

ON GRADIENT FLOWS INITIALIZED NEAR MAXIMA

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Abstract. Let (M, g) be a closed Riemannian manifold, and let $F : M \rightarrow \mathbb{R}$ be a smooth function on M . We show the following holds generically for the function F : for each maximum p of F , there exist two minima, denoted by $m_+(p)$ and $m_-(p)$, so that the gradient flow initialized at a random point close to p converges to either $m_-(p)$ or $m_+(p)$ with high probability. The statement also holds for $F \in C^\infty(M)$ fixed and a generic metric g on M . We conclude by associating to a generic pair (F, g) what we call its max-min graph, which captures the relation between minima and maxima derived in the main result.

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AMS subject classifications. 37B35, 34D30, 37D15, 49Q99

1. Introduction. Gradient-based methods are some of the most widely used methods to find minima of differentiable functions. Amongst the reasons for this widespread use is the fact that a gradient flow of a smooth function defined on a compact space always converges to one of its critical points. When the optimized function is known to have a unique minimum, gradient (descent) flows then converge to this minimum, and the major open problems relate to the speed of this convergence. If the function has more than one isolated minimum—which implies it is non-convex—and the gradient flow is Morse-Smale [3], then generically with respect to its initialization, the flow will converge to a (local) minimum of the function. Indeed, the set of initial conditions for which the flow converges to a saddle point or maxima is the union of their *stable manifolds* (see below for a definition). Since these manifolds are of codimension at least 1, the Lebesgue measure of their union is zero.

The major challenge is thus to understand to *which* minimum the gradient flow will converge. This problem is in general difficult, as it requires a global analysis of the flow that is often intractable. To sidestep these difficulties, stochastic methods such as simulated annealing [17] have been put forward, with the goal of using stochasticity to decouple the initialization of the flow from its convergence point [6, 18]. However, this comes at the cost of slower convergence times and reliance on heuristics to set the value of some parameters. In this paper, we do not rely on stochasticity. Instead, we consider deterministic gradient flows and investigate what information about the convergence point can be obtained by restricting the set of possible initializations of the gradient flow. More precisely, we show that regardless of the number of minima of F , for each maximum p of F , there exists *two* minima, not necessarily distinct, so that the gradient flow initialized at an arbitrary point *near* p converges to these minima with very high probability.

An area of application of this characterization of gradient flows relates to recent advances in learning theory and, more specifically, to the so-called implicit regularization phenomenon. In this context, *learning* amounts to tuning parameters by minimizing an appropriately defined cost function. Gradient methods initialized randomly but *near* a given point, though slower than some alternative methods, have been observed to yield models which generalize better to out of sample data. The reason behind this fact is widely believed to be the so-called implicit regularization phenomenon [7]: some gradient flows, when initialized at a random initial condition near a given point,

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converge to a rather enviable minimum out of a large set of local minima, enviable in that it displays properties not encoded in the cost function, and not shared by the other minima. A concrete example is given by the following flow [7]: let $A_i \in \mathbb{R}^{n \times n}$ be positive definite matrices, $y_i > 0$, $i = 1, \dots, q$, and consider the flow on the set of positive definite matrices $\dot{X} = -\sum_{i=1}^q (\text{tr}(A_i X) - y_i)(A_i X + X A_i)$, which is easily seen to be the gradient flow of $F(X) = \sum_{i=1}^q (\text{tr}(A_i X) - y_i)^2$ for the trace inner product. It was observed, through extensive numerical studies, that when initialized near $X = 0$, the flow often converges to a matrix X^* solving the constrained optimization problem: $\min \text{tr} X$ such that $\text{tr} A_i X = y_i$. However, the gradient flow has many more local minima which do not solve the above problem. It is easy to observe that $X = 0$ is in fact a maximum of F , and the result of this paper thus explains why the above flow initialized randomly near zero shows a predictable behavior¹. The fact that the minima of the flow corresponding to the maximum $X = 0$ solve the above constrained optimization problem can then be shown through an analysis of the flow itself [4].

We mention two potential areas of future application: the first is in the *design* of gradient flows that converge to an equilibrium satisfying a set of desired properties, i.e., in the design of implicitly regularized gradient flows. Such designs for model predictive control are currently being explored by the author. Another area of application pertains to the qualitative study of dynamical systems, and its appearance in control under the name of *system abstractions*. Indeed, based on the characterization provided by the results of this paper, we can naturally assign a graph to each generic pair function/metric (F, g) ; we refer to it as a max-min graph and discuss some of its basic properties, as well as an open problem, in the last section.

1.1. Statement of the main result. Let (M, g) be a smooth closed Riemannian manifold and $F : M \rightarrow \mathbb{R}$ be a smooth function. We denote by $\nabla^g F$ the *gradient vector field* of M for the inner product g , which is defined by the equation

$$(1.1) \quad g(\nabla^g F, X) = dF \cdot X \text{ for all } x \in M, X \in T_x M,$$

see [5, 9] for examples. We omit the exponent g when the metric is clear from the context. Given a differentiable vector field $f(x)$ on M , we denote by $e^{tf}x$ the one-parameter group of diffeomorphisms with infinitesimal generator f . Namely, we set

$$(1.2) \quad e^{f \cdot} : \mathbb{R} \times M \rightarrow M : (t, x) \mapsto e^{tf}x$$

to be the solution at time t of the Cauchy problem

$$(1.3) \quad \dot{y} = f(y), \quad y(0) = x.$$

The **gradient flow** of F at time t for the metric g is the map $x \mapsto e^{-t\nabla^g F}x$. We also write $e^{[-t, t]f}x$ to denote the solution of (1.3) between time $-t$ and t . For a subset $B \subset M$, we let $e^{[-t, t]f}B := \bigcup_{x \in B} e^{[-t, t]f}x$.

We denote by $\mathcal{M}$ the space of smooth Riemannian metrics on M and by $C^\infty(M)$ the space of smooth real-valued functions on M . We endow these spaces with the Whitney C^k -topology, for any $k \geq 3$ fixed [10]. Given a topological space X , we say that a subset $Y \subseteq X$ is **residual** if it is a countable intersection of open dense subsets Y_i of X , i.e., $Y = \bigcap_{i=1}^\infty Y_i$. A subset $A \subseteq X$ is called **generic** if it contains a residual

¹We note that the gradient flow is in fact Morse-Bott, but one can show that it possesses invariant subspaces over which it is Morse.

set. Finally, we say that X is a **Baire** space if generic subsets of X are also dense in X . The sets $\mathcal{M}$ and $C^\infty(M)$, equipped with the Whitney C^k -topology, are Baire spaces. **Minima and maxima of a function F refer to local minima and maxima.**

We are now in a position to state the main result of this paper. Let $d : M \times M \rightarrow \mathbb{R}$ be **any distance function uniformly comparable to the Euclidean metric** (for example, any Riemannian distance function satisfies this requirement) and denote by $B_\delta(p)$ the ball of radius δ centered at p for the distance d :

$$(1.4) \quad B_\delta(p) := \{x \in M \mid d(x, p) \leq \delta\}.$$

Note that d is not necessarily the distance induced by the metric g . For $m_1, m_2 \in M$, let $W_\delta(p, m_1, m_2)$ be the set of points in $B_\delta(p)$ belonging to trajectories converging to either m_1 or m_2 :

$$W_\delta(p, m_1, m_2) := \left\{x \in B_\delta(p) \mid \lim_{t \rightarrow \infty} e^{-t\nabla F} x \in \{m_1, m_2\}\right\}.$$

In terms of the stable manifolds $W^s(m_i)$ (see [3] or below for a definition), we have $W_\delta(p, m_1, m_2) = B_\delta(p) \cap (W^s(m_1) \cup W^s(m_2))$. The main theorem is:

THEOREM 1.1. *Let $F \in C^\infty(M)$ a Morse function on a smooth closed Riemannian manifold (M, g) . Let μ be a measure on M induced by a smooth positive density and let $d : M \times M \rightarrow [0, \infty)$ be any Riemannian distance function. Then generically for g (resp. generically for F), the following holds: For any maximum p , there exists two minima $m_+(p), m_-(p)$ with the property that for all $\varepsilon > 0$, there is $\delta > 0$ such that*

$$(1.5) \quad \mu(W_\delta(p, m_+(p), m_-(p))) \geq (1 - \varepsilon)\mu(B_\delta(p)).$$

We make a few comments on the theorem. The minima $m_-(p)$ and $m_+(p)$ are not necessarily distinct; the gradient flow of the height function on a sphere provides a simple example of this fact. The proofs below hold for F of class C^3 and g of class C^2 . The minimal differentiability requirement stem from the use of a linearization theorem of Hartman, see Th. 2.1 below. In fact, since we use this theorem locally, one could even relax the hypotheses to include functions and metrics that are of class C^3 and C^2 around local maxima only. The results also hold for Morse functions $F : K \subset \mathbb{R}^n \rightarrow \mathbb{R}$, where K is any compact set so that ∇F evaluated on ∂K points outside of K (said more precisely, $e^{-t\nabla F} K \subset K$ for $t \geq 0$.)

1.2. Overview of the proof. The first step of the proof is to exhibit a necessary condition on the gradient of F ensuring that (1.5) holds for a maximum and some pair of minima of F . To this end, we introduce the notion of *principal flow lines* of a maximum of F . After having defined the principal flow lines, we show in Proposition 3.9 that if they meet a condition described below, then (1.5) holds—we will say that a maximum of F is simple if its principal flow lines meet this condition. Finally, we will show in Proposition 3.14 that gradient flows with simple maxima are generic. We will prove genericity in terms of the choice of g for a fixed Morse function F , and reciprocally genericity for a smooth F given a metric g .

1.3. Terminology and conventions. We denote by $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$. We let $S_r^{n-1}(p) \subset \mathbb{R}^n$ be the unit sphere of dimension $n - 1$, radius r and centered at p . We let $D_r^n(p) \subset \mathbb{R}^n$ be the closed ball of radius r centered at p and $D_r^{n,+}(p)$ be the upper “half-ball”

$$D_r^{n,+}(p) := \{x \in \mathbb{R}^n \mid \|x - p\| \leq r \text{ and } e_1^\top(x - p) \geq 0\}.$$

For $x = (x_1, \dots, x_n)$, we define the projections

$$\pi_1 : \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto x_1 \text{ and } \pi_{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} : x \mapsto (x_2, \dots, x_n).$$

For a Morse function F with a critical point p , we denote by $\text{ind}(p)$ the Morse index of F at p . Given a map $\varphi : M \rightarrow N$, we denote by φ_* its pushforward [12].

