Can quasicircular mergers of charged black holes produce extremal black holes?

Gabriele Bozzola [*]
Department of Astronomy, University of Arizona, Tucson, Arizona 85721, USA

Vasileios Paschalidis [†]
Department of Astronomy, University of Arizona, Tucson, Arizona 85721, USA and Department of Physics, University of Arizona, Tucson, Arizona 85721, USA

(Received 11 July 2023; accepted 22 August 2023; published 7 September 2023)

In contrast to energy and angular momentum, electric charge is conserved in mergers of charged black holes. This opens up the possibility for the remnant to have Kerr-Newman parameter $\chi^2 + \lambda^2$ greater than 1 (with χ and λ being the black hole dimensionless spin and dimensionless charge, respectively), which is forbidden by the cosmic censorship conjecture. In this paper, we investigate whether a naked singularity can form in quasicircular mergers of charged binary black holes. We extend a theoretical model to estimate the final properties of the remnant left by quasicircular mergers of binary black holes to the charged case. We validate the model with numerical-relativity simulations, finding agreement at the percent level. We then use our theoretical model to argue that while naked singularities cannot form following quasicircular mergers of nonspinning charged binary black holes, it is possible to produce remnants that are arbitrarily close to the extremal limit.

DOI: 10.1103/PhysRevD.108.064010

I. INTRODUCTION

This paper is concerned with quasicircular mergers of electrically charged black holes. To set the stage of our work, it is convenient to first consider the more familiar case without charge. In 2006, it was observed that the latest stages of the inspiral of two highly spinning black holes are significantly different compared to the nonspinning counterpart [1]. The main difference is that inspiral of spinning black holes takes substantially longer than the nonspinning case—an effect that is referred to as the *orbital hang-up*. One way to understand why this happens involves conservation of angular momentum and the cosmic censorship conjecture. If all the angular momentum available in the system (orbital + spins) were to end up in the remnant, the object would be over-extremal, i.e. its dimensionless spin χ would be larger than 1. Such a black hole is not possible in general relativity, and Kerr spacetimes with $\chi > 1$ are not black holes, but naked singularities (see, e.g. [2]). Given that the formation of naked singularities is forbidden by the cosmic censorship conjecture,[1] the binary has to radiate away all the excess angular momentum to be able to merge. To do so, the black holes inspiral for longer so that the gravitational waves can carry away the excess angular momentum.

Now, consider mergers of charged black holes. While energy and angular momentum can be radiated away, electric charge is always conserved. For this reason, a natural question to ask is whether it is possible to start from charged black holes with individual charge-to-mass ratio $\lambda < 1$ and form an extremal remnant.[2] If this does not happen, how is the formation of a naked singularity avoided? Is there a charge-induced orbital hang-up? Does the system inspiral or does it outspiral after sufficient energy has been radiated away? This paper aims to answer these questions by extending the method described in [5,6] to charged black hole binaries. We are going to refer to this method as "BKL" (Buonanno-Kidder-Lehner)[3] from the initials of the original authors [5].[4] The approach is based on conservation arguments and analogy with point

*gabrielebozzola@arizona.edu
†vpaschal@arizona.edu

[1]Note, however that the formation of naked singularities in fine-tuned dynamical scenarios is possible [3].

[2]In [4], we investigated a similar question checking whether ultra-relativistic head-on collisions of black holes can lead to extremal configurations. We found that there is no indication that this can happen. In that case, the formation of a naked singularity was avoided by the large kinetic energy in the system.

[3]Not to be confused with the BKL singularity studied by Belinski-Khalatnikov-Lifshitz.

[4]The contribution of [6] is to include the loss of energy due to gravitational wave. As we will discuss later, this is needed to match the to reach percent-level agreement with the numerical relativity simulations.

particles. Previous studies have shown that this simple argument is surprisingly effective at capturing the remnant properties to within a percent [6]. We validate our extended model via numerical relativity simulations of quasicircular mergers of nonspinning, charged binary black holes, and use it to argue that quasicircular inspirals of charged binary black holes cannot form naked singularities. We focus on configurations in which the black holes have charge with the same sign so that the total charge is greater than the individual black hole charges, which give the remnant black hole the possibility to maximize the Kerr-Newman parameter through its charge.

