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Stability of the Non-Critical Spectral Properties I: Arithmetic Absolute Continuity of the Integrated Density of States

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Abstract: We develop a new method to prove absolute continuity of the integrated density of states of quasiperiodic operators, leading to the absolute continuity result for frequency-independent analytic perturbations of the non-critical almost Mathieu operator under arithmetic conditions on frequency.

1. Introduction

Analytic one-frequency Schrödinger operators on $\ell^2(\mathbb{Z})$ are given by,

$$(H_{V,\alpha,x}u)_n = u_{n+1} + u_{n-1} + V(x + n\alpha)u_n, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is the frequency, $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the phase, and $V \in C^\omega(\mathbb{T}, \mathbb{R})$[1] is the potential.

The central and most extensively studied such operator is the almost Mathieu operator (AMO),

$$(H_{\lambda,\alpha,x}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(x + n\alpha)u_n, \quad n \in \mathbb{Z}. \tag{1.2}$$

Sometimes called the drosophila of the subject. It is a model that is responsible for both the origins of the field and much of its ongoing significance in physics [14,44,57,61]. The almost Mathieu family is prototypical in the sense of Avila's global theory, with its separate regimes $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$ lending names to global classification of analytic $SL(2, \mathbb{C})$ cocycles [3].

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[1] For a bounded 1-periodic (possibly matrix valued) function F analytic on $\{x||\Im x| < h\}$, and continuous on $\{x||\Im x| \le h\}$ let $|F|_h = \sup_{|\Im x|<h} \|F(x)\|$, and denote by $C^\omega_h(\mathbb{T}, *)$ the Banach space of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $sl(2, \mathbb{R})$, $SL(2, \mathbb{R})$). $C^\omega(\mathbb{T}, \mathbb{R})$ is the locally convex space given by the union $\cup_{h>0} C^\omega_h(\mathbb{T}, \mathbb{R})$.

At the same time this family has a very special symmetry, self-duality with respect to Aubry duality (e.g. [40]), that links $\{H_{\lambda,\alpha,x}\}_x$ and $\{H_{\lambda^{-1},\alpha,x}\}_x$ and stems from the gauge invariance of the underlying two-dimensional model [56]. A number of remarkable results have been obtained, exploiting this symmetry, thus with methods not extendable to the general analytic class. This includes the ten martini problem [8,10,59], non-critical dry ten martini problem [9,13], the absolute continuity of the integrated density of states in the non-critical case [6,9] and others. See also [2,47] for some other recent progresses. Aubry duality enables one to combine the reducibility ($|\lambda| < 1$) and the localization ($|\lambda| > 1$) techniques. However, already for small perturbations of the AMO, most of the proofs involving self-duality of the AMO family break down. It is therefore an interesting question whether self-duality is just a convenient tool or of intrinsic importance to the AMO results. Indeed, a few properties of the AMO are destroyed after perturbations, for example the Lyapunov exponent is no longer a constant on the spectrum, in general, making it particularly significant to identify those that are stable.

Another (intersecting) group of important results exploits the fact that the potential of the AMO is given by a first degree trigonometric polynomial which allows for some powerful considerations not available in the general case. This includes metal-insulator transitions [12,33,46,48,49], and was also exploited in the ten martini proofs [8,9,59] and other arithmetic results, e.g. [39]. Here, by *arithmetic* we mean results under an explicit arithmetic condition on the frequency.

In contrast, for general analytic one-frequency Schrödinger operators, the current state-of-the-art results for all the above problems in the positive Lyapunov exponent regime are measure-theoretic in α [17,37,38]. The biggest issue is that one needs to eliminate frequencies α in a highly implicit way, technically due to the need to get rid of the so-called "double resonances".

At the same time, the almost Mathieu operator coming from physics, it is natural to expect that its physically relevant properties hold at least for its small perturbations. In this respect it is particularly important that the allowed perturbations are uniform in α within the Diophantine class.[2] Results with uniform dependence on Diophantine α are often called *non-perturbative* (e.g. [19]), even when the parameters involved are otherwise small, while the ones without such dependence are called *perturbative*.

The paper is the first of a multi-part project to extend various spectral properties of the almost Mathieu operator that have so far been proved in an AMO-specific way, to the analytic neighborhood of the almost Mathieu operators in a *non-perturbative* way, so that, in particular, to confirm their relevance to physics. Namely, we consider

$$(H^{\varepsilon}_{\lambda,\alpha,x}u)_n = u_{n+1} + u_{n-1} + (2\lambda\cos 2\pi(x+n\alpha) + \varepsilon v(x+n\alpha))u_n, \quad n \in \mathbb{Z}, \quad (1.3)$$

where v is a 1-periodic real analytic function. An additional aim is to develop different techniques to investigate the spectral properties of operators (1.3) in the zero/positive Lyapunov exponent regimes in a way that does not use self-duality or low degree of the potential, with an expectation that some of the techniques will also turn out to be useful for global results.