Recall that two submanifolds $M_1, M_2 \subset M$ intersect transversally at $x \in M_1 \cap M_2$ in M if $T_x M_1 \oplus T_x M_2 = T_x M$. For a vector field f on M , we say that M_1 and f are transversal at $x \in M_1$ if $T_x M_1 \oplus \text{span}\{f(x)\} = T_x M$. We shall use transversality and appeal to the jet transversality theorem at various places in the proof. We refer to [10] for an introduction. We will use throughout the paper the letter c to denote a real constant, with the understanding that the value of c can change during a derivation.

2. Preliminaries. We let $F \in C^\infty(M)$; a **critical point** of F is a point x so that $dF(x) = 0$. Their set is denoted by $\text{Crit } F$. We say that a critical point is *non-degenerate* if the symmetric matrix $\frac{\partial^2 F}{\partial z^2}(p)$, where z are coordinates around p , is invertible. A function with non-degenerate critical points is called a **Morse function** [13]. We call the **Morse index** or index of a critical point p the number of *negative* eigenvalues of $\frac{\partial^2 F}{\partial z^2}(p)$. If F is Morse, it is easy to show that its critical points are isolated (see, e.g., [3, Lemma 3.2]) and thus, since M is compact, they are finite in number. We denote by $\text{Crit}_i F$ the set of critical points of F of index i . Consequently, the set $\text{Crit}_n F$ is the set of maxima of F , and $\text{Crit}_0 F$ the set of minima.

Given a metric $g \in \mathcal{M}$ (resp. $F \in C^\infty(M)$) and a property S (e.g. F being Morse), we say that there exist $h \in \mathcal{M}$ with property S **arbitrarily close** to g if every Whitney open set containing g also contains an element h with property S . For example, if F is a smooth function, it is well-known that there exist Morse functions arbitrarily close to F [3].

The **stable manifold** $W^s(p, g)$ of a critical point p is defined as

$$(2.1) \quad W^s(p, g) := \{x \in M \mid \lim_{t \rightarrow \infty} e^{-t\nabla^g F}(x) = p\};$$

when the metric is obvious from the context, we omit it and simply write $W^s(p)$. Similarly, we define the **unstable manifold** of p as

$$W^u(p, g) := \{x \in M \mid \lim_{t \rightarrow -\infty} e^{-t\nabla^g F}(x) = p\}.$$

The stable manifold theorem (for Morse functions) states (e.g., [3, Theorem 4.2]) that $W^s(p)$ is a smoothly *embedded* open-ball of dimension $n - \text{ind}(p)$ in M . We furthermore have the following decomposition of M afforded by stable (resp. unstable) manifolds of the critical points of a Morse function F :

$$M = \coprod_{p \in \text{Crit } F} W^u(p) = \coprod_{p \in \text{Crit } F} W^s(p).$$

We will use a result of Hartman [8, 14] which generalizes the Poincaré-Dulac theorem on the linearization of analytic vector fields near a singularity [2]. It provides conditions under which a diffeomorphism is locally C^1 -conjugate to its linearization at a fixed point:

THEOREM 2.1 (Hartman). *Let U be an open subset of $\mathbb{R}^n$, $0 \in U$ and $f : U \rightarrow \mathbb{R}^n$ be a C^2 vector field with $f(0) = 0$. Assume that all the eigenvalues of $A := \frac{\partial f}{\partial x}(0)$ have a negative real part. Then there exists open neighborhoods $V \subset U$, and W of the origin, and a C^1 diffeomorphism $\psi : V \rightarrow W$ so that for $z = \psi(x)$, the differential equation $\dot{x} = f(x)$ is conjugate to $\dot{z} = Az$.*

We will rely on the following two simple results, whose proofs are omitted, to apply Theorem 2.1 to gradient vector fields.

LEMMA 2.2. *Let F be a smooth Morse function and $p \in \text{Crit } F$. Let (φ, U) be a chart so that $\varphi(p) = 0$. Denote by $H_\varphi^g(x) = d(\varphi_* \nabla^g F)$ the Jacobian matrix of $\nabla^g F$ expressed in the coordinate chart (φ, U) . Then $H_\varphi^g(0)$ is diagonalizable and has real eigenvalues, which are independent from φ . Furthermore, the number of negative eigenvalues of $H_\varphi^g(0)$ is equal to the Morse index of p .*

Proof. We work in the coordinates $z = \varphi(x)$, with $0 = \varphi(p)$. Denote by $G(z) \in \mathbb{R}^{n \times n}$ the expression of g in these coordinates. Then $\varphi_* \nabla F = G^{-1} \frac{\partial F}{\partial z}$ and $d\varphi_* \nabla F = d(G^{-1}) \cdot \frac{\partial F}{\partial z} + G^{-1} \frac{\partial^2 F}{\partial z^2}$. Since $\frac{\partial F}{\partial z}(0) = 0$, we obtain that

$$d\varphi_* \nabla F(0) = G^{-1}(0) \frac{\partial^2 F}{\partial z^2}(0).$$

Since g is Riemannian, $G^{-1}(0)$ is positive definite. It is easy to see that $d\varphi_* \nabla F(0)$ is similar to the symmetric matrix $G^{-1/2}(0) \frac{\partial^2 F}{\partial z^2}(0) G^{-1/2}(0)$. Hence, it is diagonalizable with real eigenvalues. Furthermore, it is a direct consequence of Sylvester's inertia theorem that the index of $\frac{\partial^2 F}{\partial z^2}(0)$ is independent of the choice of coordinates and equal to the Morse index of F at p . It is then clear from the previous relation that it is also the number of negative eigenvalues of $d\varphi_* \nabla F$. Finally, it is well-known that the eigenvalues of the linearization of a vector field at an equilibrium are independent of the choice of coordinates. $\square$

Since the eigenvalues of $H_\varphi^g(0)$ are independent of the chart φ , we will simply refer to the eigenvalues of $H^g(0)$.

The following Corollary provides a normal form for gradient flows around maxima (or minima):

COROLLARY 2.3. *Let F be a smooth Morse function on the Riemannian manifold (M, g) and let ∇F be its gradient. For any $p \in \text{Crit}_n F$, there exists a chart (φ, U) with $\varphi(p) = 0$ so that the gradient flow equation $\dot{x} = -\nabla F$ is C^1 -conjugate to $\dot{z} = -\Lambda z$ in the coordinates $z = \varphi(x)$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n < 0$.*

3. Proof of the main result. We start by describing the intersection of stable manifolds of ∇F with submanifolds of M . The result is needed for the proofs of Propositions 3.9 and 3.14. The topology on subspaces of M is the usual subspace topology.

LEMMA 3.1. *Let (M, g) be a closed Riemannian manifold and F a smooth function. Let S be an embedded submanifold of codimension one in M that is everywhere transversal to $\nabla^g F$ and set $M_0 := \bigsqcup_{q \in \text{Crit}_0 F} W^s(q)$. Then $M_0^S := M_0 \cap S$ is open dense in S .*

Proof. Recall the stable manifold decomposition of M :

$$M = \bigsqcup_{q \in \text{Crit } F} W^s(q)$$

where each stable manifold $W^s(q)$ is a smoothly embedded open ball of dimension $n - \text{ind}(q)$. When $\text{ind}(q) = 0$, the embedding is also a submersion and thus an open map. Hence, for $q \in \text{Crit}_0 F$, $W^s(q)$ is open in M and M_0 is also open, since it is a union of open sets. Set

$$M_1 := M - M_0 = \bigsqcup_{q \in \text{Crit } F | \text{ind}(q) \geq 1} W^s(q).$$

Then $M = M_0 \sqcup M_1$ and M_1 is closed. Set $M_1^S := M_1 \cap S$, then M_1^S is closed in S and we have $S = M_0^S \sqcup M_1^S$. Hence M_0^S is open in S as claimed.

It remains to show that M_0^S is dense in S or, equivalently, that M_1^S has an empty interior in S . To see this, first recall that M_1 is the disjoint union of embedded open balls of dimension at most $n - 1$, and thus by Sard's theorem, M_1 's interior in M is empty. Now assume by contradiction that there exists a non-empty open set $U \subset M_1^S$, and let $x_0 \in U$. Let $B \subset U$ be an embedded closed ball of dimension $n - 1$ properly containing x_0 . Because $\nabla^g F$ is transversal to S , for $\varepsilon > 0$ small enough,

$$B_1 := e^{[-\varepsilon, \varepsilon] \nabla^g F} \cdot B$$

is diffeomorphic to $[-\varepsilon, \varepsilon] \times B$. Thus there exists an open neighborhood of x_0 in M contained in B_1 . But since $B \subset M_1^S \subset M_1$ and M_1 is invariant under the gradient flow, then $B_1 \subset M_1$ and M_1 has a non-empty interior in M , which is a contradiction. In conclusion, M_1^S is a closed set with empty interior in S . Its complement M_0^S is then open dense in S as claimed. $\square$

Remark 3.2. Lemma 3.1 can be simplified under the additional assumption that $\nabla^g F$ is a *Morse-Smale* vector field, i.e., under the additional assumption that the stable and unstable manifolds of $\nabla^g F$ intersect transversally. With this additional assumption, one can obtain as a consequence of the λ -Lemma [15, Lemma 2.7.1] that the closure of M_0 is equal to M_1 (see also [19, Chapter 2]).

3.1. Principal flow lines and simple gradients. A smooth curve $\gamma_t : \mathbb{R} \rightarrow M$ is a trajectory of the gradient flow of F (resp. gradient ascent flow of F) if it satisfies $\dot{\gamma}(t) = -\nabla F(\gamma(t))$ (resp. $\dot{\gamma}_t = \nabla F(\gamma_t)$) for all $t \in \mathbb{R}$. Since F is Morse, it is well known that $\lim_{t \rightarrow \pm\infty} \gamma_t \in \text{Crit } F$ [13]. We introduce the following definition:

DEFINITION 3.3. Let (M, g) be a smooth Riemannian manifold and γ_t a smooth curve in M . We say that γ_t **reaches** $p \in M$ **tangentially to** $v \in T_p M$ if

1. $\lim_{t \rightarrow \infty} \gamma_t = p$.
2. $\lim_{t \rightarrow \infty} \frac{\dot{\gamma}_t}{\|\dot{\gamma}_t\|}$ exists and is equal to v

The existence of the limit in condition 2 of Def. 3.3, under the assumption that ∇F be analytic, is the content of *Thom's generalized gradient conjecture* [11]. While we can construct smooth gradients for which this limit does not exist, we show below in Lemma 3.5 that when F is Morse, its existence can easily be shown along what we call the principal flow lines.

We now define a class of gradient vector fields for which the main inequality (1.4) holds. We call them gradients with simple maxima. In order to define them, we first introduce the notion of *principal flow line* of a maximum of ∇F .

DEFINITION 3.4 (Principal flow lines). Let F be a smooth Morse function with gradient vector field $\nabla^g F$ and $p \in \text{Crit}_n F$. Denote by $H^g(p)$ the linearization of ∇F at p and let $v \in T_p M$ be a vector in the eigenspace of the smallest eigenvalue of $H^g(p)$. We say that a trajectory is a **principal flow line of ∇F at p** if it is a trajectory of the gradient ascent flow that reaches p tangentially to v .

