WHEELS: A NEW CRITERION FOR NON-CONVEXITY OF NEURAL CODES

ALEXANDER RUYS DE PEREZ, LAURA FELICIA MATUSEVICH, AND ANNE SHIU

Abstract. We introduce new geometric and combinatorial criteria that preclude a neural code from being convex, and use them to tackle the classification problem for codes on six neurons. Along the way, we give the first example of a code that is non-convex, has no local obstructions, and has simplicial complex of dimension two. We also characterize convexity for neural codes for which the simplicial complex is pure of low or high dimension.

1. Introduction

A *neural code* on n neurons is a subset $\mathcal{C} \subseteq 2^{[n]}$, where $[n] = \{1, 2, \dots, n\}$. Of interest are neural codes that describe the regions of a configuration of convex open sets $U_1, U_2, \dots, U_n$ in Euclidean space. Such *convex codes* arise from neurons in the hippocampus, specifically, *place cells*. In this case, each region U_i is called a *place field* and represents the subset of an animal's environment where the corresponding neuron fires. Convex codes, therefore, allow the brain to map out an environment. Place cells were found by John O'Keefe in 1971 [21], for which he was awarded a joint (with May-Britt Moser and Edvard Moser) Nobel Prize in Physiology or Medicine in 2014.

The central open question in the mathematical theory of neural codes is to determine which codes are convex. Some families of codes are known to be convex [5, 6, 15, 24], and convex codes on up to 5 neurons are well characterized [6, 7, 9]. As for precluding convexity, the main tool is a combinatorial criterion known as a local obstruction [6, 8] and its generalizations [4, 14].

Possessing a local obstruction is sufficient, but not necessary, for a code to be non-convex [18]. Accordingly, this article introduces a new tool for ruling out convexity. We show that this obstruction – which we call a wheel – captures all non-convex codes (with no local obstructions) on up to 5 neurons (Theorem 3.8). Additionally, we give combinatorial criteria for possessing a wheel, which we use to classify many codes on 6 neurons.

In our classification, a crucial step is to eliminate codes that are essentially equivalent to codes on fewer neurons. These include *decomposable* codes for which, at least intuitively, a realization is obtained by placing a realization of a smaller *embedded code* within an *ambient code*. The key result is that a decomposable code on up to 6 neurons is convex if and only if its embedded and ambient codes are convex (Theorem 6.6). The proof relies on prior classifications of codes on up to 5 neurons [6, 7, 9] as well as results on *nondegenerate convexity* due to Cruz *et al.* [5].

We also prove results pertaining to a code and the dimension of its simplicial complex. We present a code that is non-convex, has no local obstructions, and has simplicial complex of dimension two (Proposition 4.13). This example answers a question posed by Chen, Frick, and Shiu [4]. Finally, we characterize convexity for neural codes for which the simplicial complex is pure of low or high dimension (Theorem 8.2).

The outline of this article is as follows. We begin with background material on neural codes in Section 2. In Section 3, we introduce wheels, and then give combinatorial criteria for them in

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Section 4. Next, Sections 5 and 6 – which the reader might wish to skip on a first reading – contain results that allow us to search computationally for wheels, and explain how codes can be decomposed into smaller codes. These results are then used in Section 7 to obtain a partial classification of codes on 6 neurons. Section 8 presents our results on codes with simplicial complexes that are pure, and we end with a discussion in Section 9.

2. Preliminaries

In this section we introduce basic notions and review necessary background.

A *code on n neurons* is a subset $\mathcal{C} \subseteq 2^{[n]}$, where $[n] = \{1, 2, \ldots, n\}$. Elements of a code $\mathcal{C}$ are called *codewords*, and a *maximal codeword* is a codeword in $\mathcal{C}$ that is maximal with respect to inclusion.

Assumption 2.1. All codes in this article are assumed to contain the empty codeword $\varnothing$. Whether or not a code contains $\varnothing$ does not affect convexity, defined below [4, Remark 2.19].

A *realization* of a code $\mathcal{C}$ in $\mathbb{R}^d$ is a collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of open subsets of a *stimulus space* $X \subseteq \mathbb{R}^d$ such that $c \in \mathcal{C}$ if and only if $(\cap_{i\in c} U_i) \smallsetminus (\cup_{j\in[n]\smallsetminus c} U_j) \neq \varnothing$. The set $(\cap_{i\in c} U_i) \smallsetminus (\cup_{j\in[n]\smallsetminus c} U_j)$ is the *atom of c in the realization $\mathcal{U}$*. We denote $U_\sigma := \cap_{i\in\sigma} U_i$, and, by convention, $U_\varnothing := X$. If the sets $U_1, U_2, \ldots, U_n$ are convex, then $\mathcal{U}$ is a *convex realization* of $\mathcal{C}$. Codes that possess convex realizations are known as *convex codes.* As we are assuming that all codes contain $\varnothing$, we can always take $X = \mathbb{R}^d$ as the stimulus space; cf. [4, Remark 2.19].

In what follows, we write a codeword without the braces, e.g., "123" rather than "$\{1, 2, 3\}$". We also indicate maximal codewords by boldface.

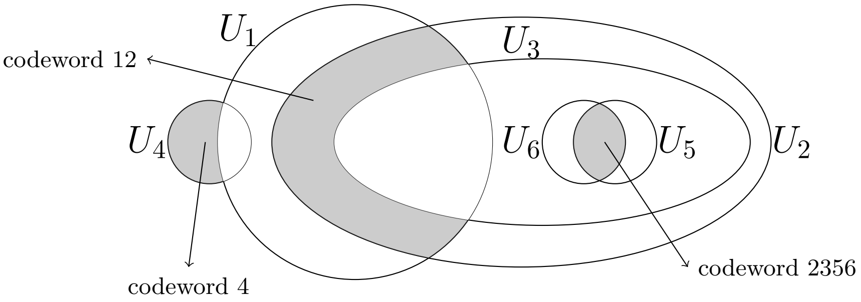


Figure 1. Displayed is a realization of the code in Example 2.2; each set U_i is the open region inside the labeled ellipse. The atoms of three codewords are shaded.

Example 2.2. The code $\mathcal{C} = \{\mathbf{2356}, \mathbf{123}, \mathbf{14}, 235, 236, 12, 23, 1, 2, 4, \varnothing\}$ on 6 neurons is convex, as shown by the convex realization depicted in Figure 1.

Given a code $\mathcal{C}$ on n neurons and a subset $\sigma \subseteq [n]$, the *trunk* of σ in $\mathcal{C}$, denoted by $\mathrm{Tk}_{\mathcal{C}}(\sigma)$, is the set of all codewords containing σ:

$$\mathrm{Tk}_{\mathcal{C}}(\sigma) := \{c \in \mathcal{C} \mid \sigma \subseteq c\} \ .$$

Next, every code $\mathcal{C}$ on n neurons gives rise to a simplicial complex on $[n]$, called the *neural complex* of $\mathcal{C}$ (or the *simplicial complex of* $\mathcal{C}$), and denoted by $\Delta(\mathcal{C})$, as follows:

$$\Delta(\mathcal{C}) := \{\sigma \in 2^{[n]} \mid \sigma \subseteq c \text{ for some } c \in \mathcal{C}\}.$$

Note that the facets of $\Delta(\mathcal{C})$ are the maximal codewords of $\mathcal{C}$. If σ is a face of a simplicial complex Δ, we write $\dim(\sigma) := |\sigma| - 1$, and $\dim(\Delta) := \max\{\dim(\sigma) \mid \sigma \in \Delta\}$, where dim stands for *dimension.*

We remark that if $\mathcal{U}$ is any realization of a code $\mathcal{C}$, then $\Delta(\mathcal{C})$ is the *nerve* of $\mathcal{U}$. By definition, this means that, for $\sigma \in 2^{[n]}$,

$$\sigma \in \Delta(\mathcal{C}) \quad \text{if and only if} \quad \bigcap_{i \in \sigma} U_i \neq \varnothing.$$

As $\Delta(\mathcal{C})$ is simultaneously the nerve of all realizations of $\mathcal{C}$, it captures important topological and geometric characteristics of $\mathcal{C}$.

It has been a fruitful approach to explore convexity of neural codes based on a study of the associated neural complexes. The key notion is that of a *local obstruction*, which is given below. Codes with local obstructions are known to be *not* convex (Proposition 2.5). Local obstructions have been generalized recently [4, 14], but for simplicity we do not present these generalizations here.

Definition 2.3. Let $\mathcal{C}$ be a code on n neurons, and assume that $\mathcal{C}$ is realized by a collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of open subsets of some $\mathbb{R}^d$. A *local obstruction* of $\mathcal{C}$ is a pair (σ, τ) of nonempty, disjoint subsets of $[n]$ such that

$$U_\sigma \ \subseteq \ \bigcup_{j \in \tau} U_j$$

and the link $\text{Lk}_\sigma(\Delta(\mathcal{C})|_{\sigma \cup \tau})$ is not contractible.

Links are usually written as $\text{Lk}_\Delta(\sigma)$, rather than $\text{Lk}_\sigma(\Delta)$; however, we follow the notation of [6].

Checking for local obstructions directly from the definition implies analyzing not just $\Delta(\mathcal{C})$ but its restriction to every subset of the vertices. However, obstructions can always be detected at the level of $\Delta(\mathcal{C})$, a phenomenon we call *bubbling up* (see Proposition 2.5(2) below). We first introduce some necessary terminology.

Definition 2.4. Let Δ be a simplicial complex on $[n]$, and let $\sigma \in \Delta$.

(i) σ is a *max-intersection face* of Δ if it is the intersection of two or more facets of Δ.
(ii) σ is a *mandatory face* of Δ if $\sigma \neq \varnothing$ and the link $\text{Lk}_\sigma(\Delta)$ is not contractible.

Let $\mathcal{C}$ be a code on n neurons.

(i) A *max-intersection face* of $\mathcal{C}$ is a max-intersection face of $\Delta(\mathcal{C})$. We denote by $\text{Max}_\cap(\Delta(\mathcal{C}))$ the collection of all such faces. The code $\mathcal{C}$ is *max-intersection-complete* if $\text{Max}_\cap(\Delta(\mathcal{C})) \subseteq \mathcal{C}$.
(ii) A *mandatory face* of $\mathcal{C}$ is a mandatory face of $\Delta(\mathcal{C})$. We denote by $\text{Man}(\Delta(\mathcal{C}))$ the collection of all such faces.

The following result summarizes several prior results on convexity (see [5, Theorem 1.2], [6, Lemma 1.4, Theorem 1.3, Proposition 1.7, Supplementary-Information Lemma 6.2], and [8, Theorem 5]).

Proposition 2.5. *Let $\mathcal{C}$ be a code on n neurons.*

(1) Every mandatory face of $\mathcal{C}$ is the intersection of two or more maximal codewords of $\mathcal{C}$ (that is, $Man(\Delta(\mathcal{C})) \subseteq Max_\cap(\Delta(\mathcal{C}))$).
(2) $\mathcal{C}$ is max-intersection-complete (i.e., $Max_\cap(\Delta(\mathcal{C})) \subseteq \mathcal{C}$) $\Rightarrow$ $\mathcal{C}$ is convex $\Rightarrow$ $\mathcal{C}$ has no local obstructions $\Leftrightarrow$ $\mathcal{C}$ contains all mandatory faces of $\mathcal{C}$ (i.e., $Man(\Delta(\mathcal{C})) \subseteq \mathcal{C}$).
(3) If $n \leq 4$, then both implications in (2) are equivalences.

It follows that the *minimal code* of a simplicial complex Δ, defined as follows:

$$\mathcal{C}_{\min}(\Delta) \ := \ \{\text{facets of } \Delta\} \cup \{\text{mandatory faces of } \Delta\} \cup \{\varnothing\} \ , \tag{1}$$

is the unique minimal (with respect to inclusion) code among all codes with neural complex Δ and no local obstructions.

But what of those codes that contain the mandatory faces but not all the max-intersection faces? The next examples show that such codes may or may not be convex. Indeed, we see that two implications in Proposition 2.5(2) are, in general, *not* equivalences.

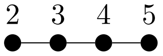
Example 2.6 (A code that is convex but not max-intersection-complete)**.** The following neural code is not max-intersection-complete ($1 = 123 \cap 134 \cap 145$ is missing):

$$\mathcal{C} = \{\mathbf{123}, \mathbf{134}, \mathbf{145}, 13, 14, \varnothing\} .$$

However, $\mathcal{C}$ is convex, as shown in [6, Figure 3c].

Example 2.7 (A non-convex code with no local obstructions)**.** The following neural code was the first example of a non-convex code with no local obstructions [18, Theorem 3.1]:

$$\mathcal{C}^\star = \{\mathbf{2345}, \mathbf{123}, \mathbf{134}, \mathbf{145}, 13, 14, 23, 34, 45, 3, 4, \varnothing\} .$$

To see that $\mathcal{C}^\star$ indeed has no local obstructions, first observe that the intersections of the maximal codewords are 23, 34, 45, 13, 14, 1, 3, 4, and $\varnothing$, each of which – except 1 – is a codeword of $\mathcal{C}^\star$. However, 1 is not mandatory, as the neural complex has facets 2345, 123, 134, and 145, and so the link of 1 in $\Delta(\mathcal{C}^\star)$ is the following path graph, which is contractible: 

3. Wheels

The following definition abstracts the type of non-convexity appearing in the code in Example 2.7.

Definition 3.1. Let $\mathcal{C}$ be a code on n neurons, and let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a realization of $\mathcal{C}$. A tuple $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (2^{[n]})^4$ is a *wheel of the realization* $\mathcal{U}$ if it satisfies:

W(i): $U_{\sigma_j} \cap U_{\sigma_k} = U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3} \neq \varnothing$ for all $1 \leq j < k \leq 3$,
W(ii): $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3} \cap U_\tau = \varnothing$,
W(iii): if U_τ and $U_{\sigma_j} \cap U_\tau$ are convex for $j = 1, 2, 3$, then there exists a line segment whose endpoints lie one in $U_{\sigma_1} \cap U_\tau$ and the other in $U_{\sigma_3} \cap U_\tau$, that also meets $U_{\sigma_2} \cap U_\tau$.

