



Almost Sure Well-Posedness for Hall MHD

Mimi Dai

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Abstract. We consider the magnetohydrodynamics system with Hall effect accompanied with initial data in supercritical Sobolev space. Via an appropriate randomization of the supercritical initial data, both local and small data global well-posedness for the system are obtained almost surely in critical Sobolev space.

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1. Introduction

A mathematical model for the incompressible magnetohydrodynamics (MHD) with Hall effect is given by

$$\begin{aligned} u_t + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla \Pi &= \Delta u, \\ B_t + (u \cdot \nabla)B - (B \cdot \nabla)u + \nabla \times ((\nabla \times B) \times B) &= \Delta B, \\ \nabla \cdot u &= 0 \end{aligned} \quad (1.1)$$

on $\mathbb{T}^3 \times [0, \infty)$. The unknowns u , B and Π are the velocity field, magnetic field and scalar pressure respectively. Note that $\nabla \cdot B(x, t) = 0$ remains true for all the time $t > 0$ if $\nabla \cdot B(x, 0) = 0$. The nonlinear term with the highest derivative is the Hall term $\nabla \times ((\nabla \times B) \times B)$ which is posed to capture the rapid magnetic reconnection phenomena in plasma physics. The presence of the Hall term makes (1.1) a quasilinear system which is usually more challenging than semilinear systems.

The well-posedness of the Hall MHD system (1.1) in various functional spaces has been studied extensively, for instance, see [1, 6, 7, 10, 11]. In these works the initial data and solution reside in the same spaces. In this paper, we are interested in the Cauchy problem for the Hall MHD with rough initial data yielding solutions with higher regularity. This can be achieved by randomizing the initial data properly. Such scheme for Cauchy problem with rough initial data was first developed in [3–5, 16] for treating dispersive equations. It has been applied to the Navier–Stokes equation in [17] to obtain global weak solutions when the initial data is in Sobolev space with negative index. Other applications can be found in [8, 12–14, 18] for both dispersive and dissipative systems. It is notable that the randomization strategy has the advantage to study Cauchy problems with supercritical feature, either the system being supercritical or the initial data being supercritical.

In the author’s previous work [9], we investigated the electron MHD with generalized diffusion $(-\Delta)^\alpha$ for suitable $\alpha > 1$ in the supercritical regime. By randomizing the initial data in H^s with $s < 0$, we established global existence of weak solutions. When $\alpha > 1$, the generalized electron MHD is no longer quasilinear, but semilinear. One major observation in [9] is that there are obstructions to apply the approach of randomization of initial data for quasilinear equations, although the method is robust in

analyzing supercritical equations. In this paper, we continue to study the full system of the Hall MHD with generalized diffusion,

$$\begin{aligned} u_t + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla \Pi &= -(-\Delta)^\alpha u, \\ B_t + (u \cdot \nabla)B - (B \cdot \nabla)u + \nabla \times ((\nabla \times B) \times B) &= -(-\Delta)^\alpha B, \\ \nabla \cdot u &= 0. \end{aligned} \quad (1.2)$$

We only consider $1 \leq \alpha < \frac{7}{4}$, in which regime system (1.2) is still supercritical, but restrict to the case of initial data in H^s with $s \geq 0$ as to distinguish the current work from [9]. We point out that the coefficients of the diffusion terms in (1.2) are set to be 1 for the purpose of simplifying notations; the diffusion coefficients do not play a role in the analysis of the paper. Unlike the classical MHD without Hall term, system (1.2) does not have a natural scaling. We extend the discussion on this aspect in the following.

When $B \equiv 0$, system (1.2) reduces to the hyperdissipative Navier–Stokes equation (NSE)

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla \Pi &= -(-\Delta)^\alpha u, \\ \nabla \cdot u &= 0 \end{aligned} \quad (1.3)$$

which has the scaling property: if $(u(x, t), \Pi(x, t))$ solves (1.3) with initial data $u_0(x)$, the rescaled pair (u_λ, Π_λ) as

$$u_\lambda(x, t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t), \quad \Pi_\lambda(x, t) = \lambda^{2(2\alpha-1)}\Pi(\lambda x, \lambda^{2\alpha}t)$$

solves (1.3) with initial data $u_{0,\lambda} = \lambda^{2\alpha-1}u_0(\lambda x)$. Among other scaling invariant (critical) spaces, the critical Sobolev space for (1.3) in 3D is $\mathcal{H}^{\frac{5}{2}-2\alpha}$. In view of the prior energy estimate in $L^2 = \mathcal{H}^0$, (1.3) is critical for $\alpha = \frac{5}{4}$; it is supercritical for $\alpha < \frac{5}{4}$ and subcritical for $\alpha > \frac{5}{4}$.

With static background flow $u \equiv 0$ the Hall MHD system (1.1) reduces to the so-called electron MHD

$$\begin{aligned} B_t + \nabla \times ((\nabla \times B) \times B) &= -(-\Delta)^\alpha B, \\ \nabla \cdot B &= 0. \end{aligned} \quad (1.4)$$

System (1.4) has the scaling

$$B_\lambda(x, t) = \lambda^{2\alpha-2}B(\lambda x, \lambda^{2\alpha}t).$$

The critical Sobolev space for (1.4) on $\mathbb{T}^3$ is $\mathcal{H}^{\frac{7}{2}-2\alpha}$. Again it follows from the basic energy law that (1.4) is critical for $\alpha = \frac{7}{4}$, supercritical for $\alpha < \frac{7}{4}$ and subcritical for $\alpha > \frac{7}{4}$.

In the full system (1.2), the scaling of the magnetic field equation plays a dominant role since it contains the highest degree nonlinear term. Thus it is referred to be supercritical when $\alpha < \frac{7}{4}$. If $\alpha \geq \frac{7}{4}$, global regular solution is expected for (1.2) through standard energy method. The purpose of the paper is to study the Cauchy problem of (1.2) in the regime $1 \leq \alpha < \frac{7}{4}$ and with initial data $(u_0, B_0) \in \mathcal{H}^{s_1} \times \mathcal{H}^{s_2}$, $0 \leq s_1 < \frac{5}{2} - 2\alpha$ and $0 \leq s_2 < \frac{7}{2} - 2\alpha$. By randomizing the initial data, we show the existence and uniqueness of solution to (1.2) in the space $\mathcal{H}^{\frac{5}{2}-2\alpha} \times \mathcal{H}^{\frac{7}{2}-2\alpha}$. As mentioned previously, the Hall term has the highest derivative in the system and poses the most challenges in our analysis. Therefore, we first investigate the hyperdissipative electron MHD (1.4) and tackle the difficulties caused by the Hall term separately. We then study the full coupled system (1.2), in which step the major difficulty comes from the coupling terms $(u \cdot \nabla)B$ and $(B \cdot \nabla)u$. The key ingredient to overcome the obstacles coming from the Hall term and coupling terms is the improved L^p estimate for the free evolution of the randomized initial data. Details will be unfolded in later sections. We state the main results respectively on the standard probability space $(\Omega, \mathcal{A}, P)$ for the electron MHD (1.4), the hyperdissipative NSE (1.3) and the full Hall MHD system (1.2) below.

Theorem 1.1. *Let $\alpha \in [1, \frac{7}{4})$. Let $B_0 = f \in \mathcal{H}^s(\mathbb{T}^3)$ with $s \geq \max\{\frac{11}{2} - 4\alpha, 0\}$. For any $m \in (0, 1)$, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) \geq m$ such that for any $\omega \in \Sigma$ the electron MHD (1.4) with the randomized initial data f^ω on $\mathbb{T}^3$ has a unique solution $B = B_{f^\omega} + H$ with $B_{f^\omega} = e^{-t(-\Delta)^\alpha} f^\omega$ and*

$$H \in C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3))$$

for some time $T > 0$. Moreover, there exists a constant $c > 0$ such that if $\|f\|_{\mathcal{H}^s(\mathbb{T}^3)} < c$, T can be arbitrarily large.

Remark 1.2. The gain of the derivative for the solution due to randomization of the initial data is

$$\begin{cases} (\frac{7}{2} - 2\alpha) - (\frac{11}{2} - 4\alpha) = 2\alpha - 2, & \text{if } 1 \leq \alpha \leq \frac{11}{8}, \\ (\frac{7}{2} - 2\alpha), & \text{if } \frac{11}{8} < \alpha < \frac{7}{4}. \end{cases}$$

In particular, when $\alpha \rightarrow 1$, the gain of the derivative is $2\alpha - 2 \rightarrow 0$. This corresponds to the classical well-posedness result in the critical space $\mathcal{H}^{\frac{3}{2}}$ with initial data in the same space for the electron MHD with $\alpha = 1$. Therefore the approach of randomization of the initial data seems not yield improvement in the quasilinear situation.

Theorem 1.3. Let $\alpha \in [1, \frac{7}{4})$. Let $u_0 = g \in \mathcal{H}^s(\mathbb{T}^3)$ with $s \geq \max\{\frac{7}{2} - \frac{7}{2}\alpha, 0\}$. For any $m \in (0, 1)$, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) \geq m$ such that for any $\omega \in \Sigma$ the NSE (1.3) with the randomized initial data g^ω has a unique solution u of the form

$$u = u_{g^\omega} + V$$

and

$$V \in C([0, T]; \mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)).$$

for some time $T > 0$. Additionally, there exists a constant c such that if $\|g\|_{\mathcal{H}^s(\mathbb{T}^3)} < c$, T can be arbitrarily large.

Remark 1.4. When $\alpha = 1$, Theorem 1.3 demonstrates almost sure well-posedness of the NSE in space $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ for initial data in $L^2(\mathbb{T}^3)$. It recovers the result of [12].

Theorem 1.5. Let $\alpha \in [1, \frac{7}{4})$. Let $u_0 = g \in \mathcal{H}^{s_1}(\mathbb{T}^3)$ and $B_0 = f \in \mathcal{H}^{s_2}(\mathbb{T}^3)$ with $s_1 \geq \max\{\frac{7}{2} - \frac{7}{2}\alpha, 0\}$ and $s_2 \geq \max\{\frac{11}{2} - 4\alpha, 0\}$. For any $m \in (0, 1)$, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) \geq m$ such that for any $\omega \in \Sigma$ the Hall MHD (1.2) with the randomized initial data (g^ω, f^ω) has a unique solution (u, B) on $[0, T]$ for some $T > 0$ in the form

$$u = u_{g^\omega} + V, \quad B = B_{f^\omega} + H$$

with $u_{g^\omega} = e^{-t(-\Delta)^\alpha} g^\omega$ and $B_{f^\omega} = e^{-t(-\Delta)^\alpha} f^\omega$, and

$$V \in C([0, T]; \mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)), \quad H \in C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)).$$

In addition, there exists a constant $c > 0$ such that if $\|g\|_{\mathcal{H}^{s_1}(\mathbb{T}^3)} + \|f\|_{\mathcal{H}^{s_2}(\mathbb{T}^3)} < c$, T can be arbitrarily large.

Remark 1.6. A special case of Theorem 1.5 is the almost sure well-posedness of (1.2) in the critical space $\mathcal{H}^{\frac{5}{2}-2\alpha} \times \mathcal{H}^{\frac{7}{2}-2\alpha}$ with initial data $(u_0, B_0) \in \mathcal{H}^{s-1} \times \mathcal{H}^s$ for $s \geq \max\{\frac{11}{2} - 4\alpha, 0\}$. When $\alpha = 1$, that indicates almost sure well-posedness of the classical Hall MHD (1.1) in the critical space $\mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{\frac{3}{2}}$ with initial data $(u_0, B_0) \in \mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{\frac{3}{2}}$. In the deterministic case in [10], well-posedness of (1.1) was established in nearly critical space $\mathcal{H}^{\frac{1}{2}+\varepsilon} \times \mathcal{H}^{\frac{3}{2}}$ with arbitrarily small $\varepsilon > 0$. As pointed out in [10], there is some essential obstacle to remove ε due to the coupling feature of the Hall MHD system and different scalings for the velocity and magnetic field. The current result thus suggests that the randomization of initial data can be employed to remove such obstacle.

Evidence of randomness providing certain form of regularization for the Hall-MHD system is also seen in previous work [15, 19]. In [15] the author proved a stochastic law of flux conservation for various models including the Hall-MHD system with diffusion; in [19] the authors showed that the noise in the Hall-MHD system induces damping effect.

The rest of the paper is organized as: (i) in Sect. 2 we lay out the notations to be used and recall the standard randomization procedure; (ii) Sect. 3 is devoted to establishing improved estimates for the free evolution u_{g^ω} and B_{f^ω} ; (iii) we present a proof for Theorems 1.1, 1.3 and 1.5 in the last three sections respectively.