We have the following result:

LEMMA 3.5. If the algebraic multiplicity of the smallest eigenvalue of $H^g(p)$ is equal to one, then $\nabla^g F$ has exactly two principal flow lines at p .

Proof. Let (φ, U) be the chart of Corollary 2.3, and set $z = \varphi(x)$. The gradient ascent flow is then

$$\frac{d}{dt} z = \Lambda z,$$

for $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n < 0$. Let $r > 0$ be so that $S_r(0) \subset V' := \varphi(U)$. Note that $S_r(0)$ parametrizes the set of gradient ascent flow lines that reach p ; indeed, every such flow lines intersects $S_r(0)$ at a unique $z_0 \in S_r(0)$, and is thus of the form $z(t) = \exp(\Lambda t)z_0$.

We can write $z_0 = \sum_{i=1}^n \zeta_i e_i$ for some coefficients $\zeta_i \in \mathbb{R}$, and $\exp(\Lambda t) = \sum_{i=1}^n e^{\lambda_i t} e_i e_i^\top$. Since $e_i^\top e_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta, we have that $z(t) = \sum_{i=1}^n e^{\lambda_i t} \zeta_i e_i$ and thus

$$\Lambda z(t) = \sum_{i=1}^n \lambda_i \zeta_i e^{\lambda_i t} e_i = e^{\lambda_1 t} \left(\lambda_1 \zeta_1 e_1 + \sum_{i=2}^n \lambda_i \zeta_i e^{(\lambda_i - \lambda_1)t} e_i \right).$$

The norm of the above vector is

$$\|\Lambda z(t)\| = \left(\sum_{i=1}^n \lambda_i^2 \zeta_i^2 e^{2\lambda_i t} \right)^{1/2} = e^{\lambda_1 t} \left(\lambda_1^2 \zeta_1^2 + \sum_{i=2}^n \lambda_i^2 \zeta_i^2 e^{2(\lambda_i - \lambda_1)t} \right)^{1/2}.$$

From the above two equations, we conclude that

$$\lim_{t \rightarrow \infty} \frac{\Lambda z(t)}{\|\Lambda z(t)\|} = \lim_{t \rightarrow \infty} \frac{\lambda_1 \zeta_1 e_1 + \sum_{i=2}^n \lambda_i \zeta_i e^{(\lambda_i - \lambda_1)t} e_i}{(\lambda_1^2 \zeta_1^2 + \sum_{i=2}^n \lambda_i^2 \zeta_i^2 e^{2(\lambda_i - \lambda_1)t})^{1/2}}.$$

Recall that by assumption, $\lambda_i - \lambda_1 > 0$, $2 \leq i \leq n$. Since the e_i are linearly independent, we conclude that the above limit is $\pm e_1$ if and only if $\zeta_i = 0$ for $2 \leq i \leq n$, and thus $\zeta_1 = \pm r$. This concludes the proof, with the vector $v \in T_p M$ obtained by tracing back the changes of variable used. $\square$

If the conditions of Lemma 3.5 are not met, a maximum of a Morse function can have more than two principal flow lines. For example, consider $F(x) = -x^\top Q x$ on $\mathbb{R}^n$, where Q is a positive definite matrix. Then F has a maximum at the origin. If $Q = I$, then every flow line is a principal flow line.

Remark 3.6 (Intrinsic definition of principal flow lines). In view of Lemma 3.5, we can define the tangent vector to a principal flow line $v \in T_p M$ intrinsically as follows. For vector fields X, Y , denote by $\mathcal{L}_X Y$ the Lie derivative of Y along X . If p is a zero of X , i.e., $X(p) = 0$, then $(\mathcal{L}_X Y)(p)$ depends on the value of Y at p only. Hence, we conclude that if $p \in \text{Crit } F$, we can define the linear map $\mathcal{L}_{\nabla F} : T_p M \rightarrow T_p M : w \mapsto \mathcal{L}_{\nabla F} W$ where W is any differentiable vector field with $W(p) = w$. Then a short calculation shows that $\mathcal{L}_{\nabla F}$ has $H_\varphi(p)$ as matrix representation in the coordinates φ . The principal flow lines at p are thus the trajectories of the gradient ascent flow that reach p tangentially to $v \in T_p M$, where v is an eigenvector of $\mathcal{L}_{\nabla F}$ corresponding to the smallest eigenvalue.

We will denote the **principal flow lines** of $\nabla^g F$ at p by $\gamma_t^+(p, g)$ and $\gamma_t^-(p, g)$. Equipped with the above Lemma, we define gradient vector fields with simple maxima:

DEFINITION 3.7 (Gradient vector fields with simple maxima). *Let (M, g) be a Riemannian manifold and $F \in C^\infty(M)$ be Morse function with a maximum at p . We say that p is a simple maximum of ∇F if $H^g(p)$ has a unique smallest eigenvalue and its principal flow lines belong to the stable manifolds of some minima of ∇F . If all the maxima of ∇F are simple maxima, we say that ∇F is simple.*

The above definition can be reformulated as follows. Let $p \in \text{Crit}_n F$ and fix a choice v_p of eigenvector spanning the eigenspace of $H^g(p)$ corresponding to the smallest

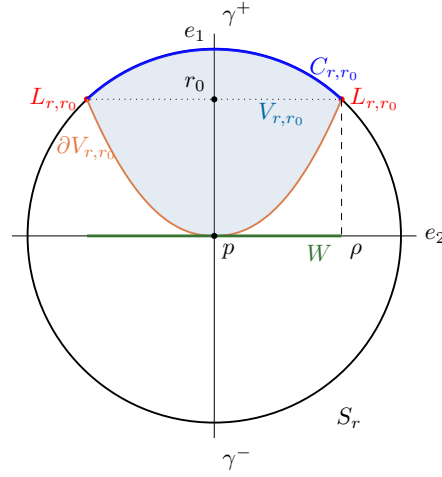


Fig. 1: The simple maximum p of ∇F for a two-dimensional M has two principal flow lines, aligned with the e_1 axis. The sphere S_r is centered at p and $C_{r,x}$ is the spherical cap with the e_1 coordinate larger than x . The set $L_{r,x}$ is the boundary of $C_{r,x}$; it is a sphere of dimension $n - 2$. Its image via the gradient flow is $\partial V_{r,x}$. We express, in Lemma 3.8, the set $\partial V_{r,x}$ as a function from W (here, the e_2 axis) to $\mathbb{R}$ (the e_1 axis.)

eigenvalue. Then ∇F is simple if for some (and thus all) $t \in \mathbb{R}$,

$$(3.1) \quad \bigcup_{p \in \text{Crit}_n F} \{\gamma_t^+, \gamma_t^-\} \subset \bigcup_{q \in \text{Crit}_0 F} W^s(q).$$

3.2. Proof of the main theorem for simple gradients. We now show that under the condition that ∇F is simple, the inequality (1.4) holds. We start with expressing an invariant set of ∇F as the epigraph of a differentiable function locally around a maximum p . To describe this set, denote by C_{r,r_0} , for $0 < r_0 < r$ the *top cap* of $S_r^{n-1}(0)$, where top cap refer to the first coordinate (i.e., along the e_1 axis) being greater than r_0 (see Fig. 1). Its boundary, which we denote by L_{r,r_0} , is a sphere of dimension $n - 2$ centered at $r_0 e_1$ given by:

$$(3.2) \quad L_{r,r_0} := \{(z_1, z_2, \dots, z_n) \mid z_1 = r_0 \text{ and } \sum_{i=2}^n z_i^2 = \rho^2\} = S_{\rho}^{n-2}(r_0 e_1)$$

where $\rho = \sqrt{r^2 - r_0^2}$. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n < 0$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n \times n}$ and define the diagonal system

$$(3.3) \quad \dot{z}_i = \lambda_i z_i, \quad \text{for } 1 \leq i \leq n.$$

We let V_{r,r_0} be the image of $C_{r,x}$ under the flow of Eq. (3.3):

$$(3.4) \quad V_{r,r_0} := e^{[0,\infty]\Lambda} \cdot C_{r,r_0} = \{z \in \mathbb{R}^n \mid z = e^{\Lambda t} y \text{ for } t \in [0, \infty], y \in C_{r,r_0}\}.$$

The boundary of V_{r,r_0} is then $\partial V_{r,r_0} = e^{\Lambda t} \cdot L_{r,r_0}$. The following result expresses this boundary as the graph of a function from $\mathbb{R}^{n-1} \rightarrow \mathbb{R}$, where by convention the domain $\mathbb{R}^{n-1}$ is the space spanned by $\{e_2, \dots, e_n\}$, and the codomain is spanned by e_1 .

LEMMA 3.8. Let $V_{r,r_0} \subset \mathbb{R}^n$ be as in (3.4) for the dynamics of (3.3) and let $W := \{w \in \mathbb{R}^{n-1} \mid \|w\| \leq \rho\}$. Then $\partial V_{r,r_0}$ is the graph of a positive differentiable function $F_l : W \rightarrow \mathbb{R}$, i.e., $\partial V_{r,r_0} = \{(F_l(w), w) \mid w \in W\}$. Furthermore, for $F_u : W \rightarrow \mathbb{R} : w \mapsto \left(\frac{\|w\|}{\rho}\right)^{\frac{\lambda_1}{\lambda_2}} r_0$, it holds that

$$F_l(w) \leq F_u(w).$$

The case $n = 2$ is proven: we have that $L_{r,r_0} = \{(r_0, \rho), (r_0, -\rho)\}$ and $\partial V_{r,r_0} = \{(e^{\lambda_1 t} r_0, \pm e^{\lambda_2 t} \rho) \mid t \in [0, \infty]\}$. A short calculation yields that $\partial V_{r,r_0} = (F_l(w), w)$ for the function

$$F_l : w \mapsto \left(\frac{|w|}{\rho}\right)^{\frac{\lambda_1}{\lambda_2}} r_0, \quad w \in [-\rho, \rho].$$

270 We now prove the general case:

Proof of Lemma 3.8. Denote a point in $\mathbb{R}^n$ as $z_1 e_1 + z_2 e_2 + \dots + z_n e_n$ and recall the definition of L_{r,r_0} in Eq. (3.2). Set $S_\rho^{n-2} := \{(z_2, \dots, z_n) \mid \sum_{i=2}^n z_i^2 = \rho^2\}$. From Eq. (3.3), we obtain

$$\partial V_{r,r_0} = \{(e^{\lambda_1 t} z_1, e^{\lambda_2 t} z_2, \dots, e^{\lambda_n t} z_n) \mid (z_1, \dots, z_n) \in L_{r,r_0}, t \in [0, \infty]\}.$$

Set $W_0 := W - \{0\}$. The map

$$\Phi : [0, \infty) \times S_\rho^{n-2} \rightarrow W_0 : (t, z_2, \dots, z_n) \mapsto (e^{\lambda_2 t} z_2, \dots, e^{\lambda_n t} z_n)$$

is a diffeomorphism onto its image. Recalling that π_1 is the projection onto the first coordinate, we see that $\partial V_{r,r_0} - \{0\}$ is the graph of

$$F_l(w) := \exp(\lambda_1 \pi_1(\Phi^{-1}(w))) r_0,$$

271 which is differentiable and can be differentially extended by 0 at 0.

