TORELLI-TYPE THEOREMS FOR GRAVITATIONAL INSTANTONS WITH QUADRATIC VOLUME GROWTH

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Abstract

We prove Torelli-type uniqueness theorems for both ALG^* *gravitational instantons and ALG gravitational instantons which are of order* 2*. That is, the periods uniquely characterize these types of gravitational instantons up to diffeomorphism. We define a period mapping* $\mathscr{P}$*, which we show is surjective in the ALG cases, and has open image in the* ALG^* *cases. We also construct some new degenerations of hyper-Kähler metrics on the K3 surface which exhibit bubbling of* ALG^* *gravitational instantons.*

1. Introduction

We begin with the following definitions.

Definition 1.1
A hyper-Kähler 4-manifold (X, g, I, J, K) is a Riemannian 4-manifold (X, g) with a triple of Kähler structures (g, I), (g, J), (g, K) such that $IJ = K$.

We denote by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the Kähler forms associated to I, J, K, respectively. It is easy to see that ω_i satisfies

$$\omega_i \wedge \omega_j = 2\delta_{ij}\, \mathrm{dvol}_g, \tag{1.1}$$

where dvol_g is the Riemannian volume element. Conversely, any triple of symplectic forms ω_i satisfying (1.1) determines a hyper-Kähler structure if we replace ω_3 by $-\omega_3$ if necessary.

Definition 1.2
A *gravitational instanton* $(X, g, \boldsymbol{\omega})$ is a noncompact complete nonflat hyper-Kähler 4-manifold X such that $|\mathrm{Rm}_g| \in L^2(X)$.

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If X is a *compact* nonflat hyper-Kähler 4-manifold, then it must be the K3 surface (see [28]). If X is a gravitational instanton, there are many known types of asymptotic geometry of X near infinity: ALE, ALF-A_k, ALF-D_k, ALG, ALH, ALG*, ALH*. We refer the reader to [1], [6]–[8], [26], [29], and [32] and the references therein for more background on gravitational instantons.

There is a well-known Torelli theorem for hyper-Kähler metrics on the K3 surface, and one may ask whether there is an analogue for gravitational instantons. This is known to hold in several cases: such a Torelli-type theorem was proved by [29] in the ALE case, by [33] in the ALF-A_k case, by [6] in the ALF-D_k case, and by [8] in the ALH case. In this paper, we are interested in an analogous result assuming that the metric is of type ALG or ALG*. In the ALG case, it was observed in [8] that the natural period mapping may not be injective, and a modified version of the Torelli theorem was conjectured there. In this paper, we prove the uniqueness part of this conjecture, which gives the Torelli uniqueness in the ALG case (see Theorem 1.5). We also prove a Torelli-type uniqueness theorem in the ALG* case (see Theorem 1.10). We note that recently, a Torelli-type uniqueness theorem in the ALH* case was proved (see [13]). We will also define a refined period mapping $\mathscr{P}$ in both the ALG and ALG* cases, which we will show to be surjective in the ALG cases, and open in the ALG* cases (see Theorem 1.7 and Theorem 1.12).

Previously, gravitational instantons of type ALE, ALF, ALG, ALH, and ALH* have been shown to bubble off of the K3 surface (see, e.g., [8], [10], [16], [18], [27], [30]). In this paper, we also show that there exist families of Ricci-flat hyper-Kähler metrics on the K3 surface which have ALG* gravitational instantons occurring as bubbles (see Theorem 1.13). These are the first known examples of this type of degeneration. These examples are produced via a gluing theorem which is actually the crucial tool in proving the aforementioned Torelli uniqueness in the ALG* case.

1.1. ALG gravitational instantons

For background on analysis on ALG gravitational instantons, related classification results, and relations to moduli spaces of monopoles and Higgs bundles, we refer the reader to [3], [4], [8], [12], [20], [24], [26], and [31] and the references therein.

In Definition 2.3 below, we will define the standard ALG model space $(\mathcal{C}_{\beta,\tau,L}(R), g^{\mathcal{C}}, \omega^{\mathcal{C}})$ for parameters $L, R \in \mathbb{R}_+$, and (β,τ) as in Table 1. Here we just note that $\mathcal{C}_{\beta,\tau,L}(R)$ is diffeomorphic to $(R,\infty)\times N^3_\beta$, where N^3_β is a torus bundle over a circle, and the metric $g^{\mathcal{C}}$ as well as the induced metric on the 3-manifold N^3_β are flat (the explicit formulas are given in Section 2.3). We let r denote the coordinate on (R,∞).

Table 1. Invariants of ALG spaces.

∞	I_0^*	II	II*	III	III*	IV	IV*
$\beta \in (0,1]$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
$\tau \in \mathbb{H}$	Any	$e^{\sqrt{-1}\cdot\frac{2\pi}{3}}$	$e^{\sqrt{-1}\cdot\frac{2\pi}{3}}$	$\sqrt{-1}$	$\sqrt{-1}$	$e^{\sqrt{-1}\cdot\frac{2\pi}{3}}$	$e^{\sqrt{-1}\cdot\frac{2\pi}{3}}$
$b_2(X_\beta)$	5	9	1	8	2	7	3

Definition 1.3 (ALG gravitational instanton)
A complete hyper-Kähler 4-manifold $(X, g, \boldsymbol{\omega})$ is called an *ALG gravitational instanton* of order $\mathfrak{n} > 0$ with parameters (β, τ) as in Table 1, and $L > 0$ if there exist $R > 0$, a compact subset $X_R \subset X$, and a diffeomorphism $\Phi : \mathcal{C}_{\beta,\tau,L}(R) \to X \setminus X_R$ such that

$$\left|\nabla^k_{g^{\mathcal{C}}}(\Phi^* g - g^{\mathcal{C}})\right|_{g^{\mathcal{C}}} = O(r^{-k-\mathfrak{n}}), \tag{1.2}$$

$$\left|\nabla^k_{g^{\mathcal{C}}}(\Phi^* \omega_i - \omega_i^{\mathcal{C}})\right|_{g^{\mathcal{C}}} = O(r^{-k-\mathfrak{n}}), \quad i = 1, 2, 3, \tag{1.3}$$

as $r \to \infty$, for any $k \in \mathbb{N}_0$.

Remark 1.4
It was proved in [6, Theorem A] that there exist ALG coordinates so that the order $\mathfrak{n}$ is 2 in the I_0^*, II, III, IV cases ($\beta \le \frac{1}{2}$) and $\mathfrak{n} = 2 - \frac{1}{\beta}$ in the II*, III*, IV* cases ($\beta > \frac{1}{2}$).

It was shown in [9, Theorem 1.10] that any two ALG gravitational instantons with the same β are diffeomorphic. So without loss of generality we can view any ALG gravitational instanton as living on a fixed space X_β. The first author and Chen proved that the naive version of the Torelli-type theorem fails when $\beta > 1/2$ (see [8]). Furthermore, it was shown in [9, Theorem 1.12] that when $\beta > 1/2$, each ALG gravitational instanton lives in a two-parameter family of ALG gravitational instantons with the same periods $[\boldsymbol{\omega}]$, with exactly one element of this family being of order 2. This reduces the general case to proving a Torelli uniqueness theorem for ALG gravitational instantons of order 2, which is our next theorem.

THEOREM 1.5 (ALG Torelli uniqueness)
Let $(X_\beta, g, \boldsymbol{\omega})$ and $(X_\beta, g', \boldsymbol{\omega}')$ be two ALG *gravitational instantons with the same τ and L, which are both* ALG *of order* 2 *with respect to a fixed* ALG *coordinate system on X_β. If*

$$[\boldsymbol{\omega}] = [\boldsymbol{\omega}'] \in H^2_{\mathrm{dR}}(X_\beta) \otimes \mathbb{R}^3, \tag{1.4}$$

then there is a diffeomorphism $\Psi : X_\beta \to X_\beta$ *which induces the identity map on* $H^2_{\mathrm{dR}}(X_\beta)$ *such that* $\Psi^* g' = g$ *and* $\Psi^* \boldsymbol{\omega}' = \boldsymbol{\omega}$.

This will be proved using a modification of the gluing construction in [10] (see Theorem 6.2 below), and then invoking the Torelli theorem for K3 surfaces. We remark that the order-2 condition is essential to control the error term in the gluing construction. Moreover, the assumption that both hyper-Kähler structures are ALG of order 2 in a *fixed* coordinate system is also crucial for the proof. However, it is superfluous in the following sense: it was proved in [9, Theorem 1.11] that any two ALG gravitational instantons of order 2 with the same (β, τ) and L can be pulled back to a *fixed* space X_β such that they are both ALG of order 2 in a *fixed* ALG coordinate system Φ_{X_β} (after possibly modifying one of the ALG coordinate systems). This motivates the following definition.

Definition 1.6
Let $\mathcal{M}_{\beta,\tau,L}$ be the collection of all gravitational instantons on X_β with parameters β, τ, and L which are ALG of order 2 with respect to a fixed ALG coordinate system Φ_{X_β}. For $(X_\beta, g^0, \boldsymbol{\omega}^0) \in \mathcal{M}_{\beta,\tau,L}$, the period mapping

$$\mathscr{P} : \mathcal{M}_{\beta,\tau,L} \to [\boldsymbol{\omega}^0] + \mathscr{H}^2 \otimes \mathbb{R}^3 \subset H^2_{\mathrm{dR}}(X_\beta) \otimes \mathbb{R}^3 \tag{1.5}$$

is defined by

$$\mathscr{P}(\boldsymbol{\omega}) = \big([\omega_1], [\omega_2], [\omega_3]\big), \tag{1.6}$$

where $\mathscr{H}^2 \equiv \mathrm{Im}(H^2_{\mathrm{cpt}}(X_\beta) \to H^2(X_\beta))$.

We will show that $\mathscr{P}$ is well defined in Section 7. The following is our main result about the period mapping in the ALG cases.

THEOREM 1.7
If $(X_\beta, g, \boldsymbol{\omega}) \in \mathcal{M}_{\beta,\tau,L}$, *then*

$$[\boldsymbol{\omega}] \cdot [C] \neq (0,0,0) \quad \textit{for all } [C] \in H_2(X_\beta; \mathbb{Z}) \textit{ satisfying } [C]^2 = -2. \tag{1.7}$$

Furthermore, the period mapping $\mathscr{P}$ *is surjective onto cohomology triples in* $[\boldsymbol{\omega}^0] + \mathscr{H}^2 \otimes \mathbb{R}^3$ *satisfying (1.7).*

We will prove this in Section 7. In particular, we see that the image of the period mapping has dimension $3(b_2(X_\beta) - 1)$, where $b_2(X_\beta)$ is given in Table 1.

1.2. ALG* *gravitational instantons*

In Section 2, we will define the *standard* ALG* *model space*, which is denoted by

$$\big(\mathfrak{M}_{2\nu}(R), g^{\mathfrak{M}}_{\kappa_0,L}, \boldsymbol{\omega}^{\mathfrak{M}}_{\kappa_0,L}\big) \equiv \big(\mathfrak{M}_{2\nu}(R), L^2 g^{\mathfrak{M}}_{\kappa_0}, L^2\boldsymbol{\omega}^{\mathfrak{M}}_{\kappa_0}\big), \tag{1.8}$$

which depends on parameters $\nu \in \mathbb{Z}_+$, $\kappa_0 \in \mathbb{R}$, $R > 0$, and an overall scaling parameter $L > 0$. Here we just note that the manifold $\mathfrak{M}_{2\nu}(R)$ is diffeomorphic to $(R,\infty) \times \mathcal{I}^3_\nu$, where $\mathcal{I}^3_\nu$ is an infra-nilmanifold, which is a circle bundle of degree ν over a Klein bottle. We will let r denote the coordinate on (R,∞), V denote the function $\kappa_0 + \frac{\nu}{\pi}\log r$, and $\mathfrak{s}$ denote the function $rV^{1/2}$. The hyper-Kähler structure is obtained via a Gibbons–Hawking ansatz. (See Section 2 for explicit formulas.)

*Definition 1.8 (*ALG* *gravitational instanton)*

A complete hyper-Kähler 4-manifold $(X, g, \boldsymbol{\omega})$ is called an ALG* *gravitational instanton* of order $\mathfrak{n} > 0$ with parameters $\nu \in \mathbb{Z}_+$, $\kappa_0 \in \mathbb{R}$, and $L > 0$ if there exist an ALG* model space $(\mathfrak{M}_{2\nu}(R), g^{\mathfrak{M}}_{\kappa_0,L}, \boldsymbol{\omega}^{\mathfrak{M}}_{\kappa_0,L})$ with $R > 0$, a compact subset $X_R \subset X$, and a diffeomorphism $\Phi : \mathfrak{M}_{2\nu}(R) \to X \setminus X_R$ such that

$$\big|\nabla^k_{g^{\mathfrak{M}}}(\Phi^* g - g^{\mathfrak{M}}_{\kappa_0,L})\big|_{g^{\mathfrak{M}}} = O(\mathfrak{s}^{-k-\mathfrak{n}}), \tag{1.9}$$

$$\big|\nabla^k_{g^{\mathfrak{M}}}(\Phi^* \omega_i - \omega^{\mathfrak{M}}_{i,\kappa_0,L})\big|_{g^{\mathfrak{M}}} = O(\mathfrak{s}^{-k-\mathfrak{n}}), \quad i = 1,2,3, \tag{1.10}$$

as $\mathfrak{s} \to \infty$, for any $k \in \mathbb{N}_0$.

Remark 1.9

It was proved in [11, Theorem 1.10] that there exist ALG* coordinates on X so that the order satisfies $\mathfrak{n} \geq 2$. This decay order will be crucial in the proof of Theorem 1.10 below.

It was proved in [9, Theorem 1.6] that any two ALG* gravitational instantons with the same ν, where $1 \leq \nu \leq 4$, are diffeomorphic to each other. So without loss of generality we can view any ALG* gravitational instanton as living on a fixed space X_ν. With this understood, our next theorem is a Torelli uniqueness theorem for ALG* gravitational instantons.

THEOREM 1.10 (ALG* Torelli uniqueness)

Let $1 \leq \nu \leq 4$, *and let* $(X_\nu, g, \boldsymbol{\omega})$, $(X_\nu, g', \boldsymbol{\omega}')$ *be two* ALG* *gravitational instantons with the same parameters* κ_0 *and* L, *which are both* ALG* *of order* 2 *with respect to a fixed* ALG* *coordinate system on* X_ν. *If*

$$[\boldsymbol{\omega}] = [\boldsymbol{\omega}'] \in H^2_{\mathrm{dR}}(X_\nu) \otimes \mathbb{R}^3, \tag{1.11}$$

then there is a diffeomorphism $\Psi : X_\nu \to X_\nu$ *which induces the identity map on* $H^2_{\mathrm{dR}}(X_\nu)$ *such that* $\Psi^* g' = g$ *and* $\Psi^* \boldsymbol{\omega}' = \boldsymbol{\omega}$*.*

This will be proved using a new gluing construction: we obtain hyper-Kähler metrics on the K3 surface using ALG* gravitational instantons (see Section 1.3 below), and then we invoke the Torelli theorem for K3 surfaces. In our proof, the requirement that both metrics are ALG* of order 2 with respect to a *fixed* ALG* coordinate system is crucial. However, this assumption is actually superfluous in the following sense. It was proved in [9, Theorem 1.7] that if $(X, g, \boldsymbol{\omega})$ and $(X', g', \boldsymbol{\omega}')$ are any two ALG* gravitational instantons of order 2 with the same parameters ν, κ_0, and L, then after possibly changing the ALG* coordinate system Φ' on X', we can arrange that the diffeomorphism map commutes with Φ and Φ'. So we can actually view any ALG* gravitational instanton with parameters ν, κ_0, and L as a gravitational instanton of order 2 on a *fixed* space X_ν with a *fixed* ALG* coordinate system Φ_{X_ν}. Similar to the ALG case, we make the following definition.

Definition 1.11
Define $\mathcal{M}_{\nu,\kappa_0,L}$ to be the collection of all gravitational instantons on X_ν with parameters ν, κ_0, and L which are ALG* of order 2 with respect to a fixed ALG* coordinate system Φ_{X_ν}. For $(X_\nu, g^0, \boldsymbol{\omega}^0) \in \mathcal{M}_{\nu,\kappa_0,L}$, the period mapping based at $\boldsymbol{\omega}^0$,

$$\mathscr{P} : \mathcal{M}_{\nu,\kappa_0,L} \to [\boldsymbol{\omega}^0] + \mathscr{H}^2 \otimes \mathbb{R}^3 \subset H^2_{\mathrm{dR}}(X_\nu) \otimes \mathbb{R}^3, \tag{1.12}$$

is defined as in (1.6), where $\mathscr{H}^2 \equiv \mathrm{Im}(H^2_{\mathrm{cpt}}(X_\nu) \to H^2(X_\nu))$.

The following is our main result about the period mapping in the ALG* cases.

THEOREM 1.12
If $(X_\nu, g, \boldsymbol{\omega}) \in \mathcal{M}_{\nu,\kappa_0,L}$*, then*

$$\boldsymbol{\omega}[C] \neq (0,0,0) \quad \textit{for all } [C] \in H_2(X_\nu;\mathbb{Z}) \textit{ satisfying } [C]^2 = -2. \tag{1.13}$$

Furthermore, the image of the period mapping $\mathscr{P}$ *is an open subset of space of cohomology triples* $[\boldsymbol{\omega}^0] + \mathscr{H}^2 \otimes \mathbb{R}^3$ *satisfying (1.13).*

We will prove this in Section 7. In particular, we see that the image of the period mapping has dimension $3(b_2(X_\nu) - 1) = 12 - 3\nu$. We conjecture that the period mapping $\mathscr{P}$ is also surjective in the ALG* cases.

1.3. ALG* *bubbles from the K3 surface*
In [23], hyper-Kähler metrics were constructed on elliptic K3 surfaces with 24 I_1-fibers, which have a 2-dimensional Gromov–Hausdorff limit $(\mathbb{P}^1, d_{\mathrm{ML}})$, where d_{ML}

is called the *McLean metric*. This was generalized to arbitrary elliptic K3 surfaces in [22] (see also [34]). Subsequently, the authors gave a new construction on arbitrary elliptic K3 surfaces in [10], which also allowed for a detailed description of the degeneration near the singular fibers, which we briefly describe next. Away from singular fibers, the degeneration was modeled by a semiflat metric, which was introduced by Greene et al. in [21]. A generalization of the Ooguri–Vafa metric (see [35]), which we called a *multi-Ooguri–Vafa metric* (with b monopole points), was used to describe the degeneration near singular fibers of type I_b. ALG metrics were used to describe the degeneration near fibers with finite monodromy. In the case of I_ν^*-fibers, the model used was a $\mathbb{Z}_2$-quotient of certain multi-Ooguri–Vafa metrics with 2ν monopole points, together with four Eguchi–Hanson metrics due to the four orbifold singularities of the resulting quotient. It was moreover shown in [10] that such degenerations exist for metrics which are Kähler with respect to the fixed elliptic complex structure.

In this paper, let $\mathcal{K}$ be an elliptic K3 surface with a singular fiber D^* of type I_b^* and $(18-b)$ singular fibers of type I_1, where $1 \leq b \leq 14$ (recall that an elliptic K3 surface can have up to an I_{14}^*-fiber and such elliptic K3 surfaces do exist; see [38]). Let X be an ALG^* gravitational instanton of order 2 with parameters $1 \leq \nu \leq 4$, $\kappa_0 \in \mathbb{R}$, and $L > 0$. Near I_1-fibers, we use the Ooguri–Vafa metric as before. Near D^*, we cut out a neighborhood of D^* in $\mathcal{K}$ and, as a new method, glue it with a neck region and a rescaling of X. We call the glued manifold $\mathcal{M}_\lambda$.

THEOREM 1.13
There exists a family of hyper-Kähler metrics g_λ on the K3 surface $\mathcal{M}_\lambda$ such that $(\mathcal{M}_\lambda, g_\lambda) \xrightarrow{\mathrm{GH}} (\mathbb{P}^1, d_{\mathrm{ML}})$ as $\lambda \to 0$, and such that near D^*, the rescaling limits are X together with $b+\nu$ Taub-NUT bubbles.

In this case of an I_b^*-fiber, the construction in [10] was done to preserve the elliptic complex structure. In this new gluing construction, the original elliptic complex structure is not preserved. An interesting question is to describe more precisely the complex structure degeneration of this new family. We also point out that this construction is somewhat analogous to [27] in that we construct a neck region with nontrivial topology which interpolates between different degree infra-nilmanifolds (versus nilmanifolds in [27]), and which is responsible for the Taub-NUT bubbles. The proof of Theorem 1.13 is contained in Sections 3, 4, and 5.

2. The model hyper-Kähler structures

In this section, we explain some properties of ALG and ALG^* gravitational instantons in more detail.