2. Preliminaries

2.1. Notations

We denote a general constant by c which may differ in different estimates. Conventionally we denote $f \lesssim g$ for an inequality $f \leq cg$ with some constant $c > 0$. For brevity, the Lebesgue space $L^p(\mathbb{T}^3)$ is sometimes denoted by L_x^p . It applies to Lebesgue spaces with respect to other variables as well, for instance, $L^p(0, T) = L_t^p$.

We introduce the weighted (in time) Lebesgue space

$$L_t^{(r,s)} L_x^p = \left\{ f : \left(\int_0^T \|t^r f(\cdot, t)\|_{L_x^p}^s dt \right)^{\frac{1}{s}} < \infty \right\}$$

equipped with the norm

$$\|f\|_{L_t^{(r,s)} L_x^p} = \left(\int_0^T \|t^r f(\cdot, t)\|_{L_x^p}^s dt \right)^{\frac{1}{s}}.$$

2.2. Randomization

We recall the probabilistic estimates obtained in [5], which are valid for both Gaussian and Bernoulli variables.

Lemma 2.1. *Let $(l_i(\omega))_{i=1}^\infty$ be a sequence of real-valued, zero-mean and independent random variables on a probability space $(\Omega, \mathcal{A}, P)$ with associated distributions $(\mu_i)_{i=1}^\infty$. Assume that there exists $c > 0$ such that*

$$\left| \int_{-\infty}^{\infty} e^{\gamma x} d\mu_i(x) \right| \leq e^{c\gamma^2} \quad \forall \gamma \in \mathbb{R} \quad \forall i \geq 1.$$

Then there exists $\beta > 0$ such that

$$P \left(\omega : \left| \sum_{i=1}^{\infty} c_i l_i(\omega) \right| > \lambda \right) \leq 2e^{-\frac{\beta \lambda^2}{\sum_{i=1}^{\infty} c_i^2}} \quad \forall \lambda > 0 \quad \forall (c_i)_{i=1}^\infty \in \ell^2.$$

Consequently, there exists another constant $c > 0$ such that

$$\left\| \sum_{i=1}^{\infty} c_i l_i(\omega) \right\|_{L^q(\Omega)} \leq c\sqrt{q} \left(\sum_{i=1}^{\infty} c_i^2 \right)^{\frac{1}{2}} \quad \forall q \geq 2 \quad \forall (c_i)_{i=1}^\infty \in \ell^2.$$

We adapt the standard diagonal randomization on the Sobolev space $\mathcal{H}^s(\mathbb{T}^n)$.

Definition 2.2. Let $(l_k(\omega))_{k \in \mathbb{Z}^n}$ be a sequence of real-valued and independent random variables on the probability space $(\Omega, \mathcal{A}, P)$ as in Lemma 2.1. Let $e_k(x) = e^{ik \cdot x}$ for any $k \in \mathbb{Z}^n$. For a vector field $f = (f_1, f_2, \dots, f_n) \in \mathcal{H}^s(\mathbb{T}^n)$ with Fourier coefficients $(a_k)_{k \in \mathbb{Z}^n}$ and $a_k = (a_k^1, a_k^2, \dots, a_k^n)$, the map

$$\mathcal{R} : (\Omega, \mathcal{A}) \longrightarrow \mathcal{H}^s(\mathbb{T}^n)$$

$$\omega \longrightarrow f^\omega, \quad f^\omega(x) = \left(\sum_{k \in \mathbb{Z}^n} l_k(\omega) a_k^1 e_k(x), \dots, \sum_{k \in \mathbb{Z}^n} l_k(\omega) a_k^n e_k(x) \right) \quad (2.1)$$

equipped with the Borel sigma algebra is introduced. The map $\mathcal{R}$ is called randomization.

It is worth to mention that the Leray projection $\mathbb{P}$ commutes with the randomization map $\mathcal{R}$, see [17]. In view of Lemma 2.1, we see that $\mathcal{R}$ is measurable and $f^\omega \in \mathcal{H}^s(\mathbb{T}^n)$ if $f \in \mathcal{H}^s(\mathbb{T}^n)$. We also have

$$f^\omega \in L^2(\Omega; \mathcal{H}^s(\mathbb{T}^n)), \quad \|f^\omega\|_{\mathcal{H}^s} \sim \|f\|_{\mathcal{H}^s}.$$

3. Estimates of the Free Evolution with Randomized Initial Data

Let $u(x, 0) = g$ and $B(x, 0) = f$ and their randomization g^ω and f^ω are given by

$$\begin{aligned} g^\omega(x) &= \left(\sum_{k \in \mathbb{Z}^n} l_k(\omega) b_k^1 e_k(x), \dots, \sum_{k \in \mathbb{Z}^n} l_k(\omega) b_k^n e_k(x) \right), \\ f^\omega(x) &= \left(\sum_{k \in \mathbb{Z}^n} l_k(\omega) a_k^1 e_k(x), \dots, \sum_{k \in \mathbb{Z}^n} l_k(\omega) a_k^n e_k(x) \right). \end{aligned}$$

We denote the free evolution of g^ω and f^ω according to the operator $-(-\Delta)^\alpha$ by $u_{g^\omega} = e^{-t(-\Delta)^\alpha} g^\omega$ and $B_{f^\omega} = e^{-t(-\Delta)^\alpha} f^\omega$ respectively.

We first recall the Hardy—Littlewood—Sobolev lemma.

Lemma 3.1. [2] *Let $K(x) = |x|^{-\zeta}$ for $x \in \mathbb{R}^n$ and $\zeta \in (0, n)$. Let $g \in L^p(\mathbb{R}^n)$. Then we have*

$$\|K * g\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}$$

with $1 < p < q < \infty$ and $\zeta = n \left(1 - \frac{1}{p} + \frac{1}{q}\right)$.

The following two estimates will be used extensively in later sections.

Lemma 3.2. *Let $\alpha > 0$, $p > 0$ and $m > 0$. Then*

$$\left\| e^{-t|\xi|^{2\alpha}} |\xi|^m \right\|_{L_\xi^p(\mathbb{R}^n)} \lesssim t^{-\frac{m}{2\alpha} - \frac{n}{2p\alpha}}.$$

Proof. Straightforward computation shows

$$\begin{aligned} \left\| e^{-t|\xi|^{2\alpha}} |\xi|^m \right\|_{L_\xi^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} e^{-pt|\xi|^{2\alpha}} |\xi|^{pm} d\xi \right)^{\frac{1}{p}} \\ &= t^{-\frac{m}{2\alpha} - \frac{n}{2p\alpha}} \left(\int_{\mathbb{R}^n} e^{-pt|\xi|^{2\alpha}} t^{\frac{pm}{2\alpha}} |\xi|^{pm} d(t^{\frac{1}{2\alpha}} \xi) \right)^{\frac{1}{p}} \\ &\lesssim t^{-\frac{m}{2\alpha} - \frac{n}{2p\alpha}} \end{aligned}$$

since the integral $\int_{\mathbb{R}^n} e^{-py^{2\alpha}} y^{pm} dy$ is bounded for $\alpha > 0$, $p > 0$ and $m > 0$. It is obvious that the estimate holds on $\mathbb{Z}^n$ as well since $\mathbb{Z}^n \subset \mathbb{R}^n$. □

Lemma 3.3. *Let $0 < r, s < 1$ and $r + s = 1$. There exists a constant $c > 0$ independent of the time t such that*

$$\int_0^t (t - \tau)^{-r} \tau^{-s} d\tau \leq c.$$

Proof. Changing variable $\tau = t\tau'$ in the integral gives

$$\int_0^t (t - \tau)^{-r} \tau^{-s} d\tau = \int_0^1 (1 - \tau')^{-r} (\tau')^{-s} d\tau' = B(1 - s, 1 - r) \leq c$$

where $B(1 - r, 1 - s)$ is the Beta function. □

We establish some probabilistic estimates for B_{f^ω} and u_{g^ω} in the following.

Lemma 3.4. *Let $r \geq q \geq p \geq 2$ and $s \geq -\frac{2\alpha}{q}$. The free evolution B_{f^ω} satisfies*

$$\|B_{f^\omega}\|_{L_\omega^r L_t^q L_x^p} \lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^s}.$$

Proof. Recall

$$B_{f^\omega}(x, t) = e^{-t(-\Delta)^\alpha} f^\omega(x) = \sum_{k \in \mathbb{Z}^n} e^{-t|k|^{2\alpha}} l_k(\omega) a_k e_k(x).$$

By Minkowski's inequality and Lemma 2.1, we have

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^q L_x^p} &\lesssim \|B_{f^\omega}\|_{L_t^q L_x^p L_\omega^r} \\ &\lesssim \sqrt{r} \left\| \left(\sum_{k \in \mathbb{Z}^n} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_x^p} \\ &= \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}}^{\frac{1}{2}}. \end{aligned}$$

We continue the estimate with Minkowski's inequality again

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^q L_x^p} &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} |k|^{-\frac{4\alpha}{q}} |a_k|^2 |e_k|^2 \right\|_{L_x^{\frac{p}{2}}}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-\frac{4\alpha}{q}} |a_k|^2 \|e_k\|_{L_x^p}^2 \right)^{\frac{1}{2}} \end{aligned}$$

where we used the estimate

$$\begin{aligned} \|e^{-2t|k|^{2\alpha}}\|_{L_t^{\frac{q}{2}}} &= \left(\int_0^t e^{-q\tau|k|^{2\alpha}} d\tau \right)^{\frac{2}{q}} \\ &= |k|^{-\frac{4\alpha}{q}} \left(\int_0^t e^{-q\tau|k|^{2\alpha}} d(\tau|k|^{2\alpha}) \right)^{\frac{2}{q}} \\ &\lesssim |k|^{-\frac{4\alpha}{q}}. \end{aligned}$$

Note that for the basis $\{e_k\}$ we have

$$\|e_k\|_{L_x^p}^2 \lesssim \|e_k\|_{L_x^2}^2.$$

Thus we further deduce

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^q L_x^p} &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-\frac{4\alpha}{q}} |a_k|^2 \|e_k\|_{L_x^p}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-\frac{4\alpha}{q}} |a_k|^2 \|e_k\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \|f\|_{\mathcal{H}_x}^{-\frac{2\alpha}{q}}. \end{aligned}$$

□

Lemma 3.5. Let $r \geq \frac{1}{2\beta} \geq p \geq 2$ and $\frac{1}{\alpha} + \frac{3}{2p\alpha} + 2\beta = 1$. Assume $s \geq 2 - 2\alpha + \frac{3}{p}$. The estimate

$$\|B_{f^\omega}\|_{L_\omega^r L_t^{\frac{1}{2\beta}} L_x^p} \lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^s}$$

is valid.

Proof. It is a special case of Lemma 3.4. Indeed, taking $q = \frac{1}{2\beta}$, the assumptions of Lemma 3.4 become $r \geq \frac{1}{2\beta} \geq p \geq 2$ and

$$s \geq -\frac{2\alpha}{q} = -4\alpha\beta. \quad (3.1)$$

On the other hand, it follows from the assumption $\frac{1}{\alpha} + \frac{3}{2p\alpha} + 2\beta = 1$ that

$$-4\alpha\beta = 2 - 2\alpha + \frac{3}{p}.$$

Thus (3.1) is valid under the assumption $s \geq 2 - 2\alpha + \frac{3}{p}$; and the estimate follows from Lemma 3.4. $\square$

Lemma 3.6. *Let $r \geq q \geq p \geq 2$ and $\eta \geq 0$. Then the estimate*

$$\|B_{f^\omega}\|_{L_\omega^r L_t^{(\eta,q)} L_x^p} \lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^s}$$

holds for $s \geq -2\alpha\eta - \frac{2\alpha}{q}$.