272 We now show that F_u dominates F_l over W_0 . To see this, it is easier to work
273 in the coordinates afforded by Φ^{-1} : in these coordinates, $w = \Phi(t, z_2, \dots, z_n)$ and,
274 recalling that π_{-1} is the projection $(z_1, z_2, \dots, z_n) \mapsto (z_2, \dots, z_n)$, we have

$$\begin{aligned} 275 \quad F_u(w) &= F_u(\pi_{-1}(e^{\lambda t} z)) = \left(\frac{\sqrt{\sum_{i=2}^n e^{2\lambda_i t} z_i^2}}{\rho}\right)^{\lambda_1/\lambda_2} r_0 \\ 276 \quad &= \left(e^{\lambda_2 t} \frac{\sqrt{z_2^2 + \sum_{i=3}^n e^{2(\lambda_i - \lambda_2)t} z_i^2}}{\rho}\right)^{\lambda_1/\lambda_2} r_0 \\ 277 \quad &\geq e^{\lambda_1 t} r_0 = F_l(w) \end{aligned}$$

279 where we used the facts that $\lambda_1 < \lambda_2 \leq \lambda_n < 0$, $3 \leq i \leq n$ and $\sum_{i=2}^n z_i^2 = \rho^2$ to obtain
280 the inequality. $\square$

281 We are now ready to prove that inequality (1.4) holds for simple gradient flows.

PROPOSITION 3.9. Let M be a closed manifold, and μ and d as in Theorem 1.1. Let (F, g) be so that $\nabla^g F$ is simple, Then for $p \in \text{Crit}_n(F)$, there exists $m_+(p), m_-(p) \in \text{Crit}_0(F)$, not necessarily distinct, with the property that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mu\left(x \in B_\delta(p) \mid \lim_{t \rightarrow \infty} e^{-t \nabla F} x \in \{m_+(p), m_-(p)\}\right) \geq (1 - \varepsilon) \mu(B_\delta(p)).$$

Proof. Fix $\varepsilon > 0$. Let (φ, U) be a chart as in Corollary 2.3. The gradient ascent flow in the coordinates given by $z = \varphi(x)$ has the form

$$\dot{z} = \Lambda z$$

in $\varphi(U)$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. The principal flow lines γ^+ and γ^- of p are aligned with the half-lines $\{z_1 e_1 \mid z_1 > 0\}$ and $\{z_1 e_1 \mid z_1 < 0\}$, respectively.

Because p is simple, the principal flow lines γ^+ and γ^- of p belong to the stable manifold of some minima of F ; denote them $m_+(p), m_-(p) \in \text{Crit}_0 F$ respectively. Let $K \subset \psi(U)$ be a compact, contractible set containing the origin in its interior. Since the distance d is uniformly comparable to the Euclidean distance in K , i.e., there exists constants $\beta > \alpha > 0$ such that

$$(3.5) \quad \alpha \|z\| \leq d(0, z) \leq \beta \|z\|, \text{ for all } z \in K,$$

where with a slight abuse of notation, we write $d(0, z)$ for $d(p, \varphi^{-1}(z))$. Fix $r > 0$ such that $S_{r/\alpha} \subset K$ and $S_r \subset K$. For any $0 < \delta < r$, let $B_\delta := \{z \mid d(0, z) \leq \delta\}$ be the ball of radius δ centered at 0 for the distance d , and by $D_\delta := \{z \mid \|z\| \leq \delta\}$ the ball of radius δ centered at 0 for the Euclidean distance. We also let $B_\delta^+ := \{z \in B_\delta \mid z_1 \geq 0\}$ (and B_δ^- is defined in the obvious way), and define the half-balls $D_\delta^\pm$ for the Euclidean distance similarly.

Because p is simple, we have $re_1 \in W^s(m_+(p))$, and because $W^s(m_+(p))$ is open in M , there exists $r_0 \in (0, r)$ such that the closed spherical cap C_{r,r_0} of S_r is contained in $W^s(m_+(p))$; see Fig. 2-left. Hence $V_{r,r_0} \subseteq W^s(m_+(p))$, where we recall that V_{r,r_0} is the image of C_{r,r_0} under the flow as we defined in (3.4).

We claim that

$$(3.6) \quad \lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^+ \cap V_{r,r_0})}{\mu(B_\delta^+)} = 1$$

and similarly, that $\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^+ \cap V_{r,r_0}^-)}{\mu(B_\delta^+)} = 1$, where V_{r,r_0}^- is the image of a lower spherical cap under the flow. Assuming the claim holds, using elementary properties of measures, we have that (see Lemma 5.3 in the Appendix for a proof)

$$\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta \cap (V_{r,r_0} \cup V_{r,r_0}^-))}{\mu(B_\delta)} = 1$$

Since $(V_{r,r_0} \cup V_{r,r_0}^-) \subseteq (W^s(m_+(p)) \cup W^s(m_-(p)))$, we conclude that for all $\varepsilon > 0$, there exists δ so that

$$(3.7) \quad \mu([W^s(m_-(p)) \cup W^s(m_+(p))] \cap B_\delta) \geq (1 - \varepsilon)\mu(B_\delta),$$

as announced.

It now remains to prove the claim, i.e. prove that (3.6) holds. Let W and $F_u(w), F_l(w)$ be as in Lemma 3.8 and define the graph of $F : W \rightarrow \mathbb{R}$ as the set $\{(F(w), w) \in \mathbb{R}^n \mid w \in W\}$. We denote by $\text{Epi}(f)$ the epigraph of a function f , and by $\text{Hyp}(f)$ its hypograph. Since $F_u \geq F_l$, we have that (see Fig. 2-right)

$$\text{Epi}(F_u) \cap D_\delta^+ \subseteq \text{Epi}(F_l) \cap D_\delta^+ = V_{r,r_0} \cap D_\delta^+,$$

for any $0 < \delta < r$. Passing to hypographs, we have

$$(3.8) \quad V_{r,r_0} \cap D_\delta^+ = D_\delta^+ - (D_\delta^+ \cap \text{Hyp}(F_l)) \supseteq D_\delta^+ - (D_\delta^+ \cap \text{Hyp}(F_u)).$$

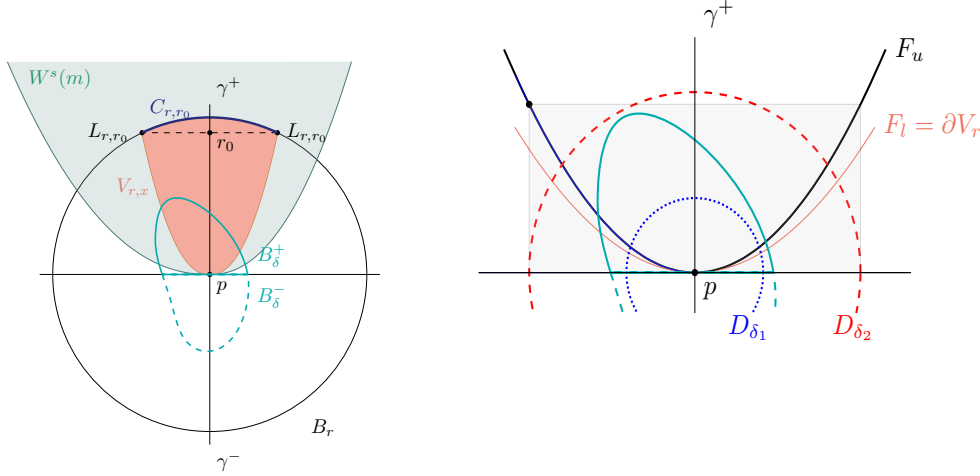


Fig. 2: *Left:* In the coordinates of Cor. 2.3, the local principal flow lines are the positive and negative e_1 (vertical) axis. The spherical cap $C_{r,x}$ is contained in a stable manifold, thus so is its image $V_{r,x}$ under the gradient ascent flow. *Right:* We can express $V_{r,x}$ as the epigraph of $F_l(w)$, which is dominated by $F_u(w)$ and thus $\text{Hyp}(F_l) \subset \text{Hyp } F_u$. Furthermore, $D_{\delta}^+ \cap \text{Hyp}(F_u)$ is contained in $B_{\delta_2} \cap \text{Hyp}(F_u)$ which is itself contained into the cylinder with base a ball of radius δ_2 and height $F_u(\delta_2)$ (light shaded rectangle).

From (3.5), we have the inclusions

$$(3.9) \quad D_{\delta_1} \subseteq B_{\delta} \subseteq D_{\delta_2}$$

for $\delta_1 := \frac{\delta}{\beta}$ and $\delta_2 := \frac{\delta}{\alpha}$. Hence,

$$(3.10) \quad B_{\delta}^+ \cap \text{Hyp}(F_u) \subseteq D_{\delta_2}^+ \cap \text{Hyp}(F_u).$$

From (3.9) and (3.10), we have that

$$(3.11) \quad \frac{\mu(B_{\delta}^+ \cap \text{Hyp}(F_u))}{\mu(B_{\delta}^+)} \leq \frac{\mu(D_{\delta_2}^+ \cap \text{Hyp}(F_u))}{\mu(D_{\delta_1}^+)}.$$

Because $F_u(w)$ is rotationally symmetric about e_1 and strictly increasing as $\|w\|$ increases, we have

$$\mu(D_{\delta_2}^+ \cap \text{Hyp}(F_u)) \leq c\delta_2^{n-1}F_u(\delta_2) \leq c\delta^{n-1+\lambda_1/\lambda_2},$$

whereas $\mu(D_{\delta_1}^+) = c\delta^n$. Since $\lambda_1/\lambda_2 > 1$, we conclude from the previous relation together with (3.11) that

$$(3.12) \quad 0 \leq \lim_{\delta \rightarrow 0} \frac{\mu(B_{\delta}^+ \cap \text{Hyp}(F_u))}{\mu(B_{\delta}^+)} \leq \lim_{\delta \rightarrow 0} \frac{\mu(D_{\delta_2}^+ \cap \text{Hyp}(F_u))}{\mu(D_{\delta_1}^+)} = 0.$$

From (3.8), we have that

$$(3.13) \quad \frac{\mu(V_{r,r_0} \cap B_{\rho}^+)}{\mu(B_{\delta}^+)} \geq 1 - \frac{\mu(B_{\delta}^+ \cap \text{Hyp}(F_u))}{\mu(B_{\delta}^+)}.$$

Taking the limit as $\delta \rightarrow 0$, using (3.12) and recalling that $V_{r,r_0} \subseteq W^s(m^+(p))$ we get that

$$\lim_{\delta \rightarrow 0} \frac{\mu(W^s(m^+(p)) \cap B_\delta^+)}{\mu(B_\delta^+)} = 1,$$

thus proving (3.6) as claimed. Applying the same reasoning to the stable manifold of $m^-(p)$ and B_δ^- , we get that similarly $\lim_{\delta \rightarrow 0} \frac{\mu(W^s(m^-(p)) \cap B_\delta^-)}{\mu(B_\delta^-)} = 1$. $\square$

3.3. Simple gradients are generic. We now prove that gradient vector fields with simple maxima are generic. There are two requirements to being simple: (1) the smallest eigenvalue of the linearized vector field has geometric multiplicity one, and (2) the principal flow lines need to be contained in stable manifolds of minima of F . We treat the two requirements separately.

