



Decorated one-dimensional cobordisms and tensor envelopes of noncommutative recognizable power series



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ABSTRACT

The paper explores the relation between noncommutative power series and topological theories of one-dimensional cobordisms decorated by labelled zero-dimensional submanifolds. These topological theories give rise to several types of tensor envelopes of noncommutative recognizable power series, including the categories built from the syntactic algebra and syntactic ideals of the series and the analogue of the Deligne category.

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1. Introduction

In the *universal construction* approach to low-dimensional topological theories [5,23,44] one starts with an evaluation of closed n -dimensional objects M taking values in a ground commutative ring or a field and then defines state spaces $A(N)$ for $(n-1)$ -dimensional objects N via the bilinear pairing on n -dimensional objects M with a given boundary, $\partial M \cong N$, by coupling two such objects M_1, M_2 along the boundary and evaluating the resulting closed object $M_1 \cup_N M_2$. The n -dimensional objects may be manifolds, manifolds with decorations, embedded manifolds or foams, or one of many other variations of these examples. The universal pairing theory of Freedman, Kitaev, Nayak, Slingerland, Walker and Wang [19], further developed by Calegari, Freedman, Walker and others [6,53], is closely related to the universal construction. Some other examples of the universal construction for $n = 2$ were recently considered in [24,31,28]. Vector spaces or modules $A(N)$ that one assigns to $(n-1)$ -dimensional objects in universal constructions usually do not satisfy the Atiyah tensor product axiom $A(N_1 \sqcup N_2) \cong A(N_1) \otimes A(N_2)$, see [1]. Instead, there are maps

$$A(N_1) \otimes A(N_2) \longleftrightarrow A(N_1 \sqcup N_2) ,$$

which one can think of as a sort of a lax tensor structure.

In this note we explain that the universal construction approach is interesting even in dimension one. Studying the universal construction for one-manifolds decorated by dots labelled by elements of a finite set S , we recover the notion of noncommutative recognizable (equivalently, rational) power series in the alphabet S as developed by Schützenberger [51], Fliess [17], Eilenberg [13,14], Conway [8], Reutenauer, Carlyle and Paz, and others. A full set of references and introductions to this theory can be found in the textbooks by Berstel and Reutenauer [3], Salomaa and Soittola [47], Ésik and Kuich [15], Kuich and Salomaa [34], also see [50,21]. For short introductions to noncommutative rational power series we refer to Reutenauer [40–42].

Theory of noncommutative recognizable power series has its roots in the theory of rational languages and finite state automata [8,13–15], and can be viewed as a linearization of the latter [3,21]. We briefly review the basics of noncommutative rational (recognizable) power series in Section 2.2 and Proposition 2.1 stated there. Part of the motivation for this theory comes from an earlier theorem of Kleene that rational languages are precisely those recognizable by FSA (finite state automata). A language L is a subset of S^* (the set of words in the letters of the alphabet S) and gives rise to series $\alpha(L)$ with the coefficient of w one if $w \in L$ and zero otherwise. Coefficients of $\alpha(L)$ belong to the Boolean semiring $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$. Kleene's theorem and the theory of finite state automata can be found in many textbooks on the field, see for instance [7] and foundational work of Conway [8] and Eilenberg [13,14]. Alternatively, we refer to Underwood [52, Chapter 2] for a brief introduction to finite state automata, regular languages, and their relation to bialgebras.

We consider various flavors of the category of S -decorated one-dimensional cobordisms. S -labelled dots placed along a one-dimensional cobordism can also be thought of as codimension one *defects* on it.

In the first example, the category \mathcal{C} of oriented one-dimensional cobordisms with labels from S is considered in Section 2. We work over a ground field \mathbf{k} for simplicity, but the construction extends to an arbitrary commutative ring R and at least parts of it extend to commutative semirings, mirroring the theory of recognizable power series over semirings.

To build an evaluation one needs a number (an element of the ground field \mathbf{k}) associated to each circle carrying a collection of S -labelled dots. This collection is determined by a finite sequence w of elements of S up to a cyclic order. Consequently, to build various evaluation categories, we need to assign a number $\alpha(w) \in \mathbf{k}$ to each such sequence or word $w \in S^*$, subject to the condition $\alpha(uv) = \alpha(vu)$ for all words u, v .

An evaluation of this type is encapsulated by a formal expression

$$Z_\alpha = \sum_{w \in S^*} \alpha(w) w, \quad \alpha = \{\alpha(w)\}_{w \in S^*}, \quad (1)$$

known as a noncommutative power series Z_α , an element of the vector space $\mathbf{k}\langle\langle S \rangle\rangle$ dual to the free associative algebra $\mathbf{k}\langle S \rangle$ generated by elements of S :

$$\mathbf{k}\langle\langle S \rangle\rangle := \mathbf{k}\langle S \rangle^* = \text{Hom}_{\mathbf{k}}(\mathbf{k}\langle S \rangle, \mathbf{k}).$$

Recognizable noncommutative power series are singled out by the condition that their *syntactic algebra* A_α , see Section 2.2, is finite-dimensional. The syntactic algebra [38] is the quotient of $\mathbf{k}\langle S \rangle$ by the largest two-sided ideal I_α of $\mathbf{k}\langle S \rangle$ that lies in the hyperplane $\ker(\alpha)$, when α is considered as a linear map $\mathbf{k}\langle S \rangle \rightarrow \mathbf{k}$.

Property $\alpha(uv) = \alpha(vu)$ for all words $u, v \in S^*$ describes a particular type of series that we refer to as *symmetric* series. Reutenauer [38] calls such series *central*.

In Section 2 we show that for recognizable symmetric series α there is a satisfactory theory of tensor envelopes [33], that is, tensor categories associated to α , that mirrors the theory of the Deligne categories associated to symmetric groups and of negligible quotients of these categories [10, 9, 16]. Similar theories have recently been introduced for evaluations of two-dimensional cobordisms in [24, 31], two-dimensional cobordisms with corners [28], and two-dimensional cobordisms with dots (codimension two defects) [27]. One can also compare our construction with tensor envelopes of the “one-sided inverse” algebras and Leavitt path algebras considered in [32] for categorifications of rings of fractions and with the diagrammatic categorification of the polynomial ring in [29].

There are several categories and functors between them associated to rational symmetric noncommutative series α , defined throughout Section 2.4 and summarized in Section 2.5 and diagram (23) there. Various skein and quotient categories that one obtains extend the notion of the syntactic algebra A_α of α (the quotient of noncommutative polynomials $\mathbf{k}\langle S \rangle$ by the largest two-sided ideal contained in $\ker \alpha$, see above) and can be thought of as forming various tensor and Karoubi closures of the latter. The theory of syntactic algebras of noncommutative recognizable (or rational) power series was introduced and developed by Reutenauer [38]. Syntactic algebra A_α appears as the endomorphism algebra of the generating object (+) in several categories associated to α .

In Section 3 we go beyond the restriction that noncommutative power series be symmetric by enlarging our category of cobordisms. We consider category $\tilde{\mathcal{C}}$ of S -decorated cobordisms M that may have endpoints strictly “inside” the cobordism, that is, not on the top or bottom boundary $\partial_1 M$ and $\partial_0 M$. We call these *inner* or *floating* endpoints. Such floating endpoints appear in diagrammatical calculi in [29, 32], for instance. Cobordisms of this type between empty 0-manifolds (closed or floating cobordisms) have connected components that are either S -decorated oriented intervals or circles. A multiplicative evaluation on such cobordisms assigns an element $\alpha^\bullet(w) \in \mathbf{k}$ to an oriented interval with word w written along it via labelled dots, and an element $\alpha^\circ(v) \in \mathbf{k}$ to an oriented circle, with word v , well-defined up to cyclic rotation, written along it.

Consequently, the analogue of noncommutative power series in this case is a pair

$$\alpha = (\alpha^\bullet, \alpha^\circ) \quad (2)$$

where α^\bullet is a noncommutative power series and α° is a symmetric noncommutative power series. There does not have to be any relation between α^\bullet and α° .

Pair α of series as above allows to evaluate S -decorated floating intervals (via α^\bullet) and floating circles (via α°). In Section 3 to the evaluation data $\alpha = (\alpha^\bullet, \alpha^\circ)$ we assign several tensor categories similar to

those for the symmetric series. The resulting categories have the best behavior when both α^\bullet and α° are recognizable series, and we specialize to this case early. We say that α is recognizable if both α^\bullet and α° are recognizable.

We follow the path familiar from Section 2 and papers [31,28] and assign several categories and functors between them to each recognizable pair α , including the following categories:

- The category \mathcal{VC}_α of viewable cobordisms, where any closed (floating) component is reduced via evaluation α .
- The skein category \mathcal{SC}_α , where, additionally, elements of two-sided and one-sided syntactic ideals I_α and $I_{\alpha^\bullet}, I_{\alpha^\circ}$ evaluate to zero when placed in the middle of the strand or by its floating endpoint, respectively.
- The category $\tilde{\mathcal{C}}_\alpha$, the quotient of either \mathcal{VC}_α or \mathcal{SC}_α by the ideal of negligible morphisms.
- Additive Karoubi closure \mathcal{DC}_α of \mathcal{SC}_α , which is the analogue of the Deligne category.
- Additive Karoubi closure $\underline{\mathcal{DC}}_\alpha$ of $\tilde{\mathcal{C}}_\alpha$, equivalent to the quotient of \mathcal{DC}_α by the ideal of negligible morphisms.

These four categories have finite-dimensional hom spaces (again, assuming α is recognizable), see diagram (31) and Section 3. They can be thought of as various tensor envelopes of α and the syntactic algebra A_α .

Categories built out of a single symmetric recognizable series in Section 2 can be considered a special case of this construction, given by setting the first series α^\bullet to zero. Setting the second series α° to zero, instead, results in another specialization of the theory, with all decorated circles evaluating to zero, while decorated intervals evaluating to coefficients of α^\bullet , see the remark at the end of Section 3.

In this paper we use *rational* and *recognizable* interchangeably to refer to noncommutative power series over a field with the syntactic ideal of finite codimension. Coincidence of rational and recognizable power series with coefficients in an arbitrary semiring is a result of Schützenberger [51], see also [3,47,15,34] for more details and references. For more general monoids, beyond the free monoid on a finite set S , the sets of recognizable and rational series may differ, see [11,49] and references therein. The difference between rational and recognizable series is also visible in examples in [28], where a recognizable series in two or more commuting variables needs to be rational with denominators restricted to polynomials in single generating variables.

The theory of recognizable noncommutative power series makes sense over non-commutative semirings [3, 47]. One can look to generalize the theory of tensor envelopes of such series from series over a field or a commutative ring to series over a semiring. Note that closed cobordisms would then evaluate to elements of the ground semiring. Components of a closed cobordism “commute”, in the sense of sliding past each other, as elements of the commutative monoid of endomorphisms of the unit object of the tensor category of cobordisms, the empty zero-manifold. For this reason, it is natural to restrict to commutative semirings in this fuller extension of the theory of tensor envelopes of noncommutative power series. We do not consider the general case of a ground commutative semiring K in this paper, though, limiting ourselves to a ground field, but it may be interesting to develop. The case when $K = \mathbb{B}$ is the boolean semiring, gives, in particular, the notion of tensor envelopes of a rational language L or, equivalently, tensor envelopes of a finite state automaton. To get the definition, run the constructions of Section 3 with \mathbb{B} in place of field \mathbf{k} and the pair $\alpha = (\alpha^\bullet, 0)$ of series with the zero symmetric series $\alpha^\circ = 0$ and α^\bullet the series of a regular language L . To test whether this notion is useful, one may study examples of quotients $\tilde{\mathcal{C}}_\alpha$ of the skein category \mathcal{SC}_α for such α .

In the follow-up paper, we will consider one-dimensional cobordisms with more general decorations, by edges and vertices of an oriented graph (or a quiver) Γ . The graph Γ may be finite or infinite. Dots on

a cobordisms are labelled by oriented edges of Γ . Intervals of the cobordisms separated by dots along a connected component are labelled by vertices of Γ . A dot labelled by an edge $s : a \rightarrow b$ is surrounded by intervals labelled by vertices a and b , respectively, in the order that matches the orientation of the corresponding connected component. Such decorations are possible for both interval and circle connected components of a cobordism. There are suitable monoidal categories $\mathcal{C}(\Gamma)$ and $\tilde{\mathcal{C}}(\Gamma)$ of Γ -decorated cobordisms generalizing categories \mathcal{C} and $\tilde{\mathcal{C}}$ in this paper. Cobordisms with floating endpoints are allowed in $\tilde{\mathcal{C}}(\Gamma)$ but not in $\mathcal{C}(\Gamma)$. Objects of $\mathcal{C}(\Gamma)$ and $\tilde{\mathcal{C}}(\Gamma)$ are finite sequences of vertices of Γ or, equivalently, finite sequences of objects of category $S(\Gamma)$, see next.

To Γ one assigns the small category $S(\Gamma)$ of paths in Γ , with vertices of Γ being the objects of $S(\Gamma)$ and paths in Γ – morphisms, with concatenation of paths as the composition. Traveling along a connected component of a Γ -decorated cobordism one encounters a path in Γ , that is, a morphism in $S(\Gamma)$. If the component is a circle, the path, in addition, must be closed, that is, start and end at the same vertex of Γ .

An evaluation α , in the case of $\tilde{\mathcal{C}}(\Gamma)$, where floating endpoints are allowed, consists of two maps:

- Map α^\bullet from the set of morphisms in $S(\Gamma)$ (paths in Γ) to the ground field \mathbf{k} or, more generally, a commutative ring or a semiring,
- Map α° from the set of *circular morphisms*, that is, closed paths in Γ without a choice of the basepoint to \mathbf{k} .

The pair $\alpha = (\alpha^\bullet, \alpha^\circ)$ is the analogue of the pair in (2), generalizing the special case considered in this paper where Γ has a single vertex and oriented loops from the vertex to itself are enumerated by elements of S .

