

THE UNGAR GAMES

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ABSTRACT. Let L be a finite lattice. Inspired by Ungar’s solution to the famous *slopes problem*, we define an *Ungar move* to be an operation that sends an element $x \in L$ to the meet of $\{x\} \cup T$, where T is a subset of the set of elements covered by x . We introduce the following *Ungar game*. Starting at the top element of L , two players—Atniss and Eeta—take turns making nontrivial Ungar moves; the first player who cannot do so loses the game. Atniss plays first. We say L is an *Atniss win* (respectively, *Eeta win*) if Atniss (respectively, Eeta) has a winning strategy in the Ungar game on L . We first prove that the number of principal order ideals in the weak order on S_n that are Eeta wins is $O(0.95586^n n!)$. We then consider a broad class of intervals in Young’s lattice that includes all principal order ideals, and we characterize the Eeta wins in this class; we deduce precise enumerative results concerning order ideals in rectangles and type- A root posets. We also characterize and enumerate principal order ideals in Tamari lattices that are Eeta wins. Finally, we conclude with some open problems and a short discussion of the computational complexity of Ungar games.

1. INTRODUCTION

1.1. Poset Games. In Gale’s game **Chomp** [14], we begin with a rectangular chocolate bar whose northwestmost *carré*¹ has been removed. Two players alternately take nonempty bites, where each bite consists of choosing a *carré* and eating all *carrés* that lie weakly southeast of the chosen one. The first player who is left with nothing to eat is designated the loser. See [Figure 1](#) for an example. Although it is easy to see (using a strategy-stealing argument, as described in Gale’s original paper) that the first player can always guarantee a win in this game, describing an explicit winning strategy is open even for 3-row chocolate bars [31].

More generally, one can play **Chomp** on a chocolate bar of an arbitrary skew partition shape; at this level of generality, the first player does not always have a winning strategy. In fact, **Chomp** generalizes even further to finite posets (without mention to chocolate). In the *poset game* played on the finite poset P , two players start with P and then alternately remove nonempty principal upward-closed sets; the first player who is unable to make a move (i.e., who is left with the empty set) loses. **Nim**, another notable example of a poset game, corresponds to the case where P is a disjoint union of chains.

1.2. Nibble. Consider now the following more genteel version of **Chomp**, which we call **Nibble**. Instead of taking a boorishly large mouthful, a player may only politely nibble away at any number of exposed corner *carrés* of the chocolate bar. An example is illustrated in [Figure 1](#). A corollary of one of our main results ([Theorem 1.5](#)) is a complete characterization of which player has a winning strategy when **Nibble** is played on a chocolate bar in the shape of an arbitrary Young diagram.

Just as **Chomp** generalizes to arbitrary finite posets, **Nibble** generalizes to arbitrary finite lattices; our next order of business is explaining this generalization.

¹It appears that there is no generally accepted English word for this concept. Hershey’s has attempted to popularize the word “pip,” but this has not caught on. The word “square” may be the closest approximation in standard English usage.

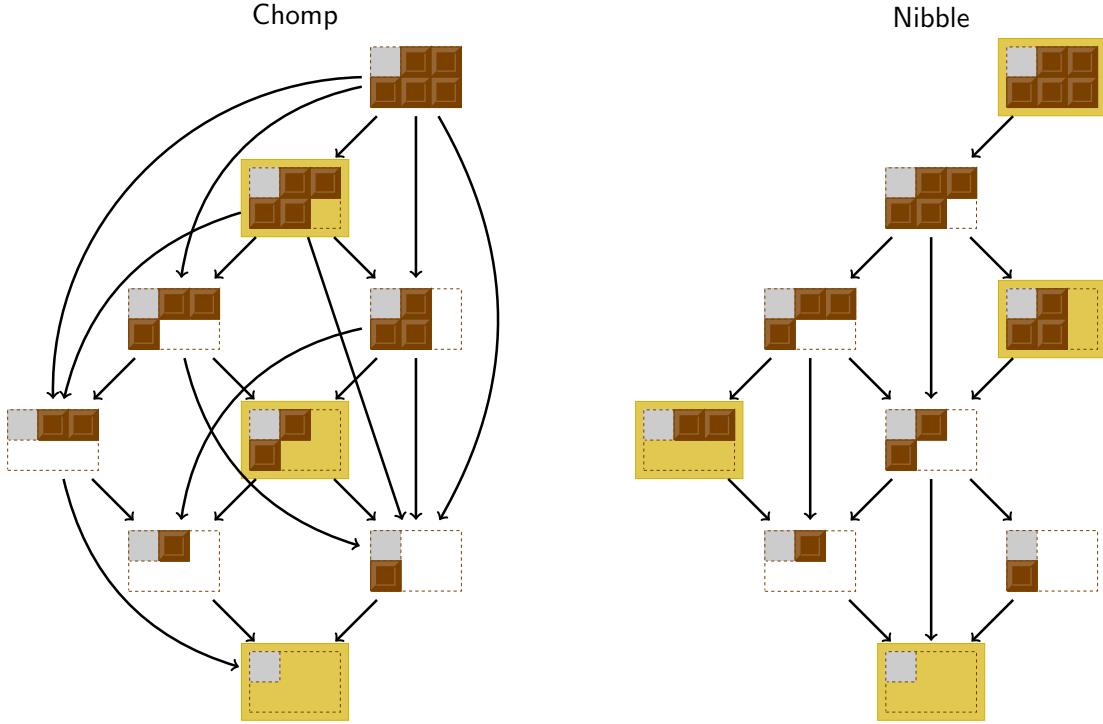


FIGURE 1. Allowable moves in Chomp (left) and in Nibble (right). Starting positions for which the second player has a winning strategy are indicated in gold.

1.3. Ungar Moves. In 1970, Scott [26] asked for the minimum possible number of distinct slopes determined by a collection of $n \geq 4$ points in the plane that do not all lie on a single line. Ungar [30] solved this problem in 1982 by showing that the answer is $2\lfloor n/2 \rfloor$. Building on an approach suggested by Goodman and Pollack [15], Ungar considered projecting the collection of points onto a rotating line. At each point in time, the ordering of the projected points along the line yields a permutation of the set $[n] = \{1, \dots, n\}$. As the line rotates, the projected points sometimes swap positions in the ordering. (See Figure 2.) This idea allowed Ungar to work in a purely combinatorial setting in which he analyzed certain *moves* that can be performed on permutations. Each such move reverses some disjoint consecutive decreasing subsequences of a permutation. For instance, we could reverse the consecutive decreasing subsequences 53 and 641 in the permutation 853297641 to obtain the new permutation 835297146.

Every poset P in this article is assumed to have the property that $\{y \in P : y \leq x\}$ is finite for every $x \in P$. Given a poset P and an element $x \in P$, we write $\text{cov}_P(x)$ for the set of elements of P that are covered by x . There is an equivalent way of formulating the moves that Ungar studied if we view the symmetric group S_n as a lattice under the (right) weak order: a move sends a permutation $w \in S_n$ to the meet $\bigwedge(\{w\} \cup T)$, where $T \subseteq \text{cov}_{S_n}(w)$. This observation leads to the following much more general definition from [10].

Definition 1.1 ([10]). Let L be a meet-semilattice. An *Ungar move* is an operation that sends an element $x \in L$ to $\bigwedge(\{x\} \cup T)$ for some set $T \subseteq \text{cov}_L(x)$. We say this Ungar move is *trivial* if $T = \emptyset$, and we say it is *maximal* if $T = \text{cov}_L(x)$.

Given a meet-semilattice L and an element $x \in L$, we write $\text{Ung}(x)$ for the set of elements of L that can be obtained by applying an Ungar move to x .

Suppose $n \geq 4$, and, as before, view S_n as a lattice under the weak order. Consider starting with the decreasing permutation with one-line notation $n(n-1)\cdots 1$ and applying nontrivial Ungar

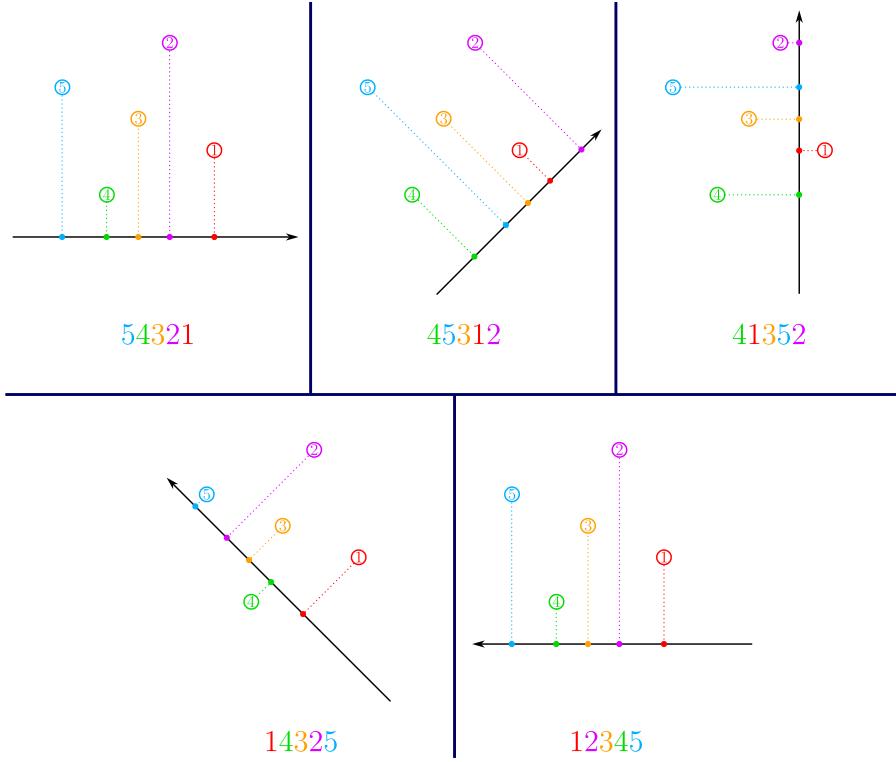


FIGURE 2. Five points in the plane are numbered $1, 2, 3, 4, 5$. One can project the points onto a line and read the ordering of the projections along the line to obtain a permutation. When the line rotates, the associated permutation changes via an Ungar move.

moves until reaching the identity permutation $12 \cdots n$. Ungar proved that if the first Ungar move in this process is not maximal, then the total number of Ungar moves needed is at least $2\lfloor n/2 \rfloor$ [30]. This allowed him to resolve Scott's original geometric problem about slopes. See [1, Chapter 12] for additional exposition about this result.

A meet-semilattice L has an associated *pop-stack sorting operator* $\text{Pop}_L: L \rightarrow L$, which acts on each element of L by applying a maximal Ungar move. The nomenclature comes from the fact that Pop_{S_n} coincides with a map that sends a permutation through a data structure called a *pop-stack*—this map has been the object of considerable study in combinatorics and theoretical computer science [2, 3, 6, 7, 19, 22], and numerous recent articles have investigated pop-stack sorting operators on other interesting lattices [5, 8, 9, 11, 17, 24]. In his original paper [30], Ungar also proved that the maximum number of iterations of Pop_{S_n} needed to send a permutation in S_n to the identity is $n - 1$.

In [10], the first author and Li studied *Ungarian Markov chains*, which are random processes on lattices in which Ungar moves are applied randomly.

1.4. Ungar Games. We can now describe our generalization of Nibble. Let L be a finite lattice. Starting at the top element $\hat{1} \in L$, two players—Atniss and Eeta—take turns making nontrivial Ungar moves; the first player who cannot make a nontrivial Ungar move loses the game. We assume that Atniss goes first. Note that the game ends precisely when a player reaches the bottom element $\hat{0}$ of L . In particular, Eeta wins if $|L| = 1$. Observe that exactly one of the two players has a winning strategy in the Ungar game on L .

Definition 1.2. We say a finite lattice L is an *Atniss win* if Atniss has a winning strategy in the Ungar game played on L ; otherwise, we say L is an *Eeta win*.

Slightly abusing terminology, we also make the following definition when we have a fixed meet-semilattice.

Definition 1.3. Given a meet-semilattice L with a minimal element $\hat{0}$, we say an element $x \in L$ is an *Atniss win* in L if the interval $[\hat{0}, x]$ in L is an Atniss win; otherwise, we say x is an *Eeta win* in L . Let $\mathbf{A}(L)$ and $\mathbf{E}(L)$ denote the set of Atniss wins in L and the set of Eeta wins in L , respectively.