To this end, for $F \in C^\infty(M)$, we denote by $\mathcal{M}_{0,F}$ the set of Riemannian metrics g on M with the property that the smallest eigenvalue of the linearization of $\nabla^g F(p)$ has geometric multiplicity one when evaluated at any maximum $p \in \text{Crit}_n(F)$. We write the requirement simply as $\lambda_1(H^g(p)) < \lambda_2(H^g(p))$ for all $p \in \text{Crit}_n F$. We further denote by $\mathcal{M}_F$ the subset of $\mathcal{M}_{0,F}$ consisting of metrics g for which $\nabla^g F$ has *simple* maxima. Given $g \in \mathcal{M}$, we similarly let $\mathcal{F}_{0,g}$ be the set of Morse functions $F \in C^\infty(M)$ on (M, g) so that for each $p \in \text{Crit}_n F$, $\lambda_1(H^g(p)) < \lambda_2(H^g(p))$ and $\mathcal{F}_g$ the subset of $\mathcal{F}_{0,g}$ consisting of functions F for which $\nabla^g F$ has simple maxima. We will show that $\mathcal{M}_F$ is residual in $\mathcal{M}$ and that $\mathcal{F}_g$ is residual in $C^\infty(M)$.

3.3.1. Geometric multiplicity of the smallest eigenvalue. We prove that the set of metrics for which the linearization of $\nabla^g F$ has a smallest eigenvalue of multiplicity one at each maximum p is open-dense:

PROPOSITION 3.10. *The set $\mathcal{M}_{0,F}$ is open and dense in $\mathcal{M}$.*

Proof. We first show the set is open. Let F be a Morse function so that for each $p \in \text{Crit}_n F$, $\lambda_1(H^g(p)) < \lambda_2(H^g(p))$. Since the eigenvalues of $H^g(p)$ depend continuously on g , there exists an open set $\mathcal{U}_p \subset \mathcal{G}$ so that for all $h \in \mathcal{U}_p$, $\lambda_1(H^h(p)) < \lambda_2(H^h(p))$. Since $|\text{Crit}_n F|$ is finite, $\mathcal{U} := \bigcap_{p \in \text{Crit}_n F} \mathcal{U}_p \subset \mathcal{M}$ is an open set containing g . Hence $\mathcal{M}_{0,F}$ is open.

To show that $\mathcal{M}_{0,F}$ is dense, assume that g is so that there exists $p \in \text{Crit}_n F$ with $\lambda_1(H^g(p)) = \lambda_2(H^g(p))$. We show that we can find, in any open set containing g , a metric h so that $\lambda_1(H^h(p)) < \lambda_2(H^h(p))$. Recall that in coordinates around p sending p to $0 \in \mathbb{R}^n$, we can write $H^h(p) = H^{-1}(0) \frac{\partial^2 F}{\partial x^2}$, where $H(x)$ is a positive definite matrix defined in a neighborhood of 0. Using a bump function around p , the fact that the map $X \mapsto X^{-1}$ is a diffeomorphism around $X = H^{-1}(0)$, and Lemma 5.1 (which states that if a product AB of two positive definite matrices has repeated eigenvalues, there exists A' positive definite and arbitrarily close to A so that $A'B$ has *distinct* eigenvalues), we can obtain a metric h arbitrarily close to g and so that $H^{-1}(0) \frac{\partial^2 F}{\partial x^2}$ has distinct eigenvalues. $\square$

We now show the equivalent result for a fixed metric g and arbitrary Morse function F . Just as above, we in fact prove the stronger statement that the set of Morse function so that $H^g(p)$ has distinct eigenvalues at each of the critical points of F is open dense. The proof relies on the notion of jet transversality – we refer to [10] for an introduction.

PROPOSITION 3.11. *The set $\mathcal{F}_{0,g}$ is open dense in $C^\infty(M)$.*

Proof. We know that Morse functions are an open dense subset of $C^\infty(M)$ [3]. We show that Morse functions for which $H^g(p)$ has distinct eigenvalues at $p \in \text{Crit}_n F$ form an open dense subset of the set of Morse functions, and thus are open dense in $C^\infty(M)$.

Given $A \in \mathbb{R}^{n \times n}$, denote by $p_A(s)$ its characteristic polynomial in the indeterminate s and let $p'_A(s) = \frac{d}{ds} p_A(s)$. Denote by $r_A \in \mathbb{R}^{2n \times 2n}$ the Sylvester resultant of p_A and p'_A . It is well known that $\det(r_A) = 0$ if and only if p_A has a double root. Let $Z \subset \mathbb{R}^{n \times n}$ be the zero set of $\det(r_A)$. Relying on Whitney's stratification theorem [20], we can show that Z is a finite union of closed manifolds.

Denote by $J^2(M, \mathbb{R})$ the second jet-space of maps $F : M \rightarrow \mathbb{R}$ and define

$$C = \{(x, y, 0_n, H) \in J^2(M, \mathbb{R}) \mid x \in M, y \in \mathbb{R}, G^{-1}(x)H \in Z\},$$

where $G(x)$ is the matrix expression of g . Then C is a finite union of submanifolds of $J^2(M, \mathbb{R})$ of codimension $\geq n + 1$ (since we restrict the first derivative to be zero, and Z is the union of submanifolds of codimension at least one.) Consequently, the second jet prolongation of F , $j^2(F)$, and C are transversal *only* at points at which they do not intersect. Furthermore, C is easily seen to be closed in $J^2(M, \mathbb{R})$. Hence, from the jet-transversality theorem [10], we conclude that the set of real-valued functions without critical points for which $H^g(p)$ has repeated eigenvalues is open and dense in $C^\infty(M)$. $\square$

3.3.2. Continuity of principal flow lines with respect to F/g . We now address the second part of the simplicity of ∇F requirement: the principal flow lines of each maxima belong to the stable manifolds of minima of F . The first step is to establish that principal flow lines depend continuously on the metric/function.

LEMMA 3.12. *Let (M, g) be a closed Riemannian manifold. Let F be a smooth Morse function, and $p \in M$ a simple maximum of $\nabla^g F$. Then, there exists a C^1 -embedded closed ball $B_p \ni p$ in M and an open set $\mathcal{U} \subset \mathcal{M}$ containing g with the following properties:*

1. B_p contains no other critical points of F
2. the principal flow line $\gamma^+(p, h)$ (resp. $\gamma^-(p, h)$) intersect ∂B_p at one point, and the intersection $\gamma^+ \cap \partial B_p$ (resp. $\gamma^- \cap \partial B_p$) depends continuously on h , $h \in \mathcal{U}$.
3. the boundary ∂B_p is everywhere transversal to $\nabla^h F$, $h \in \mathcal{U}$
4. B_p is an invariant set for the gradient ascent flow of $\nabla^h F$, $h \in \mathcal{U}$.

Proof. We work in the chart (φ, U) afforded by Corollary 2.3 sending p to $0 \in \mathbb{R}^n$, and for which the gradient flow differential equation is $\dot{z} = \Lambda z$, with Λ a diagonal matrix with diagonal entries $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n < 0$.

Since $\nabla^h F$ depends continuously on h , from the proof of Hartman's theorem [8], we know that there exists a neighborhood $V \ni 0$, a neighborhood $\mathcal{U}_0 \subset \mathcal{M}$ of g and a continuous mapping $\psi : \mathcal{U}_0 \rightarrow \text{Diff}(V, \mathbb{R}^n)$ such that for any metric $h \in \mathcal{U}_0$, the diffeomorphism $\psi_h : V \rightarrow \mathbb{R}^n$ linearizes $\nabla^h F$ around 0 (see also [14, p. 215], the author calls the continuous dependence of the linearizing diffeomorphism with respect to the vector field *robust linearization*). Note that in the coordinates used, $\psi_g = \text{Id}$.

The principal flow lines of $\nabla^g F$ in the z -coordinates are locally given by the half-lines starting at the origin and spanned by the vectors $\pm e_1$. Let $r > 0$ be such that $B_r := B_r(0) \subset V$. The half-lines intersect $\partial B_r = S_r$ at exactly two points, denote them $z_+(g), z_-(g)$, and these intersections are clearly transversal.

Taking a subset $\mathcal{U}_1 \subset \mathcal{U}_0$, we can ensure that for all $h \in \mathcal{U}_1$, $\lambda_1(H^h(p)) <$

$\lambda_2(H^h(p))$, since the eigenvalues depend continuously on h . Similarly, in the (linearizing) coordinates ψ_h , the principal flow lines of $\nabla^h F$ are half-lines starting at the origin and spanned by an eigenvector $v_1(H_\psi^h(p))$ associated with $\lambda_1(H_\psi^h(p))$ and, from Lemma 5.2, we know that the eigenspace $v_1(H_\psi^h(p))$ depends continuously on h as well. The principal flow lines of $\nabla^h F$ in the z -coordinates are given by the image under ψ_h^{-1} of the half-line starting at zero and parallel to $v_1(H_\psi^h(p))$, and thus depend continuously on h . Now since the principal flow lines of $\nabla^g F$ intersect S_r transversally, by taking a subset $\mathcal{U}_2 \subset \mathcal{U}_1$, we can ensure that for all $h \in \mathcal{U}_2$, the principal flow lines of $\nabla^h F$ in z -coordinates intersect S_r transversally and the intersections $z_+(h), z_-(h)$ are continuous in h .