The goal of our paper is to explore and understand better cosmic censorship in the nonlinear regime and see if it introduces novel effects. Several linear arguments argued that black holes cannot be overcharged [7], but the question is still open for the nonlinear case. Moreover, while this paper focuses on fundamental physics implications, the analytic model predicting the remnant properties that is developed here has direct astrophysical applications. Charge in the astrophysical context of binary mergers has recently received some attention as "charge" can acquire different meanings from magnetic monopoles to modified gravity (see, e.g., Introduction in [8]). Additionally, remnant properties can be useful when studying populations of magnetically charged primordial black holes (see, e.g., [9]) and in gravitational-wave astronomy, where they are the starting point to study the ringdown signal (see, e.g. [10]). The analytic model described in this paper is computationally efficient, which makes it optimally suited for quick estimates.

This paper is structured as follows. In Sec. II, we describe the formalism we developed for predicting the properties of the remnant of charged binary black holes. In Sec. III, we show and discuss our results. We conclude with Sec. IV. We work in geometrized and gaussian units with $G = c = (4\pi\varepsilon_0)^{-1} = 1$, with G being Newton's constant, c the speed of light in vacuum, and ε_0 the permittivity of vacuum.

II. SETUP

We approach the problem of quasicircular mergers of charged black holes with an analytical model that we validated with numerical relativity simulations. In Sec. II A, we discuss the analytical method, and in Sec. II B we present our framework for the full nonlinear calculations.

A. Analytical model

To estimate the properties of the remnant left by the merger of two charged black holes, we follow Ref. [6], which extended the approach outlined of Ref. [5]. This method is based on conservation principles and an effective one-body treatment. The core assumption of the model is that energy and angular momentum lost can be determined by looking at the properties of the innermost stable circular orbit (ISCO) of a properly computed effective background spacetime [5,6,11]. This is because the system loses the vast majority of its initial energy and angular momentum during the inspiral, and when it reaches the ISCO, the plunge is so rapid that there is no significant loss of energy and angular momentum (i.e., the emission is small compared to the rest of the inspiral). In a nutshell, the method consists of finding a suitable background spacetime and computing the properties of its innermost-stable circular orbit.

1. The effective one-body problem

BKL [5] propose to treat the general relativistic two body problem as if it was in the limit of extreme mass ratio, where the system is equivalent to a test mass in a background spacetime. Strictly speaking, this approach is invalid for comparable masses, but previous work found that the agreement with the full nonlinear solution is excellent [5,6]. Hence, we adopt the same basic idea and extend it to include the charge in black hole spacetimes.

Consider two black holes that are separated by a distance that is large enough so that we can give them a well-defined mass and charge m_1, m_2, and q_1, q_2, with total mass and charge $M = m_1 + m_2$, $Q = q_1 + q_2$. The equivalent one-body problem has a test-mass with mass $m_{\rm red}$, charge $q_{\rm red}$ in a Kerr-Newman spacetime with mass M, charge Q, and angular momentum J. Here, $m_{\rm red}$ and $q_{\rm red}$ are the reduced mass and charge, defined as

$$m_{\rm red} = \frac{m_1 m_2}{M}, \tag{1a}$$

$$q_{\rm red} = \frac{q_1 q_2}{Q}. \tag{1b}$$

As we will see later, it is more convenient to work with dimensionless charge $\lambda = Q/M$, $\mathsf{q} = q_{\rm red}/m_{\rm red}$, and dimensionless spin $\chi = J/M^2$.