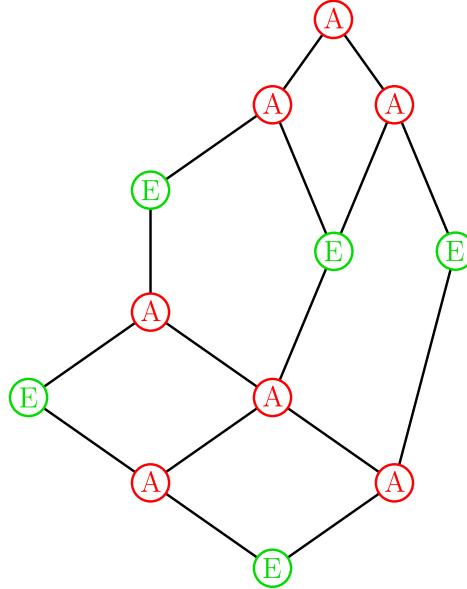


FIGURE 3. A lattice with 7 Atniss wins (labeled **A**) and 5 Eeta wins (labeled **E**). The entire lattice is an Atniss win.

One can determine the sets $\mathbf{A}(L)$ and $\mathbf{E}(L)$ recursively. First, the bottom element $\hat{0}$ is an Eeta win. In general, an element $x \in L$ is an Atniss win if there exists an Eeta win in $\text{Ung}(x) \setminus \{x\}$, while x is an Eeta win if the elements of $\text{Ung}(x) \setminus \{x\}$ are all Atniss wins. See Figure 3.

Our main focus in this article will be the characterization and the asymptotic and/or exact enumeration of Eeta (equivalently, Atniss) wins in various interesting lattices.

1.5. The Weak Order. We begin by considering the weak order on S_n since that is, after all, the context in which Ungar moves first arose.

Theorem 1.4. *We have $|\mathbf{E}(S_n)| = O(0.95586^n n!)$.*

Although the preceding theorem is not as precise as the results that we will derive for other lattices, it still shows that asymptotically almost all elements of S_n are Atniss wins.

1.6. Intervals in Young's Lattice. We write $J(P)$ for the lattice of finite order ideals of a poset P , ordered by containment. *Young's lattice* is $J(\mathbb{N}^2)$; equivalently, it is the lattice of integer partitions ordered by containment of Young diagrams. We tacitly identify integer partitions and skew partitions² with their Young diagrams (which we draw using English conventions). Suppose

²Although it is customary to write λ/μ for the skew shape obtained by removing a Young diagram μ from a Young diagram λ , we will break with this convention and write $\lambda \setminus \mu$ instead. This is because we view λ and μ as posets.

μ and λ are partitions with $\mu \leq \lambda$. We can view $\lambda \setminus \mu$ as a poset whose elements are the boxes of $\lambda \setminus \mu$; the order relation is such that $\square \leq \square'$ if and only if \square lies weakly northwest of \square' . The interval $[\mu, \lambda]$ in Young's lattice is naturally isomorphic to $J(\lambda \setminus \mu)$, the lattice of order ideals of $\lambda \setminus \mu$. The Ungar game on $J(\lambda \setminus \mu)$ is equivalent to the game **Nibble** played on a chocolate bar of shape $\lambda \setminus \mu$.

A *lattice path* is a finite path that starts at a point in \mathbb{Z}^2 and uses unit north (i.e., $(0, 1)$) steps and unit east (i.e., $(1, 0)$) steps. We denote north steps by N and east steps by E, and we identify lattice paths with finite words over the alphabet $\{N, E\}$. A *block* of a lattice path is a maximal consecutive string of steps that have the same direction. For example, the blocks of the lattice path EENENNNNN are EE, N, E, and NNNN (in that order).

Associated to a partition λ is the lattice path $\text{path}(\lambda)$ obtained by traversing the southeast boundary of λ . More precisely, if $\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \dots \geq \lambda_k \geq 1$, then

$$\text{path}(\lambda) = E^{\lambda_k} N E^{\lambda_{k-1} - \lambda_k} N \dots E^{\lambda_1 - \lambda_2} N.$$

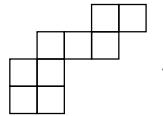
The n -th *staircase* (for $n \geq 0$) is the partition $\delta_n = (n, n-1, \dots, 2, 1)$; its associated lattice path is $\text{path}(\delta_n) = (EN)^n$.

The following theorem treats a very large class of intervals in Young's lattice and characterizes which of them are Eeta wins. In particular, the case $\mu = \emptyset$ (and $n = 0$) completely characterizes which elements of Young's lattice are Eeta wins.

Theorem 1.5. *Consider an interval $[\mu, \lambda]$ in Young's lattice. Let n be the smallest integer such that $\mu \leq \delta_n$. If $\delta_{n+1} \leq \lambda$, then the interval $[\mu, \lambda]$ is an Eeta win if and only if $\text{path}(\lambda)$ does not contain an odd-length block of east steps immediately followed by an odd-length block of north steps.*

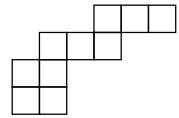
Example 1.6. Let μ be the partition $(3, 1)$. Then $\mu \leq \delta_3$, but $\mu \not\leq \delta_2$. Therefore, we can apply Theorem 1.5 whenever $\delta_4 \leq \lambda$.

If $\lambda = (5, 4, 2, 2)$, then the Young diagram of $\lambda \setminus \mu$ is



In this case, Theorem 1.5 tells us that $[\mu, \lambda]$ is an Atniss win because $\text{path}(\lambda) = \text{EENNEENEN}$ contains a block consisting of a single east step immediately followed by a block consisting of a single north step.

On the other hand, if $\lambda = (6, 4, 2, 2)$, then the Young diagram of $\lambda \setminus \mu$ is



In this case, $\text{path}(\lambda) = \text{EENNEENEEN}$, so Theorem 1.5 guarantees that $[\mu, \lambda]$ is an Eeta win.

Theorem 1.5 has a somewhat surprising corollary. Namely, whether the lattice $[\mu, \lambda] \cong J(\lambda \setminus \mu)$ is an Atniss win or an Eeta win is independent of μ so long as λ is “deep enough” in Young's lattice relative to μ . This is actually a special case of the following much more general result. Let us write $\max(P)$ for the set of maximal elements of a poset P .

Theorem 1.7. *Let P be a poset. Suppose $\delta, \lambda \in J(P)$ are such that $\delta \subseteq \lambda$ and every non-maximal element of δ is less than at least 2 maximal elements of δ . For every $\mu \in J(P)$ such that $\mu \subseteq \delta \setminus \max(\delta)$, the lattice $J(\lambda \setminus \mu)$ is an Atniss win if and only if the lattice $J(\lambda)$ is an Atniss win.*

Let us highlight two families of intervals in Young's lattice for which we will obtain especially nice enumerative results. Let $\rho_{a \times b}$ be the rectangular Young diagram that consists of a rows of size

b. Let $\Phi^+(A_n)$ denote the root poset of type A_n . Then $\Phi^+(A_n)$ is isomorphic as a poset to the skew shape $\rho_{n \times n} \setminus \delta_{n-1}$.

Theorem 1.8. *We have*

$$\sum_{a \geq 0} \sum_{b \geq 0} |\mathbf{E}(J(\rho_{a \times b}))| x^b y^a = \frac{(1+x)(1+y)}{1 - (1+x)y^2 - (1+y)x^2}.$$

Theorem 1.9. *We have*

$$\sum_{n \geq 1} |\mathbf{E}(J(\Phi^+(A_n)))| z^n = \frac{-1 - 2z + \sqrt{z^2 - 4z + 2 - 2\sqrt{1 - 4z + 4z^2 - 4z^3}}}{2z}.$$

Consequently,

$$|\mathbf{E}(J(\Phi^+(A_n)))| \sim \frac{\gamma}{\sqrt{\pi}} n^{-3/2} \rho^{n+1},$$

where

$$\rho = \frac{6}{2 - 8(3\sqrt{57} - 1)^{-1/3} + (3\sqrt{57} - 1)^{1/3}} \approx 3.13040$$

and

$$\gamma = \frac{1}{4\sqrt{291}} \sqrt{576 + (1726130304 - 69393024\sqrt{57})^{1/3} + 12(998918 + 40158\sqrt{57})^{1/3}} \approx 0.79594.$$

Since $|J(\Phi^+(A_n))|$ is the $(n+1)$ -th Catalan number (which grows as $(4 - o(1))^n$), the preceding theorem shows that $|\mathbf{E}(J(\Phi^+(A_n)))|/|J(\Phi^+(A_n))|$ is decays exponentially in n .

1.7. Tamari Lattices. Let Tam_n denote the n -th Tamari lattice. These lattices, which were introduced by Tamari [29] in 1962, are fundamental objects in algebraic combinatorics with connections to several other areas [20]; they differ from the lattices discussed in the previous subsection because they are not distributive. We will characterize Eeta wins in Tamari lattices in [Propositions 5.2](#) and [5.3](#), and this will lead to the following exact enumeration.

Theorem 1.10. *The generating function $F(z) = \sum_{n \geq 1} |\mathbf{E}(\text{Tam}_n)| z^n$ is algebraic of degree 4: it satisfies the equation $Q(F(z), z) = 0$, where*

$$Q(y, z) = z + (-1 + 3z + z^2)y + (-2 + 2z + 3z^2)y^2 + 3z^2y^3 + z^2y^4.$$

Consequently,

$$|\mathbf{E}(\text{Tam}_n)| \sim \frac{\gamma}{\sqrt{\pi}} n^{-3/2} \rho^n,$$

where $\rho \approx 2.90511$ is the unique positive real root of the polynomial

$$32z^7 - 32z^6 - 155z^5 - 20z^4 - 148z^3 + 60z^2 - 8z - 4$$

and $\gamma \approx 1.04240$ is a root of the polynomial

$$\begin{aligned} & 17348952064z^{14} - 11927404544z^{12} - 6678731520z^{10} \\ & - 886278144z^8 - 33824320z^6 - 516144z^4 + 4048z^2 + 11. \end{aligned}$$

Since $|\text{Tam}_n|$ is the n -th Catalan number (which grows as $(4 - o(1))^n$), the preceding theorem shows that $|\mathbf{E}(\text{Tam}_n)|/|\text{Tam}_n|$ is decays exponentially in n .

1.8. Outline. In [Section 2](#), we discuss some basic properties of lattices and Ungar moves. [Section 3](#) concerns the weak order on S_n ; it is in this section that we establish [Theorem 1.4](#). In [Section 4](#), we prove the results from [Section 1.6](#) about Young's lattice. [Section 5](#) is devoted to analyzing Ungar games on principal order ideals of Tamari lattices; it is in this section that we prove [Theorem 1.10](#). Finally, in [Section 6](#), we mention potential directions for future research; we also give a short argument showing that Ungar games are NC^1 -hard.

2. BASICS

We assume familiarity with the theory of posets (partially ordered sets); a standard reference is [28, Chapter 3]. As mentioned in Section 1, we assume that every poset P in this article is such that $\{y \in P : y \leq x\}$ is finite for every $x \in P$.

Let P be a poset. We tacitly view subsets of P as subposets of P . If $u, v \in P$ are such that $u \leq v$, then the *interval* from u to v is the set $[u, v] = \{w \in P : u \leq w \leq v\}$. If $|[u, v]| = 2$, then we say v *covers* u . For $x \in P$, we write $\text{cov}_P(x)$ for the set of elements of P that x covers. We write $\max(P)$ for the set of maximal elements of P . An *order ideal* of P is a subset $I \subseteq P$ such that if $x, y \in P$ are such that $x \leq y$ and $y \in I$, then $x \in I$. An order ideal is *principal* if it is of the form $\{y \in P : y \leq x\}$ for some $x \in P$. Let $J(P)$ denote the set of finite order ideals of P , ordered by containment.

A *meet-semilattice* is a poset L such that any two elements $x, y \in L$ have a greatest lower bound, which is called their *meet* and denoted $x \wedge y$. Because the meet operation is commutative and associative, it makes sense to write $\bigwedge X$ for the meet of a nonempty finite set $X \subseteq L$. Our running assumption about posets (that principal order ideals are finite) guarantees that L has a unique minimal element $\hat{0}$. We say L is a *lattice* if any two elements $x, y \in L$ also have a least upper bound, which is called their *join* and denoted $x \vee y$. If L is a finite lattice, then it has a unique maximal element $\hat{1}$.