Finally, for the last two items, since $\nabla^g F$ is linearized by φ as $\dot{z} = \Lambda z$, with Λ diagonal and with negative, real eigenvalues, then $\nabla^g F$ evaluated on S_r points inward, toward B_r : indeed, the inward pointing normal to S_r at z is $-z$ and its inner product with $\nabla^g F$ is $-z^\top \Lambda z > 0$. Because S_r is compact, the same conclusion holds for vector fields close enough to $\nabla^g F$. Hence B_r is invariant for $\nabla^h F$, for h close to g . Setting B_p to be the inverse image under the chart

$$(3.14) \quad B_p := \varphi^{-1}(B_r(0)),$$

we obtain a set with the required properties. $\square$

Remark 3.13. The above result transposes immediately to the case where the Riemannian metric g is fixed, and we consider an open set of function $\mathcal{U}_0 \subset C^\infty(M)$ containing F where $\nabla^g F$ has a simple maximum at p . The continuous dependence of $\nabla^g F$ on F is obvious. The only point of demarcation is that when varying F to a nearby F_1 , the critical points of $\nabla^g F_1$ may move. It is easy to see though that for a $\mathcal{U}_0$ small enough, they move continuously and their index remains the same: there exists a continuous map $P : \mathcal{U}_0 \rightarrow V \subset M$ so that $P(F_1)$ is a critical point of F_1 . (See, e.g., [15, Lemma 3.2.1] or [14]).

3.3.3. Genericity of simple gradients. We now prove the second part of the main theorem, namely that simple gradient flows are generic.

PROPOSITION 3.14. *Let $F \in C^\infty(M)$ be a Morse function. The set $\mathcal{M}_F$ of Riemannian metrics for which $\nabla^g F$ is simple is residual. Similarly, for a Riemannian manifold (M, g) , the set $\mathcal{F}$ of smooth functions for which $\nabla^g F$ is simple is residual.*

We prove the first statement, and then indicate the minor changes needed to obtain the second statement. The idea of the proof is to consider the sets of metrics h for which the principal flow lines of $\nabla^h F$ do not intersect an increasing sequence of nested compact subsets of the stable manifolds of the saddle points of F . We show that for each compact subset, the set of such metrics is open and dense in the set of Riemannian metrics. Since the limit of the sequence of these compact subsets is the union of the stable manifolds of the saddle points, this will prove that the set of such metrics is generic.

Proof. Pick a Morse function $F \in C^\infty(M)$ and metric $g \in \mathcal{M}_{0,F}$. We denote by $p_1, \dots, p_m$ and by $s_1, \dots, s_l$ the maxima and saddle points of F , respectively. We have shown that $\mathcal{M}_{0,F}$ is open dense in $\mathcal{M}$, it thus remains to show that metrics g in $\mathcal{M}_{0,F}$ for which the principal flow lines of $\nabla^g F$ at p_i , $1 \leq i \leq m$, belong to the stable manifolds of some minima form a generic set. Owing to the stable manifold decomposition of M and the fact that $W^s(p) = \{p\}$ for $p \in \text{Crit}_n F$, it is equivalent to show that, generically for g , the principal flow lines of $\nabla^g F$ at p_i do not belong to the stable manifold of some saddle points.

To this end, we will make use of the following straightforward characterization of generic sets: given that $\mathcal{M}_{0,F}$ is dense in $\mathcal{M}$, the subset $\mathcal{M}_F \subseteq \mathcal{M}$ is generic if and only if for each $g \in \mathcal{M}_{0,F}$, there exists a neighborhood $\mathcal{N}_g$ of g in $\mathcal{M}$ so that $\mathcal{M}_F \cap \mathcal{N}_g$ is generic in $\mathcal{N}_g$. For a proof of this statement, we refer to, e.g., [15, Lemma 3.3.3]. The statement allows us to consider only elements of $\mathcal{M}_{0,F}$, which is an easier task than considering any element of $\mathcal{M}_F$.

For each s_i , we let $W_0^s(s_i, g)$ be a compact neighborhood of s_i in the stable manifold $W^s(s_i, g)$. Let Σ_i^s be a codimension one submanifold of M that is (1) transversal to $\nabla^g F$ and to $W_0^s(s_i, g)$ and (2) meets $W_0^s(s_i, g)$ at the boundary $\partial W_0^s(s_i, g)$. The construction of the set Σ_i^s appears in the proof of the Kupka-Smale theorem [16], and we refer to, e.g., [15, p.107] for a constructive proof of its existence.

Because $\nabla^h F$ depends continuously on h , we know from the stable manifold theorem [15, Th. 2.6.2] that for h in a small enough neighborhood $\mathcal{N}_g \subset \mathcal{M}_{0,F}$, the maps $h \mapsto W_0^s(s_i, h)$, $1 \leq i \leq l$, are continuous and so that $W_0^s(s_i, h)$ intersects Σ_i^s transversally at $\partial W_0^s(s_i, h)$, $1 \leq i \leq l$. Note that since $\mathcal{M}_{0,F}$ is open in $\mathcal{M}$, $\mathcal{N}_g$ is also a neighborhood of g in $\mathcal{M}$.

Let $k \geq 1$ be a positive integer. Define

$$W_k^s(s_i, h) := e^{-k\nabla^h F} \cdot W_0^s(s_i, h),$$

i.e., the image of $W_0^s(s_i, h)$ by applying the gradient flow for a time of k (or the gradient *ascent* flow for a time $-k$.) Since $e^{-k\nabla^h F} : M \rightarrow M$ is a diffeomorphism for each k , $W_k^s(s_i, h)$ is a compact subset of M that depends continuously on h . Finally, we have by definition that

$$W^s(s_i, h) = \bigcup_{k \geq 0} W_k^s(s_i, h).$$

Let $\mathcal{M}_{k,i}(\mathcal{N}_g) \subseteq \mathcal{M}_{0,F} \cap \mathcal{N}_g$ be the set of metrics h in $\mathcal{N}_g$ for which the local principal flow lines of $\nabla^h F$ at p_i do *not* intersect $W_k^s(s_j, h)$ for all $1 \leq j \leq l$, $1 \leq i \leq m$. Let

$$\mathcal{M}_k(\mathcal{N}_g) = \bigcap_{i=1}^m \mathcal{M}_{k,i}(\mathcal{N}_g).$$

We will show that for all $k \geq 0$, $\mathcal{M}_k(\mathcal{N}_g)$ is open dense in $\mathcal{N}_g$. Since $\bigcap_{k=1}^\infty \mathcal{M}_k(\mathcal{N}_g) = \mathcal{M}_F \cap \mathcal{N}_g$, this shows that $\mathcal{M}_F \cap \mathcal{N}_g$ is generic and, using the characterization of generic sets described above, proves the result.

$\mathcal{M}_{k,i}(\mathcal{N}_g)$ is open in $\mathcal{N}_g$: We show that for any $h \in \mathcal{M}_{k,i}(\mathcal{N}_g)$, there exists an open neighborhood $\mathcal{U}_h$ of h contained in $\mathcal{M}_{k,i}(\mathcal{N}_g)$.

To this end, let $B_{p_i} \subset M$ and $\mathcal{U}^i \subset \mathcal{M}_{0,F}$ be the closed ball and open set, respectively, from Lemma 3.12 for the metric h . Since $\nabla^h F$ is transversal to ∂B_{p_i} and $\text{codim } \partial B_{p_i} = 1$, then $W_k(s_j, h)$ and ∂B_{p_i} intersect transversally. Additionally, because the map $h' \mapsto W_i^s(s_j, h')$ is continuous for $h' \in \mathcal{M}_{k,i}(\mathcal{N}_g)$, so are the intersections of $W_i^s(s_j, h')$ with B_{p_i} as a function of h' . From the same Lemma, denoting by $\gamma_{i,0}^{h'}$ the (positive) local principal flow line of $\nabla^{h'} F$ at p_i , we know that the map $h' \mapsto \gamma_{i,0}^{h'} \cap \partial B_{p_i}$ is continuous as well.

Putting the above two facts together, we conclude that there exists a neighborhood $\mathcal{U}_h$ of h in $\mathcal{M}_{k,i}(\mathcal{N}_g)$ so that for all $h' \in \mathcal{U}_h$, the principal flow lines $\gamma_{i,0}^{h'}$ do not intersect $W_k^s(s_i, h')$. Hence $\mathcal{M}_{k,i}(\mathcal{N}_g)$ is open.

$\mathcal{M}_{k,i}(\mathcal{N}_g)$ is dense in $\mathcal{N}_g$: We will show that for any element $h \in \mathcal{N}_g$, there exists an element $h' \in \mathcal{M}_{k,i}$ arbitrarily close to h . If $h \in \mathcal{M}_{k,i}$, there is nothing to prove. Hence assume, to fix ideas, that the positive principal flow line $\gamma_{i,0}^h$ intersects $\cup_{j=1}^l W_n^s(s_j, h)$.

Using the local change of variables $\varphi_h : (U, p_i) \rightarrow (\mathbb{R}^n, 0)$ afforded by Hartman's Theorem (Theorem 2.1) around the maximum p_i , the system follows the dynamics $\dot{z} = \Lambda z$ and after potentially another linear change of variables, we can assume that $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n < 0$ the eigenvalues of $H^h(p)$. From Eq. (3.14), we know that B_{p_i} in the z coordinates is a ball $B_r(0)$ of given radius $r > 0$ and centered at 0.

Denote by γ_0^+ the segment $(te_1, 0, \dots, 0) \in \mathbb{R}^n$, $0 \leq t \leq r$. It is a compact subset of the (positive) principal flow line of $\nabla^h F$ at p_i . Let $z_0 = (r/2, 0, \dots, 0)$. Since z_0 is not a critical point of $\nabla^h F$, by the flowbox theorem [15, p. 93], we know there exists a neighborhood U_0 of z_0 , which we take to be included in the ball of radius $r/2$ around z_0 , and a local diffeomorphism $\varphi_0 : U_0 \rightarrow \mathbb{R}^n$ under which the dynamics is, in the new variables induced by φ_0 (which we denote by y) given by

$$(3.15) \quad \dot{y} = (1, 0, \dots, 0).$$

Without loss of generality, we can assume that $\varphi_0(z_0) = (r/2, 0, \dots, 0) =: y_0$. See Fig. 3 for an illustration.

Working in the y -coordinates, let K be a box (unit ball for $\|\cdot\|_\infty$ norm) centered at y_0 and of width $0 < r' < r/2$ small enough so that $\varphi_0^{-1}(K) \subset B_r$. For any $\tilde{h} \in \mathcal{M}$ which agrees with h outside of K , because $\nabla^h F$ and $\nabla^{\tilde{h}} F$ then also agree outside of $M - B_r$ and this set is invariant under the flow $-\nabla^h F$ by Lemma 3.12, we have that

$$(3.16) \quad W_k^s(s_j, \tilde{h}) \cap \partial B_r = W_k^s(s_j, h) \cap \partial B_r.$$

Let $y_1 := y_0 - (r'/2, 0, \dots, 0) \in \partial K$. Let $\theta : M \rightarrow \mathbb{R}$ be a smooth positive function with support K and such that

$$\int_0^{r'} \theta(y_1 + te_1) dt = 1.$$

Now define the following smooth vector field with support in K : for $v \in \mathbb{R}^n$,

$$Y_v(y) := \theta(y)v.$$

Let $\phi_t^v(y_1)$ be the solution at time t of the Cauchy problem

$$(3.17) \quad \dot{y} = \nabla^h F(y) + Y_v(y), y(0) = y_1.$$

To proceed, we show that we can always find a metric for which the vector field in Eq. (3.17) is the gradient of F :

LEMMA 3.15. *For $\delta > 0$ small enough, there exists a metric-valued function h_v for all $v \in \mathbb{R}^n$ with $\|v\| < \delta$, depending continuously on v , agreeing with h outside of K , so that*

$$\nabla^{h_v} F(y) = \nabla^h F(y) + Y_v(y).$$

Proof. Because K does not contain any critical points of F , we have that $dF \cdot \nabla^h F > 0$. Thus, for δ small enough, we have that $dF \cdot (\nabla^h F + Y_v) > 0$ for all v with $\|v\| < \delta$, $y \in K$. Set $Z_v := \nabla^h F + Y_v$.

