THE INFORMATION PREMIUM ON A FINITE PROBABILITY SPACE

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Abstract. On a finite probability space, we consider a problem of *fair pricing* of contingent claims in the sense of [8] and its sensitivity to a distortion of information, where we follow the *weak information* modeling approach from [4]. We show that, in complete models, or more generally, for replicable contingent claims, the weak information does not affect the fair price. For incomplete models, this is not the case for non-replicable claims, where we obtain explicit formulas for the *information premium* and *correction to an optimal trading strategy*. We illustrate our results by an example, where we demonstrate that under weak information, the fair price can increase, stay the same, or decrease. Finally, we perform the stability analysis for the information premium and the correction of the optimal trading strategy to perturbations of the contingent claim payoff, stock price dynamics, and the reference probability measure.

1. Introduction

While in complete markets, every contingent claim admits a unique arbitrage-free price, in incomplete markets, this is not the case. To assign a unique number (price) in such settings, alternative approaches have been used, including the ones that are based on the preferences of a given economic agent. This leads to the notions of utility-based pricing (see [10]) and closely related fair-pricing (see [8]). Note that the question of consistency between the utility-based and arbitrage-free pricing methodologies typically has an affirmative answer (see [16]).

Information is another crucial ingredient that affects the pricing and hedging of non-replicable contingent claims. There are different approaches

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to modeling information (and information asymmetry) in financial markets. They include immersions and enlargements of filtrations and distortions of the underlying probability measure. We also note that this area of research is very active. We refer to [1] and [2] for overviews of recent developments. While immersions and enlargements of filtrations are quite technical mathematically, an approach based on conditioning from [3] and [4] allows us to work on the same probability space; and yet it is closely connected to the theory of the initial enlargement of filtration; see the discussion in [3].

The goal of this paper is to understand and quantify the effect of information on the fair-pricing and hedging methodologies. We adopt a *weak information* modeling approach from [4] and investigate its impact on fair pricing and hedging in the sense of [8]. More specifically, we suppose that an investor is trading in a discrete-time, arbitrage-free market with zero *bid-ask spread* (i.e., no transaction cost). If possible, the investor seeks to match the payoff of a derivative security via gains from the trade using some initial capital value up to the terminal time period, at which the payoff is attained. If perfect matching, also known as replication, is not possible, the investor wants to find a trading strategy, such that the associated wealth process minimizes the expectation of the square of the difference between the payoff of the contingent claim and the wealth process associated with the trading strategy. The initial value of such a minimizing wealth process is known as a fair price, see [8].

Our results include explicit formulas for the change of the fair price under weak information, that is, an information premium, and for the change of the trading strategy. By means of examples, we show that the fair price under weak information can increase, stay the same, or decrease. We also perform the stability analysis of the information premium and the correction of the optimal trading strategy with respect to simultaneous small perturbations of the payoff of the contingent claim, the subjective probability, and the dynamics of the risky asset. For this, we establish the stability of the family of the Föllmer-Schweizer decompositions under such perturbations, a result of independent usefulness.

The remainder of this paper is organized as follows. In Section 2, we review the fair pricing methodology in the sense of [8], and in Section 3, we discuss the weak information approach. In Section 4, we show that in complete markets, there is no information premium, whereas in Section 5, we provide explicit formulas for the information premium and corrections to the optimal trading strategy.

2. Fair pricing in the sense of [8]

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a finite probability space, where for some integer $N > 0$, and the filtration, $\mathcal{F} = (\mathcal{F}_n)_{n=0,1,\ldots,N}$, is an increasing family of sub-algebras each containing $\emptyset$ and Ω. We suppose that $\mathcal{F}_0$ is trivial, and $\mathcal{F}_N$ is the power set of Ω. Let the probability measure $\mathbb{P}$ be such that $\mathbb{P}[\omega] > 0$ for all $\omega \in \Omega$. Let $S = (S_n)_{n=0,1,\ldots,N}$, the discounted stock price process (i.e., if the undiscounted stock price at time $n = 0, 1, \ldots, N$, is denoted $\overline{S}_n$, for some fixed one-period market interest rate $r > -1$, $S_n = \frac{\overline{S}_n}{(1+r)^n}$), be a real-valued $\mathcal{F}$-adapted process. Furthermore, we denote

$$\Delta S_n := S_n - S_{n-1}, \qquad \text{for } n = 1, \ldots, N,$$

to be the incremental discounted stock price process. Let $\xi = (\xi_n)_{n=1,2,\ldots,N}$ be a predictable (i.e., ξ_n is $\mathcal{F}_{n-1}$-measurable for all $n = 1, \ldots, N$) trading strategy that describes the number of shares of the stock held in the portfolio. We also suppose that there is a money market account, but since we are working in the discounted terms, the price process for the money market account equals to 1 at all times.

Definition 2.1. Let Θ be the set of all predictable trading strategies ξ. For $\xi \in \Theta$, let $G(\xi)$ be defined by

$$G_n(\xi) := \sum_{j=1}^{n} \xi_j \Delta S_j.$$

Then, for a random variable V_N and some $V_0 \in \mathbb{R}$, one can define the following problem from [8]:

$$\text{minimize } \mathbb{E}\left[(V_N - V_0 - G_N(\xi))^2\right] \text{ over all } \xi \in \Theta \text{ and } V_0 \in \mathbb{R}. \qquad (1)$$

2.1. **Interpretation.** $V_0 + G_n(\xi)$, $n = 0, \ldots, N$, can be viewed as the *gains from the trade* process, starting from the initial capital V_0. We also interpret V_N as the payoff of a contingent claim or a derivative security with maturity N and whose underlying asset does not have to be S, but it can be. The most common derivative security is the *European Call Option*, which gives its owner the right, but not the obligation, to buy one share of stock at the maturity time for the *strike price* K. If the $S_N > K$, the owner can exercise the option and makes a profit of $S_N - K$, while the option would be worthless if $S_N < K$, in which case the owner does not buy a share of the stock. Therefore, the payoff of the European Call Option is said to be $(S_N - K)^+ := \max\{S_N - K, 0\}$. Generally, V_N can represent the payoff of any security, an option on a different stock, in particular.

The solution to (1), consisting of an optimal trading strategy $\hat{\xi}$, is developed and given in [8] using a method known as (backward in time) sequential regression. We need the following assumption in order for $\hat{\xi}$ to be well-defined in (3) below.

Assumption 2.2. We suppose that the process

$$v_n^{\mathbb{P}} := \mathrm{Var}_{\mathcal{F}_{n-1}}^{\mathbb{P}}[\Delta S_n], \quad n \in \{1, \ldots, N\}, \tag{2}$$

is strictly positive with probability 1.

