APPLYING A COMMOGNITIVE-BASED FRAMEWORK TO PROMOTE TEACHERS' COMMUNICATION ABOUT REASONING AND PROVING

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This study presents how the commognitive-based Opportunities for Reasoning and Proving (ORP) Framework, developed for research purposes to analyze mathematical tasks, was applied as a learning tool for teachers. Seven novice secondary teachers, who participated in a professional learning community around integrating reasoning and proving, were introduced to the ORP Framework and engaged in a sorting tasks activity. We show how the ORP Framework helped teachers to focus on the ORP embedded in tasks, to attend to student mathematical work, and to communicate about ORP coherently and unambiguously. We discuss the affordances of using a framework, which relies on the operationalized discursive language of commognition, to promote teachers' communication around reasoning and proving.

REASONING AND PROVING IN MATHEMATICS TEACHING

Mathematics educators and policymakers outline the vision of mathematics classrooms in which students develop proficiency with reasoning and proving (e.g., Hanna & de Villiers, 2012; NCTM, 2014). In this vision, teachers have a critical role in designing instructional activities that involve reasoning and proving, and teacher educators in preparing prospective teachers to design such activities (Buchbinder & McCrone, 2020; AMTE, 2017). However, what constitutes "reasoning" and "proving" has been an elusive topic (Reid & Knipping, 2010). For example, Stylianides (2008) defined "reasoning-and-proving" as a set of processes such as identifying patterns, making conjectures, and justifying, while others (e.g., Cirillo & May, 2020) focus on deductive reasoning and the logical structure of theorems and proofs. Jeannotte and Kieran (2017) argued: "what mathematical reasoning consists of is not always clear [and] it is generally assumed that everyone has a sense of what it is" (p. 1). Clarifying the notions of reasoning and proving in the school setting may aid teachers in providing their students with richer opportunities for reasoning and proving.

In our previous work (Weingarden et al., 2022), we utilized the discursive perspective of commognition to develop the *Opportunities for Reasoning and Proving* (ORP) Framework (described below) with which we conceptualized and operationalized the notion of reasoning and proving in mathematics classrooms by the opportunities provided to students to participate in certain types of discourses. In this study, we apply the ORP Framework in the context of teacher education and explore how it can support novice teachers in their communication about reasoning and proving.

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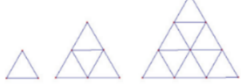
THEORETICAL PERSPECTIVE

Weingarden et al. (2022) developed the ORP Framework for characterizing mathematical tasks according to the ORP embedded in them. The framework draws on the robust theoretical tools of the commognitive perspective (Sfard, 2008), which views learning mathematics as a special type of discourse, and thus mathematical tasks can be characterized according to the type of discourse afforded to students by the task. The ORP in a task are determined by the objects at the core of the task: school-based (e.g., equation) or logic-based (e.g., conditional statement), and by the processes needed for solving the task: school-based (e.g., formulating an equation), logic-based (e.g., writing a conditional statement), or reasoning processes (e.g., generalizing, justifying). Table 1 presents the four types of ORP revealed in the previous study: Limited, Mixed, Logic-based, and Fully-Integrated ORP. The examples of tasks for each type of ORP are numbered in the order in which they were used in this study's intervention (see the Method section).

Tasks' characteristics	Examples [emphasis added]
Limited ORP	
Tasks that focus solely on school-based mathematical objects and include school-based processes.	3. *Solve* and *graph* the equation: $x^2 + 4x - 12 = 0$. 7. *Find a perimeter* of a rectangle whose one side is 5" and whose length is twice its width. 11. A farmer had some chickens and some cows. She counted 40 heads and 126 legs. How many chickens and how many cows were there?
Mixed ORP	
Tasks that involve the enactment of mathematical reasoning processes such as pattern identification, conjecturing, justifying, etc., on school-based mathematical objects.	1. *Create equations* that one can use to find the number of smaller triangles and the number of sticks for any given number triangle and *explain your reasoning*.  8. *Explain* how many solutions a quadratic equation can have. 9. *Make a conjecture* about the relationship between isosceles triangles and equilateral triangles and *justify* your thinking.
Logic-based ORP	
Tasks that are characterized by a logic-based object and can engage students with	2. *Underline the hypothesis* and *circle the conclusion* in the given statement: If both roots of quadratic function are positive then a>0.

