \text{or} \quad \frac{c_p}{2} + \alpha \leq 0, \quad \omega \neq 0. \tag{B.24}$$

In conclusion, if $(\alpha + i\omega)$ is an eigenvalue of $\mathcal{A}$ with $\omega \neq 0$, then $\alpha \leq -\frac{c_p}{2}$, as desired and part (iii) is proved. Moreover, $\|\nabla h\|^2 \leq |\alpha|$ and (B.22) imply $k|\omega|\|h\|^2 \leq |\alpha|$.

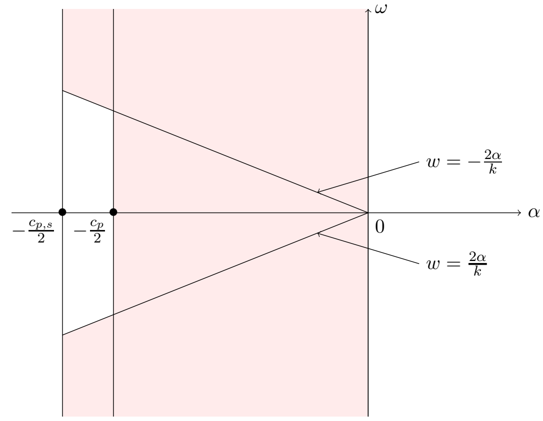
□

Appendix C: A look at the operator $\mathcal{A}$ in (4.18) from a perturbation view point: $\mathcal{A} = \mathcal{A}_0 + \mathcal{P}$

Consider the operator [31, Chapter 3, Appendix B, p.288]

$$\mathcal{A}_0\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A_{D,s}^2 & -A_{D,s} & 0 \\ 0 & 0 & A_{D,f} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -A_{D,s}\left[A_{D,s}v_1 + v_2\right] \\ A_{D,f}h \end{bmatrix} \in \mathbf{H} \tag{C.1}$$

Fig. 4 No eigenvalues in shaded area, $\omega \neq 0$



$$\mathbf{H} = \mathcal{D}(A_{D,s}) \times \mathbf{L}^2(\Omega_s) \times \mathbf{L}^2(\Omega_f); \tag{C.2}$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{v_1, v_2 \in \mathcal{D}(A_{D,s}) : \left[A_{D,s}v_1 + v_2\right] \in \mathcal{D}(A_{D,s}); h \in \mathcal{D}(A_{D,f})\right\}. \tag{C.3}$$

Let $\mathcal{P}$ be the 'perturbation' operator

$$\mathcal{P} = \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -A_{D,s}D_s\left(\left.\frac{\partial\,\cdot}{\partial\nu}\right|_{\Gamma_s}\right) \\ 0 & A_{D,f}\widetilde{D}_f\left(\left.\frac{\partial\,\cdot}{\partial\nu}\right|_{\Gamma_s}\right) & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ -A_{D,s}D_s\left(\left.\frac{\partial h}{\partial\nu}\right|_{\Gamma_s}\right) \\ A_{D,f}\widetilde{D}_f\left(\left.\frac{\partial v_2}{\partial\nu}\right|_{\Gamma_s}\right) \end{bmatrix}, \tag{C.4}$$

so that $\mathcal{A} = \mathcal{A}_0 + \mathcal{P}$, recalling (4.18). One may ask the question whether $\mathcal{P}$ is $\mathcal{A}_0$-bounded, i.e., whether with $\mathbf{x} = [v_1, v_2, h] \in \mathcal{D}(\mathcal{A}_0)$

$$\|\mathcal{P}\mathbf{x}\|_{\mathbf{H}} \leq \text{const}\|\mathcal{A}_0\mathbf{x}\|_{\mathbf{H}}, \tag{C.5a}$$

equivalently with $\mathbf{x} = \mathcal{A}_0^{-1}\mathbf{y} \in \mathcal{D}(\mathcal{A}_0)$, $\mathbf{y} \in \mathbf{H}$

$$\|\mathcal{P}\mathcal{A}_0^{-1}\mathbf{y}\|_{\mathbf{H}} \leq \text{const}\|\mathbf{y}\|_{\mathbf{H}}. \tag{C.5b}$$

Claim: The answer is in the negative.

Proof First, verify that

$$\mathcal{A}_0^{-1} = \begin{bmatrix} -A_{D,s}^{-1} & -A_{D,s}^{-2} & 0 \\ I & 0 & 0 \\ 0 & 0 & -A_{D,f}^{-1} \end{bmatrix}; \qquad \mathcal{A}_0^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -A_{D,s}^{-1}\left(A_{D,s}^{-1}y_2 + y_1\right) \\ y_1 \\ -A_{D,f}^{-1}y_3 \end{bmatrix}; \tag{C.6}$$

hence, by (C.4) and (C.6)

$$\mathcal{P}\mathcal{A}_0^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ A_{D,s}D_s\left(\left.\frac{\partial A_{D,f}^{-1}y_3}{\partial\nu}\right|_{\Gamma_s}\right) \\ A_{D,f}\widetilde{D}_f\left(\left.\frac{\partial y_1}{\partial\nu}\right|_{\Gamma_s}\right) \end{bmatrix}. \tag{C.7}$$

Then, (C.7) shows that assertion (C.5b) is false. With $y_3 \in \mathbf{L}^2(\Omega_f)$, then $A_{D,f}^{-1} y_3 \in \mathcal{D}(A_{D,f}) = \mathbf{H}^2(\Omega_f) \cap \mathbf{H}_0^1(\Omega_f)$, hence $\left.\frac{\partial A_{D,f}^{-1} y_3}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$, hence $D_s \left(\left.\frac{\partial A_{D,f}^{-1} y_3}{\partial \nu}\right|_{\Gamma_s} \right) \in \mathbf{H}^1(\Omega_s)$, by (1.22b) with $r = \frac{1}{2}$. Then, by (1.22c)

$$A_{D,s} D_s \left(\left.\frac{\partial A_{D,f}^{-1} y_3}{\partial \nu}\right|_{\Gamma_s} \right) \equiv A_{D,s}^{\frac{3}{4}+\epsilon} A_{D,s}^{\frac{1}{4}-\epsilon} D_s \left(\left.\frac{\partial A_{D,f}^{-1} y_3}{\partial \nu}\right|_{\Gamma_s} \right) \in [\mathcal{D}(A_{D,s}^{3/4+\epsilon})]', \text{ but } \notin \mathbf{L}^2(\Omega_s).$$

Similarly, $y_1 \in \mathcal{D}(A_{D,s}) = \mathbf{H}^2(\Omega_s) \cap \mathbf{H}_0^1(\Omega_s)$, implies $\left.\frac{\partial y_1}{\partial \nu}\right|_{\Gamma_s} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$, hence $\widetilde{D}_f \left(\left.\frac{\partial y_1}{\partial \nu}\right|_{\Gamma_s} \right) \in \mathbf{H}^1(\Omega_f)$ by (1.19c) for $r = \frac{1}{2}$, and then $A_{D,f} \widetilde{D}_f \left(\left.\frac{\partial y_1}{\partial \nu}\right|_{\Gamma_s} \right) = A_{D,f}^{\frac{3}{4}+\epsilon} A_{D,f}^{\frac{1}{4}-\epsilon} \widetilde{D}_f \left(\left.\frac{\partial y_1}{\partial \nu}\right|_{\Gamma_s} \right) \notin \mathbf{L}^2(\Omega_f)$. □

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