Assumption 2.2, in the settings of this paper, is equivalent to *nondegeneracy condition* from [14, p. 4]. The formula for the optimal trading strategy (recursively, backward in time) is given by:

$$\hat{\xi}_n = \frac{\mathrm{Cov}_{\mathcal{F}_{n-1}}\left[V_N - \sum_{j=n+1}^{N} \hat{\xi}_j \Delta S_j, \Delta S_n\right]}{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n]}, \quad n = N, \ldots, 1, \tag{3}$$

where $\mathrm{Cov}_{\mathcal{F}_{n-1}}[.,.]$ and $\mathrm{Var}_{\mathcal{F}_{n-1}}[.]$ denote the conditional covariance and variance conditioned on $\mathcal{F}_{n-1}$, respectively.

Furthermore, given the optimal trading strategy $\hat{\xi}$, one can find the fair derivative security price $\hat{V}_0$ from (1) given by:

$$\hat{V}_0 = \mathbb{E}[V_N - G_N(\hat{\xi})]. \tag{4}$$

These solutions to (1) gives one the discrete Föllmer-Schweizer decomposition of the payoff of the non-traded derivative security, V_N, defined below:

Definition 2.3. The discrete Föllmer-Schweizer decomposition, following [14], is defined as follows. Let $S = M + A$ be the semimartingale decomposition of S into a martingale M and a predictable process A. Then the random variable V_N admits the discrete Föllmer-Schweizer decomposition:

$$V_N = \hat{V}_0 + \sum_{j=1}^{N} \hat{\xi}_j \Delta S_j + L_N \tag{5}$$

for some $\hat{V}_0 \in \mathbb{R}$, optimal trading strategy $\hat{\xi} \in \Theta$, and a $\mathbb{P}$-martingale L such that:

(1) L and M are orthogonal (i.e., LM is a $\mathbb{P}$-martingale),
(2) $\mathbb{E}[L_0] = 0$,

where the second condition is just $L_0 = 0$ for a trivial $\mathcal{F}_0$.

3. The Weak Information Approach

Background. The weak information approach has been initiated in [3] in the context of complete markets. In [4], it is further developed for incomplete financial models. It corresponds to a certain change of a probability measure, and the name contrasts it with the strong information modeling approach, which is based on enlargements of filtration. We recall some notions from [4, p. 61]. For some random variable[1] Y (e.g., Y can be the value of the stock price at time $T > 0$), let us consider an insider who is only *weakly informed* on Y, which means that he or she has knowledge of the filtration $\mathcal{F}$ and of the law of Y. More precisely, with Y we associate a probability measure ν on $\mathbb{R}$. We assume that ν is equivalent to $\mathbb{P}$ ([4] additionally supposes the boundedness of the density that holds trivially on finite spaces). The probability ν should be interpreted as the law of Y under the effective probability of the market. A typical example of Y is the stock price at maturity.

In [4], the weak information was placed into a utility maximization problem from terminal wealth and *without contingent claims (or random endowment)*, in continuous time. Let $\mathcal{A}(x)$ be the set of all *admissible* trading strategies Θ, that is the set of θ's that are integrable with respect to S and are such that the associated wealth processes $x + \int_0^t \theta_u dS_u$, $t \in [0, T]$, where $x > 0$ is an initial wealth, stays nonnegative. Then, for some increasing, strictly concave utility function $U\colon (0, \infty) \to \mathbb{R}$, [4, p. 74] defines the financial value of the weak information on the optimal portfolio problem as

$$u(x, \nu) = \inf_{\mathbb{Q} \in \mathcal{E}^{\nu}} \sup_{\theta \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{Q}} \left[U \left(x + \int_0^T \theta_u dS_u \right) \right], \quad x > 0. \tag{6}$$

Further, for a given initial wealth $x > 0$, [4, p. 76] sets

$$u(x, \nu) - U(x) \tag{7}$$

to be the value of the *additional* information.

Our formulation. Combining the weak information on the outputs to (1) as it is done below in (8) has the spirit of [4]; however, it differs from (6) in [4] by

- switching from maximization to minimization problem (this change is fairly straightforward),
- more importantly, it allows us to incorporate the contingent claim V_N in a tractable way, which allows us to give affirmative answers to natural questions of what is the information premium and what the corrections to the optimal trading strategy are under extra information.

In contrast to [3] and [4], where continuous-time models are considered, we work in discrete-time settings.

[1] To be more precise, in [4, p. 46], $Y : \Omega \to \mathcal{P}$, an $\mathcal{F}_T$-measurable random element, where $\mathcal{P}$ is a Polish space. We suppose that $\mathcal{P} = \mathbb{R}$ for simplicity of the presentation.

Remark 3.1. The utility maximization problem with a contingent claim or a random endowment in the general settings of incomplete markets is substantially more difficult than the one without a contingent claim, and the existing results are obtained later; compare [11] and [12] (without random endowment) with [7], [9], and [13] (with endowment), where [7] is the earliest known result.

We suppose that an insider has weak information about some random variable Y (e.g., the value of the stock price at time N), given via its distribution ν as well as about the filtration $\mathcal{F}$. We let $\mathcal{E}^\nu$ set of probability measures $\mathbb{Q}$ on Ω that are equivalent to $\mathbb{P}$ and are such that the law of Y under $\mathbb{Q}$ is ν (exactly as in the construction of (6)). Then we consider the following problem:

$$\sup_{\mathbb{Q}\in\mathcal{E}^\nu} \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}}\left[(V_N - V_0 - G_N(\xi))^2\right]. \tag{8}$$

We call the *information premium* the change in the fair price $\hat{V}_0$ that comes from formulations (1) and (8). This concept has some connection to the value of the additional information in (7); however, it captures the changes in the fair price directly.

Discussion. This problem may be interpreted as the greatest decrease to the least squares difference between some non-traded derivative security payoff, dependent on S, and some initial capital plus the gains from the trade under some trading strategy up to time N, under the weak information on S_N. Comparing the solutions of (1) and (8), one may quantify the financial value of the weak information through the Föllmer-Schweizer decomposition of the derivative security payoff as well as the information premium on the fair price of the security.

The novel problem (8) holds similarities to (6). One can interpret (8) as the discrete-time, quadratic analog to (6), but with some additional contingent claim with a payoff V_N. Therefore, (8) is a novel yet natural problem to investigate.

Mathematically, using sequential regression, one can obtain the solutions to (1), and identify the Föllmer-Schweizer decomposition of V_N under some measure $\mathbb{P}$. Note that, the fair price and the optimal trading strategy, and, as a result, the Föllmer-Schweizer decomposition of the derivative security payoff depend on which probability measure that defines the weak information is used. We now investigate whether one can identify the Föllmer-Schweizer decomposition of V_N or its analog based on (8), where the objective in (8) can be interpreted as considering the worst case scenario, given the uncertainty about the choice of $\mathbb{Q} \in \mathcal{E}^\nu$.