Technically, from the point of view of Avila's global theory [3], the AMO family has one more important feature: the acceleration (see (1.4)) is bounded by 1 on the spectrum. An important goal of the present project is to show that it is exactly this feature that governs many of the spectral properties for operators (1.3), that prevents, in

[2] A result where the strength of the allowed perturbation depends, say, on the Diophantine constants of α is clearly not robust with respect to small changes of the Diophantine frequency and requires various positive measure exclusions if a.e. frequency is fixed.

particular, the occurrence of the double resonances, thus confirming the importance of the notion of acceleration in the spectral theory of analytic quasiperiodic operators.

The *acceleration* is defined as

$$\omega(E) = \lim_{y\to 0^+} \frac{L_y(E) - L_0(E)}{2\pi y}, \tag{1.4}$$

where $L_y(E)$ is the complexified Lyapunov exponent:

$$L_y(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|A(x+iy+(n-1)\alpha)\cdots A(x+iy)\| dx, \tag{1.5}$$

with

$$A(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

One of the key conclusions in [3] is that the acceleration is always an integer. Moreover, for the almost Mathieu operator, for all E in the spectrum, we have

1. $|\lambda| < 1$: $L(E) = 0$ and $\omega(E) = 0$.
2. $|\lambda| = 1$: $L(E) = 0$ and $\omega(E) = 1$.
3. $|\lambda| > 1$: $L(E) = \ln|\lambda| > 0$ and $\omega(E) = 1$.

For general one-frequency Schrödinger operators, one can similarly divide the spectrum into three regimes:

1. The subcritical regime: $L(E) = 0$ and $\omega(E) = 0$.
2. The critical regime: $L(E) = 0$ and $\omega(E) > 0$.
3. The supercritical regime: $L(E) > 0$ and $\omega(E) > 0$.

Now, for the analytic perturbations of the non-critical almost Mathieu operator (1.3), as was also proved in [3], for all E in the spectrum, one has

1. $|\lambda| < 1$ and ε small enough: $L(E) = 0$ and $\omega(E) = 0$.
2. $|\lambda| > 1$ and ε small enough: $L(E) > 0$ and $\omega(E) = 1$.

We would like to mention that the spectral properties of non-critical operators (1.3) were well studied in the perturbative regime, i.e., assuming $|\lambda|$ sufficiently large depending on α. For a *fixed* Diophantine frequency, we refer readers to [28,29,63] for the proofs of almost sure Anderson localization, to [32] for the arithmetic version of Anderson localization in the case the potential is even, to [28,65] for the proofs of Cantor spectrum, and to [15,55,64,66] for the positivity and (Hölder) continuity of the Lyapunov exponent, all even holding for much rougher C^2 potentials as long as they stay cos-type.

In the present paper, we want to study the regularity of the integrated density of states (IDS) in the global sense ($|\lambda|$ does not need to be large) and non-perturbatively.

The IDS is defined in a uniform way for the above one-frequency Schrödinger operators $(H_{V,\alpha,x})_{x\in\mathbb{T}}$ by

$$N(E) = \int_{\mathbb{T}} \mu_x(-\infty, E] dx,$$

where μ_x is the spectral measure associated with $H_{V,\alpha,x}$ and δ_0. Roughly speaking, the density of states measure $N([E_1, E_2])$ gives the "number of states per unit volume" with energy between E_1 and E_2.

Regularity of the IDS is a popular subject in the spectral theory of quasiperiodic operators, especially the absolute continuity [4,6,9] and the Hölder regularity [1,9,36, 37]. It is also closely related to many other topics. For example, absolute continuity of the IDS is closely related to purely absolutely continuous spectrum in the regime of zero Lyapunov exponent [23,52]. Hölder continuity of the IDS is closely related to homogeneity of the spectrum [24,25,54]. Before formulating our results, we first give precise arithmetic assumptions on α. A frequency $\alpha \in \mathbb{R}$ will be called (κ, τ)-*strongly Diophantine* (denoted by $\alpha \in \mathrm{SDC}(\kappa, \tau)$) where $\kappa > 0$, $\tau > 1$ if

$$\mathrm{dist}(k\alpha, \mathbb{Z}) \geq \frac{\kappa}{|k|(\ln |k|)^{\tau}}, \quad \forall k \in \mathbb{Z}\backslash\{0\}. \tag{1.6}$$

We will use the notation

$$\mathrm{SDC} := \bigcup_{\kappa>0;\, \tau>1} \mathrm{SDC}(\kappa, \tau).$$

Clearly, SDC is a set of full Lebesgue measure.

We have

Theorem 1.1. *Let $\alpha \in$ SDC, $|\lambda| \neq 1$ and v be real analytic. There is $\varepsilon_0(\lambda, v) > 0$, such that if $|\varepsilon| < \varepsilon_0$, then the integrated density of states of operator* (1.3) *is absolutely continuous.*

Remark 1.1. The strong Diophantine condition on α can be relaxed to the usual Diophantine condition, or even to the Bruno condition where we believe the method in this paper still works. We only use it to invoke the existing homogeneity results and to keep the paper short.