One can then define the analogues of all the categories in the diagram (31), including \mathcal{VC}_α , \mathcal{SC}_α , $\tilde{\mathcal{C}}_\alpha$, in this case. In particular, the category $\tilde{\mathcal{C}}(\Gamma)_\alpha$ is the quotient of the \mathbf{k} -linearization $\mathbf{k}\tilde{\mathcal{C}}(\Gamma)$ by the two-sided ideal of negligible morphisms, defined via the trace given by evaluation α .

The pair α is called *recognizable* or *locally-recognizable* (when Γ is infinite) if the “gligible quotient” category $\tilde{\mathcal{C}}(\Gamma)_\alpha$ has finite-dimensional hom spaces. Note that the boundary points and floating endpoints of Γ -decorated cobordisms are labelled by objects of $S(\Gamma)$, that is, by vertices of Γ , the label inherited from the label of the adjacent edge of the cobordism.

These constructions can be further generalized to cobordisms between finite sets of boundary points given by graphs and Γ -decorated graphs rather than by Γ -decorated one-manifolds. Cobordisms given by graphs can still be viewed as one-dimensional cobordisms between zero-dimensional objects (finite sets of points, possibly decorated by vertices of Γ and orientations, as necessary).

A natural open problem is to extend the universal construction, for decorated cobordisms (or cobordisms with defects), beyond dimension one. Parts of this extension are visible in

- [27], where two-dimensional cobordisms are decorated by dots labelled by elements of a commutative monoid or a commutative algebra, with non-trivial interactions between these dots and topology of cobordisms coming from the handle cobordism equated to a nontrivial element of the monoid or algebra.
- [28], where side boundaries of two-dimensional cobordisms with corners may be colored by elements of a finite set.
- Foam theory [23,4,12,46,44], see more references in [25], where rather particular evaluations of two-dimensional decorated CW-complexes with generic singularities embedded in \mathbb{R}^3 (foams) are used as an intermediate step to build homology theories of links that categorify various one-variable specializations of the HOMFLYPT polynomial. Soergel bimodules, singular Soergel bimodules, and some other structures in representation theory admit a foam description as well [54,45].

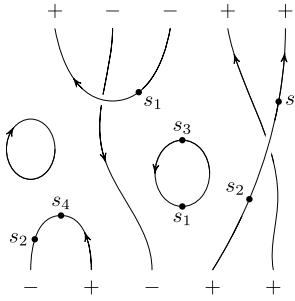


Fig. 2.1.1. A morphism from $(- - + +)$ to $(+ - - +)$ in \mathcal{C} . It has two closed (or floating) components, one undecorated, the other decorated by s_1s_3 . The two U-turns are decorated by s_2s_4 and s_1 , respectively. There are three through arcs: two undecorated, one decorated by s_2s_2 . Whether a crossing is over- or under-crossing is irrelevant.

- Evaluation theory for two-dimensional cobordisms and evaluations of overlapping foams [24], pointing towards further connections to arithmetic topology, representation theory, and the Heegaard-Floer theory.
- [26], which considers evaluations in the two-dimensional planar case with one-dimensional defects.

Freedman et al. [19] mention possible decorations on low-dimensional cobordisms for their universal pairings.

Studying evaluations not just for n -manifolds but for decorated n -manifolds, n -manifolds and their foam analogues embedded in \mathbb{R}^{n+1} , and other such refinements should ease one's way into understanding recognizable evaluations in dimension $n+1$. This program makes sense at least in dimensions $n=1, 2, 3$.

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2. Decorated one-dimensional cobordisms and their evaluations

2.1. Categories \mathcal{C} and \mathcal{C}'

Fix a finite set S of cardinality $r \geq 0$, which we often write as $S = \{s_1, s_2, \dots, s_r\}$.

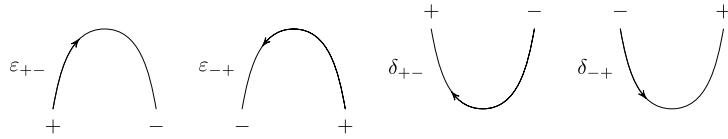
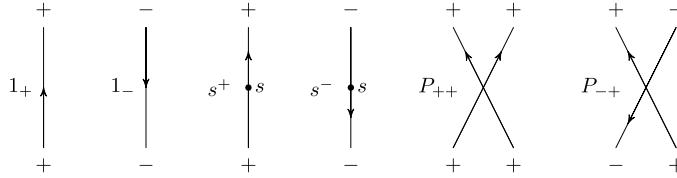
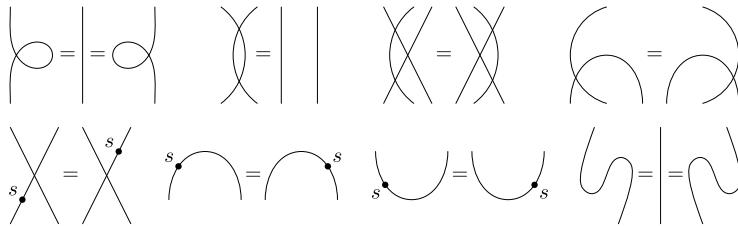
Consider the category $\mathcal{C} = \mathcal{C}_S$ of S -decorated compact oriented one-dimensional cobordisms. Its objects are oriented zero-dimensional manifolds N , that is, finite sets with a sign assignment $+$ or $-$ to each element (signed finite sets). A morphism from N_0 to N_1 is an oriented one-dimensional manifold M decorated by finitely many dots labelled by elements of S , with $\partial M = N_1 \sqcup (-N_0)$, see Fig. 2.1.1 for an example, which also sets the orientation convention for the boundary.

Dots can move along a connected component of M where they are placed but without crossing through other dots or moving to a boundary point. Two morphisms are equal if they are diffeomorphic rel boundary and keeping track of dots and their labels.

Each component c of M is either an oriented circle or an oriented interval. Going along c in the direction of its orientation, one can read off the labels of marked points. When c is an interval, the sequence of labels is an invariant of c . When c is a circle, the sequence of labels is an invariant up to a cyclic rotation of the sequence. A component may carry no dots; the corresponding sequence is empty then.

Composition of morphisms is given by their concatenation.

To reduce to fewer objects, we take the objects to be sequences of signs $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{+, -\}$. To $\underline{\epsilon}$ we associate an ordered signed zero-manifold with one point for each term in the sequence, with the signs given by $\epsilon_1, \dots, \epsilon_n$. We may alternatively write $\epsilon_i = 1$ or -1 instead of $+$ or $-$.

Fig. 2.1.2. Duality morphisms for + and - in \mathcal{C} .Fig. 2.1.3. Generating morphisms in \mathcal{C} : identity 1_+ of $(+)$, identity 1_- of $(-)$, morphism $s^+ : (+) \rightarrow (+)$ for $s \in S$, morphism $s^- : (-) \rightarrow (-)$, permutation morphism P_{++} . Other permutation morphisms, such as P_{-+} , can be obtained as compositions of these morphisms and those in Fig. 2.1.2.Fig. 2.1.4. Some relations in \mathcal{C} . They hold for any choice of orientations. Orientation of the LHS of each equation determines the orientation of the RHS and vice versa. A set of these relations with some restrictions on orientations can be taken for a defining set of relations. Commutativity relations on generators in horizontally separated regions are not shown, since they are built into the axioms of a tensor category.

Permutation cobordisms show that permuting signs in a sequence $\underline{\epsilon}$ leads to an isomorphic object, and that isomorphism classes of objects are parametrized by pairs of non-negative integers $\mathbf{n} = (n_0, n_1)$, counting the number of plus and minus signs.

When restricting to a skeleton category (one object for each isomorphism class), we thus reduce objects to pairs $\mathbf{n} = (n_0, n_1)$, where n_0 is the number of plus points and n_1 is the number of minus points. In the sequence of signs that \mathbf{n} represents, we put plus signs first, and can also write $\mathbf{n} = (+^{n_0} -^{n_1})$.

Denote by $\mathcal{C} = \mathcal{C}_S$ the category of S -decorated cobordisms with objects—finite sign sequences $\underline{\epsilon}$ as above. The skeleton category of S -decorated cobordisms, with objects \mathbf{n} , is denoted \mathcal{C}' . Category \mathcal{C} is slightly larger than the equivalent category \mathcal{C}' .

These categories are rigid symmetric tensor, with the tensor product in \mathcal{C} given on morphisms by placing their diagrams next to each other. On objects, the tensor product is the concatenation of sequences. In \mathcal{C}' when forming the tensor product of morphisms, we group plus points together and minus points together. Tensor product on objects is given by $(n_0, n_1) \otimes (m_0, m_1) = (n_0 + m_0, n_1 + m_1)$.

In both categories \mathcal{C} and \mathcal{C}' object $(+)$ has object $(-)$ as its dual, with the duality morphisms in \mathcal{C} shown in Fig. 2.1.2.

The empty sequence \emptyset is the unit object $\mathbf{1}$ of the tensor category \mathcal{C} . The pair $\mathbf{1} := (0, 0)$ is the unit object of \mathcal{C}' . Generating morphisms in \mathcal{C}' are shown in Figs. 2.1.2 and 2.1.3.

Some defining relations in \mathcal{C} are shown in Fig. 2.1.4. We do not list a full set of defining relations and will not need it. These relations can be hidden in the definition of \mathcal{C} , where morphisms are declared equal if the corresponding decorated cobordisms are diffeomorphic rel boundary.

A morphism M from ϵ to ϵ' in \mathcal{C} consists of some number of oriented circles and oriented intervals. Boundaries of oriented intervals and their orientations match entries of ϵ and ϵ' in pairs.

Denote by $|\epsilon|$ the difference of the number of plus and minus signs in ϵ and call it the weight of the sequence. For instance, $|(+ - - + + + -)| = 5 - 3 = 2$. A morphism from ϵ to ϵ' exists iff the two sequences have the same weight, $|\epsilon| = |\epsilon'|$. The weight is additive under the tensor product of objects (concatenation of sequences). Denote by $\|\epsilon\|$ the length of the sequence ϵ .

Connected components c of a morphism M are circles and intervals (arcs). Circles are closed components, also called *floating* components. Arcs have boundary and, borrowing terminology from [30], separate into *U-turns* and *through* arcs. A U-turn has both endpoints on the same side of a morphism (either on the source one-manifold or on the target, while a *through* arc has one endpoint on the source and one on the target, also see types (1)-(3) of components in Fig. 3.1.3. The endpoints of a U-turn have opposite signs, while the endpoints of a through arc carry the same sign, see Figs. 2.1.2, 2.1.3.

By analogy with [31,28], we can also call arcs *viewable* or *visible* components, since they have endpoints on the boundary of the cobordism (either top or bottom or both), and call circles *floating* components [29,28], since they are disjoint from the boundary of the cobordism.

Denote by S^* the set of finite sequences of elements of S , including the empty sequence \emptyset (also see Section 2.2). Going along an arc c of a cobordism M gives us a word $w(c) \in S^*$. Going along a circle c in x gives a word $w(c)$ well defined up to a cyclic rotation or *conjugation* of words, $w_1 w_2 \sim w_2 w_1$.

In this paper we encounter sequences $\underline{\epsilon}$ of signs, which are objects of \mathcal{C} , and sequences $w \in S^*$, which are sequences of labels encountered along connected components of a cobordism, the latter a morphism in \mathcal{C} .

2.2. Noncommutative power series

For simplicity we work over a ground field \mathbf{k} , although the theory of noncommutative power series and rational and recognizable series makes sense over an arbitrary semiring R , not necessarily commutative [3,47]. For definitive treatments we refer the reader to books [3,47] and to [40–42] for quick introductions and reviews.

Let $S^* = \emptyset \sqcup S \sqcup S^2 \sqcup \dots$ be the set of sequences $w = t_1 \dots t_n$ of elements of a finite set $S = \{s_1, \dots, s_r\}$. We call elements of S *letters* and elements of S^* *words* or *sequences* in S . The empty word \emptyset is allowed. A noncommutative power series α over \mathbf{k} is any function

$$\alpha : S^* \longrightarrow \mathbf{k}, \quad \alpha(w) \in \mathbf{k}, \quad w \in S^*. \quad (3)$$

We formally write this series as

$$Z_\alpha = \sum_{w \in S^*} \alpha_w w, \quad \alpha = (\alpha_w)_{w \in S^*}, \quad \alpha_w = \alpha(w) \in \mathbf{k}, \quad (4)$$

using either α_w or $\alpha(w)$ to denote the value of α on a noncommutative monomial or word w . Denote by $\mathbf{k}\langle\langle S \rangle\rangle$ the \mathbf{k} -vector space of noncommutative power series and by $\mathbf{k}\langle S \rangle$ the free noncommutative \mathbf{k} -algebra on generators in S (the algebra of noncommutative polynomials).

Given two series α, β , their product is the series $\alpha\beta$ that on w evaluates to

$$\alpha\beta(w) = \sum_{w=w_1 w_2} \alpha(w_1)\beta(w_2), \quad (5)$$

the sum over all decompositions of w . There are $\ell(w) + 1$ terms in the sum, where $\ell(w)$ is the length of w . This product turns $\mathbf{k}\langle\langle S \rangle\rangle$ into a \mathbf{k} -algebra, noncommutative if S has more than one element. The inclusion $\mathbf{k}\langle S \rangle \subset \mathbf{k}\langle\langle S \rangle\rangle$ is a ring homomorphism.

We say that series $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle$ is *recognizable* iff there is a homomorphism $\psi : \mathbf{k}\langle S \rangle \longrightarrow \text{End}(\mathbf{k}^n)$ of the free algebra into the algebra of $n \times n$ matrices, a vector and a dual vector $\lambda, \mu^T \in \mathbf{k}^n$ such that

$$\alpha(w) = \mu \psi(w) \lambda \quad (6)$$

for all words w . That is, the number $\alpha(w)$ is the product of the $1 \times n$ matrix μ , $n \times n$ matrix $\psi(w)$ and $n \times 1$ matrix λ . Denote by $\mathbf{k}\langle\langle S \rangle\rangle^{rec}$ the set of all recognizable series.