We interpret having weak information on S_N as an insider having extra information on the outcomes of the asset's value during the payoff period. As such, using this weak anticipation the insider may gain some knowledge on the fair price of a non-traded derivative security whose value is derived from S, as well as the optimal trading strategy required to reduce the least-squares difference between the payoff and the initial capital plus the gains from the trade up to the terminal time period, under this optimal trading strategy.

The main drawback of the considered model is that, for non-replicable contingent claims, the mathematical problem (8) governing the information premium is not well-posed in the sense of Hadamard, in general. This drawback can potentially be addressed by adjusting the modeling framework and working under some particular probability measure in $\mathcal{E}^\nu$ that can be chosen by additional considerations.

4. The Information Premium in Complete Markets is Zero

In the settings of Section 2, a market is complete, if every contingent claim V_N can be represented as

$$V_N = V_0 + G_N(\xi), \tag{9}$$

for some $\xi \in \Theta$ and some $V_0 \in \mathbb{R}$. This corresponds to the case when $L_N = 0$ in (5): the payoff of the non-traded security V_N can be exactly replicated via the gains from trade from some $\xi \in \Theta$, starting from the initial capital V_0.

Example 4.1 (The Binomial Model)**.** Suppose that we have an arbitrage-free market in which for each time step $n = 0, 1, \ldots, N-1$, where, for every realization of the undiscounted stock price S_n at time n, the value of S_{n+1} may take on one of two possible values, either uS_n or dS_n for some u, d satisfying $0 < d < 1 + r < u$, for a one-period interest rate $r > -1$. This is known as the *no-arbitrage binomial market model*. We suppose that $\mathbb{P}[S_{n+1} = uS_n | S_n] = p \in (0,1)$, and $\mathbb{P}[S_{n+1} = dS_n | S_n] = q := 1 - p$. One may interpret $S_{n+1} = uS_n$ as the stock increasing in value from time n to $n+1$, and $S_{n+1} = dS_n$ as the stock decreasing in value from time n to $n+1$. Suppose $X = (X_n)_{n=0,1,\ldots,N}$ is a wealth process that replicates the payoff V_N, i.e., a self-financing process that begins with initial capital X_0 such that

$$X_N(\omega) = V_N(\omega), \quad \omega \in \Omega.$$

Then the backward induction approach leads to the *Delta-Hedging formula* given by [15]. It provides the number of stocks to trade at each time period, the self-financing trading strategy $\hat{\xi}$, in order to replicate V_N:

$$\hat{\xi}_n(\omega) = \frac{X_n(\omega H) - X_n(\omega T)}{uS_{n-1} - dS_{n-1}},$$

where $\omega = \omega_1 \dots \omega_n$, where each $\omega_i \in \{H, T\}$. Therefore, if an insider is given some weak information $\nu(S_N)$, the problem of

$$\sup_{\mathbb{Q} \in \mathcal{E}^\nu} \inf_{V_0 \in \mathbb{R}, \xi \in \Theta} \mathbb{E}^{\mathbb{Q}} \left[(V_N - V_0 - G_N(\xi))^2 \right],$$

in an arbitrage-free binomial asset pricing model is trivial. For any $\mathbb{Q} \in \mathcal{E}^\nu$, (9) implies that the trading strategy ξ and an initial wealth V_0 result in

$$V_N - V_0 - G_N(\xi) = 0. \tag{10}$$

Thus, ξ and V_0 are minimizers to both (8) and (1). These minimizers depend on neither $\mathbb{P}$, nor $\mathcal{E}^\nu$. Equality (10) is consistent with (4) with $\hat{\xi} = \xi$ and $\hat{V}_0 = V_0$, and (10) *also reinforces the consistency between arbitrage-free prices and fair prices in the current settings*. Note that this is also addressed in [16] for utility-based prices, and that in complete binomial settings, the consistency of the hedging by sequential regression and the delta-hedging formula in demonstrated in [5].

Comparing (3), (4) with (9), and by picking

$$\hat{\xi} = \xi \quad \text{and} \quad \hat{V}_0 = V_0$$

we deduce that the weak information premium is zero (i.e., (10) holds) not only in binomial settings but in general complete models. This, in turn, implies the consistency of arbitrage-free pricing and fair pricing in complete models. Therefore, in complete markets, the weak information does not affect the fair price of a contingent claim, as the price is formed by a different mechanism: replication.

5. The Information Premium in Incomplete Markets

Let us now consider more realistic incomplete models, that is, the ones where not all contingent claims are replicable in the sense of (9). As mentioned above, in complete markets, one can price assets through the replication and subsequently reduce the Föllmer-Schweizer decomposition of a security payoff V_N to

$$V_N = \hat{V}_0 + G_N(\hat{\xi}),$$

which is consistent with (9). Instead, for incomplete markets, for non-replicable (in the sense (9)) contingent claims, we can only ask what is the optimal trading strategy and fair price of the security under which the risk of loss is minimized, as with the least squares minimization problem (1).

Therefore, in incomplete markets for non-replicable claims, if an investor has some weak information, the objective

$$\sup_{\mathbb{Q}\in\mathcal{E}^{\nu}} \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}}\left[(V_N - V_0 - G_N(\xi))^2\right] > 0,$$

becomes much more pertinent. For replicable claims, that is for the ones that admit representation (9), the situation is exactly the same as in complete markets though, and the information premium is 0.

5.1. **The information premium for non-replicable claims.** For every probability measure $\mathbb{Q}$, which is absolutely continuous with respect to $\mathbb{P}$, let us define

$$v_n^{\mathbb{Q}} := \mathrm{Var}_{\mathcal{F}_{n-1}}^{\mathbb{Q}}[\Delta S_n], \quad n \in \{1,\dots,N\}. \tag{11}$$

In this section, we need to strengthen Assumption 2.2 to make it uniform over $\mathbb{Q}\in\mathcal{E}^{\nu}$.

Assumption 5.1. We suppose that there exists $\delta > 0$, such that, for every $\mathbb{Q}\in\mathcal{E}^{\nu}$, with probability 1, we have

$$v_n^{\mathbb{Q}} \geq \delta, \quad n \in \{1,\dots,N\}.$$

As in (8) we deal with Föllmer-Schweizer decompositions under multiple probability measures, it will be convenient to emphasize the dependence of $\hat{V}_0$, $\hat{\xi}$, and L appearing in (5) on the probability measure by writing $\hat{V}_0^{\mathbb{Q}}$, $\hat{\xi}^{\mathbb{Q}}$, and $L^{\mathbb{Q}}$, where $\mathbb{Q}$ is the probability measure under which the expectation in (5) is considered, that is

$$V_N = \hat{V}_0^{\mathbb{Q}} + \sum_{j=1}^{N} \hat{\xi}_j^{\mathbb{Q}} \Delta S_j + L_N^{\mathbb{Q}}. \tag{12}$$

With $\overline{\mathcal{E}}^{\nu}$ denoting the closure of $\mathcal{E}^{\nu}$, we note that from Assumption 5.1 one can show that (12) is well-defined for every $\mathbb{Q}\in\overline{\mathcal{E}}^{\nu}$. To state Theorem 5.2, we need to consider

$$\sup_{\mathbb{Q}\in\overline{\mathcal{E}}^{\nu}} \mathbb{E}[(L_T^{\mathbb{Q}})^2]. \tag{13}$$

The following theorem gives the formulas for information premium and the correction of the optimal trading strategy.