The sets σ_1, σ_2, and σ_3 are the *spokes* and τ is the *rim*. Also, $\mathcal{W}$ is a *wheel of the code* $\mathcal{C}$ if it is a wheel of every realization of $\mathcal{C}$.

We remark that W(iii) is related to but does not imply (see Remark 3.2) the following condition:

W(iii)$_\circ$: $U_{\sigma_j} \cap U_\tau \neq \varnothing$ for $j = 1, 2, 3$.

For brevity, we say that $\mathcal{U}$ (or $\mathcal{C}$) *has a wheel* if there exists a wheel of $\mathcal{U}$ (respectively, $\mathcal{C}$).

Remark 3.2 (Condition W(iii)$_\circ$)**.** Some wheels do not satisfy W(iii)$_\circ$ (see Example 4.17 below). However, requiring this condition – which, unlike condition W(iii), is easy to check – often makes it easier to work with wheels (see the next section).

Remark 3.3. In Definition 3.1, the spoke regions U_{σ_1} and U_{σ_3} seem to play a different role than U_{σ_2}, but this is not really the case. A more symmetric way of stating W(iii) would be to ask for a line segment that intersects all three sets $U_{\sigma_j} \cap U_\tau$, for $j = 1, 2, 3$. We have adopted the current numbering convention to simplify the writing in some of the proofs below.

In the next section, we show that every realization of the code $\mathcal{C}^\star$ from Example 2.7 has a wheel (see Proposition 4.16).

Wheels are relevant because they forbid convexity:

Theorem 3.4. *Let* $\mathcal{U}$ *be a realization of a neural code* $\mathcal{C}$*. If* $\mathcal{U}$ *has a wheel, then* $\mathcal{U}$ *is not a convex realization. Consequently, if* $\mathcal{C}$ *has a wheel, then* $\mathcal{C}$ *is non-convex.*

Proof. Let $(\sigma_1, \sigma_2, \sigma_3, \tau)$ be a wheel of the realization $\mathcal{U} = \{U_i\}_{i=1}^n$ of $\mathcal{C}$ in $\mathbb{R}^d$, and assume for contradiction that $\mathcal{U}$ is convex. Let L be the line segment from W(iii). Since U_τ is convex, $L \subseteq U_\tau$. The sets U_{σ_i}, for $i = 1, 2, 3$, are convex and open. Condition W(i) implies that L intersects $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3}$ by [18, Lemma 3.2]. As $L \subseteq U_\tau$, it follows that $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3} \cap U_\tau \neq \varnothing$, which contradicts W(ii). □

The geometric intuition behind the proof of Theorem 3.4 (and the reason for our chosen terminology) is that in a realization that has a wheel $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$, the U_{σ_i}'s force U_τ to be non-convex by bending U_τ around their intersection $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3}$. See Figure 2.

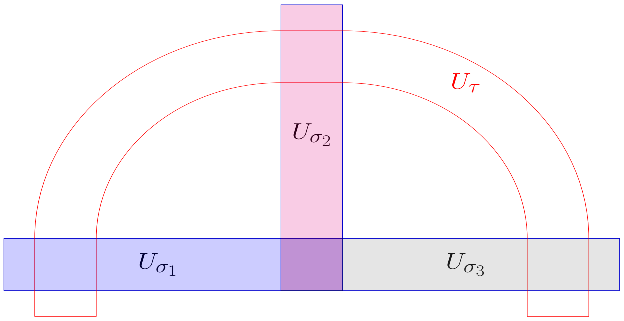


FIGURE 2. Depicted is a conceptual picture of a wheel, with intuition as follows: If U_{σ_1}, U_{σ_2}, and U_{σ_3} are convex, then the region of the rim, U_τ, "bends around" the intersection of the spoke regions, $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3}$, and so is non-convex.

Remark 3.5 (Open versus closed sets)**.** Theorem 3.4 shows that wheels prevent codes from being realized by convex, open sets. However, some codes with wheels can be realized by convex, closed sets. Such a realization is shown in [5] for the code in Example 2.7.

Remark 3.6 (Relation to sunflowers)**.** Wheels are closely related to ideas in recent work of Jeffs [10]. As a start, in a realization $\mathcal{U}$ that has a wheel (or at least satisfies W(i)), the sets U_{σ_1}, U_{σ_2}, and U_{σ_3} form what Jeffs calls a *3-petal sunflower*. Jeffs also uses sunflowers to construct an infinite family of non-convex codes $\mathcal{C}_2, \mathcal{C}_3, \ldots$ [10, Definition 4.1]. We show in Example 3.7 that $\mathcal{C}_2$ contains a wheel. The codes $\mathcal{C}_m$, for $m \geq 3$, have sunflowers with $m+1$ petals. The concept of a wheel has not yet been generalized to such higher-dimensional cases, where, instead of a line intersecting the 3 petals of a sunflower, a $(d-1)$-dimensional affine space intersects $d+1$ petals (cf. [10, Theorem 1.1]). Indeed, we checked that the codes $\mathcal{C}_m$, for small $m \geq 3$, do not have the combinatorial wheels introduced in the next section. Moreover, we expect that all $\mathcal{C}_m$, for $m \geq 3$, lack wheels; however, checking condition W(iii) is difficult. Finally, we note that, unlike prior work on using sunflowers to preclude convexity, our work provides algorithms (see Procedure 7.1) for doing so.

Example 3.7. The following neural code was introduced by Jeffs [10, Definition 4.1]:

$$\mathcal{C}_2 \ = \ \{\mathbf{1236}, \mathbf{234}, \mathbf{135}, \mathbf{456}, 13, 23, 4, 5, 6, \varnothing\} \ .$$

This code is non-convex [10, Theorem 4.2] and, among all non-convex codes, is minimal with respect to a certain partial order (relabeling the neurons via the permutation $(1,4,2,6,3,5)$ yields the code $\mathcal{C}_0$ in [11, Theorem 5.10]). We will see in Example 4.6 that $\mathcal{C}_2$ contains a wheel.

Finally, we highlight that wheels, together with local obstructions, completely characterize non-convexity in codes on up to 5 neurons (cf. [9, Theorem 3.1]):

Theorem 3.8. *A code $\mathcal{C}$ on up to 5 neurons is convex if and only if $\mathcal{C}$ has no local obstructions and no wheel frames.*

A "wheel frame", which is defined in the next section, is a combinatorial object whose presence implies the existence of wheels (Theorem 4.15). We prove Theorem 3.8 at the end of Section 6.

4. Combinatorics of wheels

We introduced wheels using realizations (Definition 3.1), because they are convenient for proving non-convexity. That being said, the true goal of this article is to obtain non-convexity criteria in terms of $\mathcal{C}$ and $\Delta(\mathcal{C})$, in a similar way to how the mandatory faces of $\Delta(\mathcal{C})$ determine whether $\mathcal{C}$ has local obstructions (recall Proposition 2.5(2)).

It turns out that conditions W(i), W(ii), and W(iii)$_\circ$ from Definition 3.1 can be restated combinatorially, and depend only on the code $\mathcal{C}$ and not on the specific realization $\mathcal{U}$ (Proposition 4.3 below). The geometric condition W(iii), however, is more subtle. Indeed, it is an open problem to recast W(iii) completely in combinatorial terms, or show that no such characterization exists. Nevertheless, we are able to provide combinatorial criteria that imply the existence of wheels in every realization of $\mathcal{C}$ (see Section 4.2).

4.1. Combinatorial versions of W(i), W(ii), and W(iii)$_\circ$.

We start with a useful technical result.

Lemma 4.1. *Consider a code $\mathcal{C}$ on n neurons, a realization $\mathcal{U} = \{U_i\}_{i=1}^n$ of $\mathcal{C}$, and subsets $\varphi, \psi_1, \psi_2, \dots, \psi_r \subseteq [n]$. Then $U_\varphi \subseteq \bigcup_{t=1}^r U_{\psi_t}$ if and only if $\mathrm{Tk}_\mathcal{C}(\varphi) \subseteq \bigcup_{t=1}^r \mathrm{Tk}_\mathcal{C}(\psi_t)$.*

Proof. ($\Rightarrow$) Suppose $U_\varphi \subseteq \bigcup_{t=1}^r U_{\psi_t}$. Let $c \in \mathrm{Tk}_\mathcal{C}(\varphi)$, that is, $\varphi \subseteq c \in \mathcal{C}$. (We must show that $\psi_s \subseteq c$ for some $1 \le s \le r$.) The containment $\varphi \subseteq c$ yields the first containment below:

$$U_c \subseteq U_\varphi \subseteq \bigcup_{i=1}^r U_{\psi_t} \, , \tag{2}$$

and the second containment is by hypothesis. Next, $c \in \mathcal{C}$ implies that $U_c \smallsetminus (\bigcup_{i \notin c} U_i) \neq \varnothing$. Combining this inequality with (2) yields $U_{\psi_s} \smallsetminus (\bigcup_{i \notin c} U_i) \neq \varnothing$ for some s, which implies $\psi_s \subseteq c$.

($\Leftarrow$) Assume that $\mathrm{Tk}_\mathcal{C}(\varphi) \subseteq \bigcup_{t=1}^r \mathrm{Tk}_\mathcal{C}(\psi_t)$, that is, $\varphi \subseteq c \in \mathcal{C}$ implies that $\psi_s \subseteq c$ for some s. Let $p \in U_\varphi$; we must show that $p \in U_{\psi_s}$ for some s. Define $c = \{i \in [n] \mid p \in U_i\}$. By construction, $p \in U_c \smallsetminus \bigcup_{i \notin c} U_i$; so, $c \in \mathcal{C}$. Also by construction (of p and c), $\varphi \subseteq c$. Hence, by hypothesis, there exists s such that $\psi_s \subseteq c$. So, $p \in U_i$ for all $i \in \psi_s$. In other words, $p \in U_{\psi_s}$. □

We are now ready to combinatorially recast part of Definition 3.1.

Definition 4.2. A tuple $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (\Delta(\mathcal{C}))^4$ is a *partial-wheel* of a neural code $\mathcal{C}$ if it satisfies the following conditions:

P(i): $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in \Delta(\mathcal{C})$, and $\mathrm{Tk}_\mathcal{C}(\sigma_j \cup \sigma_k) = \mathrm{Tk}_\mathcal{C}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$ for every $1 \le j < k \le 3$,
P(ii): $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \notin \Delta(\mathcal{C})$, and
P(iii)$_\circ$: $\sigma_j \cup \tau \in \Delta(\mathcal{C})$ for $j = 1, 2, 3$.

Proposition 4.3 (Equivalence of W(i) and P(i), etc.)**.** *Let $\mathcal{C}$ be a neural code, and let $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (\Delta(\mathcal{C}))^4$. Using the notation from Definition 4.2, the following are equivalent:*

1. *$\mathcal{W}$ satisfies W(i) (respectively W(ii), W(iii)$_\circ$) for some realization $\mathcal{U}$ of $\mathcal{C}$,*
2. *$\mathcal{W}$ satisfies W(i) (respectively W(ii), W(iii)$_\circ$) for all realizations $\mathcal{U}$ of $\mathcal{C}$, and*

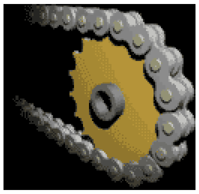


FIGURE 3. A sprocket. This image is a still of an animation of the point of contact between a chain and a sprocket, which was produced by the US Department of Labor and is in the public domain [2]. Conceptually, the witnesses ρ_1 and ρ_3 fill the role of aligning the spokes σ_1, σ_2, and σ_3, similar to how the links of the chain keep the physical sprocket in place.

(3) $\mathcal{W}$ satisfies P(i) (respectively P(ii), P(iii)$_\circ$).

Proof. The fact that $\Delta(\mathcal{C})$ is the nerve of every realization of $\mathcal{C}$ directly implies the equivalences involving W(ii) and W(iii)$_\circ$. For the equivalences involving W(i), use Lemma 4.1 with $r = 1$, $\varphi = (\sigma_i \cup \sigma_j)$, and $\psi = (\sigma_1 \cup \sigma_2 \cup \sigma_3)$. □

4.2. **Combinatorial wheels.** Our next goal is to give (combinatorial) criteria in terms of $\mathcal{C}$ and $\Delta(\mathcal{C})$ that imply the existence of wheels (and therefore non-convexity of $\mathcal{C}$). We introduce three such criteria, and we call the resulting wheel a sprocket, wire wheel, or wheel frame (see Propositions 4.5, 4.10, and Theorem 4.15).

Definition 4.4. A *sprocket* of a neural code $\mathcal{C}$ is a partial-wheel (Definition 4.2) $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ of $\mathcal{C}$ that in addition satisfies:

S(iii): There exist ρ_1, $\rho_3 \in \Delta(\mathcal{C})$ such that:
S(iii)(1): $\mathrm{Tk}_{\mathcal{C}}(\sigma_j \cup \tau) \subseteq \mathrm{Tk}_{\mathcal{C}}(\rho_j)$ for $j = 1, 3$,
S(iii)(2): $\mathrm{Tk}_{\mathcal{C}}(\tau) \subseteq \mathrm{Tk}_{\mathcal{C}}(\rho_1) \cup \mathrm{Tk}_{\mathcal{C}}(\rho_3)$, and
S(iii)(3): $\mathrm{Tk}_{\mathcal{C}}(\rho_1 \cup \rho_3 \cup \tau) \subseteq \mathrm{Tk}_{\mathcal{C}}(\sigma_2)$.

The sets ρ_1 and ρ_3 are *witnesses* for the sprocket.

To explain the terminology, a sprocket is a toothed wheel, such as the gear wheel on a bicycle (see Figure 3); we imagine the sets U_{ρ_1} and U_{ρ_3} as overlapping links in a roller chain set on a sprocket.

The following result shows that sprockets are wheels.

