How novice teachers recontextualize the teaching of mathematics via reasoning proving – a dual case study

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This study is part of a larger project exploring how beginning teachers learn to teach mathematics via reasoning and proving. The study followed two beginning secondary mathematics teachers for two years. First, as students in a capstone course in which they learned to integrate reasoning and proving into teaching mathematics, and then as full-time interns in secondary schools. The culminating part of the internship was an action research / inquiry project devoted to reasoning and proving. This exploratory multi-case study examined how conducting such an inquiry project affected interns' discourses and practices for teaching mathematics via reasoning and proving. The results show that both beginning teachers successfully recontextualized what they learned in the capstone course in their inquiry projects. Yet, there were substantial differences between the two interns, which affected their conclusions about continuing integrating reasoning and proving in their classrooms.

Keywords: Reasoning and proving, secondary mathematics teachers, interns, inquiry project.

Introduction

Mathematics teachers are tasked with creating learning environments in which students have multiple opportunities to engage with reasoning and proving. This involves a variety of reasoning processes such as identifying patterns, conjecturing, explaining, and justifying; learning about the logical structure of theorems and proofs; and applying deductive reasoning to validate and prove mathematical results – all at the level which is conceptually accessible to students of a particular age – (Stylianides, 2008). The importance of these processes for student mathematical learning is universally recognized (e.g., Hanna & de Villiers, 2012), as well as the fact that the quality of student learning experiences with proof depends on the learning opportunities teachers create in their classrooms (Stylianides et al., 2017). To provide students with rich opportunities for reasoning and proving, teachers themselves need to have strong knowledge of proof, robust proof-related teaching practices, and productive views on the nature of proof and on students as being capable of proving.

Despite these critical connections, little is known about how teacher knowledge and dispositions regarding integrating proof in teaching mathematics develops over time, especially in the early years of teacher practice (Stylianides et al., 2017). Internationally, many teacher education programs involve structured clinical experiences (*internships*, hereafter), when future teachers spend significant time in classrooms under the supervision of a mentor teacher (Strutchens et al., 2018). Internship experiences intend to help beginning teachers connect what they learned in their coursework to authentic classroom experiences, reflect on student learning and their own teaching, and develop ambitious teaching practices, like engaging students with reasoning and proving. To support these processes, many programs have their interns conduct an action research or *inquiry project* in which future teachers systematically investigate their own practice. This involves identifying a research question, designing methods of inquiry, which may include intervention, collecting, analyzing, interpreting data, and drawing conclusions (Mertler, 2009).

This study focuses on two interns, Olive and Diane (pseudonyms), who devoted their inquiry projects to studying the impact of reasoning and proving on students' mathematical confidence, engagement, and performance. This study, in turn, examines how doing this inquiry project affected the interns' professional learning with respect to teaching mathematics via reasoning and proving.

Theoretical Perspectives

While proof in mathematics serves multiple functions, such as verification, explanation, discovery, systematization, and communication (deVilliers, 1990), in mathematics classrooms, its primary role is an explanation, providing insight into why something is true. Hence, the broader goal of engaging students with proving is advancing their mathematical learning rather than teaching them how to write proofs. Buchbinder and McCrone (2022) call this Teaching Mathematics via Reasoning and Proving (TMvRP), akin to Reid and Vargas' (2019) Proof-Based Teaching.

Learning to teach mathematics via reasoning and proving is a complex process. Teacher candidates need to develop content and pedagogical knowledge specific proof, productive dispositions towards proof, and classroom practices for supporting student engagement with proof (Buchbinder & McCrone, 2020). Next, teacher candidates encounter additional challenges when transitioning from their university program to school-based internship and autonomous teaching practice (Fayne, 2007). Learning to teach occurs across multiple contexts – grade school, university, internship, and teaching jobs, each characterized by its own culture, values, discourses, and practices. Teachers learn by adopting values, contextual discourses (i.e., messages from the social, institutional, and cultural environments) and practices of these contexts and by adapting them as they transition between various settings. This view of learning is grounded in Lave and Wenger's (1991) situative perspective, which conceptualizes individuals' learning as participating in social practices of communities situated in social and physical contexts. As novice teachers begin their journey in classrooms, they need to *recontextualize* what they learned in their teacher preparation program to their new classroom setting. Recontextualizing is a process by which discourses and practices constructed in one setting are reinterpreted and reproduced in a different setting (Conner & Marchant, 2022).