If P is a poset, then $J(P)$ is a lattice whose meet and join operations are given by intersection and union, respectively. A finite lattice is *distributive* if it is isomorphic to $J(P)$ for some finite poset P . Ungar moves in distributive lattices have a simple description. For each $I \in J(P)$, we have $\text{cov}_{J(P)}(I) = \{I \setminus \{x\} : x \in \max(I)\}$. Thus, applying an Ungar move to I results in an order ideal $I \setminus T$ for some $T \subseteq \max(I)$.

If L is a meet-semilattice, then every element $x \in L$ is either an Eeta win or an Atniss win. If x is an Atniss win, then there is a nontrivial Ungar move that sends x to an Eeta win. This yields the following lemma, which we record for future reference.

Lemma 2.1. *Let L be a meet-semilattice. For every $x \in L$, the set $\text{Ung}(x) \cap \mathbf{E}(L)$ is nonempty.*

We denote the Cartesian product of sets X_1, \dots, X_m by $X_1 \times \dots \times X_m$. If L_1, \dots, L_m are lattices, then there is a natural partial order on $L_1 \times \dots \times L_m$ in which $(x_1, \dots, x_m) \leq (y_1, \dots, y_m)$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq m$; this turns $L_1 \times \dots \times L_m$ into a lattice called the *product* of L_1, \dots, L_m . The following simple lemma, which allows us to analyze the Ungar game on $L_1 \times \dots \times L_m$ in terms of the Ungar games on L_1, \dots, L_m , will be very useful for us in the sequel.

Lemma 2.2. *Let L_1, \dots, L_m be lattices. An element (x_1, \dots, x_m) is an Eeta win in the product $L_1 \times \dots \times L_m$ if and only if x_i is an Eeta win in L_i for every $i \in [m]$. That is,*

$$\mathbf{E}(L_1 \times \dots \times L_m) = \mathbf{E}(L_1) \times \dots \times \mathbf{E}(L_m).$$

Proof. We proceed by induction on $L_1 \times \dots \times L_m$. Choose $(x_1, \dots, x_m) \in L_1 \times \dots \times L_m$. The key observation is that $\text{Ung}(x_1, \dots, x_m) = \text{Ung}(x_1) \times \dots \times \text{Ung}(x_m)$. Thus, applying a nontrivial Ungar move to (x_1, \dots, x_m) consists of applying Ungar moves to x_1, \dots, x_m individually, where at least one of these Ungar moves is nontrivial.

Suppose (x_1, \dots, x_m) is such that $x_i \in \mathbf{E}(L_i)$ for all i . Applying any nontrivial Ungar move to (x_1, \dots, x_m) produces an element (y_1, \dots, y_m) such that $y_i \in \mathbf{A}(L_i)$ for some i ; by the induction hypothesis, $(y_1, \dots, y_m) \in \mathbf{A}(L_1 \times \dots \times L_m)$. This shows that $(x_1, \dots, x_m) \in \mathbf{E}(L_1 \times \dots \times L_m)$.

To prove the reverse direction, suppose (x_1, \dots, x_m) is such that $x_i \in \mathbf{A}(L_i)$ for some i . Let $K \subseteq [m]$ be the (necessarily nonempty) set of indices i such that $x_i \in \mathbf{A}(L_i)$. For each $i \in K$, there is some $y_i \in (\text{Ung}(x_i) \setminus \{x_i\}) \cap \mathbf{E}(L_i)$. For $i \notin K$, set $y_i = x_i$. Then $(y_1, \dots, y_m) \in \text{Ung}(x_1, \dots, x_m) \setminus \{(x_1, \dots, x_m)\}$ is an Eeta win by induction, so $(x_1, \dots, x_m) \in \mathbf{A}(L_1 \times \dots \times L_m)$. \square

3. THE WEAK ORDER

Consider the symmetric group S_n , whose elements are the permutations of $[n]$. An *inversion* of a permutation $w \in S_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $w^{-1}(i) > w^{-1}(j)$. The (right) weak order is the partial order on S_n in which $u \leq v$ if and only if every inversion of u is also an inversion of v . It is well known that the weak order on S_n is a lattice. We will henceforth simply write S_n for this lattice.

The Ungar moves on S_n are precisely those described in Section 1.3: each such move reverses some disjoint consecutive decreasing subsequences of a permutation.

Given a word x of length k whose entries are distinct positive integers, we define the *standardization* of x to be the permutation in S_k obtained by replacing the i -th smallest entry in x with i for all $i \in [n]$. For example, the standardization of 36582 is 24351. Given $v \in S_k$, we say a permutation $w \in S_n$ *consecutively contains* v if there exists an index $i \in [n - k + 1]$ such that the standardization of $w(i)w(i + 1) \cdots w(i + k - 1)$ is v . For example, w consecutively contains 1324 if and only if there exists $i \in [n - 3]$ such that $w(i) < w(i + 2) < w(i + 1) < w(i + 3)$. We say w *consecutively avoids* v if w does not consecutively contain v .

The following lemma will allow us to prove Theorem 1.4, which tells us that as $n \rightarrow \infty$, asymptotically almost all permutations in S_n are Atniss wins. Rather than demonstrate explicit winning strategies for Atniss, we will employ a strategy-stealing argument.

Lemma 3.1. *Let $B = \{1324, 14325, 154326, 1654327, \dots\}$ be the set of permutations of the form $1(m-1)(m-2) \cdots 2m$ for $m \geq 4$. If $w \in S_n$ is a permutation that consecutively contains one of the permutations in B , then w is an Atniss win in S_n .*

Proof. Suppose $m \geq 4$ and $i \in [n - m + 1]$ are such that $w(i)w(i + 1) \cdots w(i + m - 1)$ has standardization $1(m-1)(m-2) \cdots 2m$. Let v be the permutation obtained from w by reversing the consecutive decreasing subsequence $w(i + 1)w(i + 2) \cdots w(i + m - 2)$. The maximal consecutive decreasing subsequences of v are exactly the same as the maximal consecutive decreasing subsequences of w other than $w(i + 1)w(i + 2) \cdots w(i + m - 2)$. Therefore, $\text{Ung}(v)$ is equal to the set of permutations that can be obtained by applying an Ungar move to w that involves reversing the subsequence $w(i + 1)w(i + 2) \cdots w(i + m - 2)$. We know by Lemma 2.1 that there exists an Eeta win in $\text{Ung}(v)$. This Eeta win is in $\text{Ung}(w) \setminus \{w\}$, so w is an Atniss win. \square

Another way of phrasing the above proof of Lemma 3.1 is that Atniss can reverse the run $w(i + 1)w(i + 2) \cdots w(i + m - 2)$ as a “throwaway” move and then choose whether or not to play further.

Proof of Theorem 1.4. It follows from Lemma 3.1 that every Eeta win in S_n consecutively avoids 1324. It is known (see [21]) that the number of permutations in S_n that consecutively avoid 1324 is $O(0.95586^n n!)$. \square

4. INTERVALS IN YOUNG’S LATTICE

4.1. Intervals in Distributive Lattices. Before we specialize our attention to Young’s lattice, let us prove Theorem 1.7, which is much more general in scope because it deals with arbitrary finite distributive lattices. We begin with a simple but useful lemma that is analogous to Lemma 3.1.

Lemma 4.1. *Let P be a poset. Suppose $\lambda \in J(P)$ and $x \in \max(\lambda)$ are such that*

$$\max(\lambda \setminus \{x\}) = \max(\lambda) \setminus \{x\}.$$

Then $\lambda \in \mathbf{A}(J(P))$.

Proof. By Lemma 2.1, there exists $\nu \in \text{Ung}(\lambda \setminus \{x\}) \cap \mathbf{E}(J(P))$. Then $\nu = \lambda \setminus (\{x\} \cup T)$ for some $T \subseteq \max(\lambda \setminus \{x\}) = \max(\lambda) \setminus \{x\}$. Since $(\{x\} \cup T) \subseteq \max(\lambda)$, we have $\nu \in \text{Ung}(\lambda) \setminus \{\lambda\}$. Because $\nu \in \mathbf{E}(J(P))$, this proves that $\lambda \in \mathbf{A}(J(P))$. \square

Proof of Theorem 1.7. Let P be a poset, and suppose $\delta, \lambda, \mu \in J(P)$ are such that every non-maximal element of δ is less than at least 2 maximal elements of δ and $\mu \subseteq (\delta \setminus \max(\delta)) \subseteq \delta \subseteq \lambda$. For $I \in J(P)$, let $\tilde{I} = I \setminus \mu$. We will prove by induction on $|\lambda|$ that $\lambda \in \mathbf{A}(J(P))$ if and only if $\tilde{\lambda} \in \mathbf{A}(J(\tilde{P}))$.

First, suppose there exists $x \in \max(\delta) \cap \max(\lambda)$. It follows from our hypotheses on δ and λ that $\max(\lambda \setminus \{x\}) = \max(\lambda) \setminus \{x\}$, so Lemma 4.1 guarantees that $\lambda \in \mathbf{A}(J(P))$. On the other hand, the hypothesis that $\mu \subseteq (\delta \setminus \max(\delta))$ implies that $x \in \max(\tilde{\lambda})$ and $\max(\tilde{\lambda} \setminus \{x\}) = \max(\tilde{\lambda}) \setminus \{x\}$. Appealing to Lemma 4.1 again, we find that $\tilde{\lambda} \in \mathbf{A}(J(\tilde{P}))$ as well.

Next, suppose we have $\max(\delta) \cap \max(\lambda) = \emptyset$. Because $\max(\lambda) = \max(\tilde{\lambda})$, we know that $\text{Ung}(\tilde{\lambda}) = \{\tilde{\nu} : \nu \in \text{Ung}(\lambda)\}$. For each $\nu \in \text{Ung}(\lambda) \setminus \{\lambda\}$, we have $\delta \subseteq \nu$, so we know by induction that $\nu \in \mathbf{A}(J(P))$ if and only if $\tilde{\nu} \in \mathbf{A}(J(\tilde{P}))$. This implies that there exists an element of $\mathbf{E}(J(P))$ in $\text{Ung}(\lambda) \setminus \{\lambda\}$ if and only if there exists an element of $\mathbf{E}(J(\tilde{P}))$ in $\text{Ung}(\tilde{\lambda}) \setminus \{\tilde{\lambda}\}$. In other words, $\lambda \in \mathbf{A}(J(P))$ if and only if $\tilde{\lambda} \in \mathbf{A}(J(\tilde{P}))$. \square

4.2. Atniss and Eeta Wins in Young's Lattice. Note that every non-maximal element of the staircase partition δ_{n+1} is less than at least 2 maximal elements of δ_{n+1} . Moreover, we have $\delta_{n+1} \setminus \max(\delta_{n+1}) = \delta_n$. Appealing to Theorem 1.7, we find that in order to prove Theorem 1.5, it suffices to prove it when $\mu = \emptyset$.

We will find it helpful to define a **J-path**³ to be a lattice path of the form $E^a N^b$ for some positive integers a and b . Given parities α and β , we say that such a lattice path is (α, β) if a is α and b is β . For example, the J-path $E^2 N^5$ is (even, odd). The maximal elements of a partition λ are called the **corners** of λ . If λ has k corners, then $\text{path}(\lambda)$ can be written uniquely in the form $J_1 \cdots J_k$, where J_1, \dots, J_k are J-paths; we call these the **maximal J-paths** of λ .

Proof of Theorem 1.5. As mentioned above, we may assume $\mu = \emptyset$. Then $n = 0$. Let λ be a partition, and let J_1, \dots, J_k be the maximal J-paths of λ so that $\text{path}(\lambda) = J_1 \cdots J_k$. Let c_1, \dots, c_k be the corners of λ , listed from southwest to northeast (so c_i corresponds naturally to J_i). Our goal is to show that λ is an Eeta win in Young's lattice if and only if none of its maximal J-paths are (odd, odd). If $\lambda = \emptyset$, then this is vacuously true because λ has no maximal J-paths. Thus, we may assume λ is nonempty and proceed by induction on Young's lattice. Observe that λ is an Atniss win in Young's lattice if and only if the transpose of λ is an Atniss win in Young's lattice.