2.1. Gibbons–Hawking construction

In this subsection, we review the Gibbons–Hawking construction of the ALG* model metric. (See [11] for more details.) For any positive integer ν, the Heisenberg nilmanifold $\mathrm{Nil}^3_{2\nu}$ of degree 2ν is the quotient of $\mathbb{R}^3$ by the following actions:

$$\sigma_1(\theta_1,\theta_2,\theta_3) \equiv (\theta_1+2\pi,\theta_2,\theta_3), \tag{2.1}$$

$$\sigma_2(\theta_1,\theta_2,\theta_3) \equiv (\theta_1,\theta_2+2\pi,\theta_3+2\pi\theta_1), \tag{2.2}$$

$$\sigma_3(\theta_1,\theta_2,\theta_3) \equiv (\theta_1,\theta_2,\theta_3+2\pi^2\nu^{-1}). \tag{2.3}$$

Define

$$\Theta \equiv \frac{\nu}{\pi}(d\theta_3-\theta_2 d\theta_1), \qquad V \equiv \kappa_0 + \frac{\nu}{\pi}\log r, \tag{2.4}$$

for $r \in (R,\infty)$, $\kappa_0 \in \mathbb{R}$, and $R > e^{\frac{\pi}{\nu}(1-\kappa_0)}$ on the manifold

$$S^1 \to \widehat{\mathfrak{M}}_{2\nu}(R) \equiv (R,\infty)\times \mathrm{Nil}^3_{2\nu} \to \widetilde{U} \equiv \big(\mathbb{R}^2 \setminus \overline{B_R(0)}\big)\times S^1. \tag{2.5}$$

Then the Gibbons–Hawking metric on $\widehat{\mathfrak{M}}_{2\nu}(R)$ is given by

$$\begin{aligned} g^{\widehat{\mathfrak{M}}}_{\kappa_0} &= V(dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1}\frac{\nu^2}{\pi^2}(d\theta_3-\theta_2 d\theta_1)^2 \\ &= V(dx^2+dy^2+d\theta_2^2) + V^{-1}\Theta^2, \end{aligned} \tag{2.6}$$

where $x+\sqrt{-1}y \equiv r\cdot e^{\sqrt{-1}\theta_1}$. The model hyper-Kähler forms on the manifold $\widehat{\mathfrak{M}}_{2\nu}(R)$ are given by

$$\omega_I = \omega^{\widehat{\mathfrak{M}}}_{1,\kappa_0} = E^1\wedge E^2 + E^3\wedge E^4 = V dx\wedge dy + d\theta_2\wedge\Theta, \tag{2.7}$$

$$\omega_J = \omega^{\widehat{\mathfrak{M}}}_{2,\kappa_0} = E^1\wedge E^3 - E^2\wedge E^4 = V dx\wedge d\theta_2 - dy\wedge\Theta, \tag{2.8}$$

$$\omega_K = \omega^{\widehat{\mathfrak{M}}}_{3,\kappa_0} = E^1\wedge E^4 + E^2\wedge E^3 = dx\wedge\Theta + V dy\wedge d\theta_2, \tag{2.9}$$

where

$$\{E^1,E^2,E^3,E^4\} = \{V^{1/2}dx, V^{1/2}dy, V^{1/2}d\theta_2, V^{-1/2}\Theta\}. \tag{2.10}$$

The $\mathbb{Z}_2$-action $\iota(r,\theta_1,\theta_2,\theta_3) \equiv (r,\theta_1+\pi,-\theta_2,-\theta_3)$ induces an automorphism of the hyper-Kähler structure, and we define the ALG$^*_\nu$ model space as

$$\big(\mathfrak{M}_{2\nu}(R), g^{\mathfrak{M}}_{\kappa_0}, \omega^{\mathfrak{M}}_{1,\kappa_0}, \omega^{\mathfrak{M}}_{2,\kappa_0}, \omega^{\mathfrak{M}}_{3,\kappa_0}\big) \equiv \big(\widehat{\mathfrak{M}}_{2\nu}(R), g^{\widehat{\mathfrak{M}}}_{\kappa_0}, \omega^{\widehat{\mathfrak{M}}}_{1,\kappa_0}, \omega^{\widehat{\mathfrak{M}}}_{2,\kappa_0}, \omega^{\widehat{\mathfrak{M}}}_{3,\kappa_0}\big)/\langle\iota\rangle.$$

By rescaling, we have $(\mathfrak{M}_{2\nu}(R), g^{\mathfrak{M}}_{\kappa_0,L}, \omega^{\mathfrak{M}}_{1,\kappa_0,L}, \omega^{\mathfrak{M}}_{2,\kappa_0,L}, \omega^{\mathfrak{M}}_{3,\kappa_0,L})$ for any scaling parameter $L>0$, where

$$g^{\mathfrak{M}}_{\kappa_0,L} \equiv L^2\cdot g^{\mathfrak{M}}_{\kappa_0}, \qquad \omega^{\mathfrak{M}}_{i,\kappa_0,L} \equiv L^2\cdot\omega^{\mathfrak{M}}_{i,\kappa_0}, \quad i=1,2,3,$$

Remark 2.1
The model space has the following properties. The cross section $r = r_0$ is an *infranil* 3-manifold. There is a holomorphic map $u_{\mathfrak{M}} : \mathfrak{M}_{2\nu}(R) \to \mathbb{C}$ defined as $u_{\mathfrak{M}} = r^2 e^{2\sqrt{-1}\theta_1}$, with torus fibers. The infinite end of the model space compactifies complex analytically by adding a singular fiber of type I_ν^*.

2.2. *Choice of connection form*

In this subsection, we make some important remarks about our choice of connection form. The connection form satisfies

$$d\Theta = \frac{\nu}{\pi} d\theta_1 \wedge d\theta_2 \tag{2.11}$$

and $\iota^*\Theta = -\Theta$. Since $\dim(H^1_{\mathrm{dR}}(\widetilde{U})) = 2$ and is generated by $d\theta_1$ and $d\theta_2$, more generally we could have chosen

$$\widetilde{\Theta} = \frac{\nu}{\pi}(d\theta_3 - \theta_2 d\theta_1 + df + p d\theta_1 + q d\theta_2), \tag{2.12}$$

where $f : \widetilde{U} \to \mathbb{R}$, and $p, q \in \mathbb{R}$. Note that $\iota^*\widetilde{\Theta} = -\widetilde{\Theta}$ implies that $p = 0$ and

$$f(r, \theta_1, \theta_2) + f(r, \theta_1 + \pi, -\theta_2) = c \tag{2.13}$$

for a constant $c \in \mathbb{R}$. The mapping

$$\varphi_f(r, \theta_1, \theta_2, \theta_3) \equiv \left(r, \theta_1, \theta_2, \theta_3 + \frac{c}{2} - f\right) \tag{2.14}$$

commutes with $\sigma_1, \sigma_2, \sigma_3$ and ι. Moreover, we have

$$\varphi_f^* \widetilde{\Theta} = \frac{\nu}{\pi}(d\theta_3 - \theta_2 d\theta_1 + q d\theta_2). \tag{2.15}$$

Next, define the mapping

$$\varphi_q(\theta_1, \theta_2, \theta_3) \equiv (\theta_1 - q, \theta_2, \theta_3 - q\theta_2). \tag{2.16}$$

It is straightforward to compute that φ_q also commutes with $\sigma_1, \sigma_2, \sigma_3$ and ι. Clearly, we have $\varphi_q^* \varphi_f^* \widetilde{\Theta} = \Theta$, so the mapping $\varphi_f \circ \varphi_q$ is an isometry of the Gibbons–Hawking metric $g_{\kappa_0}^{\widehat{\mathfrak{M}}}$ with respect to the two different choices of connection form. Since the mapping $\varphi_f \circ \varphi_q$ induces a diffeomorphism $\varphi_f \circ \varphi_q : \widehat{\mathfrak{M}}_{2\nu}(R)/\iota \to \widehat{\mathfrak{M}}_{2\nu}(R)/\iota$, this mapping is an isometry of the quotient metric. Therefore, we may assume without loss of generality that $f = 0$ and $p = q = 0$, so any choice of connection form is equivalent to Θ up to diffeomorphism.

Remark 2.2
If we replace Θ in (2.7), (2.8), and (2.9) by

$$\widetilde{\Theta} = \frac{\nu}{\pi}(d\theta_3 - \theta_2 d\theta_1 + df + q d\theta_2) \tag{2.17}$$

to get $\widetilde{\omega}_I, \widetilde{\omega}_J, \widetilde{\omega}_K$, then

$$\varphi_q^* \varphi_f^* (\widetilde{\omega}_I, \widetilde{\omega}_J, \widetilde{\omega}_K) = (\omega_I, \cos q \cdot \omega_J + \sin q \cdot \omega_K, \cos q \cdot \omega_K - \sin q \cdot \omega_J).$$

In other words, we can use the standard Θ after a hyper-Kähler rotation.

2.3. *ALG model space*

In the ALG case, we have the following definition of the model space.

Definition 2.3 (Standard ALG model)
Let $\beta \in (0,1]$ and $\tau \in \mathbb{H} \equiv \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$ be parameters in Table 1, and let $L > 0$ be a scaling parameter. Consider the space

$$\big\{(\mathscr{U}, \mathscr{V}) \mid \arg \mathscr{U} \in [0, 2\pi\beta]\big\} \subset (\mathbb{C} \times \mathbb{C})/(\mathbb{Z} \oplus \mathbb{Z}), \tag{2.18}$$

where $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{C} \times \mathbb{C}$ by

$$(m,n) \cdot (\mathscr{U}, \mathscr{V}) = \big(\mathscr{U}, \mathscr{V} + (m + n\tau) \cdot L\big), \quad (m,n) \in \mathbb{Z} \oplus \mathbb{Z}. \tag{2.19}$$

We can further identify $(\mathscr{U}, \mathscr{V})$ with $(e^{\sqrt{-1}\cdot 2\pi\beta}\mathscr{U}, e^{-\sqrt{-1}\cdot 2\pi\beta}\mathscr{V})$ to obtain a manifold $\mathcal{C}_{\beta,\tau,L}$. Define

$$\mathcal{C}_{\beta,\tau,L}(R) \equiv \big\{|\mathscr{U}| > R\big\} \subset \mathcal{C}_{\beta,\tau,L}. \tag{2.20}$$

Then there is a flat hyper-Kähler metric

$$g^{\mathcal{C}} = \frac{1}{2}(d\mathscr{U} \otimes d\bar{\mathscr{U}} + d\bar{\mathscr{U}} \otimes d\mathscr{U} + d\mathscr{V} \otimes d\bar{\mathscr{V}} + d\bar{\mathscr{V}} \otimes d\mathscr{V}) \tag{2.21}$$

on $\mathcal{C}_{\beta,\tau,L}(R)$ with a hyper-Kähler triple

$$\omega_1^{\mathcal{C}} = \frac{\sqrt{-1}}{2}(d\mathscr{U} \wedge d\bar{\mathscr{U}} + d\mathscr{V} \wedge d\bar{\mathscr{V}}),$$
$$\omega_2^{\mathcal{C}} = \operatorname{Re}(d\mathscr{U} \wedge d\mathscr{V}), \qquad \omega_3^{\mathcal{C}} = \operatorname{Im}(d\mathscr{U} \wedge d\mathscr{V}).$$

Each flat space $(\mathcal{C}_{\beta,\tau,L}(R), g^{\mathcal{C}}, \boldsymbol{\omega}^{\mathcal{C}})$ given as the above is called a *standard ALG model*.

Let X_β be an ALG gravitational instanton with an ALG model $\mathcal{C}_{\beta,\tau,L}$. Table 1 exhibits some important geometric invariants of X_β. By [8], X_β is biholomorphic to a rational elliptic surface minus the fiber at infinity and the first line of Table 1 is the Kodaira type of the fiber at infinity. The second and third lines are the parameters in $\mathcal{C}_{\beta,\tau,L}$. By [9, Theorem 1.10], any two ALG gravitational instantons with the same ALG model $\mathcal{C}_{\beta,\tau,L}$ are diffeomorphic to each other, so the second Betti number in the last line of Table 1 is well defined.

Remark 2.4
The model space has the following properties. Letting $r = |\mathscr{U}|$, the cross section $\{r = r_0\}$ is a *flat* 3-manifold. There is a holomorphic map $u_{\mathcal{C}} : \mathcal{C}_{\beta,\tau,L}(R) \to \mathbb{C}$ defined as $u_{\mathcal{C}} = \mathscr{U}^{\frac{1}{\beta}}$, with torus fibers, which have area $L^2 \cdot \operatorname{Im}\tau$. As mentioned above, the infinite end of the model space compactifies complex analytically by adding a singular fiber of the specified type in the first row of Table 1.

3. Building blocks and approximate metrics

In this section, we will describe the construction of the "approximate" hyper-Kähler triple, using a gluing construction. We will divide the K3 surface into the following regions: the ALG^*_ν bubbling region, the Gibbons–Hawking neck transition region, the Ooguri–Vafa regions, and the collapsing semi-flat hyper-Kähler structure away from singular fibers.

We start with an elliptic K3 surface $\pi_{\mathcal{K}} : \mathcal{K} \to \mathbb{P}^1$ with an I^*_b-fiber for some $1 \le b \le 14$ and I_1-fibers of number $(18-b)$. Away from all singular fibers, we choose the hyper-Kähler structure as $\boldsymbol{\omega}^{\mathrm{sf}}$, given by a semi-flat metric (see [10, Section 2.2]). Near the I_1-fibers, we glue in Ooguri–Vafa metrics as in [10] and [23]. These regions contribute exponentially small error terms to the weighted estimates, so in the following we will take this as understood, and will not consider those regions in any detail. We will denote this region of the K3 surface by $\mathcal{K}^* = \mathcal{K} \setminus D^*$, where D^* is the I^*_b-fiber, and we will continue to denote the "approximately hyper-Kähler" definite triple on this region by $\boldsymbol{\omega}^{\mathrm{sf}}$ even though it is not semi-flat near the I_1-fibers.

Near the I^*_b-fiber, as in [10], we consider the local double cover, which is an I_{2b}-fiber. We choose local coordinate $\mathscr{Y}$ on the base of the local double cover and local coordinate $\mathscr{X} \in \mathbb{C}/(\mathbb{Z}\tau_1(\mathscr{Y}) \oplus \mathbb{Z}\tau_2(\mathscr{Y}))$ on the fiber of the local double cover such that $\Omega = d\mathscr{X} \wedge d\mathscr{Y}$, and for some holomorphic function $h(\mathscr{Y})$,

$$\tau_1(\mathscr{Y}) = 1 \qquad \text{and} \qquad \tau_2(\mathscr{Y}) = \frac{b}{\pi\sqrt{-1}} \log \mathscr{Y} + h(\mathscr{Y}). \tag{3.1}$$

3.1. ALG_ν^* *bubbling region*

Given a fixed $\nu \in \{1,2,3,4\}$, let $(X, g^X, \boldsymbol{\omega}^X)$ be an ALG_ν^* gravitational instanton with parameters ν, κ_0, and L. Without loss of generality, by scaling we can assume that $L = 1$. Recall that the model space is the $\mathbb{Z}_2$-quotient of the Gibbons–Hawking model $\mathfrak{M}_{2\nu}(R)$, where the Riemannian metric $g^{\widehat{\mathfrak{M}}}$ and hyper-Kähler triple $\boldsymbol{\omega}^{\widehat{\mathfrak{M}}}$ of the $\mathbb{Z}_2$-covering space $\widehat{\mathfrak{M}}_{2\nu}(R)$ are given by the following explicit formulas (as in Section 2) when r is sufficiently large:

$$\begin{aligned} g^{\widehat{\mathfrak{M}}} &= V(dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1}\Theta^2, \\ \omega_1^{\widehat{\mathfrak{M}}} &= V dx \wedge dy + d\theta_2 \wedge \Theta, \\ \omega_2^{\widehat{\mathfrak{M}}} &= V dx \wedge d\theta_2 - dy \wedge \Theta, \qquad \omega_3^{\widehat{\mathfrak{M}}} = dx \wedge \Theta + V dy \wedge d\theta_2, \end{aligned} \tag{3.2}$$

where $V = \frac{\nu}{\pi}\log r + \kappa_0$ and $\kappa_0 \in \mathbb{R}$. To perform the gluing construction, we will take a large region in X and appropriately scale down both $(g^X, \boldsymbol{\omega}^X)$ and $(g^{\mathfrak{M}}, \boldsymbol{\omega}^{\mathfrak{M}})$. We will fix parameters λ and $\mathfrak{t}$ such that

$$\lambda \to 0, \qquad \mathfrak{t} \to 0, \qquad \sigma \equiv \frac{\lambda}{\mathfrak{t}} \to 0. \tag{3.3}$$

Let us consider the rescaled coordinates $\tilde{x} \equiv \lambda \cdot x$, $\tilde{y} \equiv \lambda \cdot y$ for $(x,y) \in B_{\sigma^{-1}}(0^2) \subset \mathbb{R}^2$. Immediately, $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{\frac{1}{2}} = \lambda \cdot r$. We will work with the cutoff region $X \setminus \{r > 2\sigma^{-1}\}$ with the rescaled ALG_ν^* hyper-Kähler structure $(\tilde{g}^X, \tilde{\boldsymbol{\omega}}^X) = (\lambda^2 \cdot g^X, \lambda^2 \cdot \tilde{\omega}^X)$. Then the rescaled metric and hyper-Kähler triple on the asymptotic model can be written in terms of the rescaled coordinates:

$$\widetilde{V} = T + \frac{\nu}{\pi} \cdot \log \tilde{r} + \kappa_0, \qquad T \equiv \frac{\nu}{\pi} \log\Big(\frac{1}{\lambda}\Big) \gg 1, \tag{3.4}$$

$$\lambda^2 \cdot g^{\widehat{\mathfrak{M}}} = \widetilde{V}(d\tilde{x}^2 + d\tilde{y}^2 + \lambda^2 d\theta_2^2) + \lambda^2 \cdot \widetilde{V}^{-1} \cdot \Theta^2, \tag{3.5}$$

$$\lambda^2 \cdot \omega_1^{\widehat{\mathfrak{M}}} = \widetilde{V} \cdot d\tilde{x} \wedge d\tilde{y} + \lambda^2 \cdot d\theta_2 \wedge \Theta, \tag{3.6}$$

$$\lambda^2 \cdot \omega_2^{\widehat{\mathfrak{M}}} = \lambda \cdot \widetilde{V} \cdot d\tilde{x} \wedge d\theta_2 - \lambda \cdot d\tilde{y} \wedge \Theta, \tag{3.7}$$

$$\lambda^2 \cdot \omega_3^{\widehat{\mathfrak{M}}} = \lambda \cdot d\tilde{x} \wedge \Theta + \lambda \cdot \widetilde{V} \cdot d\tilde{y} \wedge d\theta_2. \tag{3.8}$$

Note that the cutoff region becomes $\mathcal{O}_{2\mathfrak{t}}(p) \equiv X \setminus \{\tilde{r} > 2\mathfrak{t}\}$ in terms of $\tilde{r}$.

3.2. Neck transition region

The next building block is the neck transition region. To begin with, we take a flat product metric on $Q \equiv \mathbb{R}^2 \times S^1 = \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$ with $0^* \equiv (0^2, 0) \in \mathbb{R}^2 \times S^1$,

$$g^Q = dx^2 + dy^2 + d\theta_2^2 = dr^2 + r^2 d\theta_1^2 + d\theta_2^2, \qquad \theta_2 \in [0, 2\pi]. \tag{3.9}$$

For fixed κ_0 in (3.2) and small parameter $\lambda \ll 1$, let

$$\widetilde{P} = \{\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{2\nu+2b}\} \subset \left(\mathbb{R}^2 \setminus \{0^2\}\right) \times \{0\} \subset Q \tag{3.10}$$

be a fixed set such that the following properties hold.

(1) (Balancing condition) Let $\tilde{d}_m \equiv d^Q(0^*, \tilde{p}_m)$ for any $1 \le m \le 2\nu + 2b$. Then

$$\sum_{m=1}^{2\nu+2b} \log(1/\tilde{d}_m) + 2\pi \operatorname{Im} h(0) = 2\pi\kappa_0, \tag{3.11}$$

where h is the holomorphic function in (3.1).

(2) ($\mathbb{Z}_2$-invariance) $\iota(\tilde{p}_m) = \tilde{p}_{2\nu+2b+1-m}$ for any $1 \le m \le 2\nu + 2b$.

Let P be the dilation of the set $\widetilde{P}$ by λ^{-1}. More specifically, we define

$$p_m \equiv (\lambda^{-1} \cdot \tilde{x}_m, \lambda^{-1} \cdot \tilde{y}_m, 0) \in \left(\mathbb{R}^2 \setminus \{0^2\}\right) \times S^1, \quad 1 \le m \le 2\nu + 2b, \tag{3.12}$$

where $\tilde{p}_m = (\tilde{x}_m, \tilde{y}_m, 0)$. Then there are constants $\iota_0 > 0$ independent of λ such that

$$\iota_0 \cdot \lambda^{-1} \le d^Q(p_\alpha, 0^*) \le \iota_0^{-1} \cdot \lambda^{-1}, \tag{3.13}$$

$$\iota_0 \cdot \lambda^{-1} \le d^Q(p_\alpha, p_\beta) \le \iota_0^{-1} \cdot \lambda^{-1}, \quad 1 \le \alpha < \beta \le 2\nu + 2b. \tag{3.14}$$

For every $p_m \in P$ with $1 \le m \le 2\nu + 2b$, there exists a unique Green's function G_m on $(\mathbb{R}^2 \times S^1, g^Q)$ that satisfies $-\Delta_{g^Q} G_m = 2\pi\delta_{p_m}$ and has the asymptotics

$$\left| G_m - \frac{1}{2\pi} \log \frac{1}{d^Q(\underline{\boldsymbol{x}}, p_m)} \right| \le C \cdot e^{-d^Q(\underline{\boldsymbol{x}}, p_m)} \quad \text{as } d^Q(\underline{\boldsymbol{x}}, P) \to \infty. \tag{3.15}$$

The proof is standard and we omit it. The above Green's function was also used in [10] to construct the neck transition region (see [10, Lemma 4.1]). Let $G_0 \equiv \sum_{m=1}^{2\nu+2b} G_m$ be the superposition that solves the equation $-\Delta_{g^Q} G_0 = 2\pi \sum_{m=1}^{2\nu+2b} \delta_{p_m}$. We also take

$$G_\lambda \equiv \frac{2\nu + b}{\pi} \log\left(\frac{1}{\lambda}\right) + \frac{\nu}{\pi} \cdot \log \tilde{r} + G_0 + \operatorname{Im} h\left(\tilde{\lambda} \cdot (\tilde{x} + \sqrt{-1}\tilde{y})\right), \tag{3.16}$$

where $\tilde{\lambda} \equiv \lambda^{\frac{\nu}{b}}$, $r \equiv \lambda^{-1} \cdot \tilde{r}$, and h is the holomorphic function defined in (3.1). Letting $T \equiv \frac{\nu}{\pi} \log(\frac{1}{\lambda})$, we have $\tilde{\lambda} \equiv e^{-\frac{\pi T}{b}}$. Switching to the rescaled metric $\tilde{g}^Q = \lambda^2 g^Q$, let us discuss the asymptotic behavior of the Green's function G_λ in terms of the distance function $\tilde{r}$, which will be used in the discussions of the rescaling geometry in the later subsections. Applying (3.11) and (3.16), then the following holds for any sufficiently small $\lambda \ll 1$ and $\underline{\boldsymbol{x}} \in Q$.

(1) (Near the origin) If $\tilde{r}(\underline{\boldsymbol{x}}) \to 0$, then G_λ has the expansion

$$G_\lambda(\underline{\boldsymbol{x}}) = T + \kappa_0 + \frac{\nu}{\pi} \cdot \log \tilde{r}(\underline{\boldsymbol{x}}) + E(\underline{\boldsymbol{x}}), \tag{3.17}$$

where $|E(\underline{\boldsymbol{x}})| \leq C \cdot \tilde{\lambda} \cdot \tilde{r}(\underline{\boldsymbol{x}}) = C \cdot \lambda^{\frac{\nu}{b}} \cdot \tilde{r}(\underline{\boldsymbol{x}})$ for some constants $C > 0$ independent of λ.

(2) (Near the infinity of Q) If $\tilde{r}(\underline{\boldsymbol{x}}) \to \infty$, then

$$\begin{aligned} &\big|G_\lambda(\underline{\boldsymbol{x}}) - \operatorname{Im}\tau_2(\tilde{\lambda}\cdot\tilde{\zeta})\big| \\ &\quad = \Big|G_\lambda(\underline{\boldsymbol{x}}) - \Big(T - \frac{b}{\pi}\cdot\log\tilde{r}(\underline{\boldsymbol{x}}) + \operatorname{Im} h(\tilde{\lambda}\cdot\tilde{\zeta})\Big)\Big| \leq \frac{C\cdot\lambda^2}{\tilde{r}(\underline{\boldsymbol{x}})^2}, \end{aligned} \tag{3.18}$$

where $\tilde{\zeta} \equiv \tilde{x} + \sqrt{-1}\tilde{y}$, $C > 0$ is independent of λ, and τ_2 is the function in (3.1).