Remark 1.2. We expect that Theorem 1.1 holds for all irrational α. However, for extremely Liouvillean α, the method in this paper is not effective since the spectrum is not homogenous [11]. We expect to develop techniques to approach the case of the Liouville α in the future work.

Remark 1.3. This paper mainly deals with the perturbations of the supercritical AMO. For the perturbations of the subcritical AMO, absolute continuity of the IDS is a corollary of the almost reducibility conjecture (ARC), announced by Avila [3]. However, to keep the paper self-contained, we give a short proof of it, independent of the ARC.

Remark 1.4. If $\lambda = 0$, Theorem 1.1 follows from [9].

Remark 1.5. Note that ε_0 actually depends only on λ and the analytic norm of v.

In fact, we only use a single special feature of the AMO, and effectively prove the following theorem:

Theorem 1.2. *Suppose every E in the spectrum of operator $H_{V,\alpha,x}$ with real-analytic V given by* (1.1) *is non-critical, and satisfies $\omega(E) \leq 1$. Let W be real analytic. There is $\varepsilon_0(V, W) > 0$, such that if $\alpha \in$ SDC and $|\varepsilon| < \varepsilon_0$, then the integrated density of states of operator $H_{V+\varepsilon W,\alpha,x}$ is absolutely continuous.*

Remark 1.6. Proof of Theorem 1.1 works verbatim to prove the supercritical part of Theorem 1.2. As for the subcritical part of Theorem 1.2, one needs to invoke the ARC, because our short self-contained argument in Sect. 4.1 only works for perturbations of the AMO.

Finally, we briefly introduce the main ideas of our proof. In the subcritical regime, the absolute continuity of the IDS is non-trivial, but not surprising and can be obtained, as mentioned, as a corollary of the absolutely continuous spectrum for individual spectral measures, which is in turn a corollary of the almost reducibility conjecture (although we do provide a self-contained proof, not using the almost reducibility conjecture). The main novelty of this paper is in the new method for the supercritical case where the individual spectral measures are not absolutely continuous.

Previously a method to prove absolute continuity of the IDS in the supercritical regime was developed in [37] through the use of the large deviation estimate and avalanche principle for the determinants of the truncated operator. Very involved technically, this method however cannot avoid some singularities for any fixed frequency α and thus only works in a measure-theoretic sense. Other methods [6,9] are very almost Mathieu specific. We note that even establishing non-perturbative positivity of the measure of the spectrum in the supercritical regime is a highly non-trivial argument [16].

Here we present a new approach that deals with this problem in an elegant yet general way, provided the spectrum is a homogeneous set in the Carleson sense. The method involves the properties of the non-tangential maximal function to study the normal boundary of the averaged Green's function (see Sect. 3 for the definition). Previously, integrability of the $s > 1$ power of the *imaginary* part of the normal boundary of the Borel transform has been used fruitfully to establish absolute continuity of spectral measures in certain settings [51,62]. Here we use a criterion of absolute continuity of a measure supported on a homogeneous compact set based on the integrability of the *real* part of the normal boundary of the Borel transform [35].[3] This has also previously been used to establish absolute continuity of certain spectral measures.

How can one apply it to establish absolute continuity of the *IDS*? For operators with homogeneous spectrum, it would follow from the integrability of the real part of the normal boundary of the Borel transform of the IDS, i.e., the real part of the normal boundary of the *averaged* Green's function. Thouless formula connects the averaged Green's function to the derivative of the Lyapunov exponent, but only for *complex* energies. However, the integrability problem itself is hard to deal with. Until now it was only known in the zero Lyapunov exponent regime, via the celebrated Kotani theory. In the positive Lyapunov exponent regime, whether the normal boundary of the averaged Green's is integrable is a difficult problem since Kotani theory is no longer applicable. Of course, for the almost Mathieu operator, one can study the normal boundary of the averaged Green's function in the positive Lyapunov exponent regime using Aubry duality and Kotani theory for the dual operator, however this is no longer applicable even for small perturbations because their duals are infinite-range quasiperiodic operators for which one does not even know how to build the Kotani theory.

Here we develop harmonic analysis tools to show that the study of the normal boundary of the averaged Green's function can be reduced to certain properties of the Lyapunov exponent *on the spectrum*, this reduction (see Sect. 3) being the key technical part of the paper. Based on this reduction, it remains to show that the Lyapunov exponent is differentiable *on the spectrum* in the Whitney sense. We note that the spectrum is expected to be a Cantor set, and the Lyapunov exponent is not a differentiable function on the whole real line even for the almost Mathieu operator [60]. However, Avila's global theory [3] comes to the rescue, allowing to show that the Lyapunov exponent is even analytic on the spectrum under our assumptions.

[3] with the argument apparently going back to Zinsmeister [67] and also discovered by different means in [58].