Proof. Thanks to Minkowski's inequality and Lemma 2.1 again, we infer

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^{(\eta,q)} L_x^p} &= \left\| \sum_{k \in \mathbb{Z}^n} t^\eta e^{-t|k|^{2\alpha}} l_k(\omega) a_k e_k(x) \right\|_{L_\omega^r L_t^q L_x^p} \\ &\lesssim \sqrt{r} \left\| \left(\sum_{k \in \mathbb{Z}^n} t^{2\eta} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_x^p} \\ &= \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}}^{\frac{1}{2}}. \end{aligned}$$

The norm in time can be estimated as

$$\begin{aligned} \|t^{2\eta} e^{-2t|k|^{2\alpha}}\|_{L_t^{\frac{q}{2}}} &= \left(\int_0^t t^{q\eta} e^{-q\tau|k|^{2\alpha}} d\tau \right)^{\frac{2}{q}} \\ &= |k|^{-4\alpha\eta - \frac{4\alpha}{q}} \left(\int_0^t (\tau|k|^{2\alpha})^{q\eta} e^{-q\tau|k|^{2\alpha}} d(\tau|k|^{2\alpha}) \right)^{\frac{2}{q}} \\ &\lesssim |k|^{-4\alpha\eta - \frac{4\alpha}{q}}. \end{aligned}$$

Therefore we continue the estimate as

$$\begin{aligned}
 \|B_{f^\omega}\|_{L_\omega^r L_t^{(\eta,q)} L_x^p} &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_x^{\frac{p}{2}} L_t^{\frac{q}{2}}}^{\frac{1}{2}} \\
 &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\eta - \frac{4\alpha}{q}} |a_k|^2 |e_k|^2 \right\|_{L_x^{\frac{p}{2}}}^{\frac{1}{2}} \\
 &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\eta - \frac{4\alpha}{q}} |a_k|^2 \|e_k\|_{L_x^p}^2 \right)^{\frac{1}{2}} \\
 &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\eta - \frac{4\alpha}{q}} |a_k|^2 \|e_k\|_{L_x^p}^2 \right)^{\frac{1}{2}} \\
 &\lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^{-2\alpha\eta - \frac{2\alpha}{q}}}.
 \end{aligned}$$

□

Lemma 3.7. Let $r \geq \frac{1}{\beta} \geq p \geq 2$ and $\frac{1}{\alpha} + \frac{3}{2p\alpha} + 2\beta = 1$. Let $s \geq 2 - 2\alpha + \frac{3}{p}$. We have the estimate

$$\|B_{f^\omega}\|_{L_\omega^r L_t^{(\beta, \frac{1}{\beta})} L_x^p} \lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^s}.$$

Proof. This is a special case of Lemma 3.6 with $\eta = \beta$ and $q = \frac{1}{\beta}$. □

Lemma 3.8. Let $r \geq 2$ and $\eta \geq 0$. Assume $s \geq -2\alpha\eta - 3\alpha + \frac{11}{2}$. The estimate

$$\|B_{f^\omega}\|_{L_\omega^r L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} \lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^s}$$

holds.

Proof. It follows from Minkowski's inequality and Lemma 2.1 that

$$\begin{aligned}
 \|B_{f^\omega}\|_{L_\omega^r L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} &= \|t^\eta (1 - \Delta)^{\frac{11}{4}-\alpha} B_{f^\omega}\|_{L_\omega^r L_t^2 L_x^2} \\
 &= \left\| \sum_{k \in \mathbb{Z}^n} t^\eta (1 + |k|^2)^{\frac{11}{4}-\alpha} e^{-t|k|^{2\alpha}} l_k(\omega) a_k e_k(x) \right\|_{L_\omega^r L_t^2 L_x^2} \\
 &\lesssim \sqrt{r} \left\| \left(\sum_{k \in \mathbb{Z}^n} t^{2\eta} (1 + |k|^2)^{\frac{11}{2}-2\alpha} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right) \right\|_{L_t^2 L_x^2}^{\frac{1}{2}} \\
 &= \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} (1 + |k|^2)^{\frac{11}{2}-2\alpha} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_t^1 L_x^1}^{\frac{1}{2}} \\
 &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} (1 + |k|^2)^{\frac{11}{2}-2\alpha} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_x^1 L_t^1}^{\frac{1}{2}}.
 \end{aligned}$$

We estimate the norm in time as before

$$\begin{aligned}
 \|t^{2\eta} e^{-2t|k|^{2\alpha}}\|_{L_t^1} &= \int_0^t \tau^{2\eta} e^{-2\tau|k|^{2\alpha}} d\tau \\
 &= |k|^{-4\alpha\eta-2\alpha} \int_0^t (\tau|k|^{2\alpha})^{2\eta} e^{-2\tau|k|^{2\alpha}} d(\tau|k|^{2\alpha}) \\
 &\lesssim |k|^{-4\alpha\eta-2\alpha}.
 \end{aligned}$$

Thus we infer

$$\begin{aligned}
\|B_{f\omega}\|_{L_\omega^r L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\eta} (1+|k|^2)^{\frac{11}{2}-2\alpha} e^{-2t|k|^{2\alpha}} |a_k|^2 |e_k|^2 \right\|_{L_x^1 L_t^1}^{\frac{1}{2}} \\
&\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\eta-2\alpha+11-4\alpha} |a_k|^2 |e_k|^2 \right\|_{L_x^1}^{\frac{1}{2}} \\
&\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\eta-6\alpha+11} |a_k|^2 \|e_k\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^{-2\alpha\eta-3\alpha+\frac{11}{2}}}.
\end{aligned}$$

It completes the proof of the lemma. $\square$

Remark 3.9. Recall $\mathcal{H}^{\frac{7}{2}-2\alpha}$ is a critical space for (1.4). We observe for $p \geq 2$ and $\eta \geq 0$,

$$\begin{cases} 2 - 2\alpha + \frac{3}{p} \leq \frac{7}{2} - 2\alpha, & \text{for } \alpha \geq 1, \\ -2\alpha\eta - 3\alpha + \frac{11}{2} \leq \frac{7}{2} - 2\alpha, & \text{for } \alpha \geq \frac{2}{2\eta+1}. \end{cases}$$

Therefore Lemmas 3.5, 3.7 and 3.8 hold for initial data f in supercritical and critical regime.

Denote

$$\begin{aligned}
E_1(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|B_{f\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p} \geq \lambda \right\}, \\
E_2(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \geq \lambda \right\}, \\
E_3(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|B_{f\omega}\|_{L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} \geq \lambda \right\}.
\end{aligned}$$

Lemma 3.10. *Let α, β, η and p satisfy the parameter conditions in Lemmas 3.5, 3.7 and 3.8. Assume $f \in \mathcal{H}_x^s$ for $s \geq \max\{2 - 2\alpha + \frac{3}{p}, \frac{11}{2} - 2\alpha\eta - 3\alpha\}$. There exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$P(E_i(f, \beta, \alpha, \lambda)) \leq c_1 e^{-\frac{c_2 \lambda^2}{\|f\|_{\mathcal{H}_x^s}^2}}, \quad \text{for } i = 1, 2, 3.$$

Proof. We only show the estimate for $E_1(f, \beta, \alpha, \lambda)$, since the other two can be handled analogously. In view of Bienaymé-Tchebishev's inequality, we deduce

$$P(E_1(f, \beta, \alpha, \lambda)) = P\left(\left\{\omega \in \Omega : \|B_{f\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p} \geq \lambda\right\}\right) \leq (c_0 \sqrt{r} \lambda^{-1} \|f\|_{\mathcal{H}_x^s})^r \quad (3.2)$$

for some constant $c_0 > 0$. If

$$\left(\frac{\lambda}{c_0 e \|f\|_{\mathcal{H}_x^s}}\right)^2 \geq \frac{1}{2\beta},$$

we take $r = \left(\frac{\lambda}{c_0 e \|f\|_{\mathcal{H}_x^s}}\right)^2$. It then follows from (3.2) directly

$$P(E_i(f, \beta, \alpha, \lambda)) \leq e^{-\frac{\lambda^2}{c_0^2 \|f\|_{\mathcal{H}_x^s}^2}}.$$

If otherwise

$$\left(\frac{\lambda}{c_0 e \|f\|_{\mathcal{H}_x^s}}\right)^2 < \frac{1}{2\beta},$$

then there exists a constant c_1 such that we have from (3.2)

$$P(E_1(f, \beta, \alpha, \lambda)) \leq c_1 e^{-\frac{\lambda^2}{c_0^2 \|f\|_{\mathcal{H}_x^s}^2}},$$

by using the fact that $y^{-r} \leq c_1 e^{-y^2}$ for small $y > 0$. $\square$

Note that the estimates in the lemmas above hold for u_{g^ω} with slight modifications. Applying Lemmas 3.4 and 3.6 gives

Lemma 3.11. *Let $\frac{1}{2\alpha} + \frac{3}{2q\alpha} + 2\gamma = 1$. Assume $s \geq 1 - 2\alpha + \frac{3}{q}$. We have*

$$\begin{aligned} \|u_{g^\omega}\|_{L_\omega^r L_t^{\frac{1}{2\gamma}} L_x^q} &\lesssim \sqrt{r} \|g\|_{\mathcal{H}_x^s}, \quad \text{for } r \geq \frac{1}{2\gamma} \geq q \geq 2, \\ \|u_{g^\omega}\|_{L_\omega^r L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} &\lesssim \sqrt{r} \|g\|_{\mathcal{H}_x^s}, \quad \text{for } r \geq \frac{1}{\gamma} \geq q \geq 2. \end{aligned}$$

Lemma 3.12. *Let $r \geq 4$, $\zeta \geq 0$ and $s \geq -2\alpha\zeta - \frac{5}{2}\alpha + \frac{7}{2}$. Then*

$$\|u_{g^\omega}\|_{L_\omega^r L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \lesssim \sqrt{r} \|g\|_{\mathcal{H}_x^s}.$$

Proof. It follows from Minkowski's inequality and Lemma 2.1 that

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} &= \|t^\zeta (1 - \Delta)^{\frac{7}{4}-\alpha} B_{f^\omega}\|_{L_\omega^r L_t^4 L_x^2} \\ &= \left\| \sum_{k \in \mathbb{Z}^n} t^\zeta (1 + |k|^2)^{\frac{7}{4}-\alpha} e^{-t|k|^2\alpha} l_k(\omega) a_k e_k(x) \right\|_{L_\omega^r L_t^4 L_x^2} \\ &\lesssim \sqrt{r} \left\| \left(\sum_{k \in \mathbb{Z}^n} t^{2\zeta} (1 + |k|^2)^{\frac{7}{2}-2\alpha} e^{-2t|k|^2\alpha} |a_k|^2 |e_k|^2 \right)^{\frac{1}{2}} \right\|_{L_t^4 L_x^2} \\ &= \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\zeta} (1 + |k|^2)^{\frac{7}{2}-2\alpha} e^{-2t|k|^2\alpha} |a_k|^2 |e_k|^2 \right\|_{L_t^2 L_x^1}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\zeta} (1 + |k|^2)^{\frac{7}{2}-2\alpha} e^{-2t|k|^2\alpha} |a_k|^2 |e_k|^2 \right\|_{L_x^1 L_t^2}^{\frac{1}{2}}. \end{aligned}$$

We estimate the norm in time as before

$$\|t^{2\zeta} e^{-2t|k|^2\alpha}\|_{L_t^2} \lesssim |k|^{-4\alpha\zeta-\alpha}.$$

Thus we infer

$$\begin{aligned} \|B_{f^\omega}\|_{L_\omega^r L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} t^{2\zeta} (1 + |k|^2)^{\frac{7}{2}-2\alpha} e^{-2t|k|^2\alpha} |a_k|^2 |e_k|^2 \right\|_{L_x^1 L_t^2}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left\| \sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\zeta-\alpha+7-4\alpha} |a_k|^2 |e_k|^2 \right\|_{L_x^1}^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \left(\sum_{k \in \mathbb{Z}^n} |k|^{-4\alpha\zeta-5\alpha+7} |a_k|^2 \|e_k\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{r} \|f\|_{\mathcal{H}_x^{-2\alpha\zeta-\frac{5}{2}\alpha+\frac{7}{2}}}. \end{aligned}$$

$\square$

Denote

$$\begin{aligned} E_4(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|u_{g^\omega}\|_{L_t^{\frac{1}{2\gamma}} L_x^q} \geq \lambda \right\}, \\ E_5(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \geq \lambda \right\}, \\ E_6(f, \beta, \alpha, \lambda) &= \left\{ \omega \in \Omega : \|u_{g^\omega}\|_{L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \geq \lambda \right\}. \end{aligned}$$

Lemma 3.13. *Let α, γ, ζ and q satisfy the parameter conditions in Lemmas 3.11 and 3.12. Assume $f \in \mathcal{H}_x^s$ for $s \geq \max\{1 - 2\alpha + \frac{3}{q}, \frac{7}{2} - 2\alpha\zeta - \frac{5}{2}\alpha\}$. There exist constants $c_3 > 0$ and $c_4 > 0$ such that*

$$P(E_i(f, \beta, \alpha, \lambda)) \leq c_3 e^{-\frac{c_4 \lambda^2}{\|g\|_{\mathcal{H}_x^s}^2}}, \quad \text{for } i = 4, 5, 6.$$

Remark 3.14. Note that $\mathcal{H}^{\frac{5}{2}-2\alpha}$ is a critical space for the hyperdissipative NSE (1.3). We have for $q \geq 2$ and $\zeta \geq 0$,

$$\begin{cases} 1 - 2\alpha + \frac{3}{q} \leq \frac{5}{2} - 2\alpha, & \text{for } \alpha \geq 0, \\ \frac{7}{2} - 2\alpha\zeta - \frac{5}{2}\alpha \leq \frac{5}{2} - 2\alpha, & \text{for } \alpha \geq \frac{2}{4\zeta+1}. \end{cases}$$

Thus Lemmas 3.11, 3.12 and 3.13 are valid for initial data g in supercritical and critical regime.