First, suppose none of the maximal J-paths of λ are (odd, odd). Consider $\nu \in \text{Ung}(\lambda) \setminus \{\lambda\}$. Then $\nu = \lambda \setminus T$, where $T \subseteq \{c_1, \dots, c_k\}$ is nonempty. Let us write $T = \{c_{i_1}, \dots, c_{i_m}\}$, where $i_1 < \dots < i_m$. We may assume without loss of generality that at least one of J_{i_1}, \dots, J_{i_m} is (even, odd) or (even, even); if not, then simply replace λ and ν by their transposes. Let j be the smallest index such that c_{i_j} is (even, odd) or (even, even). Say $J_{i_j} = E^a N^b$. When we delete the corners in T to obtain ν , J_{i_j} transforms into $E^{a-1} N E^{b-1}$. If $i_j = 1$ or $c_{i_j-1} \notin T$, then it is straightforward to see that $E^{a-1} N$ is an (odd, odd) maximal J-path of ν . If instead $i_j > 1$ and $c_{i_j-1} \in T$ (so $i_j - 1 = i_{j-1}$), then $J_{i_{j-1}}$ is (odd, even), so it contains at least 2 north steps. This implies that $E^{a-1} N$ is an (odd, odd) maximal J-path of ν in this case as well. In either case, we have shown that ν has an (odd, odd) maximal J-path, so we can use our induction hypothesis to see that ν is an Atniss win in Young's lattice. As ν was an arbitrary element of $\text{Ung}(\lambda) \setminus \{\lambda\}$, this proves that λ is an Eeta win in Young's lattice.

To prove the converse, suppose λ has at least one (odd, odd) maximal J-path. We consider a few cases.

Case 1. Suppose J_k has at least 2 north steps. Let $\lambda^\#$ be the partition obtained by removing the first two rows from λ . Then $\lambda^\#$ has at least one (odd, odd) maximal J-path, so it is an Atniss win in Young's lattice by induction. This means that there is a nonempty set $T^\#$ of corners of $\lambda^\#$ such

³The symbol J is pronounced “le” (or “lle”).

that $\lambda^\# \setminus T^\#$ is an Eeta win. By induction, $\lambda^\# \setminus T^\#$ has no (odd, odd) maximal \mathbb{J} -paths. The set $T^\#$ corresponds naturally to a set T of corners of λ , and $\lambda \setminus T$ is the partition obtained by adding the first two rows of λ to the top of $\lambda^\# \setminus T^\#$. Then $\lambda \setminus T$ has no (odd, odd) maximal \mathbb{J} -paths, so it is an Eeta win by induction. Since $(\lambda \setminus T) \in \text{Ung}(\lambda) \setminus \{\lambda\}$, this shows that λ is an Atniss win.

Case 2. Suppose $\mathbb{J}_k = \text{EN}$. In this case, $\max(\lambda \setminus \{c_k\}) = \max(\lambda) \setminus \{c_k\}$. Setting $P = \mathbb{N}^2$ in [Lemma 4.1](#), we find that λ is an Atniss win.

Case 3. Suppose $\mathbb{J}_k = \text{E}^a\text{N}$ for some $a \geq 2$. Let $\lambda^\#$ be the partition obtained by removing the first row from λ . By [Lemma 2.1](#), there is a (possibly empty) set $T^\#$ of corners of $\lambda^\#$ such that $\lambda^\# \setminus T^\#$ is an Eeta win. By induction, $\lambda^\# \setminus T^\#$ has no (odd, odd) maximal \mathbb{J} -paths. The set $T^\#$ corresponds naturally to a set T of corners of λ , and $\lambda \setminus T$ is the partition obtained by adding the first row of λ to the top of $\lambda^\# \setminus T^\#$. Then $\lambda \setminus T$ has no (odd, odd) maximal \mathbb{J} -paths except for possibly the northeastmost \mathbb{J} -path (call this $\tilde{\mathbb{J}}$). Notice that $\tilde{\mathbb{J}} \in \{\text{E}^a\text{N}, \text{E}^{a+1}\text{N}\}$. If $\tilde{\mathbb{J}}$ is (odd, odd), then $\lambda \setminus (\{c_k\} \cup T)$ has no (odd, odd) maximal \mathbb{J} -paths and hence is an Eeta win by induction. Since $(\lambda \setminus (\{c_k\} \cup T)) \in \text{Ung}(\lambda) \setminus \{\lambda\}$, this shows that λ is an Atniss win if $\tilde{\mathbb{J}}$ is (odd, odd). Now suppose $\tilde{\mathbb{J}}$ is instead (even, odd). Notice that T is nonempty since, if it were empty, then $\lambda \setminus T = \lambda$ would have no (odd, odd) maximal \mathbb{J} -paths, contrary to our standing assumption. So T is nonempty, and, since $\lambda \setminus T$ has no (odd, odd) maximal \mathbb{J} -paths, again we are done by induction. \square

4.3. Rectangles. Let us now prove [Theorem 1.8](#), which enumerates Eeta wins in $J(\rho_{a \times b})$, where $\rho_{a \times b}$ is the $a \times b$ rectangle poset.

Proof of Theorem 1.8. For fixed $a, b \geq 0$, we can append extra north steps to the beginning and extra east steps to the end of the path associated to an order ideal in $J(\rho_{a \times b})$ so that the resulting path uses a total of a north steps and b east steps. Then such a path can be written uniquely in the form $\text{N}^s \mathbb{J}_1 \cdots \mathbb{J}_k \text{E}^t$, where $s, t \geq 0$ and $\mathbb{J}_1, \dots, \mathbb{J}_k$ are \mathbb{J} -paths that use a total of $a - s$ north steps and $b - t$ east steps. It follows from [Theorem 1.5](#) that such an order ideal is an Eeta win in $J(\rho_{a \times b})$ if and only if none of $\mathbb{J}_1, \dots, \mathbb{J}_k$ are (odd, odd). We will consider generating functions that count \mathbb{J} -paths, with the variable x keeping track of the number of east steps and the variable y keeping track of the number of north steps. The generating function for (odd, odd) \mathbb{J} -paths is

$$(x + x^3 + x^5 + \cdots)(y + y^3 + y^5 + \cdots) = \frac{xy}{(1-x^2)(1-y^2)},$$

so the generating function for \mathbb{J} -paths that are not (odd, odd) is

$$(x + x^2 + x^3 + \cdots)(y + y^2 + y^3 + \cdots) - \frac{xy}{(1-x^2)(1-y^2)} = \frac{xy}{(1-x)(1-y)} - \frac{xy}{(1-x^2)(1-y^2)}.$$

The generating function that counts sequences of \mathbb{J} -paths that are not (odd, odd) is then

$$\frac{1}{1 - \left(\frac{xy}{(1-x)(1-y)} - \frac{xy}{(1-x^2)(1-y^2)} \right)}.$$

Hence,

$$\begin{aligned} \sum_{a \geq 0} \sum_{b \geq 0} |\mathbf{E}(J(\rho_{a \times b}))| x^b y^a &= \sum_{s \geq 0} y^s \sum_{t \geq 0} x^t \cdot \frac{1}{1 - \left(\frac{xy}{(1-x)(1-y)} - \frac{xy}{(1-x^2)(1-y^2)} \right)} \\ &= \frac{1}{(1-x)(1-y)} \cdot \frac{1}{1 - \left(\frac{xy}{(1-x)(1-y)} - \frac{xy}{(1-x^2)(1-y^2)} \right)} \\ &= \frac{(1+x)(1+y)}{1 - (1+x)y^2 - (1+y)x^2}. \end{aligned} \quad \square$$

4.4. Type-A Root Posets. The root poset $\Phi^+(A_n)$ —which is isomorphic to the skew shape $\rho_{n \times n} \setminus \delta_{n-1}$ —is an important poset in algebraic combinatorics with several interesting properties. For example, the number of order ideals of $\Phi^+(A_n)$ is the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2(n+1)}{n+1}$. In this subsection, we prove [Theorem 1.9](#), which enumerates Eeta wins in $J(\Phi^+(A_n))$.

As before, we view an order ideal in $J(\rho_{n \times n} \setminus \delta_{n-1})$ as a skew shape $\lambda \setminus \delta_{n-1}$ such that $\lambda \subseteq \rho_{n \times n}$, and we consider the associated lattice path $\text{path}(\lambda)$. Suppose $\text{path}(\lambda)$ uses s north steps and t east steps. Then $s, t \in \{n-1, n\}$. Let $\text{path}'(\lambda) = N^{n-s} \text{path}(\lambda) E^{n-t}$. If we delete from $\text{path}'(\lambda)$ all steps that lie on the boundary of δ_{n-1} or on the x -axis or y -axis, then we will break $\text{path}'(\lambda)$ into lattice paths $\eta^{(1)}, \dots, \eta^{(r)}$ that represent order ideals of smaller type-A root posets. That is, for each $1 \leq i \leq r$, there is a positive integer n_i such that $\eta^{(i)} = \text{path}(\lambda^{(i)})$ for some partition $\lambda^{(i)}$ satisfying $\delta_{n_i-1} \subseteq \lambda^{(i)} \subseteq \rho_{n_i \times n_i}$. In fact, this construction is designed so that $\lambda^{(i)}$ contains the slightly larger staircase δ_{n_i} . Setting $\mu = \delta_{n_i-1}$ in [Theorem 1.5](#), we find that the interval $[\delta_{n_i-1}, \lambda^{(i)}]$ is an Eeta win if and only if $\eta^{(i)}$ does not contain an odd-length block of east steps immediately followed by an odd-length block of north steps. It is straightforward to see that

$$J(\lambda \setminus \delta_{n-1}) \cong [\delta_{n_1-1}, \lambda^{(1)}] \times \dots \times [\delta_{n_r-1}, \lambda^{(r)}],$$

so it follows from [Lemma 2.2](#) that $\lambda \setminus \delta_{n-1}$ is an Eeta win in $J(\rho_{n \times n} \setminus \delta_{n-1})$ if and only if none of $\eta^{(1)}, \dots, \eta^{(r)}$ contains an odd-length block of east steps immediately followed by an odd-length block of north steps.

Example 4.2. Let $n = 12$, and let $\lambda = (11, 11, 11, 10, 10, 6, 6, 4, 3, 3, 3, 1)$. Then

$$\text{path}'(\lambda) = \text{ENEENN} \text{NENEEN} \text{NEEEENNENNNE}$$

is drawn in [Figure 4](#). The steps lying on the boundary of δ_{10} or the x -axis or y -axis are colored red. If we delete those steps, then we are left with the lattice paths

$$\eta^{(1)} = \text{ENEENN}, \quad \eta^{(2)} = \text{EN}, \quad \eta^{(3)} = \text{EEENNENN}.$$

Then $n_1 = 3$, $n_2 = 1$, and $n_3 = 4$. The corresponding partitions are

$$\lambda^{(1)} = (3, 3, 1), \quad \lambda^{(2)} = (1), \quad \lambda^{(3)} = (4, 4, 3, 3).$$

For each $1 \leq i \leq 3$, the skew shape $\lambda^{(i)} \setminus \delta_{n_i-1}$ is an order ideal of $\rho_{n_i \times n_i} \setminus \delta_{n_i-1}$. Notice that each $\lambda^{(i)}$ actually contains the staircase δ_{n_i} . Since the intervals $[\delta_2, \lambda^{(1)}]$ and $[\delta_0, \lambda^{(2)}]$ are Atniss wins, the lattice $J(\lambda \setminus \delta_{10})$ is also an Atniss win.

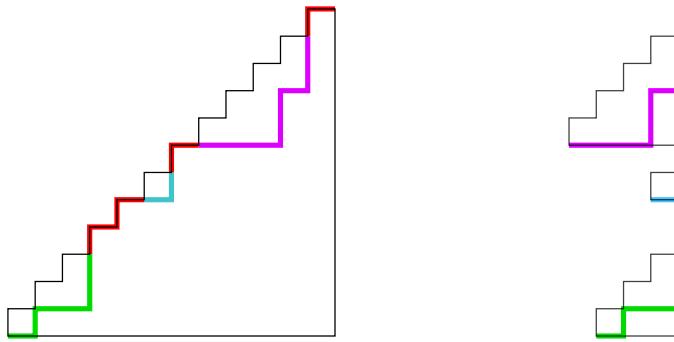


FIGURE 4. Deleting the (red) steps that lie on the boundary of δ_{10} or the x -axis or y -axis breaks a lattice path into 3 smaller lattice paths.

In our enumeration of Eeta wins in $J(\rho_{n \times n} \setminus \delta_{n-1})$, it will be convenient to use the language of Dyck paths. A [Dyck path](#) of semilength n is a path in \mathbb{R}^2 consisting of up (i.e., $(1, 1)$) steps and down (i.e., $(1, -1)$) steps that starts at $(0, 0)$, ends at $(2n, 0)$, and never passes below the x -axis.