This study is part of a larger longitudinal project which utilizes situative perspective to explore how beginning teachers learn to teach mathematics via reasoning and proving over time and across contexts. The first context is a capstone course, *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020) in the secondary mathematics teacher preparation program at a public university in the Northeastern part of the United States. The term 'capstone' describes a course taken in the last year of the program, intended as a culminating experience linking academic and professional competence. The second context is a full-time school-based internship involving an inquiry project as a culminating experience. The overarching research question is: *How did the interns recontextualize what they learned about TMvRP in the capstone course in their inquiry projects?*

Method

This multi-case study (Yin, 2017) focuses on two beginning secondary mathematics teachers, Olive and Diane, who participated in the study for two years, first as undergraduates and then as interns. Both participants completed their teacher preparation program with excellence, including the course *Mathematical Reasoning and Proving for Secondary Teachers,* in which they showed strong evidence for emerging content and pedagogical knowledge for teaching proof, and productive dispositions toward proof (Buchbinder & McCrone, in press). Olive and Diane started the following school year as interns teaching in their mentors' classrooms. Due to their exceptional performance, after one semester, they were offered full-time teaching positions in their respective schools. Both interns started their autonomous classroom teaching in the middle of the year but were still required to complete all tasks related to the internship, like weekly meetings with other interns and the internship supervisor, classroom observations, and written assignments. The culminating part of the internship was an inquiry project (Mertler, 2009) in which interns had to come up with their own research questions, review educational literature, develop interventions for their classrooms, collect and analyze data, and then present their findings to a group of peers and faculty.

Data sources for each intern included classroom observations, a completed inquiry project, video recordings of the project presentation, and a reflection interview. The participants' verbal discourse and written artifacts were analyzed using case-study analytic techniques: pattern identification, explanation building, and cross-case comparisons (Yin, 2017). The process of interns' learning and their recontextualization of TMvRP was operationalized through examination of their discourses about the role of proof in mathematics classrooms, the types of tasks and learning opportunities they designed, their inquiry goals and methods, and, importantly, what conclusions with respect to TMvRP did the interns draw from their inquiry projects to their own teaching practices.

Results

Conceptualizing the roles of proof and TMvRP during the capstone course

During the capstone course, Olive and Diane each developed and taught in 10^{th} grade classrooms four lessons that integrated proof topics like conditional statements, argument evaluation, quantified statements, and indirect reasoning. Diane successfully integrated these proof themes with algebra topics (e.g., exponent rules, linear equations) and Olive with geometry topics (e.g., similarity, coordinate proofs). Both completed the course with excellent grades, performing well on assessments of content and pedagogical knowledge, thus showing emergent teaching skills and knowledge for TMvRP. The analysis of their discourse revealed that their dispositions toward proof, while generally positive, varied by what they saw as the purpose of proof in mathematics classrooms.

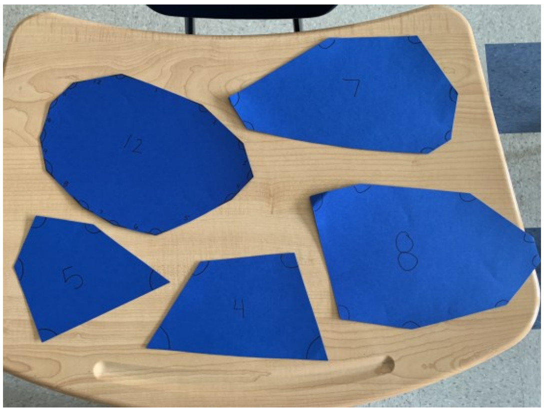
Diane attended to the verification and explanation functions of proof. For her, proof was how students justify "what steps they took to get the right answer, why they took those steps and why the answer they came to was correct." Her role as a teacher was to "train students to be able to answer the question why" and nurture students' natural curiosity for why things work. When asked about challenges to TMvRP, Diane replied that "coming up with enough ideas of ways to incorporate proof into the math that I already need to teach the students" and designing such lessons "seems kind of daunting."

Olive, on the other hand, interpreted proof as discovery and explanation, which "encourages students to explore claims in an entirely new way." Proof represents a means for students to verify that something is *always* true, without relying on the teacher's authority. The emphasis on generality was profound in Olive's discourse: "I hope that students take away the importance of being to generalize an argument", which is connected to "logical reasoning and problem solving." She acknowledged that this poses challenges to students and saw her role in supporting students' transition from example-based reasoning to "recognizing [the] need [for] a general proof."

