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Two new families of finitely generated simple groups of homeomorphisms of the real line ☆

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ABSTRACT

The goal of this article is to exhibit two new families of finitely generated simple groups of homeomorphisms of **R**. These families are strikingly different from existing families owing to the nature of their actions on **R**, and exhibit surprising algebraic and dynamical features. The first construction provides the first examples of finitely generated simple groups of homeomorphisms of **R** that also admit minimal actions by homeomorphisms on the torus. The second construction provides the first examples of finitely generated simple groups of homeomorphisms of **R** which also admit a minimal action by homeomorphisms on the circle. This also provides new examples of finitely generated simple groups that admit nontrivial homogeneous quasimorphisms (and therefore have

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infinite commutator width), also being the first such left orderable examples.

1. Introduction

Whether finitely generated infinite simple groups of homeomorphisms of $\mathbf{R}$ exist had been a longstanding open question of Rhemtulla [10] (also asked by Clay and Rolfsen in [2], by Navas in [13], and in the "Kourovka Notebook" [7]) In [4] the first two authors constructed the first examples, answering the question in the affirmative. In a subsequent article [11], Matte Bon and Triestino demonstrated that certain groups of piecewise linear homeomorphisms of flows are also examples of this phenomenon. The groups of Matte Bon and Triestino generalize the construction of [4]. The connection has been formally explained in a paper by Le Boudec and Matte Bon [9]. The goal of this article is to exhibit two new families of examples that exhibit new, strikingly different dynamical and algebraic features, compared to existing families.

The question as stated originally asks whether finitely generated simple left orderable groups exist. However, note that left orderability for countable groups is equivalent to requiring that they admit a faithful action by orientation preserving homeomorphisms of the real line. Achieving the combination of finite generation and simplicity for such groups presents certain technical challenges owing to the lack of compactness of $\mathbf{R}$. Moreover, there are also certain natural obstructions to simplicity for various finitely generated groups of homeomorphisms of $\mathbf{R}$. If such a group is amenable, then it admits a homomorphism onto $\mathbf{Z}$ (see [15]). The same holds if the group admits a nontrivial action by C^1-diffeomorphisms on a closed interval (or even $[0,1)$, see [14]). For a more detailed discussion around these issues, we refer the reader to [4].

One key motivation for the construction of these new examples in the present article is to prove the following theorem. Note that any countable group of orientation preserving homeomorphisms of the real line admits faithful actions by homeomorphisms on any given manifold of dimension one or above. However, it is more desirable to search for actions that do not admit fixed points or proper, closed invariant subsets of the manifold. Recall that a group action on a topological space by homeomorphisms is *minimal* if all orbits are dense. Indeed, minimality is a desirable dynamical condition that one may require of an action on a manifold of choice. In this article we focus on the case of the torus and the circle. We prove the following.

Theorem 1.0.1. *Denote by $\mathcal{G}$ the class of finitely generated simple groups of homeomorphisms of the real line. The following holds:*

- *There exist $G \in \mathcal{G}$ that admit a minimal action by homeomorphisms on the torus.*

- *There exist $G \in \mathcal{G}$ that admit a minimal action by homeomorphisms on the circle.*
- *There exist $G \in \mathcal{G}$ that admit nontrivial homogeneous quasimorphisms, and that have infinite commutator width: for each $n \in \mathbf{N}$ there is an element that cannot be expressed as a product of fewer than n commutators.*
- *There exist $G \in \mathcal{G}$ that admit a faithful action by C^∞-diffeomorphisms of the circle.*
- *There exist $G \in \mathcal{G}$ that admit a faithful action by piecewise linear homeomorphisms of the circle.*

The examples that witness the above are divided in two families. Groups in the first family are finitely generated by definition, however it is surprising that they are simple, and the proof of simplicity involves an intricate analysis of the group action. The groups in the second family emerge as the derived subgroups of examples called *fast n-ring groups* (defined independently by Brin, Bleak, Kassabov, Moore and Zaremsky in [1] and by the second author with Kim and Koberda in [8]). The simplicity of these examples is less surprising, however it is surprising that they are finitely generated and left orderable. The proof of finite generation involves an intricate analysis of the group action.

We now present the first family. Recall that Thompson's group T is the group of piecewise linear orientation preserving homeomorphisms of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ such that:

(1) Each linear part is of the form $2^n + d$ for $n \in \mathbf{Z}, d \in \mathbf{Z}[\frac{1}{2}]/\mathbf{Z}$.
(2) There are finitely many points where the slopes do not exist, and they lie in $\mathbf{Z}[\frac{1}{2}]$.

The group $\overline{T} < \mathrm{Homeo}^+(\mathbf{R})$ is the central extension obtained by "lifting" this action to the real line. In particular, there is a short exact sequence

$$1 \to \mathbf{Z} \to \overline{T} \to T \to 1$$

Here the group $\mathbf{Z}$ is the group of integer translations of the real line, and it lies in the center of $\overline{T}$. It is easily seen that $\overline{T}$ is finitely presented, since T is finitely presented. The group $\overline{T}$ was first studied by Ghys and Sergiescu in [3], and it has several remarkable features.

One may modify the "lift", $\overline{T}$, as follows. Let $\mathbf{S}^1$ be as above, and consider the map

$$\phi_\lambda : \mathbf{R} \to \mathbf{S}^1 \qquad \mathbf{R} \to \mathbf{R}/\lambda\mathbf{Z}$$

for each $\lambda > 0$. The map ϕ_λ provides the alternative lift $\overline{T}_\lambda < \mathrm{Homeo}^+(\mathbf{R})$, which as an abstract group is isomorphic to $\overline{T} = \overline{T}_1$. Note that the center of $\overline{T}_\lambda$ is the group $\langle t \to t + n\lambda \mid n \in \mathbf{Z}\rangle$. In spite of the fact that $\overline{T}, \overline{T}_\lambda$ are not simple, we prove the following:

Theorem 1.0.2. *Let $\lambda > 1$ be irrational. The group $G_\lambda = \langle \overline{T}, \overline{T}_\lambda \rangle$ is simple.*

This provides a family of finitely generated simple groups of homeomorphisms of the real line which are very elementary to define. Moreover, they are shown to admit minimal actions on the torus by homeomorphisms. To explain this, we use a very nice description of the action suggested to us by an anonymous referee. This description also admits a natural generalization to higher dimensions.

For each interval $I \subset \mathbf{S}^1$, we define the subgroup F_I of Thompson's group T consisting of elements which pointwise fix the complement of I. Take the linear foliation on the 2-torus, generated by an irrational direction $\lambda \in \mathbf{R}^2$. For any interval $I \subset \mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ with dyadic endpoints and length $|I| < 1$, we let $F_{I\times\mathbf{S}^1}$ be the subgroup of homeomorphisms of $\mathbf{T}^2$ supported on the annulus $I \times \mathbf{S}^1$ which preserve the linear foliation, and the action on any segment obtained by considering the intersection of the linear foliation with the annulus $I \times \mathbf{S}^1$ projects on I to the action of the group F_I. Similarly one defines the groups $F_{\mathbf{S}^1\times I}$ by considering horizontal annuli $\mathbf{S}^1 \times I$. The group generated by the subgroups of the form $F_{\mathbf{S}^1\times I}$ and $F_{I\times\mathbf{S}^1}$ is the group G_λ. As the linear foliation is preserved by the action of G_λ, the restriction of the action on $\mathbf{T}^2$ on any leaf of the foliation gives an action of G_λ on the real line.