(3) (Near a pole $p_m \in P$) If $\tilde{d}^Q(\underline{\boldsymbol{x}}, p_m) \leq \frac{\iota_0}{4}$ for some $p_m \in P$, then there exists a constant $C > 0$ independent of λ such that

$$\begin{aligned} &\Big|G_\lambda(\underline{\boldsymbol{x}}) - \Big(G_m(\underline{\boldsymbol{x}}) + T^\flat + \frac{\nu}{\pi}\log\tilde{d}_m\Big)\Big| \leq C, \\ &\qquad\qquad T^\flat \equiv \frac{2\nu+1}{2\pi}\cdot\log\Big(\frac{1}{\lambda}\Big). \end{aligned} \tag{3.19}$$

(4) (Bounded region) If there exist $R_0 > 0$ and $d_0 > 0$ such that $R_0^{-1} \leq \tilde{r}(\underline{\boldsymbol{x}}) \leq R_0$ and $\tilde{d}^Q(\underline{\boldsymbol{x}}, P) \geq \frac{d_0}{4}$, then $|G_\lambda(\underline{\boldsymbol{x}}) - T| \leq C$, where $C = C(R_0, d_0) > 0$ is independent of λ.

Now we apply the Gibbons–Hawking construction using the Green's function G_λ. Let $\mathring{\mathcal{N}}$ be the total space of the circle bundle $S^1 \to \mathring{\mathcal{N}} \xrightarrow{\pi} Q \setminus (P \cup \{0^*\})$, where Θ_λ is an S^1-connection form that satisfies the monopole equation $d\Theta_\lambda = *_{g^Q} \circ dG_\lambda$. Then we have a family of hyper-Kähler metrics $g^{\mathring{\mathcal{N}}}$ and hyper-Kähler triples $\boldsymbol{\omega}^{\mathring{\mathcal{N}}}$ when $G_\lambda > 0$:

$$\begin{aligned} g^{\mathring{\mathcal{N}}} &= \lambda^2(G_\lambda\cdot g^Q + G_\lambda^{-1}\Theta_\lambda^2) = G_\lambda(d\tilde{x}^2 + d\tilde{y}^2 + \lambda^2 d\theta_2^2) + \lambda^2 G_\lambda^{-1}\Theta_\lambda^2, \\ \omega_1^{\mathring{\mathcal{N}}} &= \lambda^2(G_\lambda dx\wedge dy + d\theta_2\wedge\Theta_\lambda) = G_\lambda d\tilde{x}\wedge d\tilde{y} + \lambda^2 d\theta_2\wedge\Theta_\lambda, \\ \omega_2^{\mathring{\mathcal{N}}} &= \lambda^2(G_\lambda dx\wedge d\theta_2 - dy\wedge\Theta_\lambda) = \lambda G_\lambda\cdot d\tilde{x}\wedge d\theta_2 - \lambda d\tilde{y}\wedge\Theta_\lambda, \\ \omega_3^{\mathring{\mathcal{N}}} &= \lambda^2(dx\wedge\Theta_\lambda + G_\lambda dy\wedge d\theta_2) = \lambda d\tilde{x}\wedge\Theta_\lambda + \lambda G_\lambda d\tilde{y}\wedge d\theta_2. \end{aligned} \tag{3.20}$$

It is easy to check that the completion $(\mathcal{N}, g^{\mathcal{N}}, \boldsymbol{\omega}^{\mathcal{N}})$ of $(\mathring{\mathcal{N}}, g^{\mathring{\mathcal{N}}}, \boldsymbol{\omega}^{\mathring{\mathcal{N}}})$ along the set P of monopole points, called the *neck transition region*, is smooth and hyper-Kähler. Moreover, the neck transition region $(\mathcal{N}, g^{\mathcal{N}}, \boldsymbol{\omega}^{\mathcal{N}})$ is invariant under the involution $\langle\iota\rangle \cong \mathbb{Z}_2$, and hence it descends to a hyper-Kähler manifold $(\mathfrak{N}, g^{\mathfrak{N}}, \boldsymbol{\omega}^{\mathfrak{N}})$, where $\mathfrak{N} \equiv \mathcal{N}/\langle\iota\rangle$.

3.3. Attaching the pieces

Let $(X, g^X, \boldsymbol{\omega}^X)$ be an ALG_ν^* gravitational instanton of order 2. We will next glue the end of X onto the neck transition region $\mathcal{N}$ near the origin. By definition, there exist a compact subset $X_R \subset X$ and a diffeomorphism $\Psi : \mathfrak{M} \to X \setminus X_R$ such that for any $k \in \mathbb{N}$,

$$\left|\nabla^k_{g^{\mathfrak{M}}}(\Psi^* \boldsymbol{\omega}^X - \boldsymbol{\omega}^{\mathfrak{M}})\right| \leq C_k \cdot \left(r \cdot V(r)^{\frac{1}{2}}\right)^{-2-k}. \tag{3.21}$$

Thanks to the following lemma, we are able to compare the two hyper-Kähler triples $\lambda^2 \cdot \boldsymbol{\omega}^{\widehat{\mathfrak{M}}}$ and $\boldsymbol{\omega}^{\mathcal{N}}$ as $\tilde{r} \to 0$.

LEMMA 3.1

There exists a diffeomorphism

$$\Psi^{\mathcal{N}} : \left\{\boldsymbol{x} \in \mathcal{N} \mid \mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\right\} \longrightarrow \left\{\boldsymbol{x} \in \widehat{\mathfrak{M}} \mid \mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\right\} \tag{3.22}$$

such that $(\Psi^{\mathcal{N}})^* dr = dr$, $(\Psi^{\mathcal{N}})^* d\theta_1 = d\theta_1$, $(\Psi^{\mathcal{N}})^* d\theta_2 = d\theta_2$, *and*

$$(\Psi^{\mathcal{N}})^* \Theta = \Theta_\lambda + \pi^* \zeta \tag{3.23}$$

for some 1*-form* ζ *on* $\{\underline{\boldsymbol{x}} \in Q \mid \mathfrak{t} \leq \tilde{r}(\underline{\boldsymbol{x}}) \leq 2\mathfrak{t}\}$ *that satisfies* $\iota^* \pi^* \zeta = -\pi^* \zeta$, *and*

$$\left|\nabla^k_{g^{\mathcal{N}}}(\pi^* \zeta)\right| \leq C_k \cdot \tilde{\lambda} \cdot \mathfrak{t}^{1-k} \cdot V(\sigma^{-1})^{-\frac{1+k}{2}}, \tag{3.24}$$

$$\left|\nabla^k_{g^{\mathcal{N}}}\left((\Psi^{\mathcal{N}})^*(\lambda^2 \cdot \boldsymbol{\omega}^{\widehat{\mathfrak{M}}}) - \boldsymbol{\omega}^{\mathcal{N}}\right)\right| \leq C_k \cdot \tilde{\lambda} \cdot \mathfrak{t}^{1-k} \cdot V(\sigma^{-1})^{-\frac{2+k}{2}}, \tag{3.25}$$

for any $k \in \mathbb{N}_0$. *Moreover,* $\Psi^{\mathcal{N}}$ *descends to a diffeomorphism*

$$\Psi^{\mathfrak{N}} : \left\{\boldsymbol{x} \in \mathfrak{N} \mid \mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\right\} \longrightarrow \left\{\boldsymbol{x} \in \mathfrak{M} \mid \mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\right\}. \tag{3.26}$$

Proof

The proof is the same as that of [27, Lemma 6.1]. Here we only mention the major difference. First, both $\mathcal{N}$ and $\widehat{\mathfrak{M}}$ can be viewed as principal S^1-bundles over $\widetilde{U} \subset \mathbb{R}^2 \times S^1$ with the connections Θ_λ and Θ, respectively, where $\widetilde{U} \equiv \mathbb{R}^2 \setminus \overline{B_R(0^2)}$. One can easily check that they have the same Euler number 2ν when $\mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}$.

Therefore, there exists a bundle isomorphism $F : \mathcal{N} \to \widehat{\mathfrak{M}}$ which covers the identity map on $\widetilde{U} \times S^1$. Moreover, the curvature difference is given by

$$F^*(d\Theta) - d\Theta_\lambda = *_Q \circ d(E), \tag{3.27}$$

where $E \in C^\infty(Q)$ is the function given by the expansion (3.17). Applying the asymptotic estimate in (3.17), we have that

$$\left|*_Q \circ d(E)\right|_{g^{\mathcal{N}}} \leq C \cdot \tilde{\lambda} \cdot V(\sigma^{-1})^{-1}. \tag{3.28}$$

Standard Hodge theory implies that there exist a diffeomorphism $\Psi^{\mathcal{N}}$, a flat connection Θ_{flat}, and a 1-form ζ on $\widetilde{U} \times S^1$ such that

$$(\Psi^{\mathcal{N}})^* \Theta - \Theta_\lambda = \Theta_{\mathrm{flat}} + \pi^* \zeta, \tag{3.29}$$

$$|\nabla^k_{g^{\mathcal{N}}} \zeta| \le C_k \cdot \tilde{\lambda} \cdot \mathfrak{t}^{1-k} \cdot V(\sigma^{-1})^{-\frac{1+k}{2}}. \tag{3.30}$$

As discussed in Section 2.2, the flat connection Θ_{flat} can be removed by appropriately choosing a bundle diffeomorphism. So the proof is done. □

LEMMA 3.2
There exists a triple of 1*-forms* $\boldsymbol{\xi}$ *on* $\{\boldsymbol{x} \in \mathcal{N} \mid \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\}$ *such that* $\iota^* \boldsymbol{\xi} = \boldsymbol{\xi}$ *and*

$$(\Psi^{\mathcal{N}})^* (\lambda^2 \cdot \boldsymbol{\omega}^{\widehat{\mathfrak{M}}}) - \boldsymbol{\omega}^{\mathcal{N}} = d\boldsymbol{\xi}. \tag{3.31}$$

Moreover, $\boldsymbol{\xi}$ *satisfies the estimate*

$$|\nabla^k_{g^{\mathcal{N}}} \boldsymbol{\xi}| \le C_k \cdot \tilde{\lambda} \cdot \mathfrak{t}^3 \cdot \big(\mathfrak{t} \cdot V(\sigma^{-1})^{\frac{1}{2}}\big)^{-1-k} \tag{3.32}$$

for any $k \in \mathbb{N}_0$. *Moreover,* $\Psi^{\mathcal{N}}$ *descends to a diffeomorphism*

$$\Psi^{\mathfrak{N}} : \big\{\boldsymbol{x} \in \mathfrak{N} \,\big|\, \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\big\} \longrightarrow \big\{\boldsymbol{x} \in \mathfrak{M} \,\big|\, \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\big\}. \tag{3.33}$$

Once we have Lemma 3.1, the proof of Lemma 3.2 follows from the same arguments as in [27, Proposition 6.2]. We omit the details.

Next, we glue the cutoff region $\{r \le 2\sigma^{-1}\} \subset X$ as introduced above into the neck region $\mathfrak{N}$. We define the diffeomorphism

$$\Phi \equiv (\Psi \circ \Psi^{\mathfrak{N}})^{-1} \tag{3.34}$$

from $\{\sigma^{-1} \le r \le 2\sigma^{-1}\} \subset X$ to a subset $\{\mathfrak{t} \le \tilde{r} \le 2\mathfrak{t}\} \subset \mathfrak{N}$. Combining Lemma 3.2 and the asymptotic estimate of an ALG* gravitational instanton, we have the following.

LEMMA 3.3
There exists a triple of 1*-forms* $\boldsymbol{\eta}^X$ *on* $\{\boldsymbol{x} \in \mathfrak{N} \mid \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\}$ *such that* $(\Phi^{-1})^* (\lambda^2 \cdot \boldsymbol{\omega}^X) - \boldsymbol{\omega}^{\mathfrak{N}} = d\boldsymbol{\eta}^X$ *and satisfies the estimate*

$$|\nabla^k_{g^{\mathfrak{N}}} \boldsymbol{\eta}^X|_{g^{\mathfrak{N}}} \le C_k \cdot (\lambda^2 + \tilde{\lambda} \cdot \mathfrak{t}^3) \cdot \big(\mathfrak{t} \cdot V(\sigma^{-1})^{\frac{1}{2}}\big)^{-1-k} \tag{3.35}$$

for any $k \in \mathbb{N}_0$.

Next, we will glue a subset of $\mathcal{K}^*$ onto the end of the neck region with $\tilde{r}$ large. As shown in [23, Construction 2.6] and [9, Proposition 2.3], the hyper-Kähler triple

$\tilde{\lambda}^{-2} \cdot \boldsymbol{\omega}^{\mathrm{sf}}$ of the rescaled semi-flat metric on $\mathcal{K}^*$, up to a $\mathbb{Z}_2$-covering, can be written in terms of the Gibbons–Hawking ansatz by applying the harmonic function

$$\mathrm{V}_{\mathrm{sf}} \equiv \operatorname{Im} \tau_2(\tilde{\lambda} \cdot \tilde{\zeta}) = T - \frac{b}{\pi} \log \tilde{r} + \operatorname{Im} h(\tilde{\lambda} \cdot \tilde{\zeta}) \tag{3.36}$$

which is the leading term of (3.18). Then we have the following lemma.

LEMMA 3.4
For any sufficiently small parameter $\lambda \ll 1$, let r_λ be a large number such that $1 \leq G_\lambda(\boldsymbol{x}) \leq 100$ as $r_\lambda \leq \tilde{r}(\boldsymbol{x}) \leq 2r_\lambda$. There exist a triple of 1-forms $\boldsymbol{\eta}^{\mathrm{sf}}$ on $\{\boldsymbol{x} \in \mathfrak{N} \mid r_\lambda \leq \tilde{r}(\boldsymbol{x}) \leq 2r_\lambda\}$ and a diffeomorphism $\Phi^\flat$ from $\{\boldsymbol{x} \in \mathfrak{N} \mid r_\lambda \leq \tilde{r}(\boldsymbol{x}) \leq 2r_\lambda\}$ to a subset of $\mathcal{K}^*$ such that for all $k \in \mathbb{N}_0$,

$$\boldsymbol{\omega}^{\mathfrak{N}} - (\Phi^\flat)^*(\tilde{\lambda}^{-2} \cdot \boldsymbol{\omega}^{\mathrm{sf}}) = d\boldsymbol{\eta}^{\mathrm{sf}}, \tag{3.37}$$

$$|\nabla^k_{g^{\mathfrak{N}}} \boldsymbol{\eta}^{\mathrm{sf}}|_{g^{\mathfrak{N}}} \leq C_k \cdot \lambda^2 \cdot \tilde{\lambda}^{1+k}. \tag{3.38}$$

Notice that (3.38) follows from (3.18), and r_λ is comparable to $\tilde{\lambda}^{-1}$.

With the above preparations, we are ready to define the closed glued manifold on which we will construct a family of collapsing hyper-Kähler metrics with a given ALG* gravitational instanton bubbling out. Now let us take the neck transition region $\mathfrak{N}$ equipped with the hyper-Kähler triple $\boldsymbol{\omega}^{\mathfrak{N}}$ for any $\lambda \ll 1$, as constructed in Section 3.2. In the region $\{\boldsymbol{x} \in \mathfrak{N} \mid \mathfrak{t} \leq \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\}$, we glue $\mathfrak{N}$ with the finite part $\{r \leq 2\sigma^{-1}\}$ of an ALG* gravitational instanton X using the diffeomorphism Φ. In the region $\{\boldsymbol{x} \in \mathfrak{N} \mid r_\lambda \leq \tilde{r}(\boldsymbol{x}) \leq 2r_\lambda\}$, we attach $\mathfrak{N}$ with $\mathcal{K}^*$ using the diffeomorphism $\Phi^\flat$ as in Lemma 3.4. Using the above gluing maps, we obtain a closed smooth 4-manifold $\mathcal{M}_\lambda$. Now we construct a family of approximately hyper-Kähler triples $\tilde{\boldsymbol{\omega}}_\lambda$ on $\mathcal{M}_\lambda$.

LEMMA 3.5 (Approximate hyper-Kähler triple)
For any sufficiently small parameter $\lambda \ll 1$, let r_λ be a large number such that $1 \leq G_\lambda(\boldsymbol{x}) \leq 100$ as $r_\lambda \leq \tilde{r}(\boldsymbol{x}) \leq 2r_\lambda$. Then there exist two triples of 1-forms $\boldsymbol{\eta}^X$ and $\boldsymbol{\eta}^{\mathrm{sf}}$ such that the glued definite triple

$$\tilde{\boldsymbol{\omega}}_\lambda \equiv \begin{cases} \lambda^2 \cdot \boldsymbol{\omega}^X & \tilde{r} \leq \mathfrak{t}, \\ \boldsymbol{\omega}^{\mathfrak{N}} + d(\varphi \cdot \boldsymbol{\eta}^X - \psi \cdot \boldsymbol{\eta}^{\mathrm{sf}}) & \mathfrak{t} \leq \tilde{r} \leq 2r_\lambda, \\ \tilde{\lambda}^{-2} \cdot \boldsymbol{\omega}^{\mathrm{sf}} & \tilde{r} \geq 2r_\lambda \end{cases} \tag{3.39}$$

satisfies the following estimates with respect to the associated Riemannian metric $\tilde{g}_\lambda$ for any $k \in \mathbb{N}_0$:

$$\sup_{\mathfrak{t}\le\mathfrak{r}\le 2\mathfrak{t}}\left|\nabla^k_{\tilde{g}_\lambda}\left(\tilde{\boldsymbol{\omega}}_\lambda-(\Phi^{-1})^*(\lambda^2\cdot\boldsymbol{\omega}^X)\right)\right|_{\tilde{g}_\lambda}$$
$$\le C_k\cdot(\lambda^2+\tilde{\lambda}\cdot\mathfrak{t}^3)\cdot\left(\mathfrak{t}\cdot V(\sigma^{-1})^{\frac{1}{2}}\right)^{-2-k}, \tag{3.40}$$
$$\sup_{r_\lambda\le\mathfrak{r}\le 2r_\lambda}\left|\nabla^k_{\tilde{g}_\lambda}\left(\tilde{\boldsymbol{\omega}}_\lambda-(\Phi^{\mathfrak{b}})^*(\tilde{\lambda}^{-2}\cdot\boldsymbol{\omega}^{\mathrm{sf}})\right)\right|_{\tilde{g}_\lambda}$$
$$\le C_k\cdot\lambda^2\cdot\tilde{\lambda}^{2+k}, \tag{3.41}$$

where $\tilde{\lambda}\equiv e^{-\frac{\pi T}{b}}=\lambda^{\frac{\nu}{b}}$*,* φ *and* ψ *are smooth cutoff functions satisfying*

$$\varphi=\begin{cases}1 & \tilde{r}\le\mathfrak{t},\\ 0 & \tilde{r}\ge 2\mathfrak{t},\end{cases}\qquad \text{and}\qquad \psi=\begin{cases}1 & \tilde{r}\ge 2r_\lambda,\\ 0 & \tilde{r}\le r_\lambda,\end{cases} \tag{3.42}$$

and $\boldsymbol{\omega}^{\mathrm{sf}}$ *is the hyper-Kähler triple of the semi-flat metric with area of each fiber equal to* $\tilde{\lambda}\cdot\lambda$ *and diameter comparable to* 1.

Proof
The proof is straightforward. The error estimate in the region $\{\mathfrak{t}\le\tilde{r}\le 2\mathfrak{t}\}$ is given by Lemma 3.3, and the error estimate in $\{r_\lambda\le\tilde{r}\le 2r_\lambda\}$ is due to Lemma 3.4. □

It turns out that the manifold is indeed diffeomorphic to the K3 surface, but for now we do not need this fact, we only need the following calculation of the Betti numbers.

COROLLARY 3.6
For λ *sufficiently small, the smooth* 4*-manifold* $\mathcal{M}_\lambda$ *satisfies*

$$b^1(\mathcal{M}_\lambda)=0,\qquad b^2_+(\mathcal{M}_\lambda)=3,\qquad b^2_-(\mathcal{M}_\lambda)=19,\qquad \chi(\mathcal{M}_\lambda)=24. \tag{3.43}$$

Proof
This is proved using a Mayer–Vietoris argument and the estimates in Lemma 3.5, which show that $\Lambda^2_+(\mathcal{M}_\lambda)$ is a trivial bundle if λ is small. We omit the details which are similar to [27, Proposition 6.6]. □

4. Metric geometry and regularity scales

To begin with, we list the notation. We will always fix a small parameter $\lambda\ll 1$.

(1) Let us denote $g_\lambda\equiv\tilde{\lambda}^2\cdot\tilde{g}_\lambda$. Then it holds that there is some constant $C_0>0$ independent of λ such that $C_0^{-1}\le \mathrm{Diam}_{g_\lambda}(\mathcal{M}_\lambda)\le C_0$.
(2) We define the smoothing function $\mathfrak{r}$ of the distance function $\tilde{r}$ by

$$\mathfrak{r}(\boldsymbol{x}) = \begin{cases} \lambda \cdot R_0 & \tilde{r}(\underline{\boldsymbol{x}}) \leq \lambda \cdot R_0, \\ \tilde{r}(\underline{\boldsymbol{x}}) & 2\lambda \cdot R_0 \leq \tilde{r}(\underline{\boldsymbol{x}}) \leq r_\lambda, \\ 2r_\lambda & \tilde{r}(\underline{\boldsymbol{x}}) \geq 2r_\lambda, \end{cases}$$

where R_0 is the constant R in Definition 1.8, and r_λ is the constant in Lemma 3.4.

(3) Given $\nu \in \mathbb{Z}_+$, let $T^\flat \equiv \frac{2\nu+1}{2\pi} \cdot \log(\frac{1}{\lambda})$, and let $\mathfrak{d}$ be the following:

$$\mathfrak{d}(\boldsymbol{x}) = \begin{cases} (T^\flat)^{-\frac{1}{2}} & d^{Q}(\underline{\boldsymbol{x}}, p_m) \leq (T^\flat)^{-1} \\ & \text{for some } 1 \leq m \leq 2\nu + 2b, \\ (T^\flat)^{\frac{1}{2}} \cdot d^{Q}(\underline{\boldsymbol{x}}, p_m) & 2(T^\flat)^{-1} \leq d^{Q}(\underline{\boldsymbol{x}}, p_m) \leq 1 \\ & \text{for some } 1 \leq m \leq 2\nu + 2b, \\ (T^\flat + \frac{1}{2\pi} \log \frac{1}{d^{Q}(\underline{\boldsymbol{x}}, p_m)})^{\frac{1}{2}} & 2 \leq d^{Q}(\underline{\boldsymbol{x}}, p_m) \leq \frac{\iota_0}{4} \cdot \lambda^{-1} \\ \quad \cdot d^{Q}(\underline{\boldsymbol{x}}, p_m) & \text{for some } 1 \leq m \leq 2\nu + 2b. \end{cases}$$

(4) Let us define $T \equiv \frac{\nu}{\pi} \log(\frac{1}{\lambda})$ and a smooth function $\mathfrak{L}_T$ that satisfies

$$\mathfrak{L}_T(\boldsymbol{x}) \equiv \begin{cases} 1 & \tilde{r}(\underline{\boldsymbol{x}}) \leq \lambda \cdot R_0, \\ T + \kappa_0 + \frac{\nu}{\pi} \log \tilde{r}(\underline{\boldsymbol{x}}) & 2\lambda \cdot R_0 \leq \tilde{r}(\underline{\boldsymbol{x}}) \leq \frac{\iota_0}{4}, \\ T + \operatorname{Im} h(0) - \frac{b}{\pi} \log \tilde{r}(\underline{\boldsymbol{x}}) & 2\iota_0^{-1} \leq \tilde{r}(\underline{\boldsymbol{x}}) \leq r_\lambda, \\ 1 & \tilde{r}(\underline{\boldsymbol{x}}) \geq 2r_\lambda. \end{cases}$$

Next, we describe the $C^{k,\alpha}$-regularity scale.