Proposition 4.5 (Every sprocket is a wheel). *Let $\mathcal{C}$ be a code on n neurons, and assume that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (2^{[n]})^4$ satisfies W(iii)$_\circ$ (or equivalently P(iii)$_\circ$). If $\mathcal{W}$ satisfies condition S(iii) from Definition 4.4, then $\mathcal{W}$ satisfies W(iii). In particular, if $\mathcal{W}$ is a sprocket of $\mathcal{C}$, then $\mathcal{W}$ is a wheel of $\mathcal{C}$. Consequently, codes with sprockets are non-convex.*

Proof. We must show that S(iii) and P(iii)$_\circ$ together imply W(iii). Using Lemma 4.1, we see that S(iii) is equivalent to the following condition in terms of realizations ("G" here stands for "geometric"):

G(iii): There exist $\rho_1, \rho_3 \subseteq [n]$ such that in every realization $\mathcal{U} = \{U_i\}_{i=1}^n$ of $\mathcal{C}$,
G(iii)(1): $U_{\sigma_j} \cap U_\tau \subseteq U_{\rho_j}$ for $j = 1, 3$,
G(iii)(2): $U_\tau \subseteq U_{\rho_1} \cup U_{\rho_3}$, and
G(iii)(3): $U_{\rho_1} \cap U_{\rho_3} \cap U_\tau \subseteq U_{\sigma_2}$.

To complete the proof, we now show that G(iii) and P(iii)$_\circ$ imply W(iii).

Let $\mathcal{U}$ be a realization of $\mathcal{C}$ such that U_τ and $U_\tau \cap U_{\sigma_j}$ are convex for $j = 1, 2, 3$. Then, for $j = 1, 3$, condition P(iii)$_\circ$ and the fact that $\Delta(\mathcal{C})$ is the nerve of $\mathcal{U}$ imply the inequality here:

$$\varnothing \ \neq \ (U_{\sigma_j} \cap U_\tau) \ \subseteq \ U_{\rho_j} \ ,$$

and the containment follows from G(iii)(1). So, for $j = 1, 3$, there exists $p_j \in U_{\sigma_j} \cap U_\tau \subseteq U_{\rho_j}$. As p_1 and p_3 are in the convex set U_τ, so too is the line segment, denoted by L, between p_1 and p_3:

$$L \ \subseteq \ U_\tau \ \subseteq \ (U_{\rho_1} \cup U_{\rho_3}) \ , \tag{3}$$

and the second containment is by G(iii)(2). Thus, the (connected) set L is covered by two nonempty sets $U_{\rho_1} \cap L$ and $U_{\rho_3} \cap L$, which are open in the subspace topology of L. So,

$$\varnothing \ \neq \ (U_{\rho_1} \cap U_{\rho_3} \cap L) \ \subseteq \ (U_{\rho_1} \cap U_{\rho_3} \cap U_\tau) \ \subseteq \ U_{\sigma_2} \ , \tag{4}$$

where the containments are by (3) and G(iii)(3), respectively. Now it follows from (3) and (4) that L meets $U_{\sigma_2} \cap U_\tau$, and so W(iii) holds. □

In the previous proof, we showed that *every* line segment whose endpoints are one in $U_{\sigma_1} \cap U_\tau$ and the other in $U_{\sigma_3} \cap U_\tau$, meets $U_{\sigma_2} \cap U_\tau$. This is, on its face, stronger than W(iii), which requires the existence of only one such line segment.

Example 4.6 (A code with a sprocket)**.** Recall the code $\mathcal{C}_2 = \{\mathbf{1236}, \mathbf{234}, \mathbf{135}, \mathbf{456}, 13, 23, 4, 5, 6, \varnothing\}$ from Example 3.7. We show that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) = (5, 6, 4, 3)$ is a sprocket, with witnesses $\rho_1 = 13$ and $\rho_3 = 23$. First, $\sigma_1 \cup \sigma_2 \cup \sigma_3 = 456 \in \Delta(\mathcal{C})$, and $\mathrm{Tk}_{\mathcal{C}}(\sigma_i \cup \sigma_j) = \{456\} = \mathrm{Tk}_{\mathcal{C}}(456)$ for all $1 \leq i < j \leq 3$. So P(i) holds. Next, $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau = 3456 \notin \Delta(\mathcal{C})$, so P(ii) is satisfied. Also, $\sigma_1 \cup \tau = 35$, $\sigma_2 \cup \tau = 36$, and $\sigma_3 \cup \tau = 34$ are all faces of $\Delta(\mathcal{C})$; this verifies P(iii)$_\circ$. Next, $\mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \tau) = \mathrm{Tk}_{\mathcal{C}}(35) = \{135\} \subseteq \{1236, 13, 135\} = \mathrm{Tk}_{\mathcal{C}}(13) = \mathrm{Tk}_{\mathcal{C}}(\rho_1)$ and $\mathrm{Tk}_{\mathcal{C}}(\sigma_2 \cup \tau) = \mathrm{Tk}_{\mathcal{C}}(34) = \{234\} \subseteq \{23, 235\} = \mathrm{Tk}_{\mathcal{C}}(23) = \mathrm{Tk}_{\mathcal{C}}(\rho_2)$, which verifies S(iii)(1). Checking S(iii)(2) and S(iii)(3) is similarly straightforward.

However, not every wheel is a sprocket.

Example 4.7 (A wheel that is not a sprocket)**.** Consider the neural code

$$\mathcal{C}_{\mathrm{TL}} = \{\mathbf{123}, \mathbf{145}, \mathbf{245}, \mathbf{246}, \mathbf{346}, 24, 45, 46, 1, 2, 3, \varnothing\} \ ,$$

and consider $\mathcal{W}_{\mathrm{TL}} = (\sigma_1, \sigma_2, \sigma_3, \tau) = (1, 2, 3, 4)$. Here, "TL" stands for "tree link", and the meaning will be apparent in the proof of Proposition 4.13, where we show that $\mathcal{W}_{\mathrm{TL}}$ is a wheel. For now, we show that $\mathcal{W}_{\mathrm{TL}}$ is not a sprocket by proving there is no eligible pair ρ_1 and ρ_3.

Suppose for contradiction that ρ_1 and ρ_3 satisfy S(iii). The codewords 145 and 346 contain, respectively, $\sigma_1 \cup \tau = 14$ and $\sigma_3 \cup \tau = 34$, so S(iii)(1) implies that $\rho_1 \subseteq 145$ and $\rho_3 \subseteq 346$. Next, 24 is a codeword that contains $\tau = 4$, so, by S(iii)(2), we have $\rho_1 \subseteq 24$ or $\rho_3 \subseteq 24$. The above constraints imply that $\rho_1 \subseteq \{4\}$ or $\rho_3 \subseteq \{4\}$. Hence, $\rho_1 \cup \rho_3 \cup \tau = \rho_1 \cup \rho_3 \cup \{4\}$ is a subset of 145 or 346, and so $\mathrm{Tk}_{\mathcal{C}}(\rho_1 \cup \rho_3 \cup \tau)$ contains 145 or 346. However, neither 145 nor 346 is in $\mathrm{Tk}_{\mathcal{C}}(\{2\}) = \mathrm{Tk}_{\mathcal{C}}(\sigma_2)$, contradicting S(iii)(3). Thus, $\mathcal{W}_{\mathrm{TL}}$ is not a sprocket of $\mathcal{C}_{\mathrm{TL}}$.

As noted in the above example, we will show that the code $\mathcal{C}_{\mathrm{TL}}$ has a wheel. To do so, we need the following definition.

Definition 4.8. Let $\mathcal{C}$ be a neural code. A tuple $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) \in (\Delta(\mathcal{C}))^4$ is a *wire wheel* of $\mathcal{C}$ if the following hold:

- $\mathcal{W}$ satisfies P(i) and P(ii),
- $\tau \notin \mathcal{C}$,

- $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ is a tree,
- for $i = 1, 2, 3$, the set $\sigma_i \smallsetminus \tau$ has size one and is a vertex of the tree $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$,
- the unique path in the tree $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ between $\sigma_1 \smallsetminus \tau$ and $\sigma_3 \smallsetminus \tau$ contains $\sigma_2 \smallsetminus \tau$.

The intuition behind this name is that wire wheels have thin spokes.

To prove that wire wheels are wheels, we need the following lemma.

Lemma 4.9. *If $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wire wheel of a neural code $\mathcal{C}$, then the sets $\sigma_1 \smallsetminus \tau$, $\sigma_2 \smallsetminus \tau$, and $\sigma_3 \smallsetminus \tau$ are distinct.*

Proof. Assume for contradiction that $(\sigma_j \smallsetminus \tau) = (\sigma_k \smallsetminus \tau)$, for some $1 \leq j < k \leq 3$. It follows that $\sigma_j \cup \tau = \sigma_k \cup \tau$. We also know that $\sigma_j \cup \tau \in \Delta(\mathcal{C})$, because $\sigma_j \smallsetminus \tau$ is a vertex of the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$. We conclude that $\sigma_j \cup \sigma_k \cup \tau = \sigma_j \cup \tau \in \Delta(\mathcal{C})$. Hence, there exists a codeword $c \in \mathcal{C}$ such that $c \in \mathrm{Tk}_\mathcal{C}(\sigma_j \cup \sigma_k \cup \tau)$. This trunk $\mathrm{Tk}_\mathcal{C}(\sigma_j \cup \sigma_k \cup \tau)$ is contained in $\mathrm{Tk}_\mathcal{C}(\sigma_j \cup \sigma_k)$, which, by P(i), equals $\mathrm{Tk}_\mathcal{C}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$. We conclude that $c \in \mathrm{Tk}_\mathcal{C}(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau)$. But this implies that $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \in \Delta(\mathcal{C})$, which contradicts P(ii). □

Proposition 4.10 (Every wire wheel is a wheel)**.** *If $\mathcal{W}$ is a wire wheel of a neural code $\mathcal{C}$, then $\mathcal{W}$ is a wheel of $\mathcal{C}$. Consequently, codes with wire wheels are non-convex.*

Proof. Let $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ be a wire wheel of $\mathcal{C}$. By the assumption that $\mathcal{W}$ satisfies P(i) and P(ii), and Proposition 4.3, we have that $\mathcal{W}$ satisfies W(i) and W(ii). Thus it suffices to show that $\mathcal{W}$ satisfies W(iii).

By Lemma 4.9, the vertices $\sigma_\ell \smallsetminus \tau$ of $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ are distinct. We can therefore relabel the neurons, if necessary, so that $\sigma_\ell \smallsetminus \tau = \{\ell\}$ for $\ell = 1, 2, 3$. Now let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a realization of $\mathcal{C}$ such that U_τ and $U_\tau \cap U_{\sigma_j}$ are convex for $j = 1, 2, 3$. Note that since $\{\ell\} \cup \tau = \sigma_\ell \cup \tau$ for $\ell = 1, 2, 3$, we have that $U_{\sigma_\ell} \cap U_\tau = U_\ell \cap U_\tau$. Thus it suffices to show that there is a line segment from a point in $U_1 \cap U_\tau$ to a point in $U_3 \cap U_\tau$ that meets $U_2 \cap U_\tau$.

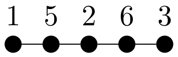
First, as $1, 3 \in \mathrm{Lk}_\tau(\Delta(\mathcal{C}))$, the sets $U_1 \cap U_\tau$ and $U_3 \cap U_\tau$ are nonempty, so let L be a line segment from a point in one set to a point in the other. The endpoints of L are in the convex set U_τ and so L is contained in U_τ. As $\tau \notin \mathcal{C}$, the line segment L is covered by the relatively open intervals $U_j \cap L$, where $j \in \mathrm{Lk}_\tau(\Delta(\mathcal{C}))$. Additionally, as the link is one-dimensional, these intervals have only pairwise intersections. Hence, for each such interval $U_j \cap L$ that is nonempty, there exists a point p_j that is in that interval and no other intervals. We conclude that the intersection patterns of the intervals correspond to a path in the link, which by assumption contains the vertex 2. Hence, the desired point p_2 in $U_j \cap L$ exists. □

Remark 4.11. The approach used at the end of the proof of Proposition 4.10 is called *order-forcing*, and is described further in [13]. Another application of order-forcing is found in [3].

Remark 4.12 (One-dimensional links)**.** Wire wheels involve a non-codeword $\tau \in \Delta(\mathcal{C}) \smallsetminus \mathcal{C}$ for which the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ is a tree (which is contractible). If this link is not a tree but still is one-dimensional, then the link is non-contractible and so, by Proposition 2.5, the code $\mathcal{C}$ is non-convex due to a local obstruction. Proposition 4.10 is therefore notable for being able to detect cases in which some τ fails to generate a local obstruction and yet the code is still non-convex.

Next, we use Proposition 4.10 to answer a question prompted by recent work on neural codes. A *3-sparse* code is a code for which the neural complex has dimension at most 2. The authors of [4] asked whether every 3-sparse code with no local obstructions, is convex. (The answer is “yes” for codes on up to 5 neurons [6, 9]; see Proposition 6.11 in a later section.) We answer this question in the negative, by providing an example of a code on 6 neurons that has a wire wheel.

Proposition 4.13. *There is a non-convex 3-sparse code with no local obstructions.*

Proof. We return to the code $\mathcal{C}_{\mathrm{TL}} := \{\mathbf{123}, \mathbf{145}, \mathbf{245}, \mathbf{246}, \mathbf{346}, 24, 45, 46, 1, 2, 3, \varnothing\}$ from Example 4.7. (Recall that TL stands for "tree link".) The maximal codewords have length 3, so $\dim(\Delta(\mathcal{C}_{\mathrm{TL}})) = 2$. The max-intersection faces are $\varnothing, 1, 2, 3, 4, 24, 45$, and 46. With the exception of 4, all of these intersections are codewords of $\mathcal{C}_{\mathrm{TL}}$. While $4 \notin \mathcal{C}$, the link $\mathrm{Lk}_{\{4\}}(\Delta(\mathcal{C}_{\mathrm{TL}}))$ is the following path, which is contractible: 

. Thus, by Proposition 2.5, the code $\mathcal{C}_{\mathrm{TL}}$ has no local obstructions.