We can represent a Dyck path as a word over the alphabet $\{U, D\}$, where U stands for an up step and D stands for a down step.

An *ascending run* (respectively, *descending run*) of a Dyck path is a maximal consecutive string of up (respectively, down) steps. Say a run is *odd* (respectively, *even*) if it has an odd (respectively, even) number of steps. Say a run is *weird* if it is odd and does not touch the x -axis or it is even and does touch the x -axis. Say a run is *strange* if it is odd and does not contain the first or last step of the Dyck path or it is even and contains the first or last step of the Dyck path.

Example 4.3. Consider the Dyck path

$$\overbrace{U|\overline{D}|UU|\overline{D}|\overline{U}|DD|UUUU|\overline{DDDD}} = \text{Diagram: A zigzag line on a grid, starting at (0,0), going up-right, down-right, up-right, down-right, up-right, down-right, up-right, down-right, up-right, ending at (10,0).}$$

Odd runs are in [light blue](#), while even runs are in [lavender](#). In the word representation of this Dyck path, we have separated the runs by bars for clarity, and we have underlined the weird runs and overlined the strange runs.

Given adjectives α and β that describe runs, let us say a Dyck path is (α, β) -*avoiding* if it does not contain an α ascending run immediately followed by a β descending run. For example, a Dyck path is (odd, strange)-avoiding if it does not contain an odd ascending run immediately followed by an strange descending run.

Given an order ideal $\lambda \setminus \delta_{n-1}$ of $\rho_{n \times n} \setminus \delta_{n-1}$, let $\text{path}^*(\lambda)$ be the word obtained from $\text{path}'(\lambda)$ by replacing each E with U and replacing each N with D. Then $U \text{path}^*(\lambda) D$ is a Dyck path of semilength $n + 1$. For example, if λ is the partition from [Example 4.2](#), then $U \text{path}^*(\lambda) D$ is the Dyck path

UUDUUDDUDUUDUUUUUDDUDDDUUD.

It follows from the above discussion that $\lambda \setminus \delta_{n-1}$ is an Eeta win in $J(\rho_{n \times n} \setminus \delta_{n-1})$ if and only if U path* (λ) D is (weird, weird)-avoiding. This allows us to prove [Theorem 1.9](#).

Proof of Theorem 1.9. Let \mathcal{F}_n be the set of (odd, odd)-avoiding Dyck paths of semilength n . Let $\underline{\mathcal{F}}_n$ and $\overline{\mathcal{F}}_n$ be the set of (weird, weird)-avoiding Dyck paths of semilength n and the set of (strange, strange)-avoiding Dyck paths of semilength n , respectively. Let

$$F(z) = \sum_{n \geq 0} |\mathcal{F}_n| z^n, \quad \underline{F}(z) = \sum_{n \geq 0} |\underline{\mathcal{F}}_n| z^n, \quad \overline{F}(z) = \sum_{n \geq 0} |\overline{\mathcal{F}}_n| z^n.$$

Let \mathcal{G}_n and \mathcal{H}_n be the set of (odd, strange)-avoiding Dyck paths of semilength n and the set of (strange, odd)-avoiding Dyck paths of semilength n , respectively. Let

$$G(z) = \sum_{n \geq 0} |\mathcal{G}_n| z^n \quad \text{and} \quad H(z) = \sum_{n \geq 0} |\mathcal{H}_n| z^n.$$

If Λ is a nonempty Dyck path, then there are unique Dyck paths Λ' and Λ'' such that $\Lambda = \text{U}\Lambda'\text{D}\Lambda''$. For example, if $\Lambda = \text{UUUDDUDDUD}$, then $\Lambda' = \text{UUDDUD}$ and $\Lambda'' = \text{UD}$. We call Λ' and Λ'' the *primary part* of Λ and the *secondary part* of Λ , respectively. A nonempty Dyck path is (weird, weird)-avoiding if and only if its primary part is (odd, odd)-avoiding and its secondary part is (weird, weird)-avoiding. Therefore,

$$(1) \quad \underline{F}(z) - 1 = zF(z)\underline{F}(z).$$

A nonempty Dyck path is (odd, odd)-avoiding if and only if its primary part is nonempty and (strange, strange)-avoiding and its secondary part is (odd, odd)-avoiding. Therefore,

$$(2) \quad F(z) - 1 = z(\overline{F}(z) - 1)F(z).$$

A nonempty Dyck path Λ is (strange, strange)-avoiding if and only if one of the following holds:

- The primary part Λ' is (odd, odd)-avoiding, and the secondary part Λ'' is empty.

- The primary part Λ' is (odd, strange)-avoiding, and the secondary part Λ'' is nonempty and (odd, strange)-avoiding.

Therefore,

$$(3) \quad \bar{F}(z) - 1 = zF(z) + zG(z)(G(z) - 1).$$

A nonempty Dyck path Λ is (odd, strange)-avoiding if and only if one of the following holds:

- The primary part Λ' is (strange, odd)-avoiding, and the secondary part Λ'' is empty.
- The primary part Λ' is nonempty and (strange, strange)-avoiding, and the secondary part Λ'' is nonempty and (odd, strange)-avoiding.

Therefore,

$$(4) \quad G(z) - 1 = zH(z) + z(\bar{F}(z) - 1)(G(z) - 1).$$

There is a simple bijection $\mathcal{G}_n \rightarrow \mathcal{H}_n$ that acts by simply reversing a Dyck path and swapping U's and D's (i.e., reflecting the path through the line $x = n$), so

$$(5) \quad G(z) = H(z).$$

Equations (1) to (5) form a system in the unknowns $F(z)$, $\underline{F}(z)$, $\bar{F}(z)$, $G(z)$, $H(z)$. We can solve this system using a computer algebra program to find that

$$\underline{F}(z) = 1 + z + \frac{-1 - 2z + \sqrt{z^2 - 4z + 2 - 2\sqrt{1 - 4z + 4z^2 - 4z^3}}}{2}.$$

For $n \geq 1$, the poset $\Phi^+(A_n)$ is isomorphic to $\rho_{n \times n} \setminus \delta_{n-1}$. As discussed above, there is a bijection from $\mathbf{E}(J(\rho_{n \times n} \setminus \delta_{n-1}))$ to \underline{F}_{n+1} given by $\lambda \setminus \delta_{n-1} \mapsto \text{U path}^*(\lambda) \text{D}$. Hence,

$$\sum_{n \geq 1} |\mathbf{E}(J(\Phi^+(A_n)))| z^n = \frac{1}{z} (-1 - z + \underline{F}(z)) = \frac{-1 - 2z + \sqrt{z^2 - 4z + 2 - 2\sqrt{1 - 4z + 4z^2 - 4z^3}}}{2z},$$

as desired.

The method used to derive the asymptotics in the statement of the theorem is routine and is discussed in [13, Chapter VII]; we will just sketch the details. The constant ρ is determined by noting that $1/\rho$ is the complex singularity of $\frac{1}{z}(-1 - z + \underline{F}(z))$ closest to the origin (Pringsheim's theorem guarantees that ρ is positive and real). One can use a computer algebra software such as Maple to expand $\frac{1}{z}(-1 - z + \underline{F}(z))$ as a Puiseux series centered at $1/\rho$; the result is $\beta_0 + \beta_1(z - 1/\rho)^{1/2} + o((z - 1/\rho)^{1/2})$ for some explicitly computable algebraic numbers β_0 and β_1 . Following the discussion in [13, Chapter VII], this expansion transfers into an asymptotic formula of the form

$$|\mathbf{E}(J(\Phi^+(A_n)))| \sim \frac{\gamma}{\sqrt{\pi}} n^{-3/2} \rho^{n+1},$$

and one can use a computer algebra software to find that γ is as stated in the theorem. \square

5. TAMARI LATTICES

A permutation $w \in S_n$ is called **312-avoiding** if there do not exist indices $i_1 < i_2 < i_3$ such that $w(i_2) < w(i_3) < w(i_1)$. The set of 312-avoiding permutations in S_n forms a sublattice of the weak order that we denote by Tam_n ; this is one of the many combinatorial realizations of the n -th **Tamari lattice**. Our goal in this section is to prove [Theorem 1.10](#), which enumerates Eeta wins in Tamari lattices both exactly and asymptotically. Our first order of business is to describe Ungar moves in Tamari lattices.

Suppose $w \in S_n$. If there exist indices i and i' such that $i + 1 < i'$ and $w(i + 1) < w(i') < w(i)$, then we can perform an **allowable swap** by swapping the entries $w(i)$ and $w(i + 1)$. Let $\pi_\downarrow(w)$ be the permutation obtained from w by repeatedly performing allowable swaps until no more allowable swaps can be performed. The element $\pi_\downarrow(w)$ is well defined (i.e., does not depend on the sequence

of allowable swaps) and is 312-avoiding [23]. Hence, we obtain a map $\pi_\downarrow: S_n \rightarrow \text{Tam}_n$. Note that $\pi_\downarrow(w) = w$ if and only if $w \in \text{Tam}_n$.

The first author showed [9, Equation (1)] that applying a maximal Ungar move within Tam_n to a 312-avoiding permutation w is equivalent to applying a maximal Ungar move to w within the weak order on S_n and then applying π_\downarrow . The exact same argument (which we omit) shows that applying an arbitrary nontrivial Ungar move to w within Tam_n is equivalent to applying an arbitrary nontrivial Ungar move to w within S_n and then applying π_\downarrow . In what follows, we give an equivalent description of Tamari lattice Ungar moves that will be more suitable for our purposes.

The *plot* of a permutation $w \in S_n$ is the diagram showing the points $(i, w(i))$ for all $i \in [n]$. We often identify permutations with their plots. Suppose $u \in S_m$ and $v \in S_n$. The *direct sum* $u \oplus v$ and the *skew sum* $u \ominus v$ are the permutations in S_{m+n} defined by

$$(u \oplus v)(i) = \begin{cases} u(i) & \text{if } 1 \leq i \leq m; \\ m + v(i - m) & \text{if } m + 1 \leq i \leq m + n \end{cases}$$

and

$$(u \ominus v)(i) = \begin{cases} n + u(i) & \text{if } 1 \leq i \leq m; \\ v(i-m) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

The plot of $u \oplus v$ (respectively, $u \ominus v$) is obtained by placing the plot of v to the northeast (respectively, southeast) of the plot of u . If U and V are sets of permutations, then we let

$$U \oplus V = \{u \oplus v : u \in U, v \in V\} \quad \text{and} \quad U \ominus V = \{u \ominus v : u \in U, v \in V\}.$$

A permutation is called *decomposable* if it can be written as the direct sum of two smaller permutations; otherwise, it is *indecomposable*. Every permutation w can be written uniquely in the form $u_1 \oplus \cdots \oplus u_k$ for some indecomposable permutations u_1, \dots, u_k ; these indecomposable permutations are called the *components* of w . Note that a permutation is 312-avoiding if and only if all of its components are 312-avoiding. Moreover, a 312-avoiding permutation in S_n is indecomposable if and only if its last entry is 1.

Suppose $w = u_1 \oplus \cdots \oplus u_k \in \text{Tam}_n$, where u_1, \dots, u_k are the components of w . Applying an Ungar move to w is equivalent to applying Ungar moves to u_1, \dots, u_k independently and then taking the direct sum of the resulting permutations. In symbols,

$$\mathrm{Ung}(w) = \mathrm{Ung}(u_1) \oplus \cdots \oplus \mathrm{Ung}(u_k).$$

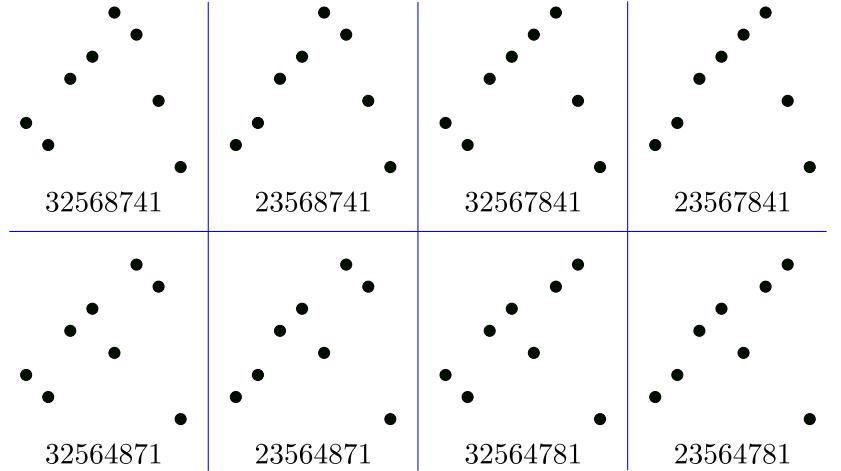
This shows that in order to describe Ungar moves, we can restrict our attention to indecomposable 312-avoiding permutations.