Vector space $\mathbf{k}\langle\langle S \rangle\rangle$ is a $\mathbf{k}\langle S \rangle$ -bimodule with $f \otimes g \in \mathbf{k}\langle S \rangle \otimes \mathbf{k}\langle S \rangle^{op}$ acting on $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle$ by

$$(f \otimes g)(\alpha)(w) = \alpha(gw f), \quad w \in S^*.$$

We write $f\alpha g := (f \otimes g)(\alpha)$. This action gives rise to the left, right, and two-sided ideals $I_\alpha^\ell, I_\alpha^r$ and I_α in $\mathbf{k}\langle S \rangle$:

- Left ideal I_α^ℓ consists of all $f \in \mathbf{k}\langle S \rangle$ such that $f\alpha = 0$, that is, all f such that $\alpha(wf) = 0$ for any word $w \in S^*$. It is the largest left ideal contained in the hyperplane $\ker(\alpha) \subset \mathbf{k}\langle S \rangle$.
- Right ideal I_α^r consists of all $g \in \mathbf{k}\langle S \rangle$ such that $\alpha g = 0$, that is, all g such that $\alpha(gw) = 0$ for all $w \in S^*$. It is the largest right ideal contained in $\ker(\alpha)$.
- Ideal I_α consists of all $f \in \mathbf{k}\langle S \rangle$ such that $\alpha(wfv) = 0$ for all $w, v \in S^*$. It is the largest two-sided ideal contained in the hyperplane $\ker(\alpha)$.

Ideal I_α^ℓ has finite codimension in $\mathbf{k}\langle S \rangle$ iff the series α is recognizable. Given triple as in (6), ideal I_α^ℓ contains the finite codimension subspace $\{x \in \mathbf{k}\langle S \rangle | \psi(x)\lambda = 0\}$. Vice versa, if I_α^ℓ has finite codimension, it is straightforward to produce the data in (6) by taking $\mathbf{k}^n \cong \mathbf{k}\langle S \rangle / I_\alpha^\ell$, $\lambda = 1$ and $\mu = \alpha$. Given a triple as in (5), $\ker \alpha$ contains two-sided ideal of finite codimension $\{f \in \mathbf{k}\langle S \rangle | \psi(f) = 0 \in \text{End}(\mathbf{k}^n)\}$. Vice versa, if I_α has finite codimension, ideals $I_\alpha^\ell \supset I_\alpha$ and $I_\alpha^r \supset I_\alpha$ have finite codimension too.

Consequently, if one of $I_\alpha^\ell, I_\alpha^r, I_\alpha$ have finite codimension in $\mathbf{k}\langle S \rangle$, the other two have finite codimension as well.

Two-sided ideal I_α of $\mathbf{k}\langle S \rangle$ is called the *syntactic ideal* of α . Denote by

$$A_\alpha := \mathbf{k}\langle S \rangle / I_\alpha \quad (7)$$

the quotient algebra, the *syntactic algebra* of α , see [38]. It is defined for any α , but we mostly restrict to considering it for recognizable α , when A_α is finite-dimensional. We also call I_α^ℓ and I_α^r the *left* and *right syntactic ideals* of α .

Algebra $\mathbf{k}\langle S \rangle$ acts on $\mathbf{k}\langle\langle S \rangle\rangle$ on the left and on the right, and $\mathbf{k}\langle S \rangle$ -bimodule generated by α (a subbimodule of $\mathbf{k}\langle\langle S \rangle\rangle$) is naturally isomorphic to the syntactic algebra A_α , the latter equipped with $\mathbf{k}\langle S \rangle$ -bimodule structure via left and right multiplications:

$$A_\alpha \cong \mathbf{k}\langle S \rangle \otimes \mathbf{k}\langle S \rangle^{op}(\alpha). \quad (8)$$

The quotient $\mathbf{k}\langle S \rangle / I_\alpha^\ell$ is naturally a faithful left A_α -module via the left multiplication action. Likewise, $\mathbf{k}\langle S \rangle / I_\alpha^r$ is a faithful right A_α -module via the right multiplication action.

We see that $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle$ is recognizable iff the cyclic $\mathbf{k}\langle S \rangle \otimes \mathbf{k}\langle S \rangle^{op}$ -module generated by α in $\mathbf{k}\langle\langle S \rangle\rangle$ is finite-dimensional, or, equivalently,

$$\dim_{\mathbf{k}} A_\alpha < \infty.$$

The *Hankel matrix* M_α of α is the infinite square matrix with rows and columns enumerated by elements of S^* with the (w_1, w_2) -entry $\alpha(w_1 w_2)$.

Given any series α with $\alpha(\emptyset) = 0$ (called *proper series*), we can form the *Kleene plus* series α^+ as the formal sum

$$\alpha^+ = \alpha + \alpha^2 + \dots, \quad (9)$$

where $\alpha^n = \alpha\alpha\dots\alpha$ is the product of n copies of α . The term α^n evaluates to 0 on any word of length less than n . Consequently, a given word evaluates nontrivially only on finitely many terms in the sum, and α^+ makes sense as an element of $\mathbf{k}\langle\langle S \rangle\rangle$. The series $1 + \alpha^+$ is the inverse of the series $1 - \alpha$ in the ring $\mathbf{k}\langle\langle S \rangle\rangle$.

A series is *finite* if it contains finitely many terms. Finite series are those in the ring of noncommutative polynomials $\mathbf{k}\langle S \rangle \subset \mathbf{k}\langle\langle S \rangle\rangle$.

Denote by $\mathbf{k}\langle\langle S \rangle\rangle^{rat}$ the smallest subset of series that

- Contains all finite series.
- Closed under the product and finite \mathbf{k} -linear combinations of series.
- Contains α^+ for any proper series α in the subset.

Series in $\mathbf{k}\langle\langle S \rangle\rangle^{rat}$ are called *rational* series.

Proposition 2.1. *The following properties of series α are equivalent.*

- (1) α is rational.
- (2) α is recognizable.
- (3) The Hankel matrix M_α of α has finite rank.
- (4) The syntactic ideal I_α has finite codimension in $\mathbf{k}\langle S \rangle$.
- (5) The left ideal I_α^ℓ has finite codimension in $\mathbf{k}\langle S \rangle$.
- (6) The right ideal I_α^r has finite codimension in $\mathbf{k}\langle S \rangle$.
- (7) α can be computed by a weighted finite automaton.

Equivalence of (2), (4), (5), (6) is explained above.

For a proof of all equivalences see Sections 1 and 2 of [3], Salomaa-Soittola [47], or references there to the original work of Schützenberger [51], Fliess [17], Eilenberg [14] and others. Most of these equivalences hold in much greater generality than over a field, in many cases over an arbitrary semiring. The Hankel matrix of noncommutative series was introduced by Fliess [17].

The notion of *weighted finite automaton* linearizes the concept of finite state automaton and, over a field \mathbf{k} , is equivalent to the triple (λ, ψ, μ) as in (6), see [3, Section 1.6], for instance. \square

We have $\mathbf{k}\langle\langle S \rangle\rangle^{rec} = \mathbf{k}\langle\langle S \rangle\rangle^{rat}$, since rational and recognizable series coincide.

Assume that α is recognizable. The trace form α on the finite-dimensional algebra A_α has the following nondegeneracy property:

$$\text{for any } a \in A_\alpha, a \neq 0 \text{ there are } b, c \in A_\alpha \text{ such that } \alpha(bac) \neq 0. \quad (10)$$

This is a much weaker condition than the usual Frobenius condition on a linear form β on a finite-dimensional algebra B :

$$\text{for any } a \in B, a \neq 0 \text{ there exist } b \text{ such that } \beta(ab) \neq 0. \quad (11)$$

In the latter case β equips B with the structure of a Frobenius algebra.

Given any finite-dimensional algebra B with a linear form $\alpha : B \rightarrow \mathbf{k}$, the condition that

$$\text{for any } a \in B, a \neq 0 \text{ there are } b, c \in B \text{ such that } \alpha(bac) \neq 0 \quad (12)$$

is equivalent to the zero ideal (0) being the only two-sided ideal in $\ker(\alpha)$. Let us call a pair (B, α) with this property a *syntactic pair*. A finite set of generators b_1, \dots, b_m of B gives rise to a surjective homomorphism

$$\rho : \mathbf{k}\langle S \rangle \longrightarrow B, \quad \rho(s_i) = b_i, \quad S = \{s_1, \dots, s_m\} \quad (13)$$

from the free algebra $\mathbf{k}\langle S \rangle$ to B and induced noncommutative power series in the set of variables S , also denoted α . This gives a bijection between recognizable power series in S and isomorphism classes of syntactic pairs (B, α) as above with a choice of generators (b_1, \dots, b_m) of B .

An algebra is called *syntactic* if it admits a presentation (7) for some S and α .

Examples:

- (1) Any Frobenius algebra B with a non-degenerate form β gives a syntactic pair (B, β) .
- (2) Take the matrix algebra $B = M_n(\mathbf{k})$ and define $\alpha(x) = x_{1,1}$ to pick the first diagonal coefficient of the matrix x . In this example the form α satisfies the weaker property (10), so that (B, α) is a syntactic pair, but α is not a Frobenius trace. Algebra B is Frobenius for a different linear form on it (for example, for the usual trace on matrices).
- (3) Take the path algebra B of the quiver with two vertices $0, 1$ and the edge (01) connecting them, with the multiplication given by concatenation of paths: $(0)(01) = (01)$, $(01)(1) = (01)$, etc. Algebra B has a basis $\{(0), (1), (01)\}$. Take any linear form α with $\alpha((01)) \neq 0$. Then (B, α) is a syntactic algebra with this linear form. It can be generated by two elements. B is neither Frobenius nor quasi-Frobenius.
- (4) A finite-dimensional commutative algebra is Frobenius iff it is syntactic.

See Reutenauer [38] and Perrin [36] for more results on syntactic algebras and the latter also for another brief introduction to the subject.

2.3. Evaluations and symmetric series

We say that series $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle$ is *symmetric* if $\alpha(w_1 w_2) = \alpha(w_2 w_1)$ for any $w_1, w_2 \in S^*$. Transformation $w_1 w_2 \mapsto w_2 w_1$ is also called *conjugation*, so one can say that α is conjugation invariant. An evaluation α is symmetric iff it only depends on a sequence up to cyclic order.

We use the word “symmetric” to define such series, since the word “cyclic” is already taken, see [2, 22, 39] and [3, Section 12.2]. A series α is called *cyclic* if, in addition to the conjugation invariance condition, it satisfies $\alpha(w^n) = \alpha(w)$ for any non-empty w . Thus, a cyclic series is symmetric but most symmetric series are not cyclic. Reutenauer [38] uses *central* instead of our *symmetric*.

Denote the set of symmetric series by $\mathbf{k}\langle\langle S \rangle\rangle^s$ and by $\mathbf{k}\langle\langle S \rangle\rangle^{s, rec}$ the set of recognizable symmetric series. A series $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle$ can be averaged out to a series $\text{av}(\alpha)$ given by

$$\text{av}(\alpha)(w) = \sum_{uv=w, v \neq \emptyset} \alpha(vu), \quad \text{if } w \neq \emptyset, \quad \text{av}(\alpha)(\emptyset) = \alpha(\emptyset). \quad (14)$$

That is, take the sum over all possible ways to split w into the product uv and evaluate α on vu . Series $\text{av}(\alpha)$ is symmetric. Only one of the two degenerate splittings $\emptyset w$ and $w\emptyset$ is used to avoid having $\alpha(w)$ twice in the sum.

Proposition 2.2. $\text{av}(\alpha) \in \mathbf{k}\langle\langle S \rangle\rangle^{s, rec}$ if $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle^{rec}$.

In other words, averaging out a recognizable series produces a symmetric recognizable series. This result is proved in Rota [43], see also [39]. It gives a large supply of symmetric recognizable series. \square

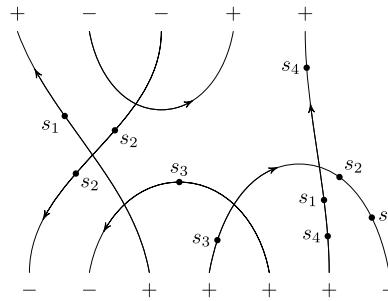


Fig. 2.4.1. A basis element in the hom space in \mathcal{VC}_α , with $S = \{s_1, s_2, s_3, s_4\}$. Floating components (circles) are absent.

Symmetric series with semisimple syntactic algebra A_α are studied in [38,36].

2.4. Tensor envelopes of series α

(1) *Category $\mathbf{k}\mathcal{C}$.* We fix a base field \mathbf{k} and form the \mathbf{k} -linearization $\mathbf{k}\mathcal{C}$ of \mathcal{C} . Category $\mathbf{k}\mathcal{C}$ has the same objects as \mathcal{C} , that is, finite sequences $\underline{\epsilon}$ of plus and minus signs. Morphisms in $\mathbf{k}\mathcal{C}$ are finite linear combinations of morphisms in \mathcal{C} , with the composition rules extended \mathbf{k} -bilinearly from those of \mathcal{C} .

(2) *Category \mathcal{VC}_α of viewable cobordisms.* Next, choose a symmetric power series $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle^s$. Define the category \mathcal{VC}_α as the quotient of $\mathbf{k}\mathcal{C}$ by the relations that a circle \widehat{w} with a sequence w written on it evaluates to $\alpha(w)$. Since $\widehat{w_1 w_2} = \widehat{w_2 w_1}$, we need the condition that α is symmetric to define this quotient.

Another way to define \mathcal{VC}_α is to say that it has the same objects as \mathcal{C} : sequences $\underline{\epsilon}$ of elements of S . A morphism in \mathcal{VC}_α from $\underline{\epsilon}$ to $\underline{\epsilon}'$ is a finite \mathbf{k} -linear combination of viewable cobordisms in \mathcal{C} from $\underline{\epsilon}$ to $\underline{\epsilon}'$. Recall that a cobordism is viewable if it has no floating connected components, that is, components homeomorphic to circles.