Consider $\mathcal{W}_{\mathrm{TL}} := (\sigma_1, \sigma_2, \sigma_3, \tau) = (1, 2, 3, 4)$. First, $\mathcal{W}_{\mathrm{TL}}$ satisfies P(i), as $123 \in \Delta(\mathcal{C}_{\mathrm{TL}})$ and $\mathrm{Tk}_{\mathcal{C}_{\mathrm{TL}}}(\sigma_i \cup \sigma_j) = \{123\} = \mathrm{Tk}_{\mathcal{C}_{\mathrm{TL}}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$ for $1 \leq i < j \leq 3$. Next, $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau = 1234 \notin \mathcal{C}_{\mathrm{TL}}$, so P(ii) also holds. Finally, we already saw that the link $\mathrm{Lk}_{\{4\}}(\Delta(\mathcal{C}_{\mathrm{TL}}))$ is a path (and thus a tree) in which the unique path from vertex $(\sigma_1 \smallsetminus \tau) = 1$ to $(\sigma_3 \smallsetminus \tau) = 3$ passes through the vertex $(\sigma_2 \smallsetminus \tau) = 2$. Hence, $\mathcal{W}_{\mathrm{TL}}$ is a wire wheel, and so Proposition 4.10 implies that $\mathcal{C}_{\mathrm{TL}}$ is non-convex. □

Recall that, in Definition 3.1, we distinguished between a wheel of a code and a wheel of a realization. Being a wheel of a code means being a common wheel of every realization of the code. However, for proving that certain codes are not convex, this is too strong a requirement. Indeed, it suffices to show that every realization has a wheel, which may vary from one realization to another. Accordingly, we now introduce a different type of combinatorial wheel using this more flexible approach.

Definition 4.14. Let $\mathcal{C}$ be a code with neural complex $\Delta(\mathcal{C})$. A triple $(\sigma_1, \sigma_3, \tau) \in (\Delta(\mathcal{C}))^3$ is a *wheel frame* of $\mathcal{C}$ if it satisfies the following conditions:

F(i): $\sigma_1 \cup \sigma_3 \in \Delta(\mathcal{C})$, and for all $\omega \subseteq \sigma_1 \cup \sigma_3$ such that
(1) neither $\omega \subseteq \sigma_1$ nor $\omega \subseteq \sigma_3$, and
(2) $\omega \cup \tau \in \Delta(\mathcal{C})$,
it follows that $\mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \omega) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3)$ and $\mathrm{Tk}_{\mathcal{C}}(\sigma_3 \cup \omega) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3)$;
F(ii): $\tau \cup \sigma_1 \cup \sigma_3 \notin \Delta(\mathcal{C})$;
F(iii): $\sigma_1 \cup \tau \in \Delta(\mathcal{C})$ and $\sigma_3 \cup \tau \in \Delta(\mathcal{C})$; and
F(iv): $\sigma_1 \cap \sigma_3 = \varnothing$ and $\mathrm{Tk}_{\mathcal{C}}(\tau) \subseteq \cup_{i \in \sigma_1 \cup \sigma_3} \mathrm{Tk}_{\mathcal{C}}(\{i\})$.

Theorem 4.15 (Wheel frames generate wheels)**.** *Let $(\sigma_1, \sigma_3, \tau)$ be a wheel frame of a code $\mathcal{C}$ on n neurons. Then, for every realization $\mathcal{U}$ of $\mathcal{C}$, there exists $\sigma_2 \subseteq [n]$ such that $(\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wheel of $\mathcal{U}$. Consequently, codes with wheel frames are non-convex.*

Proof. Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a realization of $\mathcal{C}$. Our first task is to construct σ_2.

We consider two cases. If U_τ is non-convex, set $\sigma_2 := (\sigma_1 \cup \sigma_3)$. (In this case, the non-convexity of U_τ implies that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ vacuously satisfies W(iii) with respect to $\mathcal{U}$.)

Now consider the remaining case, that is, when U_τ is convex. As $\Delta(\mathcal{C})$ is the nerve of $\mathcal{U}$, F(ii) and F(iii) imply that $U_{\sigma_1 \cup \tau}$ and $U_{\sigma_3 \cup \tau}$ are disjoint, nonempty sets. Let $p_1 \in U_{\sigma_1 \cup \tau}$ and $p_3 \in U_{\sigma_3 \cup \tau}$, and let L denote the line segment with endpoints p_1 and p_3. Then $L \subseteq U_\tau$, because p_1 and p_3 are both in the convex set U_τ.

We claim that there exists some $p_2 \in L$ between p_1 and p_3 such that $p_2 \notin U_{\sigma_j}$ for $j = 1, 3$. Set $L_j = L \cap (\bigcup_{i \in \sigma_j} U_i)$ for $j = 1, 3$. By F(iv), we have that U_τ (and thus L) is covered by $\{U_i\}_{i \in \sigma_1 \cup \sigma_3}$, so $L = L_1 \cup L_3$. Both L_1 and L_3 are relatively open subsets of the connected set L, and $L_1, L_3 \neq \varnothing$ (because $p_1 \in L_1$ and $p_3 \in L_3$), so it follows that $L_1 \cap L_3 \neq \varnothing$. We conclude that there exist $i_1 \in \sigma_1$, $i_3 \in \sigma_3$, and $p_2 \in L \subseteq U_\tau$ such that $p_2 \in U_{i_1} \cap U_{i_3}$. We remark that $i_1 \in (\sigma_1 \smallsetminus \sigma_3)$ and $i_3 \in (\sigma_3 \smallsetminus \sigma_1)$, because $\sigma_1 \cap \sigma_3 = \varnothing$ (by F(iv)). Let $\sigma_2 := \{i_1\} \cup \{i_3\}$.

We now claim that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wheel of $\mathcal{U}$. To see this, we first observe that, by construction, $\mathcal{W}$ satisfies W(iii) with respect to $\mathcal{U}$. To complete the proof, we show that $\mathcal{W}$ also satisfies P(i) and P(ii) (recall Proposition 4.3).

We first show P(i). By construction, $\sigma_2 \subseteq (\sigma_1 \cup \sigma_3)$. So, $\sigma_1 \cup \sigma_3 = \sigma_1 \cup \sigma_2 \cup \sigma_3$ holds, which implies the following:

(a): $\sigma_1 \cup \sigma_2 \cup \sigma_3$ is in $\Delta(\mathcal{C})$ (due to F(i)), and
(b): $\mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$.

Hence, only two trunk conditions in P(i) are left to verify, and by (b) these conditions are equivalent to the following:

$$\mathrm{Tk}_{\mathcal{C}}(\sigma_i \cup \sigma_2) \; = \; \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3) \quad \text{for } i = 1, 3 \; . \tag{5}$$

for $i = 1, 3$. If U_τ is non-convex, then the equalities (5) follow from $\sigma_2 = (\sigma_1 \cup \sigma_3)$. Now consider the case when U_τ is convex. We prove (5) by applying F(i) with $\omega = \sigma_2$, as follows. We already saw that $\sigma_2 \subseteq (\sigma_1 \cup \sigma_3)$. Also, $\sigma_2 \nsubseteq \sigma_1$ and $\sigma_2 \nsubseteq \sigma_3$ hold, because $i_1 \in (\sigma_2 \smallsetminus \sigma_3)$ and $i_3 \in (\sigma_2 \smallsetminus \sigma_1)$. Lastly, since $p_2 \in U_{\sigma_2} \cap U_\tau$ we see that $U_{\sigma_2} \cap U_\tau \neq \varnothing$, and so (as $\Delta(\mathcal{C})$ is the nerve of $\mathcal{U}$) we have $\sigma_2 \cup \tau \in \Delta(\mathcal{C})$. Thus, F(i) implies the equalities (5). Hence, $\mathcal{W}$ satisfies P(i).

Finally, $\mathcal{W}$ satisfies P(ii) because of F(ii), since $\sigma_2 \subseteq (\sigma_1 \cup \sigma_3)$ implies $\tau \cup \sigma_1 \cup \sigma_3 = \tau \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$. □

Next, we show that the code $\mathcal{C}^\star$ from Example 2.7 has a wheel frame.

Proposition 4.16 ($\mathcal{C}^\star$ has a wheel frame)**.** *Let $\mathcal{C}^\star$ be the neural code from Example 2.7. If $\mathcal{D}$ is a code such that*

(1): $\mathcal{C}^\star \subseteq \mathcal{D} \subseteq \Delta(\mathcal{C}^\star)$ *and*
(2): *$\mathcal{D}$ does $\underline{\text{not}}$ contain any of the following codewords:* 1, 234, *and* 245,

then $\mathcal{D}$ contains a wheel frame and thus is non-convex. In particular, $\mathcal{C}^\star$ has a wheel frame.

Proof. Assume that $\mathcal{D}$ satisfies (1) and (2). It follows that $\Delta(\mathcal{D}) = \Delta(\mathcal{C}^\star)$.

We show that $(\sigma_1, \sigma_3, \tau) = (23, 45, 1)$ is a wheel frame, as follows. It is straightforward to check that conditions F(ii) and F(iii) hold. Condition F(iv) is also easy to check (here, the assumption $1 \notin \mathcal{D}$ is used). Finally we consider F(i). The only set $\omega \subseteq (\sigma_1 \cup \sigma_3) = 2345$ for which $\omega \nsubseteq \sigma_1 = 23$, $\omega \nsubseteq \sigma_3 = 45$, and $\omega \cup \tau = \omega \cup \{1\} \in \Delta(\mathcal{C}^\star)$ is the set $\omega = 34$. For this set ω, the trunk conditions in F(i), namely, $\mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \omega) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3)$ and $\mathrm{Tk}_{\mathcal{C}}(\sigma_3 \cup \omega) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_3)$, are readily seen to hold, as (respectively) $234 \notin \mathcal{D}$ and $345 \notin \mathcal{D}$. □

We end this section with another example of a code with a wheel frame.

Example 4.17. Consider the neural code $\mathcal{C} = \{\mathbf{12}, \mathbf{13}, \mathbf{23}, 1, 2, \emptyset\}$. It is straightforward to check that $(\sigma_1, \sigma_3, \tau) := (1, 2, 3)$ is a wheel frame of $\mathcal{C}$, and so $\mathcal{C}$ is non-convex. Moreover, by following the proof of Theorem 4.15, we see that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau) = (1, 12, 2, 3)$ is a wheel of *every* realization of $\mathcal{C}$ and hence is a wheel of $\mathcal{C}$ itself.

We make several observations about this wheel $\mathcal{W}$. First, P(iii)$_\circ$ (or equivalently W(iii)$_\circ$) does not hold: $\sigma_2 \cup \tau = 123 \notin \Delta(\mathcal{C})$. Thus, the non-convexity of $\mathcal{C}$ comes from the fact that $\mathcal{W}$ only vacuously satisfies W(iii) (recall Remark 3.2). Indeed, in every realization $\mathcal{U} = \{U_1, U_2, U_3\}$ of $\mathcal{C}$, the set $U_\tau = U_3$ is disconnected and hence non-convex: $U_3 = (U_3 \cap U_1) \cup (U_3 \cap U_2)$, where $(U_3 \cap U_1)$ and $(U_3 \cap U_2)$ are open, nonempty, and disjoint. This non-convexity can also be seen through local obstructions: the mandatory face 3 of $\Delta(\mathcal{C})$ is a non-codeword of $\mathcal{C}$, and so $\mathcal{C}$ has a local obstruction and hence is non-convex (Proposition 2.5).

We revisit the code in Example 4.17 in the next section.

5. Refining the search for wheels

In the process of checking whether a code $\mathcal{C}$ has a wheel, it is natural to ask whether we must consider every quadruple $(\sigma_1, \sigma_2, \sigma_3, \tau)$ of faces in $\Delta(\mathcal{C})$. Fortunately, the answer is "no". This section features several results in this direction. First, for sprockets and wire wheels, there are no containments among the spokes σ_i (Corollary 5.2) and the rim τ is never a codeword of $\mathcal{C}$ (Propositions 5.4 and 5.8). Further constraints on rims are motivated by local obstructions, which we recall can be detected at the level of $\Delta(\mathcal{C})$, rather than a restriction of $\Delta(\mathcal{C})$, through some max-intersection face. We show that combinatorial wheels also "bubble up", that is, a code with a sprocket, wire wheel, or wheel frame also has a sprocket, wire wheel, or wheel frame in which the rim τ is a max-intersection face (Propositions 5.6, 5.8, and 5.9). Finally, we conjecture that this bubble-up property generalizes to all wheels (Conjecture 5.10).

5.1. **Precluding containments among spokes.** Consider a wheel of a code $\mathcal{C}$ that violates P(iii)$_\circ$ (for instance, the wheel in Example 4.17). We have seen that such a wheel is "extraneous" in the sense that, in every realization of $\mathcal{C}$, condition W(iii) holds vacuously (recall Remark 3.2). We now show that, for non-extraneous wheels, there are no containments among the three spokes σ_i.

Proposition 5.1 (No containments among spokes if P(iii)$_\circ$ holds)**.** *Let $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ be a wheel of a neural code $\mathcal{C}$. If $\mathcal{W}$ satisfies P(iii)$_\circ$, then $\sigma_j \not\subseteq \sigma_k$ for distinct $j, k \in \{1, 2, 3\}$.*

Proof. Assume for contradiction that $\mathcal{W}$ is a wheel of $\mathcal{C}$ such that P(iii)$_\circ$ holds and also $\sigma_j \subseteq \sigma_k$ for some $j \neq k$. Note that P(i) and P(ii) also hold, by Proposition 4.3.

By P(iii)$_\circ$, we have that $\sigma_k \cup \tau \in \Delta(\mathcal{C})$. So, there exists $c \in \mathcal{C}$ such that $\sigma_k \cup \tau \subseteq c$, and so $\sigma_j \cup \tau \subseteq c$ also holds (because $\sigma_j \subseteq \sigma_k$). Thus, c contains $\sigma_j \cup \sigma_k$. So, $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_j \cup \sigma_k) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$, where the equality is by P(i). Hence, $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau)$, which implies that $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \in \Delta(\mathcal{C})$. But this contradicts P(ii). □

Next, we apply Proposition 5.1 to sprockets and wire wheels.

Corollary 5.2 (No containments among spokes of sprockets and wire wheels)**.** *If $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a sprocket or a wire wheel of a neural code $\mathcal{C}$, then $\sigma_j \not\subseteq \sigma_k$ for distinct $j, k \in \{1, 2, 3\}$.*

Proof. Sprockets and wire wheels are wheels (Proposition 4.5 and 4.10). So, by Proposition 5.1, we need only show that sprockets and wire wheels satisfy P(iii)$_\circ$. Sprockets satisfy P(iii)$_\circ$ by definition. Now assume $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wire wheel. By definition, $\sigma_\ell \smallsetminus \tau$, for $\ell = 1, 2, 3$, is a vertex of the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$. Hence, $\sigma_\ell \cup \tau$ is a face of $\Delta(\mathcal{C})$, and so P(iii)$_\circ$ holds. □

Corollary 5.2 does not extend to wheels generated by wheel frames. Indeed, we saw such a wheel in Example 4.17 (in that wheel, $\sigma_1 = 1 \subseteq \sigma_2 = 12$). Moreover, that example showed that, for wheels generated by wheel frames, P(iii)$_\circ$ need not hold and so Proposition 5.1 does not apply. Nevertheless, the "outer spokes" σ_1, σ_3 of a wheel frame are guaranteed to not contain each other, as follows.