logic-based processes, such as identifying the hypothesis and conclusion of a given statement, or formulating the converse of a given statement.	4. Explain in your own words *what a counterexample is*. 6. Given a statement: A quadrilateral with two pairs of opposite congruent sides is a parallelogram. *Identify the hypothesis and the conclusion* of the statement and *determine if the statement is universal or existential*.
	Fully-integrated ORP
Tasks that sensibly integrate both school-based and logic-based objects, such that students must operate on both, applying two types of processes: school-based and logic-based.	*5.* Come up with an example of a *conditional statement* that has to do with *linear functions* and equations and *determine whether the statement is true or false.* 10. *Prove or refute* the following statement: A quadrilateral with two pairs of opposite congruent sides is a parallelogram.

Table 1: The four types of ORP in mathematical tasks

With the ORP Framework providing a concrete operationalization for how reasoning and proving can be integrated in mathematical tasks, we hypothesized that it could be helpful for teachers aspiring to implement reasoning and proving in their classrooms. This assumption was anchored in two research strands. First, is a strand of research that explores the use of research-designed tools (e.g., observation protocols or teaching assessments) as pedagogical tools for teacher learning. For example, Candela and Boston (2022) examined how teachers using the Instructional Quality Assessment tool helped them to reflect on their practice and improve their teaching. The second strand relates to teachers' pedagogical discourse around learning and teaching. While pedagogical terms such as "high-level thinking" and "conceptual understanding" became ubiquitous in the discourse of teachers and teacher educators, their meaning and how it is manifested in mathematics classrooms often have been vague and elusive. Thus, the communication about these terms is often incoherent or ambiguous (Weingarden & Heyd-Metzuyanim, 2023). This ambiguity is also recognized with respect to reasoning and proving, as mentioned above. Thus, introducing teachers to the ORP Framework may be beneficial for creating a common language to talk about reasoning and proving. This paper begins to explore this assumption, and attends to the research question: *How does the ORP Framework contribute to novice teachers' communication around integrating reasoning and proving in their teaching?*

METHODS

This study is part of a larger project investigating how beginning teachers learn to integrate reasoning and proving in their teaching. The first stage of this project designed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020) and examined how prospective secondary teachers' (PSTs') expertise toward reasoning and proving develops as a result of their participation in the course (Buchbinder & McCrone, in press). The second stage of the project followed the PSTs, who took the capstone course, into a year-long supervised internship; and the third stage followed the same teachers for the first two years of autonomous teaching. During the third stage, the teachers participated in an online Professional Learning Community (PLC). The PLC met four times per year, each meeting lasting 90 minutes. One of these meetings was devoted to identifying ORP in mathematical tasks. Eight teachers participated in this PLC meeting, which included three parts. First, the pre-ORP task sorting activity, where teachers worked in three groups on sorting 11 tasks (shown in Table 1) in any way they see fit and naming their categories. Sorting, modifying and characterizing tasks has been shown to be beneficial to teachers' professional learning (e.g., Swan, 2007). Second, we introduced the ORP Framework by describing and exemplifying the objects (school-based and logic-based) and the processes (school-based, logic-based, and reasoning) in a separate set of tasks, and introduced the four types of ORP (Limited, Mixed, Logic-based, and Fully-integrated). In the third part, the post-ORP task sorting activity, teachers sorted the same 11 tasks again, according to the types of ORP. Data includes the video-recording and the transcript of the PLC meeting, and the pre-ORP and post-ORP sortings made by each of the three groups. In the pre-ORP and post-ORP episodes, we identified what categories of tasks the teachers created, how they named the categories, what sorting criteria they used, and their dilemmas or disagreements. We analyzed and compared teachers' pre-ORP and post-ORP discourse, including how they talked about the tasks, what they focused on, and whether and how they referred to reasoning and proving.