Composition of morphisms in \mathcal{VC}_α is given by concatenating cobordisms and removing each closed circle \widehat{w} from the composition simultaneously with multiplying the remaining diagram by $\alpha(w)$.

The hom space $\text{Hom}_{\mathcal{VC}_\alpha}(\underline{\epsilon}, \underline{\epsilon}')$ has a basis given by a choice of orientation-respecting matching of the elements in the pair of sequences $\underline{\epsilon}, \underline{\epsilon}'$ together with a choice of word in S for each pair in the matching. An orientation-respecting matching consists of a bijection between pluses and minuses in the sequence $(-\underline{\epsilon})\underline{\epsilon}'$, which is the concatenation of $-\underline{\epsilon}$ and $\underline{\epsilon}'$, with the sequence $-\underline{\epsilon}$ given by reversing the signs of $\underline{\epsilon}$. An example in Fig. 2.4.1 shows a basis element in one such hom space, with $\underline{\epsilon} = (- - + + + -)$ and $\underline{\epsilon}' = (+ - - + +)$. Note that the size of hom spaces in \mathcal{VC}_α does not depend on α , only the composition of morphisms does.

Each word $w = t_1 \dots t_m$, $t_i \in S$, $i = 1, \dots, m$ defines a cobordism $\text{cob}(w)$ given by putting letters t_1, \dots, t_m along the interval, with the orientation going towards decreasing the index, see Fig. 2.4.2 left. Extended by linearity, this assignment is an algebra isomorphism

$$\mathbf{k}\langle S \rangle \longrightarrow \text{End}_{\mathcal{VC}_\alpha}((+)) \quad (15)$$

from the algebra of noncommutative polynomials to the algebra of endomorphisms of the sequence $(+)$ in \mathcal{VC}_α .

To a noncommutative polynomial

$$u = \sum_{i=1}^k a_i w_i \in \mathbf{k}\langle S \rangle, \quad a_i \in \mathbf{k}, \quad w_i \in S^*$$

we assign the endomorphism

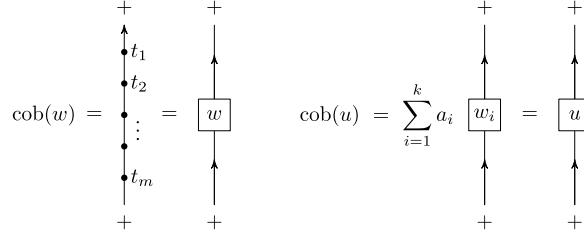


Fig. 2.4.2. Left: cobordism $\text{cob}(w)$ for a word $w = t_1 \dots t_m \in S^*$ is given by placing dots labelled by letters of word w along the oriented interval. Alternatively, $\text{cob}(w)$ can be denoted by a box labelled w on an interval. Right: a linear combination $\text{cob}(u)$ of such cobordisms and its shorthand box notation.

$$\text{cob}(u) = \sum_{i=1}^k a_i \text{cob}(w_i) \in \text{End}_{\mathcal{VC}_\alpha}((+))$$

of the sequence $(+)$ given by the linear combination of words w_i written on an upward oriented interval, see Fig. 2.4.2 right. It can be compactly denoted by a box on a strand with u written in it.

Monomials in $\mathbf{k}\langle S \rangle$ and their linear combinations can be placed along any component of a cobordism. Taking the union over all viewable cobordisms with a given boundary (one cobordism for each diffeomorphism class rel boundary) and then over all ways of placing monomials in $\mathbf{k}\langle S \rangle$ along each component of the cobordism gives a basis in the hom space in the category \mathcal{VC}_α between two objects. Recall that objects of \mathcal{VC}_α are sequences of $+$ and $-$.

The hom space between the objects $\underline{\epsilon}, \underline{\epsilon}'$ is non-zero if the objects have the same weight, $|\underline{\epsilon}| = |\underline{\epsilon}'|$. Assuming the latter, the hom space $\text{Hom}(\underline{\epsilon}, \underline{\epsilon}')$ is infinite-dimensional unless $\underline{\epsilon} = \underline{\epsilon}' = \emptyset$ is the empty sequence or if the set S of labels is empty. In the latter case $\mathbf{k}\langle S \rangle \cong \mathbf{k}$ is the ground field. Another special case is when S consists of a single element, $S = \{s\}$, for then $\mathbf{k}\langle S \rangle \cong \mathbf{k}[s]$ is commutative. The endomorphism algebra of $(+)$ in the category \mathcal{VC}_α is $\mathbf{k}\langle S \rangle$, see (15).

Category \mathcal{VC}_α is a \mathbf{k} -linear pre-additive category.

(3) *The skein category \mathcal{SC}_α .* Consider the syntactic ideal $I_\alpha \subset \mathbf{k}\langle S \rangle$ associated to the symmetric series $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle^s$. This ideal has finite codimension iff $\alpha \in \mathbf{k}\langle\langle S \rangle\rangle^{s, \text{rec}}$, that is, if α is, in addition, a recognizable series. Denote by

$$A_\alpha := \mathbf{k}\langle S \rangle / I_\alpha \tag{16}$$

the quotient algebra (the syntactic algebra) of the algebra of noncommutative polynomials by the syntactic ideal. Algebra A_α is finite-dimensional iff α is recognizable. In the latter case, let

$$d_\alpha = \dim_{\mathbf{k}}(A_\alpha) = \text{codim}_{\mathbf{k}}(I_\alpha) \tag{17}$$

be the dimension of the syntactic algebra.

We quotient the category \mathcal{VC}_α of viewable cobordisms by the relation that elements of I_α are zero along any component of a cobordism. Namely, an element of I_α is a finite linear combination

$$u = \sum_{i=1}^k a_i w_i, \quad a_i \in \mathbf{k}, \quad w_i \in S^* \tag{18}$$

of words in the alphabet S . Element $\text{cob}(u)$, see Fig. 2.4.2, can be inserted along any component of a cobordism x . We impose the condition that any such insertion results in the zero morphism in \mathcal{SC}_α between the corresponding sequences $\underline{\epsilon}, \underline{\epsilon}'$. Equivalently, we can set $\text{cob}(u) \in \text{End}((+))$ to zero for all $u \in I_\alpha$ and

Fig. 2.4.3. $\alpha(\widehat{uv}) = \alpha(\widehat{vu}) = 0$ for $u \in I_\alpha$, $v \in \mathbf{k}\langle S \rangle$ since $uv, vu \in I_\alpha$.

take the monoidal closure of the relations $\text{cob}(u) = 0$ for all such u , which is equivalent to the previous condition. Alternatively, we can choose generators $\{u_j\}, j \in J$, for the 2-sided ideal I_α , impose relation $\text{cob}(u_j) = 0, j \in J$ and take their monoidal closure.

Note that relations $\text{cob}(u) = 0$ for $u \in I_\alpha$ are compatible with the evaluation of closed components (circles). Namely, for any $v \in \mathbf{k}\langle S \rangle$, the closures \widehat{uv} and \widehat{vu} define the same element in $\text{End}_{\mathbf{k}\mathcal{C}}(\emptyset)$, namely the circle that carries the box uv or vu , and $\alpha(uv) = \alpha(vu) = 0$, see Fig. 2.4.3. Consequently, no contradiction in evaluation of closed components happens upon introducing these relations.

Denote by \mathcal{SC}_α the resulting quotient category. It has the same objects as \mathcal{VC}_α and additional relations $\text{cob}(u) = 0$ for $u \in I_\alpha$ placed anywhere along one-dimensional S -decorated cobordisms that span hom spaces in \mathcal{VC}_α .

Since relations in the syntactic ideal are imposed along each connected component of a cobordism, an element along a component can be reduced accordingly. Choose a set of elements $B_\alpha \subset \mathbf{k}\langle S \rangle$ that descend to a basis of A_α (if needed, one can choose monomials in S). Modulo I_α , an element of $\mathbf{k}\langle S \rangle$ can be reduced to a linear combination of elements of B_α . Accordingly, we can reduce a morphism in \mathcal{SC}_α to a linear combination of viewable morphisms such that along each component an element of B_α is placed. Call these morphisms *basic* and denote the set of basic morphisms from $\underline{\epsilon}$ to $\underline{\epsilon}'$ by $B_\alpha(\underline{\epsilon}, \underline{\epsilon}')$.

Recall that a morphism from $\underline{\epsilon}$ to $\underline{\epsilon}'$ exists in \mathcal{C} if the two sequences have the same weight, that is, the difference between the number of plus and minus signs in them: $|\underline{\epsilon}| = |\underline{\epsilon}'|$. In the latter case, the number of viewable morphisms (i.e., without circle components) is the number of ways to pair up elements of $\underline{\epsilon}$ and $\underline{\epsilon}'$ in an orientation-respecting way. Reverse the signs in one of the sequences, say in $\underline{\epsilon}$, and concatenate with the other to get $\underline{\epsilon}'(-\underline{\epsilon})$. This sequence has the same number n of plus and minus signs, equal to half the length of the sequence: $2n = ||\underline{\epsilon}|| + ||\underline{\epsilon}'||$. Isomorphism classes of viewable cobordisms from $\underline{\epsilon}$ to $\underline{\epsilon}'$ are in a one-to-one correspondence with bijections between plus and minus signs in $\underline{\epsilon}'(-\underline{\epsilon})$. There are $n!$ such bijections. For each bijection, there are d_α^n ways to assign an element of B_α to each component of a cobordism. The following proposition and corollary result.

Proposition 2.3. *The set of basic morphisms $B_\alpha(\underline{\epsilon}, \underline{\epsilon}')$ is a basis of the hom space $\text{Hom}_{\mathcal{SC}_\alpha}(\underline{\epsilon}, \underline{\epsilon}')$.*

Corollary 1. *Dimensions of hom spaces in \mathcal{SC}_α are given by:*

$$\dim \text{Hom}_{\mathcal{SC}_\alpha}(\underline{\epsilon}, \underline{\epsilon}') = \begin{cases} n! d_\alpha^n & \text{if } |\underline{\epsilon}| = |\underline{\epsilon}'|, 2n = ||\underline{\epsilon}|| + ||\underline{\epsilon}'||, \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

In particular, hom spaces in the category \mathcal{SC}_α are finite-dimensional.

For the endomorphism algebra of the sequence (+) we have (compare with (15))

$$\text{End}_{\mathcal{SC}_\alpha}((+)) \cong A_\alpha, \quad (20)$$

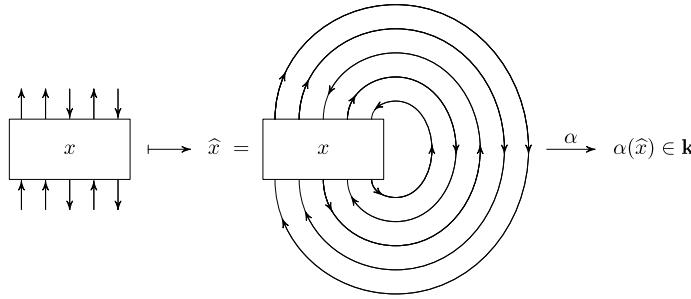


Fig. 2.4.4. The trace map: closing endomorphism x of $\underline{\epsilon}$ into \hat{x} and applying α . In this example $\underline{\epsilon} = (+ + - + -)$.

and the endomorphism algebra of $(+)$ has dimension d_α . Skein category \mathcal{SC}_α is similar to the oriented Brauer category [37], but with lines decorated by elements of S , leading to many choices for evaluations of floating components, one for each sequence in S^* up to the cyclic equivalence.

(4) *Negligible morphisms and gligible quotient \mathcal{C}_α .*

The *trace* $\text{tr}_\alpha(x)$ of a cobordism x from $\underline{\epsilon}$ to $\underline{\epsilon}$ is an element of \mathbf{k} given by closing x via $||\underline{\epsilon}||$ suitably oriented arcs connecting n top with n bottom points of x into a floating cobordism \hat{x} and applying α ,

$$\text{tr}_\alpha(x) := \alpha(\hat{x}).$$

This operation is depicted in Fig. 2.4.4. The trace is extended to all endomorphisms of $\underline{\epsilon}$ in $\mathbf{k}\widetilde{\mathcal{C}}$ by linearity. It is well-defined on trace of endomorphisms of objects $\underline{\epsilon}$ in categories \mathcal{VC}_α and \mathcal{SC}_α as well.

The trace is symmetric, that is $\text{tr}_\alpha(yx) = \text{tr}_\alpha(xy)$ for a morphism x from $\underline{\epsilon}$ to $\underline{\epsilon}'$ and y from $\underline{\epsilon}'$ to $\underline{\epsilon}$. The ideal $J_\alpha \subset \mathcal{SC}_\alpha$ is defined as follows.

A morphism $y \in \text{Hom}(\underline{\epsilon}, \underline{\epsilon}')$ is called *negligible* and belongs to the ideal J_α if $\text{tr}_\alpha(zy) = 0$ for any morphism $z \in \text{Hom}(\underline{\epsilon}', \underline{\epsilon})$. Negligible morphisms constitute a two-sided ideal in the pre-additive category \mathcal{SC}_α . We call J_α the *ideal of negligible morphisms*, relative to the trace form tr_α . Define the quotient category

$$\mathcal{C}_\alpha := \mathcal{SC}_\alpha / J_\alpha.$$

The quotient category \mathcal{C}_α has finite-dimensional hom spaces, as does \mathcal{SC}_α (recall that α is recognizable). The trace is nondegenerate on \mathcal{C}_α and defines perfect bilinear pairings

$$\text{Hom}(\underline{\epsilon}, \underline{\epsilon}') \otimes \text{Hom}(\underline{\epsilon}', \underline{\epsilon}) \longrightarrow \mathbf{k}$$

on its hom spaces. We may call \mathcal{C}_α the *gligible quotient* of \mathcal{SC}_α , having modded out by the ideal of negligible morphisms.