Definition 4.1 (Local regularity)
Let (M^n, g) be a Riemannian manifold. Given $r, \epsilon > 0$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, (M^n, g) is said to be $(r, k+\alpha, \epsilon)$-*regular* at $x \in M^n$ if g is at least $C^{k,\alpha}$ in $B_{2r}(x)$ such that the following holds. Let $(\widehat{B_{2r}(x)}, \hat{g}, \hat{x})$ be the Riemannian universal cover of $B_{2r}(x)$. Then $B_r(\hat{x})$ is diffeomorphic to a Euclidean disk $\mathbb{D}^n$ such that for any $1 \leq i, j \leq n$,

$$|\hat{g}_{ij} - \delta_{ij}|_{C^0(B_r(\hat{x}))} + \sum_{|m| \leq k} r^{|m|} \cdot |\partial^m \hat{g}_{ij}|_{C^0(B_r(\hat{x}))} + r^{k+\alpha}[\hat{g}_{ij}]_{C^{k,\alpha}(B_r(\hat{x}))} < \epsilon,$$

where m is a multi-index, and the last term is the Hölder seminorm.

Definition 4.2 ($C^{k,\alpha}$-regularity scale)
Let (M^n, g) be a smooth Riemannian manifold. The $C^{k,\alpha}$-regularity scale $r_{k,\alpha}(x)$ at $x \in M^n$ is defined to be the supremum of all $r > 0$ such that M^n is $(r, k+\alpha, 10^{-9})$-regular at x.

Remark 4.3
Note that $r_{k,\alpha}$ is 1-Lipschitz on any Riemannian manifold (M^n, g), that is,

$$\left|r_{k,\alpha}(x) - r_{k,\alpha}(y)\right| \le d_g(x, y), \quad \forall x, y \in M^n. \tag{4.1}$$

Let $\mathcal{S}_b$ be the subset of $\mathcal{M}_\lambda$ which consists of a small annular region in $\mathcal{K}$ centered around the I_b^*-fiber, the neck region $\mathfrak{N}$, and the ALG* manifold X. The following proposition gives the regularity scale estimates and bubble limits of g_λ on $\mathcal{S}_b$.

PROPOSITION 4.4
Let $\mathfrak{s}$ be a smooth function that satisfies

$$\mathfrak{s}(\boldsymbol{x}) = \begin{cases} \tilde{\lambda} \cdot (\mathfrak{L}_T(\boldsymbol{x}))^{\frac{1}{2}} \cdot \mathfrak{r}(\boldsymbol{x}) & \tilde{d}^{\mathcal{Q}}(\boldsymbol{x}, p_m) \ge 2\iota_0 \\ & \textit{for all } 1 \le m \le 2\nu + 2b, \\ \tilde{\lambda} \cdot \lambda \cdot \mathfrak{d}(\boldsymbol{x}) & \tilde{d}^{\mathcal{Q}}(\boldsymbol{x}, p_m) \le \frac{1}{4}\iota_0 \\ & \textit{for some } 1 \le m \le 2\nu + 2b. \end{cases} \tag{4.2}$$

Then the following properties hold.

(1) *Given $k \in \mathbb{N}$ and $\alpha \in (0,1)$, there exists $v_0 = v_0(k,\alpha)$ such that for any sufficiently small parameter $\lambda \ll 1$ and $\boldsymbol{x} \in \mathcal{S}_b$, the (k,α)-regularity scale $r_{k,\alpha}$ at $\boldsymbol{x}$ satisfies*

$$v_0^{-1} \le \frac{r_{k,\alpha}(\boldsymbol{x})}{\mathfrak{s}(\boldsymbol{x})} \le v_0. \tag{4.3}$$

(2) *There is a uniform constant $C_0 > 0$ such that for every $\lambda \ll 1$ and $\boldsymbol{x} \in \mathcal{S}_b$, we have*

$$C_0^{-1} \le \frac{\mathfrak{s}(\boldsymbol{y})}{\mathfrak{s}(\boldsymbol{x})} \le C_0, \quad \boldsymbol{y} \in B_{\mathfrak{s}(\boldsymbol{x})/4}(\boldsymbol{x}).$$

(3) *Let $\lambda_j \to 0$ be a sequence, and let $\boldsymbol{x}_j \in \mathcal{S}_b$ be a sequence of reference points. Then the rescaled spaces $(\mathcal{S}_b, \mathfrak{s}(\boldsymbol{x}_j)^{-2} \cdot g_{\lambda_j}, \boldsymbol{x}_j)$ converge in the Gromov–Hausdorff topology to one of the following spaces as $\lambda_j \to 0$:*
- *the Taub-NUT space $(\mathbb{C}^2, g^{\mathrm{TN}})$ and the* ALG$_\nu^*$ *gravitational instanton (X, g^X),*
- *the flat manifolds $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$, $\mathbb{R}^2$, and the flat cone $\mathbb{R}^2/\mathbb{Z}_2$,*
- *$\mathbb{P}^1$ equipped with the McLean metric d_{ML} with bounded diameter.*

Proof
We will prove (4.3) by contradiction. Suppose that there does not exist a uniform constant v_0 with respect to fixed constants $k \in \mathbb{Z}_+$ and $\alpha \in (0,1)$. That is, there is a sequence $\lambda_j \to 0$ and a sequence of points $\boldsymbol{x}_j \in \mathcal{S}_b$ such that

$$\frac{r_{k,\alpha}(\boldsymbol{x}_j)}{\mathfrak{s}(\boldsymbol{x}_j)} \to 0 \qquad \text{or} \qquad \frac{r_{k,\alpha}(\boldsymbol{x}_j)}{\mathfrak{s}(\boldsymbol{x}_j)} \to \infty. \tag{4.4}$$

Let us work with the rescaled sequence $(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j)$ with $\hat{g}_{\lambda_j} \equiv \mathfrak{s}(\boldsymbol{x}_j)^{-2} \cdot g_{\lambda_j}$ as $\lambda_j \to 0$. In the proof, we will show that the $C^{k,\alpha}$-regularity scale at $\boldsymbol{x}_j$ with respect to $\hat{g}_{\lambda_j}$ is uniformly bounded from above and below as $\lambda_j \to 0$, which contradicts (4.4). We will derive a contradiction in each of the following cases depending upon the location of $\boldsymbol{x}_j$. Denote $\underline{\boldsymbol{x}}_j \equiv \pi(\boldsymbol{x}_j) \in Q/\mathbb{Z}_2$ for any $\boldsymbol{x}_j \in \mathfrak{N}$.

Case (I). There exists a constant $\sigma_0 \geq 0$ such that $\tilde{r}(\boldsymbol{x}_j) \cdot \lambda_j^{-1} \to \sigma_0$ as $j \to \infty$. Let us consider the $\sigma_0 \leq R_0$ case first. By definition, we have $\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot R_0$. We consider the rescaled metric

$$\hat{g}_{\lambda_j} \equiv (\tilde{\lambda}_j \cdot \lambda_j \cdot R_0)^{-2} g_{\lambda_j}. \tag{4.5}$$

By the gluing construction, we have that, for any $k \in \mathbb{Z}_+$,

$$(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j) \xrightarrow{C^k} (X, R_0^{-2} \cdot g^X, \boldsymbol{x}_\infty). \tag{4.6}$$

Notice that the ALG* gravitational instanton $(X, R_0^{-2} \cdot g^X)$ is a Ricci-flat but nonflat space, which implies that for any $k \in \mathbb{Z}_+$ and $\alpha \in (0,1)$ there exists a constant $v_0 > 0$ such that $\frac{2}{v_0} \leq r_{k,\alpha}(\boldsymbol{x}_\infty) \leq \frac{v_0}{2}$. Therefore, for any $k \in \mathbb{Z}_+$ and $\alpha \in (0,1)$, $v_0^{-1} \leq r_{k,\alpha}(\boldsymbol{x}_j) \leq v_0$ holds with respect to $\hat{g}_{\lambda_j}$, which contradicts (4.4). Therefore, the proof in the case $\sigma_0 \leq R_0$ is complete. The proof in the $\sigma_0 > R_0$ case is the same.

Case (II_1). There exists some constant $\gamma_0 > 0$ such that

$$\lambda_j^{-1} \cdot \tilde{d}^Q(\underline{\boldsymbol{x}}_j, p_m) \cdot T_j^\flat \to \gamma_0 \quad \text{as } j \to \infty, \tag{4.7}$$

where $T_j^\flat \equiv \frac{2\nu+1}{2\pi} \log(\frac{1}{\lambda_j})$. We first assume that $\gamma_0 \leq 1$. By definition, $\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T_j^\flat)^{-\frac{1}{2}}$. Recall that the metric g_{λ_j} satisfies $g_{\lambda_j} = \tilde{\lambda}_j^2 \cdot g^{\mathcal{N}}$ near $\boldsymbol{x}_j$ and by (3.20),

$$g^{\mathcal{N}} = \lambda_j^2 \cdot (G_{\lambda_j} \cdot g^Q + G_{\lambda_j}^{-1} \Theta^2), \tag{4.8}$$

where

$$\left| G_{\lambda_j}(\underline{\boldsymbol{x}}) - T_j^\flat - \frac{1}{2 d^Q(\underline{\boldsymbol{x}}, p_m)} \right| \leq C \quad \text{for } d^Q(\underline{\boldsymbol{x}}, p_m) \leq r_0 \equiv \frac{1}{4} \operatorname{InjRad}_{g^Q}(Q).$$

Let (u_1, u_2, u_3) be a fixed coordinate system in $B_{r_0}(p_m)$ with respect to the metric g^Q. Consider the rescaled metric $\hat{g}_{\lambda_j} \equiv \mathfrak{s}(\boldsymbol{x}_j)^{-2} \cdot g_{\lambda_j}$ and the rescaled coordinates centered at $p_m = (p_{1,m}, p_{2,m}, p_{3,m}) \in P$,

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3) \equiv T_j^\flat (u_1 - p_{1,m}, u_2 - p_{2,m}, u_3 - p_{3,m}).$$

Then by the explicit computations, $(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j)$ C^k-converges for any $k \in \mathbb{Z}_+$, to the Taub-NUT space $(\mathbb{C}^2, g^{\mathrm{TN}}, \boldsymbol{x}_\infty)$, where the Taub-NUT metric g^{TN} can be written explicitly in terms of the Gibbons–Hawking ansatz

$$g^{\mathrm{TN}} = V_0 g^{\mathbb{R}^3} + V_0^{-1}\Theta^2, \qquad V_0 = 1 + (2r)^{-1}, \tag{4.9}$$

and r is the Euclidean distance to the origin of $\mathbb{R}^3$. Therefore, there exists a constant v_0 such that $v_0^{-1} \le r_{k,\alpha}(\boldsymbol{x}_j) \le v_0$ with respect to the rescaled metric $\hat{g}_{\lambda_j}$. Rescaling back to g_{λ_j}, we find that the above estimate contradicts (4.4). This completes the proof under the assumption $\gamma_0 \le 1$. The proof in the case $\gamma_0 > 1$ is the same.

Case (II$_2$). For some $p_m \in P$, the points $\boldsymbol{x}_j$ satisfy

$$\lambda_j^{-1} \cdot \tilde{d}^{Q}(\underline{\boldsymbol{x}}_j, p_m) \cdot T_j^{\flat} \to \infty \qquad \text{and} \qquad \lambda_j^{-1} \cdot \tilde{d}^{Q}(\underline{\boldsymbol{x}}_j, p_m) \to 0 \tag{4.10}$$

as $j \to \infty$. In this case, by definition

$$\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T_j^{\flat})^{\frac{1}{2}} \cdot d_j, \tag{4.11}$$

where $d_j \equiv d^{Q}(\underline{\boldsymbol{x}}_j, p_m)$. We will work with $\hat{g}_{\lambda_j} \equiv \mathfrak{s}(\boldsymbol{x}_j)^{-2} \cdot g_{\lambda_j}$ and the rescaled coordinates centered at $p_m = (p_{1,m}, p_{2,m}, p_{3,m}) \in P$,

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3) \equiv d_j^{-1} \cdot (u_1 - p_{1,m}, u_2 - p_{2,m}, u_3 - p_{3,m}),$$

where (u_1, u_2, u_3) is a fixed coordinate system in $B_{r_0}(p_m)$. One can verify that

$$(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j) \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, g^{\mathbb{R}^3}, \boldsymbol{x}_\infty) \tag{4.12}$$

with $d^{\mathbb{R}^3}(\boldsymbol{x}_\infty, 0^3) = 1$. The detailed and explicit rescaling computations can be found in [27] and [10]. Moreover, if we lift the metric to the local universal cover around $\boldsymbol{x}_j$, then a ball of definite size radius has uniformly bounded $C^{k,\alpha}$-geometry. This implies that $r_{k,\alpha}(\boldsymbol{x}_j) \ge v_0 > 0$. The upper bound for $r_{k,\alpha}(\boldsymbol{x}_j)$ follows from (4.1) and the calculation in Case (II$_1$). We get a contradiction with (4.4) as rescaling back to g_{λ_j}.

Notice that $\mathbb{R}^3$ is precisely the asymptotic cone of the Taub-NUT space.

Case (II$_3$). There is some constant d_0 such that for some $p_m \in P$,

$$\lambda_j^{-1} \cdot \tilde{d}^{Q}(\underline{\boldsymbol{x}}_j, p_m) \to d_0 \quad \text{as } j \to \infty. \tag{4.13}$$

By the definition of $\mathfrak{s}$, we have that

$$\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T_j^{\flat})^{\frac{1}{2}} \cdot d_0 \cdot \big(1 + o(1)\big). \tag{4.14}$$

It suffices to work with the rescaled metric $(\tilde{\lambda}_j \cdot \lambda_j \cdot (T_j^{\flat})^{\frac{1}{2}} \cdot d_0)^{-2} \cdot g_{\lambda_j}$, still denoted by $\hat{g}_{\lambda_j}$, and prove that the regularity scale $r_{k,\alpha}(\boldsymbol{x}_j)$ is uniform bounded from above and below. Then the contradiction arises.

Straightforward computations imply that $\hat{g}_\infty = d_0^{-2} \cdot g^Q$, which is a rescaling of the flat base metric g^Q on $\mathbb{R}^2 \times S^1$. Since $d^Q(P, 0^*) \geq \underline{B}_0 \cdot \lambda_j^{-1} \to \infty$, it follows that the origin $0^2 \in \mathbb{R}^2$ translates to infinity and the $\mathbb{Z}_2$-action limits to the identity. Therefore, the rescaled limit is isometric to $\mathbb{R}^2 \times S^1$. The collapsing keeps curvature uniformly bounded away from P. Then there is some uniform constant $v_0 > 0$ such that $r_{k,\alpha}(\boldsymbol{x}_j) \geq v_0 > 0$. The upper bound for $r_{k,\alpha}(\boldsymbol{x}_j)$ follows from (4.1) and the calculation in Case (II_2).

Notice that $\mathbb{R}^2 \times S^1$ is the flat base of the metric $g^{\mathcal{N}}$.

Case (II_4*).* For some $p_m \in P$, we have

$$\lambda_j^{-1} \cdot \tilde{d}^Q(\underline{\boldsymbol{x}}_j, p_m) \to \infty \qquad \text{and} \qquad \tilde{d}^Q(\underline{\boldsymbol{x}}_j, p_m) \to 0 \quad \text{as } j \to \infty. \tag{4.15}$$

Let us denote $d_j \equiv \lambda_j^{-1} \cdot \tilde{d}^Q(\underline{\boldsymbol{x}}_j, p_m)$. In this case, the definition of $\mathfrak{s}$ implies that

$$\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot \left(T_j^\flat + \frac{1}{2\pi} \log \frac{1}{d_j}\right)^{\frac{1}{2}} \cdot d_j. \tag{4.16}$$

We will prove that in terms of the rescaled metric $\hat{g}_{\lambda_j} \equiv \mathfrak{s}(\boldsymbol{x}_j)^{-2} g_{\lambda_j}$, the regularity scale $r_{k,\alpha}(\boldsymbol{x}_j)$ has a uniform lower bound and upper bound, which contradicts (4.4).

The flat product metric g^Q can be written as $g^Q = dx^2 + dy^2 + d\theta_2^2$ in coordinates. We also rescale the above coordinate system of $\mathbb{R}^2$ centered around $\underline{\boldsymbol{x}}_j = (x_j, y_j)$ by letting

$$(\hat{x}, \hat{y}) \equiv d_j^{-1} \cdot (x - x_j, y - y_j). \tag{4.17}$$

Explicit tensorial computations show that

$$(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j) \xrightarrow{\mathrm{GH}} (\mathbb{R}^2, g^{\mathbb{R}^2}, 0^2), \tag{4.18}$$

where the Euclidean metric $g^{\mathbb{R}^2}$ has the expression $g^{\mathbb{R}^2} \equiv d\hat{x}_\infty^2 + d\hat{y}_\infty^2$. By assumption, $d^Q(\underline{\boldsymbol{x}}_j, 0^*)/d_j \to \infty$, which implies that the origin $0^2 \in \mathbb{R}^2$ translates to infinity and hence the $\mathbb{Z}_2$-action limits to the identity as $j \to \infty$.

The finite set P converges to a single point $p_0 \in \mathbb{R}^2$ and $d^{\mathbb{R}^2}(p_0, 0^2) = 1$. The above collapsing keeps curvature uniformly bounded away from the point p_0. So there is some uniform constant $v_0 > 0$ such that $r_{k,\alpha}(\boldsymbol{x}_j) \geq v_0 > 0$. The upper bound for $r_{k,\alpha}(\boldsymbol{x}_j)$ follows from (4.1) and the calculation in Case (II_3).

Case (III*).* There exists some constant d_0 such that

$$\tilde{d}^Q(\underline{\boldsymbol{x}}_j, P) \geq d_0 > 0, \qquad \tilde{r}(\underline{\boldsymbol{x}}_j) \cdot \lambda_j^{-1} \to \infty, \qquad L_j \equiv \mathfrak{L}_{T_j}(\underline{\boldsymbol{x}}_j) \to +\infty. \tag{4.19}$$

There are the following subcases to analyze.

First, assume that $\tilde{r}(\underline{\boldsymbol{x}}_j) \to 0$ as $j \to \infty$. In this case, by definition,

$$\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot L_j^{\frac{1}{2}} \cdot \tilde{r}_j. \tag{4.20}$$

We will prove that under the rescaling $\hat{g}_{\lambda_j} = (\mathfrak{s}(\boldsymbol{x}_j))^{-2} \cdot g_{\lambda_j}$ and $(\hat{x}, \hat{y}) = \tilde{r}_j^{-1} \cdot (\tilde{x}, \tilde{y})$, the convergence

$$(\mathcal{S}_b, \hat{g}_{\lambda_j}, \boldsymbol{x}_j) \xrightarrow{\mathrm{GH}} (\mathbb{R}^2/\mathbb{Z}_2, d^{\mathbb{R}^2/\mathbb{Z}_2}, \underline{\boldsymbol{x}}_\infty) \tag{4.21}$$

holds, where $d^{\mathbb{R}^2/\mathbb{Z}_2}(\underline{\boldsymbol{x}}_\infty, 0^2) = 1$, and the flat metric on $\mathbb{R}^2/\mathbb{Z}_2$ can be written in terms of the limit coordinate system of $(\hat{x}, \hat{y})$. Notice that $\mathbb{R}^2/\mathbb{Z}_2$ is the asymptotic cone of the ALG$_\nu^*$ space $(X, g^X, \boldsymbol{\omega}^X)$. Moreover, we will show that the rescaled metrics $\hat{g}_{\lambda_j}$ has uniformly bounded curvature away from the cone tip. This suffices to produce the desired contradiction because the upper bound on $r_{k,\alpha}(\boldsymbol{x}_j)$ follows from (4.1) and the calculation in Case (II$_4$).

To prove the above claim, let us choose a domain

$$\mathcal{U}_{\xi_j} \equiv \left\{\boldsymbol{x} \in \mathfrak{N} \subset \mathcal{S}_b \subset \mathcal{M}_{\lambda_j} \,\middle|\, \xi_j^{-1} \le \hat{r}(\boldsymbol{x}) \le \xi_j\right\} \tag{4.22}$$

for a sequence ξ_j that satisfies $\lim_{j\to\infty} \frac{\xi_j}{L_j} = 0$. Then for any $\boldsymbol{x}_j \in \mathcal{U}_{\xi_j}$,

$$\frac{\widetilde{L}_{2\nu}(\underline{\boldsymbol{x}}_j)}{L_j} = 1 + o(1) \quad \text{as } j \to \infty. \tag{4.23}$$

By explicit tensorial computations on the Gibbons–Hawking metric $g^{\mathcal{N}}$, we can check that the $\mathbb{Z}_2$-covering of $(\mathcal{U}_{\xi_j}, \hat{g}_{\lambda_j})$ will smoothly converge to the flat metric $d\hat{x}_\infty^2 + d\hat{y}_\infty^2$ on $\mathbb{R}^2$, where $(\hat{x}_\infty, \hat{y}_\infty)$ is the limit of $(\hat{x}, \hat{y})$. Also notice that the limiting reference point $\underline{\boldsymbol{x}}_\infty$ satisfies $d^{\mathbb{R}^2}(\underline{\boldsymbol{x}}_\infty, 0^2) = 1$. Then the $\mathbb{Z}_2$-quotient metric $\hat{g}_{\lambda_j}$ converges to the flat metric on $\mathbb{R}^2/\mathbb{Z}_2$.

Next, we consider the case $\tilde{r}(\underline{\boldsymbol{x}}_j) \to d_0' > 0$. By definition,

$$\mathfrak{s}(\boldsymbol{x}_j) = \tilde{\lambda}_j \cdot T_j^{\frac{1}{2}} \cdot d_0' \cdot \big(1 + o(1)\big) \tag{4.24}$$

as $j \to \infty$. It suffices to work with the rescaled metric $(\tilde{\lambda}_j \cdot T_j^{\frac{1}{2}} \cdot d_0')^{-2} \cdot g_{\lambda_j}$, still denoted by $\hat{g}_{\lambda_j}$, and we can show that the regularity scale $r_{k,\alpha}(\boldsymbol{x}_j)$ has a uniform lower bound. Moreover, the rescaled limit in this case is isometric to $\mathbb{R}^2/\mathbb{Z}_2$ as well. We skip the detailed computations since the arguments are the same.