Theorem 5.2. *On a finite probability space, where $\mathbb{P}(\omega) > 0$ for every $\omega\in\Omega$, let us suppose that Assumption 5.1 holds and there exists a unique* *solution to* (13). *Then, the maximizer* $\hat{\mathbb{Q}}$ *to* (13) *satisfies*

$$\sup_{\mathbb{Q}\in\mathcal{E}^\nu}\inf_{V_0\in\mathbb{R},\xi\in\Theta}\mathbb{E}^{\mathbb{Q}}\left[\left(V_N - V_0 - \sum_{j=1}^N \xi_j\Delta S_j\right)^2\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(V_N - \hat{V}_0^{\hat{\mathbb{Q}}} - G_N(\xi^{\hat{\mathbb{Q}}})\right)^2\right]$$
$$= \inf_{V_0\in\mathbb{R},\xi\in\Theta}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(V_N - V_0 - \sum_{j=1}^N \xi_j\Delta S_j\right)^2\right]. \tag{14}$$

The information-based correction to the optimal strategy is well-defined and recursively, backward in time, is given by

$$\begin{aligned}\hat{\xi}_n^{\mathbb{P}} - \hat{\xi}_n^{\hat{\mathbb{Q}}} =& \frac{Cov^{\mathbb{P}}_{\mathcal{F}_{n-1}}\left[V_N - \sum_{j=n+1}^N \hat{\xi}_j^{\mathbb{P}}\Delta S_j, \Delta S_n\right]}{Var^{\mathbb{P}}_{\mathcal{F}_{n-1}}[\Delta S_n]}\\ &- \frac{Cov^{\hat{\mathbb{Q}}}_{\mathcal{F}_{n-1}}\left[V_N - \sum_{j=n+1}^N \hat{\xi}_j^{\hat{\mathbb{Q}}}\Delta S_j, \Delta S_n\right]}{Var^{\hat{\mathbb{Q}}}_{\mathcal{F}_{n-1}}[\Delta S_n]}, \quad n = N,\dots,1.\end{aligned} \tag{15}$$

The information premium (that is the correction to the fair price due to the weak information) is given by

$$V_0^{\mathbb{P}} - V_0^{\hat{\mathbb{Q}}} = \mathbb{E}\left[V_N - \sum_{j=1}^N \hat{\xi}_j^{\mathbb{P}}\Delta S_j\right] - \mathbb{E}^{\hat{\mathbb{Q}}}\left[V_N - \sum_{j=1}^N \hat{\xi}_j^{\hat{\mathbb{Q}}}\Delta S_j\right]. \tag{16}$$

The probability measure $\hat{\mathbb{Q}}$ and the outputs of the Föllmer-Schweizer decomposition can be approximated by elements of $\mathcal{E}^\nu$. This is the subject of the following proposition, and the approximation argument will be further used in the stability analysis.

Proposition 5.3. *Let us assume that the conditions of Theorem 5.2 hold, and consider* $\hat{\mathbb{Q}}$ *specified in Theorem 5.2. Then, there exists a sequence* $\mathbb{Q}^m \in \mathcal{E}^\nu$, $m \in \mathbb{N}$, *such that*

$$\begin{aligned}\lim_{k\to\infty}\hat{\xi}_j^{\mathbb{Q}^m}(\omega) &= \hat{\xi}_j^{\hat{\mathbb{Q}}}(\omega), \quad j = 1,\dots,N, \quad \omega\in\Omega,\\ \lim_{k\to\infty} G_j(\hat{\xi}^{\mathbb{Q}^m})(\omega) &= G_j(\hat{\xi}^{\hat{\mathbb{Q}}})(\omega), \quad j = 0,\dots,N, \quad \omega\in\Omega,\\ \lim_{k\to\infty}\hat{V}_0^{\mathbb{Q}^m} &= \hat{V}_0^{\hat{\mathbb{Q}}},\\ \lim_{k\to\infty} L_j^{\mathbb{Q}^m}(\omega) &= L_j^{\hat{\mathbb{Q}}}(\omega), \quad j = 0,\dots,N, \quad \omega\in\Omega.\end{aligned} \tag{17}$$

Proof. Let us consider the objective

$$F := \sup_{\mathbb{Q}\in\mathcal{E}^\nu} \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}}\left[\left(V_N - V_0 - \sum_{j=1}^{N}\xi_j\Delta S_j\right)^2\right] > 0,$$

and let us define

$$f(\mathbb{Q}) := \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}}\left[\left(V_N - V_0 - \sum_{j=1}^{N}\xi_j\Delta S_j\right)^2\right], \quad \mathbb{Q}\in\mathcal{E}^\nu.$$

Let $\mathbb{Q}^k$, $k\in\mathbb{N}$, be a sequence of probability measures in $\mathcal{E}^\nu$, such that

$$f(\mathbb{Q}^k) \geq F - 1/k. \tag{18}$$

As we work on a finite probability space, each $\mathbb{Q}$ can be represented by a finite-dimensional vector (where the dimension corresponds to the number of states, and thus such a dimension does not depend on the choice of $\mathbb{Q}$). The components of such a vector take values in $(0,1)$, and in particular are bounded from above and below. Consequently, there exists a convergent subsequence $\mathbb{Q}^{n_k}$, $k\geq 1$, such that

$$[0,1] \ni \lim_{k\to\infty} \mathbb{Q}^{n_k}(\omega) =: \hat{\mathbb{Q}}(\omega), \quad \omega\in\Omega. \tag{19}$$

Boundedness of $\mathbb{Q}^{n_k}$, $k\geq 1$, implies that $\hat{\mathbb{Q}}$ is also a probability measure[2]. We denote this subsequence by $\mathbb{Q}^m$, $m\geq 1$.