4. Well-Posedness of the Electron MHD

We prove Theorem 1.1 for the electron MHD (1.4) in this section. As discussed earlier, the electron MHD contains the nonlinear term with the highest derivative from the Hall MHD system. Hence we encounter the most challenging estimates in this part.

To take the advantage of the improved L^p estimates for the free evolution $B_{f^\omega} = e^{-t(-\Delta)^\alpha} f^\omega$, we look for a solution of (1.4) with initial data f^ω in the form

$$B = B_{f^\omega} + H$$

with the nonlinear part H solving the Cauchy problem

$$\begin{aligned} H_t + \nabla \times \nabla \cdot ((B_{f^\omega} + H) \otimes (B_{f^\omega} + H)) &= -(-\Delta)^\alpha H, \\ \nabla \cdot H &= 0, \\ H(x, 0) &= 0. \end{aligned} \tag{4.1}$$

Here we used the rewriting

$$\nabla \times [(\nabla \times (B_{f^\omega} + H)) \times (B_{f^\omega} + H)] = \nabla \times \nabla \cdot ((B_{f^\omega} + H) \otimes (B_{f^\omega} + H))$$

since $\nabla \cdot H = 0$ and $\nabla \cdot B_{f^\omega} = 0$.

Obviously Theorem 1.1 follows from the well-posedness of (4.1) in a suitable subspace of $C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3))$. Therefore we only need to show:

Theorem 4.1. *Let $1 \leq \alpha < \frac{7}{4}$, $0 \leq \beta < \frac{1}{4}$, $0 < \eta < 1$ and $p \geq 2$. Moreover, the parameters satisfy $\frac{1}{\alpha} + \frac{3}{2p\alpha} + 2\beta = 1$. Assume f has zero-mean and*

$$f \in \mathcal{H}^s \quad \text{for } s \geq \max\{2 - 2\alpha + \frac{3}{p}, \frac{11}{2} - 2\alpha\eta - 3\alpha\}. \tag{4.2}$$

For any $m \in (0, 1)$, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) \geq m$ such that for any $\omega \in \Sigma$ system (4.1) has a unique solution H satisfying

$$H \in C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)) \cap L^{(\beta, \frac{1}{\beta})}(0, T; L^p(\mathbb{T}^3)) \cap L^{\frac{1}{2\beta}}(0, T; L^p(\mathbb{T}^3))$$

for some $T > 0$. Moreover, there exists a constant $c > 0$ such that if $\|f\|_{\mathcal{H}^s} \leq c$, T can be arbitrarily large.

Proof. We proceed by employing a fixed point argument. Denote

$$Q(x, t) = (B_{f\omega} + H) \otimes (B_{f\omega} + H)(x, t).$$

The integral form of (4.1) is given by

$$H(x, t) = - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot Q(x, \tau) d\tau. \quad (4.3)$$

Denote the map

$$\Phi(H)(t) = - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot Q(x, \tau) d\tau.$$

Define the subspace $\mathcal{Y} \subset C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3))$ as

$$\mathcal{Y} = C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)) \cap L^{(\beta, \frac{1}{\beta})}(0, T; L^p(\mathbb{T}^3)) \cap L^{\frac{1}{2\beta}}(0, T; L^p(\mathbb{T}^3)).$$

We claim that the map Φ is a contraction on $\mathcal{Y}$ by showing that:

- (i) Φ maps $\mathcal{Y}$ onto itself;
- (ii) For any $H_1 \in \mathcal{Y}$ and $H_2 \in \mathcal{Y}$, we have

$$\|\Phi(H_1) - \Phi(H_2)\|_{\mathcal{Y}} \leq c\|H_1 - H_2\|_{\mathcal{Y}}.$$

In order to show (i), we estimate $\|\Phi(H)\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)}$, $\|\Phi(H)\|_{L^{(\beta, \frac{1}{\beta})}(0, T; L^p(\mathbb{T}^3))}$ and $\|\Phi(H)\|_{L^{\frac{1}{2\beta}}(0, T; L^p(\mathbb{T}^3))}$ respectively in the following. We first expand $\Phi(H)$ as

$$\begin{aligned} \Phi(H)(t) &= - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H(x, \tau) \otimes H(x, \tau)) d\tau \\ &\quad - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H(x, \tau) \otimes B_{f\omega}(x, \tau)) d\tau \\ &\quad - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (B_{f\omega}(x, \tau) \otimes H(x, \tau)) d\tau \\ &\quad - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) d\tau \\ &=: -\Phi_1 - \Phi_2 - \Phi_3 - \Phi_4. \end{aligned}$$

We estimate Φ_1 in $\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)$ as

$$\begin{aligned} &\|\Phi_1\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \\ &= \left\| (1 - \Delta)^{\frac{7}{4}-\alpha} \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_x^2} \\ &= \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(1 - \Delta)^{\frac{7}{4}-\alpha} \frac{\nabla \times \nabla \cdot}{(-\Delta)} (H(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_x^2} \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(1 - \Delta)^{\frac{7}{4}-\alpha} (H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^2} d\tau \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^2 \mathcal{F} \left((1 - \Delta)^{\frac{7}{4}-\alpha} (H(x, \tau) \otimes H(x, \tau)) \right) \right\|_{L_\xi^2} d\tau \end{aligned}$$

where Plancherel's theorem was applied in the last step. Using Hölder's inequality and Lemma 3.2 we obtain

$$\begin{aligned}
& \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2\alpha} \mathcal{F} \left((1-\Delta)^{\frac{7}{4}-\alpha} (H(x, \tau) \otimes H(x, \tau)) \right) \right\|_{L_\xi^2} \\
& \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2\alpha} \right\|_{L_\xi^p} \left\| \mathcal{F} \left((1-\Delta)^{\frac{7}{4}-\alpha} (H(x, \tau) \otimes H(x, \tau)) \right) \right\|_{L_\xi^{\frac{2p}{p-2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{\alpha}-\frac{3}{2p\alpha}} \left\| (1-\Delta)^{\frac{7}{4}-\alpha} (H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^{\frac{2p}{p+2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{\alpha}-\frac{3}{2p\alpha}} \left\| H \nabla^{\frac{7}{2}-2\alpha} H \right\|_{L_x^{\frac{2p}{p+2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{\alpha}-\frac{3}{2p\alpha}} \|H\|_{L_x^p} \left\| \nabla^{\frac{7}{2}-2\alpha} H \right\|_{L_x^2}.
\end{aligned}$$

It follows from the last two inequalities that

$$\begin{aligned}
& \|\Phi_1\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \\
& \lesssim \int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{3}{2p\alpha}} \|H\|_{L_x^p} \left\| \nabla^{\frac{7}{2}-2\alpha} H \right\|_{L_x^2} d\tau \\
& \lesssim \|H\|_{L_t^\infty \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{3}{2p\alpha}} \tau^{-\beta} (\tau^\beta \|H\|_{L_x^p}) d\tau \\
& \lesssim \|H\|_{L_t^\infty \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \left(\int_0^t (t-\tau)^{(-\frac{1}{\alpha}-\frac{3}{2p\alpha})\frac{1}{1-\beta}} \tau^{-\frac{\beta}{1-\beta}} d\tau \right)^{1-\beta} \left(\int_0^t \tau \|H\|_{L_x^p}^{\frac{1}{\beta}} d\tau \right)^\beta.
\end{aligned}$$

Based on the assumptions on the parameters, we observe that for $0 < \beta < \frac{1}{2}$

$$\left(\frac{1}{\alpha} + \frac{3}{2p\alpha} \right) \frac{1}{1-\beta} + \frac{\beta}{1-\beta} = 1,$$

and

$$0 < \left(\frac{1}{\alpha} + \frac{3}{2p\alpha} \right) \frac{1}{1-\beta} < 1, \quad 0 < \frac{\beta}{1-\beta} < 1.$$

Hence Lemma 3.3 implies the time integral is bounded. Therefore we conclude

$$\|\Phi_1\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \lesssim \|H\|_{L_t^\infty \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}. \quad (4.4)$$

We continue to estimate Φ_1 in $L_t^{(\beta, \frac{1}{\beta})} L_x^p$,

$$\begin{aligned}
& \|\Phi_1\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\
& = \left\| t^\beta \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{\beta}} L_x^p} \\
& \lesssim \left\| t^\beta \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}}.
\end{aligned}$$

Applying Hölder's inequality and Lemma 3.2 yields

$$\begin{aligned}
& \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} \\
& \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^2 \mathcal{F}((H(x, \tau) \otimes H(x, \tau))) \right\|_{L_\xi^{\frac{p}{p-1}}} \\
& \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^2 \right\|_{L_\xi^p} \left\| \mathcal{F}((H(x, \tau) \otimes H(x, \tau))) \right\|_{L_\xi^{\frac{p}{p-2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H(x, \tau) \otimes H(x, \tau)\|_{L_x^{\frac{p}{2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2.
\end{aligned}$$

Combining the last two inequalities we infer

$$\begin{aligned}
& \|\Phi_1\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\
& \lesssim \left\| \int_0^t t^\beta (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \lesssim \left\| \int_0^{\frac{t}{2}} t^\beta (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \quad + \left\| \int_{\frac{t}{2}}^t t^\beta (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& =: I_1 + I_2.
\end{aligned}$$

The term I_1 can be estimated by using Young's inequality

$$\begin{aligned}
& \left\| \int_0^{\frac{t}{2}} t^\beta (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \lesssim \left\| \int_0^{\frac{t}{2}} (t-\tau)^{\beta - \frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \lesssim \left(\int_0^{\frac{t}{2}} (t-\tau)^{(\beta - \frac{1}{\alpha} - \frac{3}{2p\alpha})(\frac{1}{1-3\beta})} d\tau \right)^{1-3\beta} \left(\int_0^{\frac{t}{2}} \|H\|_{L_x^p}^{2 \cdot \frac{1}{4\beta}} d\tau \right)^{4\beta} \\
& \lesssim \left(\int_0^{\frac{t}{2}} (t-\tau)^{-1} d\tau \right)^{1-3\beta} \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2 \\
& \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2.
\end{aligned}$$

While the term I_2 is estimated by employing Hölder's inequality and Hardy–Littlewood–Sobolev lemma

$$\begin{aligned}
& \left\| \int_{\frac{t}{2}}^t t^\beta (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \lesssim \left\| \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \tau^\beta \|H\|_{L_x^p} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
& \lesssim \left\| \left(\int_{\frac{t}{2}}^t \tau \|H\|_{L_x^p}^{\frac{1}{\beta}} d\tau \right)^\beta \left(\int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{\alpha} - \frac{3}{2p\alpha}) \frac{1}{1-\beta}} \|H\|_{L_x^p}^{\frac{1}{1-\beta}} d\tau \right)^{1-\beta} \right\|_{L_t^{\frac{1}{\beta}}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \left\| \int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{\alpha} - \frac{3}{2p\alpha}) \frac{1}{1-\beta}} \|H\|_{L_x^p}^{\frac{1}{1-\beta}} d\tau \right\|_{L_t^{\frac{1-\beta}{\beta}}}^{1-\beta} \\
&\lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \left\| \|H\|_{L_x^p}^{\frac{1}{1-\beta}} \right\|_{L_t^{\frac{1-\beta}{\beta}}}^{1-\beta} \\
&\lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}.
\end{aligned}$$

Putting together the estimates above we get

$$\|\Phi_1\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2 + \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}. \quad (4.5)$$

While to estimate Φ_1 in $L_t^{\frac{1}{2\beta}} L_x^p$, we start with

$$\begin{aligned}
&\|\Phi_1\|_{L_t^{\frac{1}{2\beta}} L_x^p} \\
&= \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\beta}} L_x^p} \\
&\lesssim \left\| \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{2\beta}}}.
\end{aligned}$$

Similarly as before, we deduce by applying Hölder's inequality and Lemma 3.2

$$\begin{aligned}
&\left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)(H(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} \\
&\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^2 \mathcal{F}(H(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^{\frac{p}{p-1}}} \\
&\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^2 \right\|_{L_\xi^p} \|\mathcal{F}(H(x, \tau) \otimes H(x, \tau))\|_{L_\xi^{\frac{p}{p-2}}} \\
&\lesssim (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H(x, \tau) \otimes H(x, \tau)\|_{L_x^{\frac{p}{2}}} \\
&\lesssim (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2.
\end{aligned}$$

Invoking Hardy–Littlewood–Sobolev lemma again, the last two inequalities together imply

$$\|\Phi_1\|_{L_t^{\frac{1}{2\beta}} L_x^p} \lesssim \left\| \int_0^t (t-\tau)^{-\frac{1}{\alpha} - \frac{3}{2p\alpha}} \|H\|_{L_x^p}^2 d\tau \right\|_{L_t^{\frac{1}{2\beta}}} \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2. \quad (4.6)$$