Proposition 5.3 (No containments among spokes σ_1 and σ_3 of wheel frames)**.** *If $(\sigma_1, \sigma_3, \tau)$ is a wheel frame of a code $\mathcal{C}$, then $\sigma_1 \not\subseteq \sigma_3$ and $\sigma_3 \not\subseteq \sigma_1$.*

Proof. F(iv) implies that $\sigma_1 \cap \sigma_3 = \emptyset$, so (by symmetry) it suffices to show that $\sigma_1 \neq \emptyset$. Indeed, if $\sigma_1 = \emptyset$, then F(ii) and F(iii) can not both hold. □

5.2. **Constraints on sprockets.** The results in this subsection simplify the search for sprockets. Our first result states that we need only check those quadruples $(\sigma_1, \sigma_2, \sigma_3, \tau)$ for which $\tau \notin \mathcal{C}$.

Proposition 5.4 (Rims of sprockets are non-codewords)**.** *If $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a sprocket of a neural code $\mathcal{C}$, then $\tau \notin \mathcal{C}$.*

Proof. Assume for contradiction that $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a sprocket of $\mathcal{C}$ with $\tau \in \mathcal{C}$. By definition and Proposition 4.3, $\mathcal{W}$ satisfies P(i), P(ii), and P(iii)$_\circ$. Let ρ_1, ρ_3 be witnesses for $\mathcal{W}$, as in S(iii).

Next, $\tau \in \mathcal{C}$ implies that $\tau \in \mathrm{Tk}_{\mathcal{C}}(\tau)$. Hence, by S(iii)(2), we have $\tau \in \mathrm{Tk}_{\mathcal{C}}(\rho_1)$ or $\tau \in \mathrm{Tk}_{\mathcal{C}}(\rho_3)$. By symmetry, we may assume that $\tau \in \mathrm{Tk}_{\mathcal{C}}(\rho_1)$ or, equivalently, $\rho_1 \subseteq \tau$.

By P(iii)$_\circ$, there exists a codeword $c \in \mathcal{C}$ such that $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_3 \cap \tau)$. Then by S(iii)(1) we have $c \in \mathrm{Tk}_{\mathcal{C}}(\rho_3)$, and thus $c \in \mathrm{Tk}_{\mathcal{C}}(\rho_3 \cup \tau)$. It follows that $c \in \mathrm{Tk}_{\mathcal{C}}(\rho_1 \cup \rho_3 \cup \tau)$ (because $\rho_1 \subseteq \tau$), which, by S(iii)(3), implies that $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_2)$.

Thus, $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_2) \cap \mathrm{Tk}_{\mathcal{C}}(\sigma_3) = \mathrm{Tk}_{\mathcal{C}}(\sigma_2 \cup \sigma_3)$. Then P(i) implies that $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$. But then $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cap \mathrm{Tk}_{\mathcal{C}}(\tau) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau)$, which contradicts P(ii). □

The next result states that the witnesses ρ_1 and ρ_3 for a sprocket, from S(iii) in the definition of a sprocket, can not be equal.

Proposition 5.5 (Witnesses for a sprocket are distinct)**.** *Let $\mathcal{W}$ be a sprocket of a neural code $\mathcal{C}$. If $\rho_1, \rho_3 \in \Delta(\mathcal{C})$ are witnesses for the sprocket $\mathcal{W}$ (as in S(iii)), then $\rho_1 \neq \rho_3$.*

Proof. Assume for contradiction that ρ_1, ρ_3 are witnesses for the sprocket $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$, and that $\rho_1 = \rho_3$. Let $\rho := \rho_1 = \rho_3$. By P(iii)$_\circ$, there exists $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \tau)$. Thus, by S(iii)(1), we have that $c \in \mathrm{Tk}_{\mathcal{C}}(\rho)$. So, $c \in \mathrm{Tk}_{\mathcal{C}}(\rho \cup \tau)$, and thus by S(iii)(3) (and the fact that $\rho = \rho_1 = \rho_3$) we have that $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_2)$. Furthermore, $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1) \cap \mathrm{Tk}_{\mathcal{C}}(\sigma_2) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$, where the last equality is from P(i). Hence, $c \in \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cap \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \tau) = \mathrm{Tk}_{\mathcal{C}}(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau)$, and thus $(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau) \subseteq c$, which contradicts P(ii). □

The next result is inspired by the theory of local obstructions. Recall that a code $\mathcal{C}$ has no local obstruction if and only if $\mathcal{C}$ contains all mandatory faces of $\Delta(\mathcal{C})$ (Proposition 2.5). The reason behind this is a bubbling-up property: if $\mathcal{C}$ has a local obstruction (σ, τ), then it also has a local obstruction of the form $(\widetilde{\sigma}, [n] \smallsetminus \widetilde{\sigma})$ where $\sigma \subseteq \widetilde{\sigma}$ and $\widetilde{\sigma}$ is a mandatory face (and thus a max-intersection face) of $\Delta(\mathcal{C})$ [6]. In a similar manner, the following result shows that if a code $\mathcal{C}$ has a sprocket, then it also has a sprocket in which the rim τ is a max-intersection face of $\Delta(\mathcal{C})$.

Proposition 5.6 (Bubble-up property for sprockets)**.** *If a neural code $\mathcal{C}$ has a sprocket, then $\mathcal{C}$ has a sprocket $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ in which τ is a max-intersection face of $\Delta(\mathcal{C})$.*

Proof. Let $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ be a sprocket of a code $\mathcal{C}$, and let $\widetilde{\tau}$ be the intersection of all maximal codewords of $\mathcal{C}$ that contain τ. It suffices to show that $\widetilde{\mathcal{W}} := (\sigma_1, \sigma_2, \sigma_3, \widetilde{\tau})$ is also a sprocket of $\mathcal{C}$. First, P(i) depends only on σ_1, σ_2, σ_3; so the fact that $\mathcal{W}$ satisfies P(i) implies that $\widetilde{\mathcal{W}}$ does too.

Next, we show that $\widetilde{\mathcal{W}}$ satisfies P(ii). The containment $\tau \subseteq \widetilde{\tau}$ implies that $(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cup \tau \subseteq (\sigma_1 \cup \sigma_2 \cup \sigma_3) \cup \widetilde{\tau}$. Then, from $(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cup \tau \notin \Delta(\mathcal{C})$ (because $\mathcal{W}$ satisfies P(ii)), we have that $(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cup \widetilde{\tau} \notin \Delta(\mathcal{C})$.

To show that $\widetilde{\mathcal{W}}$ satisfies P(iii)$_\circ$, we must show that $\sigma_j \cup \widetilde{\tau} \in \Delta(\mathcal{C})$ for $j = 1, 2, 3$. Because $\mathcal{W}$ satisfies P(iii)$_\circ$, there exists a maximal codeword c_j of $\mathcal{C}$ such that $\sigma_j \cup \tau \subseteq c_j$. By construction of $\widetilde{\tau}$, we have that $\widetilde{\tau} \subseteq c_j$. Hence, $\sigma_j \cup \widetilde{\tau} \subseteq c_j$ and so, as desired, $\sigma_j \cup \widetilde{\tau} \in \Delta(\mathcal{C})$.

To show that $\widetilde{\mathcal{W}}$ satisfies S(iii), we begin by choosing witnesses ρ_1, ρ_3 for the sprocket $\mathcal{W}$ (as in S(iii)). Next, the containment $\tau \subseteq \widetilde{\tau}$ yields the following:

- $\mathrm{Tk}_{\mathcal{C}}(\sigma_j \cup \widetilde{\tau}) \subseteq \mathrm{Tk}_{\mathcal{C}}(\sigma_j \cup \tau)$,
- $\mathrm{Tk}_{\mathcal{C}}(\widetilde{\tau}) \subseteq \mathrm{Tk}_{\mathcal{C}}(\tau)$, and
- $\mathrm{Tk}_{\mathcal{C}}(\rho_1 \cup \rho_3 \cup \widetilde{\tau}) \subseteq \mathrm{Tk}_{\mathcal{C}}(\rho_1 \cup \rho_3 \cup \tau)$.

Combining these three containments with, respectively, the containments in S(iii)(1–3) satisfied by $\mathcal{W}$ by way of ρ_1, ρ_3, we conclude that ρ_1, ρ_3 are also witnesses for $\widetilde{\mathcal{W}}$. That is, $\widetilde{\mathcal{W}}$ satisfies S(iii). □

Propositions 5.4 and 5.6 together imply the following result.

Corollary 5.7. *If a neural code $\mathcal{C}$ has a sprocket, then $\mathcal{C}$ has a sprocket $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ such that $\tau \notin \mathcal{C}$ and τ is a max-intersection face of $\Delta(\mathcal{C})$.*

5.3. Beyond sprockets.

A natural question is whether Corollary 5.7 extends beyond sprockets. In fact, a slightly stronger result holds for wire wheels (Proposition 5.8), and a bubble-up property holds for wheel frames (Proposition 5.9). However, for wheel frames we do not know whether the rim τ is always a non-codeword.

Proposition 5.8 (Properties of wire wheels)**.** *If $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wire wheel of a neural code $\mathcal{C}$, then $\tau \notin \mathcal{C}$ and τ is a max-intersection face of $\Delta(\mathcal{C})$.*

Proof. Let $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ be a wire wheel of $\mathcal{C}$. By definition, $\tau \notin \mathcal{C}$. Now we must show that τ is a max-intersection face of $\Delta(\mathcal{C})$.

By definition, the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ is a tree, and the unique path in this tree between $\sigma_1 \smallsetminus \tau$ and $\sigma_3 \smallsetminus \tau$ contains $\sigma_2 \smallsetminus \tau$. Denote this path by $p_1 = (\sigma_1 \smallsetminus \tau), p_2, \ldots, p_r = (\sigma_2 \smallsetminus \tau), p_{r+1}, \ldots, p_s = (\sigma_3 \smallsetminus \tau)$. By Lemma 4.9, we know that $1 < r < s$.

Next, we claim that $2 < r$ (and so, by symmetry, $r + 1 < s$). Assume for contradiction that $2 = r$. Then $\tau \cup \sigma_1 \cup \sigma_2 = \tau \cup \{p_1, p_2\}$, which is a face of $\Delta(\mathcal{C})$ because $\{p_1, p_2\}$ is an edge of the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$. We also know that $\tau \cup \sigma_1 \cup \sigma_2 = \tau \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$ (by P(i)). We conclude that $\tau \cup \sigma_1 \cup \sigma_2 \cup \sigma_3 \in \Delta(\mathcal{C})$, which contradicts P(ii). Therefore, as claimed, $2 < r$ (and so $r + 1 < s$). In particular, $s > 4$.

Let $F_1 := \tau \cup \{p_1, p_2\}$ and $F_2 := \tau \cup \{p_{s-1}, p_s\}$. Both F_1 and F_2 are facets of $\Delta(\mathcal{C})$, because $\{p_1, p_2\}$ and $\{p_{r-1}, p_r\}$ are edges of the link $\mathrm{Lk}_\tau(\Delta(\mathcal{C}))$ (which is a tree). Also, as $s > 4$, we have $F_1 \cap F_2 = \tau$. So, τ is the intersection of two facets of $\Delta(\mathcal{C})$ and therefore is a max-intersection face. □

Proposition 5.9 (Bubble-up property for wheel frames)**.** *If a neural code $\mathcal{C}$ has a wheel frame, then $\mathcal{C}$ has a wheel frame $(\sigma_1, \sigma_3, \tau)$ in which τ is a max-intersection face of $\Delta(\mathcal{C})$.*

Proof. Let $\mathcal{F} = (\sigma_1, \sigma_3, \tau)$ be a wheel frame of $\mathcal{C}$. Let $\widetilde{\tau}$ be the intersection of all facets of $\Delta(\mathcal{C})$ that contain τ (in particular, $\tau \subseteq \widetilde{\tau}$). We must show that $\widetilde{\mathcal{F}} = (\sigma_1, \sigma_3, \widetilde{\tau})$ is also a wheel frame of $\mathcal{C}$. First, F(i) for $\widetilde{\mathcal{F}}$ is implied by the same condition for $\mathcal{F}$ (because $\tau \subseteq \widetilde{\tau}$). Similarly, $\widetilde{\mathcal{F}}$ also inherits F(ii) from $\mathcal{F}$, because $\tau \subseteq \widetilde{\tau}$ and simplicial complexes are closed with respect to containment. Next, we know that $\sigma_1 \cup \tau \in \Delta(\mathcal{C})$ and $\sigma_3 \cup \tau \in \Delta(\mathcal{C})$, because $\mathcal{F}$ satisfies F(iii). So, there exist facets F_1 and F_3 of $\Delta(\mathcal{C})$ such that $\sigma_1 \cup \tau \subseteq F_1$ and $\sigma_3 \cup \tau \subseteq F_3$. By construction, $\widetilde{\tau} \subseteq F_1$ and $\widetilde{\tau} \subseteq F_3$. We conclude that $\sigma_1 \cup \widetilde{\tau} \subseteq F_1$ and $\sigma_3 \cup \widetilde{\tau} \subseteq F_3$. Hence, $\sigma_1 \cup \widetilde{\tau} \in \Delta(\mathcal{C})$ and $\sigma_3 \cup \widetilde{\tau} \in \Delta(\mathcal{C})$; in other words, $\widetilde{\mathcal{F}}$ satisfies F(iii). Finally, $\tau \subseteq \widetilde{\tau}$ implies that $\mathrm{Tk}_{\mathcal{C}}(\widetilde{\tau}) \subseteq \mathrm{Tk}_{\mathcal{C}}(\tau)$, and this containment implies that F(iv) is inherited from $\mathcal{F}$ to $\widetilde{\mathcal{F}}$. □

So far, we have shown that for certain types of wheels, rims can not be codewords, but always bubble up to a max-intersection face. We conjecture that these results generalize to all wheels.