Let us recall that for a fixed $\mathbb{Q}$, such that the process $v^{\mathbb{Q}}$, defined in (2), is strictly positive with probability 1, the optimal $\hat{\xi}$ in (5) is given by

$$\hat{\xi}_n^{\mathbb{Q}} = \frac{\mathrm{Cov}_{\mathcal{F}_{n-1}}^{\mathbb{Q}}\left[V_N - \sum_{j=n+1}^{N}\hat{\xi}_j^{\mathbb{Q}}\Delta S_j, \Delta S_n\right]}{v_n^{\mathbb{Q}}}, \quad n=1,\ldots,N, \tag{20}$$

see (3). Assumption 5.1 ensures that $v^{\hat{\mathbb{Q}}}$ is strictly positive. Therefore, from (19) and (20), we get

$$\lim_{m\to\infty} \hat{\xi}_N^{\mathbb{Q}^m}(\omega) = \hat{\xi}_N^{\hat{\mathbb{Q}}}(\omega), \quad \omega\in\Omega.$$

Consequently, we obtain

$$\lim_{m\to\infty} \hat{\xi}_N^{\mathbb{Q}^m}(\omega)\Delta S_N(\omega) = \hat{\xi}_N^{\hat{\mathbb{Q}}}(\omega)\Delta S_N(\omega), \quad \omega\in\Omega.$$

In turn, this allows to deduce that

$$\lim_{m\to\infty} \hat{\xi}_{N-1}^{\mathbb{Q}^m}(\omega) = \hat{\xi}_{N-1}^{\hat{\mathbb{Q}}}(\omega), \quad \omega\in\Omega.$$

[2]We note that this implication can be though as an application of Prokhorov's theorem.

Continuing the backward recursion procedure, we deduce the first two equalities in (17).

From the first two equalities in (17), (4), and (19) we deduce that

$$\lim_{m\to\infty} \hat{V}_0^{\mathbb{Q}^m} = \hat{V}_0^{\hat{\mathbb{Q}}},$$

that is the third equality in (17) holds. Now, from (12), we deduce that

$$\lim_{m\to\infty} L_n^{\mathbb{Q}^m}(\omega) = L_n^{\hat{\mathbb{Q}}}(\omega), \quad n = 0, \dots, N, \quad \omega \in \Omega,$$

which is the last equality in (17). $\square$

Remark 5.4. The argument in the proof of Proposition 5.3 can be applied to every element $\tilde{\mathbb{Q}}$ in the closure of $\mathcal{E}^\nu$, where Assumption 5.1 ensures that $v^{\tilde{\mathbb{Q}}}$ is strictly positive, and this implies representations analogous to (17) for the components of (12) under $\tilde{\mathbb{Q}}$.

Proof of Theorem 5.2. First, we observe that Assumption 5.1 implies that

$$\mathbb{Q}[|\Delta S_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S_n]| = 0] = 0, \quad n \in \{1, \dots, N\}, \quad \mathbb{Q} \in \mathcal{E}^\nu. \tag{21}$$

The equivalence of $\mathbb{P}$ to every probability measure in $\mathcal{E}^\nu$ and (21) implies that $v^{\mathbb{P}}$ is a strictly positive process, with probability 1.

To show (14), let us consider and approximating sequence $\mathbb{Q}^m \in \mathcal{E}^\nu$, $m \in \mathbb{N}$, as in the statement of Proposition 5.3. From (17), we deduce that

$$\lim_{m\to\infty} \left(V_N - \hat{V}_0^{\mathbb{Q}_m} - G_N(\hat{\xi}^{\mathbb{Q}_m})\right)^2(\omega) = \left(V_N - \hat{V}_0^{\hat{\mathbb{Q}}} - G_N(\hat{\xi}^{\hat{\mathbb{Q}}})\right)^2(\omega), \quad \omega \in \Omega,$$

and therefore from (19), we get

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{Q}_m}\left[\left(V_N - \hat{V}_0^{\mathbb{Q}_m} - G_N(\hat{\xi}^{\mathbb{Q}_m})\right)^2\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(V_N - \hat{V}_0^{\hat{\mathbb{Q}}} - G_N(\hat{\xi}^{\hat{\mathbb{Q}}})\right)^2\right]. \tag{22}$$

Consequently, we obtain

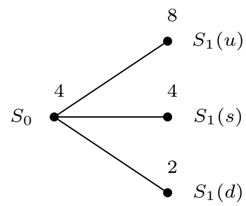
$$\begin{aligned}
&\sup_{\mathbb{Q}\in\mathcal{E}^{\nu}} \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}}\left[\left(V_N - V_0 - \sum_{j=1}^{N}\xi_j\Delta S_j\right)^2\right] \\
=& \lim_{k\to\infty} \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\mathbb{Q}_{m_k}}\left[\left(V_N - V_0 - \sum_{j=1}^{N}\xi_j\Delta S_j\right)^2\right] \\
=& \lim_{k\to\infty} \mathbb{E}^{\mathbb{Q}_{m_k}}\left[\left(V_N - \hat{V}_0^{\mathbb{Q}_{m_k}} - G_N(\hat{\xi}^{\mathbb{Q}_{m_k}})\right)^2\right] \\
=& \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(V_N - \hat{V}_0^{\hat{\mathbb{Q}}} - G_N(\hat{\xi}^{\hat{\mathbb{Q}}})\right)^2\right] \\
=& \inf_{V_0\in\mathbb{R},\xi\in\Theta} \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left(V_N - V_0 - \sum_{j=1}^{N}\xi_j\Delta S_j\right)^2\right],
\end{aligned} \tag{23}$$

where in the first equality, we used (18), in the second and fourth equalities, we have used the optimality of $\hat{V}_0$'s and $\hat{\xi}$'s, and in the third equality, we have used (22). Now, (23) gives (14). In turn (14) and (4) imply the representation (16), and finally (14) and (3) give (15).

□

5.2. **Example: the information premium in a one-period trinomial model.** A classical example of a discrete-time arbitrage-free incomplete market model is based on the trinomial asset pricing model. Here, we model a discounted risky asset S such that at each time period $n = 0, 1, \ldots, N-1$, S_{n+1} may take on three different values. That is, $S_{n+1} = uS_n$, or the asset value increase by factor u from time n, $S_{n+1} = sS_n = S_n$, or the asset value remains the same as in time n, or $S_{n+1} = dS_n$, or the asset value decreases by factor d from time n.

For example, consider a 1-step trinomial asset pricing model with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = 0$, and a *European Call Option*, V_1, expiring at time $N = 1$ with strike price $K = 7$. Note that in this market, the payoff $V_1(\omega)$ is 1 if and only if $\omega = u$. In the following subsection, we will illustrate that extra information can increase, decrease or keep the fair time-zero price $\hat{V}_0$ the same.