The estimate of Φ_4 is analogous to that of Φ_1 . We sketch some details below. Regarding the estimate in $\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)$, we split Φ_4 as

$$\begin{aligned}
&\|\Phi_4\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \\
&= \left\| (1-\Delta)^{\frac{7}{4}-\alpha} \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) d\tau \right\|_{L_x^2} \\
&\lesssim \int_0^\delta \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2+\frac{7}{2}-2\alpha} \mathcal{F}(B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right\|_{L_\xi^2} d\tau \\
&\quad + \int_\delta^t \left\| e^{-(t-\tau)|\xi|^{2\alpha}} \mathcal{F} \left((-\Delta)(1-\Delta)^{\frac{7}{4}-\alpha} (B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right) \right\|_{L_\xi^2} d\tau \\
&=: I_3 + I_4.
\end{aligned}$$

where $\delta \in (0, t)$ is a fixed small constant. Applying Lemma 3.2 we infer for $2 \leq p \leq 4$

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2+\frac{7}{2}-2\alpha} \mathcal{F}(B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right\|_{L_\xi^2} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2+\frac{7}{2}-2\alpha} \right\|_{L_\xi^{\frac{2p}{4-p}}} \left\| \mathcal{F}(B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right\|_{L_\xi^{\frac{p}{p-2}}} \\ & \lesssim (t-\tau)^{-\frac{11}{4\alpha}+1-\frac{3}{2\alpha}\cdot\frac{4-p}{2p}} \|B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)\|_{L_x^{\frac{p}{2}}} \\ & \lesssim (t-\tau)^{-\frac{11}{4\alpha}+1-\frac{3(4-p)}{4p\alpha}} \|B_{f\omega}\|_{L_x^p}^2. \end{aligned}$$

Thus, applying Hölder's inequality and Lemma 3.3 leads to

$$\begin{aligned} I_3 & \lesssim \int_0^\delta (t-\tau)^{-\frac{11}{4\alpha}+1-\frac{3(4-p)}{4p\alpha}} \|B_{f\omega}\|_{L_x^p}^2 d\tau \\ & = \int_0^{\frac{t}{2}} \left((t-\tau)^{-\frac{11}{4\alpha}+1-\frac{3(4-p)}{4p\alpha}} \tau^{-2\beta} \right) (\tau^\beta \|B_{f\omega}\|_{L_x^p})^2 d\tau \\ & \lesssim \left(\int_0^\delta (t-\tau)^{(-\frac{11}{4\alpha}+1-\frac{3(4-p)}{4p\alpha})\frac{1}{1-2\beta}} \tau^{-2\beta\cdot\frac{1}{1-2\beta}} d\tau \right)^{1-2\beta} \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}^2 \\ & = \left(\int_0^\delta (t-\tau)^{-1+\frac{2\beta}{1-2\beta}} \tau^{-\frac{2\beta}{1-2\beta}} d\tau \right)^{1-2\beta} \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}^2 \\ & \lesssim \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}^2. \end{aligned}$$

On the other hand, for $p > 4$ we have

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2+\frac{7}{2}-2\alpha} \mathcal{F}(B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right\|_{L_\xi^2} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{2+\frac{7}{2}-2\alpha} \right\|_{L_\xi^\infty} \left\| \mathcal{F}(B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)) \right\|_{L_\xi^2} \\ & \lesssim (t-\tau)^{-\frac{11}{4\alpha}+1} \|B_{f\omega}(x, \tau) \otimes B_{f\omega}(x, \tau)\|_{L_x^2} \\ & \lesssim (t-\tau)^{-\frac{11}{4\alpha}+1} \|B_{f\omega}\|_{L_x^4}^2 \end{aligned}$$

and hence

$$\begin{aligned} I_3 & \lesssim \int_0^\delta (t-\tau)^{-\frac{11}{4\alpha}+1} \|B_{f\omega}\|_{L_x^4}^2 d\tau \\ & = \int_0^\delta \left((t-\tau)^{-\frac{11}{4\alpha}+1} \tau^{-2\beta} \right) (\tau^\beta \|B_{f\omega}\|_{L_x^4})^2 d\tau \\ & \lesssim \left(\int_0^\delta (t-\tau)^{(-\frac{11}{4\alpha}+1)\frac{1}{1-2\beta}} \tau^{-\frac{2\beta}{1-2\beta}} d\tau \right)^{1-2\beta} \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^4}^2 \\ & \lesssim \|B_{f\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^4}^2. \end{aligned}$$

In the estimate above, the time integral is handled as

$$\begin{aligned} & \int_0^\delta (t-\tau)^{(-\frac{11}{4\alpha}+1)\frac{1}{1-2\beta}} \tau^{-\frac{2\beta}{1-2\beta}} d\tau \\ & \lesssim t^{(-\frac{11}{4\alpha}+1)\frac{1}{1-2\beta}} \int_0^{\frac{t}{2}} \tau^{-\frac{2\beta}{1-2\beta}} d\tau \\ & \lesssim 1 \end{aligned}$$

since $t > \delta > 0$ and observing that for $1 \leq \alpha < \frac{7}{4}$ and $0 \leq \beta < \frac{1}{4}$

$$\left(-\frac{11}{4\alpha} + 1\right) \frac{1}{1-2\beta} \leq 0, \quad 0 \leq \frac{2\beta}{1-2\beta} < 1.$$

We continue to estimate I_4 . Again invoking Hölder's inequality and Lemma 3.2 we obtain

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} \mathcal{F} \left((-\Delta)(1-\Delta)^{\frac{7}{4}-\alpha} (B_{f^\omega}(x, \tau) \otimes B_{f^\omega}(x, \tau)) \right) \right\|_{L_\xi^2} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} \right\|_{L_\xi^p} \left\| \mathcal{F} \left((-\Delta)(1-\Delta)^{\frac{7}{4}-\alpha} (B_{f^\omega}(x, \tau) \otimes B_{f^\omega}(x, \tau)) \right) \right\|_{L_\xi^{\frac{2p}{p-2}}} \\ & \lesssim (t-\tau)^{-\frac{3}{2p\alpha}} \left\| (-\Delta)(1-\Delta)^{\frac{7}{4}-\alpha} (B_{f^\omega}(x, \tau) \otimes B_{f^\omega}(x, \tau)) \right\|_{L_x^{\frac{2p}{p+2}}} \\ & \lesssim (t-\tau)^{-\frac{3}{2p\alpha}} \|\nabla^{\frac{11}{2}-2\alpha} B_{f^\omega}\|_{L_x^2} \|B_{f^\omega}\|_{L_x^p}. \end{aligned}$$

Therefore we have for $1 \leq \alpha < \frac{7}{4}$ and $0 < \eta < 1$

$$\begin{aligned} I_4 & \lesssim \int_\delta^t (t-\tau)^{-\frac{3}{2p\alpha}} \|\nabla^{\frac{11}{2}-2\alpha} B_{f^\omega}\|_{L_x^2} \|B_{f^\omega}\|_{L_x^p} d\tau \\ & = \int_\delta^t (t-\tau)^{-\frac{3}{2p\alpha}} \tau^{-\eta-\beta} \left(\tau^\eta \|\nabla^{\frac{11}{2}-2\alpha} B_{f^\omega}\|_{L_x^2} \right) \left(\tau^\beta \|B_{f^\omega}\|_{L_x^p} \right) d\tau \\ & \lesssim \left(\int_\delta^t (t-\tau)^{-\frac{3}{2p\alpha} \cdot \frac{2}{1-2\beta}} \tau^{(-\eta-\beta) \frac{2}{1-2\beta}} d\tau \right)^{\frac{1-2\beta}{2}} \|B_{f^\omega}\|_{L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\ & \lesssim \|B_{f^\omega}\|_{L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \end{aligned}$$

where the time integral is estimated as

$$\begin{aligned} & \int_\delta^t (t-\tau)^{-\frac{3}{2p\alpha} \cdot \frac{2}{1-2\beta}} \tau^{(-\eta-\beta) \frac{2}{1-2\beta}} d\tau \\ & \lesssim \delta^{-\frac{2(\eta+\beta)}{1-2\beta}} \int_\delta^t (t-\tau)^{-\frac{3}{2p\alpha} \cdot \frac{2}{1-2\beta}} d\tau \\ & \lesssim 1 \end{aligned}$$

since $t > \delta > 0$ and $p > \frac{3}{\alpha(1-2\beta)}$ (hence $0 < \frac{3}{p\alpha(1-2\beta)} < 1$). Summarizing the estimates above leads to

$$\begin{aligned} \|\Phi_4\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} & \lesssim \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}^2 + \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^4}^2 \\ & \quad + \|B_{f^\omega}\|_{L_t^{(\eta,2)} \mathcal{H}_x^{\frac{11}{2}-2\alpha}} \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}. \end{aligned} \quad (4.7)$$

The estimates of Φ_4 in $L_t^{(\beta, \frac{1}{\beta})} L_x^p$ and $L_t^{\frac{1}{2\beta}} L_x^p$ are identical to that of Φ_1 . Hence we collect the estimates

$$\begin{aligned} \|\Phi_4\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} & \lesssim \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2 + \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p}, \\ \|\Phi_4\|_{L_t^{\frac{1}{2\beta}} L_x^p} & \lesssim \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p}^2. \end{aligned} \quad (4.8)$$

Obviously Φ_2 and Φ_3 can be estimated similarly. Moreover, we observe that $\|\Phi_2\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)}$ shares an analogous estimate with $\|\Phi_4\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)}$, while $\|\Phi_2\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}$ and $\|\Phi_2\|_{L_t^{\frac{1}{2\beta}} L_x^p}$ enjoy analogous estimates

with $\|\Phi_1\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}$ and $\|\Phi_1\|_{L_t^{\frac{1}{2\beta}} L_x^p}$ respectively. Therefore we claim

$$\begin{aligned}
 & \|\Phi_2\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} + \|\Phi_3\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \\
 & \lesssim \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} + \|B_{f^\omega}\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^4} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\
 & \quad + \|B_{f^\omega}\|_{L_t^{(\eta, 2)} \mathcal{H}_x^{\frac{1}{2}-2\alpha}} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}, \\
 & \|\Phi_2\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} + \|\Phi_3\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\
 & \lesssim \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p} + \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}, \\
 & \|\Phi_2\|_{L_t^{\frac{1}{2\beta}} L_x^p} + \|\Phi_3\|_{L_t^{\frac{1}{2\beta}} L_x^p} \lesssim \|B_{f^\omega}\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p}.
 \end{aligned} \tag{4.9}$$

It follows from the estimates (4.4)–(4.9) and Lemma 3.10 that for almost every $\omega \in \Omega$

$$\|\Phi(H)\|_{\mathcal{Y}} \lesssim \|H\|_{\mathcal{Y}}^2 + \lambda^2 \tag{4.10}$$

which verifies condition (i) that Φ maps the subspace $\mathcal{Y}$ onto $\mathcal{Y}$.

Regarding (ii), straightforward algebra shows that for $H_1, H_2 \in \mathcal{Y}$

$$\begin{aligned}
 \Phi(H_1)(t) - \Phi(H_2)(t) = & - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (H_1 \otimes (H_1 - H_2)) d\tau \\
 & - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot ((H_1 - H_2) \otimes H_2) d\tau \\
 & - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot (B_{f^\omega} \otimes (H_1 - H_2)) d\tau \\
 & - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot ((H_1 - H_2) \otimes B_{f^\omega}) d\tau.
 \end{aligned}$$

One notices that $\Phi(H_1) - \Phi(H_2)$ can be estimated similarly as $\Phi(H)$ in the space $\mathcal{Y}$ and

$$\|\Phi(H_1) - \Phi(H_2)\|_{\mathcal{Y}} \lesssim (\|H_1\|_{\mathcal{Y}} + \|H_2\|_{\mathcal{Y}} + \lambda) \|H_1 - H_2\|_{\mathcal{Y}}. \tag{4.11}$$

We are ready to finish the proof the theorem by applying a fixed point argument. Indeed, following from (4.10) and (4.11), there exists a constant $C > 0$ such that

$$\begin{aligned}
 \|\Phi(H)\|_{\mathcal{Y}} & \leq C (\|H\|_{\mathcal{Y}}^2 + \lambda^2), \\
 \|\Phi(H_1) - \Phi(H_2)\|_{\mathcal{Y}} & \leq C (\|H_1\|_{\mathcal{Y}} + \|H_2\|_{\mathcal{Y}} + \lambda) \|H_1 - H_2\|_{\mathcal{Y}}.
 \end{aligned} \tag{4.12}$$

For such C , we then choose λ such that

$$C((2C\lambda^2)^2 + \lambda^2) \leq 2C\lambda^2, \quad C(4C\lambda^2 + \lambda) < 1$$

which are satisfied for $C^2\lambda^2 \leq \frac{1}{8}$. Thus a suitable choice is $\lambda = \bar{\lambda} = \frac{1}{3C}$. Therefore thanks to (4.12), for such C and $\bar{\lambda}$, the map Φ is a contraction on the ball $B(0, 2C\bar{\lambda}^2) = B(0, \frac{2}{9C}) \subset \mathcal{Y}$. Denote

$$\begin{aligned}
 & E(f, \beta, \alpha, \bar{\lambda}, T) \\
 & = \left\{ \omega \in \Omega : \|B_{f^\omega}\|_{L_T^{\frac{1}{2\beta}} L_x^p} + \|B_{f^\omega}\|_{L_T^{(\beta, \frac{1}{\beta})} L_x^p} + \|B_{f^\omega}\|_{L_T^{(\eta, 2)} \mathcal{H}_x^{\frac{1}{2}-2\alpha}} \geq \bar{\lambda} \right\}.
 \end{aligned}$$

Take $\Omega_T = E^c(f, \beta, \alpha, \bar{\lambda}, T)$ and $\Sigma = \cup_{-\infty < j < \infty} \Omega_{2^j}$. One can see that

$$E(f, \beta, \alpha, \bar{\lambda}, T_1) \subset E(f, \beta, \alpha, \bar{\lambda}, T_2) \quad \text{for } T_1 \leq T_2$$

and hence $\Omega_{T_2} \subset \Omega_{T_1}$.