2. Preliminaries

2.1. Quasiperiodic cocycles and the Lyapunov exponent. Given $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, we define the *quasiperiodic* $SL(2,\mathbb{R})$*-cocycle* (α, A):

$$(\alpha, A): \begin{cases} \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2 \\ (x, v) \mapsto (x+\alpha, A(x)\cdot v) \end{cases}.$$

The iterates of (α, A) are of the form $(\alpha, A)^n = (n\alpha, \mathcal{A}_n)$, where

$$\mathcal{A}_n(x) := \begin{cases} A(x+(n-1)\alpha)\cdots A(x+\alpha)A(x), & n \geq 0 \\ A^{-1}(x+n\alpha)A^{-1}(x+(n+1)\alpha)\cdots A^{-1}(x-\alpha), & n < 0 \end{cases}.$$

The *Lyapunov exponent* is defined by $L(\alpha, A) := \lim\limits_{n\to\infty} \dfrac{1}{n}\displaystyle\int_{\mathbb{T}} \ln \|\mathcal{A}_n(x)\| dx$.

A basic fact about quasiperiodic $SL(2,\mathbb{R})$-cocycle is the continuity of the Lyapunov exponent:

Theorem 2.1 ([18]). *The functions* $\mathbb{R}\times C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R})) \ni (\alpha, A) \mapsto L(\alpha, A) \in [0,\infty)$ *are continuous at any* (α', A') *with* $\alpha' \in \mathbb{R}\backslash\mathbb{Q}$.

2.2. The rotation number. Assume that $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ is homotopic to the identity. (α, A) induces the projective skew-product $F_A: \mathbb{T}\times\mathbb{S}^1 \to \mathbb{T}\times\mathbb{S}^1$

$$F_A(x, w) := \left(x+\alpha, \frac{A(x)\cdot w}{|A(x)\cdot w|}\right),$$

which is also homotopic to the identity. Thus we can lift F_A to a map $\widetilde{F}_A: \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ of the form $\widetilde{F}_A(x, y) = (x+\alpha, y+\psi_x(y))$, where for every $x \in \mathbb{T}$, ψ_x is $\mathbb{Z}$-periodic. The map $\psi: \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ is called a *lift* of A. Let μ be any probability measure on $\mathbb{T}\times\mathbb{R}$ which is invariant by $\widetilde{F}_A$, and whose projection on the first coordinate is given by the Lebesgue measure. The number

$$\rho(\alpha, A) := \int_{\mathbb{T}\times\mathbb{R}} \psi_x(y)\, d\mu(x, y) \bmod \mathbb{Z}$$

depends neither on the lift ψ nor on the measure μ, and is called the *fibered rotation number* of (α, A) (see [43,50] for more details).

A typical example is represented by the *Schrödinger cocycles* (α, S_E^V), where

$$S_E^V(x) := \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}, \quad E \in \mathbb{R}.$$

Schrödinger cocycles are a dynamical equivalent of the eigenvalue equations $H_{V,\alpha,x}u = Eu$. Indeed, any formal solution $u = (u_n)_{n\in\mathbb{Z}}$ of $H_{V,\alpha,x}u = Eu$ satisfies

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E^V(x+n\alpha)\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall\, n \in \mathbb{Z}.$$

The spectral properties of $H_{V,\alpha,x}$ and the dynamics of (α, S_E^V) are closely related by the Johnson's theorem: $E \in \Sigma_{V,\alpha}$[4] if and only if (α, S_E^V) is *not* uniformly hyperbolic. Throughout the rest of the paper, we set $L(E) := L(\alpha, S_E^V)$[5] and $\rho(E) := \rho(\alpha, S_E^V)$ for brevity. It is also well known that $\rho(E) \in [0, \frac{1}{2}]$ relates to the integrated density of states as follows:

$$N(E) = 1 - 2\rho(E). \tag{2.1}$$

3. Proof for the Supercritical Case

The proof of Theorem 1.1 contains two parts, based on two different methods. In this section, we deal with the perturbations of the supercritical almost Mathieu operators.

Theorem 3.1. *Let $\alpha \in$ SDC, $|\lambda| > 1$ and v be real analytic. Then for ε small enough, depending on λ and v, the IDS of operator (1.3) is absolutely continuous.*

Define

$$\delta_j(n) = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases}.$$

For any $E \in \mathbb{R}$ and $\eta > 0$, one can define the averaged Green's function of operator (1.1) as

$$G(0, 0, E + i\eta) = \int_{\mathbb{T}} \langle \delta_0, (H_{V,\alpha,x} - (E + i\eta))^{-1} \delta_0 \rangle dx = \int \frac{1}{E' - (E + i\eta)} dN(E').$$

Let us recall that a *Herglotz function* is a holomorphic mapping of $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ to itself. One can easily check that $G(0, 0, z)$ is a Herglotz function. Thus for almost every $E \in \mathbb{R}$, the normal boundary of $G(0, 0, z)$ exists and one can define

$$G(0, 0, E + i0) = \lim_{\eta \to 0^+} G(0, 0, E + i\eta).$$

Before we give the proof of Theorem 3.1, let's recall two interesting theorems. Given a compact set $\mathcal{S} \subset \mathbb{R}$, we say $\mathcal{S}$ is homogenous if there is $\sigma_0 > 0$ such that for any $\sigma < \sigma_0$ and $E \in \mathcal{S}$, we have

$$|(E - \sigma, E + \sigma) \cap \mathcal{S}| \geq \frac{1}{2}\sigma.$$

Theorem 3.2 (Theorem H of [25]). *Assume $\alpha \in$ SDC, V is real analytic and $L(E) > 0$ for all $E \in \mathbb{R}$, then $\Sigma_{V,\alpha}$ is homogeneous.*

Remark 3.1. As explained in [25], the strong Diophantine condition on α can be relaxed to the usual Diophantine condition.