In the BKL framework, the energy (angular momentum) radiated by gravitational waves is the orbital energy (angular momentum) of the test particle up to the innermost-stable circular orbit.[5] We make the same assumption and compute final mass, spin, and charge for a binary merger by studying the ISCO. If $\varepsilon^{\mathsf{q}}_{\rm ISCO}(\lambda, \chi)$ is the specific energy at the ISCO for a particle with reduced charge q in a Kerr-Newman spacetime with charge-to-mass ratio $\lambda_{\rm final}$ and dimensionless spin $\chi_{\rm final}$ (note, ε is independent of the mass M), the energy radiated is

$$E_{\rm GW} = m_{\rm red} - m_{\rm red}\varepsilon^{\mathsf{q}}_{\rm ISCO}(\lambda_{\rm final}, \chi_{\rm final}), \tag{2}$$

[5]Note that there are two ISCOs, prograde and retrograde with respect to the rotation of the black hole. In this paper, we only consider prograde ones.

where λ_{final} and χ_{final} have yet to be determined. Conservation of energy implies that

$$M_{\mathrm{final}} = M - E_{\mathrm{rad}} = M(1 - \nu(1 - \varepsilon^{\mathsf{q}}_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}}))), \quad (3)$$

where E_{rad} is the total energy emitted in gravitational and electromagnetic waves, and $\nu = m_{\mathrm{red}}/M$ is the symmetric mass ratio. Similarly, all the angular momentum is radiated away except for the amount available at the ISCO, which is $J_{\mathrm{ISCO}} = m_{\mathrm{red}} M l^{\mathsf{q}}(\lambda_{\mathrm{final}}, \chi_{\mathrm{final}})$, where ℓ is the dimensionless angular momentum at the ISCO of a Kerr-Newman spacetime with charge λ_{final} and spin χ_{final}.[6] Therefore, we have that

$$\chi_{\mathrm{final}} = \frac{J_{\mathrm{ISCO}}}{{M_{\mathrm{final}}}^2} = \frac{\nu \ell_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}})}{[1 - \nu(1 - \varepsilon_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}}))]^2}. \quad (4)$$

Charge is conserved, so $Q_{\mathrm{final}} = Q = q_1 + q_2$, and

$$\lambda_{\mathrm{final}} = \frac{Q}{M_{\mathrm{final}}} = \frac{\lambda}{1 - \nu(1 - \varepsilon_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}}))}. \quad (5)$$

Finally, we have the coupled system of nonlinear algebraic equations

$$\chi_{\mathrm{final}} = \frac{\nu \ell^{\mathsf{q}}_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}})}{[1 - \nu(1 - \varepsilon^{\mathsf{q}}_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}}))]^2}, \quad (6a)$$

$$\lambda_{\mathrm{final}} = \frac{\lambda}{1 - \nu(1 - \varepsilon^{\mathsf{q}}_{\mathrm{ISCO}}(\chi_{\mathrm{final}}, \lambda_{\mathrm{final}}))}. \quad (6b)$$

The unknowns in these equations are χ_{ISCO} and λ_{ISCO}, that we find numerically with the Levenberg-Marquardt algorithm [12,13] as implemented in the `root` function in `scipy.optimize` [14].

2. *ISCO for a charged particle in a Kerr-Newman spacetime*

To solve the system defined by Eq. (6), we need to compute the dimensionless energy ε and angular momentum ℓ for particles with charge-to-mass ratio q on the ISCO of Kerr-Newman spacetimes. If we focus on the equatorial plane and work in Boyer-Lindquist coordinates (t, r, θ, ϕ), these quantities can be calculated using an effective potential $V_{\mathrm{eff}}(r)$. For Kerr-Newman black holes with unit mass,[7] charge λ, and spin χ, V_{eff} is given by [15–17] (see, Sec. IVA. in [17])

$$\begin{aligned} V_{\mathrm{eff}}(r) = \frac{1}{r^4}[&-\Delta(r) + (\Delta(r) - \chi^2)\tilde{l}^2(r) \\ &- 2\chi(r^2 + \chi^2 - \Delta(r))\tilde{l}(r)\tilde{\varepsilon}(r) \\ &+ ((r^2 + \chi^2)^2 - \Delta(r)\chi^2)\varepsilon^2(r)], \end{aligned} \quad (7)$$