To describe the second family, we recall the notion of a *fast n-ring group.*

Definition 1.0.3. For $n \geq 3$, let $\{J_1, ..., J_n\}$ be a set of open intervals in $\mathbf{S}^1$ that cover $\mathbf{S}^1$, and homeomorphisms $\{f_1, ..., f_n\}$ that satisfy:

(1) $J_i \cap J_j = \emptyset$ if $|i - j| \notin \{0, 1\}$ **mod** n and is a nonempty, proper, subinterval of both J_i, J_j if $|i - j| = 1$ **mod** n.
(2) $J_i = Supp(f_i) = \{x \in \mathbf{S}^1 \mid x \cdot f_i \neq x\}$ for each $1 \leq i \leq n$.

The aforementioned configuration is called an *n-ring of intervals and homeomorphisms.* For $n \geq 3$, the group $G_n = \langle f_1, ..., f_n\rangle$ is said to be a *fast n-ring group* if the following holds. In what appears below, we interpret the subscripts as modulo n. For each $1 \leq i \leq n$, let x_i be the endpoint of J_{i+1} that lies in J_i. Then we have the following dynamical condition which we refer to throughout the article as $(*)$:

$$x_i \cdot f_i f_{i+1} ... f_{i+l} \in J_{i+l+1} \qquad \forall 1 \leq l \leq n$$

It was demonstrated in [1] that the isomorphism type of G_n does not depend on the choice of homeomorphisms $f_1, ..., f_n$, provided the dynamical condition $(*)$ is satisfied. The nature of the isomorphism type of G_n for $n \geq 3$ remains mysterious to the authors. Our second family emerges from the derived subgroups of these examples. First, we observe that these groups in fact admit actions on the line, by exploiting the dynamics that emerge from the condition $(*)$ above.

Proposition 1.0.4. *The lift of the given action of G_n to $\mathbf{R}$ is isomorphic to G_n.*

Finally, we show the following.

Theorem 1.0.5. *For each $n \geq 3$, the group $H_n = G_n'$ is finitely generated and simple.*

To provide another dynamical motivation for this second family, we recall the following dynamical trichotomy for group actions on the real line. We recall that an action $\Phi : G \to \text{Homeo}^+(\mathbf{R})$ is *proximal* if for every open interval $I \subseteq \mathbf{R}$ and bounded interval $J \subset \mathbf{R}$, there is a group element $f \in G$ such that $J \cdot \Phi(f) \subset I$. Also, we recall the notion of *semiconjugacy* of actions. Two actions $\Phi_1, \Phi_2 : G \to \text{Homeo}^+(\mathbf{R})$ are said to be semiconjugate if there is a non-decreasing map $\phi : \mathbf{R} \to \mathbf{R}$ that is also proper (the preimage of every compact set is bounded), and satisfy:

$$\phi \circ \Phi_1(g) = \Phi_2(g) \circ \phi$$

For every action of a finitely generated group G by orientation preserving homeomorphisms of the real line without global fixed points, there are three possibilities:

(i) There is a σ-finite measure μ that is invariant under the action.
(ii) The action is semiconjugate to a minimal action for which every small enough interval is sent into a sequence of intervals that converge to a point under well chosen group elements, however, this property does not hold for every bounded interval.
(iii) The action is globally contracting; more precisely, it is semiconjugate to a proximal action.

(For details, we refer the reader to [12]). Note that if a group admits a faithful action of type (i), then it is *indicable*: it admits a homomorphism onto $\mathbf{Z}$. Therefore, finitely generated simple groups of homeomorphisms of the real line may only admit actions of type (ii) or (iii). It was shown in [5] that the groups G_ρ constructed by the first two authors in [4] have the property that every action on the real line by homeomorphisms without global fixed points is of type (iii). The same was shown by Matte Bon and Triestino for their examples in [11]. The following is a corollary of Theorem 1.0.5, which illustrates a striking new phenomenon associated with the groups $H_n, n \geq 3$. Note that the fact that these group actions are of type (ii) follows from the fact that the action is minimal, locally contracting (as a consequence of Lemma 3.2.1), and each element commutes with integer translations.

Corollary 1.0.6. *There exist finitely generated simple left orderable groups which admit actions by orientation preserving homeomorphisms on the real line which are of type (ii).*

The reader should compare this with the results in [5] and [11], where the groups G_ρ were shown to be uniformly perfect. Given a group that admits an action of type (ii) on $\mathbf{R}$, it is easy to see that the action of the group on the orbit of 0 provides an unbounded homogenous quasimorphism. As a consequence, we have the following.

Corollary 1.0.7. *There exist finitely generated simple left orderable groups that admit non-trivial (unbounded) homogeneous quasimorphisms into the reals.*

This also has a nice algebraic consequence.

Corollary 1.0.8. *The groups $H_n, n \geq 3$ have infinite commutator width: for each $n \in \mathbf{N}$ there is an element that cannot be expressed as a product of fewer than n commutators.*

Note that the homeomorphisms that generate the group G_n can be realized as elements of Thompson's group T. It follows that the groups H_n are subgroups of T. It was shown in [3] that the given action of T on $\mathbf{S}^1$ is conjugate to an action on $\mathbf{S}^1$ by C^∞-diffeomorphisms.

We conclude the following.

Corollary 1.0.9. *There exists a finitely generated simple left orderable group that admits:*

(1) *a faithful action by C^∞-diffeomorphisms of the circle.*
(2) *a faithful action by piecewise linear homeomorphisms of the circle (with finitely many allowable breakpoints for each element).*
(3) *an embedding into Thompson's group T.*

Note that one may show (1) directly without appealing to [3], since we can simply choose the homeomorphisms that generate G_n to be smooth.

Convention 1.0.10. In this article, all group actions will be right actions. We will use the notation $[f, g] = fgf^{-1}g^{-1}$ and $f^g = g^{-1}fg$. For $f \in \text{Homeo}^+(\mathbf{R})$, we define $Supp(f) = \{x \in \mathbf{R} \mid x \cdot f \neq x\}$.

2. The first family

The goal of this section is to prove Theorem 1.0.2. We first state and discuss a few preliminaries.

2.1. Preliminaries

Throughout the section we denote by F the standard piecewise linear action of Thompson's group $F \leq \text{Homeo}^+[0, 1]$. This coincides with the stabilizer of 0 in the standard action of Thompson's T on the circle $\mathbf{R}/\mathbf{Z}$. Also, we fix $\lambda \in \mathbf{R} \setminus \mathbf{Q}, \lambda > 1$. Recall that $G_\lambda = \langle \overline{T}, \overline{T}_\lambda \rangle$ where $\overline{T}_\lambda < \text{Homeo}^+(\mathbf{R})$ is the lift of the action of T on $\mathbf{S}^1$ with the identification $\mathbf{R} \to \mathbf{R}/\lambda\mathbf{Z}$. We denote the center of $\overline{T}, \overline{T}_\lambda$ by $Z(\overline{T}), Z(\overline{T}_\lambda)$, respectively. Note that

$$Z(\overline{T}) = \{t \mapsto t + n \mid n \in \mathbf{Z}\} \qquad Z(\overline{T}_\lambda) = \{t \mapsto t + n\lambda \mid n \in \mathbf{Z}\}$$

Recall that the pointwise stabilizer of $\mathbf{Z}$ in $\overline{T}$, which we denote by F_1, is naturally isomorphic to Thompson's group F, which we define as the stabilizer of 0 in T. Indeed the restriction of this action of F_1 to each interval $[n, n+1]$ for each $n \in \mathbf{Z}$ is conjugate to the standard piecewise linear action of F on $[0,1]$ by the translation $t \to t+n$. The only normal subgroups of $\overline{T}$ are the subgroups of the center. Indeed, for any element $f \in \overline{T} \setminus Z(\overline{T})$, $\langle\langle f \rangle\rangle = \overline{T}$. The analogous statement holds for $\overline{T}_\lambda$.