[4] By minimality, the spectrum of $H_{V,\alpha,x}$ denoted by $\Sigma_{\alpha,V}$, is a compact subset of $\mathbb{R}$, independent of x if $(1, \alpha)$ is rationally independent.

[5] We sometimes identify $L_0(E)$ (see (1.5)) and $L(E)$.

Theorem 3.3 (Theorem 3.3 of [35]). *Let $\mathcal{E} \subset \mathbb{R}$ be a compact homogenous set and f a Herglotz function with representation*

$$f(z) = \int_{\mathcal{E}} \frac{d\mu(E)}{E - z}, \quad z \in \mathbb{C}_+,$$

where $d\mu$ is a finite measure with $supp(d\mu) \subset \mathcal{E}$. Let $f(E + i0) = \lim_{\eta \to 0^+} f(E + i\eta)$ be the a.e. normal boundary of f and assume that

$$\Re f(\cdot + i0) \in L^1(\mathcal{E}, dE).$$

Then $d\mu$ is absolutely continuous.

Theorems 3.2 and 3.3 indicate that to prove absolute continuity of the IDS, we need to study the regularity of the real part of the normal boundary of the averaged Green's function. It's easy to see that $\Re G(0, 0, E+i0)$ is well-defined and real analytic outside the spectrum. Beyond that, the key statement of the celebrated Kotani theory [52], says that $G(0, 0, z)$ is *averaged reflectionless* in the zero Lyapunov exponent regime. This means that for almost every E in the zero Lyapunov exponent regime, $\Re G(0, 0, E + i0) = 0$.

However, in the positive Lyapunov exponent regime, the regularity of $\Re G(0, 0, E+i0)$ remains widely open. Indeed, in this case $\Re G(0, 0, E + i0)$ as a function of E may in principle be as bad as possible, since the spectrum is purely singular. In particular, before the integration, we have $|(H_{V,\alpha,x} - E)^{-1}(0, 0, E + i0)| \to \infty$ a.e. with respect to the spectral measure. It turns out that this common intuition is completely wrong. Indeed, we have the following surprising theorem.

Theorem 3.4. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $|\lambda| \neq 0, 1$ and v be real analytic. Then for ε small enough, $\Re G_\varepsilon(0, 0, E + i0)$*[6] *almost surely coincides with an analytic function on the spectrum.*

Here we say f is a real analytic function on a set S if it is the restriction of some real analytic function defined on an open neighborhood of S.

Remark 3.2. Let $\Sigma^\varepsilon_{\lambda,\alpha}$ be the spectrum of $H^\varepsilon_{\lambda,\alpha,x}$. We actually effectively prove that the Whitney derivative of $L(E)$ on $\Sigma^\varepsilon_{\lambda,\alpha}$, denoted by $L'_W(E)$, exists and is analytic where

$$L'_W(E) = \lim_{E' \to E} \frac{L(E') - L(E)}{E' - E}, \quad E', E \in \Sigma^\varepsilon_{\lambda,\alpha}. \tag{3.1}$$

Moreover,

$$\Re G_\varepsilon(0, 0, E + i0) = L'_W(E)$$

for a.e. $E \in \Sigma^\varepsilon_{\lambda,\alpha}$.

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$$G_\varepsilon(0, 0, E + i0) = \lim_{\eta \to 0^+} G_\varepsilon(0, 0, E + i\eta),$$

and

$$G_\varepsilon(0, 0, E + i\eta) = \int_{\mathbb{T}} \langle \delta_0, (H^\varepsilon_{\lambda,\alpha,x} - (E + i\eta))^{-1} \delta_0 \rangle dx.$$

Remark 3.3. The property of averaged reflectionless is just the same result, but with the analytic function being 0. In this sense, almost sure analyticity can be viewed as the generalized notion of *averaged reflectionless*.

Note that Theorem 3.1 follows from Theorem 3.4.

Proof of Theorem 3.1. For the supercritical AMO, it is well known that the Lyapunov exponent is positive for all $E \in \mathbb{R}$ [3,18]. Thus by Theorem 2.1, there is $\varepsilon_0(\lambda, v) > 0$, such that if $|\varepsilon| < \varepsilon_0$, then

$$L(\alpha, S_E^{2\lambda \cos + \varepsilon v}) > 0$$

for any $E \in \mathbb{R}$. By Theorem 3.2, if $\alpha \in$ SDC, then $\Sigma_{\lambda,\alpha}^{\varepsilon}$ is homogenous where $\Sigma_{\lambda,\alpha}^{\varepsilon}$ is the spectrum of $H_{\lambda,\alpha,x}^{\varepsilon}$. On the other hand, by Theorem 3.4, there is $\varepsilon_1(\lambda, v) > 0$, such that if $|\varepsilon| < \varepsilon_1$, then $\Re G_\varepsilon(0, 0, E + i0)$ almost surely equals to an analytic function on the spectrum. Thus for $|\varepsilon| < \min\{\varepsilon_0, \varepsilon_1\}$, we have $\Sigma_{\lambda,\alpha}^{\varepsilon}$ is homogenous and $\Re G_\varepsilon(0, 0, E + i0) \in L^1(\Sigma_{\lambda,\alpha}^{\varepsilon})$. Note that

$$G_\varepsilon(0, 0, z) = \int \frac{1}{E - z} dN_\varepsilon(E), \quad z \in \mathbb{C}_+.$$