RESULTS

Pre-ORP sorting task activity: Overlooking the logic-based ORP

In the pre-ORP sorting task activity, teachers mainly focused on the level of thinking required from students, the complexity of the tasks, the extent to which the tasks involved multiple solution paths or a factual answer, and other pedagogical elements such as whether the task belongs to beginning or end of a unit (Table 2 shows the sorting of each of the three groups). For example, group 1 (Diane and Olive) sorted the tasks according to the assumed level of thinking. The three task categories they created were: *low-level thinking, moderate-level thinking, and higher-order thinking*. The *low-level thinking* category included tasks 2, 3, and 7, which they described as "straightforward tasks," "do this tasks," and "plug-and-chug tasks." In contrast, the *higher-order thinking* tasks (# 1, 5, and 9), were assumed to require "independent thought," and "explorations," where students "actually need to think about it" rather than being "fed the answer." Olive and Diane's discourse and sorting categories did not attend to the ORP embedded in the tasks. For example, task 3, which asks students to solve an equation, and task 2, which asks to identify the hypothesis and conclusion in a conditional statement, were similarly classified as “straightforward tasks” in the *low-thinking* category. This type of sorting did not distinguish between the tasks’ topic and nature, the ORP embedded in them, and student mathematical work around them. Specifically, this categorization completely overlooked the logic-based ORP of task 2. Olive’s comment ”that's a nothing question,” suggests that she did not attend to the logic-based characteristics of the task and its importance in explicating the logical structure of arguments and proofs. When classifying task 6, Diane and Olive contemplated whether it belongs to “low-thinking” or not, since “it starts as like a simple task”, similar to task 2, but on the other hand, “it takes it a little bit further than just identifying the hypothesis and conclusion.” Eventually, they classified this task as *moderate-level thinking*, since determining if a statement is existential or universal “requires more discussion than graphing an equation,” (c.f., task 3) but did not specify what this “more discussion” involves, how these tasks are different in the mathematical work students need to do, and how these characteristics relate to reasoning and proving.

Group	Categories names and task numbers		
1: Olive, Diane	Low-level thinking (2, 3 ,7)	Moderate-level thinking (4, 6, 8, 10, 11)	Higher Order Thinking (1 ,5 ,9)
2: Bella, Nancy, Francesca	Direct Approach (3, 4, 8)	Exploration (1, 7, 9, 11)	Conditional Statements/Proving (2, 5, 6, 10)
3: Riley, Wendy	Direct (2, 3, 7, 11)	Explanation (4, 8 ,10)	Multi-step (1, 6, 9) Open-ended (5)

Table 2: Pre-ORP categories and task numbers by group

Group 2, in contrast, recognized that some of the tasks are proof-oriented. Right from the start Nancy said: “one thing that's starting to jump out at me is that there's a couple [of tasks] that look like they're all about conditional statements, like number two, five…” This led the group to create a category *conditional statements/proving* which included tasks 2, 5, 6, and 10. All these tasks included the word “statement,” and had students prove a statement, identify its hypothesis and conclusion, or produce a statement. Group 2’s teachers also suggested a category of tasks that “don’t have anything about proving.” This category was further split into *exploration* (tasks 1, 7, 9, and 11) and *direct approach* (3, 4, and 8). The *exploration* problem category included “open-ended,” and “experimental” questions, where “students have to play around with and figure out,” and “need a more solid explanation to back it up.” The *direct approach* category included questions “that have just one answer,” do not imply “multiple ways to do it,” and explicitly state “what students have to do.” When discussing task 4, which asks students to explain in their own words what a counterexample is, the opinions split. Nancy and Bella wanted to categorize it as a *conditional statement/proving* task because “you're finding a counterexample for a conditional statement,” but Francesca thought it fits better under the *direct approach* category. She explained: “I feel like counterexample is something that could be put at any level… but it's not asking you to necessarily find a counterexample. It's just asking what it is.” This dilemma, similar to group’s 1 uncertainty regarding task 6, shows that by classifying tasks into high-level (e.g., exploration problems, higher-order thinking) and low-level (direct, low-level thinking), the teachers overlooked the added value of tasks like 2, 4, and 6, that although are straightforward and require a factual answer, are important for students making arguments and proving (logical-based ORP).