We observe the following.

Lemma 2.1.1. *Every $g \in G_\lambda$ is Lipschitz. In particular, g is uniformly continuous.*

Proof. It is straightforward to see that the elements of $\overline{T}, \overline{T}_\lambda$ are Lipschitz. Since this property for homeomorphisms is closed under composition and inverses, we are done. □

2.2. The proof

The key idea in the proof of Theorem 1.0.2 is the following.

Proposition 2.2.1. *Let $f \in G_\lambda \setminus \{id\}$. For each $c \in \{1, \lambda\}$, there is an element $g \in \langle\langle f \rangle\rangle$ that satisfies the following.*

(1) *g fixes $c \cdot \mathbf{Z}$ pointwise.*
(2) *There exists a pair $x, y \in [0, c]$, $x < y$ such that*

$$(x + c \cdot n) \cdot g > y + c \cdot n \qquad \forall n \in \mathbf{Z}$$

Using Proposition 2.2.1, we can finish the proof of Theorem 1.0.2 as follows.

Proof of Theorem 1.0.2. Let $g_1, g_2 \in G_\lambda \setminus \{id\}$ be elements that satisfy the conclusion of Proposition 2.2.1 for $c = 1, c = \lambda$, respectively. We will show that:

(1) $\langle\langle g_1 \rangle\rangle_{G_\lambda} \cap (\overline{T} \setminus Z(\overline{T})) \neq \emptyset$.
(2) $\langle\langle g_2 \rangle\rangle_{G_\lambda} \cap (\overline{T}_\lambda \setminus Z(\overline{T}_\lambda)) \neq \emptyset$.

We know that the normal closure of any element in $(\overline{T} \setminus Z(\overline{T}))$ is all of $\overline{T}$. Similarly, the normal closure of any element in $(\overline{T}_\lambda \setminus Z(\overline{T}_\lambda))$ is all of $\overline{T}_\lambda$. So showing the above concludes the proof. Indeed, since the proofs for $(1), (2)$ are analogous, we shall just prove (1).

Thanks to Lemma 2.1.1, g_1^{-1} is Lipschitz, and recall that g_1 fixes $\mathbf{Z}$ pointwise. Combining this with the fact that there exists a pair $x, y \in [0,1]$, $x < y$ such that

$$(x + n) \cdot g_1 > y + n \qquad \forall n \in \mathbf{Z}$$

we obtain the following. We can find an open interval $x \in I \subset (0,1)$ such that

$$X \cdot g_1 \cap X = \emptyset \qquad \text{where } X = \bigcup_{n\in\mathbf{Z}} (I+n)$$

We proceed to find nontrivial elements $h_1, h_2 \in F_1 \leq \overline{T}$ such that

$$Supp(h_1), Supp(h_2) \subset X$$

and let $h = [h_1, h_2]$. Note that also $Supp(h) \subset X$. Since

$$Supp(g_1^{-1}h_2^{-1}g_1) = Supp(h_2^{-1}) \cdot g_1$$

it follows that

$$Supp(g_1^{-1}h_2^{-1}g_1) \cap Supp(h_1) = \emptyset$$

Let $h_3 = g_1^{-1}h_2^{-1}g_1$. By our assumption, we know that $[h_3, h_1] = id$. Therefore,

$$\begin{aligned}[h_1,[h_2,g_1^{-1}]] &= [h_1, h_2h_3] = h_1h_2(h_3h_1^{-1}h_3^{-1})h_2^{-1} = h_1h_2h_1^{-1}h_2^{-1} \\ &= [h_1,h_2] = h \in F_1 \setminus \{id\} \subseteq (\overline{T} \setminus Z(\overline{T}))\end{aligned}$$

finishing the proof. □

The rest of this section shall be devoted to proving Proposition 2.2.1. We will prove it for the case $c = 1$, the other case is completely analogous.

Definition 2.2.2. Let $G < \text{Homeo}^+(\mathbf{R})$ be a given subgroup. Given compact intervals I, J in $\mathbf{R}$ such that $|I| = |J|$, we denote by $T_{J,I} : \mathbf{R} \to \mathbf{R}$ the unique translation so that $T_{J,I}(J) = I$. Given $g, g_1, g_2 \in G$ and I, J as above, we define

$$d_g(I,J) = sup\{(|x\cdot g - x\cdot h| \mid x \in I\} \qquad \text{where } h = T_{I,J} \circ g \circ T_{J,I}$$

and

$$d_I(g_1,g_2) = sup\{|x\cdot g_1 - x\cdot g_2| \mid x \in I\}$$

Note that a direct computation gives us that $d_g(I,J) = d_g(J,I)$.

Lemma 2.2.3. *Consider an element*

$$g = u_1v_1...u_nv_n \in G_\lambda \qquad u_i \in \overline{T}, v_i \in \overline{T}_\lambda \textit{ for each } 1 \leq i \leq n$$

For each $\epsilon > 0$, there is a $\delta_1 > 0$ such that for each $\delta \in (-\delta_1, \delta_1)$, the element

$$g_\delta = u_1(f_\delta^{-1}v_1f_\delta)...u_n(f_\delta^{-1}v_nf_\delta) \qquad \textit{where } x\cdot f_\delta = x+\delta$$

satisfies that $d_{[0,1]}(g, g_\delta) < \epsilon$.

Proof. This follows from an elementary inductive argument on n, using uniform continuity of elements of T_λ (which follows from the fact that its elements are lifts of homeomorphisms of the circle, which is a compact space). $\square$

The following is a basic dynamical fact about irrational translations, we refer the reader to chapter 4 in [6].

Lemma 2.2.4. *Fix $\lambda \in \mathbf{R} \setminus \mathbf{Q}, \lambda > 1$. For each $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that for any interval I such that $|I| > N$, there are $m, k \in \mathbf{Z}$ such that*

$$[m, m+1] \subset I \qquad |m - k\lambda| < \epsilon$$

Definition 2.2.5. An element $g \in \mathrm{Homeo}^+(\mathbf{R})$ is *repetitive* if for each $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that for each interval I such that $|I| > N$, there is a subinterval $[m, m+1] \subset I, m \in \mathbf{Z}$ such that

$$d_g([0,1], [m, m+1]) < \epsilon$$

We say that a group action $G \leq \mathrm{Homeo}^+(\mathbf{R})$ is said to be *repetitive*, if every $g \in G$ is repetitive.

Proposition 2.2.6. *G_λ is repetitive.*

Proof. Consider a nontrivial element $g = u_1 v_1 \ldots u_n v_n \in G_\lambda$, where $u_i \in \overline{T}$ and $v_i \in \overline{T}_\lambda$. We will show that g is repetitive. Let $\epsilon > 0$. Applying Lemma 2.2.3, there is a $\delta_1 > 0$ such that for each $\delta \in (-\delta_1, \delta_1)$, the element

$$g_\delta = u_1(f_\delta^{-1} v_1 f_\delta) \ldots u_n(f_\delta^{-1} v_n f_\delta) \qquad \text{where } x \cdot f_\delta = x + \delta$$

satisfies that $d_{[0,1]}(g, g_\delta) < \epsilon$.