Thus by Theorem 3.3, we have $N_\varepsilon(E)$, the IDS of operator (1.3), is absolutely continuous. □

In the remaining part of this section we prove Theorem 3.4. The foundation is the following remarkable result of Avila in [3], on the analyticity of the Lyapunov exponent.

Theorem 3.5 ([3]). *Let $\lambda > 1$ and v be any real analytic function. Then for ε small enough and for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $L(\alpha, S_E^{2\lambda \cos + \varepsilon v})$ restricted to the spectrum is a positive real analytic function.*

On the other hand, there is the following relation between the Lyapunov exponent and the Green's function, see [50,52,53],

$$\frac{\partial L(E + i\eta)}{\partial E} = \Re G(0, 0, E + i\eta).$$

Roughly speaking, the real part of the normal boundary of the averaged green's function is exactly the normal boundary of the derivative of the Lyapunov exponent. We will now show that $\frac{\partial L(E+i0)}{\partial E}$ is almost surely analytic. To prove this, we will need some ideas from hard analysis.

3.1. Non-tangential Maximal Functions

Definition 3.1. The non-tangential maximal function takes a function F defined on the upper-half plane

$$\mathbb{C}_+ := \{x + iy : x \in \mathbb{R}, y > 0\}$$

and produces a function F^* defined on $\mathbb{R}$ via the expression

$$F^*(x_0) = \sup_{|x - x_0| < y} |F(x + iy)|.$$

Note that for any fixed x_0, the set $\{(x, y) : |x - x_0| < y\}$ is a cone in $\mathbb{R}^2_+$ with vertex at $(x_0, 0)$ and axis perpendicular to the boundary of the x-axis. Thus, the non-tangential maximal operator simply takes the supremum of the function F over a cone with vertex at the x-axis.

Definition 3.2. The Hardy space H^p where $0 < p < \infty$, on the upper half-plane $\mathbb{C}_+$ is defined to be the space of holomorphic functions F on $\mathbb{C}_+$ with bounded norm, the norm being given by

$$|F|_{H^p} = \sup_{y>0} \left(\int_{-\infty}^{+\infty} |F(x + iy)|^p dx \right)^{\frac{1}{p}}.$$

Proposition 3.1 (Page 1 of [41], Theorem 1 of [20]). *Let F be an analytic function on the upper-half plane, and of the class H^p where $0 < p < \infty$. Then*

$$\int_{-\infty}^{\infty} (F^*(x))^p dx \le C_p \sup_{y>0} \int_{-\infty}^{+\infty} |F(x + iy)|^p dx.$$

Note that $G(0, 0, z)$ is an analytic function on the upper-half plane. Thus one can define the corresponding non-tangential maximal function

$$G^*(E) = \sup_{|E'-E|<\eta} |G(0, 0, E' + i\eta)|.$$

Proposition 3.2. *For each $\sigma > 0$,*

$$\left|\{E : G^*(E) > \sigma\}\right| \le \frac{D}{\sigma^{\frac{3}{4}}},$$

for some $D > 0$ (does not depend on σ).

Proof. Using (3.2), one can check that $G(0, 0, \cdot) \in H^p$ for any $\frac{1}{2} < p < 1$. Thus by Proposition 3.1, there is $D > 0$ such that

$$\int_{-\infty}^{\infty} (G^*(E))^{\frac{3}{4}} dE \le D.$$

Thus

$$\begin{aligned} \sigma^{\frac{3}{4}} \left|\{E : G^*(E) > \sigma\}\right| &\le \int_{\{E:G^*(E)>\sigma\}} (G^*(E))^{\frac{3}{4}} dE \\ &\le \int_{-\infty}^{\infty} (G^*(E))^{\frac{3}{4}} dE \le D. \end{aligned}$$

□

3.2. Proof of Theorem 3.4 Now we are ready to prove Theorem 3.4. For simplicity, we will omit ε in the notations. Note that by the spectral theorem, for any $z \in \mathbb{C}\backslash\mathbb{R}$, we have

$$G(0,0,z) = \int \frac{1}{E-z} dN(E). \tag{3.2}$$

On the other hand, denoting

$$w(z) = \int \ln(E-z) dN(E).$$

The Thouless formula [21,22,42] says

$$L(z) = \int \ln|E-z| dN(E) = \Re w(z).$$

The followings are some basic facts on $G^*(E)$,

1. G^* is lower semicontinuous and in particular,

$$U_\sigma = \{E : G^*(E) > \sigma\}$$

is open.
2. By Proposition 3.2, $\mathbb{R} \doteq \cup_n U_n^c$ where we say $A \doteq B$ if $|A\backslash B| = |B\backslash A| = 0$.