The inquiry projects

Diane's inquiry project

Diane's project was titled "The benefits of reasoning and proof through experiential learning in the high school mathematics classroom." Diane designed and enacted in her 9th grade Geometry classroom three exploratory activities on the topics: sum of angles in a polygon, Pythagorean theorem, and surface area and volume. In the first activity, students worked in groups to calculate the sum of angles in paper polygons. Eventually, the students came up with the idea of dividing each polygon into non-overlapping triangles and calculating the sum of angles in all triangles. The class summarized the calculations of all groups in a table and came up with a general formula for the sum of angles in a polygon (Figure 1).

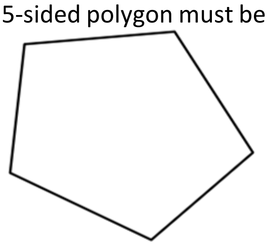


Name of Polygon	Number of Sides	Number of Triangles Formed	Number of Degrees in your shape
Triangle	3	1	180°
Quadrilateral	4	2	360°
Pentagon	5	3	540°
Hexagon	6	4	720°
Heptagon	7	5	900°
Octagon	8	6	1080°
n-gon	n	n-2	(n - 2)(180°)

Figure 1: The paper polygons (left) and summary table (right)

Diane's research questions for each activity were: (1) Can students discover a certain mathematical rule/relationship (e.g., can students derive the formula for the sum of interior angles of a polygon?), (2) Does the exploratory activity improve students' assessment performance (e.g., support retention, reduce calculation errors). The first question was answered during the in-class activity. For the second research question, Diane included in the unit test (a few weeks after each activity) one question directly related to the content of the activity (Figure 2a) and several application questions (Figure 2b).

a) Ms. Diane asked Jane what is the sum of interior angles in a 5-sided polygon? Jane argued that the sum is 180°. Explain to Jane why the sum of interior angles in a 5-sided polygon must be more than 180°.



b) Find the value of x in each polygon.

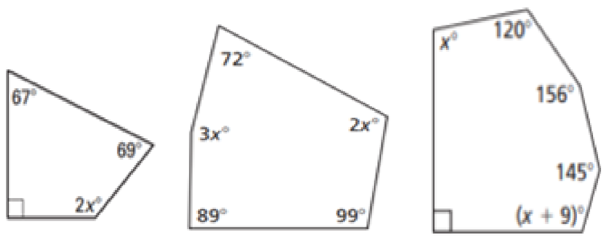


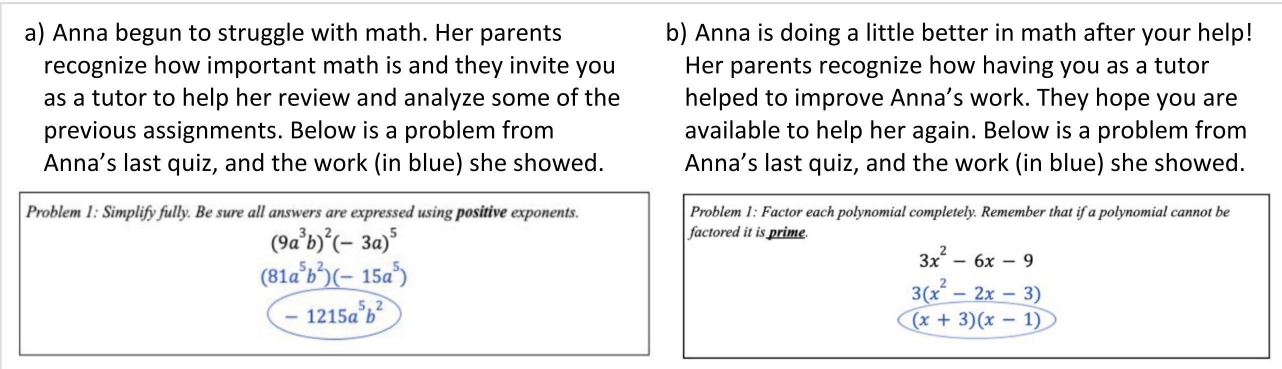
Figure 2: Assessment questions: direct (a), application (b)

Overall, Diane recorded a 72% success on the test's polygon portion, noting that most students used the reasoning strategy from the in-class activity – dividing a polygon into triangles and finding the sum of angles in all triangles to respond to the questions, or they correctly recall the formula. Students who missed the activity and could not recall the formula were unsuccessful on these questions.