Suppose $w \in \text{Tam}_n$ is indecomposable, and assume $n \geq 2$. We can write $w = w' \ominus 1$ for some $w' \in \text{Tam}_{n-1}$. Let v_1, \dots, v_k be the components of w' so that $w = (v_1 \oplus \dots \oplus v_k) \ominus 1$. Suppose $v_k \in \text{Tam}_m$. To apply an Ungar move to w , we apply an Ungar move to w' and then either keep the entry 1 in the last position or slide the 1 into position $n-m$. In symbols, we have

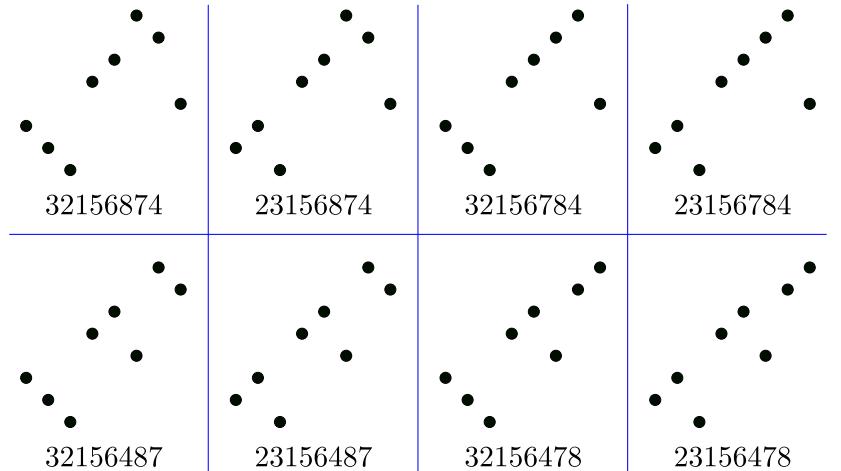
$$\text{Ung}(w) = ((\text{Ung}(v_1) \oplus \cdots \oplus \text{Ung}(v_k)) \ominus \{1\}) \sqcup (((\text{Ung}(v_1) \oplus \cdots \oplus \text{Ung}(v_{k-1})) \ominus \{1\}) \oplus \text{Ung}(v_k)).$$

Example 5.1. Suppose

The 8 indecomposable elements of $\text{Ung}(w)$ are



while the 8 decomposable elements of $\text{Ung}(w)$ are



Recall from [Section 3](#) the definition of the *standardization* of a word. Consider a permutation $w = w(1) \cdots w(n) \in \text{Tam}_n$. Let us say w is *even-districted* if one of the following conditions holds:

- $n = 1$;
- $n \geq 3$, $w = w' \ominus 1$ for some $w' \in \text{Tam}_{n-1}$ with an even number of components, and the standardization of $w(1) \cdots w(n-2)$ is an Eeta win in Tam_{n-2} .

The following two propositions provide a recursive description of Eeta wins in Tamari lattices.

Proposition 5.2. *Let $w = u_1 \oplus \cdots \oplus u_k \in \text{Tam}_n$, where u_1, \dots, u_k are the components of w . Let n_i be the size of u_i . Then $w \in \mathbf{E}(\text{Tam}_n)$ if and only if $u_i \in \mathbf{E}(\text{Tam}_{n_i})$ for all $1 \leq i \leq k$.*

Proof. The interval $[\hat{0}, w]$ in Tam_n is isomorphic to the product $[\hat{0}, u_1] \times \cdots \times [\hat{0}, u_k]$ (abusing notation, we use $\hat{0}$ to denote the bottom elements of different lattices). Therefore, the desired result follows from [Lemma 2.2](#). \square

Proposition 5.3. *An indecomposable permutation is an Eeta win in Tam_n if and only if it is even-districted.*

Our proof of [Proposition 5.3](#) will require the following lemmas. We refer the reader to [Example 5.5](#) for an illustration of the proof of [Lemma 5.4](#).

Lemma 5.4. *Let $x \in \text{Tam}_n$, and suppose there exists a permutation $y \in \text{Ung}(x) \cap \mathbf{E}(\text{Tam}_n)$ with an even number of components. Then there exists $\tilde{y} \in \text{Ung}(x)$ such that $\tilde{y} \ominus 1$ is even-districted.*

Proof. Let $v \ominus 1$ be the final component of y . Then $y = u \oplus (v \ominus 1)$, where u has an odd number of components. Let m be the size of u (so $u \in \text{Tam}_m$). Because $y \in \mathbf{E}(\text{Tam}_n)$, we know by [Proposition 5.2](#) that all of the components of u are Eeta wins in their respective Tamari lattices. The number $m+1$ is the last entry in y . Since $y \in \text{Ung}(x)$, we know that $y \leq x$ in the weak order. This implies that $m+1$ appears to the right of the entries $m+2, \dots, n$ in x . Let $r = x^{-1}(m+1)$. Because x is 312-avoiding, the entries in positions $r - (n-m) + 1, \dots, r-1$ in x are the numbers $m+2, \dots, n$ in some order; that is

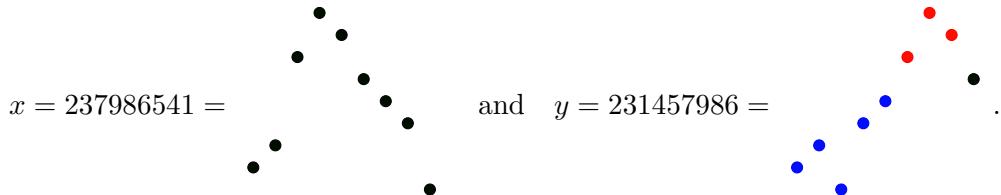
$$\{x(r - (n-m) + i) : 1 \leq i \leq n-m-1\} = \{m+2, \dots, n\}.$$

Let $z \in \text{Tam}_{n-m-1}$ be the standardization of the sequence $x(r - (n-m) + 1) \cdots x(r-1)$. According to [Lemma 2.1](#), there exists $z' \in \text{Ung}(z) \cap \mathbf{E}(\text{Tam}_{n-m-1})$. Note that $z' \ominus 1 \in \text{Ung}(z \ominus 1)$.

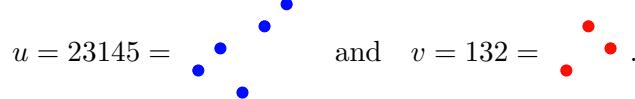
Applying an Ungar move to x amounts to moving the entries $1, \dots, m$ and then moving the entries $m+1, \dots, n$ independently. To make this more precise, let $w \in \text{Tam}_{m+1}$ be the permutation obtained from x by deleting the entries $m+2, \dots, n$, and let Z be the set of permutations of the set $\{m+1, \dots, n\}$ whose standardizations are in $\text{Ung}(z \ominus 1)$. Then $\text{Ung}(x)$ is the set of permutations that can be obtained by selecting a permutation $w' \in \text{Ung}(w)$ and then replacing the entry $m+1$ in w' with a permutation in Z . Since $y = u \oplus (v \ominus 1) \in \text{Ung}(x)$, it must be the case that $u \oplus 1 \in \text{Ung}(w)$. Also, there is a permutation in Z whose standardization is $z' \ominus 1$. It follows that $u \oplus (z' \ominus 1) \in \text{Ung}(x)$. Let $\tilde{y} = u \oplus (z' \ominus 1)$. To complete the proof, we just need to show that $\tilde{y} \ominus 1$ is even-districted.

The components of \tilde{y} are the components of u and the indecomposable permutation $z' \ominus 1$. Since u has an odd number of components, \tilde{y} has an even number of components. If we delete the last two entries from $\tilde{y} \ominus 1$ and then standardize, we obtain $u \oplus z'$. We observed above that all of the components of u are Eeta wins, and we chose z' to be an Eeta win. Therefore, it follows from [Proposition 5.2](#) that $u \oplus z'$ is an Eeta win. This demonstrates that $\tilde{y} \ominus 1$ is even-districted. \square

Example 5.5. Preserve the notation from the proof of [Lemma 5.4](#). Let $n = 9$. Suppose



Then we have



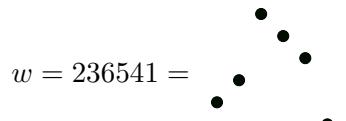
We have $m = 5$ and $r = x^{-1}(6) = 6$. The sequence

$$x(r - (n-m) + 1) \cdots x(r-1) = x(3)x(4)x(5) = 798$$

has standardization $z = 132$. We must choose a permutation

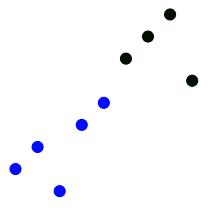
$$z' \in \text{Ung}(z) \cap \mathbf{E}(\text{Tam}_3) = \text{Ung}(132) \cap \mathbf{E}(\text{Tam}_3);$$

in this particular example, our only choice is to set $z' = 123$. The permutation obtained from x by deleting the entries 7, 8, 9 is



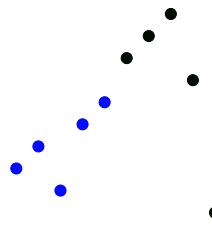
As observed in the proof of [Lemma 5.4](#), we have $u \oplus 1 = 231456 \in \text{Ung}(w)$. Finally, we set

$$\tilde{y} = u \oplus (z' \ominus 1) = 231457896 =$$



and we observe that

$$\tilde{y} \ominus 1 = (u \oplus (z' \ominus 1)) \ominus 1 = 34256891071 =$$



is indeed even-districted.

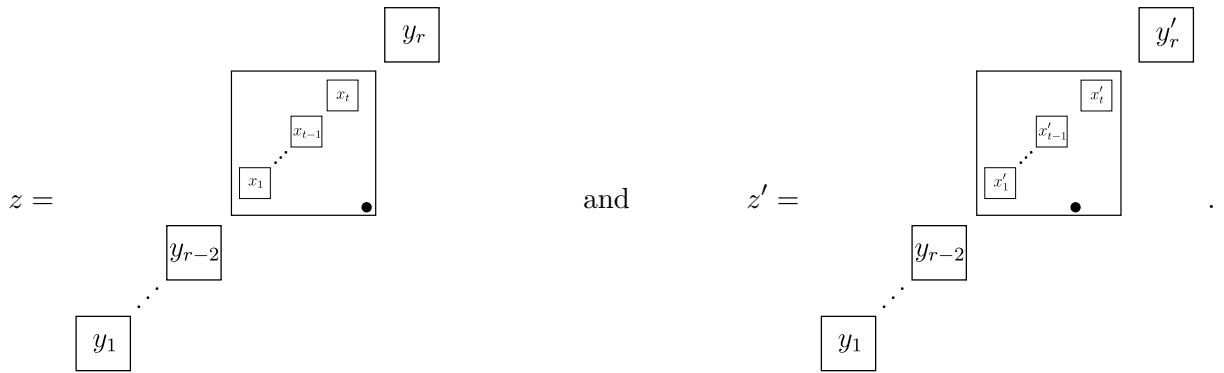
Lemma 5.6. *Suppose $z \in \text{Tam}_n$ is even-districted and z' is a decomposable element of $\text{Ung}(z)$. Then the first component of z' is not even-districted.*

Proof. Let us write $z = (y_1 \oplus \cdots \oplus y_r) \ominus 1$, where y_1, \dots, y_r are indecomposable. Let $y_r = \hat{y} \ominus 1$. The assumption that z is even-districted is equivalent to the assertion that r is even and $y_1 \oplus \cdots \oplus y_{r-1} \oplus \hat{y}$ is an Eeta win. In particular, it follows from [Proposition 5.2](#) that y_1, \dots, y_{r-1} are Eeta wins. According to our description of Tamari lattice Ungar moves, we can write $z' = ((y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1) \oplus y'_r$, where $y'_i \in \text{Ung}(y_i)$ for all $1 \leq i \leq r$. The first component of z' is $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$, so we need to show that this is not even-districted. We consider a few cases.