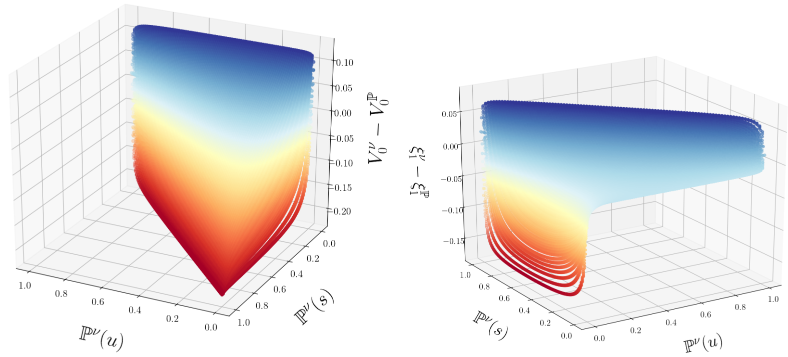


1-period Trinomial Model, with $K = 7$

Using the above one-period trinomial model, assume that an un-informed investor (one without weak information on $Y = S_1$) has a naive belief that the stock prices will be uniformly distributed at time 1 (i.e., the stock price distribution at time 1 matches the probability measure $\mathbb{P} = \{p(u), p(s), p(d)\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, where $p(u), p(s), p(d)$ represent the probability of the stock price increasing, staying the same, or decreasing respectively). Now suppose we have an insider who is given some weak information on $Y = S_1$. Note that for the one-period model case, ν defines some new probability measure $\mathbb{P}^{'} = \{p^{'}(u), p^{'}(s), p^{'}(d)\}$. We thus consider a range of possible ν that an insider may be given, and show how the fair price of the option and optimal trading strategy change under different measures that may be defined. We compute the time-zero fair price for this contingent claim using the following formulas:

$$\hat{\xi_1} = \frac{\mathrm{Cov}\Big[V_1, \Delta S_1\Big]}{\mathrm{Var}[\Delta S_1]},$$
$$\hat{V_0} = \mathbb{E}[V_1] - \hat{\xi}_1 \mathbb{E}[\Delta S_1].$$

We note that under $\mathbb{P}$, calculating $\hat{\xi_1}$ and $\hat{V_0}$ yields $\frac{5}{28}$ and $\frac{3}{14}$. Analogous results can be obtained under weak information, and the following graphs give how the fair price of the contingent claim and optimal trading strategy change when an investor is given some weak information:



Positive and negative values are shown in blue and red colors, respectively. (left) Fair time-zero option price difference under ν from $\mathbb{P}$. (right) Strategy difference under the extra information under ν from $\mathbb{P}$.

Since $\mathbb{E}[V_1] = \mathbb{P}(u)$, the option is priced higher if $\mathbb{P}(u)$ increases, or $\hat{\xi}_1$ or $\mathbb{E}[\Delta S_1]$ decreases. Notice that weak information that gives extremely small $\nu(S_1 = 2)$ and $\nu(S_1 = 8)$ results in a positive and negative change to the optimal trading strategy, $\xi_1^\nu - \xi_1^{\mathbb{P}}$, respectively. This matches with an intuition that if there is almost no chance of a stock losing (or gaining) its value, a trader should buy more (or less) shares of stock than he or she would have under the naive assumption in order to hedge a position.

We conclude this example by pointing out that it demonstrated that the weak information in an incomplete, trinomial model could change the fair price $\hat{V}_0$ and the trading strategy $\hat{\xi}$. *The fair price can become higher, lower, or stay the same.* This contrasts with the complete models considered in Section 4.

5.3. **Stability of the Föllmer-Schweizer decomposition.** For some $\varepsilon_0 > 0$, let us consider a family of adapted stock price processes parametrized by ε, $(S^\varepsilon)_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ (where an example of such a family corresponds to linear perturbations of the drift and volatility considered in [5], another closely related example corresponds to perturbations of the numéraire considered in [6]), a family of probability measures $(\mathbb{P}^\varepsilon)_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$, such that $\mathbb{P}^\varepsilon(\omega) > 0$ for every $\omega \in \Omega$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and a family of contingent claim payoffs $(V_N^\varepsilon)_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$. We note that, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the measurable space $(\Omega, \mathbb{F})$ and the filtration $\mathcal{F} = (\mathcal{F}_n)_{n=0,1,\ldots,N}$ are the same and are described in Section 2. We suppose that

$$S^\varepsilon \to S^0, \quad \mathbb{P}^\varepsilon \to \mathbb{P}^0, \quad \textit{and} \quad V_N^\varepsilon \to V_0^\varepsilon,$$

in the following sense

$$\begin{aligned}
&\lim_{\varepsilon\to 0} S^{\varepsilon}_n(\omega) = S^0_n(\omega), \quad n\in\{0,\dots,N\}, \quad \omega\in\Omega,\\
&\lim_{\varepsilon\to 0} \mathbb{P}^{\varepsilon}(\omega) = \mathbb{P}^0(\omega) > 0, \quad \omega\in\Omega,\\
&\lim_{\varepsilon\to 0} V^{\varepsilon}_N(\omega) = V^0_N(\omega), \quad \omega\in\Omega.
\end{aligned} \tag{24}$$

Next, we show stability of a family of the Föllmer-Schweizer decompositions under *joint* perturbations of the stock price dynamics, the contingent claim payoff, and the reference probability measure.

Proposition 5.5. *On a finite space, let us consider a family of stock price processes, $(S^{\varepsilon})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, a family of probability measures $(\mathbb{P}^{\varepsilon})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, such that $\mathbb{P}^{\varepsilon}(\omega)>0$ for every $\omega\in\Omega$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, and a family of payoffs of contingent claims, $(V^{\varepsilon}_N)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ satisfying (24). Let us suppose that Assumption 2.2 holds for the base model corresponding to $\varepsilon=0$. Then, there exists $\overline{\varepsilon}_0\in(0,\varepsilon_0]$, such that for every $\varepsilon\in(-\overline{\varepsilon}_0,\overline{\varepsilon}_0)$, the family of the Föllmer-Schweizer decompositions*

$$V^{\varepsilon}_N = \hat{V}^{\varepsilon}_0 + \sum_{j=1}^{N} \hat{\xi}^{\varepsilon}_j \Delta S^{\varepsilon}_j + L^{\varepsilon}_N, \quad \varepsilon\in(-\overline{\varepsilon}_0,\overline{\varepsilon}_0). \tag{25}$$

is well-defined and satisfies

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \hat{\xi}^{\varepsilon}_n = \hat{\xi}^0_n, \quad n\in\{1,\dots,N\},\\
&\lim_{\varepsilon\to 0} \hat{V}^{\varepsilon}_0 = \hat{V}^0_0,\\
&\lim_{\varepsilon\to 0} L^{\varepsilon}_n = L^0_n, \quad n\in\{0,\dots,N\},
\end{aligned} \tag{26}$$

where the equalities hold for every $\omega\in\Omega$.

Proof. We observe that the positivity of $\mathbb{P}^{\varepsilon}(\omega)$, for every $\omega\in\Omega$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, implies that

$$\mathbb{P}^{\varepsilon}\sim\mathbb{P}^0, \quad \varepsilon\in(-\varepsilon_0,\varepsilon_0).$$

From Assumption 2.2, obtain that for $\varepsilon=0$, (25) holds, where

$$\hat{\xi}^0_n = \frac{\mathrm{Cov}^{\mathbb{P}^0}_{\mathcal{F}_{n-1}}\left[V^0_N - \sum\limits_{j=n+1}^{N} \hat{\xi}_j \Delta S^0_j, \Delta S^0_n\right]}{\mathrm{Var}^{\mathbb{P}^0}_{\mathcal{F}_{n-1}}[\Delta S^0_n]}, \quad n = N,\dots,1,$$

is well-defined, where, here and below, we used the superscript $\mathbb{P}^0$ to emphasize the probability measure, under which the conditional covariance and the conditional variance are computed.