Note that $C_n := U_n^c \cap [-n, n]$ is compact. We define

$$\eta(E) = \mathrm{dist}(E, C_n).$$

One can easily verify that

$$|\eta(E) - \eta(E')| \le |E - E'|$$

for any $E, E' \in \mathbb{R}$. By Rademacher's theorem, $\eta(E)$ is an absolutely continuous function and is differentiable almost surely.

On the other hand, C_n^c is open, thus $C_n^c = \cup_i (a_i, b_i)$, and we have

$$\eta'(E) = \begin{cases} 0 & E \in C_n \backslash (\{a_i\} \cup \{b_i\}), \\ 1 & E \in \cup_i \left(a_i, \frac{a_i+b_i}{2}\right), \\ -1 & E \in \cup_i \left(\frac{a_i+b_i}{2}, b_i\right). \end{cases} \tag{3.3}$$

For any fixed $\delta > 0$, we consider the function $f_\delta(E) = L(E + i(\eta(E) + \delta))$, which is obviously Lipschitz. Thus

$$f_\delta(E_1) - f_\delta(E_2) = \int_{E_1}^{E_2} \frac{\partial L}{\partial E}(E + i(\eta(E)+\delta)) + \frac{\partial L}{\partial \eta}(E + i(\eta(E)+\delta))\, \eta'(E) dE.$$

The following Lemma plays a crucial role.

Lemma 3.1. *For any $E_1, E_2 \in C_n$, we have*

$$L(E_1) - L(E_2) = \int_{E_1}^{E_2} g(E)dE,$$

where

$$g(E) = \begin{cases} \lim\limits_{\delta\to 0^+} \frac{\partial L}{\partial E}(E + i\delta) & E \in C_n, \\ \frac{\partial L}{\partial E}(E + i\eta(E)) + \frac{\partial L}{\partial \eta}(E + i\eta(E))\,\eta'(E) & E \notin C_n. \end{cases} \tag{3.4}$$

Proof. Since $E_1, E_2 \in C_n$, for all $E \in (E_1, E_2)$, by the definition of $\eta(E)$, we have

$$E + i(\eta(E) + \delta) \in \bigcup_{E' \in C_n} \left\{x + iy \in \mathbb{C} : |x - E'| \le y\right\}.$$

Thus by the definition of $G^*(E)$, C_n and the fact that

$$\frac{dL}{dz}(E + i\eta) = G(0, 0, E + i\eta),$$

we have

$$\left|\frac{\partial L}{\partial E}(E + i(\eta(E) + \delta)) + \frac{\partial L}{\partial \eta}(E + i(\eta(E) + \delta))\,\eta'(E)\right| \le 2n, \tag{3.5}$$

uniformly for all $\delta > 0$. Thus the result follows from dominated convergence by letting $\delta \to 0$. □

Now, we apply Lebesgue's theorem on differentiation of integrals to $g \in L^\infty$,[7]

$$g(E) = \lim_{E' \to E} \frac{1}{E' - E} \int_E^{E'} g(E)dE, \tag{3.6}$$

for a.e. $E \in \mathbb{R}$.

We may assume $|C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}| > 0$ (otherwise there is nothing to say). Then by Lebesgue's density theorem, for a.e. $E \in C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}$, there is a sequence $E_j \in C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}$ such that

$$\lim_{j\to\infty} E_j = E. \tag{3.7}$$

Now combining (3.6), (3.7) with Lemma 3.1, for a.e. $E \in C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}$, we can find a sequence $E_j \in C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}$ such that

$$g(E) = \lim_{j\to\infty} \frac{L(E_j) - L(E)}{E_j - E}. \tag{3.8}$$

Note that by Theorem 3.5, there is an analytic function $f : U \to \mathbb{R}$ such that $L(E) = f(E)$ on $\Sigma^\varepsilon_{\lambda,\alpha} \subset U$ where U is open. Thus by (3.4) and (3.8), we have for a.e. $E \in C_n \cap \Sigma^\varepsilon_{\lambda,\alpha}$,

$$\Re G(0, 0, E + i0) = \lim_{\eta\to 0^+} \frac{\partial L}{\partial E}(E + i\eta) = g(E) = f'(E).$$

Note that $f'(E)$ is also analytic. This completes the proof.

[7] It follows from (3.5) and the definition of g.

4. Proof of the Subcrtical Case

Absolute continuity of the IDS in the subcritical regime is actually a corollary of the almost reducibility [4,5]. Thus the main aim of the section is to prove that the cocycle corresponding to operators (1.3) with $|\lambda| < 1$ and $\varepsilon \ll 1$ is still almost reducible. This immediately follows from the almost reducibility conjecture (ARC), announced by Avila in [3], to appear in [4,5]. However, we point out that almost reducibility of the perturbations of the subcritical AMO follows directly from the openness of almost reducibility and compactness. Here we give a self-contained proof which is independent of the ARC.