Using Lemma 2.2.4 we find an $N \in \mathbf{N}$ such that in every interval I of length at least N there are $m, k \in \mathbb{Z}$ such that $[m, m+1] \subset I$ and $|m - k\lambda| < \delta_1$. For such m, k there is a $\delta \in (-\delta_1, \delta_1)$ such that

$$g \restriction [m, m+1] = f_m^{-1} g_\delta f_m \qquad \text{where } t \cdot f_m = t + m$$

Combining this with the fact that $d_{[0,1]}(g, g_\delta) < \epsilon$, we obtain

$$d_g([0,1], [m, m+1]) < \epsilon \quad \square$$

For the rest of the section, we denote $\langle\langle g \rangle\rangle_{G_\lambda}$ as simply $\langle\langle g \rangle\rangle$. We shall now focus our attention on the pointwise stabilizer of $\mathbf{Z}$ in G_λ. Recall that the pointwise stabilizer of $\mathbf{Z}$ in $\overline{T}$ is F_1, defined in the preliminaries above.

Proposition 2.2.7. *Let $g \in G_\lambda \setminus \{id\}$. There is an open interval $J \subset (0,1)$ whose closure is also contained in $(0,1)$, and a nontrivial element $f \in \langle\langle g \rangle\rangle$ such that:*

$$Supp(f) \subset \bigcup_{n \in \mathbf{Z}} (n + J)$$

Proof. First we argue that $\langle\langle g \rangle\rangle$ must contain elements that do not lie in $\langle t \to t+n \mid n \in \mathbf{Z} \rangle$. If g is itself not in this subgroup, then we are done. Otherwise, we can find an element $f \in \overline{T}_\lambda$ such that the element $[g,f] \neq id$ and satisfies the required property.

We assume in the rest of the proof that $g \notin \langle t \to t+n \mid n \in \mathbf{Z} \rangle$. It follows that there is an $m \in \mathbf{Z}$ and an open interval J such that $\overline{J} \subset (m, m+1)$ and $g^{-1} \restriction J$ is not the restriction of an integer translation. We may assume for simplicity that $\overline{J} \subset (0,1)$, by replacing g by a conjugate of g by an integer translation. Since g is Lipschitz, for each $\epsilon > 0$ there is a $\delta > 0$ such that given an interval I, if $|I| < \delta$, then $|I \cdot g| < \epsilon$.

Since $g^{-1} \restriction J$ is not the restriction of an integer translation, combining this with the Lipschitz condition we can choose a sufficiently small $\epsilon > 0$ and an open interval $I \subset (inf(J) + \epsilon, sup(J) - \epsilon)$ such that:

$$|I| < \delta \qquad (I \cdot g^{-1}) \cap \bigcup_{n \in \mathbf{Z}} (n + I) = \emptyset \qquad \forall n \in \mathbf{Z}, |(I+n) \cdot g| < \epsilon$$

From our assumption that $I \subset (inf(J) + \epsilon, sup(J) - \epsilon)$ and the Lipschitz condition, we know that for any $n_1, n_2 \in \mathbf{Z}$, if $(n_1 + I) \cdot g \cap (n_2 + I) \neq \emptyset$, then $(n_1 + I) \cdot g \subset (n_2 + J)$.

Let $f_1, f_2 \in F_1$ be such that

$$Supp(f_1), Supp(f_2) \subset \bigcup_{n \in \mathbf{Z}} (n + I) \qquad [f_1, f_2] \neq id$$

Note that, in particular, $[f_1, f_2] \restriction I \neq id \restriction I$ and that I is f_1, f_2-invariant.

Claim. *The element $f = [f_1, [f_2, g^{-1}]] \in \langle\langle g \rangle\rangle$ is nontrivial and satisfies that:*

$$Supp(f) \subset \bigcup_{n \in \mathbf{Z}} (n + J)$$

Proof of claim. Set $\gamma = g^{-1} f_2^{-1} g$. Note that $[f_2, g^{-1}] = f_2 \gamma$, and that

$$Supp(\gamma) = Supp(f_2) \cdot g \subset (\bigcup_{n \in \mathbf{Z}} (n + I)) \cdot g = \bigcup_{n \in \mathbf{Z}} ((n + I) \cdot g)$$

First we show that $f \neq id$. Since

$$(I \cdot g^{-1}) \cap \bigcup_{n \in \mathbf{Z}} (n + I) = \emptyset$$

it follows that

$$I \cap (\bigcup_{n\in\mathbf{Z}} (n+I)\cdot g) = \emptyset$$

and so $\gamma \restriction I = id \restriction I$. Therefore,

$$[f_1, f_2\gamma] \restriction I = [f_1, f_2] \restriction I \neq id \restriction I$$

Recall from our assumption above that for any $n_1, n_2 \in \mathbf{Z}$, if $(n_1+I)\cdot g \cap (n_2+I) \neq \emptyset$, then $(n_1+I)\cdot g \subset (n_2+J)$. Let

$$X_1 = \{n \in \mathbf{Z} \mid ((n+I)\cdot g) \cap (\bigcup_{m\in\mathbf{Z}} (m+I)) \neq \emptyset\}$$
$$X_2 = \{n \in \mathbf{Z} \mid ((n+I)\cdot g) \cap (\bigcup_{m\in\mathbf{Z}} (m+I)) = \emptyset\}$$

and

$$U_1 = \bigcup_{n\in X_1} ((n+I)\cdot g) \qquad U_2 = \bigcup_{n\in X_2} ((n+I)\cdot g)$$

Let $\gamma_1, \gamma_2 \in \mathrm{Homeo}^+(\mathbf{R})$ be defined as:

$$\gamma_1 \restriction U_1 = \gamma \restriction U_1 \qquad \gamma_1 \restriction \mathbf{R}\setminus U_1 = id \restriction \mathbf{R}\setminus U_1$$
$$\gamma_2 \restriction U_2 = \gamma \restriction U_2 \qquad \gamma_2 \restriction \mathbf{R}\setminus U_2 = id \restriction \mathbf{R}\setminus U_2$$

By design, $\gamma = \gamma_1\gamma_2$, and that

$$Supp(\gamma_1) \subset \bigcup_{n\in\mathbf{Z}} (n+J) \qquad Supp(\gamma_2) \cap \bigcup_{n\in\mathbf{Z}} (n+I) = \emptyset$$

In particular, $[\gamma_2, f_2] = id$. It follows that

$$f = [f_1, f_2\gamma] = f_1 f_2 \gamma f_1^{-1} \gamma^{-1} f_2^{-1} = f_1 f_2 (\gamma_1(\gamma_2 f_2^{-1} \gamma_2^{-1})\gamma_1^{-1}) f_2^{-1}$$
$$= f_1 f_2 (\gamma_1 f_2^{-1} \gamma_1^{-1}) f_2^{-1}$$

Since

$$Supp(f_1), Supp(f_2), Supp(\gamma_1) \subset (\bigcup_{n\in\mathbf{Z}} (n+J))$$

it follows that $Supp(f) \subset (\bigcup_{n\in\mathbf{Z}}(n+J))$. $\square$

We define a map $\nu : F \to \overline{T}$ as the obvious extension of the natural map $\nu : F \to F_1 \leq \overline{T}$. For an open interval $I \subset (0,1)$ and $N \in \mathbf{N}$, an element $f \in G_\lambda$ is said to be (I, N)-*regular*, if the following holds.

(1) $Supp(f) \subset \bigcup_{n\in\mathbf{Z}}(I+n)$.
(2) There is an element $g \in F \setminus \{id\}$ such that for each interval $L, |L| > N$, there is an interval $[m, m+1] \subset L$ for $m \in \mathbf{Z}$, such that

$$\nu(g) \restriction [m, m+1] = f \restriction [m, m+1]$$

Note that the element f that emerges in the conclusion of Proposition 2.2.7 satisfies the first of the two conditions above. We show the following.