We first show the existence on $[0, \infty)$ with smallness assumption on $\|f\|_{\mathcal{H}_x^s}$. Indeed, for any $T > 0$ there exists k such that $2^{k-1} \leq T < 2^k$ and

$$\Omega_{2^k} \subset \Omega_T \subset \Omega_{2^{k-1}}.$$

Therefore we infer from Lemma 3.10 that

$$P(\Omega_T) \geq P(\Omega_{2^k}) = 1 - P(E(f, \beta, \alpha, \bar{\lambda}, 2^k)) \geq 1 - 3c_1 e^{-\frac{c_2 \bar{\lambda}^2}{\|f\|_{\mathcal{H}_x^s}^2}}.$$

It follows that for any $m \in (0, 1)$, if

$$\|f\|_{\mathcal{H}_x^s}^2 \leq \frac{c_2}{9C^2(\ln(3c_1) - \ln(1-m))}, \quad (4.13)$$

we have $P(\Omega_T) \geq m$ and hence $P(\Sigma) \geq m$.

Regarding the local existence without assumption on the size of $\|f\|_{\mathcal{H}_x^s}$, (4.13) indicates $P(\Omega_T) \geq m$ holds for

$$C^2 \leq \frac{c_2}{9\|f\|_{\mathcal{H}_x^s}^2(\ln(3c_1) - \ln(1-m))}. \quad (4.14)$$

In view of the integral form (4.3) and $H(x, 0) = 0$, we know the constant C appeared in (4.12) satisfies (4.14) for small time $T > 0$.

Noting $2 - 2\alpha + \frac{3}{p} = -4\alpha\beta \leq 0$ and taking $\eta = \frac{1}{2}$ in (4.2), we know the conclusion of Theorem 1.1 holds for $s \geq \max\{\frac{11}{2} - 4\alpha, 0\}$. □

5. Well-Posedness of the Hyperdissipative NSE

This section concerns the well-posedness for the hyperdissipative NSE (1.3). Namely we will prove Theorem 1.3. The proof is analogous to that of Theorem 1.1 presented in Sect. 4. We only include limited details to reveal the requirement on the parameters as stated in Theorem 1.3.

We consider a solution to (1.3) in the form $u = u_{g^\omega} + V$ with $u_{g^\omega} = e^{-t(-\Delta)^\alpha} g^\omega$ and V satisfying

$$\begin{aligned} V_t + (u_{g^\omega} + V) \cdot \nabla (u_{g^\omega} + V) + \nabla \Pi &= -(-\Delta)^\alpha V, \\ \nabla \cdot V &= 0, \\ V(x, 0) &= 0. \end{aligned} \quad (5.1)$$

Denote the map through the integral form of (5.1)

$$\Psi(V)(t) = - \int_0^t e^{-(t-\tau)^{2\alpha}} \mathbb{P} \nabla \cdot ((u_{g^\omega} + V) \otimes (u_{g^\omega} + V)) \, d\tau.$$

For $1 \leq \alpha < \frac{7}{4}$, define the subspace $\mathcal{X} \subset C([0, T]; \mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3))$ as

$$\mathcal{X} = C([0, T]; \mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)) \cap L^{(\gamma, \frac{1}{\gamma})}(0, T; L^q(\mathbb{T}^3)) \cap L^{\frac{1}{2\gamma}}(0, T; L^q(\mathbb{T}^3))$$

where $0 \leq \gamma < \frac{1}{2}$ and $q \geq 2$ satisfy $\frac{1}{2\alpha} + \frac{3}{2q\alpha} + 2\gamma = 1$ as in Lemma 3.11.

We only need to show that

$$\begin{aligned} \|\Psi(V)\|_{\mathcal{X}} &\lesssim \lambda^2 + \|V\|_{\mathcal{X}}^2 \quad \forall V \in \mathcal{X}, \\ \|\Psi(V_1) - \Psi(V_2)\|_{\mathcal{X}} &\lesssim (\lambda + \|V_1\|_{\mathcal{X}} + \|V_2\|_{\mathcal{X}}) \|V_1 - V_2\|_{\mathcal{X}} \quad \forall V_1, V_2 \in \mathcal{X}. \end{aligned} \quad (5.2)$$

Then a similar probability analysis combined with the fixed point argument as in the previous section provides a proof of Theorem 1.3.

Note that it is sufficient to prove the first inequality of (5.2) since the second one can be obtained similarly. Moreover, we only show details to estimate terms involving $V \otimes V$ and $u_{g^\omega} \otimes u_{g^\omega}$ as the mixed terms $u_{g^\omega} \otimes V$ and $V \otimes u_{g^\omega}$ can be estimated analogously.

We first estimate $\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau$ in $\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)$,

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)} \\ &= \left\| (1-\Delta)^{\frac{5}{4}-\alpha} \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_x^2} \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (1-\Delta)^{\frac{5}{4}-\alpha} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^2} d\tau \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F} \left((1-\Delta)^{\frac{5}{4}-\alpha} (V(x, \tau) \otimes V(x, \tau)) \right) \right\|_{L_\xi^2} d\tau \end{aligned}$$

followed by applying Hölder's inequality and Lemma 3.2

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F} \left((1-\Delta)^{\frac{5}{4}-\alpha} (V(x, \tau) \otimes V(x, \tau)) \right) \right\|_{L_\xi^2} \\ &\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \right\|_{L_\xi^q} \left\| \mathcal{F} \left((1-\Delta)^{\frac{5}{4}-\alpha} (V(x, \tau) \otimes V(x, \tau)) \right) \right\|_{L_\xi^{\frac{2q}{q-2}}} \\ &\lesssim (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \left\| (1-\Delta)^{\frac{5}{4}-\alpha} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^{\frac{2q}{q+2}}} \\ &\lesssim (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \left\| \nabla^{\frac{5}{2}-2\alpha} V \right\|_{L_x^{\frac{2q}{q+2}}} \\ &\lesssim (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|V\|_{L_x^q} \left\| \nabla^{\frac{5}{2}-2\alpha} V \right\|_{L_x^2}. \end{aligned}$$

Combining the last two inequalities yields

$$\begin{aligned} & \|\Phi_1\|_{\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)} \\ &\lesssim \int_0^t (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|V\|_{L_x^q} \left\| \nabla^{\frac{5}{2}-2\alpha} V \right\|_{L_x^2} d\tau \\ &\lesssim \|V\|_{L_t^\infty \mathcal{H}_x^{\frac{5}{2}-2\alpha}} \int_0^t (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \tau^{-\gamma} (\tau^\gamma \|V\|_{L_x^q}) d\tau \\ &\lesssim \|V\|_{L_t^\infty \mathcal{H}_x^{\frac{5}{2}-2\alpha}} \left(\int_0^t (t-\tau)^{(-\frac{1}{2\alpha}-\frac{3}{2q\alpha})\frac{1}{1-\gamma}} \tau^{-\frac{\gamma}{1-\gamma}} d\tau \right)^{1-\gamma} \left(\int_0^t \tau \|V\|_{L_x^q}^{\frac{1}{\gamma}} d\tau \right)^\gamma. \end{aligned}$$

Noticing that

$$\begin{aligned} & \left(\frac{1}{2\alpha} + \frac{3}{2q\alpha} \right) \frac{1}{1-\gamma} + \frac{\gamma}{1-\gamma} = 1, \\ & 0 < \left(\frac{1}{2\alpha} + \frac{3}{2q\alpha} \right) \frac{1}{1-\gamma} < 1, \quad 0 < \frac{\gamma}{1-\gamma} < 1, \end{aligned}$$

It follows from Lemma 3.3 that

$$\int_0^t (t-\tau)^{(-\frac{1}{2\alpha}-\frac{3}{2q\alpha})\frac{1}{1-\gamma}} \tau^{-\frac{\gamma}{1-\gamma}} d\tau \lesssim 1.$$

Therefore we have

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)} \\ &\lesssim \|V\|_{L_t^\infty \mathcal{H}_x^{\frac{5}{2}-2\alpha}} \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}. \end{aligned} \tag{5.3}$$

The estimate of $\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}$ is given by

$$\begin{aligned} & \left\| t^\gamma \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{\gamma}} L_x^q} \\ & \lesssim \left\| t^\gamma \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^q} d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \end{aligned}$$

and

$$\begin{aligned} & \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^q} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F}((V(x, \tau) \otimes V(x, \tau))) \right\|_{L_\xi^{\frac{q}{q-1}}} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \right\|_{L_\xi^q} \left\| \mathcal{F}((V(x, \tau) \otimes V(x, \tau))) \right\|_{L_\xi^{\frac{q}{q-2}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V(x, \tau) \otimes V(x, \tau)\|_{L_x^{\frac{q}{2}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2. \end{aligned}$$

Therefore it follows

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \\ & \lesssim \left\| \int_0^t t^\gamma (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\ & \lesssim \left\| \int_0^{\frac{t}{2}} t^\gamma (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\ & \quad + \left\| \int_{\frac{t}{2}}^t t^\gamma (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\ & =: I_5 + I_6. \end{aligned}$$

The term I_5 is estimated as by using Hölder's inequality

$$\begin{aligned} & \left\| \int_0^{\frac{t}{2}} t^\gamma (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\ & \lesssim \left\| \int_0^{\frac{t}{2}} (t-\tau)^{\gamma - \frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\ & \lesssim \left(\int_0^{\frac{t}{2}} (t-\tau)^{(\gamma - \frac{1}{2\alpha} - \frac{3}{2q\alpha})(\frac{1}{1-3\gamma})} d\tau \right)^{1-3\gamma} \left(\int_0^{\frac{t}{2}} \|V\|_{L_x^q}^{2 \cdot \frac{1}{4\gamma}} d\tau \right)^{4\gamma} \\ & = \left(\int_0^{\frac{t}{2}} (t-\tau)^{-1} d\tau \right)^{1-3\gamma} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2 \\ & \lesssim \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2 \end{aligned}$$

and the term I_6 is estimated by applying Hölder's inequality and Hardy–Littlewood–Sobolev lemma

$$\begin{aligned}
& \left\| \int_{\frac{t}{2}}^t t^\gamma (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|H\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\
& \lesssim \left\| \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \tau^\gamma \|V\|_{L_x^q} \|V\|_{L_x^q} d\tau \right\|_{L_t^{\frac{1}{\gamma}}} \\
& \lesssim \left\| \left(\int_{\frac{t}{2}}^t \tau \|V\|_{L_x^q}^{\frac{1}{\gamma}} d\tau \right)^\gamma \left(\int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{2\alpha}-\frac{3}{2q\alpha})\frac{1}{1-\gamma}} \|V\|_{L_x^q}^{\frac{1}{1-\gamma}} d\tau \right)^{1-\gamma} \right\|_{L_t^{\frac{1}{\gamma}}} \\
& \lesssim \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \left\| \int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{2\alpha}-\frac{3}{2q\alpha})\frac{1}{1-\gamma}} \|V\|_{L_x^q}^{\frac{1}{1-\gamma}} d\tau \right\|_{L_t^{\frac{1-\gamma}{\gamma}}}^{1-\gamma} \\
& \lesssim \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \left\| \|V\|_{L_x^q}^{\frac{1}{1-\gamma}} \right\|_{L_t^{\frac{1-\gamma}{2\gamma}}}^{1-\gamma} \\
& \lesssim \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}.
\end{aligned}$$