State spaces of recognizable series α . Recall that in the category \mathcal{VC}_α objects are sign sequences $\underline{\epsilon}$ and morphisms are finite linear combinations of viewable cobordisms. The space of homs

$$V_{\underline{\epsilon}} := \text{Hom}_{\mathcal{VC}_\alpha}(\emptyset, \underline{\epsilon}),$$

has a basis of all viewable cobordisms (no floating components) M with $\partial M = \underline{\epsilon}$. This space carries a symmetric bilinear form, given on pairs of basis elements (viewable cobordisms) by

$$(x, y)_{\underline{\epsilon}} := \alpha(\overline{y}x) \in \mathbf{k},$$

where \overline{y} is the reflection of y about a horizontal line combined with the orientation reversal on y , and $\overline{y}x$ is the closed cobordism which is the composition of \overline{y} and x .

Define $A_\alpha(\underline{\epsilon})$ as the quotient of $V_{\underline{\epsilon}}$ by the kernel of this bilinear form. Then there is a canonical isomorphism

$$A_\alpha(\underline{\epsilon}) \cong \text{Hom}_{\mathcal{C}_\alpha}(\emptyset, \underline{\epsilon})$$

as well as isomorphisms

$$A_\alpha((- \underline{\epsilon}) \sqcup \underline{\epsilon}') \cong \text{Hom}_{\mathcal{C}_\alpha}(0, (- \underline{\epsilon}) \sqcup \underline{\epsilon}') \cong \text{Hom}_{\mathcal{C}_\alpha}(\underline{\epsilon}, \underline{\epsilon}')$$

given by moving the bottom boundary $\underline{\epsilon}$ of a cobordism to the top via a cobordism with $\|\underline{\epsilon}\|$ parallel arcs. Here the sequence $(- \underline{\epsilon}) \sqcup \underline{\epsilon}'$ is the concatenation of $- \underline{\epsilon}$ and $\underline{\epsilon}'$.

Note that $\underline{\epsilon}$ must be balanced for $A_\alpha(\underline{\epsilon})$ to be nonzero, that is, $\underline{\epsilon}$ must have the same number n of pluses and minuses. We can then define

$$A_\alpha(n) := A_\alpha((+^n -^n)). \quad (21)$$

Spaces $A_\alpha(n)$ come with a lot of structure, including multiplication maps

$$A_\alpha(n) \otimes A_\alpha(m) \longrightarrow A_\alpha(n + m).$$

We have $A_\alpha(0) \cong \mathbf{k}$ and $A_\alpha(1) \cong A_\alpha$. Vector space $A_\alpha(n)$ carries an action of the symmetric group product $S_n \times S_n$ by the permutation cobordisms, as well as an action of the tensor power of the syntactic algebra $A_\alpha^{\otimes n} \otimes (A_\alpha^{\text{op}})^{\otimes n}$, with one copy of $A_\alpha \cong \text{End}_{\mathcal{C}_\alpha}((+))$ or $A_\alpha^{\text{op}} \cong \text{End}_{\mathcal{C}_\alpha}((-))$ acting at each sign of $+^n -^n$. More generally, a version of the oriented walled Brauer algebra with strands carrying S -labelled dots and closed decorated circles evaluating via α acts on $A_\alpha(n)$ and, more generally, on $\text{Hom}_{\mathcal{C}_\alpha}(\underline{\epsilon}, +^n -^n)$ for any sign sequence $\underline{\epsilon}$. This generalized walled Brauer algebra $\text{Br}_{n,\alpha}$ is straightforward to define. It is associated to any recognizable series α , finite-dimensional, and isomorphic to the endomorphism algebra $\text{End}_{\mathcal{SC}_\alpha}((+^n -^n))$ of the object $(+^n -^n)$ in the skein category \mathcal{SC}_α . The action of $\text{Br}_{n,\alpha}$ on $\text{Hom}_{\mathcal{C}_\alpha}(\underline{\epsilon}, (+^n -^n))$ descends to the action of its quotient algebra $\text{End}_{\mathcal{C}_\alpha}((+^n -^n))$ on the same space.

Multiplication maps turn the direct sum

$$A_\alpha^* := \bigoplus_{n \geq 0} A_\alpha(n) \quad (22)$$

into a graded associative \mathbf{k} -algebra, with compatible actions of S_n on $A_\alpha(n)$ over all n , making A_α^* into what Sam and Snowden call a *twisted commutative algebra* or *tca* in [48, Definition 7.2.1]. A twisted commutative algebra in that sense may be very far from being commutative: for instance, the free associative algebra (the tensor algebra of a vector space) has the obvious tca structure [48, Example 7.2.2]. More generally, given an n -dimensional topological theory α as defined in [24], perhaps for manifolds with defects, etc. and an $(n-1)$ -manifold N , the direct sum

$$A^*(N) := \bigoplus_{n \geq 0} \alpha(\sqcup_n N)$$

of state spaces of disjoint unions of n copies of N , over all n , is naturally a tca in the sense of [48].

(5) *The Deligne category \mathcal{DC}_α and its gluable quotient $\underline{\mathcal{DC}}_\alpha$.* The skein category \mathcal{SC}_α is a rigid symmetric monoidal \mathbf{k} -linear category with signed sequences $\underline{\epsilon}$ as objects and finite-dimensional hom spaces. We form the additive Karoubi closure

$$\mathcal{DC}_\alpha := \text{Kar}(\mathcal{SC}_\alpha^\oplus)$$

by allowing formal finite direct sums of objects in \mathcal{SC} , extending morphisms correspondingly, and then adding idempotents to get a Karoubi-closed category. Category \mathcal{DC}_α plays the role of the Deligne category in our construction.

The trace tr_α extends to \mathcal{DC}_α and defines a 2-sided ideal $\mathcal{DJ}_\alpha \subset \mathcal{DC}_\alpha$ of negligible morphisms relative to tr_α . Define the *glible quotient* category by

$$\underline{\mathcal{DC}}_\alpha := \underline{\mathcal{DC}}_\alpha / \mathcal{DJ}_\alpha.$$

This category is equivalent to the additive Karoubi envelope of \mathcal{C}_α . It is a Karoubi-closed rigid symmetric category with non-degenerate bilinear forms on its hom spaces.

2.5. Summary of categories and functors

Here is the summary of the categories that have been introduced.

- \mathcal{C} : the category of S -decorated one-dimensional cobordisms. Its objects are sequences $\underline{\epsilon}$ of plus and minus signs and morphisms are one-manifolds with boundary decorated by S -labelled dots. That is, the morphisms are S -decorated one-manifolds with boundary.
- $\mathbf{k}\mathcal{C}$: this category has the same objects as \mathcal{C} ; its morphisms are formal finite \mathbf{k} -linear combinations of morphisms in \mathcal{C} .
- \mathcal{VC}_α : in this quotient category of $\mathbf{k}\mathcal{C}$ we reduce morphisms to linear combinations of viewable cobordisms. Floating connected components (circles, possibly carrying S -dots) are removed by evaluating them via α .
- \mathcal{SC}_α : to define this category, specialize to rational α and add skein relations by modding out by elements of the ideal I_α in $\mathbf{k}[S]$, along each component of the cobordism. Hom spaces in this category are finite-dimensional.
- \mathcal{C}_α : the quotient of \mathcal{SC}_α by the ideal J_α of negligible morphisms. This category is also equivalent (even isomorphic) to the quotients of $\mathbf{k}\mathcal{C}$ and \mathcal{VC}_α by the corresponding ideals of negligible morphisms in them. The trace pairing in \mathcal{C}_α between $\text{Hom}(n, m)$ and $\text{Hom}(m, n)$ is perfect.
- \mathcal{DC}_α is the analogue of the Deligne category obtained from \mathcal{SC}_α by allowing finite direct sums of objects and then adding idempotents as objects to get a Karoubi-closed category.
- $\underline{\mathcal{DC}}_\alpha$: the quotient of \mathcal{DC}_α by the two-sided ideal of negligible morphisms. This category is equivalent to the additive Karoubi closure of \mathcal{C}_α and sits in the bottom right corner of the commutative square below.

We arrange these categories and functors, for recognizable α , into the following diagram:

$$\begin{array}{ccccccc}
 \mathcal{C} & \longrightarrow & \mathbf{k}\mathcal{C} & \longrightarrow & \mathcal{VC}_\alpha & \longrightarrow & \mathcal{SC}_\alpha \longrightarrow \mathcal{DC}_\alpha \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{C}_\alpha & \longrightarrow & \underline{\mathcal{DC}}_\alpha
 \end{array} \tag{23}$$

All seven categories are rigid symmetric monoidal. All but the leftmost category \mathcal{C} are \mathbf{k} -linear. Except for the two categories on the far right, the objects of each category are sequences $\underline{\epsilon}$ of plus and minus signs. The four categories on the right all have finite-dimensional hom spaces. The two categories on the far right are additive and Karoubi-closed. The four categories in the middle of the diagram are pre-additive but not additive.

The arrows show functors between these categories considered in the paper. The square is commutative. An analogous diagram of functors and categories can be found in [31] for the category of oriented 2D

cobordisms in place of \mathcal{C} and in [28] for suitable categories of oriented 2D cobordisms with side boundary and corners.

For convenience, one- or two-word summaries of these categories are provided below, in the diagram essentially identical to that in [28, Section 3.4]:

$$\begin{array}{ccccccc}
 \text{S-dotted} & \longrightarrow & \mathbf{k}\text{-linear} & \longrightarrow & \text{viewable} & \longrightarrow & \text{skein} \longrightarrow \text{Deligne (Karoubian)} \\
 \text{cobordisms} & & & & & & \\
 & & & & & \downarrow & \downarrow \\
 & & & & & \text{glible} & \longrightarrow \text{glible and Karoubian}
 \end{array} \tag{24}$$

It is possible to go directly from $\mathbf{k}\mathcal{C}$ to \mathcal{C}_α by modding out by the ideal of negligible morphisms in the former category. It is convenient to arrive at this quotient in several steps, introducing categories \mathcal{VC}_α and \mathcal{SC}_α on the way.

If α is not recognizable, we can still define categories \mathcal{VC}_α , \mathcal{C}_α and $\underline{\mathcal{DC}}_\alpha$, but then, for instance, one can potentially get two non-equivalent categories in place of $\underline{\mathcal{DC}}_\alpha$ by following along the two different paths in the square above. To justify considering these categories for some non-recognizable α one would want to find interesting examples where the glible quotient category \mathcal{C}_α has additional relations beyond those in \mathcal{SC}_α , that is, beyond the relations that elements of the syntactic ideal I_α are zero in $\text{End}(+)$ in \mathcal{SC}_α and \mathcal{C}_α .

2.6. Examples and variations of the construction

An involution. Categories \mathcal{C} and $\mathbf{k}\mathcal{C}$ carry contravariant involution $\bar{}$ that reflects the cobordism about the middle, reversing its source and target objects, and reverses the orientation of the cobordism. This involution takes the object $\underline{\mathfrak{c}}$ to $-\underline{\mathfrak{c}}$, that is, reverses the sign (orientation) of boundary zero-manifolds as well. To match this involution to evaluation α , assume that \mathbf{k} comes with an involution, also denoted $\bar{}$, and α satisfies $\overline{\alpha(w)} = \alpha(\bar{w})$, where $\bar{w} = t_n \dots t_1$ is the word $w = t_1 \dots t_n$ in reverse. Then there are induced contravariant involutions on all the categories associated to α and displayed in diagram (23), and one can, for instance, study such unitary 1D topological theories, with the set of defects S , for $\mathbf{k} = \mathbb{C}$ and $\bar{}$ the complex involution.

Examples.

(1) If the set $S = \emptyset$ is empty, there are no decorations and the series α is given by its value on the empty sequence, that is, by its constant term, and we can write $\alpha = \lambda \in \mathbf{k}$ for that value. A circle cobordism evaluates to λ . The skein category \mathcal{SC}_λ is isomorphic to the viewable category \mathcal{VC}_λ and to the oriented Brauer category B_λ for the parameter λ . Category \mathcal{C}_λ is then the quotient of B_λ by the ideal of negligible morphisms, while \mathcal{DC}_λ is the additive Karoubi closure of B_λ , etc. Note that these categories depend both on the field \mathbf{k} and $\lambda \in \mathbf{k}$.

(2) If $S = \{s\}$ is a one-element set, the series α is a one-variable series, with the generating function

$$Z_\alpha(T) = \sum_{n \geq 0} \alpha_n T^n, \quad \alpha_n = \alpha(s^n). \tag{25}$$

α is recognizable iff $Z_\alpha(T)$ is a rational function, with $Z_\alpha(T) = P(T)/Q(T)$ for some polynomials $P(T), Q(T)$.

This example is similar to the ones in [24,31], where the topological theory is 2-dimensional but there are no defects. The analogue of the Hankel matrix measuring bilinear pairing on connected cobordisms with the

boundary \mathbb{S}^1 (such cobordisms are determined by the genus g), see [24], is the Hankel matrix for evaluations of $\overline{x_m}x_n$ where x_n is an arc with n dots, viewed as a cobordism from \emptyset to $(+-)$. Cobordism $\overline{x_m}$ from $(+-)$ to \emptyset is an arc with m dots. Closed cobordism $\overline{x_m}x_n$ is a circle with $n+m$ dots, and once again the Hankel matrix H with the (n, m) -entry α_{n+m} results, as in [24]. In both cases the state space (of $(+-)$, respectively of \mathbb{S}^1) is the quotient of \mathbb{R}^N by the null space of H .

The theories diverge beyond this example, but there is another connection between the two, slightly different from the one above due to an additional shift in the dots versus handles correspondence between 1D and 2D cobordisms. Namely, the state space of $(+^k -^k)$ in the one-dimensional theory with the series α maps to the state space for the union $\sqcup_k \mathbb{S}^1$ of k circles in the two-dimensional theory for the series $\alpha' = (a, \alpha_0, \alpha_1, \dots)$ for any $a \in \mathbf{k}$. In terms of generating functions, $Z_{\alpha'} = a + Z_\alpha T$. On the topological side, an arc with n dots is mapped to an annulus with n handles, while a circle carrying n dots is mapped to the torus with additional n handles (thus a surface of genus $n+1$). This shift from n to $n+1$ accounts for the discrepancy between the series but does not change their recognizability. The map from the state spaces in the 1D theory to the state spaces in the 2D theory respects the bilinear forms on these spaces.