Proposition 2.2.8. *Let $I \subset (0,1)$ be an open interval whose closure is also contained in $(0,1)$, and let $f \in G_\lambda \setminus \{id\}$ be such that:*

$$Supp(f) \subset \bigcup_{n\in\mathbf{Z}}(n+I)$$

Then there is a nontrivial element $h \in \langle\langle f\rangle\rangle$ and an $N \in \mathbf{N}$ such that h is (I,N)-regular.

Proof. Assume without loss of generality that $f \restriction [0,1]$ is nontrivial. (Otherwise, we may replace f with a conjugate of f by an integer translation and proceed.) Let $J \subset I \subset (0,1)$ be an open interval such that either $sup(J\cdot f) < inf(J)$ or $sup(J) < inf(J\cdot f)$. It follows that there is an $\epsilon > 0$ such that for any $g \in \text{Homeo}^+[0,1]$ such that $d_{[0,1]}(g,f) < \epsilon$, we have that $J\cdot g \cap J = \emptyset$.

Since f is repetitive, there is an $N \in \mathbf{N}$ such that for each interval $L, |L| > N$ there is an interval $[k, k+1] \subset L$ for $k \in \mathbf{Z}$, such that

$$d_f([0,1],[k,k+1]) < \epsilon$$

Let $g = [g_1, g_2], g_1, g_2 \in F$ be non trivial elements such that

$$Supp(g), Supp(g_1), Supp(g_2) \subset J$$

Using an argument analogous to the one in the proof of Theorem 1.0.2, we see that the element

$$h = [\nu(g_1), [\nu(g_2), f^{-1}]] \in \langle\langle f\rangle\rangle$$

has the property that for each interval $L, |L| > N$ there is an interval $[k, k+1] \subset L$ for $k \in \mathbf{Z}$ such that

$$\nu(g) \restriction [k, k+1] = h \restriction [k, k+1]$$

Moreover,

$$Supp(h) \subset \bigcup_{k\in\mathbf{Z}}(I+k)$$

This finishes the proof. □

An element $f \in \mathrm{Homeo}^+(I)$ (for I a compact interval) is said to be *ϵ-pressive* for some $\epsilon > 0$, if for each $x \in I$ we have that $x \cdot f \geq x - \epsilon$.

Lemma 2.2.9. *Let $\alpha \in \mathrm{Homeo}^+[0,1] \setminus \{id\}$ such that $Supp(\alpha) \subseteq I$ for an open interval $I \subset (0,1)$ whose closure is also contained in $(0,1)$. For any $\epsilon > 0$ and $x, y \in (0,1), x < y$, there exist elements $g_1, ..., g_n \in F, n \in \mathbf{N}$ such that:*

(1) *The element $h = (\alpha^{g_1})^{l_1} ... (\alpha^{g_n})^{l_n}$ satisfies that $x \cdot h > y$ for some $l_1, ..., l_n \in \mathbf{Z}$.*
(2) *For any homeomorphism $\beta \in \mathrm{Homeo}^+[0,1], Supp(\beta) \subseteq I$ (where I is the same interval as above), we have that $(\beta^{g_1})^{l_1} ... (\beta^{g_n})^{l_n}$ is ϵ-pressive for any $l_1, ..., l_n \in \mathbf{Z}$.*

Proof. Recall that the action of F on $(0,1)$ has the following two features, which we shall use. The first is that for any pair of closed intervals $I_1, I_2 \subset (0,1)$ with dyadic rational endpoints, there is an element $f \in F$ such that $I_1 \cdot f = I_2$. The second is that for any triple of closed intervals $I_1 \subset I_2 \subset I_3 \subset (0,1)$ with dyadic rationals as endpoints, we can find an $f \in F$ such that $Supp(f) \subset I_3$ and that $I_2 \subset I_1 \cdot f$.

Let $K \subset I$ be an open interval such that K is a connected component of $Supp(\alpha)$. It follows that either $\forall x \in K, x \cdot \alpha > x$ or $\forall x \in K, x \cdot \alpha < x$. We use the aforementioned features of F to construct elements $g_1, ..., g_n \in F$ such that

$$J_1 = I \cdot g_1 \qquad ... \qquad J_n = I \cdot g_n$$

and

$$K_1 = K \cdot g_1 \qquad ... \qquad K_n = K \cdot g_n$$

are intervals satisfying:

(1) $x \in K_1, y \in K_n$.
(2) For each $1 \leq i \leq n-1$, we have

$$inf(K_i) < inf(K_{i+1}) < sup(K_i) < sup(K_{i+1}) \qquad sup(K_i) < inf(K_{i+2}) \text{ (for } i < n-1)$$

(3) For each $1 \leq i \leq n$, $|J_i| < \epsilon$.

We remark that $K_1, ..., K_n$ forms an n-chain of intervals, in the sense of [8]. It follows that for suitable $l_1, ..., l_n \in \mathbf{Z}$ we have

$$x \cdot (\alpha^{g_1})^{l_1} ... (\alpha^{g_n})^{l_n} > y$$

Note that condition (3) above guarantees that for any $\beta \in \mathrm{Homeo}^+[0,1]$ such that $Supp(\beta) \subseteq I$, we have that $(\beta^{g_1})^{l_1} ... (\beta^{g_n})^{l_n}$ is ϵ-pressive for any $l_1, ..., l_n \in \mathbf{Z}$. □

Proof of Proposition 2.2.1. Let $g \in G_\lambda \backslash \{id\}$. By combining Propositions 2.2.7 and 2.2.8, we obtain an element $h \in \langle\langle g \rangle\rangle$, an open interval $I \subset (0,1)$ whose closure is also contained in $(0,1)$, and an $N \in \mathbf{N}$ such that h is (I, N)-regular. Replacing h by a conjugate of h by an integer translation if necessary, we assume that for $[0, N]$, the interval $[0,1]$ realizes condition (2) of the definition of (I, N)-regular.

We apply Lemma 2.2.9 to $\alpha = h \restriction [0,1]$ for $\epsilon = \frac{1}{8N}, x = \frac{1}{8}, y = \frac{3}{4}$ to obtain elements $g_1, ..., g_n \in F, n \in \mathbf{N}$ such that:

(1) The element $\gamma = (\alpha^{g_1})^{l_1}...(\alpha^{g_n})^{l_n}$ satisfies that $\frac{1}{8} \cdot \gamma > \frac{3}{4}$ for some $l_1, ..., l_n \in \mathbf{Z}$.
(2) For any homeomorphism $\beta \in \text{Homeo}^+[0,1], Supp(\beta) \subseteq I$ (where I is the same interval as above), we have that $(\beta^{g_1})^{l_1}...(\beta^{g_n})^{l_n}$ is ϵ-pressive for any $l_1, ..., l_n \in \mathbf{Z}$.

Let

$$\zeta_1 = (h^{\nu(g_1)})^{l_1}...(h^{\nu(g_n)})^{l_n}$$

and

$$\zeta_2 = \prod_{0 \leq k \leq N, k \in \mathbf{Z}} f_k^{-1} \zeta_1 f_n \qquad \text{where } t \cdot f_k = k + 1$$

Then ζ_2 is the required element that satisfies the conditions of Proposition 2.2.1 for $c = 1$, that is:

(1) $x \cdot \zeta_2 = x$ for all $x \in \mathbf{Z}$.
(2) And

$$(\frac{1}{4} + n) \cdot \zeta_2 > \frac{1}{2} + n \qquad \forall n \in \mathbf{Z} \quad \square$$

3. The second family

Our goal in this section will be to prove Theorem 1.0.5.