with

$$\Delta(r) = r^2 - 2r + \chi^2 + \lambda^2, \quad (8a)$$

$$\tilde{l}(r) = l(r) + \mathsf{q}\frac{\lambda}{r}\chi, \quad (8b)$$

$$\tilde{\varepsilon}(r) = \varepsilon(r) + \mathsf{q}\frac{\lambda}{r}, \quad (8c)$$

where $\varepsilon(r)$ and $l(r)$ are the specific energy and dimensionless angular momentum ($\ell = a/m_{\mathrm{red}}$) for circular orbits of radius r. The properties of the ISCO are found solving the following equations simultaneously

$$V_{\mathrm{eff}}(r_{\mathrm{ISCO}}) = 0, \quad (9a)$$

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r}(r_{\mathrm{ISCO}}) = 0, \quad (9b)$$

$$\frac{\mathrm{d}^2 V_{\mathrm{eff}}}{\mathrm{d}r^2}(r_{\mathrm{ISCO}}) = 0, \quad (9c)$$

for r_{ISCO}, $\varepsilon_{\mathrm{ISCO}} = \varepsilon(r_{\mathrm{ISCO}})$ and $\ell_{\mathrm{ISCO}} = \ell(r_{\mathrm{ISCO}})$. Equations (9a) and (9b) impose circularity of the orbit, Eq. (9c) the condition of being innermost stable.

These equations can be solved analytically, but it is simpler and faster to solve them numerically. In practice, we use SymPy [18] to derive symbolically $V_{\mathrm{eff}}(r)$ and Eq. (9). Then, we solve this system numerically using the `root` function in `scipy.optimize` [14]. This gives us $\varepsilon^{\mathsf{q}}_{\mathrm{ISCO}}(\lambda, \chi)$ and $l^{\mathsf{q}}_{\mathrm{ISCO}}(\lambda, \chi)$ for a particle of charge-to-mass ratio q in a Kerr-Newman spacetime with dimensionless charge and spin λ and χ. This is what we need in order to solve Eq. (6). The Python code that implements the entire scheme is provided in the Supplemental Material [19].

B. Numerical simulations

We validate the model described in the previous section using two sets of numerical relativity simulations. First, we use the simulations of the quasicircular inspiral and merger of unequal-mass (mass ratio of 29/36) charged black holes we presented in [4,17,20,21]. Second, we perform new simulations with higher resolution and charge. The second set consists of eleven simulations with charge-to-mass ratio up to $\lambda = 0.6$ (like sign charge) and equal mass. Systems with higher λ take a significantly longer time to merge and require higher resolution and larger numerical grid. Given that our current set of simulations already took months to

[6] $M\ell$ is the specific angular momentum, so $m_{\mathrm{red}} M\ell$ is the actual angular momentum.

[7] Note, ε and ℓ are independent of the mass.

complete, the computational cost for calculations with larger change-to-mass ratio is currently prohibitive.

Our numerical relativity simulations solve the coupled Einstein-Maxwell equations in a $3+1$ decomposition of the spacetime (for more details, see, [22–24]) and use the `Einstein Toolkit` [25–28] for the numerical integration. We generate initial data with `TwoChargedPunctures` [8] for systems of two black holes with fixed charge-to-mass ratio λ. We use sixth-order finite-difference methods to evolve the spacetime with the `Lean` code [29], which implements the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation of Einstein's equations [30,31] and the electromagnetic fields are evolved with the massless version of the `ProcaEvolve` [32] code, part of Canuda suite [33,34]. We locate apparent horizons with `AHFinderDirect` [35,36], and their physical properties are measured with `QuasiLocalMeasuresEM`, a version of `QuasiLocalMeasures` [37] updated to implement the isolated horizon formalism in full Einstein-Maxwell theory (see Sec. II C in [8]).