The last possibility is when $\boldsymbol{x}_j$ satisfies $\tilde{r}(\underline{\boldsymbol{x}}_j) \to \infty$ and $L_j \to \infty$. In the proof, we still use the rescaled metric $\hat{g}_{\lambda_j} = (\mathfrak{s}(\boldsymbol{x}_j))^{-2} \cdot g_{\lambda_j}$ and the rescaled coordinates $(\hat{x}, \hat{y}) = \tilde{r}_j^{-1} \cdot (x, y)$. The computations are the same. We only mention that, as $\tilde{r}_j =$

$\tilde{r}(\underline{\boldsymbol{x}}_j)$ becomes very large, one can obtain the rescaled limit $\mathbb{R}^2/\mathbb{Z}_2$ as long as $L_j \to \infty$.

When $\tilde{r}_j$ is sufficiently large such that $L_j \to L_0 > 0$ as $j \to \infty$, we will obtain another rescaled limit. This becomes Case (IV).

*Case (*IV*).* There is some constant $L_0 > 0$ such that $L_j \to L_0 > 0$. In this case,

$$\mathfrak{s}(\boldsymbol{x}_j) \equiv \tilde{\lambda}_j \cdot L_j^{\frac{1}{2}} \cdot \tilde{r}_j = \tilde{\lambda}_j \cdot \tilde{r}_j \cdot L_0 \cdot \big(1 + o(1)\big) \tag{4.25}$$

as $j \to \infty$. In the meantime, notice that

$$L_j = \Big(T_j + \operatorname{Im} h(0) - \frac{b}{\pi} \log \tilde{r}_j\Big) \cdot \big(1 + o(1)\big) \tag{4.26}$$

as $j \to \infty$. It is easy to verify that $\mathfrak{s}(\boldsymbol{x}_j)$ is a bounded constant. Then the rescaled limit is the McLean metric on $\mathbb{P}^1$. Moreover, the convergence keeps curvature uniformly bounded away from the singular fiber.

The above covers all the points on $\mathcal{S}_b$, which completes the proof. □

5. Perturbation to hyper-Kähler metrics

In this section, we will glue an ALG* gravitational instanton into a region near an I_b^*-fiber of an elliptic K3 surface $\pi_{\mathcal{K}} : \mathcal{K} \to \mathbb{P}^1$. For our purpose, it suffices to assume that the singular fibers of $\pi_{\mathcal{K}}$ consist of an I_b^*-fiber for some $1 \le b \le 14$ and I_1-fibers of number $(18 - b)$. Following the notation in Section 6 of [10], we denote by $\mathcal{S}_b$ the subset of $\mathcal{M}_\lambda$ which consists of a small annular region in $\mathcal{K}$ centered around the I_b^*-fiber, the neck region $\mathfrak{N}$, and the ALG* manifold X. We denote by $\mathcal{S}_{\mathrm{I}_1}$ the subset of $\mathcal{M}_\lambda$ which consists of small annular regions in $\mathcal{K}$ centered around I_1-fibers and Ooguri–Vafa manifolds. Let $\mathcal{R}_\lambda$ be the regular region in $\mathcal{K}$. We will prove that the glued manifold $\mathcal{M}_\lambda$ admits collapsing hyper-Kähler metrics with prescribed behaviors. In the following weighted analysis, the weight function ρ as a global smooth function on $\mathcal{M}_\lambda$ is defined as follows:

$$\rho(\boldsymbol{x}) \equiv \begin{cases} \mathfrak{s}(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{S}_b, \\ \mathfrak{s}_1(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{S}_{\mathrm{I}_1}, \\ 1 & \boldsymbol{x} \in \mathcal{R}_\lambda, \end{cases} \tag{5.1}$$

where $\mathfrak{s}_1$ is the canonical scale function defined in Section 6.3 of [10]. With respect to the weight function ρ, we will define the following weighted Hölder norms.

Definition 5.1

For any fixed parameter $\lambda \ll 1$, let g_λ be the approximately hyper-Kähler metric defined on the glued manifold $\mathcal{M}_\lambda$. Let $U \subset \mathcal{M}_\lambda$ be a compact subset. Then the weighted Hölder norm of a tensor field $\chi \in T^{r,s}(U)$ of type (r,s) is defined as follows:

(1) The weighted $C^{k,\alpha}$-seminorm of χ is defined by

$$[\chi]_{C^{k,\alpha}_{\mu}(\boldsymbol{x})} \equiv \sup\Big\{\rho^{k+\alpha-\mu}(\boldsymbol{x}) \cdot \frac{|\nabla^k \hat{\chi}(\hat{\boldsymbol{x}}) - \nabla^k \hat{\chi}(\hat{\boldsymbol{y}})|}{(d_{\hat{g}_\lambda}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}))^{\alpha}} \;\Big|\; \hat{\boldsymbol{y}} \in B_{r_{k,\alpha}(\boldsymbol{x})}(\hat{\boldsymbol{x}})\Big\},$$

$$[\chi]_{C^{k,\alpha}_{\mu}(U)} \equiv \sup\big\{[\chi]_{C^{k,\alpha}_{\mu}(\boldsymbol{x})} \;\big|\; \boldsymbol{x} \in U\big\},$$

where $r_{k,\alpha}(\boldsymbol{x})$ is the $C^{k,\alpha}$-regularity scale at $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$ denotes a lift of $\boldsymbol{x}$ to the universal cover of $B_{2r_{k,\alpha}(\boldsymbol{x})}(\boldsymbol{x})$, the difference of the two covariant derivatives is defined in terms of parallel translation in $B_{r_{k,\alpha}(\boldsymbol{x})}(\hat{\boldsymbol{x}})$, and $\hat{\chi}$, $\hat{g}_\lambda$ are the lifts of χ, g_λ, respectively.

(2) The weighted $C^{k,\alpha}$-norm of χ is defined by

$$\|\chi\|_{C^{k,\alpha}_{\mu}(U)} \equiv \sum_{m=0}^{k} \|\rho^{m-\mu} \cdot \nabla^m \chi\|_{C^0(U)} + [\chi]_{C^{k,\alpha}_{\mu}(U)}.$$

Now let us briefly describe the perturbation scheme to produce hyper-Kähler triples from the approximate triples constructed in Section 3. This original characterization is due to Donaldson [15], which has also been used in [8], [10], [17], [19], and [27]. Let M^4 be an oriented 4-manifold with a volume form dvol_0. A triple of closed 2-forms $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is said to be *definite* if the matrix $Q = (Q_{ij})$ defined by $\frac{1}{2}\omega_i \wedge \omega_j = Q_{ij}\,\mathrm{dvol}_0$ is positive definite. A definite triple $\boldsymbol{\omega}$ is called a *hyper-Kähler triple* if $Q_{ij} = \delta_{ij}$. Given a definite triple $\boldsymbol{\omega}$, the associated volume form is defined as $\mathrm{dvol}_{\boldsymbol{\omega}} \equiv (\det(Q))^{\frac{1}{3}}\,\mathrm{dvol}_0$, and we denote by $Q_{\boldsymbol{\omega}} \equiv (\det(Q))^{-\frac{1}{3}} Q$ the normalized matrix with unit determinant. Every definite triple $\boldsymbol{\omega}$ determines a Riemannian metric $g_{\boldsymbol{\omega}}$ such that each ω_j, $j \in \{1,2,3\}$, is self-dual with respect to $g_{\boldsymbol{\omega}}$ and $\mathrm{dvol}_{g_{\boldsymbol{\omega}}} = \mathrm{dvol}_{\boldsymbol{\omega}}$.

Suppose that we have a closed definite triple $\boldsymbol{\omega}$ on $\mathcal{M}_\lambda$. We want to find a triple of closed 2-forms $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ such that $\underline{\boldsymbol{\omega}} \equiv \boldsymbol{\omega} + \boldsymbol{\theta}$ is an actual hyper-Kähler triple on $\mathcal{M}_\lambda$ satisfying

$$\frac{1}{2}(\omega_i + \theta_i) \wedge (\omega_j + \theta_j) = \delta_{ij}\,\mathrm{dvol}_{\boldsymbol{\omega}+\boldsymbol{\theta}}, \tag{5.2}$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2}(\omega_i \wedge \omega_j + \omega_i \wedge \theta_j + \omega_j \wedge \theta_i + \theta_i \wedge \theta_j)\\ &\qquad = \frac{1}{6}\delta_{ij} \sum_{k=1}^{3} (\omega_k^2 + \theta_k^2 + 2\omega_k \wedge \theta_k).\end{aligned} \tag{5.3}$$

Writing $\boldsymbol{\theta} = \boldsymbol{\theta}^+ + \boldsymbol{\theta}^-$ with $*_{g_{\omega}}\boldsymbol{\theta}^{\pm} = \pm\boldsymbol{\theta}^{\pm}$, we define the matrices $A = (A_{ij})$ and $S_{\boldsymbol{\theta}^-} = (S_{ij})$ by

$$\theta_i^+ = \sum_{j=1}^{3} A_{ij}\omega_j, \qquad \frac{1}{2}\theta_i^- \wedge \theta_j^- = S_{ij}\,\mathrm{dvol}_{\boldsymbol{\omega}}, \quad 1 \le i \le j \le 3. \tag{5.4}$$

Then (5.3) is equivalent to

$$\mathrm{tf}(Q_{\boldsymbol{\omega}}A^T + Q_{\boldsymbol{\omega}}A + AQ_{\boldsymbol{\omega}}A^T) = \mathrm{tf}(-Q_{\boldsymbol{\omega}} - S_{\boldsymbol{\theta}^-}), \tag{5.5}$$

where $\mathrm{tf}(B) \equiv B - \frac{1}{3}\mathrm{Tr}(B)\,\mathrm{Id}$ for a 3×3 real matrix B, and $Q_{\boldsymbol{\omega}}$ is the 3×3 real matrix such that $\det(Q_{\boldsymbol{\omega}}) = 1$ and

$$\frac{1}{2}\omega_i \wedge \omega_j = (Q_{\boldsymbol{\omega}})_{ij}\,\mathrm{dvol}_{\boldsymbol{\omega}}. \tag{5.6}$$

Then observe that a solution of

$$\begin{aligned} &d^+\boldsymbol{\eta} + \boldsymbol{\xi} = \mathfrak{F}_0\big(\mathrm{tf}(-Q_{\boldsymbol{\omega}} - S_{d^-\boldsymbol{\eta}})\big), \\ &\quad d^*\boldsymbol{\eta} = 0, \quad \boldsymbol{\eta} \in \Omega^1(\mathcal{M}_\lambda)\otimes\mathbb{R}^3, \qquad \boldsymbol{\xi} \in \mathcal{H}^+_{g_{\boldsymbol{\omega}}}(\mathcal{M}_\lambda)\otimes\mathbb{R}^3, \end{aligned} \tag{5.7}$$

is also a solution of (5.5). Here $\mathfrak{F}_0$ denotes the local inverse near zero of

$$\mathfrak{G}_0 : \mathscr{S}_0(\mathbb{R}^3) \to \mathscr{S}_0(\mathbb{R}^3), \qquad A \mapsto \mathrm{tf}(Q_{\boldsymbol{\omega}}A^T + AQ_{\boldsymbol{\omega}} + AQ_{\boldsymbol{\omega}}A^T)$$

on the space of trace-free symmetric (3×3)-matrices $\mathscr{S}_0(\mathbb{R}^3)$, and $d^{\pm}\boldsymbol{\eta}$ is the self-dual or anti-self-dual part of $d\boldsymbol{\eta} = \boldsymbol{\theta} - \boldsymbol{\xi}$, respectively. The linearization of the elliptic system (5.7) at $\boldsymbol{\eta} = 0$ is given by $\mathscr{L} = (\mathscr{D} \oplus \mathrm{Id})\otimes\mathbb{R}^3 : (\Omega^1(\mathcal{M}_\lambda) \oplus \mathcal{H}^+_{g_{\boldsymbol{\omega}}}(\mathcal{M}_\lambda))\otimes\mathbb{R}^3 \longrightarrow (\Omega^0(\mathcal{M}_\lambda)\oplus\Omega^2_+(\mathcal{M}_\lambda))\otimes\mathbb{R}^3$, where

$$\mathscr{D} \equiv d^* + d^+ : \Omega^1(\mathcal{M}_\lambda) \longrightarrow \big(\Omega^0(\mathcal{M}_\lambda)\oplus\Omega^2_+(\mathcal{M}_\lambda)\big).$$

For any sufficiently small $\lambda \ll 1$, we will solve the elliptic system (5.7). The proof of the existence of hyper-Kähler triples requires the following version of the implicit function theorem.

LEMMA 5.2
Let $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$ be a map between two Banach spaces with

$$\mathscr{F}(\boldsymbol{x}) = \mathscr{F}(\boldsymbol{0}) + \mathscr{L}(\boldsymbol{x}) + \mathscr{N}(\boldsymbol{x}), \tag{5.8}$$

where the operator $\mathscr{L} : \mathfrak{A} \to \mathfrak{B}$ is linear and $\mathscr{N}(\boldsymbol{0}) = \boldsymbol{0}$. Assume that
(1) *$\mathscr{L}$ is an isomorphism with $\|\mathscr{L}^{-1}\| \le C_L$ for some $C_L > 0$;*
(2) *there are constants $r > 0$ and $C_N > 0$ such that:*

(a) $r < (10C_L \cdot C_N)^{-1}$,

(b) $\|\mathscr{N}(\boldsymbol{x}) - \mathscr{N}(\boldsymbol{y})\|_{\mathfrak{B}} \leq C_N \cdot (\|\boldsymbol{x}\|_{\mathfrak{A}} + \|\boldsymbol{y}\|_{\mathfrak{A}}) \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathfrak{A}}$ *for all* $x, y \in B_r(\mathbf{0}) \subset \mathfrak{A}$,

(c) $\|\mathscr{F}(\mathbf{0})\|_{\mathfrak{B}} \leq \frac{r}{10C_L}$.

Then $\mathscr{F}(\boldsymbol{x}) = \mathbf{0}$ *has a unique solution* $\boldsymbol{x} \in \mathfrak{A}$ *such that* $\|\boldsymbol{x}\|_{\mathfrak{A}} \leq 2C_L\|\mathscr{F}(\mathbf{0})\|_{\mathfrak{B}}$.

To apply the implicit function theorem, first we fix two Banach spaces

$$\mathfrak{A} \equiv \big(C^{1,\alpha}_{\mu}\big(\mathring{\Omega}^1(\mathcal{M}_\lambda)\big) \oplus \mathcal{H}^+(\mathcal{M}_\lambda)\big) \otimes \mathbb{R}^3, \qquad \mathfrak{B} \equiv \big(C^{0,\alpha}_{\mu-1}\big(\Omega^2_+(\mathcal{M}_\lambda)\big)\big) \otimes \mathbb{R}^3,$$

where $\mu \in (-1, 0)$, $\alpha \in (0, 1)$, and $\mathring{\Omega}^1(\mathcal{M}_\lambda) \equiv \{\eta \in \Omega^1(\mathcal{M}_\lambda) \mid d^*\eta = 0\}$. The following error estimate is an immediate corollary of Lemma 3.5.

COROLLARY 5.3

There exists $C_0 > 0$ *independent of the parameters* λ *and* $\mathfrak{t}$ *such that*

$$\big\|\mathscr{F}(\mathbf{0})\big\|_{\mathfrak{B}} \leq C_0 \cdot (\lambda^2 + \tilde{\lambda} \cdot \mathfrak{t}^3) \cdot \tilde{\lambda}^{-\mu+1} \cdot \mathfrak{t}^{-\mu-1} \cdot V(\sigma^{-1})^{\frac{-\mu-1}{2}} + C_0 \cdot \lambda^2 \cdot \tilde{\lambda}^2.$$

Proof

Let $\mathfrak{N}_r \equiv \{\boldsymbol{x} \in \mathfrak{N} \mid r \leq \tilde{r}(\boldsymbol{x}) \leq 2r\}$. Then by Lemma 3.5,

$$\|Q_{\boldsymbol{\omega}} - \mathrm{Id}\|_{C^{0,\alpha}_{\mu-1}(\mathfrak{N}_{\mathfrak{t}})} \leq C_0 \cdot (\lambda^2 + \tilde{\lambda} \cdot \mathfrak{t}^3) \cdot \tilde{\lambda}^{-\mu+1} \cdot \mathfrak{t}^{-\mu-1} \cdot V(\sigma^{-1})^{\frac{-\mu-1}{2}},$$

$$\|Q_{\boldsymbol{\omega}} - \mathrm{Id}\|_{C^{0,\alpha}_{\mu-1}(\mathfrak{N}_{r_\lambda})} \leq C_0 \cdot \lambda^2 \cdot \tilde{\lambda}^2.$$

On the other hand, the error estimate near an I_1-fiber is much smaller. In fact, by Theorem 4.4 of [23] (see also [10, Proposition 8.2]),

$$\|Q_{\boldsymbol{\omega}} - \mathrm{Id}\|_{C^{0,\alpha}_{\mu-1}(T_{\delta_0,2\delta_0}(\mathcal{S}_{\mathrm{I}_1}))} \leq C_1 \cdot e^{-C_2 \cdot \tilde{\lambda}^{-1}} \tag{5.9}$$

for some constants $C_1 > 0$, $C_2 > 0$ independent of $\tilde{\lambda}$ (and hence λ), where $T_{\delta_0,2\delta_0}(\mathcal{S}_{\mathrm{I}_1})$ is an annular neighborhood of definite size $\delta_0 > 0$ independent of $\tilde{\lambda}$. □

We also need the weighted estimate on the nonlinear errors.

LEMMA 5.4 (Nonlinear estimate)

There exists some constant $K_0 > 0$ *independent of* λ *such that for any* $v_1, v_2 \in B_1(\mathbf{0}) \subset \mathfrak{A}$*, we have that*

$$\big\|\mathscr{N}_\lambda(v_1) - \mathscr{N}_\lambda(v_2)\big\|_{\mathfrak{B}} \leq K_0 \cdot (\tilde{\lambda} \cdot \lambda)^{\mu-1} (T^\flat)^{\frac{1-\mu}{2}} \big(\|v_1\|_{\mathfrak{A}} + \|v_2\|_{\mathfrak{A}}\big) \cdot \|v_1 - v_2\|_{\mathfrak{A}}.$$

Proof
For any $v_1, v_2 \in B_1(\mathbf{0}) \subset \mathfrak{A}$, by explicit computations,

$$\left|\mathscr{N}_\lambda(v_1) - \mathscr{N}_\lambda(v_2)\right| \leq K_0 \cdot \left(|d^-\eta_1| + |d^-\eta_2|\right) \cdot \left|d^-(\eta_1 - \eta_2)\right|,$$

where $K_0 > 0$ is independent of λ. Multiplying by the weight function $\mathfrak{s}(\boldsymbol{x})^{-\mu+1}$, we have that

$$\begin{aligned} &\mathfrak{s}(\boldsymbol{x})^{-\mu+1} \cdot \left|\mathscr{N}_\lambda(v_1) - \mathscr{N}_\lambda(v_2)\right| \\ &\qquad \leq K_0 \cdot \mathfrak{s}(\boldsymbol{x})^{-\mu+1} \cdot \left(|d^-\eta_1| + |d^-\eta_2|\right) \cdot \left|d^-(\eta_1 - \eta_2)\right|. \end{aligned}$$

By definition, the scale function $\mathfrak{s}(\boldsymbol{x})$ achieves the minimum $\tilde{\lambda} \cdot \lambda \cdot (T^\flat)^{-\frac{1}{2}}$ when $d^{Q}(\underline{\boldsymbol{x}}, p_m) \leq (T^\flat)^{-1}$ for some $1 \leq m \leq 2\nu + 2b$. Then we have that

$$\begin{aligned} &\left\|\mathscr{N}_\lambda(v_1) - \mathscr{N}_\lambda(v_2)\right\|_{C^0_{\mu+1}(\mathcal{M}_\lambda)} \\ &\quad \leq K_0 \cdot (\tilde{\lambda} \cdot \lambda)^{\mu-1} (T^\flat)^{\frac{1-\mu}{2}} \left(\|v_1\|_{C^1_\mu(\mathcal{M}_\lambda)} + \|v_2\|_{C^1_\mu(\mathcal{M}_\lambda)}\right) \cdot \left(\|v_1 - v_2\|_{C^1_\mu(\mathcal{M}_\lambda)}\right). \end{aligned}$$

By similar computations, we also have the desired estimate for the $C^{0,\alpha}$-seminorm. This completes the proof. □

The following is the main ingredient needed to carry out the perturbation.

PROPOSITION 5.5 (Weighted linear estimate)
Let $\mathcal{M}_\lambda$ be the glued manifold with a family of approximately hyper-Kähler metrics g_λ. Then there exists $C > 0$, independent of λ, such that for every self-dual 2-form $\xi^+ \in \mathfrak{B}$, there exists a unique pair $(\eta, \bar{\xi}^+) \in \mathfrak{A}$ such that for some $\mu \in (-1, 0)$ and $\alpha \in (0, 1)$,

$$\mathscr{L}_\lambda(\eta, \bar{\xi}^+) = \xi^+, \tag{5.10}$$

$$\|\eta\|_{C^{1,\alpha}_\mu(\mathcal{M}_\lambda)} + \|\bar{\xi}^+\|_{C^{0,\alpha}_{\mu-1}(\mathcal{M}_\lambda)} \leq C\|\xi^+\|_{C^{0,\alpha}_{\mu-1}(\mathcal{M}_\lambda)}. \tag{5.11}$$

The proof is very similar to the proof of Proposition 8.7 in [10] which follows from a contradiction argument and applying various Liouville theorems on the blowup limits. We omit the details and only mention the outline.

(1) If the blowup limit is an ALF or ALG* gravitational instanton (X, g, p) with $p \in X$, the Liouville theorem invoked in the proof is that any 1-form ω that satisfies $\Delta_H \omega = 0$ and $\lim_{d_g(\boldsymbol{x},p)\to\infty} |\omega(\boldsymbol{x})| \to 0$ has to vanish everywhere, that is, $\omega \equiv 0$ on X. Indeed, the Bochner formula and the Ricci-flatness imply that $\Delta|\omega|^2 \geq 0$, and the vanishing of ω is then a consequence of the maximum principle.

(2) If the blowup limit is a flat space in Proposition 4.4, namely, $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$, $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z}_2$, then we will quote the following Liouville theorem: if $\mu \in (-1, 0)$, then any harmonic function f that satisfies $|f| \leq C \cdot r^{\mu}$ for any $r \in (0, \infty)$ has to be identically zero, where r is the Euclidean distance to a fixed point.