3.1. Preliminaries

We recall from the introduction the n-ring configuration of intervals $\{J_1, ..., J_n\}$ and homeomorphisms $\{f_1, ..., f_n\}$ that satisfy the dynamical condition $(*)$. Note that if the above is satisfied for some $1 \leq i \leq n$, then it is satisfied for all $1 \leq i \leq n$. As before, we denote the resulting group, called the fast n-ring group, as $G_n = \langle f_1, ..., f_n \rangle$. The following was proved in [1].

Theorem 3.1.1. *Given an n-ring configuration of intervals and homeomorphisms that satisfies condition $(*)$, the (marked) isomorphism type of the group G_n (with generating set $\{f_1, ..., f_n\}$) does not depend on the choice of homeomorphisms $f_1, ..., f_n$.*

Recall the following well known result (Theorem 2.1.1 in [14]).

Theorem 3.1.2. *If $G < \mathrm{Homeo}^+(\mathbf{S}^1)$ then precisely one of the following holds:*

(1) *There is a finite orbit.*
(2) *All the orbits are dense.*
(3) *There exists a copy of the Cantor set $C \subset \mathbf{S}^1$, which is G-invariant, and such that $G \restriction C$ is minimal. In this case, the given action is semiconjugate to a minimal action, i.e. there is a degree one continuous map $\Phi : \mathbf{S}^1 \to \mathbf{S}^1$ and a group homomorphism $\psi : G \to H$ onto some group H such that*

$$\forall f \in G \qquad \Phi \circ f = \psi(f) \circ \Phi$$

The resulting minimal action of H on $\mathbf{S}^1$ is called the minimalization of the action of G.

Remark 3.1.3. Note that in case 3, the minimalization is an action of H, but since H is a quotient of G (possibly with trivial kernel), it may be viewed as an action of G. In some cases (for instance if G is the fast n-ring group), one can show that $\phi : G \to H$ must be an isomorphism.

Given a group of homeomorphisms of the circle, we say that the action is *proximal*, if for every interval $I \subset \mathbf{S}^1$ such that $\mathbf{S}^1 \setminus I$ has nonempty interior, and any nonempty open set $J \subset \mathbf{S}^1$, there is an element f in the group such that $I \cdot f \subset J$.

3.2. The proof of Theorem 1.0.5

Observe that while a given action of G_n may not be minimal, by part (3) of Theorem 3.1.2, it is semiconjugate to a minimal action on $\mathbf{S}^1$. (Clearly the group action has no finite orbit.) It is clear that the dynamical condition $(*)$ holds for the new minimal action as well. Since this dynamical condition guarantees a stable isomorphism type (by Theorem 3.1.1), it follows that this new minimal action of G_n is also faithful. Actually, the main Theorem of [1] in fact guarantees that we can choose the homeomorphisms $f_1, ..., f_n$ satisfying the dynamical condition $(*)$ such that the action of G_n on $\mathbf{S}^1$ is minimal. Therefore, for the rest of this section we shall assume that the action of G_n on $\mathbf{S}^1$ is minimal. We denote as before $H_n = G_n'$ and assume that $n \geq 3$.

Lemma 3.2.1. *The action of H_n on $\mathbf{S}^1$ is proximal.*

Proof. First we shall prove that the action of G_n on $\mathbf{S}^1$ is proximal. Let $I \subset \mathbf{S}^1$ such that $\mathbf{S}^1 \setminus I$ has nonempty interior, and consider an open set $J \subset \mathbf{S}^1$.

By minimality, we find an element $g_1 \in G_n$ such that $inf(J_1) \cdot g_1 \in J$. By continuity, there is an open interval I_1 containing $inf(J_1)$ such that $I_1 \cdot g_1 \subset J$. It is an elementary exercise using the definition of $f_1, ..., f_n$, and minimality, to construct an element $g_2 \in G_n$ such that $I \cdot g_2 \subset Supp(f_1)$. There is an $n \in \mathbf{Z}$ such that $I \cdot g_2 \cdot f_1^n \subset I_1$. It follows that $I \cdot g_2 f_1^n g_1 \subset J$, finishing the proof.

Next we show that the action of H_n is minimal. Using proximality of G_n as above, let $g_i \in G_n$ be an element such that $J_i \cdot g_i \subset J_i^c$. We define the elements

$$l_i = g_i^{-1} f_i^{-1} g_i f_i$$

Clearly, $l_i \in H_n$ and $l_i \upharpoonright Supp(f_i) = f_i \upharpoonright Supp(f_i)$. It follows that the orbits of the actions of H_n, G_n are the same, hence the action of H_n is minimal. We show that the action of H_n is proximal in a similar way as done above for G_n, replacing the elements f_i by l_i. (Note that one may modify l_i above so that $Supp(l_i) \cap J_i^c$ lies in any given open interval $J \subset J_i^c$.) $\square$

Proposition 3.2.2. *H_n is simple.*

Proof. To prove simplicity, we must show that $\langle\langle g \rangle\rangle_{H_n} = H_n$ for an arbitrary $g \in H_n \setminus \{id\}$. First we show that $\langle\langle g \rangle\rangle_{H_n}$ contains a nontrivial element f such that $Supp(f)^c$ has nonempty interior. Let J be an open interval such that $J \cap (J \cdot g) = \emptyset$. We find elements $l_1, l_2 \in H_n$ such that $[l_1, l_2] \neq id$ and $(Supp(l_1) \cup Supp(l_2))^c$ has nonempty interior. Using Proximality, we find $h_1 \in H_n$ such that

$$(Supp(l_1) \cup Supp(l_2)) \cdot h_1 \subset J$$

The element $f = [[l_1^{h_1}, g_1^{-1}], l_2^{h_2}]$ has the feature that $f \in \langle\langle g \rangle\rangle_{H_n} \setminus \{id\}$ and that $Supp(f)^c$ has nonempty interior.

Since $\langle\langle f \rangle\rangle_{H_n} \leq \langle\langle g \rangle\rangle_{H_n}$, showing that $\langle\langle f \rangle\rangle_{H_n} = H_n$ finishes the proof. It suffices to show that $[f_i, f_j]^h = [f_i^h, f_j^h] \in \langle\langle f \rangle\rangle_{H_n}$ for each $1 \leq i, j \leq n$ and $h \in G_n$.

We denote $\beta_1 = f_i^h, \beta_2 = f_j^h$ and $K_1 = Supp(\beta_1), K_2 = Supp(\beta_2)$. It is easy to see that $(K_1 \cup K_2)^c$ has nonempty interior since $Supp([f_i, f_j])^c$ has the same feature.

Let I_1, I_2, I_3 be disjoint open intervals such that

$$I_1 \cdot f \cap I_1 = \emptyset \qquad I_2 \cup I_3 \subset Supp(f)^c$$

Using proximality, we find $g_1, g_2, g_3 \in H_n$ such that

$$(K_1 \cup K_2) \cdot g_1 \subset I \qquad K_1 \cdot g_2 \subset I_2 \qquad K_2 \cdot g_3 \subset I_3$$

Let

$$\alpha_1 = [g_1^{-1}, \beta_1][\beta_1, g_2^{-1}] = \beta_1^{g_1}(\beta_1^{-1})^{g_2} \qquad \alpha_2 = [g_1^{-1}, \beta_2][\beta_2, g_3^{-1}] = \beta_2^{g_1}(\beta_2^{-1})^{g_3}$$

Note that $\alpha_1, \alpha_2 \in H_n$.