Theorem 4.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $|\lambda| < 1$ and v be real analytic. Then for ε small enough, depending on λ, v, the IDS of operator (1.3) is absolutely continuous.*

Remark 4.1. In this theorem, α can be any irrational number, it does not need to be (strongly) Diophantine.

4.1. Reducibility of quasiperiodic cocycles We will only consider cocycles (α, A) with $\deg A = 0$. A quasiperiodic C^ω-cocycle (α, A) is called C^ω*-rotations reducible* if there exists $B \in C^\omega(\mathbb{T}, SL(2, \mathbb{R}))$ and $\psi \in C^\omega(\mathbb{T}, \mathbb{R})$ such that

$$B(x+\alpha)^{-1} A(x) B(x) = R_{\psi(x)}. \tag{4.1}$$

We will call a cocycle *almost reducible* if there exist a sequence of $B_n \in C^\omega(\mathbb{T}, PSL(2, \mathbb{R}))$ and $A_n \in SL(2, \mathbb{R})$ such that

$$B_n(x+\alpha)^{-1} A(x) B_n(x) = A_n, \tag{4.2}$$

moreover, $A_n \to A$ for some $A \in SL(2, \mathbb{R})$.

Now we consider quasiperiodic Schrödinger cocycles. Define the following subset:

$$\mathcal{R}_{\alpha,V} = \{E \in \mathbb{R} \colon (\alpha, S_E^V) \text{ is } C^\omega\text{-rotations reducible}\}.$$

We will sometimes drop the indices and simply use $\mathcal{R}$, if the values of the indices are clear from the context. We mention that (almost) reducibility has been proved to be very fruitful [9,10,12,13,26,27,30,31,33,34,45,54].

For every $\tau > 1$ and $\gamma > 0$, we define

$$\Theta_\gamma^\tau = \left\{ \theta \in \mathbb{T} \| 2\theta + k\alpha \|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|k|+1)^\tau}, k \in \mathbb{Z} \right\},$$
$$\Theta = \cup_{\gamma>0, \tau>1} \Theta_\gamma^\tau.$$

Note that Θ is a subset of $\mathbb{T}$ of full Lebesgue measure.

4.2. Proof of Theorem 4.1 We first prove a theorem related to rotations reducibility.

Theorem 4.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $|\lambda| < 1$ and v be real analytic. Then for ε small enough, depending only on λ, v, $(\alpha, S_E^{2\lambda\cos+\varepsilon v})$ is rotations reducible if $\rho(\alpha, S_E^{2\lambda\cos+\varepsilon v}) \in \Theta$.*

Proof. We denote by

$$\beta = \beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}.$$

Note that if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $(\alpha, S_E^{2\lambda\cos})$ is almost reducible for all $E \in \mathbb{R}$ (see Theorem 1.4 of [9] for the case $\beta = 0$ and Theorem 1.1 of [4] for the case $\beta > 0$). We then need the following two known results,

Theorem 4.3 (Corollary 1.3 of [4]). *Almost reducibility is stable, in the sense that it defines an open set in* $\mathbb{R}\backslash\mathbb{Q} \times C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$.

Note that for $E \in \Sigma^\varepsilon_{\lambda,\alpha}$, there is $h_0 > 0$ such that

$$\left| S_E^{2\lambda\cos+\varepsilon v} - S_E^{2\lambda\cos} \right|_{h_0} \le |\varepsilon v|_{h_0}.$$

$\Sigma^\varepsilon_{\lambda,\alpha}$ is compact and by Theorem 4.3, for ε sufficiently small, we have that $(\alpha, S_E^{2\lambda\cos+\varepsilon v})$ is almost reducible if $E \in \Sigma^\varepsilon_{\lambda,\alpha}$. Finally, Theorem 4.2 follows from the following theorem.

Theorem 4.4 (Corollary 1.3 of [4]). *If* (α, A) *is almost reducible and* $\rho(\alpha, A) \in \Theta$, *then* (α, A) *is rotations reducible.* □

Proof of Theorem 4.1. Let

$$E \in \mathcal{E} = \left\{ E \in \Sigma^\varepsilon_{\lambda,\alpha} : \rho(\alpha, S_E^{2\lambda\cos+\varepsilon v}) \in \Theta \right\}.$$

By Theorem 4.2, we have $(\alpha, S_E^{2\lambda\cos+\varepsilon v})$ is rotations reducible for all $E \in \mathcal{E}$. Thus by the argument in Avila-Fayad-Krikorian [7], $N(E) = 1 - 2\rho(E)$ is Lipschitz on $\mathcal{E}$. Thus the image of non-Lipschitz $E's$ under N is of zero Lebesgue measure, therefore the IDS of operator (1.3) is absolutely continuous under the assumption that $|\lambda| < 1$ and ε sufficiently small.[8] □

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Declarations

Conflict of interest There is no conflict of interest to report.

[8] This actually follows from the definition of the absolutely continuous function and we mention that this argument was first used in [4].

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