Conjecture 5.10. *Let $\mathcal{C}$ be a neural code.*

(i): *If $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ is a wheel of $\mathcal{C}$, then $\tau \notin \mathcal{C}$.*
(ii): *If $\mathcal{C}$ has a wheel, then $\mathcal{C}$ has a wheel $\mathcal{W} = (\sigma_1, \sigma_2, \sigma_3, \tau)$ in which τ is a max-intersection face of $\Delta(\mathcal{C})$.*

Conjecture 5.10 holds for sprockets and wire wheels (Proposition 5.4, Corollary 5.7, and Proposition 5.8). Also, Conjecture 5.10(ii) – suitably adjusted to accommodate the fact that wheel frames generate but are not themselves wheels – holds for wheel frames (Proposition 5.9).

We end this section by describing the intuition behind Conjecture 5.10(ii). We know that having a wheel guarantees non-convexity (Theorem 3.4), while being max-intersection-complete guarantees convexity (Proposition 2.5). Hence, the presence of a wheel in a code $\mathcal{C}$ must somehow force a max-intersection face to be excluded from $\mathcal{C}$. Our examples and results suggest that this excluded max-intersection face is always a rim.

6. Reducible and decomposable codes

One aim of this article is to use wheels to classify codes on 6 neurons. A first step is to exclude those codes that are somehow equivalent to codes on fewer neurons. Such codes are described in this section: reducible codes – those with irrelevant neurons (Section 6.1) – and decomposable codes – those built from simpler codes (Section 6.2). We also prove some preliminary results on decomposable codes on up to 6 neurons (Section 6.3).

In what follows, we use the following definition: let $\mathcal{C}$ be a code on n neurons, and χ a subset of $[n]$. The *restricted code* $\mathcal{C}|_\chi$ is $\{c \cap \chi \mid c \in \mathcal{C}\}$. If $\mathcal{U} = \{U_i\}_{i=1}^n$ is a realization of $\mathcal{C}$, then we let $\mathcal{U}|_\chi$ denote the realization $\{U_i\}_{i\in\chi}$ of $\mathcal{C}|_\chi$.

Lemma 6.1. *Let $\mathcal{C}$ be a code on n neurons, and let $\chi \subseteq [n]$. If $\mathcal{C}$ is convex, then the restricted code $\mathcal{C}|_\chi$ is also convex.*

Proof. If $\{U_i\}_{i\in[n]}$ is a convex realization of $\mathcal{C}$, then $\{U_i\}_{i\in\chi}$ is a convex realization of $\mathcal{C}|_\chi$. □

6.1. Reducible codes.

The following terminology, introduced by Jeffs [11, §3], captures the case of a superfluous neuron.

Definition 6.2. Let $\mathcal{C}$ be a code on n neurons.

(1) A neuron i is *trivial* in $\mathcal{C}$ if i is not in any codeword of $\mathcal{C}$, that is, $\mathrm{Tk}_{\mathcal{C}}(\{i\}) = \varnothing$.
(2) A neuron i is *redundant* in $\mathcal{C}$ if i is not trivial in $\mathcal{C}$ and there exists $\sigma \subseteq [n]$ with $i \notin \sigma$ such that $\mathrm{Tk}_{\mathcal{C}}(\{i\}) = \mathrm{Tk}_{\mathcal{C}}(\sigma)$.
(3) The code $\mathcal{C}$ is *reduced* if it has no redundant neurons and no trivial neurons.

We say that a code is *reducible* if it is not reduced.

The following result is essentially due to Jeffs (see [11, Lemma 3.11 and Corollary 4.2]):

Proposition 6.3. *Let $\mathcal{C}$ be a code on n neurons. Assume j is a trivial or redundant neuron of $\mathcal{C}$. Then $\mathcal{C}$ is convex if and only if the restricted code $\mathcal{C}|_{[n]\smallsetminus\{j\}}$ is convex.*

Proof. If j is trivial, then $\mathcal{C} = \mathcal{C}|_{[n]\smallsetminus\{j\}}$ and so the result follows.

Now assume that j is redundant. It follows that $\mathrm{Tk}_{\mathcal{C}}(\{j\}) = \mathrm{Tk}_{\mathcal{C}}(\sigma)$ for some $\sigma \subseteq [n]$ with $j \notin \sigma$. The implication "$\Rightarrow$" is Lemma 6.1.

For "$\Leftarrow$", assume that the restricted code $\mathcal{C}|_{[n]\smallsetminus\{j\}}$ is convex. Let $\{U_i\}_{i\in[n]\smallsetminus\{j\}}$ be a convex realization of the restricted code. We claim that $\sigma \neq \emptyset$. Indeed, if $\sigma = \emptyset$, then because $\emptyset \in \mathcal{C}$ (recall Assumption 2.1), we have that $\emptyset \in \mathrm{Tk}_{\mathcal{C}}(\sigma) = \mathrm{Tk}_{\mathcal{C}}(\{j\})$, which is a contradiction. So, σ is nonempty and hence U_σ is a convex, open set. Let $U_j := U_\sigma$, and now it is straightforward to check that $\{U_i\}_{i\in[n]}$ is a convex realization of $\mathcal{C}$. □

6.2. **Decomposable codes.**

This subsection considers codes $\mathcal{C}$ that can be "decomposed" into two smaller codes $\mathcal{C}_1$ and $\mathcal{C}_2$ so that a convex realization of $\mathcal{C}$, if it exists, can be obtained by "embedding" a convex realization of $\mathcal{C}_1$ into that of $\mathcal{C}_2$ (see Figure 1 and Theorem 6.6).

Definition 6.4. A code $\mathcal{C}$ on n neurons is *decomposable* if there exist disjoint subsets $\varphi, \psi \subsetneq [n]$ with $\varphi \neq \varnothing$ such that:

(i) $\psi \in \mathcal{C}$, and
(ii) every $c \in \mathcal{C}$ that contains at least one neuron of φ has the form $c = \widetilde{\varphi} \cup \psi$ for some $\widetilde{\varphi} \subseteq \varphi$.

We call $\mathcal{C}|_\varphi$ the *embedded code* and $\mathcal{C}|_{[n]\smallsetminus\varphi}$ the *ambient code.* Also, ψ is the *ambient codeword.*

Example 6.5. The code $\mathcal{C} = \{\mathbf{2356}, \mathbf{123}, 235, 236, 12, 14, 23, 1, 2, 4, \varnothing\}$ from Example 2.2 is decomposable, where $\varphi = \{5, 6\}$ and $\psi = \{2, 3\}$. Indeed, the codewords $c \in \mathcal{C}$ intersecting φ are $235 = \{5\} \cup \psi$, $236 = \{6\} \cup \psi$, and $2356 = \{5, 6\} \cup \psi$. Recall the realization $\mathcal{U}$ of $\mathcal{C}$ depicted in Figure 1. Note that $\mathcal{U}$ can be obtained by first drawing a realization of $\mathcal{C}|_{[n]\smallsetminus\varphi} = \{\mathbf{123}, \mathbf{14}, 12, 23, 1, 2, 4, \varnothing\}$ and then placing a realization of $\mathcal{C}|_\varphi = \{\mathbf{56}, 5, 6, \varnothing\}$ inside the atom of the ambient codeword 23.

6.3. **Application to codes on up to 6 neurons.**

We saw that for the code $\mathcal{C}$ in Example 6.5, a convex realization was obtained by placing a convex realization of the embedded code inside that of the ambient code. This was possible because the atom of the ambient codeword has nonempty interior. Is this always possible? The main result of this subsection, Theorem 6.6, gives an affirmative answer for codes on up to 6 neurons. We do not know, however, whether this theorem extends to codes on more neurons.

Theorem 6.6. *Suppose that $\mathcal{C}$ is a decomposable code on up to 6 neurons with embedded code $\mathcal{C}|_\varphi$ and ambient code $\mathcal{C}|_{[n]\smallsetminus\varphi}$. Then $\mathcal{C}$ is convex if and only if $\mathcal{C}|_\varphi$ and $\mathcal{C}|_{[n]\smallsetminus\varphi}$ are convex.*

The proof of Theorem 6.6, which appears at the end of this section, uses the idea of nondegeneracy introduced by Cruz *et al.* [5, §2]. Indeed, condition (i) of the following definition guarantees that, for a decomposable code, the atom of the ambient codeword has full dimension, and hence we can place within it a realization of the embedded code (see Proposition 6.8 below).

Definition 6.7. [5, Definition 2.10] For a collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of subsets of $\mathbb{R}^d$, consider the following properties:

(i): For all $\sigma \subseteq [n]$, the set $(\cap_{i\in c} U_i) \smallsetminus (\cup_{j\in[n]\smallsetminus c} U_j)$ is either empty or *top-dimensional*, i.e., every nonempty intersection with an open subset of $\mathbb{R}^d$ has nonempty interior.
(ii): For all nonempty $\sigma \subseteq [n]$, we have $\bigcap_{i\in\sigma} \partial U_i \subseteq \partial(\bigcap_{i\in\sigma} U_i)$.

Then $\mathcal{U}$ is *top-dimensional* if condition (i) holds, and is *nondegenerate* if both (i) and (ii) hold.

We say that a code $\mathcal{C}$ is *top-dimensionally convex* (respectively, *nondegenerately convex*) if it has a convex realization $\mathcal{U} = \{U_i\}_{i=1}^n$ that is top-dimensional (respectively, nondegenerate).

Proposition 6.8. *Suppose that $\mathcal{C}$ is a decomposable code with embedded code $\mathcal{C}|_\varphi$ and ambient code $\mathcal{C}|_{[n]\smallsetminus\varphi}$. If $\mathcal{C}|_\varphi$ is top-dimensionally convex and $\mathcal{C}|_{[n]\smallsetminus\varphi}$ is convex, then $\mathcal{C}$ is convex.*

Proof. Assume that $\mathcal{C}|_\varphi$ is top-dimensionally convex and $\mathcal{C}|_{[n]\smallsetminus\varphi}$ is convex. Relabel the neurons so that $\varphi = \{1, 2, \ldots, k\}$ and $[n] \smallsetminus \varphi = \{k+1, k+2, \ldots, n\}$. Let $\mathcal{V} = \{V_i\}_{i=1}^k$ be a convex realization for $\mathcal{C}|_\varphi$ and $\mathcal{W} = \{W_i\}_{i=k+1}^n$ a top-dimensional convex realization for $\mathcal{C}|_{[n]\smallsetminus\varphi}$. We may assume that both $\mathcal{V}$ and $\mathcal{W}$ are realizations in $\mathbb{R}^d$, for some d. (Indeed, every realization $\mathcal{U}$ in some $\mathbb{R}^{d_1}$ can be elevated to a realization in $\mathbb{R}^{d_1+d_2}$ by taking the product of each U_i in $\mathcal{U}$ with $(0,1)^{d_2}$.)

As $\mathcal{W}$ is top-dimensional and ψ is a codeword of $\mathcal{C}$, there exists an open ball B in $\mathbb{R}^d$ that is strictly contained in the atom for the codeword ψ. Also, recall that we may assume that every V_i in the realization $\mathcal{V}$ is contained in an open ball $\widetilde{B}$ in $\mathbb{R}^d$ (cf. [4, Remark 2.19]). Let $F : \widetilde{B} \to B$ be the scaling bijection between the two balls (which preserves convexity and is a homeomorphism). Then, by construction, $\{F(V_i)\}_{i=1}^k \cup \{W_i\}_{i=k+1}^n$ is a convex realization of $\mathcal{C}$. □

In light of Proposition 6.8, we need only show that convexity and top-dimensional convexity are equivalent for codes on up to 5 neurons (see Proposition 6.13 below), and then Theorem 6.6 will follow. To prove Proposition 6.13, we need two results on nondegenerate convexity, the first of which follows directly from the results of Cruz *et al.* [5, Proposition 4.3 and Lemma 3.1]:

Lemma 6.9. *Every max-intersection-complete code is nondegenerately convex.*

Proof. [5, Proposition 4.3 and Lemma 3.1] construct a convex realization for every max-intersection-complete code. It is straightforward to verify that this realization is also nondegenerate. □

The following result is also essentially due to Cruz *et al.* [5]:

Lemma 6.10 (Monotonicity of top-dimensional and nondegenerate convexity)**.** *Let $\mathcal{C}$ be a neural code, and let $\mathcal{D}$ be a neural code such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \Delta(\mathcal{C})$.*

(i) If $\mathcal{C}$ is top-dimensionally convex, then $\mathcal{D}$ is also top-dimensionally convex.
(ii) If $\mathcal{C}$ is nondegenerately convex, then $\mathcal{D}$ is also nondegenerately convex.

Proof. Both (i) and (ii) follow directly from the proof of [5, Theorem 1.3], if we can add the following assertion to the statement of [5, Lemma 3.1]: "Also, if $\mathcal{U}$ is top-dimensional, then $\mathcal{V}$ can also be chosen to be top-dimensional". Indeed, the proof of [5, Lemma 3.1] directly accommodates this extra assertion once we add the following statement (which is identical to the prior one) to [5, Lemma A.7]: "Also, if $\mathcal{U}$ is top-dimensional, then $\mathcal{V}$ can also be chosen to be top-dimensional." The proof of this assertion is achieved by starting from the proof of [5, Lemma A.7], replacing the two occurrences of the word "non-degenerate" with "top-dimensional", and then deleting the final sentence of that proof. □

In light of Lemma 6.10, to show top-dimensional convexity for convex codes on up to 5 neurons, we can proceed by analyzing one simplicial complex Δ at a time: it suffices to check that $\mathcal{C}_{\min}(\Delta)$, the minimal code of Δ, is top-dimensionally convex. (Recall that $\mathcal{C}_{\min}(\Delta)$ is the smallest code – with respect to inclusion – with neural complex Δ that has no local obstructions.)

However, there is one simplicial complex for which a different approach must be taken, namely, the simplicial complex of the non-convex code $\mathcal{C}^\star$ (with no local obstructions) from Example 2.7:

$$\Delta(\mathcal{C}^\star) \;=\; \text{the simplicial complex with facets } 2345,\ 123,\ 134,\ 145\ . \tag{6}$$

Note that $\mathcal{C}_{\min}(\Delta(\mathcal{C}^*)) = \mathcal{C}^\star$. Part (ii) of the next result clarifies which codes with neural complex (6) are convex, while part (i), which is due to Goldrup and Phillipson [9], pertains to all other simplicial complexes on 5 vertices.