We work with Cartesian grids with Berger-Oliger adaptive mesh refinement as provided by `Carpet` [38]. The simulations use between nine and thirteen refinement levels centered on and tracking the centroid of the black hole apparent horizons. The initial separation is $12.1M$, where M is the total ADM mass of the system. Eccentricity is kept below 0.01 with the method described in [17]. For the higher values of charge, we also had to manually adjust the initial momenta to meet the target eccentricity. We did so through trial and error. The resolution of our unequal-mass simulations is $M/65$ for the unequal mass case, whereas out set of equal-mass simulations has charge-dependent resolution as follows: the finest grid spacing set to $\Delta x_{\mathrm{finest}} = \sqrt{1-\lambda^2}/320M$,[8] which ensures that the horizons are resolved with more than 80 grid points. The damping parameters η and κ in the evolution equations of the shift vector and the electric field were set to $1.5M$ and $10M$. We refer the reader to [4,17,20] for a detailed complete discussion of the methods and tools we employ.

In addition to the convergence studies described in [4,17,20], we performed more simulations at higher resolution to estimate errors an convergence properties. In all cases, we find that the quasilocal properties of the black hole (mass, spin, charge) are exceptionally well-behaved and we estimate the numerical error due to finite resolution to be at the level of 0.1%.

III. RESULTS

In Fig. 1, we show the predictions of the model for mergers of black holes with the same mass and charge-to-mass ratio

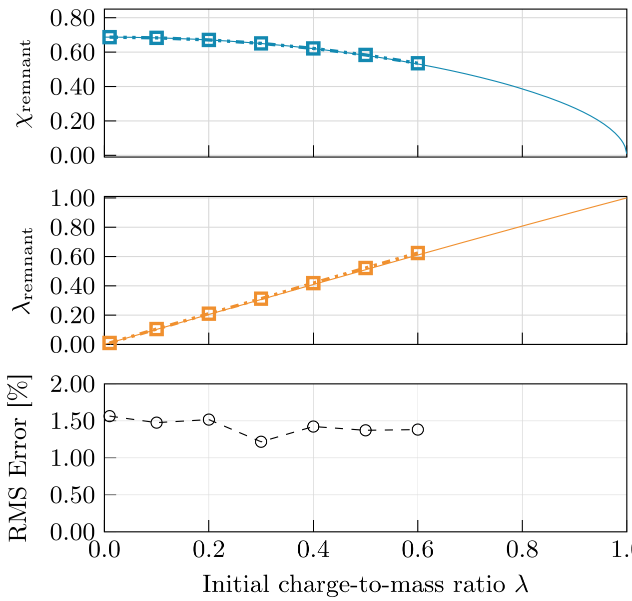

FIG. 1. Top two panels: Physical properties of the remnants left by the merger of two equal-mass, equal-charge black holes for the numerical-relativity simulations (squares) and the analytical predictions (solid lines). Bottom panel: Total relative error, measured as in Eq. (10). The error is consistently around 1.5%. The case with mass ratio 29/36 is similar.

and we plot the result from the numerical-relativity simulations (squares).

We measure the total error as

$$\text{RMS Error} = \sqrt{\left(\frac{\chi_{\mathrm{model}} - \chi_{\mathrm{sim}}}{\chi_{\mathrm{model}}}\right)^2 + \left(\frac{\lambda_{\mathrm{model}} - \lambda_{\mathrm{sim}}}{\lambda_{\mathrm{model}}}\right)^2}, \tag{10}$$

and find that the error is about 1.5% independently of the value of λ (bottom panel). By comparing the two terms in Eq. (10), we find that most of the error comes from the charge-to-mass ratio as opposed to the spin. We also find the same error level and behavior in the unequal-mass cases we consider here. Therefore, we conclude that the method described in the previous sections is effective at predicting the properties of the remnant left by the merger of two charged black holes with mass ratios close to unity, and equal charge-to-mass ratio. Considering the complex and nonlinear system under consideration, this is a remarkable agreement.