From the above, we can decompose the tangent space $T_y M = \text{span } Z_v(y) \oplus \ker dF(y)$ for $y \in K$. We now introduce a metric for which this decomposition of the tangent space is orthogonal. In coordinates, it has the matrix expression

$$h_v := \begin{pmatrix} dF \cdot Z_v & 0 \\ 0 & h|_{\ker dF} \end{pmatrix},$$

where $h|_{\ker dF}$ is the restriction of h to the $n-1$ dimensional subspace $\ker dF$ (precisely, the matrix expression for h_v is in the basis $\{Z_v, Z_1, \dots, Z_{n-1}\}$ where the Z_i are any independent system spanning $\ker dF$. In particular, note that $h_v(Z_v, Z_v) = dF \cdot Z_v$ and $h_v(Z_v, Z_i) = 0$, $1 \leq i \leq n-1$).

The above construction is such that h_v depends continuously on v , $h_0 = h$ and $h_v = h$ in $M - K$. Finally, we show that $\nabla^{h_v} F = Z_v$. To this end, let W be an arbitrary vector field; we can decompose it uniquely as $W = a_1 Z_v + W_h$, where $W_h \in \ker dF$, $a_1 \in \mathbb{R}$. We then have

$$(3.18) \quad dF \cdot W = dF \cdot (a_1 Z_v + W_h)$$

$$(3.19) \quad = a_1 dF \cdot Z_v = a_1 h_v(Z_v, Z_v)$$

$$(3.20) \quad = h_v(a_1 Z_v + W_h, Z_v) = h_v(W, Z_v),$$

which concludes the proof. $\square$

Now introduce the flow map of (3.17)

$$(3.21) \quad \Phi : \mathbb{R}^n \rightarrow M : v \mapsto \Phi(v) := \phi_{r'}^v(y_1).$$

Then, recalling that $\nabla^h F = (1, 0, \dots, 0)$ in K , we see that $\Phi(0) = y_1 + (r', 0, \dots, 0) =: y_2 \in \partial K$. Furthermore, we have the following Lemma:

LEMMA 3.16. *The map Φ defined in Eq. (3.21) is locally surjective around 0.*

Proof. We prove the statement by showing that the linearization of Φ around 0 is surjective. Denote by $w(t) = y_1 + te_1$ the solution of (3.17) with $v = 0$. It is clear that $w(t)$ is a segment of the positive principal flow line of $\nabla^h F$ at p_i , and that $w(0) = y_1 = ((r - r')/2, 0, \dots, 0)$ and $w(r') = ((r + r')/2, 0, \dots, 0) = y_2$. Recall the perturbation formula [1, Sec. 32]

$$(3.22) \quad \frac{d}{d\eta}|_{\eta=0} \Phi(\eta v) = \int_0^{r'} Y_v(w(r' - s)) ds.$$

In particular, the right-hand side depends on the value of Y_v along w only and, by construction, is equal to v . This proves that Φ is locally surjective as claimed. $\square$

To conclude the proof, we show for any $\delta > 0$, we can find v with $\|v\| < \delta$ so that the gradient of F for h_v is simple. Since $h_0 = h$ and h_v is continuous in v , this shows that there exists metric arbitrarily close to h for which the gradient of F is simple.

As above, let $\mathcal{U}_i \ni h$ be the open neighborhood of h from Lemma 3.12. By perhaps decreasing δ , we can ensure that $h_v \in \mathcal{U}_i$ for all v with $\|v\| < \delta$ (since h_v depends continuously on v , and $h_0 = h$.)

Denote by $x^*(h_v) \in \partial B_r$ the point of intersection of ∂B_r and $\gamma(p_i, h_v)$ (the intersection is not empty per Lemma 3.12). Then for each $\|v\| < \delta$, $x^*(h_v)$ is on the same flow line as y_1 since h_v agrees with h outside of K . Because Φ is locally surjective around 0, appealing to the inverse function theorem, we can find, for δ and δ_1 small

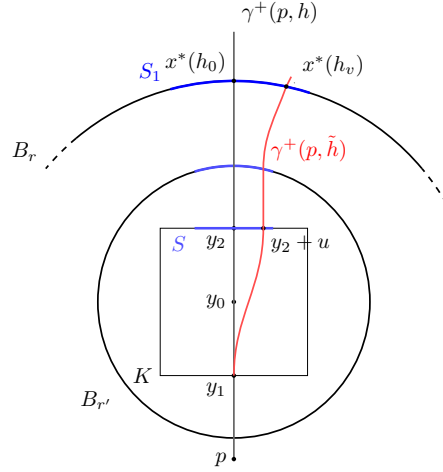


Fig. 3: The gradient flow inside $B_{r'}$ goes along vertical lines. The principal flow line for h , and for any metric agreeing with h outside of K , contains the segment (p, y_1) . Changing the metric only inside K , we can make the corresponding principal flow line go through $y_2 + u$, for any small u . The set of realizable intersections of top face of K and principal flow lines (by changing the metric to h' inside K) is denoted by S . Since S is transversal to $\nabla^h F$ (and thus to $\nabla^{h'} F$, since h and h' agree outside of K), its image under the gradient flow intersects ∂B_r to yield S' containing an open set around $x^*(h_0)$.

enough, a continuous function $\Phi^{-1} : u \mapsto v$ so that $\Phi(v) = u$, for all u with $\|u\| < \delta_1$. Let $S \in \partial K$ be the subset of the 'top face' defined as

$$S := \{y_2 + u \mid \|u\| < \delta_1 \text{ and } e_1^\top u = 0\}.$$

546 Note that S and $\nabla^h F$ are transversal by construction.

To make the notation simpler, we set $v := \Phi^{-1}(u)$. The principal flow line of $\nabla^{h_v} F$ intersects S at $y_2 + u$: every point in S can thus be made to belong to a principal flow line of a $\nabla^{h_v} F$ for an appropriate v . Using again the fact that h_v agrees with $h_0 = h$ outside of K , we see that

$$e^{-[0,\infty)\nabla^{h_v} F}(S) = e^{-[0,\infty)\nabla^h F}(S),$$

547 and thus $S_1 := e^{-[0,\infty)\nabla^{h_v} F}(S) \cap \partial B_r$ contains an open set around $x^*(h_0)$. Hence, for
 548 any $x_1^* \in \partial B_r$ near $x^*(h_0)$, we can find a v so that the principal flow line of $\nabla^{h_v} F$
 549 goes through x_1^* . Finally, since $W_n(p_i, h) = W_n(p_i, h_v)$ and $W_n(p_i, h) \cap \partial B_r$ is closed,
 550 there exists $x_1^* \in \partial B_r$ arbitrarily close to $x^*(h)$ —and thus a v arbitrarily small—so
 551 that the principal flow line of $\nabla^{h_v} F$ does not belong to $W_n(p_i, h_v) \cap \partial B_r$ and thus
 552 does not belong to $W_n(p_i, h_v)$. This concludes the proof. $\square$

553 4. Summary and outlook: max-min graphs.

554 **4.1. Max-min graphs.** From the main result of the paper, we see that given a
 555 smooth n -dimensional closed manifold M , to any generic pair $(F, g) \in C^\infty(M) \times \mathcal{M}$,
 556 there is a naturally assigned bipartite graph $G = (V, E)$, which we call *max-min graph*
 557 of (F, g)

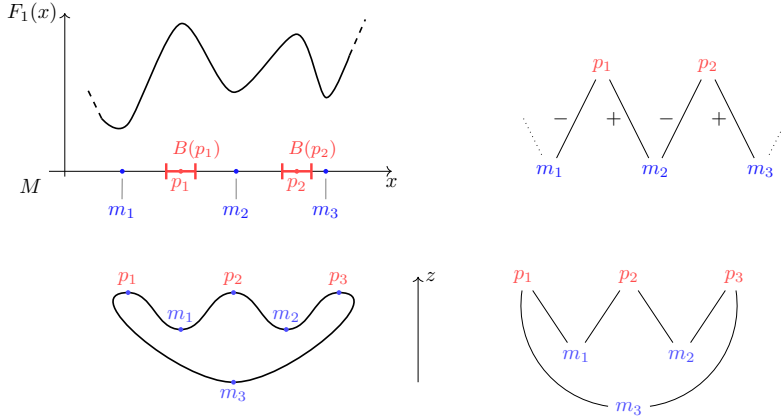


Fig. 4: *Top Left:* A Morse function on $M = \mathbb{R}$. To each maximum p_i , we can assign two minima $m_-(p), m_+(p)$ so that gradient descent for F_1 initialized in $B(p)$ converges to either $m_-(p)$ or $m_+(p)$. *Top Right:* The flow graph of the gradient of F_1 . Each minimum has degree two and maxima have degrees one or two. *Bottom Left:* A circle is embedded in the plane with vertical axis z and we consider the Morse function $F_2(x) = z$ (height function). It has three maxima and three minima. *Bottom Right:* The flow graph of the gradient of F_2 . All critical points have degree two.