Diane summarized her conclusions from the inquiry project, which involved three cycles of exploration activities and assessment, in three words: "engagement, memorable activities, retention." She was pleased that students were engaged in the activities and excited about "talking to their friends and sharing what they discovered." The activities were "memorable" for the students and "sticking in their brains." Diane noted that since she took over the class in the middle of the school year, students were hesitant to participate because they were used to passive learning. The exploratory activities were "something to look for in each unit." When asked about the implications of her project, Diane replied: "The next step is getting all the math teachers to teach like this, which is a daunting task since these activities take a lot of preparation, a lot of time in class, and both out of class."

Olive's inquiry project

Olive titled her project "The effects of peer review style reasoning and proving activities on math thinking." Olive designed three instructional activities and enacted them in her 10th grade Algebra class. The first two activities were about exponents, and the third was on factoring trinomials. Before and after each activity, there was a short assessment (five overall) and a summative reflection at the end of the project. By "peer-review style" activities, Olive meant tasks containing a fictitious student's solution, where students had to identify errors, *convince* the student that she is wrong, and provide feedback describing how to approach solving such problems (Figure 3).


a) Anna begun to struggle with math. Her parents recognize how important math is and they invite you as a tutor to help her review and analyze some of the previous assignments. Below is a problem from Anna's last quiz, and the work (in blue) she showed.

Problem 1: Simplify fully. Be sure all answers are expressed using ***positive*** *exponents.*

$(9a^3b)^2(-3a)^5$

$(81a^5b^2)(-15a^5)$

$-1215a^5b^2$

b) Anna is doing a little better in math after your help! Her parents recognize how having you as a tutor helped to improve Anna's work. They hope you are available to help her again. Below is a problem from Anna's last quiz, and the work (in blue) she showed.

Problem 1: Factor each polynomial completely. Remember that if a polynomial cannot be factored it is ***prime****.*

$3x^2 - 6x - 9$

$3(x^2 - 2x - 3)$

$(x + 3)(x - 1)$

Figure 3: Peer-review problems in exponents (a) and factoring trinomials (b)

Olive's research questions reflect her view of proof as a generalized argument as well as her curiosity about student thinking. The questions were: (a) How does students' mathematical thinking evolve when asked to make sense of someone else's reasoning? (b) At what rate are students able to identify errors, describe them, produce work that constitutes proof and extend generalized reasoning to later work? (c) What barriers prevent students from generalizing their argument? and (d) Can students become more proficient in identifying errors and generalizing math concepts with practice?

To explore these questions, Olive analyzed student success in identifying mistakes, explaining them, and generalizing solution strategies. According to her analysis, across the three activities, 70% - 82% of students were able to identify two errors, and 41% - 58% of students were able to explain the error and/or suggest how to mend it; but students' success in generalizing solutions varied between 3% - 30%. Olive's summative assessment questions examined students' preferences of certain types of justifications, according to Harel and Sowder's (2007) proof schemes: external validation, empirical evidence, and deductive reasoning (Figure 4 shows two out of three questions).

a) Which of the following best explains why $x = 1$ is a solution to the quadratic question $y = (x - 4)(x - 1)$?

- ○ Because when you set the equation to zero and solve $x = 1$ will be a solution.
- ○ Because a will be a solution to any factor $(x - a)$ when $y = 0$.
- ○ Because I have an example in my notes that looks just like this.

b) Which of the following best explains why $x = 2$ is **not** a solution to the pictured quadratic question?

- ○ Because in Desmos the solutions appear as -2 and 4
- ○ Because at $x = 2$, the y value is 8.
- ○ Because solutions occur where the graph crosses the x-axis.

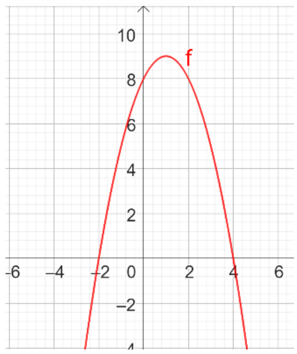


Figure 4: Sample assessment questions in Olive's questionnaire

Olive's analysis showed that on two out of three summative assessment questions students had a strong preference (~52%) toward empirical evidence, while 63% chose deductive justification for question (b) in Figure 4. Olive was intrigued by these results and wanted to investigate them further.

