Quantization of Axion-Gauge Couplings and Noninvertible Higher Symmetries

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We derive model-independent quantization conditions on the axion couplings (sometimes known as the anomaly coefficients) to the standard model gauge group $[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y]/\mathbb{Z}_q$ with $q=1,2,3,6$. Using these quantization conditions, we prove that any QCD axion model to the right of the $E/N=8/3$ line on the $|g_{a\gamma\gamma}|$-m_a plot must necessarily face the axion domain wall problem in a postinflationary scenario. We further demonstrate the higher-group and noninvertible global symmetries in the standard model coupled to a single axion. These generalized global symmetries lead to universal bounds on the axion string tension and the monopole mass. If the axion were discovered in the future, our quantization conditions could be used to constrain the global form of the standard model gauge group.

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Introduction.—Axions have long been a major target in particle phenomenology. Originally emerging as an elegant solution to the strong CP problem [1–4], axions have since been identified as one of the most well-motivated dark matter candidates [5–8], appearing in a number of extensions of the standard model (SM). They have motivated dozens of unique experimental searches bridging a wide variety of disciplines such as collider physics, astrophysics, condensed matter physics, and quantum optics [8]. Axionlike particles can provide a natural candidate for the inflaton [9,10] and may also play a central role in a variety of important cosmological effects such as the cosmological constant problem [10]. Axionlike particles also exist ubiquitously in string theory [11–16]. See, for example, [10,17–21] for various reviews.

In this Letter, we derive model-independent quantization conditions imposed on the coupling constants K_3, K_2, and K_1 [defined in (2)] between the axion field and instanton number densities of the $\mathrm{su}(3)\times\mathrm{su}(2)\times\mathrm{u}(1)_Y$ gauge fields in the SM and discuss their physical implications on the relation between the axion-photon effective coupling and domain wall number. The precise quantization conditions depend on the global form of the SM gauge group, which is summarized in Table I. It is known that a $\mathbb{Z}_6$ center subgroup of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ acts trivially on all the SM particles. Therefore, the SM gauge group need not be a product group but can be one of the following:

$$G_{\mathrm{SM}}=[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y]/\mathbb{Z}_q,\qquad q=1,2,3,6. \quad (1)$$

More physically, the global form of G_{SM} depends on what gauge charges are allowed or introduced in physics beyond the standard model. For example, in the SU(5) and many other grand unified theories (GUTs), all new particles carry vanishing $\mathbb{Z}_6$ charges, and $q=6$. In other UV models for the axion (such as the simplest version of the Kim-Shifman-Vainshtein-Zakharov model [22,23]), the heavy fermions may carry nontrivial $\mathbb{Z}_6$ charges, and the corresponding global form of G_{SM} is different. See Ref. [24] for more discussions.

Furthermore, we discuss the generalized global symmetry structure of the model, summarized in Table II, which allows us to constrain the tension of axion strings as well as the mass of monopoles. Many generalized global symmetries in models of axions have been discussed in the past, including noninvertible symmetries [25–28] as well as higher-group symmetries [29–34]. See Refs. [35–40] for reviews on generalized global symmetries.

Quantization of the axion couplings to gauge fields.—The axion field $\theta(x)$ is a periodic scalar field whose periodicity is given by $\theta\sim\theta+2\pi$, which should be viewed as a gauge symmetry. We consider a generic coupling of the axion field to the SM gauge fields given by

$$\frac{K_3}{8\pi^2}\theta\mathrm{Tr}F_3\wedge F_3+\frac{K_2}{8\pi^2}\theta\mathrm{Tr}F_2\wedge F_2+\frac{K_1}{8\pi^2}\theta F_1\wedge F_1, \quad (2)$$

where F_3, F_2, and F_1 are the field strengths for the SU(3), SU(2), and U(1) gauge fields, respectively, and we have adopted the differential form notation. Without loss of generality, we assume $K_3\geq 0$, which can always be achieved by a field redefinition $\theta\to-\theta$. Throughout,

TABLE I. Quantization conditions of the axion couplings to the gauge fields for the possible global forms of the SM gauge group, $G_{\rm SM} = [{\rm SU}(3)\times{\rm SU}(2)\times{\rm U}(1)_Y]/\mathbb{Z}_q$ with $q = 1, 2, 3, 6$.