(3) The Liouville theorem corresponding to $(\mathbb{P}^1, d_{\mathrm{ML}})$ is Proposition 7.8 in [10].

Combining the above results, we now prove the perturbation theorem.

THEOREM 5.6

Let $(X, g^X, \boldsymbol{\omega}^X)$ be an order-2 ALG^*_ν gravitational instanton for some $\nu \in \{1, 2, 3, 4\}$. Then for any integer $1 \leq b \leq 14$ and for any sufficiently small parameter $\lambda \ll 1$, there exists a family of hyper-Kähler structures $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ on the K3 surface $\mathcal{M}_\lambda$ such that the following properties hold as $\lambda \to 0$.

(1) *We have Gromov–Hausdorff convergence $(\mathcal{M}_\lambda, h_\lambda) \xrightarrow{\mathrm{GH}} (\mathbb{P}^1, d_{\mathrm{ML}})$, where d_{ML} is the McLean metric on $\mathbb{P}^1$ with a finite singular set $\mathcal{S} \equiv \{q_0, q_1, \ldots, q_{18-b}\} \subset \mathbb{P}^1$. Moreover, the curvatures of h_λ are uniformly bounded away from $\mathcal{S}$, but are unbounded around $\mathcal{S}$.*

(2) *The hyper-Kähler structures $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ satisfy the uniform error estimate for some positive number $0 < \epsilon \ll \min\{1, \frac{\nu}{b}\}$,*

$$\|h_\lambda - g_\lambda\|_{C^{0,\alpha}_0(\mathcal{M}_\lambda)} \leq C \cdot (\lambda^{2-\epsilon} + \lambda^{\frac{\nu}{b}-\epsilon}), \tag{5.12}$$

$$\|\boldsymbol{\omega}_{h_\lambda} - \boldsymbol{\omega}_\lambda\|_{C^{0,\alpha}_0(\mathcal{M}_\lambda)} \leq C \cdot (\lambda^{2-\epsilon} + \lambda^{\frac{\nu}{b}-\epsilon}), \tag{5.13}$$

where g_λ is the metric determined by the definite triple $\boldsymbol{\omega}_\lambda$, and the $C^{0,\alpha}_0$-norm is the weighted norm in Definition 5.1 when $k = 0$ and $\mu = 0$.

(3) *Rescalings of $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ around q_i for $1 \leq i \leq 18 - b$ converge in the pointed C^k-topology (for all $k \in \mathbb{Z}_+$) to a complete Taub-NUT gravitational instanton on $\mathbb{C}^2$.*

(4) *Rescalings of $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ around q_0 converge in the pointed C^k-topology (for all $k \in \mathbb{Z}_+$) to the given ALG^*_ν gravitational instanton $(X, g^X, \boldsymbol{\omega}^X)$ or one of $(\nu + b)$ copies of complete Taub-NUT gravitational instantons.*

Proof of Theorem 5.6

We will apply Lemma 5.2 to perform the perturbation. Let

$$C_{\mathrm{err}} \equiv C_0 \cdot (\lambda^2 + \tilde{\lambda} \cdot \mathfrak{t}^3) \cdot \tilde{\lambda}^{-\mu+1} \cdot \mathfrak{t}^{-\mu-1} \cdot V(\sigma^{-1})^{\frac{-\mu-1}{2}} + C_0 \cdot \lambda^2 \cdot \tilde{\lambda}^2, \tag{5.14}$$

$$C_N \equiv (\tilde{\lambda} \cdot \lambda)^{\mu-1} \cdot (T^\flat)^{\frac{1-\mu}{2}} \tag{5.15}$$

be the constants in Corollary 5.3 and Lemma 5.4. Recall that λ and $\mathfrak{t}$ are chosen such that $\sigma = \frac{\lambda}{\mathfrak{t}} \to 0$. To prove (5.13), we only need to fix the parameter $\mathfrak{t} \equiv \lambda^{\frac{\epsilon}{10}}$ for a fixed $\epsilon \ll 1$ and let $\mu = -1 + \frac{\epsilon}{10}$. Then it is obvious that $C_{\mathrm{err}} \cdot C_N \to 0$ as $\lambda \to 0$. The uniform linear estimate is given by Proposition 5.5. Then Lemma 5.2 implies that there exists a solution which satisfies the desired estimate. Moreover, (5.13) follows from (5.15). The classification of the intermediate bubbles is given by Proposition 4.4 and noticing that the solutions h_λ are sufficiently close to g_λ. ☐

6. Proofs of Torelli uniqueness theorems

In this section, we complete the proofs of Theorem 1.10 and Theorem 1.5. We also explain the reason for the order 2 assumption in Theorem 1.5.

6.1. *Proof of Theorem 1.10:* ALG* *Torelli uniqueness*

Let $(X_\nu, g, \boldsymbol{\omega})$ and $(X_\nu, g', \boldsymbol{\omega}')$ be ALG* gravitational instantons on X_ν with the same parameters κ_0, L, which are both of order 2 with respect to the coordinates Φ_{X_ν} and which satisfy (1.11). Let $\pi_{\mathcal{K}} : \mathcal{K} \to \mathbb{P}^1$ be any elliptic K3 surface with a single fiber of type I_b^*, call it D^*, but has all other singular fibers of type I_1. Let $U = \{\boldsymbol{x} \in \mathcal{M}_\lambda \mid \tilde{r}(\boldsymbol{x}) \geq \mathfrak{t}\}$ and $V = \{\boldsymbol{x} \in \mathcal{M}_\lambda \mid \tilde{r}(\boldsymbol{x}) \leq 2\mathfrak{t}\}$. Then $\mathcal{M}_\lambda = U \cup V$. The gluing procedure in Section 3.3 produces approximate hyper-Kähler triples $\tilde{\boldsymbol{\omega}}_\lambda$ and $\tilde{\boldsymbol{\omega}}'_\lambda$ on $\mathcal{M}_\lambda$. Note that $U \cap V$ deformation retracts onto the 3-manifold $\mathcal{I}^3_\nu$.

LEMMA 6.1
The manifold $\mathcal{I}^3_\nu = \mathrm{Nil}^3_{2\nu}/\mathbb{Z}_2$ is an infra-nilmanifold, which is a circle bundle of degree ν over a Klein bottle. Furthermore, we have $b^1(\mathcal{I}^3_\nu) = 1$, with $H^1_{\mathrm{dR}}(\mathcal{I}^3_\nu)$ generated by the 1-form $d\theta_1$.

Proof
The first statement follows since $\mathrm{Nil}^3_{2\nu}$ is a circle bundle over a torus, and the quotient space is then clearly a circle bundle over a Klein bottle. From [27, Proposition 2.3], we have $b^1(\mathrm{Nil}^3_{2\nu}) = 2$, with $H^1_{\mathrm{dR}}(\mathrm{Nil}^3_{2\nu})$ generated by $d\theta_1$ and $d\theta_2$. These forms are harmonic with respect to any left-invariant and $\mathbb{Z}_2$-invariant metric on $\mathrm{Nil}^3_{2\nu}$. Of these generators, only $d\theta_1$ is invariant under this action, so the lemma follows from the Hodge theorem. ☐

The Mayer–Vietoris sequence in de Rham cohomology for $\{U, V\}$ is

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \arrow[r] & H^1_{\mathrm{dR}}(\mathcal{I}^3_\nu) \arrow[dll, rounded corners] & \\ H^2_{\mathrm{dR}}(\mathcal{M}_\lambda) \arrow[r] & H^2_{\mathrm{dR}}(U) \oplus H^2_{\mathrm{dR}}(V) \arrow[r] & H^2_{\mathrm{dR}}(\mathcal{I}^3_\nu) \arrow[r] & 0. \end{tikzcd} \tag{6.1}$$

From the gluing in Section 3, we have

$$(\Phi^{-1})^*(\lambda^2 \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}^{\mathfrak{N}} = d\boldsymbol{\eta}, \qquad (\Phi^{-1})^*(\lambda^2 \cdot \boldsymbol{\omega}') - \boldsymbol{\omega}^{\mathfrak{N}} = d\boldsymbol{\eta}', \tag{6.2}$$

where $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are triples of 1-forms on $\{\boldsymbol{x} \in \mathfrak{N} \mid \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\}$. From Lemma 3.5, on the region $\{\boldsymbol{x} \in \mathfrak{N} \mid \mathfrak{t} \le \tilde{r}(\boldsymbol{x}) \le 2\mathfrak{t}\}$, the approximate hyper-Kähler triples are

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}^{\mathfrak{N}} + d(\varphi \cdot \boldsymbol{\eta}), \qquad \tilde{\boldsymbol{\omega}}' = \boldsymbol{\omega}^{\mathfrak{N}} + d(\varphi \cdot \boldsymbol{\eta}'), \tag{6.3}$$

where φ is a cutoff function which is 1 when $\tilde{r}(\boldsymbol{x}) \le \mathfrak{t}$, and is 0 when $\tilde{r}(\boldsymbol{x}) \ge 2\mathfrak{t}$. Clearly, the image of $[\tilde{\omega}_i] \in H^2_{\mathrm{dR}}(\mathcal{M}_\lambda)$ in $H^2_{\mathrm{dR}}(U) \oplus H^2_{\mathrm{dR}}(V)$ is $([\omega_i], [\omega_i^V])$. Since the two ALG* gravitational instantons have the same $[\omega_i] = [\omega_i']$ and we also use the same ω_i^V for both, we see that the image of $[\omega_i]$ and $[\omega_i']$ are the same. So their difference is in the image of $H^1_{\mathrm{dR}}(\mathcal{I}^3_\nu)$. To see the image, we start with $d\theta_1 \in H^1_{\mathrm{dR}}(\mathcal{I}^3_\nu)$. It can be written as the difference of $\varphi d\theta_1$ on U and $(\varphi - 1)d\theta_1$ on V. The form $d(\varphi d\theta_1) = d((\varphi - 1)d\theta_1)$ can be viewed as a 2-form on $\mathcal{M}_\lambda$ which is the image of $d\theta_1$ in $H^2_{\mathrm{dR}}(\mathcal{M}_\lambda)$. Therefore, $[\tilde{\omega}_i]$ and $[\tilde{\omega}_i']$ may differ by a multiple of $[d(\varphi d\theta_1)]$. Fortunately, we can modify the 1-form η_i by the same multiple of $d\theta_1$, and we then obtain $[\tilde{\omega}_i] = [\tilde{\omega}_i'] \in H^2_{\mathrm{dR}}(\mathcal{M}_\lambda)$. This modification will not affect any of the estimates in the proof of the gluing theorem. In fact, the change of η_i contributes an error term of the size of $|d\theta_1|$, which can be absorbed in the error estimate (3.40). Notice that the estimate $|d\theta_1| = O(r^{-1}(\log r)^{-1/2})$ holds for a fixed ALG* model, and simple rescaling computations justify the claim.

Then we need to perturb the approximate hyper-Kähler triples to be actually hyper-Kähler. The resulting cohomology classes will not be exactly the same any-more, but the span of them will remain the same since $\mathcal{H}^2_+(\mathcal{M}_\lambda)$ is spanned by the approximate hyper-Kähler triples. Therefore, by a rescaling and a hyper-Kähler rotation, we can get the same $[\omega_i^{\mathrm{HK}}]$ on $\mathcal{M}_\lambda$. Observe that the rescaling factor converges to 1 and the hyper-Kähler rotation matrix converges to the identity matrix as $\lambda \to 0$. By the Torelli-type theorem for K3 surfaces, there exists an isometry between them which maps the hyper-Kähler triples onto each other, and induces the identity mapping on $H^2(\mathcal{M}_\lambda)$ (see [2], [5], [36]). Therefore, the restriction of these maps to the ALG* bubbling regions will then converge to an isometry of the ALG* spaces as $\lambda \to 0$, since the isometry must map the ALG* regions to each other. Obviously, this isometry will map the hyper-Kähler triples onto each other. The homology class of a fiber generates $H_2(\mathcal{I}^3_\nu; \mathbb{R}) = \mathbb{R}$ and is nontrivial in both $H_2(U; \mathbb{R})$ and $H_2(V; \mathbb{R})$ under the natural inclusions. From the Mayer–Vietoris sequence in homology, it follows that the natural mapping $H_2(V; \mathbb{R}) \to H_2(\mathcal{M}_\lambda; \mathbb{R})$ is injective. By duality, the restriction $H^2(\mathcal{M}_\lambda; \mathbb{R}) \to H^2(V; \mathbb{R}) \cong H^2(X_\nu; \mathbb{R})$ is surjective, which implies that the isometry of the ALG* regions also induces the identity map on $H^2(X_\nu; \mathbb{R}) \cong H^2_{\mathrm{dR}}(X_\nu)$, so we are done.

6.2. *Proof of Theorem 1.5: ALG Torelli uniqueness*

The next goal is to prove Theorem 1.5, which requires the following gluing result.

THEOREM 6.2

Let (X, g^X, ω^X) be an ALG *gravitational instanton of order* 2 *with $\chi(X) = \chi_0$. Then there exists a family of hyper-Kähler structures $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ on the K3 surface $\mathcal{M}_\lambda$ such that the following holds as $\lambda \to 0$.*

(1) *We have Gromov–Hausdorff convergence $(\mathcal{M}_\lambda, h_\lambda) \xrightarrow{\mathrm{GH}} (\mathbb{P}^1, d_{\mathrm{ML}})$, where d_{ML} is the McLean metric on $\mathbb{P}^1$ with a finite singular set $\mathcal{S} \equiv \{q_0, q_1, \ldots, q_{24-\chi_0}\} \subset \mathbb{P}^1$. Moreover, the curvatures of h_λ are uniformly bounded away from $\mathcal{S}$, but are unbounded around $\mathcal{S}$.*

(2) *Rescalings of $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ around q_i for $1 \le i \le 24 - \chi_0$ converge to a complete Taub-NUT gravitational instanton on $\mathbb{C}^2$.*

(3) *Rescalings of $(\mathcal{M}_\lambda, h_\lambda, \boldsymbol{\omega}_{h_\lambda})$ around q_0 converge to the given* ALG *gravitational instanton $(X, g^X, \boldsymbol{\omega}^X)$.*

Proof

The proof is a straightforward generalization of [10, Theorem 1.1] using a general hyper-Kähler triple gluing argument as in Section 5. In [10], we assumed that the ALG gravitational instantons were isotrivial which was necessary to preserve the complex structure. Since we are not fixing the complex structure on the K3 surface, only the order 2 assumption is necessary. For this, we just need to note that [10, Proposition 5.6] holds for any order-2 ALG space; the isotrivial condition is not necessary. □

Let $(X_\beta, g, \boldsymbol{\omega})$ and $(X_\beta, g', \boldsymbol{\omega}')$ be ALG gravitational instantons on X_β with the same parameters β, τ, and L, which are both of order 2 with respect to the coordinates Φ_{X_β} and which satisfy (1.4). The parameter β determines a fiber D of type I_0^*, II, III, IV, II*, III*, IV* as in Table 1. Let $\pi_{\mathcal{K}} : \mathcal{K} \to \mathbb{P}^1$ be any elliptic K3 surface with a single fiber D^* of the dual type, which means I_0^*, II*, III*, IV*, II, III, IV, respectively, but has all other singular fibers of type I_1. We use an attaching map Ψ from $\{\lambda^{-1} \le r \le 2\lambda^{-1}\} \subset X_\beta$ to a small annular region in $\mathcal{K}$ centered around D^* to obtain a manifold $\mathcal{M}_\lambda$, where λ is sufficiently small. Let U be the subset such that $r \ge \lambda^{-1}$, and let V be the subset such that $r \le 2\lambda^{-1}$. Then $\mathcal{M}_\lambda = U \cup V$.

The gluing procedure in the proof of Theorem 6.2 produces approximate hyper-Kähler triples $\tilde{\boldsymbol{\omega}}_\lambda$ and $\tilde{\boldsymbol{\omega}}'_\lambda$ on the $\mathcal{M}_\lambda$. Note that $U \cap V$ deformation retracts onto the 3-manifold N^3_β which is the restriction of an elliptic fibration with a single fiber of type D^* to S^1.

LEMMA 6.3
The manifold N^3_β is flat and satisfies $b^1(N^3_\beta) = b^2(N^3_\beta) = 1$. Furthermore, a generator for $b^1(N^3_\beta)$ is the 1-form $d\theta_1$, where θ_1 is the angular coordinate on the cone $\mathcal{C}(2\pi\beta)$.

Proof
The 3-manifold N^3_β is a T^2-fibration over S^1. We cover $S^1 = \mathbb{R}/2\pi\beta\mathbb{Z}$ by two intervals $(0, 2\pi\beta)$ and $(\pi\beta, 3\pi\beta)$. Then we can write N^3_β as the union of $N^3_{\beta,1} \equiv (0, 2\pi\beta) \times T^2$ and $N^3_{\beta,2} \equiv (\pi\beta, 3\pi\beta) \times T^2$. The Mayer–Vietoris sequence is

$$\begin{tikzcd}
H^0_{\mathrm{dR}}(N^3_\beta) \arrow[r] & H^0_{\mathrm{dR}}(N^3_{\beta,1}) \oplus H^0_{\mathrm{dR}}(N^3_{\beta,2}) \arrow[r] & H^0_{\mathrm{dR}}(N^3_{\beta,1} \cap N^3_{\beta,2}) \arrow[dll, rounded corners] \\
H^1_{\mathrm{dR}}(N^3_\beta) \arrow[r] & H^1_{\mathrm{dR}}(N^3_{\beta,1}) \oplus H^1_{\mathrm{dR}}(N^3_{\beta,2}) \arrow[r] & H^1_{\mathrm{dR}}(N^3_{\beta,1} \cap N^3_{\beta,2}).
\end{tikzcd}$$

If the monodromy group is A, then the map

$$H^1_{\mathrm{dR}}(N^3_{\beta,1}) \oplus H^1_{\mathrm{dR}}(N^3_{\beta,2}) = \mathbb{R}^2 \oplus \mathbb{R}^2 \to H^1_{\mathrm{dR}}(N^3_{\beta,1} \cap N^3_{\beta,2}) = \mathbb{R}^2 \oplus \mathbb{R}^2 \tag{6.4}$$

is given by $(C_1, C_2) \mapsto (C_1 - C_2, C_1 - AC_2)$ for $C_1, C_2 \in \mathbb{R}^2$, whose kernel is the same as $\ker(A - \mathrm{Id})$. For singular fibers of finite monodromy, $\ker(A - \mathrm{Id}) = 0$. The map

$$H^0_{\mathrm{dR}}(N^3_{\beta,1}) \oplus H^0_{\mathrm{dR}}(N^3_{\beta,2}) \to H^0_{\mathrm{dR}}(N^3_{\beta,1} \cap N^3_{\beta,2}) \tag{6.5}$$

is a rank-1 map $(a, b) \mapsto (a - b, a - b)$. So $H^1_{\mathrm{dR}}(N^3_\beta) = \mathbb{R}$ and it is generated by the image of $(2\pi\beta, 0) \in H^0_{\mathrm{dR}}(N^3_{\beta,1} \cap N^3_{\beta,2})$. To see this image, we note that the difference of the function θ_1 on $(\pi\beta, 3\pi\beta) \times T^2$ with the function θ_1 on $(0, 2\pi\beta) \times T^2$ is exactly $2\pi\beta$ on $(0, \pi\beta) \times T^2$ and 0 on $(\pi\beta, 2\pi\beta) \times T^2$, and all their derivatives are $d\theta_1$. So the image of $(2\pi\beta, 0)$ is $d\theta_1$. In other words, we have proved that $H^1_{\mathrm{dR}}(N^3_\beta) = \langle d\theta_1 \rangle$. By Poincaré duality, $b^2(N^3_\beta) = b^1(N^3_\beta) = 1$. The flatness of N^3_β is a corollary of the fact that the flat metric on $N^3_{\beta,1}$ and $N^3_{\beta,2}$ can be glued into a flat metric on N^3_β. $\square$

The proof of Theorem 6.2 uses [10, Proposition 5.6], which implies that

$$\Phi^*_{X_\beta}(\boldsymbol{\omega}) - \boldsymbol{\omega}^{\mathcal{C}} = d\boldsymbol{\eta}, \qquad \Phi^*_{X_\beta}(\boldsymbol{\omega}') - \boldsymbol{\omega}^{\mathcal{C}} = d\boldsymbol{\eta}' \tag{6.6}$$

for some triples of 1-forms $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ defined on the end of the model space. On the region U, away from the damage zone, the approximate hyper-Kähler triples are exactly the same (they are semi-flat, with I_1-fibers resolved using Ooguri–Vafa metrics). Using the same Mayer–Vietoris sequence (6.1), and Lemma 6.3, we can adjust the 1-form η_i on the "damage zone" by a term of the form $d(\varphi\theta_1)$, to arrange that $[\omega_i] = [\omega_i']$ in $H^2_{\mathrm{dR}}(\mathcal{M}_\lambda)$. The remainder of the proof is then exactly the same as in the ALG* case above.

7. Results on the period mapping

In this section, let $(X, g, \boldsymbol{\omega})$ be an ALG or ALG* gravitational instanton of order 2. In either case, we define a smooth function $s : X \to [1, \infty)$ as follows. In the ALG case, let s be a smooth extension of r via the diffeomorphism $\Phi : \mathcal{C}_{\beta,\tau,L}(R) \to X \setminus X_R$, where Φ and r are defined as in Section 1.1. In the ALG* case, let s be a smooth extension of $\mathfrak{s}$ via the diffeomorphism $\Phi : \mathfrak{M}_{2\nu}(R) \to X \setminus X_R$, where Φ and $\mathfrak{s}$ are defined as in Section 1.2. Our analysis required the following weighted Sobolev norms.

Definition 7.1

Let $(X, g, \boldsymbol{\omega})$ be an ALG or ALG* gravitational instanton. For any fixed $\delta \in \mathbb{R}$, we define the weight function $\hat{\varrho}_\delta$ on X as

$$\hat{\varrho}_\delta \equiv s^{-\delta-1}. \tag{7.1}$$

Then the weighted Sobolev norms are defined as follows:

$$\|\omega\|_{L^2_\delta(X)} \equiv \Big(\int_X |\omega \cdot \hat{\varrho}_\delta|^2 \, \mathrm{dvol}_X \Big)^{\frac{1}{2}},$$

$$\|\omega\|_{W^{k,2}_\delta(X)} \equiv \Big(\sum_{m=0}^{k} \|\nabla^m \omega\|^2_{L^2_{\delta-m}(X)} \Big)^{\frac{1}{2}}.$$

Remark 7.2

We remark that this convention differs from the convention in [7, Definition 4.1] in the ALG cases, but agrees with the convention in [11, Definition 4.1] in the ALG* cases.

Our convention on the Sobolev weight is explained by the following important lemma.