558 DEFINITION 4.1 (Max-min graph of (F, g)). *The max-min graph of a generic pair*
 559 $(F, g) \in C^\infty(M) \times \mathcal{M}$ *is the bipartite graph* $G = (V, E)$ *with* $V = \text{Crit}_0(F) \cup \text{Crit}_n(F)$
 560 *and*

$$E = \{(p_i, m_-(p_i)), (p_i, m_+(p_i)) \mid p_i \in \text{Crit}_n(F)\}.$$

561 The set of possible max-min graphs for generic gradient vector fields for $n = 1$
 562 is easily seen to depend on the topology of M , and can be completely characterized:
 563 denote by p_i and m_j the elements of $\text{Crit}_n(F)$ and $\text{Crit}_0(F)$, respectively. Denote by
 564 $H_1(M)$ is the first homology group of M which, since $\dim M = 1$ and M is connected,
 565 has rank either 0 or 1. Recall that if $M = \mathbb{R}^n$, we assume that $\lim_{\|x\| \rightarrow \infty} F(x) = \infty$
 566 and F has a finite number of critical points. We have (see Fig. 4 for an illustration)

567 PROPOSITION 4.2 (Max-min graphs for $\dim M = 1$). *Assume* $\dim M = 1$, *then*

1. case $\text{rank } H_1(M) = 0$: $k =: |\text{Crit}_0(F)| = |\text{Crit}_n(F)| + 1$ *and there exists an ordering of* p_i, m_j *so that*

$$E = \cup_{i=1}^k \{(p_i, m_i), (p_i, m_{i+1})\}.$$

568 Thus $\deg(p_i) = 2$ for $p_i \in \text{Crit}_n(F)$.

2. case $\text{rank } H_1(M) = 1$: *then* $k =: |\text{Crit}_n(F)| = |\text{Crit}_0(F)|$ *and there exists an ordering of* p_i, m_j *so that*

$$E = \cup_{i=1}^k \{(m_i, p_i), (m_i, p_{i+1 \bmod k})\}.$$

569 Thus $\deg(m_i) = \deg(p_i) = 2$ for $m_i, p_i \in V$.

570 The proof of the proposition is an immediate consequence of the following facts: (1)
 571 F is generically Morse (and thus does not have saddle points if $\dim M = 1$); (2) the

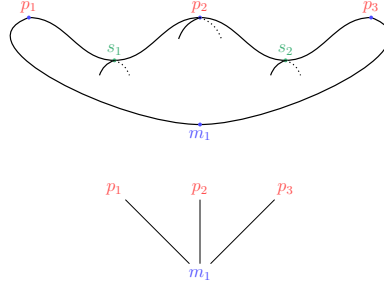


Fig. 5: *Top*: We consider the height functions of an embedded sphere in $\mathbb{R}^3$. The function has three maxima p_1, p_2, p_3 , two saddle points s_1, s_2 and a minimum m_1 . *Bottom*: Max-min graph of the gradient of the height function of the embedded sphere.

critical points of F can in this case be given a cyclic (if $\text{rank } H_1(M) = 1$) or linear (if $\text{rank } H_1(M) = 0$) order and (3) maxima and minima of F appear alternatively in this order.

4.2. Realizable max-min graphs and topology of M . This leads us to the following:

Open problem: what is the set of bipartite graphs that can be max-min graphs of generic pair (F, g) over M ?

To address this problem, we call an *abstract max-min* graph any simple bipartite graph $G = (V_0 \cup V_1, E)$ where

1. $|V_0| \geq 1, |V_1| \geq 1$
2. $1 \leq \deg(p) \leq 2$ for all $p \in V_1$

We think of V_0 as the set of minima and V_1 as the set of maxima. We say that a pair (F, g) *realizes* G on M with the max-min graph of $\nabla^g F$ is equal to G .

The set of abstract max-min graphs that can be realized depends on the topology of M , as was clear in the case $\dim M = 1$ described in Prop. 4.2. We can also easily realize max-min graphs with a single node in V_0 and an arbitrary number of nodes in V_1 , by generalizing the construction of Fig. 5 to add more maxima. These yield max-min graphs where the degree of elements in V_1 is one and the degree of the element in V_0 is unbounded. Reciprocally, we can have functions with a single node in V_1 and an arbitrary number of nodes in V_0 . For example, it suffices to consider the negative of the height function for the embedded sphere in Fig. 5. From this particular example, we also conclude that flow graphs can be disconnected: since $|V_0| = 3$ and $|V_1| = 1$ and the degree of the node in V_1 is at most 2, at least one node in V_0 has no incident edges. Furthermore, we see that reversing the direction of the gradient flow (i.e., considering the gradient ascent flow of F instead of the gradient descent flow), does *not* yield an automorphism of the corresponding flow graphs: indeed, while the elements of V_0 become the elements of V_1 and vice-versa, the edge sets of the two flow graphs do not even necessarily have the same cardinality. Finally, it should be clear that none of the examples described in the paragraph could be realized over a state-space M of dimension 1. The above leads to the question of how can one realize an abstract max-min graph, and what restriction on the topology of the underlying state-space is imposed. We will address these questions, and others, in a forthcoming publication.

4.3. Summary. Let M be a smooth closed manifold and (F, g) a generic pair where F is a smooth function and g a Riemannian metric on M . We have shown in this paper that to each maximum p of F , we can assign two minima—denoted $m_-(p), m_+(p)$ —having the following property: the gradient flow of F initialized close enough to p converges with high-probability to the set $\{m_-(p), m_+(p)\}$. In order to prove the result, we introduced the notion of *principal flow lines* of a maximum. When the linearization of the gradient flow around p has a smallest eigenvalue of algebraic multiplicity one, we showed the existence of exactly two flow lines of the gradient ascent flow that reach p tangentially to the corresponding eigenspace. These are the principal flow lines of p . If they belong to the stable manifolds of minima of F , we call the corresponding gradient vector field *simple*. We then showed in a first part that for simple gradients, most of the volume of any small ball containing a maximum p belongs to the union of the two stable manifolds to which principal flow lines belong. In a second part, we showed that simple gradient vector fields are generic.

The proof of the first part is local in nature, with the exception of the reliance on the global stable manifold decomposition theorem. The C^1 linearization result of Hartman [8] plays an important role there, and we note that it holds only if all eigenvalues of the linearized gradient vector field have real parts of the same sign. This result thus cannot be used at a saddle point of F . We also point out that the topological equivalence provided by the Hartman-Grobman theorem, which can be applied at any hyperbolic fixed point, is not sufficient to obtain our result. The second part of the proof shows that generically for (F, g) , the linearization of the gradient flow at a maximum has a smallest eigenvalue of multiplicity one, and the corresponding principal flow lines belong to the stable manifolds of some minima. The proof that the linearization of the gradient vector field at p has a unique smallest eigenvalue relies on transversality arguments. The proof that the principal flow lines belong to stable manifolds of minima goes by showing that the property holds for an increasing sequence of compact subsets of the stable manifolds, and appealing to Baire theorem. Finally, we introduced the notion of max-min graph of a generic pair (F, g) , and described some of its properties along with open questions.

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5. Appendix.

LEMMA 5.1. *Let $A, B \in \mathbb{R}^{n \times n}$ be positive definite matrices so that AB has repeated eigenvalues. Then for any $\varepsilon > 0$, there exists a positive definite Q , with $\|Q\| < \varepsilon$ and $(A + Q)B$ has distinct eigenvalues.*

Proof. We give a simple, constructive proof. The matrix AB is similar to $B^{1/2}AB^{1/2}$. The latter being symmetric, there exists an orthogonal matrix P and a diagonal matrix D so that $P^\top B^{1/2}AB^{1/2}P = D$, where the diagonal entries of D are the eigenvalues of AB . Denote the $p_i \in \mathbb{R}^n$ the i th column of P . Then $p_i^\top p_j = \delta_{ij}$ and $P^\top p_i = e_i$. Now set $v_i = B^{-1/2}p_i$. Then $P^\top B^{1/2}(A + \varepsilon_i v_i v_i^\top)B^{1/2}P = D + \varepsilon_i e_i e_i^\top$. Since $D + \varepsilon_i e_i e_i^\top$ is diagonal, it contains the eigenvalues of $P^\top B^{1/2}(A + \varepsilon_i v_i v_i^\top)B^{1/2}P$, which are the same as the eigenvalues of $(A + \varepsilon_i v_i v_i^\top)B$. It now suffices to choose the $\varepsilon_i > 0$ small enough and so that $D + \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ has distinct entries, and set $Q = \sum_{i=1}^n \varepsilon_i v_i v_i^\top$. $\square$

LEMMA 5.2. *Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$. Let $v_1 : S_n \rightarrow \mathbb{R}P^{n-1} : A \mapsto v_1(A)$ be a map assigning to A the eigenspace associated with λ_1 . Then v_1 is differentiable around A .*

Proof. Consider the map

$$V : S_n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n+1} : (X, u, \lambda) \mapsto \begin{pmatrix} (\lambda I - X)u \\ u^\top u - 1 \end{pmatrix}.$$

Let $A \in S_n$ be such that $\lambda_1 > \lambda_2$ and denote by v a unit eigenvector spanning the eigenspace of λ_1 . Then $V(A, v, \lambda_1) = 0$ and the differential of V with respect to u, λ evaluated at (A, v, λ_1) is

$$d_{u,\lambda}V(A, v, \lambda_1) = \begin{pmatrix} \lambda_1 I - A & v \\ 2v^\top & 0 \end{pmatrix}.$$

Since λ_1 is a simple eigenvalue of A , the above map is invertible. Hence, the implicit function theorem states that there is an open set $U \subset S_n$ containing A and differentiable functions $\lambda(X), u(X)$ such that $(\lambda(X)I - X)u(X) = 0$ and $\|u(X)\|^2 = 1$ for all $X \in U$, which proves the result. $\square$

LEMMA 5.3. *Let $B_\delta = B_\delta^1 \cup B_\delta^2$ and $V = V^1 \cup V^2$ with*

$$\mu(B_\delta^1 \cap B_\delta^2) = \mu(V^1 \cap V^2) = \mu(B_\delta^1 \cap V_2) = \mu(B_\delta^2 \cap V_1) = 0.$$

Assume that

$$\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^1 \cap V^1)}{\mu(B_\delta^1)} = \lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^2 \cap V^2)}{\mu(B_\delta^2)} = 1.$$

Then it holds that

$$\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta \cap V)}{\mu(B_\delta)} = 1$$

695 *Proof.* Since $B_\delta^i \subseteq B_\delta$, we have

$$696 \quad 0 = 1 - \lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^1 \cap V^1)}{\mu(B_\delta^1)} = \lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^1 - (B_\delta^1 \cap V^1))}{\mu(B_\delta^1)} = \lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^1 - (B_\delta^1 \cap V^1))}{\mu(B_\delta)} \\ 697$$

698 and, similarly, $\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta^2 - (B_\delta^2 \cap V^2))}{\mu(B_\delta^2)} = 0$. Summing the above two equalities, we get
699 in the numerator

$$700 \quad \mu(B_\delta^1 - (B_\delta^1 \cap V^1)) + \mu(B_\delta^2 - (B_\delta^2 \cap V^2)) = \mu((B_\delta^1 - (B_\delta^1 \cap V^1)) \cup (B_\delta^2 - (B_\delta^2 \cap V^2))) \\ 701 \quad = \mu(B_\delta - ((B_\delta^1 \cap V^1) \cup (B_\delta^2 \cap V^2))) \\ 703 \quad = \mu(B_\delta - (B_\delta \cap V)). \quad \blacksquare$$

704 Hence, $\lim_{\delta \rightarrow 0} \frac{\mu(B_\delta - (B_\delta \cap V))}{\mu(B_\delta)} = 0$, which concludes the proof. $\square$