Case 1. Suppose that $y'_j \neq y_j$ for some $j \in [r-2]$. Since y_j is an Eeta win, y'_j is an Atniss win. This implies (by [Proposition 5.2](#)) that the standardization of the permutation obtained by deleting the last two entries from $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$ is an Atniss win, so $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$ is not even-districted.

Case 2. Suppose that $y'_j = y_j$ for all $j \in [r-2]$ and that y'_{r-1} is indecomposable. Then $y'_1 \oplus \cdots \oplus y'_{r-1}$ has $r-1$ components, so $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$ is not even-districted because $r-1$ is odd.

Case 3. Suppose that $y'_j = y_j$ for all $j \in [r-2]$ and that y'_{r-1} is decomposable. Let us write $y_{r-1} = (x_1 \oplus \cdots \oplus x_t) \ominus 1$ for some indecomposable permutations x_1, \dots, x_t . According to our description of Tamari lattice Ungar moves, we must have $y'_{r-1} = ((x'_1 \oplus \cdots \oplus x'_{t-1}) \ominus 1) \oplus x'_t$, where $x'_i \in \text{Ung}(x_i)$ for all $1 \leq i \leq t$. Then we have



[Lemma 2.1](#) tells us that there exists an Eeta win x''_t in $\text{Ung}(x_t)$. Then $((x'_1 \oplus \cdots \oplus x'_{t-1}) \ominus 1) \oplus x''_t$ is in $\text{Ung}(y_{r-1})$ and is not equal to y_{r-1} because y_{r-1} is indecomposable. Because y_{r-1} is an Eeta win,

this implies that $((x'_1 \oplus \cdots \oplus x'_{t-1}) \ominus 1) \oplus x''_t$ is an Atniss win. But x''_t is an Eeta win, so it follows from [Proposition 5.2](#) that $(x'_1 \oplus \cdots \oplus x'_{t-1}) \ominus 1$ is an Atniss win. This shows that some non-final component of y'_{r-1} is an Atniss win, so some non-final component of $y'_1 \oplus \cdots \oplus y'_{r-1}$ is an Atniss win. By [Proposition 5.2](#), the standardization of the permutation obtained by deleting the last two entries from $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$ is an Atniss win, so $(y'_1 \oplus \cdots \oplus y'_{r-1}) \ominus 1$ is not even-districted. \square

We can now prove [Proposition 5.3](#).

Proof of Proposition 5.3. It is easy to check that the desired result holds when $n \leq 2$. Therefore, we may assume $n \geq 3$ and proceed by induction on n . Let $w \in \text{Tam}_n$ be indecomposable. We will prove that $w \in \mathbf{E}(\text{Tam}_n)$ if and only if w is even-districted. We may also apply induction on the lattice Tam_n . In other words, we may assume that the set of indecomposable Eeta wins that are less than w in Tam_n is equal to the set of even-districted permutations that are less than w in Tam_n .

Assume first that w is even-districted. Suppose $x \in \text{Ung}(w) \setminus \{w\}$; we need to show that x is an Atniss win. If x is decomposable, then we can set $z = w$ and $z' = x$ in [Lemma 5.6](#) to find that the first component of x is not even-districted. By induction, this implies that the first component of x is an Atniss win, so it follows from [Proposition 5.2](#) that x is an Atniss win.

Now assume x is indecomposable. Let $x = x' \ominus 1$. Because $x < w$ in Tam_n , we can use induction to see that x is an Atniss win if and only if it is not even-districted; thus, we need to show that x is not even-districted. Let q be the standardization of the permutation obtained by deleting the last two entries from x . It suffices to show either that x' has an odd number of components or that q is an Atniss win. Let us write $w = (u_1 \oplus \cdots \oplus u_r) \ominus 1$, where u_1, \dots, u_r are indecomposable. Then $x' = u'_1 \oplus \cdots \oplus u'_r$, where $u'_i \in \text{Ung}(u_i)$ for all $1 \leq i \leq r$. Let $u_r = y \ominus 1$. Our assumption that w is even-districted tells us that r is even and that $u_1 \oplus \cdots \oplus u_{r-1} \oplus y$ is an Eeta win. It follows from [Proposition 5.2](#) that u_1, \dots, u_{r-1}, y are Eeta wins. We now consider three cases.

Case 1. Suppose $u'_j \neq u_j$ for some $j \in [r-1]$. Because u_j is an Eeta win and $u'_j \in \text{Ung}(u_j)$, we know that u'_j is an Atniss win. It follows from [Proposition 5.2](#) that q is an Atniss win, so x is not even-districted.

Case 2. Suppose that $u'_j = u_j$ for all $j \in [r-1]$ and that u'_r is indecomposable. Then $u'_r = y' \ominus 1$ for some $y' \in \text{Ung}(y) \setminus \{y\}$. Since y is an Eeta win, y' is an Atniss win. Thus, $q = u_1 \oplus \cdots \oplus u_{r-1} \oplus y'$ is an Atniss win by [Proposition 5.2](#). This proves that x is not even-districted.

Case 3. Suppose that $u'_j = u_j$ for all $j \in [r-1]$ and that u'_r is decomposable. We can write $y = v_1 \oplus \cdots \oplus v_t$, where v_1, \dots, v_t are the components of y . Because y is an Eeta win, we know by [Proposition 5.2](#) that v_1, \dots, v_t are Eeta wins. Our induction hypothesis guarantees that v_1, \dots, v_t are even-districted. Since u'_r is decomposable, we have $u'_r = ((v'_1 \oplus \cdots \oplus v'_{t-1}) \ominus 1) \oplus v'_t$, where $v'_i \in \text{Ung}(v_i)$ for all $1 \leq i \leq t$. If v'_t is indecomposable, then u'_r has exactly 2 components, so $x' = u_1 \oplus \cdots \oplus u_{r-1} \oplus u'_r$ has exactly $r+1$ components. In this case, x is not even-districted because $r+1$ is odd. Thus, we may assume that v'_t is decomposable. Applying [Lemma 5.6](#) with $z = v_t$ and $z' = v'_t$, we find that the first component of v'_t is not even-districted. By induction, the first component of v'_t is an Atniss win. The first component of v'_t is a non-final component of u'_r , so it is also a non-final component of x' . This implies that the first component of v'_t is also a component of q , so q is an Atniss win by [Proposition 5.2](#).

We have proven that if w is even-districted, then it is an Eeta win. To prove the converse, let us now assume w is not even-districted; our goal is to show that w is an Atniss win. Hence, we need to show that there exists an Eeta win in $\text{Ung}(w) \setminus \{w\}$. Let us write $w = w' \ominus 1$, and let v be the final component of w' . Let $v = v' \ominus 1$. By [Lemma 2.1](#), there exist Eeta wins $z \in \text{Ung}(v)$ and $z' \in \text{Ung}(v')$. If w' is indecomposable, then $v = w'$, so $1 \oplus z$ is an Eeta win in $\text{Ung}(w) \setminus \{w\}$.

Thus, we may assume w' is decomposable and write $w' = u \oplus v$ for some (possibly decomposable) permutation u . We consider three cases.

Case 1. Suppose u is an Atniss win. Then there exists an Eeta win $\hat{u} \in \text{Ung}(u) \setminus \{u\}$. Note that $(\hat{u} \oplus (z' \ominus 1)) \ominus 1 \in \text{Ung}(w) \setminus \{w\}$. If \hat{u} has an odd number of components, then we can use [Proposition 5.2](#) to see that $(\hat{u} \oplus (z' \ominus 1)) \ominus 1$ is even-districted (because \hat{u} and z' are Eeta wins), so it follows by induction that $(\hat{u} \oplus (z' \ominus 1)) \ominus 1$ is an Eeta win. Now suppose \hat{u} has an even number of components. According to [Lemma 5.4](#), there exists $\hat{u}' \in \text{Ung}(u)$ such that $\hat{u}' \ominus 1$ is even-districted. By induction, $\hat{u}' \ominus 1$ is an Eeta win. Consequently, $(\hat{u}' \ominus 1) \oplus z$ is an Eeta win in $\text{Ung}(w) \setminus \{w\}$.

Case 2. Suppose u is an Eeta win with an even number of components. Since $u \in \text{Ung}(u)$, we can appeal to [Lemma 5.4](#) to find that there exists $u' \in \text{Ung}(u)$ such that $u' \ominus 1$ is even-districted. By induction, $u' \ominus 1$ is an Eeta win. Consequently, $(u' \ominus 1) \oplus z$ is an Eeta win in $\text{Ung}(w) \setminus \{w\}$.

Case 3. Suppose u is an Eeta win with an odd number of components. Then $u \oplus z'$ is an Eeta win by [Proposition 5.2](#), so $(u \oplus (z' \ominus 1)) \ominus 1$ is even-districted. Also, $(u \oplus (z' \ominus 1)) \ominus 1$ is in $\text{Ung}(w) \setminus \{w\}$ (notice that $(u \oplus (z' \ominus 1)) \ominus 1 \neq w$ by our assumption that w is not even-districted). This implies that $(u \oplus (z' \ominus 1)) \ominus 1 < w$ in Tam_n , so by induction, $(u \oplus (z' \ominus 1)) \ominus 1$ is an Eeta win. \square

Having recursively characterized Eeta wins in Tamari lattices via [Propositions 5.2](#) and [5.3](#), we can now enumerate them.

Proof of Theorem 1.10. Let $G(z) = \sum_{n \geq 1} g_n z^n$, where g_n is the number of even-districted elements of Tam_n . For $n \geq 3$, it follows from [Propositions 5.2](#) and [5.3](#) that every even-districted element of Tam_n can be written uniquely in the form $(u_1 \oplus \cdots \oplus u_k \oplus ((u_{k+1} \oplus \cdots \oplus u_r) \ominus 1)) \ominus 1$, where k is odd, $r \geq k$, and u_1, \dots, u_r are even-districted. Thus,

$$g_n = \sum_{\substack{r \geq k \geq 1 \\ k \text{ odd}}} \sum_{\substack{n_1, \dots, n_r \geq 1 \\ n_1 + \cdots + n_r = n-2}} g_{n_1} \cdots g_{n_r} = \sum_{r \geq 1} \lceil r/2 \rceil \sum_{\substack{n_1, \dots, n_r \geq 1 \\ n_1 + \cdots + n_r = n-2}} g_{n_1} \cdots g_{n_r}.$$

Translating this recurrence into generating functions yields

$$\begin{aligned} G(z) &= z + z^2 \sum_{r \geq 1} \lceil r/2 \rceil G(z)^r \\ &= z + z^2 \sum_{m \geq 1} m(G(z)^{2m-1} + G(z)^{2m}) \\ &= z + z^2(G(z) + G(z)^2) \sum_{m \geq 1} m(G(z)^2)^{m-1} \\ (6) \quad &= z + z^2 \frac{G(z) + G(z)^2}{(1 - G(z)^2)^2}. \end{aligned}$$

Let $F(z) = \sum_{n \geq 1} |\mathbf{E}(\text{Tam}_n)| z^n$. According to [Proposition 5.3](#), $G(z)$ is the generating function for indecomposable Tamari lattice Eeta wins. As a consequence, $F(z) = \frac{G(z)}{1 - G(z)}$. Equivalently, $G(z) = \frac{F(z)}{1 + F(z)}$. After substituting this into (6) and performing basic algebraic manipulations, we find that $Q(F(z), z) = 0$, where

$$Q(y, z) = z + (-1 + 3z + z^2)y + (-2 + 2z + 3z^2)y^2 + 3z^2y^3 + z^2y^4.$$

The method used to derive the asymptotics in the statement of the theorem is routine and is discussed in [\[13, Chapter VII\]](#); we will just sketch the details. Let $\rho = \lim_{n \rightarrow \infty} |\mathbf{E}(\text{Tam}_n)|^{1/n}$. The discriminant of $Q(y, z)$ with respect to y is $z^2 \hat{Q}(z)$, where

$$\hat{Q}(z) = 32 - 32z - 155z^2 - 20z^3 - 148z^4 + 60z^5 - 8z^6 - 4z^7.$$

Pringsheim's theorem states that $1/\rho$ must be a positive real root of this discriminant, so ρ is a positive real root of $z^7\widehat{Q}(1/z)$. One can check that $z^7\widehat{Q}(1/z)$ has a unique positive real root. One can then use a computer algebra software such as Maple to expand $F(z)$ as a Puiseux series centered at $1/\rho$; the result is $\beta_0 + \beta_1(z - 1/\rho)^{1/2} + o((z - \rho)^{1/2})$ for some explicitly computable algebraic numbers β_0 and β_1 . Following the discussion in [13, Chapter VII], this expansion transfers into an asymptotic formula of the form

$$|\mathbf{E}(\text{Tam}_n)| \sim \frac{\gamma}{\sqrt{\pi}} n^{-3/2} \rho^n,$$

and one can use a computer algebra software to find that the minimal polynomial of γ is as stated in the theorem. \square

6. OPEN PROBLEMS

6.1. The Weak Order. Although [Theorem 1.4](#) provides an asymptotic upper bound for the number of Eeta wins in the weak order on S_n , we are still far from fully understanding these Ungar games. It would be interesting to improve the upper bound in [Theorem 1.4](#) or find a nontrivial lower bound. For instance, does the number of Eeta wins grow more like $c^n n!$ or more like $(n!)^c$ (each for some $c < 1$)?