Our numerical relativity simulations verified that the method described in Sec. II A can correctly capture the properties of the remnant left by the merger of charged black holes. With this, we can now look at what happens when we consider the case with $\lambda \to 1$. In this, we are interested in checking whether the remnant left by the merger would be over-extremal, i.e., $\lambda^2 + \chi^2 > 1$. Kerr-Newman spacetimes are the most general axisymmetric and

[8]In isotropic coordinates, the horizon radius for a Reissner-Nordström black hole with mass $M_1 = 0.5M$ and charge $Q_1 = \lambda M_1$ is $\sqrt{1-\lambda^2}/4M$.

stationary four-dimensional electrovacuum spacetimes, and when $\lambda^2 + \chi^2 > 1$ they describe naked singularities. So, a merger of charge black holes such that the remnant has $\lambda^2 + \chi^2 > 1$ would be a good candidate to violate cosmic censorship.

At this point, it is useful to recall what happens in the case of the merger of two uncharged black holes with spin $\chi_0 \to 1$. If we assumed that all the angular momentum and mass end up in the remnant black hole, we would find that the Kerr remnant has spin (for identical black holes with prograde dimensionless spin χ_0) $2\chi_0 + \chi_{\mathrm{orbital}} > 1$, hinting to a violation of the cosmic censorship conjecture. This does not happen because the vast majority of the total angular momentum is radiated away through emission of gravitational waves. In particular, the system orbits for longer than the nonspinning case to radiate all the excess angular momentum and ensure that the remnant is a black hole and not a naked singularity. We can construct a similar thought experiment for the case of charge, with the difference that charge is conserved. Therefore, if we start with charge-to-mass ratio λ_0, the remnant must have $\lambda > 2\lambda_0$ (because there is emission of energy), and its spin must be greater than 0. So, it appears that there could be conditions that favor $\lambda^2 + \chi^2 > 1$. Given charge conservation, Nature has to find a new way to avoid the formation of a naked singularity, if cosmic censorship is not violated.

In Fig. 2, we plot the prediction for $1 - \sqrt{\lambda^2 + \chi^2}$ for the remnant left by the merger of two equal mass, equal charge binaries according to the analytical model described earlier. Cosmic censorship demands that $1 - \sqrt{\lambda^2 + \chi^2} > 0$ ($M^2 > Q^2 + a^2$). The plot shows that it is possible to have a remnant that is arbitrary close to extremality (with $\lambda^2 + \chi^2 \to 1$), but it is not possible to pass this limit. This means that the quasicircular mergers of charged black holes should not be expected to lead to naked singularities. The reason for this is different from the case of purely spinning

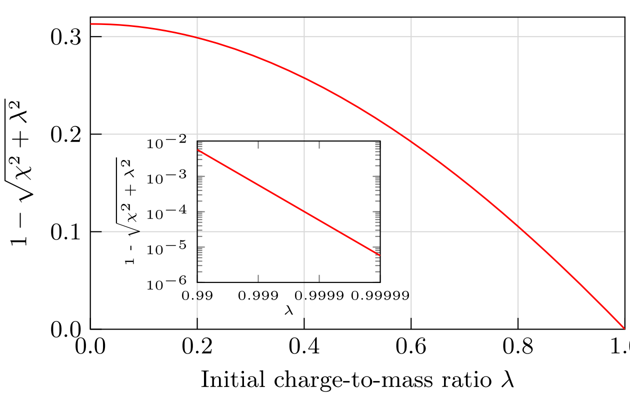

FIG. 2. Kerr-Newman parameter as a function of the initial charge-to-mass ratio for quasicircular mergers of equal-mass, equal-charge binaries, as predicted with the method described in this paper. The inset shows that even with $\lambda \to 1$, we have that the Kerr-Newman parameter is larger than 0.

black holes, where is the emission of angular momentum that prevents the over-extremality, and more similar to the results found in [39] for head-on collisions. We find is that with higher charge, the binary is less and less bound, and the orbital acceleration is smaller and smaller. With smaller acceleration, there is weaker emission of gravitational and electromagnetic waves. In the limit of $\lambda \to 1$, the binary takes an infinite amount of time to merge, emitting a vanishing amount of energy. For λ identically equal to 1, the system is in equilibrium with gravitational and electrostatic forces canceling out (this is the Majumdar-Papapetrou solution [8,40,41]). In our simulations we find that a binary with $\lambda = 0.6$ orbits twice as many times as uncharged binary before merger. Moreover, with increasing λ, the orbital angular momentum decreases (it roughly goes as $\sqrt{1 - \lambda^2}$), so that the spin of the remnant becomes arbitrarily small (as seen in the top panel of Fig. 1).