The estimates above together imply

$$\begin{aligned}
& \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \\
& \lesssim \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2 + \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}.
\end{aligned} \tag{5.4}$$

Finally we estimate $\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\gamma}} L_x^q}$,

$$\begin{aligned}
& \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\gamma}} L_x^q} \\
& \lesssim \left\| \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^q} d\tau \right\|_{L_t^{\frac{1}{2\gamma}}},
\end{aligned}$$

and since

$$\begin{aligned}
& \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes V(x, \tau)) \right\|_{L_x^q} \\
& \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F}(V(x, \tau) \otimes V(x, \tau)) \right\|_{L_\xi^{\frac{q}{q-1}}} \\
& \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \right\|_{L_\xi^q} \left\| \mathcal{F}(V(x, \tau) \otimes V(x, \tau)) \right\|_{L_\xi^{\frac{q}{q-2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|V(x, \tau) \otimes V(x, \tau)\|_{L_x^{\frac{q}{2}}} \\
& \lesssim (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|V\|_{L_x^q}^2.
\end{aligned}$$

it follows from Hardy–Littlewood–Sobolev lemma that

$$\begin{aligned}
& \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\gamma}} L_x^q} \\
& \lesssim \left\| \int_0^t (t-\tau)^{-\frac{1}{2\alpha}-\frac{3}{2q\alpha}} \|V\|_{L_x^q}^2 d\tau \right\|_{L_t^{\frac{1}{2\gamma}}} \\
& \lesssim \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2.
\end{aligned} \tag{5.5}$$

Therefore it follows from (5.3)–(5.5) that

$$\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (V(x, \tau) \otimes V(x, \tau)) d\tau \right\|_{\mathcal{X}} \lesssim \|V\|_{\mathcal{X}}^2. \quad (5.6)$$

Now we estimate $\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau$ in $\mathcal{X}$. First, we have

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau \right\|_{\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)} \\ &= \left\| (1-\Delta)^{\frac{5}{4}-\alpha} \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) d\tau \right\|_{L_x^2} \\ &\lesssim \int_0^\delta \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{1+\frac{5}{2}-2\alpha} \mathcal{F}(u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_\xi^2} d\tau \\ &\quad + \int_\delta^t \left\| e^{-(t-\tau)|\xi|^{2\alpha}} \mathcal{F} \left((-\Delta)^{\frac{1}{2}} (1-\Delta)^{\frac{5}{4}-\alpha} (u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right) \right\|_{L_\xi^2} d\tau \\ &=: I_7 + I_8 \end{aligned}$$

for a small constant $0 < \delta < t$. For $2 \leq q \leq 4$, we have from Hölder's inequality and Lemma 3.2

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{1+\frac{5}{2}-2\alpha} \mathcal{F}(u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_\xi^2} \\ &\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{1+\frac{5}{2}-2\alpha} \right\|_{L_\xi^{\frac{2q}{4-q}}} \left\| \mathcal{F}(u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_\xi^{\frac{q}{q-2}}} \\ &\lesssim (t-\tau)^{-\frac{7}{4\alpha}+1-\frac{3}{2\alpha}\cdot\frac{4-q}{2q}} \|u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)\|_{L_x^{\frac{q}{2}}} \\ &\lesssim (t-\tau)^{-\frac{7}{4\alpha}+1-\frac{3(4-q)}{4q\alpha}} \|u_{g^\omega}\|_{L_x^q}^2, \end{aligned}$$

and hence

$$\begin{aligned} I_7 &\lesssim \int_0^\delta (t-\tau)^{-\frac{7}{4\alpha}+1-\frac{3(4-q)}{4q\alpha}} \|u_{g^\omega}\|_{L_x^q}^2 d\tau \\ &= \int_0^\delta \left((t-\tau)^{-\frac{7}{4\alpha}+1-\frac{3(4-q)}{4q\alpha}} \tau^{-2\gamma} \right) (\tau^\gamma \|u_{g^\omega}\|_{L_x^q})^2 d\tau \\ &\lesssim \left(\int_0^\delta (t-\tau)^{(-\frac{7}{4\alpha}+1-\frac{3(4-q)}{4q\alpha})\frac{1}{1-2\gamma}} \tau^{-2\gamma\cdot\frac{1}{1-2\gamma}} d\tau \right)^{1-2\gamma} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}^2 \\ &= \left(\int_0^\delta (t-\tau)^{-1+\frac{2\gamma}{1-2\gamma}} \tau^{-\frac{2\gamma}{1-2\gamma}} d\tau \right)^{1-2\gamma} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}^2 \\ &\lesssim \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}^2. \end{aligned}$$

For $q > 4$, we instead have

$$\begin{aligned} & \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{1+\frac{5}{2}-2\alpha} \mathcal{F}(u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_\xi^2} \\ &\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{1+\frac{5}{2}-2\alpha} \right\|_{L_\xi^\infty} \left\| \mathcal{F}(u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_\xi^2} \\ &\lesssim (t-\tau)^{-\frac{7}{4\alpha}+1} \|u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)\|_{L_x^2} \\ &\lesssim (t-\tau)^{-\frac{7}{4\alpha}+1} \|u_{g^\omega}\|_{L_x^4}^2 \end{aligned}$$

and

$$\begin{aligned}
 I_7 &\lesssim \int_0^\delta (t-\tau)^{-\frac{7}{4\alpha}+1} \|u_{g^\omega}\|_{L_x^4}^2 d\tau \\
 &= \int_0^\delta \left((t-\tau)^{-\frac{7}{4\alpha}+1} \tau^{-2\gamma} \right) (\tau^\gamma \|u_{g^\omega}\|_{L_x^4})^2 d\tau \\
 &\lesssim \left(\int_0^\delta (t-\tau)^{(-\frac{7}{4\alpha}+1)\frac{1}{1-2\gamma}} \tau^{-\frac{2\gamma}{1-2\gamma}} d\tau \right)^{1-2\gamma} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^4}^2 \\
 &\lesssim \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^4}^2.
 \end{aligned}$$

The term I_8 is estimated as

$$\begin{aligned}
 &\left\| e^{-(t-\tau)|\xi|^{2\alpha}} \mathcal{F} \left((-\Delta)^{\frac{1}{2}} (1-\Delta)^{\frac{5}{4}-\alpha} (u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right) \right\|_{L_\xi^2} \\
 &\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} \right\|_{L_\xi^q} \left\| \mathcal{F} \left((-\Delta)^{\frac{1}{2}} (1-\Delta)^{\frac{5}{4}-\alpha} (u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right) \right\|_{L_\xi^{\frac{2q}{q-2}}} \\
 &\lesssim (t-\tau)^{-\frac{3}{2q\alpha}} \left\| (-\Delta)^{\frac{1}{2}} (1-\Delta)^{\frac{5}{4}-\alpha} (u_{g^\omega}(x, \tau) \otimes u_{g^\omega}(x, \tau)) \right\|_{L_x^{\frac{2q}{q+2}}} \\
 &\lesssim (t-\tau)^{-\frac{3}{2q\alpha}} \|\nabla^{\frac{7}{2}-2\alpha} u_{g^\omega}\|_{L_x^2} \|u_{g^\omega}\|_{L_x^q}.
 \end{aligned}$$

and

$$\begin{aligned}
 I_8 &\lesssim \int_\delta^t (t-\tau)^{-\frac{3}{2q\alpha}} \|\nabla^{\frac{7}{2}-2\alpha} u_{g^\omega}\|_{L_x^2} \|u_{g^\omega}\|_{L_x^q} d\tau \\
 &= \int_\delta^t (t-\tau)^{-\frac{3}{2q\alpha}} \tau^{-\zeta-\gamma} \left(\tau^\zeta \|\nabla^{\frac{7}{2}-2\alpha} u_{g^\omega}\|_{L_x^2} \right) (\tau^\gamma \|u_{g^\omega}\|_{L_x^q}) d\tau \\
 &\lesssim \left(\int_\delta^t (t-\tau)^{-\frac{3}{2q\alpha} \cdot \frac{4}{3-4\gamma}} \tau^{(-\zeta-\gamma)\frac{4}{3-4\gamma}} d\tau \right)^{\frac{3-4\gamma}{4}} \|u_{g^\omega}\|_{L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \\
 &\lesssim \|u_{g^\omega}\|_{L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}.
 \end{aligned}$$

The time integral above is bounded for $0 < \zeta < 1$,

$$\begin{aligned}
 &\int_\delta^t (t-\tau)^{-\frac{3}{2q\alpha} \cdot \frac{4}{4-3\gamma}} \tau^{(-\zeta-\gamma)\frac{4}{4-3\gamma}} d\tau \\
 &\lesssim \delta^{-\frac{4(\zeta+\gamma)}{4-3\gamma}} \int_\delta^t (t-\tau)^{-\frac{6}{q\alpha(4-3\gamma)}} d\tau \\
 &\lesssim 1
 \end{aligned}$$

since $0 < \delta < t$ and $\frac{6}{q\alpha(4-3\gamma)} < 1$. Combining the estimates above gives

$$\begin{aligned}
 &\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau \right\|_{\mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)} \\
 &\lesssim \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}^2 + \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^4}^2 \\
 &\quad + \|u_{g^\omega}\|_{L_t^{(\zeta, 4)} \mathcal{H}_x^{\frac{7}{2}-2\alpha}} \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q}.
 \end{aligned} \tag{5.7}$$

The estimates of $\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau$ in $L_t^{(\gamma, \frac{1}{\gamma})} L_x^q$ and $L_t^{\frac{1}{2\gamma}} L_x^q$

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau \right\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \\ & \lesssim \|u_{g^\omega}\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2 + \|u_{g^\omega}\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \|u_{g^\omega}\|_{L_t^{\frac{1}{2\gamma}} L_x^q}, \\ & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau \right\|_{L_t^{\frac{1}{2\gamma}} L_x^q} \lesssim \|u_{g^\omega}\|_{L_t^{\frac{1}{2\gamma}} L_x^q}^2 \end{aligned} \quad (5.8)$$

can be obtained in an analogy way with the estimate of Φ_1 in the previous section. Thus it follows from (5.7), (5.8) and Lemmas 3.11, 3.12 and 3.13 that

$$\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u_{g^\omega} \otimes u_{g^\omega}) d\tau \right\|_{\mathcal{X}} \lesssim \lambda^2. \quad (5.9)$$

In view of (5.7) and (5.9), we conclude the proof of the first inequality in (5.2) is complete, under the conditions on the parameters $s \geq \frac{7}{2} - \frac{5}{2}\alpha - 2\alpha\zeta$, $s \geq 1 - 2\alpha + \frac{3}{q}$ and $0 < \zeta < 1$. We observe $1 - 2\alpha + \frac{3}{q} = -4\alpha\gamma \leq 0$. Taking $\zeta = \frac{1}{2}$, we have $s \geq \max\{\frac{7}{2} - \frac{7}{2}\alpha, 0\}$. The proof of Theorem 1.3 follows from an analogous analysis as in Sect. 4.

6. Well-Posedness of the Full System of Hall MHD

We address the well-posedness of the Hall MHD system (1.2) in this final section. As before, we seek a solution of (1.2) in the form

$$u = u_{g^\omega} + V, \quad B = B_{f^\omega} + H$$

with (V, H) satisfying the system

$$\begin{aligned} & V_t + (u_{g^\omega} + V) \cdot \nabla (u_{g^\omega} + V) \\ & \quad - (B_{f^\omega} + H) \cdot \nabla (B_{f^\omega} + H) + \nabla \Pi = -(-\Delta)^\alpha V, \\ & H_t + (u_{g^\omega} + V) \cdot \nabla (B_{f^\omega} + H) - (B_{f^\omega} + H) \cdot \nabla (u_{g^\omega} + V) \\ & \quad + \nabla \times \nabla \cdot ((B_{f^\omega} + H) \otimes (B_{f^\omega} + H)) = -(-\Delta)^\alpha H, \\ & \nabla \cdot V = 0, \quad \nabla \cdot H = 0, \\ & V(x, 0) = H(x, 0) = 0. \end{aligned} \quad (6.1)$$

Without causing confusion, we use the same notations Φ and Ψ to denote the maps

$$\begin{aligned} \Psi(V, H)(t) &= - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot [(u_{g^\omega} + V) \otimes (u_{g^\omega} + V)] d\tau \\ & \quad + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot [(B_{f^\omega} + H) \otimes (B_{f^\omega} + H)] d\tau \\ &:= \Psi^1(V, H) + \Psi^2(V, H), \\ \Phi(V, H)(t) &= - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot [(u_{g^\omega} + V) \otimes (B_{f^\omega} + H)] d\tau \\ & \quad + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot [(B_{f^\omega} + H) \otimes (u_{g^\omega} + V)] d\tau \\ & \quad - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \times \nabla \cdot [(B_{f^\omega} + H) \otimes (B_{f^\omega} + H)] d\tau \\ &:= \Phi^1(V, H) + \Phi^2(V, H) + \Phi^3(V, H). \end{aligned}$$

Define the functional spaces $\mathcal{X}$ and $\mathcal{Y}$ as in previous sections

$$\begin{aligned}\mathcal{X} &= C([0, T]; \mathcal{H}^{\frac{5}{2}-2\alpha}(\mathbb{T}^3)) \cap L_t^{(\gamma, \frac{1}{\gamma})} L_x^q \cap L_t^{\frac{1}{2\gamma}} L_x^q, \\ \mathcal{Y} &= C([0, T]; \mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)) \cap L_t^{(\beta, \frac{1}{\beta})} L_x^p \cap L_t^{\frac{1}{2\beta}} L_x^p\end{aligned}$$

with

$$\frac{1}{\alpha} + \frac{3}{2p\alpha} + 2\beta = 1, \quad \frac{1}{2\alpha} + \frac{3}{2q\alpha} + 2\gamma = 1 \quad (6.2)$$

for $\beta, \gamma > 0$ and $p, q \geq 2$.