Consider the set B of permutations from the statement of [Lemma 3.1](#). We deduced [Theorem 1.4](#) from that lemma and a known asymptotic estimate for the number of permutations in S_n that consecutively avoid 1324. It could be interesting to more accurately enumerate (either exactly or asymptotically) the permutations that consecutively avoid *all* of the patterns in B ; this would immediately yield an improvement upon [Theorem 1.4](#).

An earlier version of this work included the conjecture that if a permutation w is an Eeta win in the weak order on S_n , then w has at most $\frac{n-1}{2}$ descents. This conjecture was disproved by Evan Bailey, who used a computer to find all counterexamples for $n \leq 14$. The first counterexamples appear for $n = 10$; for example, the permutation with one-line notation 3, 10, 9, 8, 4, 7, 2, 5, 1, 6 is an Eeta win with 5 descents.

6.2. Other Lattices. [Theorem 1.5](#) considers a large class of intervals in Young's lattice and characterizes which of them are Eeta wins. It would be interesting to extend this characterization to *all* intervals in Young's lattice.

Of course, it would also be interesting to study Ungar games on other lattices beyond those considered here. For example, since Young's lattice is $J(\mathbb{N}^2)$, it is natural to ask what can be said about Ungar games on principal order ideals of $J(\mathbb{N}^3)$. Another well-studied lattice that is similar in many ways to Young's lattice is the Young–Fibonacci lattice, which was introduced by Fomin [12] and Stanley [27]; note, however, that this lattice is not distributive. The number of elements of rank n in the Young–Fibonacci lattice is the Fibonacci number f_n , where we use the conventions $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Conjecture 6.1. *For $n \geq 2$, the number of Eeta wins of rank n in the Young–Fibonacci lattice is $f_{n-2} + (-1)^n$.*

The n -th *shifted staircase* is the subposet SS_n of \mathbb{N}^2 consisting of all pairs (i, j) such that $1 \leq i \leq j \leq n$. There is a natural bijection between order ideals of SS_n and binary strings of length n ; we illustrate this bijection for $n = 5$ in [Figure 5](#). A **0-block** (respectively, **1-block**) in a binary string is a maximal consecutive substring of 0's (respectively, 1's). Note that $J(\text{SS}_n)$ is generally not isomorphic to an interval in Young's lattice, so we cannot apply [Theorem 1.5](#) to understand its Atniss wins and Eeta wins. Nevertheless, the following characterization seems to hold.

Conjecture 6.2. *An order ideal of SS_n is an Eeta win in $J(\text{SS}_n)$ if and only if its corresponding length- n binary string ends with 0 and does not contain an odd-length 0-block immediately followed by an odd-length 1-block.*

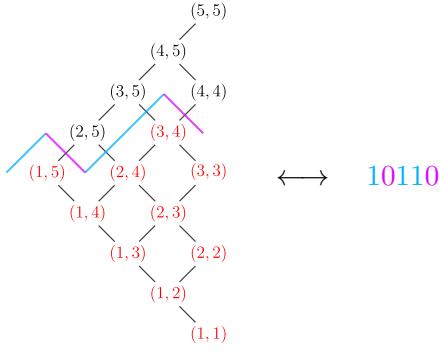


FIGURE 5. An order ideal of the shifted staircase SS_5 is shown in red. This order ideal is uniquely determined by a path of up and down steps lying just above it, and that path corresponds to the length-5 binary string 10110.

6.3. Complexity. It is natural to consider Ungar games from the point of view of complexity theory. A *boolean formula* is an expression on n boolean inputs using the usual binary operations OR and AND and the unary operation \neg . We are interested in the class of boolean formulas whose truth value can be computed with a circuit of depth $O(\log(n))$ [4]. A decision problem is called **NC¹-hard** if any such formula is linearly reducible to it.

In [18], Kalinich showed that poset games are NC¹-hard. We can adapt this argument to Ungar games.

Theorem 6.3. *Ungar games are NC¹-hard.*

Proof. As in [18], we show that we can construct Ungar games that encode the boolean formula value problem with only linear blowup—that is, we produce a lattice that is an Eta win if and only if the formula evaluates to 1 using the given inputs.

We represent posets as Hasse diagrams so that the data for a poset is polynomial in the number of its elements. Noting that the lattice with 1 element is an Eta win and the lattice with 2 elements is an Atniss win, we see that it suffices to construct the OR of two games and the \neg of a game. This is carried out in Figure 6, at the expense of 7 extra elements per OR and 1 extra element per \neg .

By our assumption that the depth of the given formula is $O(\log(n))$, the resulting poset has $n^{O(1)}$ elements. It is straightforward to see by induction that this poset is indeed a lattice. \square

By analogy with poset games, it is reasonable to consider Ungar games on distributive lattices $J(P)$, so that the game can be played on P itself. Building on work of Schaeffer [25], Grier proved that poset games are PSPACE-complete [16]. It is not so easy to adapt Grier’s argument from poset games to Ungar games on distributive lattices.

Question 6.4. *Are Ungar games on distributive lattices $J(P)$ PSPACE-complete in $|P|$?*

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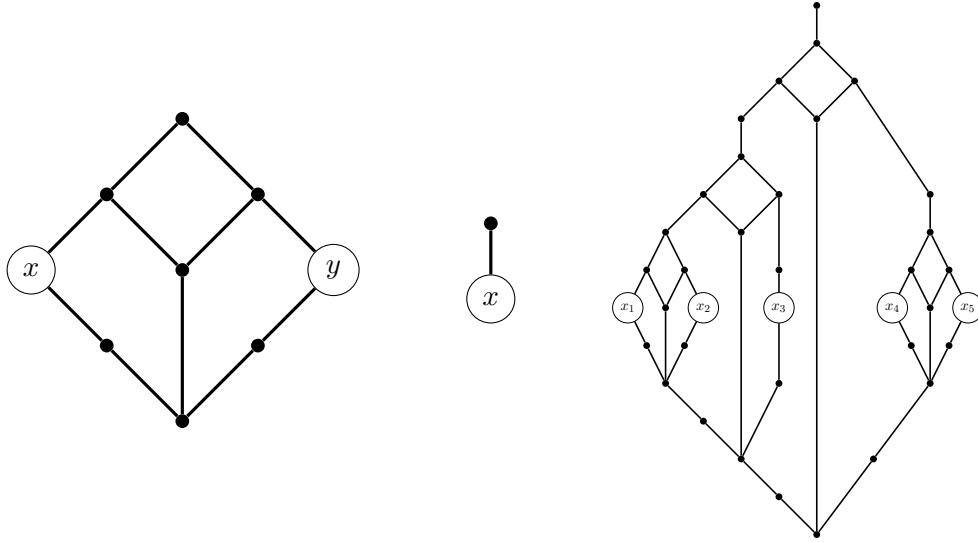


FIGURE 6. Left: a lattice encoding the boolean formula $x \text{ OR } y$; middle: a lattice encoding $\neg x$; right: a lattice encoding $(x_1 \text{ OR } x_2 \text{ OR } (\neg x_3)) \text{ AND } (x_4 \text{ OR } x_5)$. Each variable should be replaced by the 1-element lattice (corresponding to setting the variable to 1) or the 2-element lattice (corresponding to setting the variable to 0).

REFERENCES

- [1] M. Aigner and G. M. Ziegler, Proofs from the book, Fifth Edition. Springer-Verlag, 2014.
- [2] A. Asinowski, C. Banderier, and B. Hackl, Flip-sort and combinatorial aspects of pop-stack sorting. *Discrete Math. Theor. Comput. Sci.*, **22** (2021).
- [3] A. Asinowski, C. Banderier, S. Billey, B. Hackl, and S. Linusson, Pop-stack sorting and its image: permutations with overlapping runs. *Acta. Math. Univ. Comenian.*, **88** (2019), 395–402.
- [4] D. Barrington, N. Immerman, and H. Straubing, On uniformity within NC^1 . *J. Comput. System Sci.*, **41** (1990), 274–306.
- [5] Y. Choi and N. Sun, The image of the Pop operator on various lattices. *Adv. Appl. Math.*, **154** (2024).
- [6] A. Claesson and B. Á. Guðmundsson, Enumerating permutations sortable by k passes through a pop-stack. *Adv. Appl. Math.*, **108** (2019), 79–96.
- [7] A. Claesson, B. Á. Guðmundsson, and J. Pantone, Counting pop-stacked permutations in polynomial time. *Experiment. Math.*, (2021).
- [8] C. Defant, Pop-stack-sorting for Coxeter groups. *Comb. Theory*, **2** (2022).
- [9] C. Defant, Meeting covered elements in ν -Tamari lattices. *Adv. Appl. Math.*, **134** (2022).
- [10] C. Defant and R. Li, Hungarian Markov chains. *Electron. J. Probab.*, **28** (2023), 1–39.
- [11] C. Defant and N. Williams, Semidistrim lattices. *Forum Math. Sigma*, **11** (2023).
- [12] S. V. Fomin, Generalized Robinson–Schensted–Knuth correspondence. *J. Soviet Math.*, **41** (1988), 979–991.
- [13] P. Flajolet and R. Sedgewick, Analytic combinatorics. Cambridge University Press, 2009.
- [14] D. Gale, A curious Nim-type game. *Amer. Math. Monthly*, **81** (1974), 876–879.
- [15] J. E. Goodman and R. Pollack, A combinatorial perspective on some problems in geometry. *Congr. Numer.*, **32** (1981), 383–394.
- [16] D. Grier, Deciding the winner of an arbitrary finite poset game is PSPACE-complete. *Automata, Languages, and Programming: 40th International Colloquium, ICALP Proceedings* (2013), 497–503.
- [17] L. Hong, The pop-stack-sorting operator on Tamari lattices. *Adv. Appl. Math.*, **139** (2022).
- [18] A. O. Kalinich, Flipping the winner of a poset game. *Inform. Process. Lett.*, **112** (2012), 86–89.
- [19] L. Lichev, Lower bound on the running time of pop-stack sorting on a random permutation. arXiv:2212.09316(v1).
- [20] F. Müller-Hoissen, J. M. Pallo, and J. Stasheff, Associahedra, Tamari lattices and related structures: Tamari memorial festschrift, vol. 299 of Progress in Mathematics. Birkhäuser, 2012.
- [21] OEIS Foundation Inc. (2023), Entry A113228 in The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A113228>.
- [22] L. Pudwell and R. Smith, Two-stack-sorting with pop stacks. *Australas. J. Combin.*, **74** (2019), 179–195.

- [23] N. Reading, Cambrian lattices. *Adv. Math.*, **205** (2006), 313–353.
- [24] A. Sapounakis, I. Tasoulas, and P. Tsikouras, On the dominance partial ordering on Dyck paths. *J. Integer Seq.*, **9** (2006).
- [25] T. J. Schaefer, On the complexity of some two-person perfect-information games. *J. Comput. System Sci.*, **16** (1978), 185–225.
- [26] P. R. Scott, On the sets of directions determined by n points. *Amer. Math. Monthly* **77** (1970), 502–505.
- [27] R. P. Stanley, Differential posets. *J. Amer. Math. Soc.*, **1** (1988), 919–961.
- [28] R. P. Stanley, Enumerative combinatorics, vol. 1, Second Edition. Cambridge University Press, 2012.
- [29] D. Tamari, The algebra of bracketings and their enumeration. *Nieuw Archief voor Wiskunde*, **10** (1962), 131–146.
- [30] P. Ungar, $2N$ noncollinear points determine at least $2N$ directions. *J. Combin. Theory Ser. A*, **33** (1982), 343–347.
- [31] D. Zeilberger, Three-rowed CHOMP. *Adv. Appl. Math.*, **26** (2001), 168–179.

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