q	Quantization of K_i's	Quantization of N and E
1	$K_3, K_2, K_1 \in \mathbb{Z}$	$N \in \frac{1}{2}\mathbb{Z},\ E \in \frac{\mathbb{Z}}{36}$
2	$K_3, K_2 \in \mathbb{Z},\ K_1 \in 2\mathbb{Z},$ $2K_2 + K_1 \in 4\mathbb{Z}$	$N \in \frac{1}{2}\mathbb{Z},\ E \in \frac{1}{9}\mathbb{Z}$
3	$K_3, K_2 \in \mathbb{Z},\ K_1 \in 3\mathbb{Z},$ $6K_3 + K_1 \in 9\mathbb{Z}$	$N \in \frac{1}{2}\mathbb{Z},\ E \in \frac{\mathbb{Z}}{12},$ $4N + 12E \in 3\mathbb{Z}$
6	$K_3, K_2 \in \mathbb{Z},\ K_1 \in 6\mathbb{Z},$ $24K_3 + 18K_2 + K_1 \in 36\mathbb{Z}$	$N \in \frac{1}{2}\mathbb{Z},\ E \in \frac{1}{3}\mathbb{Z},$ $4N + 3E \in 3\mathbb{Z}$

we normalize our fields so that the instanton number $(1/8\pi^2)\int {\rm Tr}F_i \wedge F_i$ is a quantized number independent of the gauge coupling. See Supplemental Material [41] for the relation to the standard notation in particle physics where the fields have canonical kinetic terms. Below the electroweak symmetry breaking (EWSB) scale, (2) reduces to

$$\frac{N}{4\pi^2}\theta{\rm Tr}F_3 \wedge F_3 + \frac{E}{8\pi^2}\theta F \wedge F. \tag{3}$$

Here, we follow the standard convention in the literature [42] to denote the axion-gluon coupling by N. It is related to K_3 (which equals the number $N_{\rm DW}$ of axion domain walls) as

$$K_3 = N_{\rm DW} = 2N. \tag{4}$$

The (bare) axion-photon coupling E is related to K_1, K_2 as $E = (1/36)(K_1 + 18K_2)$, which can be obtained from the standard relations between the electromagnetic, SU(2), and $\mathrm{U}(1)_Y$ gauge fields.

The allowed values of the quantized coupling constants K_3, K_2, and K_1, as well as N and E, which depend on the global form of the SM gauge group (1), are summarized in Table I. We now provide a rigorous, model-independent derivation of the quantization conditions from the fractional instanton numbers. This derivation is universal and does not require any assumption about the UV origin of the axion. The fractional instantons on T^4 of the SM gauge group have been previously studied in [69] by imposing the 't Hooft twisted boundary conditions [70,71].

The periodic identification of the dynamical axion field $\theta \sim \theta + 2\pi$ is a gauge symmetry, and the exponentiated action e^{iS} should be gauge invariant. In the presence of the axion coupling (2), e^{iS} transforms under $\theta \sim \theta + 2\pi$ by a phase $\exp(2\pi i \sum_i K_i n_i)$, where n_i's are the instanton numbers defined by $n_3 \equiv (1/8\pi^2)\int {\rm Tr}F_3 \wedge F_3$, $n_2 \equiv (1/8\pi^2)\int {\rm Tr}F_2 \wedge F_2$, and $n_1 \equiv (1/8\pi^2)\int F_1 \wedge F_1$. The quantization of the instanton numbers (which depends on the global form of the SM gauge group) then gives the quantization of the K_i's as well as E and N. We focus on the $q = 6$ case, while the other values of q as well as alternative derivations are discussed in Supplemental Material [41]. Also, we work under the minimal setup where the only degrees of freedom are the SM fields and a single axion field. In the presence of additional topological degrees of freedom, the axion-gauge couplings K_i can be further fractionalized [43], which we review in Supplemental Material [41].

For $q = 6$, the allowed values of the fractional instanton numbers and their correlations are

$$n_3 \in \frac{1}{3}\mathbb{Z}, \qquad n_2 \in \frac{1}{2}\mathbb{Z}, \qquad n_1 \in \frac{1}{36}\mathbb{Z},$$
$$n_3 - 24n_1 \in \mathbb{Z}, \qquad n_2 - 18n_1 \in \mathbb{Z}. \tag{5}$$

One can derive (5) as follows. Since now the SM gauge group is $[{\rm SU}(3)\times{\rm SU}(2)\times{\rm U}(1)_Y]/\mathbb{Z}_6$, we may view the gauge fields A_3, A_2, and A_1 as PSU(3), SO(3), and $\mathrm{U}(1)/\mathbb{Z}_6$ gauge fields, respectively, where the various characteristic classes are related by [30,44]

$$\int c_1(F_1) = \int w_2(A_2) \mod 2,$$
$$\int c_1(F_1) = \int w_2(A_3) \mod 3. \tag{6}$$

Here, $c_1(F_1) = (6F_1/2\pi)$ is the first Chern class, and w_2 is the second Stiefel-Whitney class [45]. The conditions in (6) imply that the transition functions of PSU(3), SO(3), and $\mathrm{U}(1)/\mathbb{Z}_6$ bundles are correlated in such a way that they consistently combine into a $[{\rm SU}(3)\times{\rm SU}(2)\times{\rm U}(1)_Y]/\mathbb{Z}_6$ bundle.

On spin manifolds, $\frac{1}{2}c_1(F_1)^2$ has integral periods, which gives the condition $n_1 \in (1/36)\mathbb{Z}$. Furthermore, the fractional parts of n_2 and n_3 are [30,44–48]

$$n_2 = \frac{1}{4}\int \mathcal{P}[w_2(A_2)] \mod 1,$$
$$n_3 = \frac{1}{3}\int w_2(A_3) \cup w_2(A_3) \mod 1, \tag{7}$$

from which we obtain $n_2 \in \frac{1}{2}\mathbb{Z}$ and $n_3 \in \frac{1}{3}\mathbb{Z}$. Here, $\mathcal{P}\colon H^2(X,\mathbb{Z}_2) \to H^4(X,\mathbb{Z}_4)$ is the Pontryagin square, which is reviewed, for instance, in [44], and the value of $\int \mathcal{P}[w_2(A_2)]$ is 0 or 2 mod 4 on spin manifolds. The correlated quantization conditions $n_3 - 24n_1 \in \mathbb{Z}$ and $n_2 - 18n_1 \in \mathbb{Z}$ are obtained by taking (Pontryagin) squares of the two sides of equations in (6) and then integrating them over the spacetime manifold. To obtain the quantization condition of K_i's, we can rewrite $K_3n_3 + K_2n_2 + K_1n_1$ as $K_3(n_3 - 24n_1) + K_2(n_2 - 18n_1) + (1/36)(K_1 + 18K_2 + 24K_3)(36n_1)$. We see that the exponentiated action e^{iS} is single valued if and only if the conditions for $q = 6$ in Table I are satisfied.

In some axion models, such as the Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) model [72,73], the axion coupling (3) below the EWSB scale is a valid description, but at higher energy scales the UV theory is different from the one we consider in (2). In such cases, our results on the quantization conditions of E and N are still valid.

Constraints on the effective axion-photon coupling $g_{a\gamma\gamma}$.—We now discuss model-independent constraints on the axion physics from the quantization of the axion couplings to the SM gauge fields. We assume that we have a single QCD axion whose mass m_a comes from the coupling to QCD. We focus on the effective axion-photon coupling $g_{a\gamma\gamma}$ defined as $\mathcal{L}_{a\gamma\gamma} = -\frac{1}{4} g_{a\gamma\gamma} a F_{\mathrm{EM},\mu\nu} \tilde{F}^{\mu\nu}_{\mathrm{EM}}$. Here, $F_{\mathrm{EM},\mu\nu}$ is the field strength for the canonically normalized EM gauge field, and $a = f\theta$ is the canonically normalized axion. The effective coupling $g_{a\gamma\gamma}$ receives contribution from the bare coupling E as well as the mixing with π^0 [74]:

$$g_{a\gamma\gamma} = \left[0.203(3)\frac{E}{N} - 0.39(1)\right]\frac{m_a}{\mathrm{GeV}^2}, \qquad (8)$$

where m_a is the axion mass.

The quantization condition in Table I implies that the value of $g_{a\gamma\gamma}/m_a$ cannot be an arbitrary real number but is subject to some rationality constraint from the ratio E/N. While any given real number can be approximated arbitrarily well by a rational number, an accurate approximation requires a large denominator N, which is related to the number of axion domain walls $N_{\mathrm{DW}} = 2N$. In the absence of other mechanisms, stable domain walls formed after inflation will survive till today and are inconsistent with current observations. Therefore, to avoid the axion domain wall problem in cosmology, the value of N_{DW} is preferred to be small in realistic, postinflationary axion models. For any given upper bound on N_{DW}, there is a strict lower bound on $|g_{a\gamma\gamma}|/m_a$ from the quantization conditions on E and N.

Another cosmological constraint on UV completions of the axion is the stable relic problem [75,76]. There are typically new heavy G_{SM} charged particles in UV models. If ever in thermal equilibrium after inflation, unless they can decay to SM states, these particles would freeze out and become exotic stable relics, which would be ruled out. Since all SM particles have trivial $\mathbb{Z}_6$ charge, in order for these heavy states to decay away, they must also have trivial $\mathbb{Z}_6$ charge. Therefore, the global form of the SM gauge group is preferred to have $q = 6$ in any postinflationary axion model to avoid exotic stable relics. This is also the gauge group that is compatible with various GUT models such as SU(5), Spin(10), and E_6. See, for example, [18] and references therein for an extensive discussion on the phenomenological criteria for axion models. For this reason, we focus on the $q = 6$ case below.

The quantization conditions in Table I for E and N in the $q = 6$ case are $4N + 3E \in 3\mathbb{Z}$, $N \in (1/2)\mathbb{Z}$, and $E \in (1/3)\mathbb{Z}$. If we want to avoid the domain wall problem, i.e., $N_{\mathrm{DW}} = 2N = 1$, then $g_{a\gamma\gamma}$ in (8) cannot be arbitrarily small because of the quantization condition. This leads to the following model-independent lower bound:

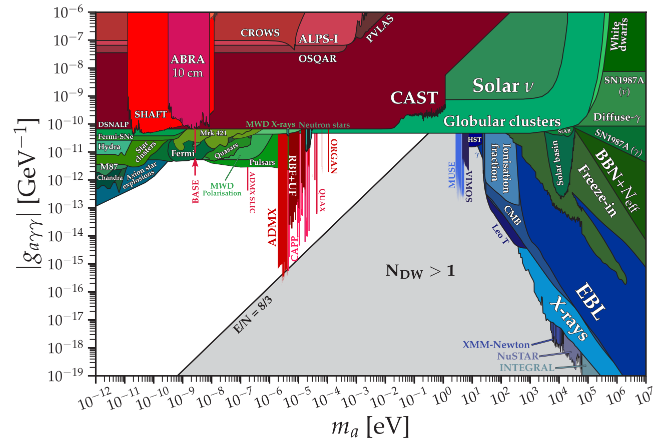

FIG. 1. Constraints on the effective axion-photon coupling $g_{a\gamma\gamma}$ versus the axion mass m_a, modified from [79]. The model-independent quantization conditions in Table I imply that any QCD axion model in the gray region below the $E/N = 8/3$ line necessarily faces the axion domain wall problem in a postinflationary scenario, i.e., $N_{\mathrm{DW}} > 1$.

$$\frac{|g_{a\gamma\gamma}|}{m_a} \geq 0.15(1)\ \mathrm{GeV}^{-2}, \quad \text{if } N_{\mathrm{DW}} = 1. \qquad (9)$$

The lower bound is saturated by

$$\frac{E}{N} = \frac{8}{3}, \qquad N_{\mathrm{DW}} = 2N = 1, \qquad E = \frac{4}{3}, \qquad (10)$$

which is the closest rational number E/N to 1.92 subject to the condition in Table I and $N_{\mathrm{DW}} = 1$. This ratio is famously realized by one of the classic DFSZ models [72,73]. (It is worthwhile noting that the standard DFSZ models have $N_{\mathrm{DW}} = 3$ or 6, while $N_{\mathrm{DW}} = 1$ can be achieved by relaxing PQ charge universality such as in [77].) It is also the ratio realized by the SU(5), Spin(10), and E_6 GUT models [42,78]. In other words, the region to the right of the $E/N = 8/3$ line on the $|g_{a\gamma\gamma}|$-m_a plot must necessarily face the domain wall problem in a postinflationary scenario (see Fig. 1). This provides an invariant meaning to the ratio $E/N = 8/3$ in the landscape of axion models.

There are many proposed solutions to the axion domain wall problems in the literature, so models with $N_{\mathrm{DW}} > 1$ are still potentially phenomenologically viable. Here, we merely point out that those QCD axion models violating (9) must necessarily have $N_{\mathrm{DW}} > 1$.

As we increase the allowed tolerance of N_{DW}, eventually the lower bound will be smaller than the uncertainty in (8) and cease to be meaningful. For instance, the lower bound with $N_{\mathrm{DW}} \leq 2$ is $(|g_{a\gamma\gamma}|/m_a) \geq 0.05(1)\ \mathrm{GeV}^{-2}$, which is saturated by $E/N = 5/3$. For other values of $q = 1, 2, 3$, the lower bounds on $|g_{a\gamma\gamma}|/m_a$ are instead saturated by $E/N = 35/18, 2, 13/6$, respectively. However, these lower bounds with $q \neq 6$ are of the same order as the uncertainty in (8).

TABLE II. Summary of the higher group for the center 1-form and winding 2-form symmetries and noninvertible 1-form symmetries of the SM coupled to an axion, written in terms of both (K_3, K_2, K_1) and (N, E). Whenever an entry in one of the columns is nonzero, there exists the corresponding symmetry. Here, $K \equiv \gcd(6, K_1) = \gcd(6, 36E)$. Note that K_1 and K are always integer multiples of q due to the quantization condition in Table I. The $q = 6$ case does not have these generalized global symmetries.

q	Higher group	Noninvertible
1	$[(24K_3 + 18K_2 + K_1)/K] \bmod K$	$K_1 \bmod 6$
	$[(48N + 36E)/K] \bmod K$	$36E \bmod 6$
2	$[(6K_3 + K_1)/(K/2)] \bmod (K/2)$	$(K_1/2) \bmod 3$
	$[(12N + 36E)/(K/2)] \bmod (K/2)$	$18E \bmod 3$
3	$[(2K_2 + K_1)/(K/3)] \bmod (K/3)$	$(K_1/3) \bmod 2$
	$[36E/(K/3)] \bmod (K/3)$	$12E \bmod 2$
6		

Higher symmetries of the standard model coupled to an axion.—The SM coupled to an axion enjoys a myriad of generalized global symmetries, which are summarized in Table II, and more detailed descriptions are presented in Supplemental Material [41].

The generalized global symmetries we discuss are typically emergent in the IR, and they are, therefore, broken as we go to higher energies. However, as a consequence of the higher-group and noninvertible symmetries, they are not all on the same footing. Sometimes a "child" symmetry G_C is subordinate to a "parent" symmetry G_P, in the sense that the former symmetry cannot exist without the latter. For instance, the child noninvertible symmetry G_C is typically tied to a parent invertible symmetry G_P, and the algebra of symmetry elements of G_C contains those of G_P. This hierarchical structure constrains the renormalization group flows, since it is not possible to have an effective field theory at an intermediate scale where the child symmetry is preserved while the parent one is broken. This gives universal inequalities on the energy scales where the symmetries become emergent. These inequalities take the form $E_C \lesssim E_P$, where E_C and E_P are the energy scales above which the child and parent symmetries are broken, respectively. Since the symmetries are emergent, the inequalities we derive are approximate.

Let us first consider the case where there is a nontrivial higher-group symmetry. There are two invertible symmetries of interest: the $\mathrm{U}(1)^{(2)}$ winding 2-form symmetry and the $\mathbb{Z}_{K/q}^{(1)}$ electric 1-form symmetry, where $K \equiv \gcd(6, K_1)$. We denote E_{winding} and E_{center} as the energy scales below which $\mathrm{U}(1)^{(2)}$ and $\mathbb{Z}_{K/q}^{(1)}$ become emergent, respectively. Now suppose we flow to a scale below E_{center}. Because of the higher-group symmetry structure, turning on the background gauge field for the $\mathbb{Z}_{K/q}^{(1)}$ electric symmetry automatically activates the background gauge field for the $\mathrm{U}(1)^{(2)}$ winding symmetry as well, which makes sense only if we are also below the scale E_{winding} [49]. We conclude that

$$E_{\mathrm{center}} \lesssim E_{\mathrm{winding}}. \tag{11}$$

When we have a noninvertible symmetry, we obtain stronger bounds. We denote the emergence scale for the noninvertible 1-form symmetry again as, by an abuse of notation, E_{center}, since it acts on the Wilson lines in the same way as the ordinary center 1-form symmetry. In addition, we also associate an emergence energy scale E_{magnetic} to the $\mathrm{U}(1)^{(1)}$ magnetic 1-form symmetry. The fusion algebra of the noninvertible 1-form symmetry operator explained in Supplemental Material [41] shows that it cannot exist without the winding 2-form symmetry and the magnetic 1-form symmetry. Therefore,

$$E_{\mathrm{center}} \lesssim \min\{E_{\mathrm{magnetic}}, E_{\mathrm{winding}}\}. \tag{12}$$

Let us now discuss the physical implications. E_{winding} physically corresponds to the energy scale at which axion strings become dynamical and may unwind. This is naturally associated with the string tension $E_{\mathrm{winding}} \sim \sqrt{T}$, although it may be quite a bit smaller than $\sqrt{T}$ depending on the UV theory [33].

E_{magnetic} is most naturally associated with the mass of the lightest hypercharge monopole $E_{\mathrm{magnetic}} \sim m_{\mathrm{monopole}}$, although, depending on the UV theory, it may again be significantly smaller. For example, if the dynamical monopole is an 't Hooft-Polyakov monopole associated with a UV non-Abelian gauge group $\mathcal{G}$, then the magnetic symmetry breaking scale is instead the Higgsing scale $v \sim E_{\mathrm{magnetic}}$ at which $\mathcal{G}$ is broken to $\mathrm{U}(1)_Y$, while $m_{\mathrm{monopole}} \sim v/g \gg v$ for a weakly coupled theory.

Similarly, E_{center} can be associated with the mass of the lightest particle charged under the center of the gauge group, which we denote as $E_{\mathrm{center}} \sim m_{\mathrm{center}}$. To be more precise, for the case of higher-group symmetry, m_{center} is the mass of the lightest particle charged under the $\mathbb{Z}_{K/q}$ subgroup of the center of the gauge group. For the case of noninvertible symmetry, m_{center} is the mass of the lightest particle charged under the $\mathbb{Z}_{6/q}$ center of the gauge group. In both cases, we refer to m_{center} loosely as the mass of the lightest "$\mathbb{Z}_6$-charged particle." We emphasize that such a particle can be either a new fundamental field with $\mathbb{Z}_6$ gauge charge or a solitonic excitation of the monopole and axion string dictated by the anomaly inflow.

Combining these interpretations, our inequalities imply

$$m_{\mathrm{center}} \lesssim \sqrt{T} \tag{13}$$

whenever the higher-group symmetry exists and

$$m_{\text{center}} \lesssim \min\left\{m_{\text{monopole}}, \sqrt{T}\right\} \quad (14)$$

whenever the noninvertible 1-form symmetry exists. In short, the tension of axion strings and the mass of hypercharge monopoles are generically bounded from below by the mass of the lightest particle charged under the center of the SM gauge group.

Conclusions.—The ambiguity in the global form of the SM gauge group generally has few experimentally observable effects and is not often discussed as a result. However, for an axion, we have shown that this global structure is of central importance in dictating the allowed values of its quantized couplings to the SM gauge fields, which has immediate phenomenological consequences. In particular, for $q = 6$—the phenomenologically preferred value from the perspective of GUTs and the nonobservation of cosmologically stable exotic relics—the smallest allowed effective coupling to photons $|g_{a\gamma\gamma}|$ for a postinflationary QCD axion with domain wall number $N_{\text{DW}} = 1$ is realized by the ratio $E/N = 8/3$, which should be taken as an experimental target. We have additionally shown that certain values of E and N admit higher-group and noninvertible symmetries, which result in model-independent constraints between the masses of $\mathbb{Z}_6$-charged particles, the masses of hypercharge magnetic monopoles, and the axion string tension.

There are several avenues for further study. We have considered the simplest case of a single axion coupled only to the SM gauge group. However, in recent years, there have been a number of axion models developed going beyond these minimal assumptions [18]. A clear next step would be to generalize these results to theories of multiple axions or extended gauge groups, which may have more complicated symmetry structures and quantization conditions. We have also neglected discussing the 0-form shift symmetry of the axion. However, it is known that the shift symmetry can also lead to a higher-group symmetry [31–33,50,51] or become noninvertible [25–27], which would then lead to additional inequalities. These may be relevant for axions that do not couple to QCD or for theories with multiple domain walls.

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