In the conclusions, she wrote that using proof schemes as a lens afforded a "greater insight into student comprehension" and "another layer of students understanding/proving ability." While recognizing individual differences, Olive adopted a positive outlook on students' proving ability: "Though not all students are capable of crafting their own deductive reasoning, some are certainly able to identify it!" Olive attested that doing this inquiry project left "lasting implications" and identified types of activities, e.g., collaborative proof writing; she wanted to try in the future.

Summary and discussion

This study sheds light on how two beginning teachers with similar backgrounds recontextualized their experiences with reasoning and proving across two contexts: a capstone course: *Mathematical Reasoning and Proving for Secondary Teachers,* and a full-time classroom internship, specifically in their inquiry projects. The capstone course supported Olive and Diane's development of content and pedagogical knowledge specific to proof. It offered opportunities to hone their proof-related classroom practices by designing and enacting lessons that integrate proof into the regular secondary curriculum (Buchbinder & McCrone, 2023). The analysis of Olive and Diane's discourses at the end of the course showed that they both recognized the importance of proof for student mathematical learning and its explanatory role in mathematics classrooms (Hanna & deVilliers, 2012). However, there were key differences in Olive and Diane's discourses on other functions of proof in mathematics classrooms. Beyond explanation, Olive focused on the discovery function of proof in supporting student thinking, exploration, and generalizing arguments. She viewed proof as intrinsically linked to problem-solving, logical reasoning, and generalization. Diane, on the other hand, interpreted the explanatory function of proof as fostering students' ability to justify why their mathematical work is correct and explain solution steps. Diane's discourse seems to be dominated by the word "daunting" when considering proof integration in teaching (e.g., “it seemed daunting to take a random secondary mathematics subject and incorporate a given proof theme into teaching the topic”).

As Olive and Diane transitioned to internship and then stepped into teaching their own classrooms, they recontextualized the discourses and practices regarding teaching mathematics via reasoning and proof (TMvRP). Both Olive and Diane designed rich tasks for engaging students with reasoning and proving. Diane’s tasks provided students with opportunities to explore and generalize patterns, discover and justify mathematical formulas like the sum of angles in a polygon. Olive’s tasks afforded students to critically examine the mathematical work of others, identify errors in it, and generalize arguments supporting correct solution strategies. Both interns supported students’ engagement with proof by devoting class time to activities that embedded multiple opportunities for reasoning and proving, encouraging students to share their arguments and justify their thinking. This itself was an important pedagogical step – by the middle of the school year, when novice teachers took over their classrooms, students were already accustomed to passively receiving information. Olive and Diane had to establish new classroom norms, which involved active engagement, reasoning and proving.

The differences in Olive and Diane’s discourses constructed in the capstone course reemerged in the various aspects of the inquiry projects. Diane’s pragmatic view of the role of proof in supporting memory retention and improved performance on assessments is evident in her research questions, assessment tasks, and the conclusions drawn from the inquiry project. While she recognized the advantages of TMvRP, the effort associated with it still felt daunting to her. Olive’s focus on student thinking and reasoning processes is also reflected in her research questions, which focus on reasoning and argument generalization. It is also evident in her assessments going beyond student success on proving tasks but evaluating the nature of their justifications in terms of students’ proof schemes (Harel & Sowder, 2007). It is worth noting that proof schemes were not emphasized in the capstone course but rather emerged from Olive’s own review of proof literature. Olive’s conclusions from conducting the inquiry project reflect her desire to continue exploring students’ reasoning and experiment further with integrating proof in her mathematics teaching.

Implications

This study contributes to the body of knowledge on the experiences of beginning teachers regarding teaching mathematics via reasoning and proving. Examining how beginning teachers’ proof-specific discourses and practices developed in a teacher preparation program become recontextualized in the internship contexts advances our collective understanding of how teacher knowledge regarding TMvRP evolves. The study findings point to the *alignment* between teachers’ discourses and practices across the two contexts, as evident in the multiple aspects of their inquiry projects. Despite having similar preparation and positive perspectives on proof and its teaching, the differences in the interns’ interpretation of the goals and functions of engaging students with proof were apparent. While individual differences in teachers’ recontextualizations of reasoning and proving are to be expected (Conner & Marchant, 2022), teacher educators may be aware of them as they support the development of future teachers’ practices around proof. Future studies should continue examining the development of teachers’ proof-specific knowledge and practices over time. Inquiry projects (Mertler, 2009) provide an important lens for examining these processes.

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