Now, (24) imply that there exists $\overline{\varepsilon}_0 \in (0, \varepsilon_0]$, such that for every $\varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0)$ and $n = 1, \ldots, N$, with probability 1, we have

$$\mathrm{Var}^{\mathbb{P}^\varepsilon}_{\mathcal{F}_{n-1}}[\Delta S^\varepsilon_n] > 0. \tag{27}$$

Now, (24) implies that $\lim_{\varepsilon \to 0} \mathrm{Var}^{\mathbb{P}^\varepsilon}_{\mathcal{F}_{n-1}}[\Delta S^\varepsilon_n] = \mathrm{Var}^{0}_{\mathcal{F}_{n-1}}[\Delta S^0_n]$, for every $n \in \{1, \ldots, N\}$ and $\omega \in \Omega$, which together with the representation of $\hat{\xi}^\varepsilon_N$ for every $\mathbb{P}^\varepsilon$ (in (3) at $n = N$ however with V^ε_N and S^ε instead of V^0_N and S^0) gives

$$\lim_{\varepsilon \to 0} \hat{\xi}^\varepsilon_N = \hat{\xi}^0_N,$$

and therefore also

$$\lim_{\varepsilon \to 0} \xi^\varepsilon_N \Delta S^\varepsilon_N = \xi^0_N \Delta S^0_N,$$

Now, proceeding recursively, backward in time, we deduce from formula (3) for every $\mathbb{P}^\varepsilon$, $\varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0)$ that

$$\begin{gathered} \lim_{\varepsilon \to 0} \hat{\xi}^\varepsilon_n = \hat{\xi}^0_n, \quad n \in \{N, \ldots, 1\}, \\ \lim_{\varepsilon \to 0} \sum_{j=1}^{n} \hat{\xi}^\varepsilon_j \Delta S^\varepsilon_j = \lim_{\varepsilon \to 0} \sum_{j=1}^{n} \hat{\xi}^0_j \Delta S^0_j, \quad n \in \{N, \ldots, 1\}, \end{gathered} \tag{28}$$

therefore, in particular, the first equation in (26) holds.

Further, by the martingale property of L^ε under the associated $\mathbb{P}^\varepsilon$ and since $\mathbb{E}^{\mathbb{P}^\varepsilon}[L^\varepsilon_0] = 0$, we have

$$V^\varepsilon_0 = \mathbb{E}^{\mathbb{P}^\varepsilon}\left[V^\varepsilon_N - \sum_{j=1}^{N} \hat{\xi}^\varepsilon_j \Delta S^\varepsilon_j \right].$$

Consequently, (24) (convergence of V^ε_N's) and (28) result in the convergence of V^ε_0's, that is the second equation in (26). By rewriting (25) as

$$L^\varepsilon_N = V^\varepsilon_N - \hat{V}^\varepsilon_0 - \sum_{j=1}^{N} \hat{\xi}^\varepsilon_j \Delta S_j, \quad \varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0),$$

and using first two equalities in (26), we obtain that $\lim_{\varepsilon \to 0} L^\varepsilon_N = L^0_N$. Finally, by taking the conditional expectations under the corresponding measures $\mathbb{P}^\varepsilon$'s, and invoking the martingale property of L^ε's under the associated measures, we deduce that the third equality in (26) holds. □

5.4. **Stability of the information premium and the correction to optimal trading strategy.** We work in the settings and notations of Section 5.3. The main result below gives the stability of the information premium and the correction to the optimal trading strategy under perturbation (24).

Let $\overline{\varepsilon}_0$ be as in Proposition 5.5. For every $\varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0)$, to emphasize the dependence of the components of the Fölmer-Schweizer decomposition on both ε and the probability measure $\mathbb{Q}$, that is, for the outputs to

$$\text{minimize } \mathbb{E}^{\mathbb{Q}}\left[(V_N^\varepsilon - V_0 - \sum_{j=1}^N \xi_j \Delta S_j^\varepsilon)^2\right] \text{ over all } \xi \in \Theta \text{ and } V_0 \in \mathbb{R}, \quad (29)$$

we denote the associated Fölmer-Schweizer decompositions as follows

$$V_N^\varepsilon = \hat{V}_0^{\mathbb{Q},\varepsilon} + \sum_{j=1}^N \hat{\xi}_j^{\mathbb{Q},\varepsilon} \Delta S_j^\varepsilon + L_N^{\mathbb{Q},\varepsilon}, \quad \varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0). \quad (30)$$

Theorem 5.6. *On a finite space, let us consider a family of stock price processes, $(S^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, a family of probability measures $(\mathbb{P}^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, such that $\mathbb{P}^\varepsilon(\omega) > 0$ for every $\omega \in \Omega$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and a family of payoffs of contingent claims, $(V_N^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ satisfying (24). Let us suppose that Assumption 5.1 holds and there exists a unique solution to (13) for the base model corresponding to $\varepsilon = 0$.*

Then, for $\overline{\varepsilon}_0 > 0$ as in Proposition 5.5 and every $\mathbb{Q}^\varepsilon \in \arg\max_{\overline{\mathcal{E}}^\nu} v(\varepsilon)$, $\varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0)$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left(V_0^{\mathbb{P}^\varepsilon,\varepsilon} - V_0^{\mathbb{Q}^\varepsilon,\varepsilon}\right) = V_0^{\mathbb{P}^0,0} - V_0^{\hat{\mathbb{Q}},0}, \\ &\lim_{\varepsilon\to 0}\left(\xi_n^{\mathbb{P}^\varepsilon,\varepsilon} - \xi_n^{\mathbb{Q}^\varepsilon,\varepsilon}\right) = \xi_n^{\mathbb{P}^0,0} - \xi_n^{\hat{\mathbb{Q}},0}, \quad n \in \{1,\dots,N\}, \quad \omega \in \Omega, \end{aligned} \quad (31)$$

where $\hat{\mathbb{Q}}$ is the optimizer to (34), for $\varepsilon = 0$, and we use double superscripts as in (30).