Partial fraction decomposition method of [27] can be applied in this case as well to understand the categories associated to α . When the set S has more than one element, recognizable power series still admit an analogue of the partial fraction decomposition, see [18] and references therein, which should lead to decompositions of associated tensor categories.

Unoriented cobordisms. There is an obvious unoriented version of the category \mathcal{C} , where one-dimensional cobordisms are unoriented and the objects, in the skeletal category case, are numbers $n \in \mathbb{Z}_+$, counting the number of top and bottom endpoints of the cobordism. Evaluation α must be \neg -invariant, that is, to satisfy $\alpha(\overline{w}) = \alpha(w)$, for any word w , in addition to being symmetric, as earlier: $\alpha(w_1 w_2) = \alpha(w_2 w_1)$, for any words w_1, w_2 . The dihedral group D_n acts on the set S^n of words of length n in the alphabet S , and the function $\alpha : S^* \rightarrow \mathbf{k}$, when restricted to these words, must be D_n -invariant. Such series can be called *d-symmetric*, for instance.

The theory then goes through and one can define the viewable category \mathcal{VC}_α , the skein category \mathcal{SC}_α , the gligible quotient \mathcal{C}_α , and so on. The interesting case, as before, is when α is *recognizable*, that is, when the category \mathcal{C}_α has finite-dimensional hom spaces. A d-symmetric series α is recognizable iff it is recognizable as noncommutative series iff the syntactic ideal I_α has finite codimension in $\mathbf{k}\langle S \rangle$.

If the set S is empty, cobordisms do not carry any dots (defects), and the category \mathcal{SC}_α is the unoriented Brauer category Br_λ^{un} for the parameter $\lambda = \alpha(1) \in \mathbf{k}$, while \mathcal{C}_α is the gligible quotient of B_λ^{un} .

3. Cobordisms with inner (floating) boundary

3.1. Category $\tilde{\mathcal{C}}$ of decorated cobordisms with inner boundary

To connect decorated one-dimensional cobordisms with noncommutative rational power series that are not necessarily symmetric we enlarge the category \mathcal{C} by allowing cobordisms M that may have additional boundary points (*floating boundary* points) strictly inside the cobordism, not being part of the top $\partial_1 M$ or bottom $\partial_0 M$ boundary of M .

Define the category $\tilde{\mathcal{C}}$ of S -labelled cobordism with floating (or inner) boundary to have the same objects as \mathcal{C} , that is, finite sequences $\underline{\epsilon}$ of plus and minus signs. A morphism in $\tilde{\mathcal{C}}$ from $\underline{\epsilon}$ to $\underline{\epsilon}'$, see Fig. 3.1.1 for an example, is a compact oriented S -decorated one-manifold M with

$$\partial M = \underline{\epsilon}' \sqcup (-\underline{\epsilon}) \sqcup \partial_{in} M, \quad (26)$$

where $\partial_{in} M$ is the *inner* or *floating* boundary of M that is disjoint from the top boundary, given by $\underline{\epsilon}'$ and from the bottom boundary, given by $-\underline{\epsilon}$. In (26) we interpret a sign sequence $\underline{\epsilon}'$ as a zero-dimensional

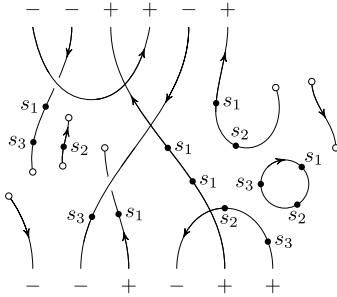


Fig. 3.1.1. A morphism from $(- - + -++)$ to $(- - + + -+)$. Over- and undercrossing and intersections are “virtual” and should be ignored. Hollow dots are not labels and show inner (floating) endpoints of the cobordism.

oriented manifold, with oriented connected components described by elements of the sequence. The sequence $-\underline{\epsilon}$ opposite to $\underline{\epsilon}$ corresponds to the orientation reversal of 0D manifold $\underline{\epsilon}$. An S -decoration is a collection of points (dots) labelled by elements of the set S inside M (not on the boundary ∂M). Labelled points can move along a connected component but not cross through each other.

Morphisms are such decorated 1D cobordisms, possibly with inner endpoints (inner boundary points) considered up to rel boundary diffeomorphisms. Fig. 3.1.1 shows an example of a morphism from $\underline{\epsilon} = (- - + -++)$ to $\underline{\epsilon}' = (- - + + -+)$.

Composition of morphisms in $\tilde{\mathcal{C}}$ is given by concatenation of cobordisms. The category $\tilde{\mathcal{C}}$ contains \mathcal{C} as the subcategory with the same objects as $\tilde{\mathcal{C}}$ and morphisms – morphisms of $\tilde{\mathcal{C}}$ that have no inner (floating) boundary points.

Connected components of a cobordism in $\tilde{\mathcal{C}}$ split into *viewable* and *floating* types. Fig. 3.1.1 cobordism has three floating components: one circle and two intervals. The same cobordism has eight viewable components: four of them have both endpoints on top or bottom boundary, while the other four have one floating endpoint. Floating components terminology was introduced in [29].

Going along a component c in the direction of its orientation we read off the labels of dots. If the component is an arc, the result is a sequence $\text{sec}(c) \in S^*$, a word in the alphabet S . If the component is a circle, the sequence $\text{sec}(c)$ is defined up to cyclic rotation. Our convention is to write the sequence from right to left as we follow the orientation. For instance, in Fig. 3.1.1 left the sequence is $s_1s_3s_2$, while in Fig. 3.1.1 right the sequence is $s_3s_2s_1$. Orientation reversal of a component corresponds to reversing the sequence.

The sequences for components of Fig. 3.1.1 cobordism are:

- The empty sequence (\emptyset) and (s_2) for the two floating arc components.
- Sequence $(s_1s_3s_2)$, up to cyclic rotation, for the unique floating circle component.
- Sequences (s_1s_1) and (s_3) for the two connected components that connect a top endpoint and a bottom endpoint.
- The empty sequence (\emptyset) for the unique component that connects two top endpoints.
- Sequence (s_2s_3) for the unique component connecting two bottom endpoints.
- Sequences (s_3s_1) and (s_1s_2) for arc components with one top and one inner boundary point.
- Sequences (\emptyset) and (s_1) for arc components with one bottom and one inner boundary point.

A floating component of a cobordism x in $\tilde{\mathcal{C}}$ is either an interval or a circle, see Fig. 3.1.2.

A viewable component has one of the five types shown in Fig. 3.1.3, with some number of dots (perhaps none) on it.

Monoidal category $\tilde{\mathcal{C}}$ has generators shown in Figs. 2.1.3 and 2.1.4 and common with its subcategory \mathcal{C} and two additional generators shown in Fig. 3.1.4 left. These are arcs with one floating and one top endpoint. Applying U-turns to them results in arcs with one floating and one bottom endpoint, see Fig. 3.1.4 right.

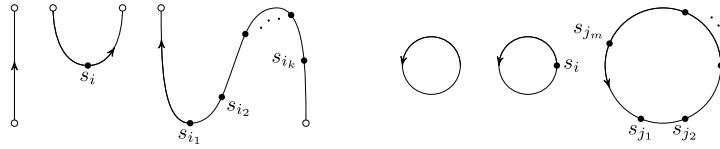


Fig. 3.1.2. Left: three interval floating components, with sequences (\emptyset) , (s_i) , and $(s_{i_1} s_{i_2} \dots s_{i_k})$. Right: three circle components, with sequences (\emptyset) , (s_i) and $(s_{j_m} \dots s_{j_2} s_{j_1})$ up to cyclic rotation.

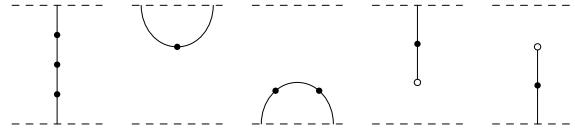


Fig. 3.1.3. Five types of viewable components, left to right: an interval connecting (1) a top and a bottom point, (2) two top points, (3) two bottom points; a interval with an inner boundary point and a (4) top endpoint, (5) bottom endpoint. Labels of dots and orientations of lines are not shown. In the subcategory \mathcal{C} viewable components are of types (1)-(3) only.

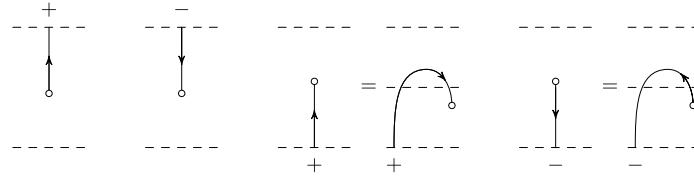


Fig. 3.1.4. Left: additional generating morphisms for monoidal category $\tilde{\mathcal{C}}$ beyond the generating morphisms common for $\tilde{\mathcal{C}}$ and \mathcal{C} shown in Figs. 2.1.3 and 2.1.4.

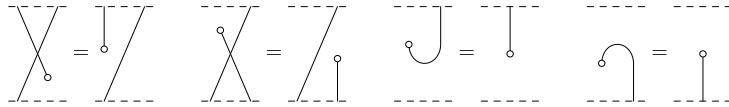


Fig. 3.1.5. Some additional relations in $\tilde{\mathcal{C}}$.

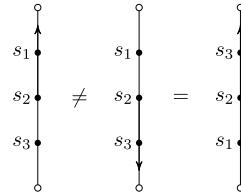


Fig. 3.1.6. Orientation matters: evaluations $\alpha(s_1 s_2 s_3)$ and $\alpha(s_3 s_2 s_1)$ are different, in general. Reversal of a sequence corresponds to orientation reversal of the corresponding floating arc or circle.

Some additional defining relations in $\tilde{\mathcal{C}}$ are shown in Fig. 3.1.5, see also Fig. 2.1.4 for defining relations in the subcategory \mathcal{C} , which also give a subset of defining relations in $\tilde{\mathcal{C}}$. We will not need a full set of defining relations for $\tilde{\mathcal{C}}$ in this paper.

3.2. Tensor envelopes of $\tilde{\mathcal{C}}$

Floating (closed) cobordisms in $\tilde{\mathcal{C}}$ (endomorphisms of the empty zero-manifold (\emptyset)) are unions of floating intervals and circles. A floating interval carries a sequence $w \in S^*$, a floating circle carries a sequence v well-defined up to cyclic rotation, $v_1 v_2 \equiv v_2 v_1$. Consequently a multiplicative evaluation of floating cobordisms in $\tilde{\mathcal{C}}$, as explained in the introduction, consists of a pair of series

$$\alpha = (\alpha^\bullet, \alpha^\circ), \quad (27)$$

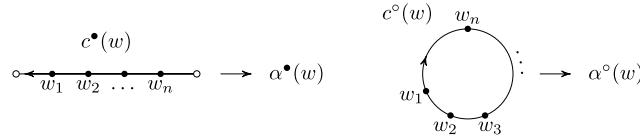


Fig. 3.2.1. Evaluation $\alpha^•(w)$ of the floating interval $c^•(w)$ and evaluation $\alpha^o(w)$ of the circle $c^o(w)$ in \mathcal{VC} , for a word $w = w_1 \dots w_n$.

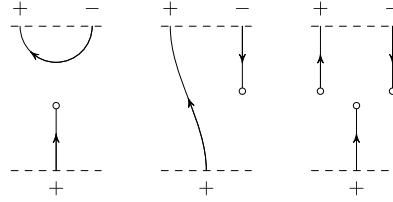


Fig. 3.2.2. Three elements of the set $\mathcal{I}(+, +-)$.

where $\alpha^• \in \mathbf{k}\langle\langle S \rangle\rangle$ is a noncommutative series and $\alpha^o \in \mathbf{k}\langle\langle S \rangle\rangle^s$ is a symmetric series.

A multiplicative evaluation on closed cobordisms in $\tilde{\mathcal{C}}$ assigns $\alpha^•(w) \in \mathbf{k}$ to an oriented interval with word w written along it via labelled dots, see Fig. 3.2.1. Element $\alpha^o(v) \in \mathbf{k}$ is assigned to an oriented circle with word v , well-defined up to a cyclic rotation, written along it.

We now proceed along a familiar route, as in [31, 28] and Section 2, to build various tensor envelopes of a pair $\alpha = (\alpha^•, \alpha^o)$.

(1) *Pre-linearization category $\mathbf{k}\tilde{\mathcal{C}}$.* Category $\mathbf{k}\tilde{\mathcal{C}}$ has the same objects as $\tilde{\mathcal{C}}$, and the morphisms are finite \mathbf{k} -linear combinations of morphisms in $\tilde{\mathcal{C}}$. This is a naive *linearization* or *pre-linearization* of $\tilde{\mathcal{C}}$.

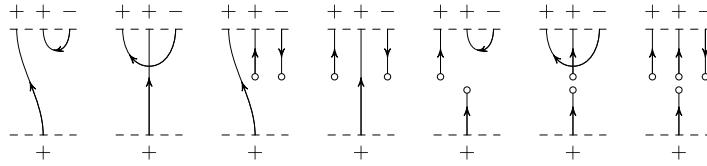
(2) *Viewable cobordisms category \mathcal{VC}_α .* To form category \mathcal{VC}_α , we mod out tensor category $\mathbf{k}\tilde{\mathcal{C}}$ by relations that evaluate floating (closed) cobordisms to elements of the ground field via α . Namely, a floating oriented interval with a sequence $w \in S^*$ on it, denoted $c^•(w)$, evaluates to $\alpha^•(w) \in \mathbf{k}$. A floating oriented w -decorated circle $c^o(w)$ evaluates to $\alpha^o(w)$, see Fig. 3.2.1. Recall that α^o is symmetric and $\alpha^o(v_1v_2) = \alpha^o(v_2v_1)$ for any words v_1v_2 , matching circle rotation, $c^o(v_1v_2) = c^o(v_2v_1)$.