Post-ORP sorting task activity: Focusing on reasoning and proving

In the post-ORP sorting activity, the teachers’ discourse changed. First, instead of talking about the tasks’ characteristics (e.g., straightforward, open-ended, high level, fits to advance students, can be part of a summative assessment), the teachers turned to talk about the mathematical work students need to do. For example, Olive suggested that task 10 has fully-integrated ORP and explained: “because you have to use the logic stuff and the school math content.” She then concisely listed the logic-based processes and the school-based processes students need to do in the task. Regarding the same task, Bella said: “You need logic because you need to know what prove or refute means. But you also need to know what a quadrilateral with two pairs of congruent sides is. So that would be fully-integrated.” Like Olive, Bella also clearly referred to the objects and the processes embedded in the task, which includes both the school-based components (“what a quadrilateral with… is”), and the logic-based components (“what prove or refute means”).

The second change identified in the teachers’ post-ORP discourse is that it became less ambiguous, and more objectified and concise compared to the pre-ORP discourse. For example, during the pre-ORP activity, Francesca described the tasks in the *conditional statement/proving* category as: “you're doing something but without it being an exploration. But you're also not proving it.” With these vague terms, she tried to capture the essence of the logic-based ORP type of tasks, that can be straightforward (“without exploration”) and not require proving, but still related to conditional statements (“you’re doing something”). On the contrary, in the post-ORP activity, when sorting task 2, which includes logic-based ORP, Francesca explained that students are “just underlining [the hypothesis] and circling [the conclusion], but they still have to have that logic of it.” That is, Francesca clearly and explicitly indicated what students need to do (underlying, circling) and use (“the logic of it”) to solve the task.

Similar observations were revealed in all other groups, where vague terms and reliance on feelings about the difficulty level of the task, were replaced with the precise language of the ORP Framework and meaningful sorting of tasks according to types of reasoning and proving activity expected from the students engaged with the task.

DISCUSSION

We examined how the ORP Framework, developed for research purposes, can be used as a learning tool for teachers in a professional development setting. Our findings show that the ORP Framework helped teachers to attend to the ORP embedded in the tasks. In the pre-ORP sorting activity, the teachers' discourse was subjectified and was lacking a unified and coherent language to describe students' mathematical work. The teachers used vague and ambiguous terms (e.g., "needs discussion," "play around," "doing something") and focused on general pedagogical aspects, such as level of thinking or task complexity. The teachers also attempted to characterize the tasks by the keywords (e.g., "explain," "find," or "conditional statement") rather than focusing on the conceptual, mathematical work students need to do in the task. In contrast, teachers' post-ORP discourse was more objectified, coherent, and focused on student mathematical work, and components of reasoning and proving. The operationalized characterization of ORP also helped teachers develop an objectified way of talking about tasks, including what the task is about, and what students need to do to solve it.

We find this outcome interesting, because our teachers, although novices, were well familiar with proof-related tasks, having developed and enacted many such tasks as PSTs in the capstone course (Buchbinder & McCrone, 2020). Yet when it came to identifying the potential of a task to engage students with reasoning and proving, the teachers lacked the common unambiguous language for describing this potential. The ORP Framework, by relying on the discursive language of commognition, provided teachers with such a common language.

The advantage of commognition (Sfard, 2008) is that it enables operationalized communication about mathematics teaching and learning. However, communicating about teaching through the commognitive lens, especially with teachers, is not a straightforward process. The ORP Framework, similar to other tools and mediators developed based on commognition (e.g., Weingarden & Heyd-Metzuyanim, 2023), can help teachers to communicate about teaching more coherently without their familiarity and expertise in the commognitive framework.

The ORP Framework, developed first as a research tool for characterizing ORP in mathematical tasks (Weingarden et al., 2022), was found in this study, to contribute to teachers' emergent development of a common language (shared with teacher educators as well) for communicating about opportunities for reasoning and proving embedded in tasks. By this, our study contributes to the growing research on using research-based tools for teacher education and professional development (e.g., Candela & Boston, 2022). Moreover, by providing teachers with a coherent and objectified language to communicate about reasoning and proving, we step forward to support teachers' practices of identifying, designing, modifying, and enacting tasks that afford students ample opportunities for reasoning and proving – a need raised by many researchers and teacher educators (e.g., Hanna & de Villiers, 2012).

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