We obtain

$$[\beta_1, \beta_2] = [\alpha_1, [\alpha_2, f^{-1}]]^{g_1^{-1}} \in \langle\langle f \rangle\rangle_{H_n}$$

This proves our claim. □

Definition 3.2.3. We say that an open interval $I \subset \mathbf{S}^1$ is *small* if there exists $1 \leq k \leq n$ such that $I \subset J_k$. We find a collection of pairwise disjoint small intervals

$$\mathcal{I} = \{L_{i,j} \subset \mathbf{S}^1 \mid 1 \leq i, j \leq n\}$$

satisfying that

$$\overline{L_{i,j}} \subset int(\mathbf{S}^1 \setminus J_j) \qquad 1 \leq i, j \leq n$$

We say that a small open interval $I \subset \mathbf{S}^1$ is $\mathcal{I}$-*small* if for each $1 \leq i, j \leq n$ the following holds:

(1) If $I \cap L_{i,j} \neq \emptyset$ then $I \cap J_j = \emptyset$.
(2) If $I \cap J_j \neq \emptyset$, then $I \cap L_{i,j} = \emptyset$.

It is an easy exercise to show that for any such $\mathcal{I}$, there is an $\epsilon > 0$ such that any interval I with $|I| < \epsilon$ is $\mathcal{I}$-small. More generally, we say that a subset of $\mathbf{S}^1$ is $\mathcal{I}$-small, if it is contained in a $\mathcal{I}$-small interval.

Using proximality from Lemma 3.2.1, we construct a set of n^2 elements

$$\{\lambda_{i,j} \mid 1 \leq i, j \leq n\} \subset G_n$$

satisfying that

$$J_i \cdot \lambda_{i,j} \subset L_{i,j} \qquad \forall 1 \leq i, j \leq n$$

We define $\nu_{i,j} = \lambda_{i,j}^{-1} f_i \lambda_{i,j}$. Note that $Supp(\nu_{i,j}) \subseteq L_{i,j}$. Since the intervals $\{L_{i,j} \mid 1 \leq i, j \leq n\}$ are pairwise disjoint, the elements of the set $\{\nu_{i,j} \mid 1 \leq i, j \leq n\}$ generate a free abelian group of rank n^2.

We define the set

$$X = \{\nu_{i,j}^{-1} f_i \mid 1 \leq i, j \leq n\} \subset H_n$$

Observe that

$$\{\nu_{i,j_1}\nu_{i,j_2}^{-1} \mid 1 \leq i, j_1, j_2 \leq n\} \subset \langle X\rangle$$

since $\nu_{i,j_1}\nu_{i,j_2}^{-1} = (\nu_{i,j_2}^{-1} f_i)(\nu_{i,j_1}^{-1} f_i)^{-1}$. Also, observe that $(f_i\nu_{i,j}^{-1}) \in \langle X\rangle$ since

$$(\nu_{i,i}^{-1}\nu_{i,j})(\nu_{i,j}^{-1} f_i)(\nu_{i,i}\nu_{i,j}^{-1}) = (\nu_{i,i}^{-1} f_i)(\nu_{i,i}\nu_{i,j}^{-1}) = (f_i\nu_{i,i}^{-1})(\nu_{i,i}\nu_{i,j}^{-1}) = (f_i\nu_{i,j}^{-1})$$

An element of the form $f\nu_{i,j}^{-1} \in \langle X\rangle$ is called a *special element* if $Supp(f) \cap Supp(\nu_{i,j}^{-1}) = \emptyset$ and $Supp(f)$ is $\mathcal{I}$-small.

Lemma 3.2.4. *Let $f\nu_{i,j}^{-1} \in \langle X\rangle$ be a special element and let $g \in \{f_1^{\pm 1}, ..., f_n^{\pm 1}\}$ be such that $Supp(f^g)$ is $\mathcal{I}$-small. Then there is a $1 \leq k \leq n$ such that $f^g\nu_{i,k}^{-1}$ is also a special element of $\langle X\rangle$.*

Proof. Assume that $g = f_l$. The proof where $g = f_l^{-1}$ is similar. If $Supp(f) \cap Supp(f_l) = \emptyset$, then $f^g\nu_{i,j}^{-1} = f\nu_{i,j}^{-1}$ and we are done. If $Supp(f) \cap Supp(f_l) \neq \emptyset$, then we consider the special element

$$f\nu_{i,l}^{-1} = (f\nu_{i,j}^{-1})(\nu_{i,j}\nu_{i,l}^{-1}) \in \langle X\rangle$$

Note that this is a special element since $Supp(f)$ is $\mathcal{I}$-small, and $Supp(f) \cap Supp(f_l) \neq \emptyset$, hence $Supp(f) \cap Supp(\nu_{i,l}^{-1}) = \emptyset$. It follows that

$$\begin{aligned}(f\nu_{i,l}^{-1})^{f_l\nu_{l,l}^{-1}} &= f^{f_l\nu_{l,l}^{-1}}\nu_{i,l}^{-1} = (f^{\nu_{l,l}^{-1}})^{f_l}\nu_{i,l}^{-1} \\ &= f^{f_l}\nu_{i,l}^{-1} \in \langle X\rangle\end{aligned}$$

is a special element. □

Lemma 3.2.5. *Let $J \subset \mathbf{S}^1$ be an open interval such that J^c has nonempty interior. For each $1 \leq i \leq n, s \in \{\pm 1\}$, there exists an element $\gamma \in \langle X\rangle$ such that $\gamma \restriction J = f_i^s \restriction J$.*

Proof. We fix $i \in \{1, ..., n\}$ and $s = -1$ (the proof for $s = +1$ is similar). Consider the element $\nu_{i,j}$ for some $1 \leq j \leq n$. We know that $Supp(\nu_{i,j}) \subset J_k$ for some $1 \leq k \leq n$. Consider the element $\nu_{i,j}\nu_{i,k}^{-1} \in \langle X\rangle$.

By minimality of the action of G_n on $\mathbf{S}^1$, we can find an element $g = g_1...g_m$ for $g_i \in \{f_1^{\pm 1}, ..., f_n^{\pm 1}\}$ such that $inf(J_k) \cdot g \subset J^c$. By continuity, we find a $\mathcal{I}$-small open interval I containing $inf(J_k)$ such that:

(1) $I \cdot g \subset J^c$.
(2) For each $1 \leq s \leq m$, the interval $I \cdot g_1...g_s$ is $\mathcal{I}$-small.

We let $h = f_k \nu_{k,k}^{-1} \in \langle X \rangle$. It follows that for some large $l \in \mathbf{N}$ the element

$$(\nu_{i,j}\nu_{i,k}^{-1})^{h^l} = \nu_{i,j}^{f_k^l}\nu_{i,k}^{-1} \in \langle X \rangle$$

has the property that if $\gamma = \nu_{i,j}^{f_k^l}$, then $Supp(\gamma) \subset I$ and $\gamma\nu_{i,k}^{-1}$ is a special element. Note that this means that for each $1 \leq s \leq m$, $Supp(\gamma) \cdot g_1 ... g_s$ is $\mathcal{I}$-small, hence $Supp(\gamma^{g_1 \cdots g_s})$ is $\mathcal{I}$-small.