Since all floating components of a cobordism reduce to elements in \mathbf{k} , the vector space of homs from $\underline{\epsilon}$ to $\underline{\epsilon}'$ in \mathcal{VC}_α has a basis of viewable cobordisms from $\underline{\epsilon}$ to $\underline{\epsilon}'$ with any sequences written on its connected components.

Denote by $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$ the set of diffeomorphism classes of viewable cobordisms (without dot decorations) from $\underline{\epsilon}$ to $\underline{\epsilon}'$. A viewable cobordism has no circles and all its connected components are intervals. Such a cobordism C may have some number of viewable components of types (4) and (5), see Fig. 3.1.3. Each such component has one floating boundary point and one boundary point among elements of $\underline{\epsilon} \sqcup \underline{\epsilon}'$. Other connected components (of types (1)-(3)) give an orientation-respecting matching of the remaining elements of $\underline{\epsilon}$ and $\underline{\epsilon}'$.

To specify an element of $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$ we select a subset I' of elements in the sequence $(-\underline{\epsilon}) \sqcup \underline{\epsilon}'$ so that the remaining sequence is *balanced*, that is, has the same number of pluses and minuses. We then choose a bijection b between pluses and minuses of $(-\underline{\epsilon}) \sqcup \underline{\epsilon}' \setminus I'$. Such pairs (I', b) are in a bijection with isomorphism classes of viewable undecorated cobordisms between $\underline{\epsilon}$ and $\underline{\epsilon}'$, that is, elements of $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$. Fig. 3.2.2 shows elements of the set $\mathcal{I}(+, +-)$. Fig. 3.2.3 shows elements of the set $\mathcal{I}(+, + + -)$.

To allow S -decorations, we consider the set $\mathcal{I}^S(\underline{\epsilon}, \underline{\epsilon}')$ which consists of a pair: an element of $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$ and a choice of word $w(c)$ in S^* for each component c of $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$. To such a pair we assign an S -decorated viewable cobordism given by the element of $\mathcal{I}(\underline{\epsilon}, \underline{\epsilon}')$ and words $w(c)$ written on components c of the cobordism. Some words may be empty (have length zero). Denote the cobordism associated with $t \in \mathcal{I}^S(\underline{\epsilon}, \underline{\epsilon}')$ by $C(t)$.

Fig. 3.2.3. Seven elements of the set $\mathcal{I}(+, + + -)$.

Proposition 3.1. *Viewable cobordisms $C(t)$, over all t in $\mathcal{I}^S(\underline{\epsilon}, \underline{\epsilon}')$, constitute a basis in the hom space $\text{Hom}(\underline{\epsilon}, \underline{\epsilon}')$ in the category \mathcal{VC}_{α} .*

Composing cobordisms from these bases sets results in cobordisms that, in general, have floating components. These components are evaluated via α , viewable components are kept, and the composition of two basis elements is a basis element in a suitable hom space, scaled by an element of \mathbf{k} .

As earlier, to each word $w \in S^*$ we associate the upward interval with w written on it, that is, cobordism $\text{cob}(w)$ from $(+)$ to $(+)$, see Fig. 2.4.2 left. Each element u of $\mathbf{k}\langle S \rangle$ gives rise to a linear combination $\text{cob}(u)$ of these cobordisms, see Fig. 2.4.2 right. The resulting map

$$\text{cob} : \mathbf{k}\langle S \rangle \longrightarrow \text{End}_{\mathcal{VC}_{\alpha}}((+))$$

is an injective homomorphism from the free algebra $\mathbf{k}\langle S \rangle$ to the ring of endomorphisms of $(+)$ in category \mathcal{VC}_{α} (in the smaller category \mathcal{VC}_{α} considered in Section 2 this map is an isomorphism). One-sided inverse homomorphism to cob is given by the surjection

$$\text{cob}' : \text{End}_{\mathcal{VC}_{\alpha}}((+)) \longrightarrow \mathbf{k}\langle S \rangle$$

that sends any cobordism with floating endpoints to zero. The latter cobordisms span a two-sided ideal in $\text{End}_{\mathcal{VC}_{\alpha}}((+))$, with the quotient isomorphic to $\mathbf{k}\langle S \rangle$. This ideal is naturally isomorphic to $\mathbf{k}\langle S \rangle \otimes \mathbf{k}\langle S \rangle$ when viewed as a $\mathbf{k}\langle S \rangle$ -bimodule. Multiplication in this ideal is given by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \alpha^{\bullet}(x_2 y_1) x_1 \otimes y_2.$$

Composition $\text{cob}' \circ \text{cob} = \text{Id}_{\mathbf{k}\langle S \rangle}$.

(3) *Skein category \mathcal{SC}_{α} .* This category has finite-dimensional hom spaces when α is recognizable, and we restrict to that case. We say that $\alpha = (\alpha^{\bullet}, \alpha^{\circ})$ is *recognizable* if both series α^{\bullet} and α° are recognizable. Series α^{\bullet} and α° has syntactic ideals $I_{\alpha^{\bullet}}, I_{\alpha^{\circ}} \subset \mathbf{k}\langle S \rangle$, respectively. Recognizability means that both ideals have finite codimension in $\mathbf{k}\langle S \rangle$. Equivalently, the two-sided ideal

$$I_{\alpha} := I_{\alpha^{\bullet}} \cap I_{\alpha^{\circ}} \subset \mathbf{k}\langle S \rangle \quad (28)$$

has finite codimension in $\mathbf{k}\langle S \rangle$. Denote by

$$A_{\alpha} := \mathbf{k}\langle S \rangle / I_{\alpha}$$

the syntactic algebra of the pair α .

Starting with the category \mathcal{VC}_{α} , we add tensor relations $\text{cob}(u) = 0$ for any $u \in I_{\alpha}$, see Fig. 3.2.4 left. These relations are consistent with the evaluation α of floating components. Consistency is due to restricting to u in the syntactic ideal, which is contained in both ideals $I_{\alpha^{\bullet}}$ and $I_{\alpha^{\circ}}$. Elements of the first ideal evaluate

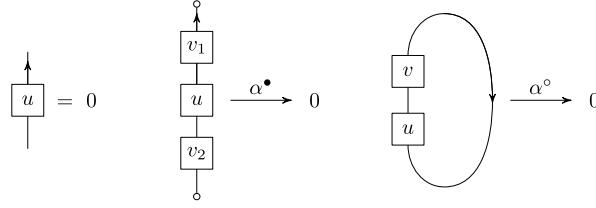


Fig. 3.2.4. Left: endomorphism $\text{cob}(u)$ of $(+)$ is set to zero in \mathcal{SC}_{α} for $u \in I_{\alpha}$. Middle: $\alpha^{\bullet}(v_1 u v_2) = 0$ for any $v_1, v_2 \in \mathbf{k}\langle S \rangle$ since $u \in I_{\alpha} \subset I_{\alpha^{\bullet}}$. Right: $\alpha^{\circ}(v u) = 0$ for any $v \in \mathbf{k}\langle S \rangle$ since $u \in I_{\alpha} \subset I_{\alpha^{\circ}}$.

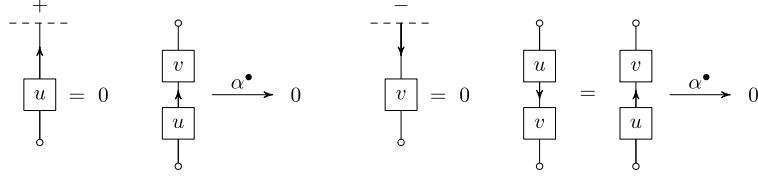


Fig. 3.2.5. Left: Element $\text{cob}^+(u)$ of $\text{Hom}(\emptyset, (+))$ is set to zero in \mathcal{SC}_{α} for $u \in I_{\alpha^{\bullet}}^{\ell}$. This relation is compatible with evaluations of floating diagrams, since $\alpha^{\bullet}(vu) = 0 \forall v \in \mathbf{k}\langle S \rangle$, second left. Right: Element $\text{cob}^-(v)$ of $\text{Hom}(\emptyset, (-))$ is set to zero in \mathcal{SC}_{α} for $v \in I_{\alpha^{\bullet}}^r$. This relation is also compatible with evaluations of floating diagrams, since $\alpha^{\bullet}(vu) = 0 \forall u \in \mathbf{k}\langle S \rangle$ for such v , see the last equation on the right.

to zero when placed anywhere on a floating interval, see Fig. 3.2.4 middle. Elements of the second ideal evaluate to zero when placed on a circle, see Fig. 3.2.4 right.

Recall that in addition to two-sided syntactic ideals $I_{\alpha^{\bullet}}, I_{\alpha^{\circ}}$ and their intersection $I_{\alpha} = I_{\alpha^{\bullet}} \cap I_{\alpha^{\circ}}$ there are one-sided syntactic ideals $I_{\alpha^{\bullet}}^{\ell}$ and $I_{\alpha^{\bullet}}^r$. Here

$$I_{\alpha^{\bullet}}^{\ell} = \{x \in \mathbf{k}\langle S \rangle \mid \alpha^{\bullet}(yx) = 0 \forall y \in \mathbf{k}\langle S \rangle\} \quad \text{and} \quad I_{\alpha^{\bullet}}^r = \{x \in \mathbf{k}\langle S \rangle \mid \alpha^{\bullet}(xy) = 0 \forall y \in \mathbf{k}\langle S \rangle\}$$

are left and right ideals in $\mathbf{k}\langle S \rangle$, respectively.

For $u \in \mathbf{k}\langle S \rangle$ denote by $\text{cob}^+(u)$ the element of $\text{Hom}(\emptyset, (+))$ given by putting u on an interval at its “out” floating endpoint, see Fig. 3.2.5 left. Define $\text{cob}^-(v)$ likewise, see Fig. 3.2.5 right. We add relations that $\text{cob}^+(u) = 0$ for $u \in I_{\alpha^{\bullet}}^{\ell}$ and $\text{cob}^-(v) = 0$ for $v \in I_{\alpha^{\bullet}}^r$. This finishes our definition of category \mathcal{SC}_{α} .

Note that ideals $I_{\alpha}, I_{\alpha^{\bullet}}^{\ell}, I_{\alpha^{\bullet}}^r$ have finite codimensions in $\mathbf{k}\langle S \rangle$, and each of these ideals is finitely-generated. In particular, one can restrict to adding finitely many relations to \mathcal{VC}_{α} to get the “skein” category \mathcal{SC}_{α} .

Due to consistency of these relations with the evaluation α on floating components we can describe a basis in the hom spaces in the category \mathcal{SC}_{α} , as follows. Choose subsets $B_{\alpha}, B_{\alpha^{\bullet}}^{\ell}, B_{\alpha^{\bullet}}^r \subset \mathbf{k}\langle S \rangle$ that descend to bases of $A_{\alpha}, \mathbf{k}\langle S \rangle/I_{\alpha^{\bullet}}^{\ell}$ and $I_{\alpha^{\bullet}}^r/\mathbf{k}\langle S \rangle$, respectively.

Recall the basis $\mathcal{I}^S(\underline{\epsilon}, \underline{\epsilon}')$ of the hom space from $\underline{\epsilon}$ to $\underline{\epsilon}'$ in \mathcal{VC}_{α} constructed earlier. It consists of a floating cobordism x from $\underline{\epsilon}$ to $\underline{\epsilon}'$ with various monomials written on components of the cobordism (all components are viewable). Define the set $B_{\alpha}(\underline{\epsilon}, \underline{\epsilon}')$ to also consist of floating cobordisms from $\underline{\epsilon}$ to $\underline{\epsilon}'$, but now we write an element of one of the three sets $B_{\alpha}, B_{\alpha^{\bullet}}^{\ell}, B_{\alpha^{\bullet}}^r$ on each component of c , depending on its type:

- If a component has no floating endpoints, thus connects two boundary points (at the top or bottom boundary, or both), put an element of B_{α} along it.
- If a component has a floating endpoint and is oriented away from this endpoint, put an element of $B_{\alpha^{\bullet}}^{\ell}$ along this component.
- If a component has a floating endpoint and is oriented towards it, put an element of $B_{\alpha^{\bullet}}^r$ along this component.

An undecorated viewable cobordism x with n_1, n_2, n_3 components of these three types, respectively, admits $|B_{\alpha}|^{n_1} |B_{\alpha^{\bullet}}^{\ell}|^{n_2} |B_{\alpha^{\bullet}}^r|^{n_3}$ possible decorations. The set $B_{\alpha}(\underline{\epsilon}, \underline{\epsilon}')$ is the union of these decorated cobordisms,

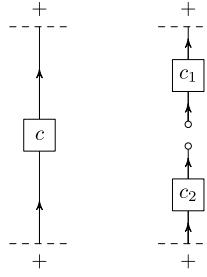


Fig. 3.2.6. There are two types of cobordisms from $(+)$ to $(+)$ in $\tilde{\mathcal{C}}$. Mirroring that decomposition, basis $B_\alpha((+),(+))$ consists of elements of $c \in B_\alpha$ placed on through strand and pairs of elements $c_1 \in B_\alpha^\ell$ and $c_2 \in B_\alpha^r$ placed on the two strands with floating endpoints.

where we start with any viewable undecorated cobordism x from $\underline{\epsilon}$ to $\underline{\epsilon}'$ and decorate it in all possible such ways. The set $B_\alpha(\underline{\epsilon}, \underline{\epsilon}')$ is finite.

For example, $B_\alpha((+),(+))$ has cardinality $|B_\alpha| + |B_\alpha^\ell| \cdot |B_\alpha^r|$ and consists of diagrams of two types, see Fig. 3.2.6.