LEMMA 7.3

Let $(X, g, \boldsymbol{\omega})$ be a gravitational instanton of type either ALG *or* ALG* *of order* 2. *For any $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$, there exists a constant $C_{k,\delta} > 0$ so that*

$$\sum_{m=0}^{k} \sup_{\boldsymbol{x} \in X} \big| \big(s(\boldsymbol{x})\big)^{m-\delta} \nabla^m \omega(\boldsymbol{x}) \big| \le C \|\omega\|_{W^{k+3,2}_\delta(X)} \tag{7.2}$$

for all $\omega \in W^{k+3,2}_\delta(X)$.

Proof
The proof is a standard rescaling argument. (See [11, Propositions 3.2 and 3.3] for more details.) □

7.1. Harmonic 2*-forms of order* 2
In order to properly define the period map, we begin with a proposition relating compactly supported de Rham cohomology and decaying harmonic 2-forms.

PROPOSITION 7.4
For any ALG *or* ALG* *gravitational instanton* $(X, g, \boldsymbol{\omega})$ *of order* 2*,*

$$\begin{aligned}&\{\omega = O(s^{-2}) \in \Omega^2(X) \mid \Delta\omega = 0\}\\&\quad = \{\omega = O(s^{-2}) \in \Omega^2(X) \mid d\omega = d^*\omega = 0\}\\&\quad = \{\omega = O(s^{-2}) \in \Omega^2_-(X) \mid d\omega = d^*\omega = 0\}\\&\quad = \mathrm{Im}\big(H^2_{\mathrm{cpt}}(X) \to H^2(X)\big) = \Big\{[\omega] \in H^2(X), \int_D \omega = 0\Big\},\end{aligned}$$

where D *is any fiber arising from the compactification of* X *to a rational elliptic surface.*

Proof
We first consider the ALG* case. If $\omega = O(s^{-2}) \in \Omega^2(X)$, and $\Delta\omega = 0$, then by standard elliptic regularity, $\omega \in W^{k,2}_\delta$ for any $k \in \mathbb{N}_0$ and $\delta > -2$. So the boundary term in

$$\int_{r<R} \big((\omega, \Delta\omega) - (d\omega, d\omega) - (d^*\omega, d^*\omega)\big) \tag{7.3}$$

goes to 0 when $R \to \infty$, which implies that $d\omega = d^*\omega = 0$. Conversely, if $d\omega = d^*\omega = 0$, then $\Delta\omega = 0$.

Then we study $\mathrm{Im}(H^2_{\mathrm{cpt}}(X) \to H^2(X))$. Define $U = \{x \in X, r(x) > R\}$. Then U deformation retracts to the 3-manifold $\mathcal{I}^3_\nu$. By Lemma 6.1, $H^1(\mathcal{I}^3_\nu)$ is generated by $d\theta_1$. By Poincaré duality, $H_2(\mathcal{I}^3_\nu)$ is generated by $[D]$, where D is any fiber, so $H_2(U)$ is also generated by $[D]$. Therefore, if $[\omega] \in H^2(X)$ and $\int_D \omega = 0$, then $\omega|_U$ is exact, so there exists $\eta \in \Omega^1(U)$ such that $\omega = d\eta$ on U. Let χ be a cutoff function which is 0 when $r \leq R$ and is 1 when $r \geq 2R$. Then $\omega - d(\chi \cdot \eta)$ is compactly supported, so $[\omega] \in \mathrm{Im}(H^2_{\mathrm{cpt}}(X) \to H^2(X))$. Conversely, if ω is compactly supported, then it is trivial to see that $\int_D \omega = 0$.

Let $\omega = O(s^{-2}) \in \Omega^2(X)$ be such that $\Delta\omega = 0$. Then for any D sufficiently far from a basepoint, we have $\int_D \omega = 0$ since the area of D is independent of the choice of D. So there is a map

$$\{\omega = O(s^{-2}) \in \Omega^2(X) : d\omega = d^*\omega = 0\} \to \Big\{[\omega] \in H^2(X) : \int_D \omega = 0\Big\}.$$

To show the surjectivity, for any compactly supported closed form ω, choose an arbitrary $0 < \epsilon < 1$ and a basis η_i of 2-forms in $W^{k,2}_{-1-\epsilon}(X) \subset L^2(X)$ such that $\Delta\eta_i = 0$. Since $(\eta_i, \eta_j)_{L^2}$ is invertible, there exist $c_i \in \mathbb{R}$ such that

$$(\omega, \eta_j)_{L^2} = \Big(\sum_i c_i \eta_i, \eta_j\Big)_{L^2}. \tag{7.4}$$

By [11, Proposition 4.5(2)], there exists $\phi \in W^{k+2,2}_{1+\epsilon}$ such that

$$\Delta\phi = \omega - \sum_i c_i \eta_i. \tag{7.5}$$

Since

$$\int_{r<R} \big((\eta_i, \Delta\eta_i) - (d\eta_i, d\eta_i) - (d^*\eta_i, d^*\eta_i)\big) \tag{7.6}$$

also decays as $R \to \infty$, η_i are closed and coclosed. So,

$$\omega - dd^*\phi = d^*d\phi + \sum_i c_i \eta_i \in W^{k,2}_{-1+\epsilon} \tag{7.7}$$

is closed and coclosed. The self-dual part is $\sum_{i=1}^3 f_i \omega_i$ for decaying harmonic functions f_i, which must be zero. By [11, Lemma A.8], the closed and coclosed anti-self-dual form $\omega - dd^*\phi$ must be $O(s^{-2})$, which implies the surjectivity.

To show the injectivity, assume that $d\phi = \omega = O(s^{-2}) \in \Omega^2(X)$ is also coclosed. We write $\omega = dr \wedge \alpha + \beta$, where α is a 1-form on $\mathcal{I}^3_\nu = \{r = r_0\}$, and β is a 2-form on $\{r = r_0\}$. Then

$$0 = d\omega = -dr \wedge d_{\mathcal{I}^3_\nu}\alpha + d_{\mathcal{I}^3_\nu}\beta + dr \wedge \frac{\partial\beta}{\partial r}. \tag{7.8}$$

Define $\gamma \equiv \int_\infty^r \alpha$ on U. Then

$$d\gamma = dr \wedge \alpha + \int_\infty^r d_{\mathcal{I}^3_\nu}\alpha = dr \wedge \alpha + \int_\infty^r \frac{\partial\beta}{\partial r} = dr \wedge \alpha + \beta = \omega. \tag{7.9}$$

So, $\gamma - \phi$ is closed on U. By Lemma 6.1, $H^1(U)$ is generated by $d\theta_1$. So there exist a constant c and a function ψ on U such that $\gamma - \phi = c\,d\theta_1 + d\psi$. Then $\omega = d\eta$, where $\eta = \phi + d(\chi \cdot \psi)$. Moreover, $\eta = \gamma - c\,d\theta_1$ when $r \geq 2R$. So $\eta \in W^{k+1,2}_{-1+\epsilon}(X)$ for any $\epsilon > 0$, which comes from the definition of γ and the assumption on the decay of α. Therefore,

$$\int_{r<R} \big((\omega, d\eta) - (d^*\omega, \eta) \big) \tag{7.10}$$

also converges to 0 as $R \to \infty$. In other words, $\omega = d\eta = 0$ since $d^*\omega = 0$.

Using the same proof, and the ALG asymptotic analysis in [7], a similar proof also holds for ALG gravitational instantons of order 2. See also [24, Section 7.1.3] and Theorems 9.3 and 9.4 of [10]. □

7.2. Definition of the period map

In this subsection, we prove that the period mappings are well defined.

PROPOSITION 7.5

The period mappings $\mathscr{P}$ in Definition 1.6 and Definition 1.11 are well defined.

Proof

We first consider the ALG case. If $(X_\beta, g, \boldsymbol{\omega}) \in \mathcal{M}_{\beta,\tau,L}$, then it is ALG with respect to the fixed ALG coordinate system Φ_{X_β}. Then ω_1 is taken to be the Kähler form which is asymptotic to the elliptic complex structure, and the choice of ω_2 and ω_3 is also determined since they are asymptotic to the model Kähler forms in the Φ_{X_β} coordinates. The point is that our Definition 1.3 removes the freedom of hyper-Kähler rotations, so we have a well-defined ordered choice of the three Kähler forms. From [7, Theorem 4.14], there is a holomorphic function $u : X_\beta \to \mathbb{C}$ which is an elliptic fibration. The level sets of u are tori. As $u \to \infty$, these level sets are close to the model holomorphic tori. Therefore the homology class $[D]$ of any fiber is well defined, the same class for all elements in $\mathcal{M}_{\beta,\tau,L}$. Since the forms ω_2 and ω_3 are orthogonal to ω_1, any torus which is holomorphic for I is Lagrangian with respect to J or K. Use Proposition 7.4 to identify $\mathscr{H}^2$ with order-2 decaying harmonic anti-self-dual 2-forms; the classes $[\omega_2]$ and $[\omega_3]$ automatically lie in $\mathscr{H}^2$. Finally, since the holomorphic tori for I and I_0 are homologous, we have $\int_D (\omega_1 - \omega_1^0) = 0$ since the areas of the holomorphic tori are the same. Using [9, Proposition 3.1], the argument in the ALG* case is exactly the same. □

7.3. The nondegeneracy condition

In this subsection, we prove the nondegeneracy condition stated in Theorems 1.7 and 1.12:

$$\boldsymbol{\omega}[C] \neq (0,0,0) \quad \text{for all } [C] \in H_2(X;\mathbb{Z}) \text{ satisfying } [C]^2 = -2. \tag{7.11}$$

To prove this, we use the gluing construction in Theorem 6.2 in the ALG case and Theorem 5.6 in the ALG* case. A basic transversality argument shows that we can represent any $[C] \in H_2(X;\mathbb{Z})$ by an embedded surface $\iota : C \to X$. If (7.11) is not satisfied by an ALG or ALG* gravitational instanton $(X, \boldsymbol{\omega}^X)$, then by choosing a small enough gluing parameter λ, we can assume that the glued closed definite triple $\boldsymbol{\omega}_\lambda = \boldsymbol{\omega}^X$ near $\iota(C)$. A Mayer–Vietoris argument in homology shows that $[C]$ is nontrivial in $H_2(\mathcal{M}_\lambda, \mathbb{Z})$. So there exists $[C] \in H_2(\mathcal{M}_\lambda, \mathbb{Z})$ such that $[C]^2 = -2$ and $[\boldsymbol{\omega}_\lambda] \cdot [C] = \mathbf{0}$. In the perturbation arguments, the span of the hyper-Kähler classes $[\boldsymbol{\omega}_\lambda^{\mathrm{HK}}]$ on the K3 surface $\mathcal{M}_\lambda$ is the same as the span of $[\boldsymbol{\omega}_\lambda]$. Therefore, $[\boldsymbol{\omega}_\lambda^{\mathrm{HK}}] \cdot [C] = \mathbf{0}$, which is a contradiction with the well-known nondegeneracy condition on the K3 surface $\mathcal{M}_\lambda$.

7.4. Proofs of Theorem 1.7 and Theorem 1.12

We follow the route map of [8, Section 7]. For any point in $[\boldsymbol{\omega}^0] + \mathscr{H}^2 \otimes \mathbb{R}^3$ satisfying (7.11), we can connect it to $[\boldsymbol{\omega}^0]$ by zigzags of the form

$$\big([\alpha_{1,0}] + t[\beta_1], [\alpha_2], [\alpha_3]\big), \tag{7.12}$$

$$\big([\alpha_1], [\alpha_{2,0}] + t[\beta_2], [\alpha_3]\big), \tag{7.13}$$

or

$$\big([\alpha_1], [\alpha_2], [\alpha_{3,0}] + t[\beta_3]\big). \tag{7.14}$$

We require that all the points in the zigzags satisfy (7.11). This assumption is clearly possible since (7.11) holds outside a set of codimension 3. Let us consider the ALG case. For the path in (7.12), we have

$$(X, \omega_{1,0} \equiv \alpha_{1,0}, \omega_2 \equiv \alpha_2, \omega_3 \equiv \alpha_3) \in \mathcal{M}_{\beta,\tau,L}. \tag{7.15}$$

Using Proposition 7.4, we choose the representative β_1 in the class $[\beta_1]$ by requiring it to be closed, coclosed, and anti-self-dual with respect to the hyper-Kähler metric determined by $(X, \omega_{1,0}, \omega_2, \omega_3)$. Since β_1 is anti-self-dual,

$$\beta_1 \wedge \omega_{1,0} = \beta_1 \wedge \omega_2 = \beta_1 \wedge \omega_3 = 0. \tag{7.16}$$

Then we choose $c_t \in \mathbb{R}$ such that

$$\omega_{1,t} \equiv \omega_{1,0} + t\beta_1 + c_t\sqrt{-1}\partial_I\bar{\partial}_I\big(\chi \cdot \log|u|\big) \tag{7.17}$$

satisfies

$$\int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = 0, \tag{7.18}$$

where I, J, K are the hyper-Kähler structures determined by $(X, \omega_{1,0}, \omega_2, \omega_3)$, $u : X \to \mathbb{C}$ is the I-holomorphic function which makes X a rational elliptic surface minus the fiber at infinity, and χ is a cutoff function which is 0 for small $|u|$ and is 1 for large $|u|$. Using (7.16), the formula $dd_I^c = 2\sqrt{-1}\partial_I\bar{\partial}_I$, and Stokes's theorem, we have

$$\int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = \int_X t^2\beta_1^2 + 2c_t \int_X \omega_{1,0} \wedge \sqrt{-1}\partial_I\bar{\partial}_I\big(\chi \cdot \log|u|\big). \tag{7.19}$$

From (7.19), we see that the constant c_t indeed exists since

$$0 \neq \int_X \omega_{1,0} \wedge \sqrt{-1}\partial_I\bar{\partial}_I\big(\chi \cdot \log|u|\big), \tag{7.20}$$

and the integral is finite. In the (7.13) case, we use $\sqrt{-1}\partial_J\bar{\partial}_J$ instead of $\sqrt{-1}\partial_I\bar{\partial}_I$, and in the (7.14) case, we use $\sqrt{-1}\partial_K\bar{\partial}_K$.

Back to the (7.12) case, consider the collection $\mathcal{S}$ of $t \in [0,1]$ for which there exist $\delta_t > 0$ and $\varphi_t \in W^{k,2}_{-\delta_t}(X, \omega_{1,0})$ for any $k \in \mathbb{N}_0$ such that

$$(X, \omega_t \equiv \omega_{1,t} + \sqrt{-1}\partial_I\bar{\partial}_I\varphi_t, \omega_2, \omega_3) \in \mathcal{M}_{\beta,\tau,L}. \tag{7.21}$$

By assumption, $0 \in \mathcal{S}$. If $t_0 \in \mathcal{S}$, then by definition of $\mathcal{S}$, applying Lemma 7.3 and using a standard elliptic regularity argument, we see that for t sufficiently close to t_0, $\omega_{1,t} + \sqrt{-1}\partial_I\bar{\partial}_I\varphi_{t_0}$ will be ALG of order 2. Furthermore,

$$\int_X (\omega_{1,t} + \sqrt{-1}\partial_I\bar{\partial}_I\varphi_{t_0})^2 - \omega_{1,0}^2 = \int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = 0. \tag{7.22}$$

By [40, Theorem 1.1], there exists a bounded solution φ_t of the equation

$$(\omega_{1,t} + \sqrt{-1}\partial_I\bar{\partial}_I\varphi_t)^2 = e^f \omega_{1,t}^2, \tag{7.23}$$

where

$$f \equiv \log\frac{\omega_t^2 - \omega_{1,t}^2}{\omega_{1,t}^2} = \log\frac{\omega_{1,0}^2 - \omega_{1,t}^2}{\omega_{1,t}^2} = O(r^{-4}), \tag{7.24}$$

and the middle equality follows from (7.16). By [26, Proposition 2.6], $\int_X |\nabla_{\omega_{1,t}}\varphi_t|^2 \times \omega_{1,t}^2 < \infty$. Then by [26, Proposition 2.9(ib)], there exists a $\delta_t > 0$ so that

$$\sup|\varphi_t| \leq Cs^{-\delta_t}. \tag{7.25}$$

Then [26, Proposition 2.9(ii)] implies that

$$\sup|\nabla^k_{\omega_{1,t}}\varphi_t| \leq C_k s^{-\delta_t - k}, \tag{7.26}$$

since these estimates are implied by Hein's weighted Hölder estimates. This implies that $\varphi_t \in W^{k,2}_{-\delta_t}(X, \omega_{1,0})$ for any $k \in \mathbb{N}$ if we slightly shrink δ_t. Consequently, $\mathcal{S}$ is open. Since the Tian–Yau and Hein estimates depend only on geometric bounds and do not depend on the choice of complex structure, the above argument also works for the paths (7.13) and (7.14). It is easy to see that this implies that the image of the period mapping is open. The above arguments hold in the ALG* case (with $\mathcal{M}_{\beta,\tau,L}$ replaced by $\mathcal{M}_{\nu,\kappa_0,L}$), so this completes the proof of Theorem 1.12.

To finish the proof of Theorem 1.7, we need to show that $\mathcal{S}$ is closed in the ALG cases. So suppose that $t_i \to t_\infty$ is a sequence in $\mathcal{S}$. Then

$$\begin{aligned}\int_X (\mathrm{tr}_{\omega_{1,0}} \omega_{t_i} - 2)\frac{\omega_{1,0}^2}{2} &= \int_X \omega_{1,0} \wedge (\omega_{t_i} - \omega_{1,0}) \\ &= \int_X \omega_{1,0} \wedge (\omega_{1,t_i} - \omega_{1,0}) \\ &= c_{t_i} \int_X \omega_{1,0} \wedge \sqrt{-1}\partial_I \bar{\partial}_I \big(\chi \cdot \log|u|\big) \\ &= -\frac{t_i^2}{2} \int_X \beta_1^2 \leq C \end{aligned} \tag{7.27}$$

for a constant independent of t_i, and

$$\begin{aligned}\int_X (\mathrm{tr}_{\omega_{t_j}} \omega_{t_i} - 2)\frac{\omega_{t_j}^2}{2} &= \int_X \omega_{t_j} \wedge (\omega_{t_i} - \omega_{t_j}) \\ &= \int_X \omega_{1,t_j} \wedge (\omega_{1,t_i} - \omega_{1,t_j}) \\ &= \int_X (\omega_{1,0} + t_j\beta_1) \wedge \big((t_i - t_j)\beta_1 \\ &\qquad + (c_{t_i} - c_{t_j})\sqrt{-1}\partial_I \bar{\partial}_I \big(\chi \cdot \log|u|\big)\big) \\ &= (c_{t_i} - c_{t_j}) \int_X \omega_{1,0} \wedge \sqrt{-1}\partial_I \bar{\partial}_I \big(\chi \cdot \log|u|\big) \\ &\qquad + t_j(t_i - t_j) \int_X \beta_1^2 \\ &= \Big(-\frac{t_i^2}{2} + \frac{t_j^2}{2} + t_j(t_i - t_j)\Big) \int_X \beta_1^2 \\ &= -\frac{(t_i - t_j)^2}{2} \int_X \beta_1^2 \to 0 \end{aligned} \tag{7.28}$$

as $i, j \to \infty$. These bounds imply the following pointwise bound.

THEOREM 7.6

The function $e(t_i) = \mathrm{tr}_{\omega_0} \omega_{t_i} = \mathrm{tr}_{\omega_{t_i}} \omega_0$ is uniformly bounded on X.

Proof

We use (7.27) and (7.28) to go through the arguments in [8, Section 7], with some minor modifications, to get the required bound. First, we cover X by balls with radius 1 in the sense of the metric determined by $(X, \omega_{1,0}, \omega_2, \omega_3)$ such that the number of balls containing any point in X is uniformly bounded. Then we use these balls to replace the sets U_N in [8, Theorem 7.3], to obtain the subsets $V_{N_i} \subset U_N$ which carry a large percentage of the volume of U_N (with respect to the background metric), and which satisfy a diameter bound (with respect to the metric ω_{t_i}). Note that the proof of [14, Lemma 1.3] is valid in the ALG case, since ALG metrics are volume noncollapsed in bounded scales at infinity.

To prove the analogue of [8, Theorem 7.4], we need to show that if there exists a sequence of cohomology classes $[\Sigma_i] \in H_2(X, \mathbb{Z})$ satisfying $[\Sigma_i]^2 = -2$ and $\int_{\Sigma_i} \omega_{t_i} \to 0$, $\int_{\Sigma_i} \omega_2 \to 0$, $\int_{\Sigma_i} \omega_3 \to 0$ as $i \to \infty$, then there are only finitely many distinct $[\Sigma_i]$. To prove this, recall that by the assumption of Theorem 1.7 based on [9, Theorem 1.10], X is in particular diffeomorphic to an isotrival ALG gravitational instanton. These compactify to an isotrivial rational elliptic surface S by adding a finite monodromy fiber D_∞ at infinity. By [26, Section 3.1], $S \setminus D_\infty$ deformation retracts onto the *dual* finite monodromy fiber. Therefore, the intersection form of $H^2(X, \mathbb{Z})$ is an extended Dynkin diagram. Next, for example, assume that the extended Dynkin diagram is $\tilde{D}_4$. Then $H^2(X, \mathbb{Z})$ is generated by $[E_i]$, $i = 1, 2, \ldots, 5$, with $[E_i]^2 = -2$ for all i, $[E_i] \cdot [E_j] = 1$ for all $\{i, j\} = \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}$, and $[E_i] \cdot [E_j] = 0$ otherwise. The homology class of each fiber is

$$[F] = 2[E_1] + [E_2] + [E_3] + [E_4] + [E_5]. \tag{7.29}$$

The intersection numbers of $[F]$ with all $[E_i]$ are zero. We write $[\Sigma_i]$ as

$$[\Sigma_i] = a_i[F] + b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4]. \tag{7.30}$$

Then the self-intersection number of $b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4]$ is -2. The extended Dynkin diagram restricted to this subset is the unextended Dynkin diagram, which has negative definite intersection form. This implies that there are only finitely many distinct $b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4]$ with self-intersection -2. Then, we use $\int_F \omega_{t_i} = \int_F \omega_{1,0} \neq 0$ to control a_i. The proofs for other extended Dynkin diagrams are similar.