IV. CONCLUSIONS

In this paper, we presented an analytical and computationally cheap method to estimate the properties of the remnant left by the merger of two charged black holes. The method is an extension of the technique developed in Refs. [5,6] which uses an effective-one-body treatment and the properties of the innermost-stable circular orbit in an equivalent Kerr-Newman spacetime. We performed numerical relativity simulations and verified that the method is accurate at the percent level. This shows that the simple argument is remarkably effective at quantitative predictions for the properties of the remnant.

While the results presented here are only for quasicircular mergers, we expect them to hold for eccentric as well, because these orbits are bound, too. The analytical method presented here essentially compares the energy at infinity and the energy at the ISCO, so it does not matter how one reaches the ISCO (as in the case without charge [42]). The only exception is that for highly eccentric mergers one would need to consider the ISCO for these orbits. The analytic model we presented works for arbitrary mass, spin, and charge configurations, but our validation only involved quasicircular mergers with nonspinning black holes of comparable mass and charge-to-mass ratio values up to 0.6. When more numerical-relativity simulations of charged inspirals will be available, the validation can be extended. This is one of the possible limitations of the argument that overextremal black holes cannot form in quasicircular mergers of charged black holes.

Our second goal was to learn more about quasicircular mergers of highly charged black holes in the context of the cosmic censorship conjecture. Charge is conserved quantity, whereas energy and angular momentum are not. When two identical black holes with charge-to-mass ratio λ_0 merge, the remnant has to have $\lambda > 2\lambda_0$ because of emission of energy. Moreover, the dimensionless spin χ of the remnant has to be larger than 0. This opens up the possibility that some initial configurations might lead to a remnant with $\lambda^2 + \chi^2 > 1$. Given that the only axisymmetric and stationary spacetime is the Kerr-Newman one and that $\lambda^2 + \chi^2 > 1$ would mean that there is no horizon, finding such a configuration would hint to a possible way of forming naked singularities. In the case of spin, this is avoided via emission of angular momentum and the orbital hang-up. Our study shows that in the charged black hole case increasing the initial black hole charge makes the system less and less dynamical. With $\lambda_0 \to 1$, the system asymptotically takes an infinite amount of time to merge and radiate a vanishing amount of gravitational and electromagnetic waves. Moreover, when $\lambda_0 \to 1$ the orbital angular momentum also vanishes. Therefore, we conclude that while it is possible to produce remnants that are arbitrarily close to extremality, it is not possible to break the limit.

One of the reasons why it is not possible to form naked singularities is that the system is bound, which sets a limit on the available orbital angular momentum. In future studies, we will test the limits of our model by considering the case of unlike charges (which increase the angular momentum). We will also treat the case of unbound orbits, such as the hyperbolic encounters [43].

ACKNOWLEDGMENTS

We used KUIBIT [44] for part of our analysis. KUIBIT [44] use NumPy [45], H5py [46], SciPy [14], and SymPy [18]. We are grateful to the developers and maintainers of the open-source codes that we used. We thank Cyrus Worley for his help with the eccentricity estimates in simulation data. This work was in part supported by NSF Grants PHY-1912619, and PHY-2145421, as well as NASA Grant No. 80NSSC20K1542 to the University of Arizona. This work used Expanse (funded by the NSF through Award No. OAC-1928224) at the San Diego Supercomputing Center (SDSC), through allocation TG-PHY190020 from the Advanced Cyberinfrastructure Coordination Ecosystem: Services and Support (ACCESS) program, which is supported by National Science Foundation Grants No. 2138259, No. 2138286, No. 2138307, No. 2137603, and No. 2138296. The work also used NASA's High End Computing clusters.

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