Proposition 3.2. *The set $B_\alpha(\underline{\epsilon}, \underline{\epsilon}')$ is a basis of the hom space $\text{Hom}_{S\tilde{\mathcal{C}}_\alpha}(\underline{\epsilon}, \underline{\epsilon}')$ in the category $S\tilde{\mathcal{C}}_\alpha$.*

This construction gives a basis in hom spaces of $S\tilde{\mathcal{C}}_\alpha$ for non-recognizable α as well, but then A_α and hom spaces are infinite-dimensional. Recall that we restrict to considering recognizable α for most of this section.

The endomorphism ring of $(+)$ in $S\tilde{\mathcal{C}}_\alpha$ contains a two-sided ideal isomorphic to the tensor product $\mathbf{k}\langle S \rangle / I_{\alpha^\bullet}^\ell \otimes I_{\alpha^\bullet}^r \setminus \mathbf{k}\langle S \rangle$, with the quotient algebra isomorphic to A_α , so there is an exact sequence of $\mathbf{k}\langle S \rangle$ -bimodules

$$0 \longrightarrow \mathbf{k}\langle S \rangle / I_{\alpha^\bullet}^\ell \otimes I_{\alpha^\bullet}^r \setminus \mathbf{k}\langle S \rangle \longrightarrow \text{End}_{S\tilde{\mathcal{C}}_\alpha}((+)) \longrightarrow A_\alpha \longrightarrow 0. \quad (29)$$

The quotient map onto A_α admits a section, and A_α is naturally a subalgebra of $\text{End}_{S\tilde{\mathcal{C}}_\alpha}((+))$. This decomposition corresponds to two types of endomorphisms of $(+)$ in $\tilde{\mathcal{C}}$ (without floating endpoints versus having two floating endpoints) and corresponding bases in endomorphisms of $(+)$ in $S\tilde{\mathcal{C}}_\alpha$, see Fig. 3.2.6.

(4) *Gligible quotient category $\tilde{\mathcal{C}}_\alpha$.* The trace $\text{tr}_\alpha(x)$ of a cobordism x from $\underline{\epsilon}$ to $\underline{\epsilon}$ is defined in the same way as for cobordisms in the smaller category \mathcal{C}_α , by closing x into a floating cobordism \hat{x} , see Fig. 2.4.4 and evaluating via α :

$$\text{tr}_\alpha(x) := \alpha(\hat{x}).$$

This operation extends to a \mathbf{k} -linear trace on $\mathbf{k}\tilde{\mathcal{C}}_\alpha$ that descends to a trace on $\mathcal{V}\tilde{\mathcal{C}}_\alpha$ and $S\tilde{\mathcal{C}}_\alpha$:

$$\text{End}_{\mathbf{k}\tilde{\mathcal{C}}}(\underline{\epsilon}) \longrightarrow \text{End}_{\mathcal{V}\tilde{\mathcal{C}}_\alpha}(\underline{\epsilon}) \longrightarrow \text{End}_{S\tilde{\mathcal{C}}_\alpha}(\underline{\epsilon}) \xrightarrow{\text{tr}_\alpha} \mathbf{k}.$$

The trace is symmetric. The two-sided ideal $\tilde{J}_\alpha \subset S\tilde{\mathcal{C}}_\alpha$ of *negligible morphisms* is defined as usual, see Section 2.4 for the definition of negligible ideal in the subcategory $S\mathcal{C}_\alpha$ of $S\tilde{\mathcal{C}}_\alpha$.

Define the quotient category

$$\tilde{\mathcal{C}}_\alpha := S\tilde{\mathcal{C}}_\alpha / J_\alpha.$$

The quotient category $\tilde{\mathcal{C}}_\alpha$ has finite-dimensional hom spaces, as does $S\tilde{\mathcal{C}}_\alpha$, since α is recognizable. The trace is nondegenerate on $\tilde{\mathcal{C}}_\alpha$ and defines perfect bilinear pairings

$$\text{Hom}(\underline{\epsilon}, \underline{\epsilon}') \otimes \text{Hom}(\underline{\epsilon}', \underline{\epsilon}) \longrightarrow \mathbf{k}$$

on its hom spaces. We call $\tilde{\mathcal{C}}_\alpha$ the *gligible quotient* of \mathcal{SC}_α , having modded out by the ideal of negligible morphisms.

Up to an isomorphism, the state space

$$A_\alpha(\underline{\epsilon}) := \text{Hom}_{\tilde{\mathcal{C}}_\alpha}(\emptyset, \underline{\epsilon}) \quad (30)$$

depends only on the number of pluses and minuses in $\underline{\epsilon}$ and $A_\alpha(\underline{\epsilon}) \cong A_\alpha(+^n -^m)$, where n and m is the number of pluses and minuses in $\underline{\epsilon}$. Summing $A_\alpha(+^n -^m)$ over $n, m \geq 0$ one get a bigraded associative algebra with $S_n \times S_m$ action on the homogeneous (n, m) component with the properties similar to that of a *tca* algebra [48].

(5) *The Deligne category and its gligible quotient.* From the skein category \mathcal{SC}_α we can pass to its additive Karoubi closure

$$\mathcal{D}\tilde{\mathcal{C}}_\alpha := \text{Kar}(\mathcal{SC}_\alpha^\oplus),$$

which is the analogue of the Deligne category. The quotient of $\mathcal{D}\tilde{\mathcal{C}}_\alpha$ by the ideal $\mathcal{D}\tilde{\mathcal{J}}_\alpha$ of negligible morphisms,

$$\underline{\mathcal{D}\tilde{\mathcal{C}}}_\alpha := \mathcal{D}\tilde{\mathcal{C}}_\alpha / \mathcal{D}\tilde{\mathcal{J}}_\alpha,$$

is equivalent to the additive Karoubi closure of $\tilde{\mathcal{C}}_\alpha$.

Summary: To summarize, the following categories are assigned to a recognizable pair α as in (27):

- The category \mathcal{VC}_α of viewable cobordisms with the α -evaluation of floating (or closed) components.
- The skein category \mathcal{SC}_α where closed (floating) S -decorated intervals and circles are evaluated via α and elements of the syntactic ideal I_α evaluate to zero when placed along any interval in a cobordism. Furthermore, elements of left and right syntactic ideals $I_{\alpha^\bullet}^\ell$ and $I_{\alpha^\bullet}^r$ evaluate to zero when placed at the beginning or end of an interval with the corresponding endpoint floating.
- The quotient $\tilde{\mathcal{C}}_\alpha$ of \mathcal{SC}_α by the two-sided ideal of negligible morphisms. We also call $\tilde{\mathcal{C}}_\alpha$ the *gligible* category or the gligible quotient. Hom spaces in $\tilde{\mathcal{C}}_\alpha$ come with nondegenerate bilinear forms

$$\text{Hom}(\underline{\epsilon}, \underline{\epsilon}') \otimes \text{Hom}(\underline{\epsilon}', \underline{\epsilon}) \longrightarrow \mathbf{k},$$

where $\underline{\epsilon}, \underline{\epsilon}'$ are objects $\tilde{\mathcal{C}}_\alpha$, sequences of pluses and minuses describing oriented zero-manifolds that are the source and the target of decorated one-cobordisms. The universal construction, in this case, assigns the vector space $\text{Hom}_{\tilde{\mathcal{C}}_\alpha}(\emptyset, \underline{\epsilon})$ of morphisms from the empty zero-manifold \emptyset to the oriented zero-manifold $\underline{\epsilon}$.

- Additive Karoubi closure $\mathcal{D}\tilde{\mathcal{C}}_\alpha$ of \mathcal{SC}_α , analogous to the Deligne category. The quotient of $\mathcal{D}\tilde{\mathcal{C}}_\alpha$ by the ideal of negligible morphisms is denoted $\underline{\mathcal{D}\tilde{\mathcal{C}}}_\alpha$.

We arrange these categories and functors, for recognizable $\alpha = (\alpha^\bullet, \alpha^\circ)$, into the following diagram, with a commutative square on the right:

$$\begin{array}{ccccccc} \tilde{\mathcal{C}} & \longrightarrow & \mathbf{k}\tilde{\mathcal{C}} & \longrightarrow & \mathcal{VC}_\alpha & \longrightarrow & \mathcal{SC}_\alpha & \longrightarrow & \mathcal{D}\tilde{\mathcal{C}}_\alpha \\ & & & & \downarrow & & \downarrow & & \\ & & & & \tilde{\mathcal{C}}_\alpha & \longrightarrow & \underline{\mathcal{D}\tilde{\mathcal{C}}}_\alpha & & \end{array} \quad (31)$$

Properties of the categories in the analogous diagram (23) in Section 2.5, as explained in the paragraph following (23), hold for the categories in (31) as well.

The category $\tilde{\mathcal{C}}$ and categories built out of it require a pair of series $\alpha = (\alpha^\bullet, \alpha^\circ)$ for evaluation. When working with the subcategory \mathcal{C} of cobordisms without floating endpoints, only circles appear as connected components of floating cobordisms, and series α° is needed for evaluation. Instead of working with \mathcal{C} , one can use $\tilde{\mathcal{C}}$ but set the connected component $\alpha^\bullet = 0$, so that $\alpha = (0, \alpha^\circ)$. Then in the viewable category \mathcal{VC}_α any cobordism that contains a floating interval evaluates to zero. Syntactic ideals $I_{\alpha^\bullet}^\ell, I_{\alpha^\bullet}^r = \mathbf{k}\langle S \rangle$, and in the skein category \mathcal{SC}_α any cobordism containing an interval (including viewable intervals, with one boundary and one floating endpoint) evaluates to 0. This results in equivalences of categories

$$\mathcal{SC}_{(0, \alpha^\circ)} \cong \mathcal{SC}_{\alpha^\circ}, \quad \tilde{\mathcal{C}}_{(0, \alpha^\circ)} \cong \mathcal{C}_{\alpha^\circ}, \quad \mathcal{DC}_{(0, \alpha^\circ)} \cong \mathcal{DC}_{\alpha^\circ}, \quad \underline{\mathcal{DC}}_{(0, \alpha^\circ)} \cong \underline{\mathcal{DC}}_{\alpha^\circ}, \quad (32)$$

that, furthermore, respect commutative squares of categories in diagrams (23) and (31).

Thus, the construction in Section 2 of the skein category \mathcal{SC}_α , the glible quotient category \mathcal{C}_α and other categories defined there and associated to symmetric series α can be considered a special case of the construction of the present section, specializing to the pair $(0, \alpha)$ with the first recognizable series in the pair being zero.

Alternatively, one can set α° to zero and consider a pair $\alpha = (\alpha^\bullet, 0)$. In the viewable cobordism category \mathcal{VC}_α for this α a circle (necessarily floating, and with any decoration) evaluates to 0. Only the ideals $I_{\alpha^\bullet}, I_{\alpha^\bullet}^\ell, I_{\alpha^\bullet}^r$ (two-sided, left and right, respectively) are used in the definition of the skein category \mathcal{SC}_α . A decorated U -turn as in Fig. 3.1.3, case (2) or (3), may be non-zero in $\tilde{\mathcal{C}}_\alpha$, since it may be coupled to two intervals on the other side with a non-zero evaluation.

When $\alpha = (\alpha^\bullet, 0)$, another approach is to restrict possible cobordisms and disallow U -turns as cobordisms. Then a cobordism c must have no critical points under the natural projection onto the interval $[0, 1]$ under which $\partial_i c$ projects onto i , for $i = 0, 1$. When components of cobordisms are unoriented, such restricted cobordisms appear in [29] in a categorification of the polynomial ring (without dot decorations) and in [32] in a categorification of $\mathbb{Z}[1/2]$ and potential categorifications of $\mathbb{Z}[1/n]$ as monoidal envelopes of certain Leavitt path algebras and the “one-sided inverse” algebra $\mathbf{k}\langle a, b \rangle / (ab - 1)$. The latter cobordisms carry dot decorations, corresponding to the generators of these algebras.

When $S = \emptyset$, the evaluation again reduces to two numbers (evaluations of the oriented interval and oriented circle), and the skein category \mathcal{SC} is the oriented partial Brauer category, see the remark below. When $S = \{s\}$ has cardinality one, recognizable series α is encoded by two rational functions $Z_{\alpha^\bullet}(T), Z_{\alpha^\circ}(T)$ in a single variable T .

Remark. Instead of power series $\alpha = (\alpha^\bullet, \alpha^\circ)$ in noncommuting variables one can instead start with an associative \mathbf{k} -algebra B and two \mathbf{k} -linear maps

$$\alpha^\bullet : B \rightarrow \mathbf{k}, \quad \alpha^\circ : B \rightarrow \mathbf{k} \quad (33)$$

such that α° is symmetric, $\alpha^\circ(ab) = \alpha^\circ(ba)$, $a, b \in B$. Two-sided syntactic ideals $I_{\alpha^\bullet}, I_{\alpha^\circ} \subset B$, their intersection $I_\alpha := I_{\alpha^\bullet} \cap I_{\alpha^\circ}$, and one-sided syntactic ideals $I_{\alpha^\bullet}^r, I_{\alpha^\bullet}^\ell$ are defined in the same way as for noncommutative series.

The nondegenerate case is that of $I_\alpha = 0$ being the zero ideal in B , but the arbitrary case can be reduced to it by passing to the quotient B/I_α . Recognizable case corresponds to finite-dimensional B . Analogues of all categories in (31) can be defined for such pair of traces on a finite-dimensional B . Defects on cobordisms are now labelled by elements of B rather than by elements of S . The difference from noncommutative

recognizable power series is that one does not pick any particular set of generators S of B , working with the entire B instead, but the resulting categories, starting with the category \mathcal{SC}_α in (31), are equivalent to the ones built from a noncommutative power series once a set S of generators of B is chosen.

Remark. It is straightforward to modify the constructions of this section to the case of unoriented one-manifolds with floating endpoints and S -decorated dots. If, in addition, $S = \emptyset$, there are no dots and floating cobordisms reduce to unions of intervals and circles. The evaluation is then a pair $(\alpha^\bullet(1), \alpha^\circ(1))$ of elements of \mathbf{k} and the unoriented skein category \mathcal{SC}_α is the partial Brauer category [35], also known as the rook-Brauer category [20].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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