Proof. We observe that, similarly to the proof of Theorem 5.2, Assumption 5.1 and equivalence of $\mathbb{P}^0$ to every probability measure in $\mathcal{E}^\nu$ imply that $v^{\mathbb{P}^0}$ is strictly positive, where $v^{\mathbb{P}^0}$ is defined in (2), that is Assumption 2.2 holds for $\mathbb{P}^0$. Next, using (24), we deduce that there exists $\overline{\varepsilon}_0 \in (0, \varepsilon_0]$, such that for every $\varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0)$ we have

$$\mathrm{Var}_{\mathcal{F}_{n-1}}^{\mathbb{P}^\varepsilon}[\Delta S_n^\varepsilon] \geq \tfrac{1}{2} v_n^{\mathbb{P}^0} > 0, \quad n \in \{1,\dots,N\}. \quad (32)$$

By Proposition 5.5, we deduce that

$$\lim_{\varepsilon\to 0} V_0^{\mathbb{P}^\varepsilon,\varepsilon} = V_0^{\mathbb{P}^0,0} \quad \textit{and} \quad \lim_{\varepsilon\to 0} \xi_n^{\mathbb{P}^\varepsilon,\varepsilon} = \xi_n^{\mathbb{P}^0,0}, \quad n \in \{1,\dots,N\}. \quad (33)$$

Next, we set

$$u(\varepsilon) := \sup_{\mathbb{Q}\in\overline{\mathcal{E}}^\nu} \mathbb{E}[(L_N^{\mathbb{Q},\varepsilon})^2], \quad \varepsilon \in (-\overline{\varepsilon}_0, \overline{\varepsilon}_0). \quad (34)$$

We claim that

$$\lim_{\varepsilon\to 0} u(\varepsilon) = u(0). \quad (35)$$

Let us consider

$$\liminf_{\varepsilon\to 0} u(\varepsilon) = \liminf_{\varepsilon\to 0} \sup_{\mathbb{Q}\in\overline{\mathcal{E}}^{\nu}} \mathbb{E}^{\mathbb{Q}}[(L_N^{\mathbb{Q},\varepsilon})^2] \geq \liminf_{\varepsilon\to 0} \mathbb{E}^{\hat{\mathbb{Q}}}[(L_N^{\hat{\mathbb{Q}},\varepsilon})^2]. \tag{36}$$

Using Proposition 5.5, we can rewrite the latter limit as

$$\liminf_{\varepsilon\to 0} \mathbb{E}^{\hat{\mathbb{Q}}}[(L_N^{\hat{\mathbb{Q}},\varepsilon})^2] = \liminf_{\varepsilon\to 0} \mathbb{E}^{\hat{\mathbb{Q}}}[(L_N^{\hat{\mathbb{Q}},0})^2] = u(0),$$

combining which with (36), we deduce that

$$\liminf_{\varepsilon\to 0} u(\varepsilon) \geq u(0). \tag{37}$$

Let us now consider a sequence ε_k, $k\in\mathbb{N}$, such that

$$\lim_{k\to\infty} u(\varepsilon_k) = \limsup_{\varepsilon\to 0} u(\varepsilon) \tag{38}$$

and we fix $\mathbb{Q}^k \in \overline{\mathcal{E}}^{\nu}$, $k\in\mathbb{N}$, such that

$$u(\varepsilon_k) \leq \mathbb{E}[(L_N^{\mathbb{Q}_k,\varepsilon_k})^2] + \frac{1}{k}, \quad k\in\mathbb{N}. \tag{39}$$

For the sequence $\mathbb{Q}^k \in \overline{\mathcal{E}}^{\nu}$, $k\in\mathbb{N}$, we can pick a convergent subsequence k_l, $l\in\mathbb{N}$, whose limit we denote by $\overline{\mathbb{Q}}$. One can see that $\overline{\mathbb{Q}} \in \overline{\mathcal{E}}^{\nu}$. Applying Proposition 5.5 (the last equality in (26)), we deduce that

$$\lim_{l\to\infty} L_N^{\mathbb{Q}_{k_l},\varepsilon_{k_l}} = L_N^{\overline{\mathbb{Q}},0},$$

for every $\omega\in\Omega$. Combining this with (38) and (40), we deduce that

$$\limsup_{\varepsilon\to 0} u(\varepsilon) = \lim_{l\to\infty} \mathbb{E}[(L_N^{\mathbb{Q}_{k_l},\varepsilon_{k_l}})^2] = \mathbb{E}[(L_N^{\overline{\mathbb{Q}},0})^2] \leq \sup_{\mathbb{Q}\in\overline{\mathcal{E}}^{\nu}} \mathbb{E}[(L_N^{\mathbb{Q},0})^2] = u(0). \tag{40}$$

Now, (37) and (40) imply (35).

Let ε_m, $m\in\mathbb{N}$, be an arbitrary sequence converging to 0 and such that $\varepsilon_m \in (-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$, $m\in\mathbb{N}$. Let us consider $\mathbb{Q}^{\varepsilon_m} \in \arg\max_{\overline{\mathcal{E}}^{\nu}} u(\varepsilon_m)$, $m\in\mathbb{N}$. We claim that

$$\lim_{m\to\infty} \mathbb{Q}^{\varepsilon_m}(\omega) = \hat{\mathbb{Q}}(\omega), \quad \omega\in\Omega. \tag{41}$$

If by contradiction we do not have such convergence, there exists a subsequence m_k, $k\in\mathbb{N}$, and a probability measure $\tilde{\mathbb{Q}}$, such that $\lim_{k\to\infty} \mathbb{Q}^{\varepsilon_{m_k}} = \tilde{\mathbb{Q}}$, for every $\omega\in\Omega$. Proposition 5.5 (see (26)) this implies that

$$\lim_{k\to\infty} L_N^{\mathbb{Q}^{\varepsilon_{m_k}},\varepsilon_{m_k}} = L_N^{\tilde{\mathbb{Q}},0},$$

for every $\omega\in\Omega$. Therefore, as $\hat{\mathbb{Q}}$ is the unique optimizer to $u(0)$ (by the assumption of this theorem), we deduce from (34) that

$$\lim_{k\to\infty} u(\varepsilon_{m_k}) = \lim_{k\to\infty} \mathbb{E}[(L_N^{\mathbb{Q}^{\varepsilon_{m_k}},\varepsilon_{m_k}})^2] = \mathbb{E}[(L_N^{\tilde{\mathbb{Q}},0})^2] < \mathbb{E}[(L_N^{\hat{\mathbb{Q}},0})^2] = u(0),$$

which contradicts to (35). Thus, (41) holds. Applying Proposition 5.5 (see (26) again), we obtain that

$$\lim_{m\to\infty} \hat{\xi}_n^{\mathbb{Q}^{\varepsilon_m},\varepsilon_m} = \hat{\xi}_n^{\hat{\mathbb{Q}},0}, \quad n \in \{1,\dots,N\}, \quad \textit{and} \quad \lim_{m\to\infty} \hat{V}_0^{\mathbb{Q}^{\varepsilon_m},\varepsilon_m} = \hat{V}_0^{\hat{\mathbb{Q}},0}. \tag{42}$$

As ε_m, $m \in \mathbb{N}$, is an arbitrary sequence, (42) combined with (33) imply (31). □

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