The existence and uniqueness of solution to (1.2) is a consequence of that the map (Ψ, Φ) is a contraction on $\mathcal{X} \times \mathcal{Y}$. Thus in order to prove Theorem 1.5, the analysis of Sect. 4 suggests that it is sufficient to show

$$\begin{aligned}\|\Psi(V, H)\|_{\mathcal{X}} + \|\Phi(V, H)\|_{\mathcal{Y}} &\lesssim \lambda^2 + \|V\|_{\mathcal{X}}^2 + \|H\|_{\mathcal{Y}}^2 \quad \forall (V, H) \in (\mathcal{X}, \mathcal{Y}), \\ \|\Psi(V_1, H_1) - \Psi(V_2, H_2)\|_{\mathcal{X}} + \|\Phi(V_1, H_1) - \Phi(V_2, H_2)\|_{\mathcal{Y}} \\ &\lesssim (\lambda + \|V_1\|_{\mathcal{X}} + \|V_2\|_{\mathcal{X}} + \|H_1\|_{\mathcal{Y}} + \|H_2\|_{\mathcal{Y}}) (\|V_1 - V_2\|_{\mathcal{X}} + \|H_1 - H_2\|_{\mathcal{Y}}) \\ &\text{for all } (V_1, H_1) \in (\mathcal{X}, \mathcal{Y}), (V_2, H_2) \in (\mathcal{X}, \mathcal{Y}).\end{aligned} \quad (6.3)$$

Note that the estimates regarding the highest derivative term Φ^3 are obtained in Sect. 4 and the estimates of the pure fluid term Ψ^1 are established in Sect. 5. The term Ψ^2 has a similar quadratic structure as Φ^3 and it is in lower order; hence the estimates of Ψ^2 to establish (6.3) are guaranteed. Among Φ^1 and Φ^2 we only need to estimate one of them, say Φ^1 . That is the task in the following.

We start with the $\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)$ norm of $V \otimes H$ from Φ^1 ,

$$\begin{aligned}&\left\| (1 - \Delta)^{\frac{7}{4}-\alpha} \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_x^2} \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (1 - \Delta)^{\frac{7}{4}-\alpha} (V(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^2} d\tau \\ &\lesssim \int_0^t \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{\frac{9}{2}-2\alpha} \mathcal{F}(V(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^2} d\tau.\end{aligned}$$

In view of Hölder's inequality and Lemma 3.2 we infer

$$\begin{aligned}&\left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{\frac{9}{2}-2\alpha} \mathcal{F}(V(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^2} \\ &\lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi|^{\frac{9}{2}-2\alpha} \right\|_{L_\xi^{\frac{2pq}{2p+2q-pq}}} \left\| \mathcal{F}(V(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^{\frac{pq}{pq-p-q}}} \\ &\lesssim (t - \tau)^{-\frac{9}{4\alpha} + 1 - \frac{3}{2\alpha}(\frac{p+q}{pq} - \frac{1}{2})} \|V(x, \tau) \otimes H(x, \tau)\|_{L_x^{\frac{pq}{p+q}}} \\ &\lesssim (t - \tau)^{-\frac{3}{2\alpha} + 1 - \frac{3}{2p\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p}\end{aligned}$$

followed by applying Hölder's inequality and Lemma 3.3

$$\begin{aligned}&\int_0^t (t - \tau)^{-\frac{3}{2\alpha} + 1 - \frac{3}{2p\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \\ &= \int_0^t (t - \tau)^{-\frac{3}{2\alpha} + 1 - \frac{3}{2p\alpha} - \frac{3}{2q\alpha}} \tau^{-\beta-\gamma} (\tau^\gamma \|V\|_{L_x^q}) (\tau^\beta \|H\|_{L_x^p}) d\tau \\ &\lesssim \left(\int_0^t (t - \tau)^{(-\frac{3}{2\alpha} + 1 - \frac{3}{2p\alpha} - \frac{3}{2q\alpha}) \frac{1}{1-\beta-\gamma}} \tau^{-(\beta+\gamma) \frac{1}{1-\beta-\gamma}} d\tau \right)^{1-\beta-\gamma} \\ &\quad \cdot \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}.\end{aligned}$$

The application of Lemma 3.3 is justified by the assumptions on the parameters (6.2) which imply

$$\left(\frac{3}{2\alpha} - 1 + \frac{3}{2p\alpha} + \frac{3}{2q\alpha}\right) \frac{1}{1 - \beta - \gamma} + \frac{\beta + \gamma}{1 - \beta - \gamma} = 1,$$

and

$$0 < \left(\frac{3}{2\alpha} - 1 + \frac{3}{2p\alpha} + \frac{3}{2q\alpha}\right) \frac{1}{1 - \beta - \gamma} < 1, \quad 0 < \frac{\beta + \gamma}{1 - \beta - \gamma} < 1.$$

Hence we have

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{\mathcal{H}^{\frac{7}{2}-2\alpha}(\mathbb{T}^3)} \\ & \lesssim \|V\|_{L_t^{(\gamma, \frac{1}{\gamma})} L_x^q} \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p}. \end{aligned} \quad (6.4)$$

Continuing to the $L_t^{(\beta, \frac{1}{\beta})} L_x^p$ norm we have

$$\begin{aligned} & \left\| t^\beta \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{\beta}} L_x^p} \\ & \lesssim \left\| t^\beta \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \end{aligned}$$

and

$$\begin{aligned} & \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F}((V(x, \tau) \otimes H(x, \tau))) \right\|_{L_\xi^{\frac{p}{p-1}}} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \right\|_{L_\xi^q} \left\| \mathcal{F}((V(x, \tau) \otimes H(x, \tau))) \right\|_{L_\xi^{\frac{pq}{pq-p-q}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V(x, \tau) \otimes H(x, \tau)\|_{L_x^{\frac{pq}{p+q}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p}. \end{aligned}$$

It follows that

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\ & \lesssim \left\| \int_0^t t^\beta (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\ & \lesssim \left\| \int_0^{\frac{t}{2}} t^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\ & \quad + \left\| \int_{\frac{t}{2}}^t t^\beta (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}}. \end{aligned}$$

By Young's inequality we have

$$\begin{aligned}
 & \left\| \int_0^{\frac{t}{2}} t^\beta (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
 & \lesssim \left\| \int_0^{\frac{t}{2}} (t-\tau)^{\beta - \frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
 & \lesssim \left(\int_0^{\frac{t}{2}} (t-\tau)^{(\beta - \frac{1}{2\alpha} - \frac{3}{2q\alpha})(\frac{1}{1-\beta-2\gamma})} d\tau \right)^{1-\beta-2\gamma} \\
 & \quad \cdot \left(\int_0^{\frac{t}{2}} \|H\|_{L_x^p}^{\frac{1}{2\beta}} d\tau \right)^{2\beta} \left(\int_0^{\frac{t}{2}} \|V\|_{L_x^q}^{\frac{1}{2\gamma}} d\tau \right)^{2\gamma} \\
 & \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}
 \end{aligned}$$

where we used the fact

$$(\beta - \frac{1}{2\alpha} - \frac{3}{2q\alpha})(\frac{1}{1-\beta-2\gamma}) = -1$$

thanks to (6.2), and hence $\int_0^{\frac{t}{2}} (t-\tau)^{-1} d\tau \lesssim 1$.

Resorting to Hölder's inequality and Hardy–Littlewood–Sobolev lemma for the parameters satisfying (6.2) we deduce

$$\begin{aligned}
 & \left\| \int_{\frac{t}{2}}^t t^\beta (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
 & \lesssim \left\| \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \tau^\beta \|V\|_{L_x^q} \|H\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{\beta}}} \\
 & \lesssim \left\| \left(\int_{\frac{t}{2}}^t \tau \|H\|_{L_x^p}^{\frac{1}{\beta}} d\tau \right)^\beta \left(\int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{2\alpha} - \frac{3}{2q\alpha})\frac{1}{1-\beta}} \|V\|_{L_x^q}^{\frac{1}{1-\beta}} d\tau \right)^{1-\beta} \right\|_{L_t^{\frac{1}{\beta}}} \\
 & \lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \left\| \int_{\frac{t}{2}}^t (t-\tau)^{(-\frac{1}{2\alpha} - \frac{3}{2q\alpha})\frac{1}{1-\beta}} \|V\|_{L_x^q}^{\frac{1}{1-\beta}} d\tau \right\|_{L_t^{\frac{1-\beta}{\beta}}}^{1-\beta} \\
 & \lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \left\| \|V\|_{L_x^q}^{\frac{1}{1-\beta}} \right\|_{L_t^{\frac{1-\beta}{2\gamma}}}^{1-\beta} \\
 & \lesssim \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}.
 \end{aligned}$$

We conclude from the estimates above that

$$\begin{aligned}
 & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \\
 & \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q} + \|H\|_{L_t^{(\beta, \frac{1}{\beta})} L_x^p} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}.
 \end{aligned} \tag{6.5}$$

In the end, we estimate the norm of $L_t^{\frac{1}{2\beta}} L_x^p$ as

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\beta}} L_x^p} \\ & \lesssim \left\| \int_0^t \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} d\tau \right\|_{L_t^{\frac{1}{2\beta}}} \end{aligned}$$

and

$$\begin{aligned} & \left\| e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^{\frac{1}{2}} (V(x, \tau) \otimes H(x, \tau)) \right\|_{L_x^p} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \mathcal{F}(V(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^{\frac{p}{p-1}}} \\ & \lesssim \left\| e^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \right\|_{L_\xi^q} \left\| \mathcal{F}(V(x, \tau) \otimes H(x, \tau)) \right\|_{L_\xi^{\frac{pq}{p-q}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|V(x, \tau) \otimes H(x, \tau)\|_{L_x^{\frac{pq}{p+q}}} \\ & \lesssim (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|H\|_{L_x^p} \|V\|_{L_x^q}. \end{aligned}$$

Again, the last two inequalities together with Hardy–Littlewood–Sobolev lemma imply

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{L_t^{\frac{1}{2\beta}} L_x^p} \\ & \lesssim \left\| \int_0^t (t-\tau)^{-\frac{1}{2\alpha} - \frac{3}{2q\alpha}} \|H\|_{L_x^p} \|V\|_{L_x^q} d\tau \right\|_{L_t^{\frac{1}{2\beta}}} \\ & \lesssim \| \|H\|_{L_x^p} \|V\|_{L_x^q} \|_{L_t^{\frac{1}{2\beta} + 2\gamma}} \\ & \lesssim \|H\|_{L_t^{\frac{1}{2\beta}} L_x^p} \|V\|_{L_t^{\frac{1}{2\gamma}} L_x^q}. \end{aligned} \tag{6.6}$$

Immediately from (6.4)–(6.6) we obtain

$$\begin{aligned} & \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \nabla \cdot (V(x, \tau) \otimes H(x, \tau)) d\tau \right\|_{\mathcal{Y}} \\ & \lesssim \|V\|_{\mathcal{X}}^2 + \|H\|_{\mathcal{Y}}^2. \end{aligned}$$

The estimates for $V \otimes B_{f^\omega}$, $u_{g^\omega} \otimes H$ and $u_{g^\omega} \otimes B_{f^\omega}$ from Φ^1 can be achieved in an analogy way. Thus we claim the first inequality of (6.3) is justified. As explained in Sect. 4, the second inequality of (6.3) can be established similarly as the first one.

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Mimi Dai
 Department of Mathematics, Statistics and Computer
 Science
 University of Illinois at Chicago
 Chicago IL60607
 USA
 e-mail: mdai@uic.edu

and

 School of Mathematics
 Institute for Advanced Study
 Princeton NJ08540
 USA

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