Proposition 6.11 (Convexity of codes on 5 neurons)**.** *Let Δ be a connected simplicial complex on 5 vertices.*

(i): *If Δ is not isomorphic to the simplicial complex* (6)*, then the code $\mathcal{C}_{\min}(\Delta)$ is top-dimensionally convex.*
(ii): *If Δ is the simplicial complex* (6)*, then for a code $\mathcal{C}$ with neural complex $\Delta(\mathcal{C}) = \Delta$, the following are equivalent:*
 (a): *$\mathcal{C}$ is convex,*
 (b): *$\mathcal{C}$ is top-dimensionally convex, and*
 (c): *$\mathcal{C}$ contains at least one of the following codewords:* 1, 234*, and* 345*.*

Proof. (i). If $\mathcal{C}_{\min}(\Delta)$ is max-intersection-complete, the result follows from Lemma 6.9. By [9, Theorem 3.1] (in that article, an isomorphic copy of $\mathcal{C}_{\min}(\Delta(\mathcal{C}^*))$ is given the name "C4"), all remaining minimal codes, besides $\mathcal{C}_{\min}(\Delta(\mathcal{C}^*))$, have convex realizations depicted in [9, Appendix B]. These realizations are easily seen to be top-dimensional.

(ii). The implication (b)⇒(a) is by definition. Next, the contrapositive of (a)⇒(c) follows directly from Proposition 4.16. Now we prove (c)⇒(b). The code $\mathcal{C}^\star \cup \{1\}$ is max-intersection-complete and thus, by Lemma 6.9, is nondegenerately convex – and so, by definition, top-dimensionally convex. The code $\mathcal{C}^\star \cup \{234\}$ is also top-dimensionally convex: such a realization will appear in the forthcoming work of Magaña and Phillipson [19]. The code $\mathcal{C}^\star \cup \{345\}$ is also top-dimensionally convex: after relabeling the neurons via the permutation (25)(34), the resulting code is the code $\mathcal{C}^\star \cup \{234\}$, which we already analyzed. Thus, by Lemma 6.10(i), every code $\mathcal{C}$ with neural complex $\Delta(\mathcal{C}) = \Delta$ that contains 1, 234, or 345 is top-dimensionally convex. □

Remark 6.12. Proposition 6.11(ii) implies that $\mathcal{C}^\star \cup \{1\}$, $\mathcal{C}^\star \cup \{234\}$, and $\mathcal{C}^\star \cup \{345\}$ are the minimal (with respect to inclusion) convex codes with neural complex equal to $\Delta(\mathcal{C}^\star)$. This corrects an error in [18, Remark 3.5], where it was asserted that $\mathcal{C}^\star \cup \{234, 345\}$ is such a minimal convex code.

We now show the equivalence of convexity and top-dimensional convexity for codes on up to 5 neurons.

Proposition 6.13. *Let $\mathcal{C}$ be a code on* 5 *or fewer neurons. Then $\mathcal{C}$ is convex if and only if $\mathcal{C}$ is top-dimensionally convex.*

Proof. The direction ⇐ holds by definition. For ⇒, let $\mathcal{C}$ be a convex code on n neurons, where $n \leq 5$. If $n \leq 4$, then Proposition 2.5(3) and Lemma 6.9 together imply that $\mathcal{C}$ is nondegenerately convex and hence, by definition, top-dimensionally convex.

Now assume $n = 5$. If the neural complex $\Delta(\mathcal{C})$ is not isomorphic to $\Delta(\mathcal{C}^\star)$, the simplicial complex shown in (6), then the result follows from Lemma 6.10(i) and Proposition 6.11(i). The only remaining case, therefore, is when $\Delta(\mathcal{C})$ is isomorphic to $\Delta(\mathcal{C}^\star)$. By relabeling neurons, if necessary, we may assume that $\Delta(\mathcal{C}) = \Delta(\mathcal{C}^\star)$. Now the result follows from Proposition 6.11(ii). □

We expect that Proposition 6.13 does not extend to codes on more neurons. Related questions were posed recently by Chan *et al.* [3, Question 2.17] and Jeffs [12, Question 10.1].

We now prove two results, stated earlier, pertaining to codes on up to 5 or 6 neurons.

Proof of Theorem 3.8. (⇒) This implication follows from Proposition 2.5(2) and Theorem 4.15.

($\Leftarrow$) Let n denote the number of neurons. For $n \leq 4$, this result follows from Proposition 2.5(3). Now assume $n = 5$. We consider two cases. If $\Delta(\mathcal{C})$ is isomorphic to $\Delta(\mathcal{C}^\star)$, the simplicial complex shown in (6), then the theorem follows from Propositions 4.16 and 6.11(ii). If $\Delta(\mathcal{C})$ is *not* isomorphic to $\Delta(\mathcal{C}^\star)$, then the theorem follows from Lemma 6.10(i) and Proposition 6.11(i). □

Proof of Theorem 6.6. The implication $\Rightarrow$ follows from Lemma 6.1, and $\Leftarrow$ is immediate from Propositions 6.8 and 6.13. □

7. Codes on 6 neurons

Having shown that wheels and local obstructions completely characterize non-convexity for codes on up to 5 neurons, we now turn our attention to codes on 6 neurons. Due to the large number of such codes, we restrict our analysis to codes with up to 7 maximal codewords. Additionally, like the classification process of [9], we focus on minimal codes, $\mathcal{C}_{\min}(\Delta)$, as in (1). Recall that when some $\mathcal{C}_{\min}(\Delta)$ is convex, then all codes with neural complex equal to Δ (and no local obstructions) are convex (Proposition 2.5).

Our results, summarized in Table 1, are quite promising. Approximately 300 codes are found to be non-convex due to a wheel. Also, the number of codes with wheels increases with the number of maximal codewords. Wheels are therefore surprisingly efficient in detecting non-convexity.

	Number of maximal codewords			
	Four	Five	Six	Seven
Reducible or decomposable	203	480	526	341
Max-intersection-complete (thus, convex)	4	79	399	909
Wheel (thus, non-convex)				
Wheel frame only	0	1	11	36
Sprocket only	2	6	14	14
Wheel frame and sprocket	1	29	92	108
Wire wheel only	0	0	1	1
Unknown	0	96	535	1169
Total	210	691	1578	2578

Table 1. Classification of 6-neuron codes with 4 to 7 maximal codewords. More precisely, the table classifies minimal codes $\mathcal{C}_{\min}(\Delta)$ of all (up to isomorphism) connected simplicial complexes on 6 vertices with 4 to 7 facets. This table was obtained using Procedure 7.1. Some codes from the "Unknown" row are listed in Appendix A. No code containing a wire wheel also contained another type of wheel.

Our workflow to produce the data in Table 1 was as follows. We first used `nauty` [20] to enumerate all connected simplicial complexes on 6 vertices (up to isomorphism). Next, we used `SageMath` [23] to compute the minimal code of each simplicial complex (see [18, Algorithm 4.1]). Finally, we classified 6-neuron minimal codes with exactly four, five, six, or seven maximal codewords, by applying Procedure 7.1.

Procedure 7.1 (Classifying codes on 6 neurons)**.**
Input: A code $\mathcal{C}$ on (up to) 6 neurons.
Output: "Reducible or decomposable", "Max-intersection-complete", "Wheel", or "Unknown".
Steps:

(1) Determine whether $\mathcal{C}$ is reducible[1] or decomposable. (We saw in the previous section – specifically, Proposition 6.3 and Theorem 6.6 – that, in terms of convexity, such codes are equivalent to codes on fewer neurons.) If not, proceed to the next step.
(2) Determine whether $\mathcal{C}$ is max-intersection-complete. (Such codes are convex, by Proposition 2.5.) If not, proceed to the next step.
(3) Determine whether $\mathcal{C}$ has a sprocket, wire wheel, or wheel frame. If so, the convexity status of $\mathcal{C}$ is "Wheel". If not, the status is "Unknown".

A package containing the `SageMath` scripts we used to apply Procedure 7.1, as well as the resulting data on codes (for instance, which have wheels and what those wheels are), is available on `GitHub` [1]. The scripts we used to perform step (3) of Procedure 7.1 incorporate several results from Section 5 to make checking for a wheel more efficient. Also, a selection of the codes whose convexity status is unresolved (those in the "Unknown" row in Table 1) are listed in Appendix A.

Remark 7.2 (Codes with up to three maximal codewords)**.** For codes with at most three maximal codewords – and any number of neurons – convexity has been fully characterized: such codes are convex if and only if they have no local obstructions [16]. We therefore do not include codes with three or fewer maximal codewords in Table 1.

In light of Remark 7.2, we draw another conclusion from Table 1: *A 6-neuron minimal code $\mathcal{C}$ with up to four maximal codewords is convex if and only if $\mathcal{C}$ has no local obstructions and no wheel obstructions (more precisely, no sprocket, wire wheel, or wheel frame).* We do not know whether this observation generalizes, either to non-minimal codes or to codes on 7 or more neurons.

Going forward, there are still many codes whose convexity status is unknown (as seen in Table 1). Additionally, we would like to extend the classification to codes with more than seven maximal codewords, but this task is currently computationally challenging. Nevertheless, we hope that results like those in Section 5 will make this approach more tractable.

8. Pure neural codes

We saw in the previous section that classifying convex codes on six (or more) neurons is challenging, due to the sheer number of such codes. A more reasonable task, therefore, is to analyze codes with special properties. One such property is having a neural complex that is pure, as follows.

Definition 8.1. A neural code $\mathcal{C}$ is *pure* if its neural complex $\Delta(\mathcal{C})$ is pure, that is, every facet of $\Delta(\mathcal{C})$ has the same dimension.

Next, we show that for pure codes of low or high dimension, being convex is equivalent to being max-intersection-complete, and thus is easy to check. For a discussion of the algorithmic aspects of checking whether a code is max-intersection-complete, see [22, §6].

Theorem 8.2. *Let $\mathcal{C}$ be a code on n neurons. If $\mathcal{C}$ is pure of dimension 0, 1, $n-2$, or $n-1$, then the following are equivalent:*

(i): *$\mathcal{C}$ is convex,*
(ii): *$\mathcal{C}$ has no local obstructions, and*
(iii): *$\mathcal{C}$ is max-intersection-complete.*

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are contained in Proposition 2.5.

[1] In our enumerations, when determining whether a code is reducible, we need only check for redundant neurons. This is because the simplicial complexes we consider are connected, and so the resulting codes lack trivial neurons.

The implication (ii) ⇒ (iii) holds for dimension 0 (trivially, as by assumption $\varnothing \in \mathcal{C}$), dimension 1 (this follows easily from [15, Theorem 1.3]), and dimension $n-1$ (by [6, Lemma 2.5]). Now assume that $\mathcal{C}$ is pure of dimension $n-2$ and has no local obstructions.

Let $F_1, F_2, \ldots, F_m$ be distinct facets of $\Delta(\mathcal{C})$ (with $m \geq 2$); we must show that $\sigma := \bigcap_{j=1}^{m} F_j$ is a codeword of $\mathcal{C}$. As $\mathcal{C}$ has dimension $n-2$, there exist distinct $i_1, i_2, \ldots, i_m$ such that $F_j = [n] \smallsetminus \{i_j\}$ (for every j). Thus, $\sigma \ = \ [n] \smallsetminus \{i_1, i_2, \ldots, i_m\}$.

We now claim that every facet of $\Delta(\mathcal{C})$ that contains σ is one of the F_j. To this end, suppose F is such a facet. As $\Delta(\mathcal{C})$ is pure, $F = [n] \smallsetminus \{i\}$ for some i. Then $i \notin \sigma$, and so $i \in \{i_1, i_2, \ldots, i_m\}$. Therefore, $i = i_j$ (for some $j = 1, \ldots, m$) and thus $F = F_j$, proving the claim.

The claim implies that the facets of $\mathrm{Lk}_\sigma(\Delta(\mathcal{C}))$ are precisely the sets

$$F_j \smallsetminus \sigma = \{i_1, i_2, \ldots, i_{j-1}, \hat{i}_j, i_{j+1}, \ldots, i_m\}, \text{ for } j = 1, \ldots, m.$$

Thus, $\mathrm{Lk}_\sigma(\Delta(\mathcal{C}))$ is the hollow simplex on m vertices (that is, it contains all possible faces on the m vertices except the top face) and so is not contractible (recall that $m \geq 2$). Thus, as $\mathcal{C}$ has no local obstructions by assumption, we have $\sigma \in \mathcal{C}$, and so (ii) ⇒ (iii) holds. □

The analogous result for pure codes of dimension 2 or $n-3$ (or any dimension in between) does not hold. Indeed, we have already seen examples demonstrating this fact. For instance, the code from Example 2.6 is a convex code on $n = 5$ neurons that fails to be max-intersection-complete, but is pure and of dimension $n - 3 = 2$. Another example is the code in Proposition 4.13, which is pure of dimension 2, has no local obstructions, and is non-convex.

We end this section by analyzing pure codes on 6 neurons. For such codes of dimension 0, 1, 4, or 5, convexity is well understood (by Theorem 8.2). We therefore focus on pure 6-neuron codes of dimension 2 or 3. We applied Procedure 7.1 to these codes, and our results are shown in Table 2. Around half of these codes reduce to smaller codes or are classified as convex or non-convex. Only 6 are found to have wheels, and many codes have unknown convexity status. A partial list of these "unknown" codes appears in Appendix B, and the full list is available at `GitHub` [1].

	dim = 2	dim = 3
Reducible or decomposable	153	36
Max-intersection-complete (convex)	944	32
Wheel (non-convex)	0	6
Unknown	1004	76
Total	2101	150

Table 2. Classification of pure 6-neuron codes of dimension 2 or 3. More precisely, the table classifies minimal codes $\mathcal{C}_{\min}(\Delta)$ of all (up to isomorphism) connected simplicial complexes on 6 vertices that are pure of dimension 2 or 3. All 6 codes with a wheel contain both a wheel frame and a sprocket, but not a wire wheel.

9. Discussion

We have now presented a new tool for inferring non-convexity of neural codes, namely, the wheel. Notably, wheels and local obstructions together completely characterize convexity for codes on up to 5 neurons – and yield a partial classification of codes on up to 6 neurons.