This proves a uniform curvature bound, and this yields a bound on the ω_i-holomorphic radius exactly as in [8, Theorem 7.4]. The proof of [8, Theorem 7.4] relies on [37, Proposition 2.1], which is valid in the ALG case since these are volume noncollapsed in bounded scales at infinity. Theorem 7.6, Lemma 7.7, and Theorem 7.8 of [8] then go through exactly the same in the ALG cases, with U_N replaced by balls of radius 1. Note that only the hyper-Kähler condition is used in [8, Theorem 7.6]. $\square$

The equation $\omega_{t_i}^2 = \omega_{1,0}^2$ and the bound on $\mathrm{tr}_{\omega_{t_i}} \omega_{1,0} = \mathrm{tr}_{\omega_{1,0}} \omega_{t_i}$ imply that there exists a constant C independent of t_i such that

$$C^{-1}\omega_{1,0} \le \omega_{t_i} \le C\omega_{1,0}. \tag{7.31}$$

Since the difference $\omega_{1,t_i} - \omega_{1,0}$ decays uniformly, there exists a constant R such that $\frac{1}{2}\omega_{1,0} \le \omega_{1,t_i} \le 2\omega_{1,0}$ for all $s \ge R$. So

$$|\Delta_{\omega_{1,0}}\varphi_{t_i}| = \left|\mathrm{tr}_{\omega_{1,0}}(\omega_{t_i} - \omega_{1,t_i})\right| = \left|\mathrm{tr}_{\omega_{1,0}} \omega_{t_i} - 2 - c_{t_i} \Delta_{\omega_{1,0}}\big(\chi \cdot \log |u|\big)\right| \le C$$

on X. Moreover,

$$\begin{aligned} &\int_X |\Delta_{\omega_{1,0}}\varphi_{t_i}| \frac{\omega_{1,0}^2}{2} \\ &\quad \le \int_X (\mathrm{tr}_{\omega_{1,0}} \omega_{t_i} - 2)\frac{\omega_{1,0}^2}{2} + |c_{t_i}| \int_X \left|\Delta_{\omega_{1,0}}\big(\chi \cdot \log |u|\big)\right| \frac{\omega_{1,0}^2}{2} \le C, \end{aligned} \tag{7.32}$$

where we have used the fact that $\omega_{t_i}^2 = \omega_{1,0}^2$, which implies that $\mathrm{tr}_{\omega_{1,0}} \omega_{t_i} \ge 2$. So for $\delta = \frac{1}{100}$, $\|\Delta_{\omega_{1,0}}\varphi_{t_i}\|_{L^2_{-1+\delta}(X,\omega_{1,0})} \le C$. Now we consider the operator $\Delta_{\omega_{1,0}} : W^{2,2}_{1+\delta}(X,\omega_{1,0}) \to L^2_{-1+\delta}(X,\omega_{1,0})$. By the ALG weighted analysis in [6]–[8], and [24], it is easy to see that any function in the kernel of this operator must be a constant, and consequently there exists another function $\tilde{\varphi}_{t_i} \in W^{2,2}_{1+\delta}(X,\omega_{1,0})$ such that $\varphi_{t_i} - \tilde{\varphi}_{t_i}$ is a constant and

$$\begin{aligned} \|\tilde{\varphi}_{t_i}\|_{W^{2,2}_{1+\delta}(X,\omega_{1,0})} &\le C\|\Delta_{\omega_{1,0}}\tilde{\varphi}_{t_i}\|_{L^2_{-1+\delta}(X,\omega_{1,0})} \\ &= C\|\Delta_{\omega_{1,0}}\varphi_{t_i}\|_{L^2_{-1+\delta}(X,\omega_{1,0})} \le C. \end{aligned} \tag{7.33}$$

This implies that $\|\tilde{\varphi}_{t_i}\|_{W^{2,p}(\{s\le 4R\},\omega_{1,0})} \le C(p)$ for any $p > 1$ using the bound on $|\Delta_{\omega_{1,0}}\varphi_{t_i}|$. For any $\alpha \in (0,1)$, by the Evans–Krylov estimate (see, e.g., [39, Section 2.4]),

$$[\partial\bar{\partial}\tilde{\varphi}_{t_i}]_{C^\alpha(\{s\le 3R\},\omega_{1,0})} \le C(\alpha). \tag{7.34}$$

By standard elliptic estimates, for any $k \in \mathbb{N}$ and any $\alpha \in (0,1)$,

$$\|\tilde{\varphi}_{t_i}\|_{C^{k,\alpha}(\{s\le 2R\},\omega_{1,0})} \le C(k,\alpha). \tag{7.35}$$

When $s \ge R$,

$$\begin{aligned} |\Delta_{\omega_{1,t_i}+\omega_{t_i}}\tilde{\varphi}_{t_i}| &\le C\left|(\omega_{1,t_i} + \omega_{t_i}) \wedge (\omega_{t_i} - \omega_{1,t_i})\right|_{\omega_{1,t_i}+\omega_{t_i}} \\ &= C|\omega_{1,0}^2 - \omega_{1,t_i}^2|_{\omega_{1,t_i}+\omega_{t_i}} \\ &\le C|\omega_{1,0}^2 - \omega_{1,t_i}^2|_{\omega_{1,0}} \le Cs^{-4}. \end{aligned} \tag{7.36}$$

Let χ_R be a cutoff function which is 1 when $s \geq 2R$ and is 0 when $s \leq R$. Then $|\Delta_{\omega_{1,t_i}+\omega_{t_i}} \xi_{t_i}| \leq C s^{-4}$, where $\xi_{t_i} \equiv \chi_R \cdot \tilde{\varphi}_{t_i}$

We use the Moser iteration technique to prove that $\|\xi_{t_i}\|_{C^0} \leq C$ for a constant C independent of t_i and p. For any $j = 0, 1, 2, 3, \ldots$ and $p = 2^j$,

$$\begin{aligned}
p^2 &\int_X \xi_{t_i}^{2p-2} |\nabla_{\omega_{1,t_i}+\omega_{t_i}} \xi_{t_i}|^2 \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2} \\
&= \int_X \big|\nabla_{\omega_{1,t_i}+\omega_{t_i}} (\xi_{t_i}^p)\big|^2 \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2} \\
&= -\int_X \xi_{t_i}^p \Delta_{\omega_{1,t_i}+\omega_{t_i}} (\xi_{t_i}^p) \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2} \\
&= -p(p-1) \int_X \xi_{t_i}^{2p-2} |\nabla_{\omega_{1,t_i}+\omega_{t_i}} \xi_{t_i}|^2 \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2} \\
&\quad - p \int_X \xi_{t_i}^{2p-1} \Delta_{\omega_{1,t_i}+\omega_{t_i}} \xi_{t_i} \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2}.
\end{aligned} \tag{7.37}$$

We have used the fact that $\xi_{t_i} - \varphi_{t_i}$ is a constant when $s \geq 2R$, and there exists $\delta_{t_i} > 0$ such that $\varphi_{t_i} \in W^{k,2}_{-\delta_{t_i}}(X, \omega_{1,0})$. Therefore,

$$\begin{aligned}
&\int_X \big|\nabla_{\omega_{1,t_i}+\omega_{t_i}} (\xi_{t_i}^p)\big|^2 \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2} \\
&\qquad = -\frac{p^2}{2p-1} \int_X \xi_{t_i}^{2p-1} \Delta_{\omega_{1,t_i}+\omega_{t_i}} \xi_{t_i} \frac{(\omega_{1,t_i} + \omega_{t_i})^2}{2}.
\end{aligned}$$

Recall that by Theorem 1.2(i) of [25], there exist a constant C and a weight function ψ with $\int_X \psi \frac{\omega_{1,0}^2}{2} = 1$ such that for any $\xi \in C_0^\infty(X)$,

$$\Big(\int_X |\xi - \xi_0|^4 s^{-4} \frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}} \leq C \int_X |\nabla_{\omega_{1,0}} \xi|^2 \frac{\omega_{1,0}^2}{2}, \tag{7.38}$$

where $\xi_0 \equiv \int_X \psi \xi \frac{\omega_{1,0}^2}{2}$. Equation (7.38) also holds for $\xi_{t_i}^p$ because (7.38) is unchanged if we add ξ by a constant, and $\xi_{t_i}^p$ can be written as a constant plus a function in $W^{k,2}_{-\delta_{t_i}}(X, \omega_{1,0})$. Then

$$\begin{aligned}
|\xi_0|^2 &\leq C \Big(\int_{s \leq 2R} |\xi_0|^4 s^{-4} \frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}} \\
&\leq C \Big(\int_X |\xi - \xi_0|^4 s^{-4} \frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}} + C \|\xi\|^2_{C^0(\{s \leq 2R\}}.
\end{aligned} \tag{7.39}$$

So

$$\Big(\int_X |\xi|^4 s^{-4}\frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}} \le C\int_X |\nabla_{\omega_{1,0}}\xi|^2\frac{\omega_{1,0}^2}{2} + C\|\xi\|^2_{C^0(\{s\le 2R\})} \tag{7.40}$$

for $\xi = \xi_{t_i}^p$ and a constant C independent of t_i and p. Therefore,

$$\begin{aligned}
&\Big(\int_X |\xi_{t_i}|^{4p} s^{-4}\frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}}\\
&\quad\le C\int_X |\nabla_{\omega_{1,0}}\xi_{t_i}^p|^2\frac{\omega_{1,0}^2}{2} + C\|\xi_{t_i}\|^{2p}_{C^0(\{s\le 2R)\}}\\
&\quad\le C\int_X \big|\nabla_{\omega_{1,t_i}+\omega_{t_i}}(\xi_{t_i}^p)\big|^2\frac{(\omega_{1,t_i}+\omega_{t_i})^2}{2} + C\|\xi_{t_i}\|^{2p}_{C^0(\{s\le 2R)\}}\\
&\quad\le \frac{Cp^2}{2p-1}\int_X |\xi_{t_i}|^{2p-1}s^{-4}\frac{\omega_{1,0}^2}{2} + C\|\xi_{t_i}\|^{2p}_{C^0(\{s\le 2R\})}\\
&\quad\le \frac{Cp^2}{2p-1}\int_X \Big(\frac{2p-1}{2p}|\xi_{t_i}|^{2p} + \frac{1}{2p}\Big)s^{-4}\frac{\omega_{1,0}^2}{2} + C\|\xi_{t_i}\|^{2p}_{C^0(\{s\le 2R)\}}\\
&\quad\le C\Big(p\int_X |\xi_{t_i}|^{2p}s^{-4}\frac{\omega_{1,0}^2}{2} + 1 + \|\xi_{t_i}\|^{2p}_{C^0(\{s\le 2R)\}}\Big).
\end{aligned} \tag{7.41}$$

For $p = 1$,

$$\begin{aligned}
\int_X |\xi_{t_i}|^2 s^{-4}\frac{\omega_{1,0}^2}{2} &\le C\Big(\int_X |\xi_{t_i}|^4 s^{-4}\frac{\omega_{1,0}^2}{2}\Big)^{\frac{1}{2}}\\
&\le C\int_X |\xi_{t_i}|s^{-4}\frac{\omega_{1,0}^2}{2} + C\|\xi_{t_i}\|^2_{C^0(\{s\le 2R)\}} + C\\
&\le C\epsilon\int_X |\xi_{t_i}|^2 s^{-4}\frac{\omega_{1,0}^2}{2} + C\epsilon^{-1}\\
&\quad + C\|\xi_{t_i}\|^2_{C^0(\{s\le 2R)\}} + C
\end{aligned} \tag{7.42}$$

for all $\epsilon > 0$. If we choose ϵ such that the coefficient $C\epsilon < \frac{1}{2}$, then

$$\int_X |\xi_{t_i}|^2 s^{-4}\frac{\omega_{1,0}^2}{2} \le C. \tag{7.43}$$

As in [8, p. 715], $\|\xi_{t_i}\|_{C^0(X)} \le C$ using (7.41). This implies that

$$\|\varphi_{t_i}\|_{C^0} \le \|\tilde{\varphi}_{t_i}\|_{C^0} + |\varphi_{t_i} - \tilde{\varphi}_{t_i}| \le 2\|\tilde{\varphi}_{t_i}\|_{C^0} \le C \tag{7.44}$$

because $\varphi_{t_i} - \tilde{\varphi}_{t_i}$ is a constant and φ_{t_i} decays. Using the Evans–Krylov estimate and standard elliptic estimates on $B(x, 1, \omega_{1,0})$ for any $x \in X$, $\|\varphi_{t_i}\|_{C^k(X,\omega_{1,0})} \le C(k)$ for constants $C(k)$ independent of t_i.

The bound on $\int_X |\xi_{t_i}|^2 s^{-4} \frac{\omega_{1,0}^2}{2}$ also implies a bound on $\int_X |\nabla_{\omega_{1,0}} \xi_{t_i}|^2 \frac{\omega_{1,0}^2}{2}$ by (7.37). This implies that

$$\int_X |\nabla_{\omega_{1,0}} \varphi_{t_i}|^2 \frac{\omega_{1,0}^2}{2} \leq C. \tag{7.45}$$

Finally, we use [26, Proposition 2.9] to prove that there exist a constant $\delta > 0$ and constants $C(k,\delta) > 0$ independent of t_i such that

$$\|s^{k+\delta} \nabla_{\omega_{1,0}}^k \varphi_{t_i}\| \leq C(k,\delta) \tag{7.46}$$

for all k. Then we use the Arzelà–Ascoli lemma, a diagonal argument, and standard elliptic estimates to finish the proof.

7.5. *Closing remarks*

There is a folklore conjecture that some examples constructed using gauge theory by Biquard and Boalch [3] are ALG and by varying parameters, achieve all possible periods satisfying (7.11). See [20] for some progress towards this conjecture. We also mention that there is a folklore conjecture that some examples constructed using gauge theory by Biquard and Boalch [3] and Cherkis and Kapustin [12] are ALG* and by varying parameters, achieve all possible periods satisfying (7.11).

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References

[1] S. BANDO, A. KASUE, and H. NAKAJIMA, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), no. 2, 313–349. MR 1001844. DOI 10.1007/BF01389045. *(228)*

[2] A. L. BESSE, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. (3) **10**, Springer, Berlin, 1987. MR 0867684. DOI 10.1007/978-3-540-74311-8. *(258)*

[3] O. BIQUARD and P. BOALCH, *Wild non-abelian Hodge theory on curves*, Compos. Math. **140** (2004), no. 1, 179–204. MR 2004129. DOI 10.1112/S0010437X03000010. *(228, 272)*

[4] O. BIQUARD and V. MINERBE, *A Kummer construction for gravitational instantons*, Comm. Math. Phys. **308** (2011), no. 3, 773–794. MR 2855540. DOI 10.1007/s00220-011-1366-y. *(228)*

[5] D. BURNS, JR. and M. RAPOPORT, *On the Torelli problem for kählerian $K3$ surfaces*, Ann. Sci. Éc. Norm. Supér. (4) **8** (1975), no. 2, 235–273. MR 0447635. *(258)*

[6] G. CHEN and X. CHEN, *Gravitational instantons with faster than quadratic curvature decay, II*, J. Reine Angew. Math. **756** (2019), 259–284. MR 4026454. DOI 10.1515/crelle-2017-0026. *(228, 229, 269)*

[7] ———, *Gravitational instantons with faster than quadratic curvature decay, I*, Acta Math. **227** (2021), no. 2, 263–307. MR 4366415. DOI 10.4310/acta.2021.v227.n2.a2. *(228, 261, 264, 269)*

[8] ———, *Gravitational instantons with faster than quadratic curvature decay, III*, Math. Ann. **380** (2021), no. 1–2, 687–717. MR 4263695. DOI 10.1007/s00208-020-01984-9. *(228, 229, 237, 252, 265, 268, 269, 271)*

[9] G. CHEN and J. VIACLOVSKY, *Gravitational instantons with quadratic volume growth*, preprint, arXiv:2110.06498v2 [math.DG], to appear in J. London Math. Soc. *(229, 230, 231, 232, 237, 242, 264, 268)*

[10] G. CHEN, J. VIACLOVSKY, and R. ZHANG, *Collapsing Ricci-flat metrics on elliptic K3 surfaces*, Comm. Anal. Geom. **28** (2020), no. 8, 2019–2133. MR 4201795. DOI 10.4310/CAG.2020.v28.n8.a9. *(228, 230, 233, 237, 239, 248, 251, 252, 254, 255, 256, 259, 260, 264)*

[11] ———, *Hodge theory on* ALG^* *manifolds*, J. Reine Angew. Math. **799** (2023), 189–227. MR 4595310. DOI 10.1515/crelle-2023-0016. *(231, 234, 261, 262, 263)*

[12] S. A. CHERKIS and A. KAPUSTIN, *Hyper-Kähler metrics from periodic monopoles*, Phys. Rev. D (3) **65** (2002), no. 8, art. ID 084015. MR 1899201. DOI 10.1103/PhysRevD.65.084015. *(228, 272)*

[13] T. COLLINS, A. JACOB, and Y.-S. LIN, *The Torelli theorem for* ALH^* *gravitational instantons*, Forum Math. Sigma **10** (2022), no. e79. MR 4589525. DOI 10.1017/fms.2022.67. *(228)*

[14] J.-P. DEMAILLY, T. PETERNELL, and M. SCHNEIDER, *Kähler manifolds with numerically effective Ricci class*, Compos. Math. **89** (1993), no. 2, 217–240. MR 1255695. *(268)*

[15] S. K. DONALDSON, "Two-forms on four-manifolds and elliptic equations" in *Inspired by S. S. Chern*, Nankai Tracts Math. **11**, World Sci., Hackensack, 2006, 153–172. MR 2313334. DOI 10.1142/9789812772688_0007. *(252)*

[16] ———, "Calabi-Yau metrics on Kummer surfaces as a model gluing problem" in *Advances in Geometric Analysis*, Adv. Lect. Math. (ALM) **21**, Int. Press, Somerville, 2012, 109–118. MR 3077251. DOI 10.14219/jada.archive.2012.0111. *(228)*

[17] J. FINE, J. D. LOTAY, and M. SINGER, *The space of hyperkähler metrics on a 4-manifold with boundary*, Forum Math. Sigma **5** (2017), no. e6. MR 3631268. DOI 10.1017/fms.2017.3. *(252)*

[18] L. FOSCOLO, *ALF gravitational instantons and collapsing Ricci-flat metrics on the $K3$ surface*, J. Differential Geom. **112** (2019), no. 1, 79–120. MR 3948228. DOI 10.4310/jdg/1557281007. *(228)*

[19] ———, "Gravitational instantons and degenerations of Ricci-flat metrics on the $K3$ surface" in *Lectures and Surveys on* G2*-Manifolds and Related Topics*, Fields Inst. Commun. **84**, Springer, New York, 2020, 173–199. MR 4295858. *(252)*

[20] L. FREDRICKSON, R. MAZZEO, J. SWOBODA, and H. WEISS, *Asymptotic geometry of the moduli space of parabolic* $\mathrm{SL}(2,\mathbb{C})$*-Higgs bundles*, J. Lond. Math. Soc. (2) **106** (2022), 590–661. MR 4477200. DOI 10.1112/jlms.12581. *(228, 272)*

[21] B. R. GREENE, A. SHAPERE, C. VAFA, and S.-T. YAU, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, Nuclear Phys. B **337** (1990), no. 1, 1–36. MR 1059826. DOI 10.1016/0550-3213(90)90248-C. *(233)*

[22] M. GROSS, V. TOSATTI, and Y. ZHANG, *Gromov-Hausdorff collapsing of Calabi-Yau manifolds*, Comm. Anal. Geom. **24** (2016), no. 1, 93–113. MR 3514555. DOI 10.4310/CAG.2016.v24.n1.a4. *(233)*

[23] M. GROSS and P. M. H. WILSON, *Large complex structure limits of $K3$ surfaces*, J. Differential Geom. **55** (2000), no. 3, 475–546. MR 1863732. *(232, 237, 242, 254)*

[24] T. HAUSEL, E. HUNSICKER, and R. MAZZEO, *Hodge cohomology of gravitational instantons*, Duke Math. J. **122** (2004), no. 3, 485–548. MR 2057017. DOI 10.1215/S0012-7094-04-12233-X. *(228, 264, 269)*

[25] H.-J. HEIN, *Weighted Sobolev inequalities under lower Ricci curvature bounds*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2943–2955. MR 2801635. DOI 10.1090/S0002-9939-2011-10799-8. *(270)*

[26] ———, *Gravitational instantons from rational elliptic surfaces*, J. Amer. Math. Soc. **25** (2012), no. 2, 355–393. MR 2869021. DOI 10.1090/S0894-0347-2011-00723-6. *(228, 266, 268, 272)*

[27] H.-J. HEIN, S. SUN, J. VIACLOVSKY, and R. ZHANG, *Nilpotent structures and collapsing Ricci-flat metrics on the K3 surface*, J. Amer. Math. Soc. **35** (2022), no. 1, 123–209. MR 4322391. DOI 10.1090/jams/978. *(228, 233, 241, 242, 244, 248, 252, 257)*

[28] K. KODAIRA, *On the structure of compact complex analytic surfaces, I*, Amer. J. Math. **86** (1964), 751–798. MR 0187255. DOI 10.2307/2373157. *(228)*

[29] P. B. KRONHEIMER, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. **29** (1989), no. 3, 685–697. MR 0992335. DOI 10.4310/jdg/1214443067. *(228)*

[30] C. LEBRUN and M. SINGER, *A Kummer-type construction of self-dual* 4*-manifolds*, Math. Ann. **300** (1994), no. 1, 165–180. MR 1289837. DOI 10.1007/BF01450482. *(228)*

[31] R. MAZZEO, *Elliptic theory of differential edge operators, I*, Comm. Partial Differential Equations **16** (1991), no. 10, 1615–1664. MR 1133743. DOI 10.1080/03605309108820815. *(228)*

[32] V. MINERBE, *On the asymptotic geometry of gravitational instantons*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 883–924. MR 2778451. DOI 10.24033/asens.2135. *(228)*

[33] ———, *Rigidity for multi-Taub-NUT metrics*, J. Reine Angew. Math. **656** (2011), 47–58. MR 2818855. DOI 10.1515/CRELLE.2011.042. *(228)*

[34] Y. ODAKA and Y. OSHIMA, *Collapsing K3 Surfaces, Tropical Geometry and Moduli Compactifications of Satake, Morgan-Shalen Type*, MSJ Mem. **40**, Math. Soc. Japan, Tokyo, 2021. MR 4290084. DOI 10.1142/e071. *(233)*

[35] H. OOGURI and C. VAFA, *Summing up Dirichlet instantons*, Phys. Rev. Lett. **77** (1996), no. 16, 3296–3298. MR 1411842. DOI 10.1103/PhysRevLett.77.3296. *(233)*

[36] I. I. PIATESKII-SHAPIRO and I. R. SHAFAREVITCH, *Torelli's theorem for algebraic surfaces of type* K3, Izv. Ross. Akad. Nauk Ser. Mat. **35** (1971), 530–572. MR 0284440. *(258)*

[37] W.-D. RUAN, *On the convergence and collapsing of Kähler metrics*, J. Differential Geom. **52** (1999), no. 1, 1–40. MR 1743466. *(268)*

[38] T. SHIODA, *The elliptic $K3$ surfaces with a maximal singular fibre*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 7, 461–466. MR 2023754. DOI 10.1016/j.crma.2003.07.007. *(233)*

[39] Y. T. SIU, *Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics*, DMV Semin. **8**, Birkhäuser, Basel, 1987. MR 0904673. DOI 10.1007/978-3-0348-7486-1. *(269)*

[40] G. TIAN and S.-T. YAU, *Complete Kähler manifolds with zero Ricci curvature, I*, J. Amer. Math. Soc. **3** (1990), no. 3, 579–609. MR 1040196. DOI 10.2307/1990928. *(266)*

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