Applying Lemma 3.2.4 to $\gamma\nu_{i,k}^{-1}$, we conclude that there is a $1 \leq k_1 \leq n$ such that $\gamma^{g_1}\nu_{i,k_1}^{-1} \in \langle X \rangle$ is a special element. Proceeding inductively, applying Lemma 3.2.4 each time, we find $1 \leq k_1, ..., k_m \leq n$ such that

$$\gamma^{g_1 \cdots g_s}\nu_{i,k_s}^{-1} \in \langle X \rangle \qquad \text{is a special element for } 1 \leq s \leq m$$

In particular, $\gamma^g \nu_{i,k_m}^{-1} \in \langle X \rangle$ and $(\gamma^g \nu_{i,k_m}^{-1})(\nu_{i,k_m} f_i^{-1}) = \gamma^g f_i^{-1} \in \langle X \rangle$ is an element which satisfies the conclusion of the Lemma, since $Supp(\gamma^g) \subset J^c$. $\square$

Now we prove our main theorem.

Theorem 3.2.6. $H_n = \langle X \rangle$, *and hence it is finitely generated.*

Proof. Let $f \in \langle X \rangle \setminus \{id\}$ be such that $Supp(f)^c$ has nonempty interior. Since H_n is simple, $\langle\langle f \rangle\rangle_{H_n} = H_n$. To prove the claim, it suffices to show that for any element $g \in H_n$, we have that $(f^{\pm 1})^g \in \langle X \rangle$. We proceed by induction on the word length of g. The base case is trivial. Now assume that $g = hf_l$ for some $1 \leq l \leq n$, and that by the induction hypothesis $f^h \in \langle X \rangle$. The proof for the case $g = hf_l^{-1}$ shall be similar.

Let $J = Supp(f^h)$. Note that J^c has nonempty interior. Applying Lemma 3.2.5, there exists an element $\gamma \in \langle X \rangle$ such that $\gamma \restriction J = f_l$. We obtain that $f^{hf_l} = f^{h\gamma} \in \langle X \rangle$. $\square$

We now prove Proposition 1.0.4.

Proof of Proposition 1.0.4. Recall that $\mathrm{Homeo}^+(\mathbf{S}^1)$ admits a central extension as follows:

$$1 \to \mathbf{Z} \to \widetilde{\mathrm{Homeo}}^+(\mathbf{S}^1) \to \mathrm{Homeo}^+(\mathbf{S}^1) \to 1$$

where $\mathbf{Z} = \langle t \to t + n \mid n \in \mathbf{Z} \rangle$ and $\widetilde{\mathrm{Homeo}}^+(\mathbf{S}^1)$ is the centralizer of $\langle t \to t + n \mid n \in \mathbf{Z} \rangle$ in $\mathrm{Homeo}^+(\mathbf{R})$.

We claim that G_n lifts to a subgroup of $\mathrm{Homeo}^+(\mathbf{R})$. We choose lifts of the generators f_i as homeomorphisms $h_i \in \widetilde{\mathrm{Homeo}}^+(\mathbf{R})$, for each $1 \leq i \leq n$, such that each h_i fixes points in $\mathbf{R}$. Let $\widetilde{G_n} = \langle h_1, ..., h_n \rangle$. It suffices to show that

$$\widetilde{G_n} \cap \langle t \to t + n \mid n \in \mathbf{Z} \rangle = \{id_{\mathbf{R}}\}$$

If this were not the case, then we can find a word

$$g = g_1 ... g_m \qquad g_i \in \{h_1^{\pm 1}, ..., h_n^{\pm 1}\}$$

with the property that $x \cdot g = x + n$ for $n \in \mathbf{Z} \setminus \{0\}$. The corresponding word $\gamma = \gamma_1 ... \gamma_m$, where each $h_i^{\pm 1}$ is replaced by $f_i^{\pm 1}$, must satisfy the following condition. For any point in $x \in \mathbf{S}^1$, the sequence

$$x, x \cdot \gamma_1, x \cdot \gamma_1 \gamma_2, ..., x \cdot \gamma_1 ... \gamma_m = x$$

must "go around the circle" a nontrivial number of times to arrive back to itself. We will show that this is impossible, i.e. the above can only happen with "backtracking".

Let $x = inf(J_1)$. Consider the sequence

$$x, x \cdot \gamma_1, x \cdot \gamma_1 \gamma_2, ..., x \cdot \gamma_1 ... \gamma_m = x$$

If

$$x \cdot \gamma_1 ... \gamma_i = x \cdot \gamma_1 ... \gamma_{i+1}$$

we delete the occurrence of γ_{i+1} from the word $\gamma_1 ... \gamma_m$, and adjust the indices (replacing j by $j-1$ for $j > i+1$) to obtain a new word $\gamma_1 ... \gamma_{m-1}$. Whenever we find *backtracking*, i.e.

$$x \cdot \gamma_1 ... \gamma_{i-1} = x \cdot \gamma_1 ... \gamma_{i+1}$$

in $\gamma_1 ... \gamma_m$, we remove $\gamma_i \gamma_{i+1}$ from the word, and adjust the indices (replacing j by $j-2$ for $j > i+1$) to obtain a new word $\gamma_1 ... \gamma_{m-2}$. At the end of this process, we obtain a new word $\gamma_1 ... \gamma_k$ such that

$$x, x \cdot \gamma_1, x \cdot \gamma_1 \gamma_2, ..., x \cdot \gamma_1 ... \gamma_k = x$$

such that for each $1 \leq i < k$

$$x \cdot \gamma_1 ... \gamma_i \neq x \cdot \gamma_1 ... \gamma_{i+1}$$

and for each $1 \leq i < k-1$

$$x \cdot \gamma_1 ... \gamma_i \neq x \cdot \gamma_1 ... \gamma_{i+2}$$

Assume without loss of generality that $\gamma_i = f_i$ (rather than f_i^{-1}) for some $1 \leq i \leq n$. By our analysis above, together with the condition $(*)$, we have that

$$(I) \qquad \gamma_1 ... \gamma_k = f_i^{p_0} f_{i+1}^{p_1} ... f_{i+m-1}^{p_{m-1}}$$

for some $1 \leq i \leq n, p_j \in \mathbf{N} \setminus \{0\}$, and where the indices of f_i are read modulo n.

Note that the dynamical condition $(*)$ can be applied in an iterative fashion and hence is in fact equivalent to the following dynamical condition (where the indices are read modulo n as usual):

$$(II) \qquad x_i \cdot f_i f_{i+1} ... f_{i+l} \in J_{i+l+1} \qquad \forall l \in \mathbf{N}, l \geq 1$$

Combining this with equation (I) immediately gives us that

$$x \cdot \gamma_1 ... \gamma_k \notin \{inf(J_i), sup(J_i) \mid 1 \leq i \leq n\}$$

This contradicts our assumption that $x = inf(J_1)$, and hence the hypothesis that

$$\widetilde{G_n} \cap \langle t \to t + n \mid n \in \mathbf{Z} \rangle = \{id_{\mathbf{R}}\}$$

must hold. □

4. A question

The following is an open question that has been driving the curiosity of the authors throughout the course of this research.

Question 4.0.1. Do there exist finitely presented infinite simple groups of homeomorphisms of $\mathbf{R}$?

We also remark that it would be particularly interesting to find finitely presented infinite simple groups of homeomorphisms of $\mathbf{R}$ that admit type (iii) actions on $\mathbf{R}$. Since most known examples of groups of homeomorphisms that are finitely presented and simple emerge as groups of homeomorphisms of the Cantor set or the circle, an example acting on $\mathbf{R}$ by means of a type (iii) action shall exhibit a fundamentally new phenomenon.

Data availability

No data was used for the research described in the article.

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