This work sheds more light on the question of which codes are convex, and also raises interesting new questions. Wheels are different from prior non-convexity criteria in that they are inherently a property of the realization studied, and not the code. Currently, we have no way to determine that a code has no wheels, short of proving the code is convex (as opposed to the situation with local obstructions). For these reasons, the foremost open question in this direction is to find a complete criterion for the existence of wheels in terms of the code, and not of its realizations. Needless to say, effective criteria, that can be combinatorially stated and verified, are the most desirable.

Another question we have posed is whether, in a wheel, the rim is always a non-codeword that bubbles up to a max-intersection face (Conjecture 5.10). A related question arising from our examples is whether every code with a wheel "bubbles down" to a wheel in which the union of the spokes, $\sigma_1 \cup \sigma_2 \cup \sigma_3$, is disjoint from the rim τ. An additional future direction is to generalize wire wheels to accommodate links beyond path graphs; one idea here is to use the concept of order-forcing [13] to guarantee the existence of the required line segment through the three spokes. Answers to these questions would make searching for wheels more efficient and potentially more fruitful.

Other questions concerning wheels arise in connection with other works in the literature. For instance, we know that detecting convexity is an NP-hard problem [17]. Is the same true for detecting wheels? We would also like to investigate the relationship between wheels, decomposable codes, and the partial order on codes introduced by Jeffs in which the convex codes form a down-set [11].

Returning to our partial classification of codes on 6 neurons, there are hundreds of codes for which convexity is not currently known. To handle so many codes, we need automatic ways of proving a code is convex – and also more ways to preclude convexity. One approach, mentioned earlier, is to generalize wheels to accommodate more spokes, inspired by Jeffs's approach on sunflowers [10] (Remark 3.6). In fact, it might be that such a generalization is needed to resolve our conjectures. Specifically, when a wheel bubbles up, perhaps what is obtained is a generalized wheel having more than three spokes.

Finally, we know that wheels preclude a code from being open-convex, but not necessarily closed-convex. A counterpart to wheels for ruling out closed-convexity (independently of local obstructions), via "rigid structures", was given recently by Chan *et al.* [3]. On the other hand, order-forcing can be used to construct codes that have no local obstructions and yet are neither open-convex nor closed-convex [13]. Going forward, we expect that all of these concepts – wheels, local obstructions, rigid structures, and order-forcing, plus reducible and decomposable codes – will aid in future classifications of neural codes, both in terms of open-convexity and closed-convexity.

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Appendix A. Codes on 6 neurons with 5 to 7 maximal codewords and unknown convexity status

Recall from Table 1 that among the minimal codes (with connected simplicial complex) on 6 neurons, many have unknown convexity status (that is, assessing whether such a code is convex can not be done automatically). There are 96 (respectively, 535 or 1169) such codes with 5 (respectively, 6 or 7) maximal codewords; we list 10 codes from this set in Section A.1 (respectively, A.2 or A.3). The complete list can be found at GitHub [1].

A.1. **Codes with 5 maximal codewords.**

(1) $\{\mathbf{2346}, \mathbf{145}, \mathbf{456}, \mathbf{12}, \mathbf{13}, 45, 46, 1, 2, 3, \emptyset\}$
(2) $\{\mathbf{1456}, \mathbf{245}, \mathbf{346}, \mathbf{12}, \mathbf{13}, 45, 46, 1, 2, 3, \emptyset\}$
(3) $\{\mathbf{245}, \mathbf{346}, \mathbf{456}, \mathbf{12}, \mathbf{13}, 45, 46, 1, 2, 3, \emptyset\}$
(4) $\{\mathbf{2456}, \mathbf{135}, \mathbf{156}, \mathbf{12}, \mathbf{34}, 15, 56, 1, 2, 3, 4, \emptyset\}$
(5) $\{\mathbf{2456}, \mathbf{156}, \mathbf{235}, \mathbf{12}, \mathbf{34}, 25, 56, 1, 2, 3, 4, \emptyset\}$
(6) $\{\mathbf{2356}, \mathbf{134}, \mathbf{135}, \mathbf{146}, \mathbf{12}, 13, 14, 35, 1, 2, 6, \emptyset\}$
(7) $\{\mathbf{2356}, \mathbf{134}, \mathbf{135}, \mathbf{246}, \mathbf{12}, 13, 26, 35, 1, 2, 4, \emptyset\}$
(8) $\{\mathbf{1456}, \mathbf{134}, \mathbf{135}, \mathbf{246}, \mathbf{12}, 13, 14, 15, 46, 1, 2, \emptyset\}$
(9) $\{\mathbf{3456}, \mathbf{134}, \mathbf{135}, \mathbf{246}, \mathbf{12}, 13, 34, 35, 46, 1, 2, 3, \emptyset\}$
(10) $\{\mathbf{134}, \mathbf{135}, \mathbf{256}, \mathbf{346}, \mathbf{12}, 13, 34, 1, 2, 5, 6, \emptyset\}$

A.2. **Codes with 6 maximal codewords.**

(1) $\{\mathbf{145}, \mathbf{246}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 45, 46, 1, 2, 3, \emptyset\}$

(2) $\{\mathbf{3456}, \mathbf{145}, \mathbf{246}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 45, 46, 1, 2, 3, \emptyset\}$

(3) $\{\mathbf{235}, \mathbf{256}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{14}, 25, 56, 1, 2, 3, 4, \emptyset\}$

(4) $\{\mathbf{2456}, \mathbf{156}, \mathbf{235}, \mathbf{12}, \mathbf{13}, \mathbf{14}, 25, 56, 1, 2, 3, 4, \emptyset\}$

(5) $\{\mathbf{2345}, \mathbf{2356}, \mathbf{156}, \mathbf{12}, \mathbf{13}, \mathbf{14}, 235, 56, 1, 2, 3, 4, \emptyset\}$

(6) $\{\mathbf{3456}, \mathbf{156}, \mathbf{235}, \mathbf{12}, \mathbf{13}, \mathbf{24}, 35, 56, 1, 2, 3, 4, \emptyset\}$

(7) $\{\mathbf{2345}, \mathbf{156}, \mathbf{256}, \mathbf{12}, \mathbf{13}, \mathbf{14}, 25, 56, 1, 2, 3, 4, \emptyset\}$

(8) $\{\mathbf{235}, \mathbf{256}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{24}, 25, 56, 1, 2, 3, 4, \emptyset\}$

(9) $\{\mathbf{1456}, \mathbf{235}, \mathbf{256}, \mathbf{12}, \mathbf{13}, \mathbf{24}, 25, 56, 1, 2, 3, 4, \emptyset\}$

(10) $\{\mathbf{235}, \mathbf{356}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{24}, 35, 56, 1, 2, 3, 4, \emptyset\}$

A.3. **Codes with 7 maximal codewords.**

(1) $\{\mathbf{146}, \mathbf{156}, \mathbf{246}, \mathbf{12}, \mathbf{13}, \mathbf{23}, \mathbf{45}, 16, 46, 1, 2, 3, 4, 5, \emptyset\}$

(2) $\{\mathbf{145}, \mathbf{146}, \mathbf{245}, \mathbf{346}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 14, 45, 46, 1, 2, 3, \emptyset\}$

(3) $\{\mathbf{145}, \mathbf{146}, \mathbf{156}, \mathbf{245}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 14, 15, 16, 45, 1, 2, 3, \emptyset\}$

(4) $\{\mathbf{145}, \mathbf{146}, \mathbf{245}, \mathbf{256}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 14, 25, 45, 1, 2, 3, 6, \emptyset\}$

(5) $\{\mathbf{145}, \mathbf{156}, \mathbf{245}, \mathbf{346}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 15, 45, 1, 2, 3, 4, 6, \emptyset\}$

(6) $\{\mathbf{145}, \mathbf{156}, \mathbf{246}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 15, 45, 46, 56, 1, 2, 3, 5, \emptyset\}$

(7) $\{\mathbf{3456}, \mathbf{145}, \mathbf{156}, \mathbf{246}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 15, 45, 46, 56, 1, 2, 3, 5, \emptyset\}$

(8) $\{\mathbf{145}, \mathbf{245}, \mathbf{346}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 45, 46, 1, 2, 3, \emptyset\}$

(9) $\{\mathbf{145}, \mathbf{246}, \mathbf{356}, \mathbf{456}, \mathbf{12}, \mathbf{13}, \mathbf{23}, 45, 46, 56, 1, 2, 3, \emptyset\}$

(10) $\{\mathbf{3456}, \mathbf{156}, \mathbf{245}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{23}, 45, 56, 1, 2, 3, 4, \emptyset\}$

Appendix B. Pure codes on 6 neurons with unknown convexity status

Recall that Table 2 classifies minimal 6-neuron codes for which the neural complex is connected and pure. Among such codes that are pure of dimension 2 (respectively, dimension 3), there are 1004 (respectively, 76) with unknown convexity status. Partial lists of these codes are appear in Sections B.1 and B.2, and the complete lists are posted to GitHub [1].

B.1. **Codes of dimension 2.** Among the 1004 "unknown" minimal 6-neuron codes that are pure of dimension 2, the number of maximal codewords is between 5 and 14. Below we list one such code for each possible number of maximal codewords (from 5 to 14).

(1) $\{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{135}, \mathbf{456}, 12, 13, 4, 5, 6, \emptyset\}$

(2) $\{\mathbf{123}, \mathbf{124}, \mathbf{135}, \mathbf{156}, \mathbf{245}, \mathbf{246}, 12, 13, 15, 24, 5, 6, \emptyset\}$

(3) $\{\mathbf{123}, \mathbf{124}, \mathbf{135}, \mathbf{145}, \mathbf{236}, \mathbf{356}, \mathbf{456}, 12, 13, 14, 15, 23, 35, 36, 45, 56, 1, 3, 5, \emptyset\}$

(4) $\{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{145}, \mathbf{156}, \mathbf{234}, \mathbf{356}, \mathbf{456}, 12, 14, 15, 16, 23, 24, 45, 56, 1, 2, 3, 5, \emptyset\}$

(5) $\{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{134}, \mathbf{136}, \mathbf{145}, \mathbf{234}, \mathbf{256}, \mathbf{345}, 12, 13, 14, 16, 23, 24, 26, 34, 45, 1, 2, 3, 4, 5, \emptyset\}$

(6) $\{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{126}, \mathbf{134}, \mathbf{135}, \mathbf{156}, \mathbf{245}, \mathbf{346}, \mathbf{356}, 12, 13, 14, 15, 16, 24, 25, 34, 35, 36, 56, 1, 2, 3, 5, \emptyset\}$

(7) $\{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{126}, \mathbf{134}, \mathbf{135}, \mathbf{136}, \mathbf{146}, \mathbf{156}, \mathbf{245}, \mathbf{346}, 12, 13, 14, 15, 16, 24, 25, 34, 36, 46, 1, 2, 3, 4, 6, \emptyset\}$

(8) $\{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{135}, \mathbf{145}, \mathbf{156}, \mathbf{235}, \mathbf{236}, \mathbf{246}, \mathbf{256}, \mathbf{345}, \mathbf{356}, 12, 13, 14, 15, 23, 24, 25, 26, 35, 36, 45, 56, 1, 2, 3, 5, 6, \emptyset\}$

(9) $\{\mathbf{123}, \mathbf{124}, \mathbf{126}, \mathbf{134}, \mathbf{135}, \mathbf{136}, \mathbf{145}, \mathbf{234}, \mathbf{235}, \mathbf{245}, \mathbf{345}, \mathbf{346}, \mathbf{356}, 12, 13, 14, 15, 16, 23, 24, 25, 34, 35, 36, 45, 1, 2, 3, 4, 5, \emptyset\}$

(10) $\{\mathbf{123}, \mathbf{124}, \mathbf{125}, \mathbf{126}, \mathbf{135}, \mathbf{136}, \mathbf{146}, \mathbf{156}, \mathbf{235}, \mathbf{236}, \mathbf{245}, \mathbf{256}, \mathbf{345}, \mathbf{356}, 12, 13, 14, 15, 16, 23, 24, 25, 26, 35, 36, 45, 56, 1, 2, 3, 5, 6, \emptyset\}$

B.2. Codes of dimension 3.

(1) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1246}, \mathbf{1256}, \mathbf{1345}, 123, 124, 125, 126, 134, 135, 12, 13, \emptyset\}$

(2) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1236}, \mathbf{1256}, \mathbf{1345}, 123, 125, 126, 134, 135, 12, 13, \emptyset\}$

(3) $\{\mathbf{1234}, \mathbf{1236}, \mathbf{1256}, \mathbf{1345}, \mathbf{2345}, 123, 126, 134, 234, 345, 15, 25, 34, 1, 2, 5, \emptyset\}$

(4) $\{\mathbf{1234}, \mathbf{1236}, \mathbf{1256}, \mathbf{1345}, \mathbf{2346}, 123, 126, 134, 234, 236, 15, 23, 1, \emptyset\}$

(5) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1256}, \mathbf{1345}, \mathbf{2356}, 123, 125, 134, 135, 235, 256, 13, 25, \emptyset\}$

(6) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1236}, \mathbf{1245}, \mathbf{1456}, \mathbf{2345}, 123, 124, 125, 145, 234, 235, 245, 12, 16, 23, 24, 25, 1, 2, \emptyset\}$

(7) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1246}, \mathbf{1256}, \mathbf{1345}, \mathbf{3456}, 123, 124, 125, 126, 134, 135, 345, 12, 13, 46, 56, 4, 5, 6, \emptyset\}$

(8) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1246}, \mathbf{1256}, \mathbf{1345}, \mathbf{2345}, 123, 124, 125, 126, 134, 135, 234, 235, 345, 12, 13, 23, 34, 35, 3, \emptyset\}$

(9) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1246}, \mathbf{1256}, \mathbf{1345}, \mathbf{2456}, 123, 124, 125, 126, 134, 135, 246, 256, 12, 13, 26, 45, 4, 5, \emptyset\}$

(10) $\{\mathbf{1234}, \mathbf{1235}, \mathbf{1236}, \mathbf{1246}, \mathbf{1256}, \mathbf{1345}, 123, 124, 125, 126, 134, 135, 12, 13, \emptyset\}$

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA