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Compact moduli of elliptic K3 surfaces

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We construct various modular compactifications of the space of elliptic K3 surfaces using tools from the minimal model program, and explicitly describe the surfaces parametrized by their boundaries. The coarse spaces of our constructed compactifications admit morphisms to the Satake–Baily–Borel compactification and the GIT compactification of Miranda.

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1 Introduction

Ever since the compactification of the moduli space of smooth curves by Deligne and Mumford was accomplished, the search for analogous compactifications in higher dimensions became an actively studied problem in algebraic geometry. While moduli in higher dimensions is highly intricate, the pioneering work of Kollár and Shepherd-Barron [31] and Alexeev [3] (see also Hacon, McKernan and Xu [19], Hacon and

Xu [20] Kollár [28] and Kovács and Patakfalvi [32] has established much of the underlying framework for modular compactifications in the (log) general type case via *KSBA stable pairs*, where semi-log canonical singularities serve as the generalization of nodal curves; see the survey by Kollár [27].

One of the most sought-after compactifications is for the space of K3 surfaces. K3 surfaces do not immediately fit into the above framework as they are not of general type, but rather Calabi–Yau varieties. On the other hand, like for abelian varieties, since the space of (polarized) K3 surfaces is a locally symmetric variety it has several natural compactifications, eg the Satake–Baily–Borel (SBB), toroidal, and semitoric compactifications of Looijenga. Unlike the KSBA approach, these compactifications do not necessarily carry a universal family or modular meaning over the boundary.

As such, one of the central questions in moduli theory is to give the aforementioned naturally arising compactifications a stronger geometric meaning by connecting them with a KSBA compactification. With this in mind, our goal is to construct modular compactifications for elliptic K3 surfaces — compactifications where the degenerate objects are K3 surfaces with controlled singularities — and understand how they compare to the Satake–Baily–Borel compactification.

By the Torelli theorem, the moduli space of polarized K3 surfaces is a 19–dimensional locally symmetric variety. Similarly, it is well known that the moduli space of elliptic K3 surfaces with a section, which we denote by $\mathcal{W}$ with coarse space W , is an 18–dimensional locally symmetric variety, corresponding to U –polarized K3 surfaces; see Dolgachev [14] and Nikulin [38]. Recall that a generic elliptic K3 surface $f: X \rightarrow \mathbb{P}^1$ with section S has 24 I_1 singular fibers. Let $F_{\mathcal{A}} = \sum a_i F_i$ denote the sum of these 24 fibers weighted by $a_i \in \mathbb{Q} \cap [0, 1]^{24}$. We consider the closure of the locus of pairs $(f: X \rightarrow C, S + F_{\mathcal{A}})$ inside the KSBA moduli space. For the moment we assume all $a_i = a$, so that we can quotient by S_{24} . Denote the closure of the resulting locus by $\overline{\mathcal{W}}_{\sigma}(a)$, and let $0 < \epsilon \ll 1$.

Theorem 1.1 (Theorems 6.13, 6.15 and 6.14, and Figure 1) *The proper Deligne–Mumford stacks $\overline{\mathcal{W}}_{\sigma}(a)$ for $a \in \mathbb{Q} \cap [0, 1]$ give modular compactifications of $\mathcal{W}$. There is an explicit classification of the broken elliptic K3 surfaces parametrized by $\overline{\mathcal{W}}_{\sigma}(\epsilon)$, and an explicit morphism from the coarse space $\overline{\mathcal{W}}_{\sigma}(\epsilon)$ to $\overline{\mathcal{W}}^*$, the SBB compactification of W . Furthermore, the surfaces parametrized by $\overline{\mathcal{W}}_{\sigma}(\epsilon)$ satisfy $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$.*

Theorem 1.1 shows that the boundary of $\overline{\mathcal{W}}_\sigma(\epsilon)$ parametrizes K3 surfaces with slc singularities. Although $\overline{\mathcal{W}}_\sigma(\epsilon)$ compactifies a moduli space of pairs, it gives a natural compactification of the space of elliptic K3s as the singular fibers are an intrinsic choice of divisor. Moreover, without choosing a divisor, the moduli space is a nonseparated Artin stack. In [Section 7](#), we present an alternative explicit description of the surfaces parametrized on the boundary of the moduli space more akin to Kulikov models. In particular, we show that we can decompose the boundary of $\overline{\mathcal{W}}_\sigma(a)$ into combinatorially described parameter spaces.

As mentioned above, viewing the moduli space of elliptic K3 surfaces as a locally symmetric variety, one naturally obtains the SBB compactification $\overline{\mathcal{W}}^*$. While a priori the SBB compactification does not have a modular meaning, it turns out that in the case of elliptic K3 surfaces, this compactification can be identified with the GIT compactification of Weierstrass models of Miranda $\overline{\mathcal{W}}^G$ (see [Section 2.6](#) and Odaka and Oshima [[39](#), Theorem 7.9]), which provides some geometric meaning. In particular, in the theorem above as well as the remainder of this section, all of our spaces admit morphisms to $\overline{\mathcal{W}}^G$.

One benefit of the SBB compactification is that all of the parametrized surfaces are irreducible. The next theorem discusses a modular compactification coming from the KSBA approach, where the boundary parametrizes irreducible surfaces. Indeed, consider pairs $(f: X \rightarrow \mathbb{P}^1, S + \epsilon F)$ for $0 < \epsilon \ll 1$, ie only one singular fiber carries a nonzero weight, and this weight is very small. We denote the closure of this locus by $\overline{\mathcal{K}}_\epsilon$.

Theorem 1.2 (Theorems [8.1](#) and [8.2](#), and [Figure 1](#)) *The compact moduli space $\overline{\mathcal{K}}_\epsilon$ parametrizes irreducible semi-log canonical Weierstrass elliptic K3 surfaces satisfying $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$. Moreover, there is an explicit generically finite morphism from the coarse space $\overline{\mathcal{K}}_\epsilon$ to $\overline{\mathcal{W}}^*$.*

In light of the above theorem, it is natural to ask how the compactifications $\overline{\mathcal{W}}_\sigma(\epsilon)$ and $\overline{\mathcal{K}}_\epsilon$ are related. In previous work (see Ascher and Bejleri [[8](#)]) we showed the existence of wall-crossing morphisms on moduli spaces of elliptic surfaces. In particular, our previous work implies that (up to a 24-to-1 base change corresponding to choosing a singular fiber) the universal families of $\overline{\mathcal{W}}_\sigma(\epsilon)$ and $\overline{\mathcal{K}}_\epsilon$ are related by an explicit series of flips and divisorial contractions as the weights of 23 of the marked fibers are reduced from ϵ to 0. This aspect is crucial to our work (see eg [Section 8.1](#))—these explicit morphisms allow us to understand how our compactifications are related to each other, and how they compare to others lacking a modular meaning.

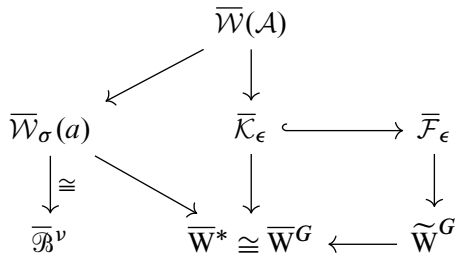


Figure 1: This diagram shows the various compactifications we introduce as well as how they are related; see also [Remark 4.10](#).

Finally, we introduce one more KSBA compactification. While in $\overline{\mathcal{K}}_\epsilon$ we mark one singular fiber with weight ϵ , it is natural to ask what happens if we mark *any* fiber, not necessarily singular, with weight ϵ . We denote this compactification by $\overline{\mathcal{F}}_\epsilon$. See [Figure 1](#) for the relations between the spaces we introduce, which are:

- $\overline{\mathcal{B}}^\nu$ The normalization of Brunyate’s compactification with small weights on both section and singular fibers; see [Section 1.1](#).
- $\overline{W}(\mathcal{A})$ The KSBA compactification with $\mathcal{A}$ -weighted singular fibers.
- $\overline{W}_\sigma(a)$ The quotient by S_{24} when $\mathcal{A} = (a, \dots, a)$.
- $\overline{\mathcal{K}}_\epsilon$ The KSBA compactification with a single ϵ -marked singular fiber (where $\epsilon \ll 1$).
- $\overline{\mathcal{F}}_\epsilon$ The KSBA compactification with *any fiber* marked by ϵ (where $\epsilon \ll 1$).
- $\overline{W}^*$ The SBB compactification of the period domain moduli space W .
- $\overline{W}^G$ Miranda’s GIT compactification of Weierstrass models; see [Section 2.6](#).
- $\widetilde{W}^G$ The GIT compactification of Weierstrass models with a chosen fiber; see the discussion after [Theorem 1.3](#).

Theorem 1.3 ([Theorem 8.8](#) and [Figure 1](#)) *There exists a smooth proper Deligne–Mumford stack $\overline{\mathcal{F}}_\epsilon$ parametrizing semi-log canonical elliptic K3 surfaces with a single marked fiber. Its coarse space is isomorphic to an explicit GIT quotient $\widetilde{W}^G$ of Weierstrass K3 surfaces and a chosen fiber. Furthermore, the surfaces parametrized by $\overline{\mathcal{F}}_\epsilon$ satisfy $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$.*

On the interior, $\overline{\mathcal{F}}_\epsilon$ is a $\mathbb{P}^1$ bundle over W . In this sense $\overline{\mathcal{F}}_\epsilon$ is similar in spirit to the KSBA compactification of Laza of degree-two K3 surfaces [\[34\]](#). The GIT problem

of Miranda can be modified to parametrize Weierstrass fibrations with a chosen fiber (see [Section 8.3](#)), denoted above by $\widetilde{W}^G$. It turns out that $\widetilde{W}^G$ is precisely the coarse moduli space of $\overline{\mathcal{F}}_\epsilon$; in particular, the morphism $\overline{\mathcal{F}}_\epsilon \rightarrow \widetilde{W}^G$ realizes $\overline{\mathcal{F}}_\epsilon$ as a smooth Deligne–Mumford stack.

Our approach combines explicit use of the theory of twisted stable maps (see eg Ascher and Bejleri [\[7\]](#)) with the minimal model program (MMP). The various compactifications are then related by an explicit series of wall-crossing morphisms. In particular, we wish to emphasize that the power of our approach lies in understanding the compactifications for various coefficients and how they are related via wall crossing morphisms. Often the spaces with very small coefficients are the smallest compactifications which are still modular, but having access to the spaces for all coefficients is helpful in understanding the geometry of compactifications obtained via different methods.

1.1 Previous results

Using Kulikov models, Brunyate’s thesis [\[12\]](#) constructs a stable pairs compactification of the space of elliptic K3 surfaces $\overline{\mathcal{B}}$ which parametrizes pairs $(X, \epsilon S + \delta F)$, where ϵ and δ are both small. In particular, Brunyate gives a classification of the surfaces appearing on the boundary, and conjectures that the normalization of $\overline{\mathcal{B}}$ is a toroidal compactification. Recently Alexeev, Brunyate and Engel [\[4\]](#) confirmed Brunyate’s conjecture, and showed that this space is isomorphic to a particular toroidal compactification using the theory of integral affine geometry and continuing the program started by Alexeev, Engel and Thompson [\[5\]](#).

One difference between our approach and the work of Brunyate is in our descriptions of the compactifications at various weights and choice of markings. Instead of using Kulikov models, we describe the steps of MMP and the induced wall-crossing morphisms that relate the stable limits of elliptic K3 surfaces for different weights to highlight the underlying geometry of the various compactifications. Brunyate’s space $\overline{\mathcal{B}}$ admits a morphism $\overline{W}_\sigma(\epsilon) \rightarrow \overline{\mathcal{B}}$ which identifies $\overline{W}_\sigma(\epsilon)$ with the normalization of $\overline{\mathcal{B}}$; see [Proposition 4.4](#) and [Remark 4.7](#). In particular, the boundary components of $\overline{\mathcal{B}}$ and $\overline{W}_\sigma(\epsilon)$ are in bijection (see [Remark 4.5](#)) and the moduli spaces parametrize essentially the same surfaces. Indeed there is a sequence of flips relating the universal family of $\overline{\mathcal{B}}$ and the universal family over $\overline{W}_\sigma(\epsilon)$ which induces this morphism.

Finally, we note that in a slightly different direction, Inchiostro constructs a KSBA compactification of the space of Weierstrass fibrations (of not necessarily K3 surfaces) with both section and fibers marked by $0 < \epsilon, \delta \ll 1$ [\[25\]](#).

1.2 Other lattice polarizations

It is natural to consider fibrations with specified singular fibers. In this case, one obtains a moduli space which is a locally symmetric variety, corresponding to a M -lattice polarization, encoding the singular fiber type. Our methods work in that case as well. Here we quickly discuss an example of this point of view.

Example 1.4 Consider the lattice $M = U \oplus D_4^{\oplus 4}$. Then M -polarized K3 surfaces correspond to $4I_0^*$ isotrivial elliptic K3 surfaces. Equivalently, these are Kummer K3 surfaces obtained from abelian surfaces of the form $E \times E'$ with the elliptic fibration induced by the projection $E \times E' \rightarrow E$. Marking the four minimal Weierstrass cusps by a single weight a gives us a moduli space whose coarse space is two copies of the j -line, one parametrizing the j -invariant of the fibration, and the other the j -invariant of the configuration of singular fibers. The stable pairs compactification has coarse space given by $\mathbb{P}^1 \times \mathbb{P}^1 = \overline{M}_{0,4} \times \overline{M}_{0,4}$. The universal family consists of $4N_1$ isotrivial j -invariant ∞ fibrations over the locus $\{\infty\} \times \mathbb{P}^1$, a union $X \cup_{I_0} X$ of two copies of the $2I_0^*$ rational elliptic surface glued along a smooth fiber over the locus $\mathbb{P}^1 \times \{\infty\}$, and a union $X \cup_{N_0} X$ of two copies of the $2N_1$ isotrivial j -invariant ∞ fibration glued along an N_0 fiber over the point (∞, ∞) .

Structure of the paper

In [Section 2](#) we discuss the background on elliptic K3 surfaces and their moduli (as a period domain, the Satake–Baily–Borel compactification, and a geometric invariant theory compactification). In [Section 3](#) we review the results from our previous works [\[6; 7; 8; 9\]](#) on KSBA compactifications of moduli spaces of elliptic fibrations and the connection with twisted stable maps. In [Section 4](#) we restrict to the case of elliptic K3 surfaces and collect the definitions of and preliminary observations on the compactifications we consider, including a discussion on isotrivial j -invariant ∞ fibrations of K3 type.

The main body of the paper begins with [Section 5](#), where we discuss the wall-crossings that occur for the compactification $\overline{W}_\sigma(a)$ as the coefficient a is lowered from 1 down to $\frac{1}{12} + \epsilon$ for $0 < \epsilon \ll 1$. In [Section 6](#) we continue the wall-crossing analysis as a is decreased down to $0 < \epsilon \ll 1$, and we prove [Theorem 1.1](#), which describes the surfaces appearing on the boundary of the moduli space $\overline{W}_\sigma(\epsilon)$. In [Section 7](#) we use [Theorem 1.1](#) and twisted stable maps (see [Section 3.2](#)) to explicitly describe the

boundary components of $\overline{\mathcal{W}}_{\sigma}(\epsilon)$. Finally, in [Section 8](#) we describe the moduli spaces with one marked fiber ($\overline{\mathcal{K}}_{\epsilon}$ and $\overline{\mathcal{F}}_{\epsilon}$) and prove [Theorems 1.2](#) and [1.3](#); the latter theorem is proven by introducing a modified version of Miranda's GIT compactification; see [Section 8.3](#).

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2 Elliptic K3 surfaces and their moduli

2.1 Elliptic surfaces

We begin with the basic definitions surrounding elliptic surfaces following [\[8\]](#); see also [\[37\]](#).

Definition 2.1 An irreducible *elliptic surface with section* $(f: X \rightarrow C, S)$ is an irreducible surface X together with a surjective proper flat morphism $f: X \rightarrow C$ to a smooth curve C and a section S such that

- (i) the generic fiber of f is a stable elliptic curve, and
- (ii) the generic point of the section is contained in the smooth locus of f .

We call the pair $(f: X \rightarrow C, S)$ *standard* if all of S is contained in the smooth locus of f .

Definition 2.2 A *Weierstrass fibration* is an elliptic surface obtained from a standard elliptic surface by contracting all fiber components not meeting the section. We call the output of this process a *Weierstrass model*. If starting with a smooth relatively minimal elliptic surface, we call the result a *minimal Weierstrass model*.

The geometry of an elliptic surface is largely influenced by the *fundamental line bundle* $\mathcal{L}$.

Definition 2.3 The *fundamental line bundle* of a standard elliptic surface is

$$(1) \quad \mathcal{L} := (f_* \mathcal{N}_{S/X})^{-1},$$

where $\mathcal{N}_{S/X}$ denotes the normal bundle of S in X . For an arbitrary elliptic surface we define $\mathcal{L}$ as the line bundle associated to its minimal semiresolution.¹

For X a standard elliptic surface, the line bundle $\mathcal{L}$ is invariant under taking a semiresolution or Weierstrass model, is independent of choice of section S , has nonnegative degree, and determines the canonical bundle of X if X is either relatively minimal or Weierstrass; see [37, III.1.1].

2.2 Singular fibers

If $(f: X \rightarrow C, S)$ is a smooth relatively minimal elliptic surface, then f has finitely many singular fibers, which are each unions of rational curves with possibly nonreduced components whose dual graphs are ADE Dynkin diagrams. The singular fibers were classified by Kodaira and Néron (see [11, Section V.7]).

An elliptic surface in Weierstrass form can be described locally by an equation of the form $y^2 = x^3 + Ax + B$, where A and B are functions of the base curve. Furthermore, the possible singular fiber types can be characterized in terms of vanishing orders of A and B by Tate's algorithm; see [43, Table 1]. Moreover, if the smooth relatively minimal model $(f: X \rightarrow C, S)$ has a singular fiber with a given Dynkin diagram, the minimal Weierstrass model will have an ADE singularity of the same type.

2.3 Elliptic K3 surfaces

By the canonical bundle formula and the observation that $\deg \mathcal{L} = 0$ if and only if the surface is a product, a smooth elliptic surface with section $(f: X \rightarrow C, S)$ is a K3 surface if and only if $C \cong \mathbb{P}^1$ and $\deg(\mathcal{L}) = 2$; see [37, III.4.6].

Definition 2.4 A standard (possibly singular) elliptic surface is of *K3 type* if $C \cong \mathbb{P}^1$ and $\deg(\mathcal{L}) = 2$.

For an elliptic surface of K3 type, the Weierstrass model is given by $y^2 = x^3 + Ax + B$, where A and B are sections of $\mathcal{O}(8)$ and $\mathcal{O}(12)$, respectively, and the *discriminant* $\mathcal{D} = 4A^3 + 27B^2$ is a section of $\mathcal{L}^{\otimes 12} \cong \mathcal{O}(24)$.

¹The seminormal version of resolution of singularities; see eg [26, Section 1.13].

Remark 2.5 The number of singular fibers of a Weierstrass elliptic K3 counted with multiplicity is 24, and a generic elliptic K3 has exactly 24 nodal (I_1) singular fibers.

2.4 Moduli of lattice polarized K3 surfaces

We now discuss lattice polarized K3 surfaces and their moduli; see [21; 15; 16]. An elliptic K3 with section $(f: X \rightarrow \mathbb{P}^1, S)$ is characterized by the fact that $\mathrm{NS}(X)$ contains a lattice U which is spanned by the classes of the fiber f and section S . The moduli of K3 surfaces with specified $\mathrm{NS}(X)$ were studied by Dolgachev [14]; see also [38]. By the Torelli theorem for polarized K3 surfaces, the moduli space of minimal Weierstrass elliptic K3 surfaces with at worst ADE singularities is an 18-dimensional locally symmetric variety $W = \Gamma \backslash D$ associated to the lattice $U_{K3}^\perp \cong U^2 \oplus E_8^2$.

2.5 The Satake–Baily–Borel compactification

One can use the techniques of Baily and Borel [10] to obtain a compactification $\overline{W}^*$ by adding some curves and points. We briefly review this compactification following [35, Section 3.1]. The boundary components of $\overline{W}^*$ are determined by rational maximal parabolic subgroups of the identity component of the orthogonal group $O(2, 18)$ of the lattice $U_{K3}^\perp$. Every boundary component of $\overline{W}^*$ has the structure of a locally symmetric variety of lower dimension. We recall the following properties:

- (i) The compactification is canonical.
- (ii) The boundary components have high codimension (as they are points and curves).
- (iii) The compactification is *minimal*: if S is a smooth variety with $\overline{S}$ a smooth simple normal crossing compactification, then any locally liftable map $S \rightarrow W$ extends to a regular map $\overline{S} \rightarrow \overline{W}^*$.

Theorem 2.6 [21, Section 2.3; 42] *The boundary of $\overline{W}^*$ is a union of 0– and 1–dimensional strata. The 0–dimensional strata correspond to K3s of type III, and the 1–dimensional strata to degenerate K3s of type II. Moreover, the 1–dimensional strata are all rational curves, each parametrizing the j –invariant of the elliptic double curves appearing in the corresponding type II degenerate K3.*

2.6 Geometric invariant theory

Miranda [36] used geometric invariant theory (GIT) to construct a compactification of the moduli space of Weierstrass fibrations, and completed an explicit classification in

the case of rational elliptic surfaces. More recently, Odaka and Oshima [39] explicitly calculated Miranda's compactification for the case of elliptic K3 surfaces. Moreover, they showed that the GIT compactification of Miranda, $\overline{W}^G$, is isomorphic to $\overline{W}^*$, the SBB compactification. In particular, using this identification, one is able to give a geometric meaning to $\overline{W}^*$ by relating the boundary of $\overline{W}^*$ with the GIT polystable orbits in $\overline{W}^G$. We review these results now.

Let $\Gamma_n = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. The surface X has a *Weierstrass equation*, and as such X can be realized as a divisor in a $\mathbb{P}^2$ -bundle over the base curve. For the Weierstrass model of an elliptic K3 surface, we think of X as being the closed subscheme of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ defined by the equation $y^2z = x^3 + Axz^2 + Bz^3$, where $A \in \Gamma_8$, $B \in \Gamma_{12}$, and

- (i) $4A(q)^3 + 27B(q)^2 = 0$ precisely at the (finitely many) singular fibers X_q , and
- (ii) for each $q \in \mathbb{P}^1$ we have $v_q(A) \leq 3$ or $v_q(B) \leq 5$.

We note that any Weierstrass elliptic K3 surface with section and ADE singularities satisfies the above conditions, and conversely, the surface defined as above is a Weierstrass elliptic K3 surface with section and ADE singularities; see [39, Theorem 7.1].

We write $V_{24} = \Gamma_8 \oplus \Gamma_{12}$ and define the GIT moduli space for Weierstrass elliptic K3 surfaces by $\overline{W}^G = V_{24}^{ss} // \mathrm{SL}_2$. By the above discussion the open locus $W^G \subset \overline{W}^G$ parametrizes the ADE Weierstrass elliptic K3 surfaces. The following theorem describes the boundary $\overline{W}^G \setminus W^G$:

Theorem 2.7 [39, Proposition 7.4] *The boundary $\overline{W}^G \setminus W^G$ consists of*

- (i) a 1-dimensional component $\overline{W}_{\mathrm{slc}}^G$ parametrizing isotrivial j -invariant ∞ slc surfaces,
- (ii) a 1-dimensional component $\overline{W}_L^G$ whose open locus $\overline{W}_{L,o}^G$ parametrizes normal surfaces with two type L type cusps.

Furthermore, the intersection of the two components is the infinity point of both $\mathbb{P}^1$ s parametrizing the unique j -invariant ∞ slc surface with two L type cusps. This point is polystable, and the strictly semistable locus is $\overline{W}_L^G$, ie $\overline{W}_{\mathrm{slc}}^G$ is part of the GIT-stable locus of $\overline{W}^G$.

It is natural to compare the GIT compactification $\overline{W}^G$ to the SBB compactification $\overline{W}^*$. This is the content of [39, Theorem 7.9], where we define $\overline{W}_{\mathrm{slc},o}^G := \overline{W}_{\mathrm{slc}}^G \setminus \overline{W}_L^G$.

Theorem 2.8 [39, Theorem 7.9] *The period map $W^G \rightarrow W$ extends to an isomorphism $\overline{W}^G \cong \overline{W}^*$, which identifies $\overline{W}_{\text{slc},o}^G \cup \overline{W}_{L,o}^G$ with the 1-dimensional cusps and identifies $\overline{W}_{\text{slc}}^G \cap \overline{W}_L^G$ with the 0-dimensional cusp.*

3 Moduli of $\mathcal{A}$ -broken elliptic surfaces and wall-crossing

In this section we review and supplement the results from our previous work on compactifications of the moduli spaces of elliptic surfaces via KSBA stable pairs.

Definition 3.1 A KSBA stable pair (X, D) is a pair consisting of a variety X and a Weil divisor D such that

- (i) (X, D) has semi-log canonical (slc) singularities, and
- (ii) $K_X + D$ is an ample $\mathbb{Q}$ -Cartier divisor.

Stable pairs are the natural higher-dimensional generalization of stable curves, and their moduli space compactifies the moduli space of log canonical models of pairs of log general type.

In [8], we defined KSBA compactifications $\mathcal{E}_{\mathcal{A}}$ of the moduli space of log canonical (lc) models $(f: X \rightarrow C, S + F_{\mathcal{A}})$ of $\mathcal{A}$ -weighted Weierstrass elliptic surface pairs. For each admissible weight vector $\mathcal{A}$, we obtained a compactification $\mathcal{E}_{\mathcal{A}}$, which is representable by a proper Deligne–Mumford stack of finite type [8, Theorems 1.1 and 1.2]. These spaces parametrize slc pairs $(f: X \rightarrow C, S + F_{\mathcal{A}})$, where $(f: X \rightarrow C, S)$ is an slc elliptic surface with section, $F_{\mathcal{A}} = \sum a_i F_i$ is a weighted sum of marked fibers with $\mathcal{A} = (a_1, \dots, a_n)$, and $0 < a_i \leq 1$, and $(X, S + F_{\mathcal{A}})$ is a stable pair.

Before stating the main result, Theorem 3.6, we must first discuss the different (singular) fiber types that appear in semi-log canonical models of elliptic fibrations as studied in [6].

Definition 3.2 Let $(g: Y \rightarrow C, S' + aF')$ be a Weierstrass elliptic surface pair over the spectrum of a DVR and let $(f: X \rightarrow C, S + F_a)$ be its relative log canonical model. We say that X has

- (i) a *twisted fiber* if the special fiber $f^*(s)$ is irreducible and $(X, S + E)$ has (semi-)log canonical singularities where $E = f^*(s)^{\text{red}}$;

- (ii) an *intermediate fiber* if $f^*(s)$ is a nodal union of an arithmetic genus-zero component A , and a possibly nonreduced arithmetic genus-one component supported on a curve E such that the section meets A along the smooth locus of $f^*(s)$ and the pair $(X, S + A + E)$ has (semi-)log canonical singularities.

Given an elliptic surface $f: X \rightarrow C$ over the spectrum of a DVR such that X has an *intermediate fiber* we obtain the *Weierstrass model* of X by contracting the component E , and we obtain the *twisted model* by contracting the component A . As such, the intermediate fiber can be seen to interpolate between the Weierstrass and twisted models.

One can consider a Weierstrass elliptic surface $(g: Y \rightarrow C, S' + aF')$ over the spectrum of a DVR, where either F' is a Kodaira singular fiber type, or g is isotrivial with constant j -invariant ∞ with F' being an N_k fiber type. Then the relative log canonical model $(f: X \rightarrow C, S + F_a)$ depends on the value of a . When $a = 1$ the fiber is in *twisted* form, when $a = 0$ the fiber is in *Weierstrass form*, and for some $0 < a_0 < 1$ the fiber enters *intermediate* form. The values a_0 were calculated for all fiber types in [8, Theorem 3.10]:

(2)

fiber type	II	III	IV	N_1	II^*	III^*	IV^*	I_n^*
a_0	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$

We now state the definition of pseudoelliptic surfaces, which appear as components of surfaces in our moduli spaces, a phenomenon first observed by La Nave [33].

Definition 3.3 A *pseudoelliptic pair* is a surface pair (Z, F) obtained by contracting the section of an irreducible elliptic surface pair $(f: X \rightarrow C, S + F')$. We call F the *marked pseudofibers* of Z . We call $(f: X \rightarrow C, S)$ the associated elliptic surface to (Z, F) .

The MMP will contract the section of an elliptic surface if it has nonpositive intersection with the log canonical divisor of the surface. There are two types of pseudoelliptic surfaces which appear, and we refer the reader to [8, Definitions 4.6 and 4.7] for the precise definitions.

Definition 3.4 A pseudoelliptic surface of *type II* is formed by the log canonical contraction of a section of an elliptic component attached along *twisted* or *stable* fibers.

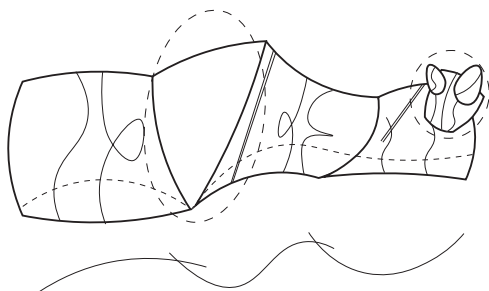


Figure 2: An $\mathcal{A}$ -broken elliptic surface. Two types of pseudoelliptic surfaces (see Definitions 3.4 and 3.5) are circled: type II (left) and type I (right).

Definition 3.5 A pseudoelliptic surface of *type I* appears in *pseudoelliptic trees*, attached by gluing an irreducible pseudofiber G_0 on the root component to an arithmetic genus-one component E of an intermediate (pseudo)fiber of an elliptic or pseudoelliptic component.

Figure 2 has a tree of pseudoelliptic surfaces of type I circled on the right, with a pseudoelliptic of type II circled on the left.

Theorem 3.6 [8, Theorem 1.6] *The boundary of the proper moduli space $\mathcal{E}_{v,\mathcal{A}}$ parametrizes $\mathcal{A}$ -broken stable elliptic surfaces, which are pairs $(f: X \rightarrow C, S + F_{\mathcal{A}})$ consisting of a stable pair $(X, S + F_{\mathcal{A}})$ with a map to a nodal curve C such that X consists of*

- *an slc union of elliptic surfaces with section S and marked fibers, as well as*
- *chains of pseudoelliptic surfaces of types I and II (see Definition 3.3) contracted by f with marked pseudofibers.*

Contracting the section of a component to form a pseudoelliptic component corresponds to stabilizing the base curve as an $\mathcal{A}$ -stable curve in the sense of Hassett; see [6, Corollaries 6.7 and 6.8]. In particular:

Theorem 3.7 [8, Theorem 1.4] *There are forgetful morphisms $\mathcal{E}_{v,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$.*

Remark 3.8 For an irreducible component with base curve $\mathbb{P}^1$ and $\deg \mathcal{L} > 0$, contracting the section of an elliptic component may *not* be the final step in the MMP—we may need to contract the entire pseudoelliptic component to a curve or a point; see [6, Proposition 7.4].

3.0.1 Wall and chamber structure We are now ready to discuss how the moduli spaces $\mathcal{E}_{\mathcal{A}}$ change as we vary $\mathcal{A}$. There are three types of walls in our wall and chamber decomposition.

- Definition 3.9** (I) A wall of *type* W_I is a wall arising from the log canonical transformations, ie the walls where the fibers of the relative log canonical model transition between fiber types.
- (II) A wall of *type* W_{II} is a wall at which the morphism induced by the log canonical transformation contracts the section of some components.
- (III) A wall of *type* W_{III} is a wall at which the morphism induced by the log canonical transformation contracts an entire rational pseudoelliptic component; see [Remark 3.8](#).

- Remark 3.10** (i) The walls of type W_{II} are precisely the walls of Hassett's wall and chamber decomposition [\[23\]](#); see discussion preceding [Theorem 3.7](#).
- (ii) There are finitely many walls; see [\[8, Theorem 6.3\]](#).

Theorem 3.11 [\[8, Theorem 1.5\]](#) Let $\mathcal{A}, \mathcal{B} \in \mathbb{Q}^r$ be weight vectors with $0 < \mathcal{A} \leq \mathcal{B} \leq 1$. Then:

- (i) If $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber, then the moduli spaces and universal families are isomorphic.
- (ii) If $\mathcal{A} \leq \mathcal{B}$ then there are reduction morphisms $\mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$ on moduli spaces which are compatible with the reduction morphisms on the Hassett spaces.
- (iii) The universal families are related by a sequence of explicit divisorial contractions and flips. More precisely, across W_I and W_{III} walls there is a divisorial contraction of the universal family, and across a W_{II} wall the universal family undergoes a log flip.

Remark 3.12 For more on [Theorem 3.11\(iii\)](#), we refer the reader to [\[8, Section 8\]](#). La Nave (see [\[33, Section 4.3 and Theorem 7.1.2\]](#)) noticed that the contraction of the section of a component is a log flipping contraction inside the total space of a one-parameter degeneration. In particular, the type I pseudoelliptic surfaces are thus attached along the reduced component of an intermediate (pseudo)fiber; see [\[8, Figure 13\]](#).

3.1 Strictly (semi-)log canonical Weierstrass models

In order to understand the stable pair degenerations of log canonical models of Weierstrass elliptic surfaces, we need to understand strictly log canonical and semi-log canonical Weierstrass fibrations. We collect some results in this direction here, beginning with the definition of a *type L* singular fiber.

Definition 3.13 [33, Section 3.3] Let $f: X \rightarrow C$ be a Weierstrass fibration with smooth generic fiber and Weierstrass data (A, B) . If $12 = \min(3v_q(A), 2v_q(B))$, where v_q denotes the order of vanishing at a point $q \in \mathbb{P}^1$, we say that f has a *type L fiber* at q .

Lemma 3.14 If F is a type L cusp of X , then X has strictly log canonical singularities in a neighborhood of F and the log canonical threshold $\text{lct}(X, 0, F)$ equals 0.

Proof After performing a weighted blowup $\mu: Y \rightarrow X$ at the cuspidal point of F , we get an exceptional divisor E (a possibly nodal elliptic curve) and strict transform $A := \mu_*^{-1}(F)$ (a rational curve meeting E transversely). Writing $\mu^*K_X = K_Y + aE$, it follows from the projection formula that $K_Y.E + aE^2 = 0$. On the other hand, $K_Y.E + E^2 = K_E = 0$ by the adjunction formula and $E^2 \neq 0$, since it is exceptional. Therefore $a = 1$, so X has a strictly log canonical singularity at the cuspidal point of F , and the discrepancy of $(X, \epsilon F)$ for any $\epsilon > 0$ will be strictly greater than 1. $\square$

Remark 3.15 The type L cusp decreases the self intersection S^2 by 1, and thus increases $\deg \mathcal{L}$ by 1; see [33, Remark 5.3.8].

We now discuss some facts on nonnormal Weierstrass fibrations with generic fiber a nodal elliptic curve. These appear as semi-log canonical degenerations of normal elliptic surfaces and as isotrivial j -invariant ∞ components of broken elliptic surfaces.

We first recall the definition of the fiber types N_k of these fibrations; see [6, Section 5] and [33, Lemma 3.2.2].

Definition 3.16 Fibers of type N_k have Weierstrass equation $y^2 = x^2(x - t^k)$.

Lemma 3.17 [33, Lemma 3.2.2] Fibers of type N_k are slc if and only if $k \in \{0, 1, 2\}$.

Remark 3.18 (i) The general fiber of an isotrivial j -invariant ∞ fibration is type N_0 .

(ii) N_2 is the j -invariant ∞ version of the L cusp; see Remark 3.19.

Remark 3.19 The N_2 fiber behaves analogously to the type L fiber. Indeed by the proof of [6, Lemma 5.1], on the normalization (X^ν, D) of a surface X with an N_2 fiber, the double locus D consists of a nodal curve with node lying over the cuspidal point of the N_2 fiber, and X^ν is smooth in a neighborhood of this point. In particular, (X^ν, D) has log canonical singularities in a neighborhood of the nodal point of D and $\text{lct}(X^\nu, D, A) = 0$ for any curve A passing through this point. Therefore by the definition of semi-log canonical, X has strictly semi-log canonical singularities in a neighborhood of the N_2 fiber F and $\text{slct}(X, 0, F) = 0$.

The local equation given above for a type N_k fiber is not a standard Weierstrass equation. One can check that the standard equation of an N_k fiber is given by

$$(3) \quad y^2 = x^3 - \frac{1}{3}t^{2k}x + \frac{2}{27}t^{3k}.$$

Proposition 3.20 *If $(f: X \rightarrow C, S)$ is an isotrivial j -invariant ∞ slc Weierstrass fibration with a_k type N_k fibers, then $-S^2 = \deg(\mathcal{L}) = \sum_k a_k \frac{1}{2}k$.*

Proof Let A and B the Weierstrass data of $(f: X \rightarrow C, S)$. If $q \in C$ lies under an N_k fiber, then A vanishes to order $2k$ and B to order $3k$ at q . Then A and B have degree $\sum 2ka_k$ and $\sum 3ka_k$, respectively. The result follows since the degrees of A and B are $4 \deg \mathcal{L}$ and $6 \deg \mathcal{L}$, respectively. $\square$

Note that for k even the N_k fiber has trivial monodromy, and for k odd it has μ_2 monodromy. This determines the twisted models of these fibers.

Corollary 3.21 *Let F be an N_k fiber. Then the twisted model of F is an N_0 (respectively twisted N_1) fiber if k is even (respectively odd).*

Proof By the local analysis of [7, Section 6.2], in the even case the twisted model must be stable since there is no base change required, and the odd case there is a μ_2 base change so the twisted model is a nodal cubic curve modulo the μ_2 action, ie a twisted N_2 fiber. $\square$

Thus, given an N_k fiber, we can cut it out and glue in an N_{k+2} fiber since the families are isomorphic to N_0 (respectively N_1) families over a punctured neighborhood. We can ask how this surgery affects $-S^2 = \deg \mathcal{L}$.

Corollary 3.22 *Let $(f: X \rightarrow C, S)$ be an isotrivial j -invariant ∞ Weierstrass fibration and let $(f: X' \rightarrow C, S')$ be the result of replacing an N_k fiber by an N_{k+2} fiber. Then $-(S')^2 = -S^2 + 1$.*

3.2 Elliptic fibrations via twisted stable maps

In [7] we used the theory of twisted stable maps, originally developed by Abramovich and Vistoli [1; 2], to understand limits of families of elliptic fibrations. The basic idea is that an elliptic surface $f: X \rightarrow C$ gives an a priori rational map $C \dashrightarrow \overline{\mathcal{M}}_{1,1}$ which extends to a morphism $\mathcal{C} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ from an orbifold curve $\mathcal{C}$ with coarse moduli space C . Now we understand limits of a family of elliptic surfaces by computing limits of the corresponding family of such maps. The twisted stable limits serve the same purpose for elliptic fibrations that *Kulikov models* serve for K3 surfaces, ie they form the starting point from which applying the MMP yields the stable limit.

3.2.1 Twisted stable maps limits We now recall structure of the limiting surfaces obtained using the twisted stable maps construction. As we will be studying slc degenerations of surfaces, the surfaces themselves will degenerate into possibly reducible surfaces. The degenerate surfaces will carry a fibration over a nodal curve whose j -map is the limit of the j -map of the degenerating family. Furthermore, there is a *balancing condition* on the stabilizers of the orbicurve $\mathcal{C}$ over nodes, which implies the action on the tangent spaces of the two branches at a node must be dual; see [1, Definition 3.2.4] and [40]. Finally, the stabilizers of a twisted stable map are concentrated either over nodes or at marked gerbes contained in the smooth locus. In particular, the limit of a map from a smooth schematic curve C can only have stabilizers over the nodes.

These observations motivate the following necessary conditions for a twisted surface to appear as a limit of a family of degenerating elliptic surfaces. We consider the case where the degenerating family of elliptic surfaces has $12dI_1$ marked singular fibers where $d = \deg \mathcal{L}$, as this is the generic situation and the relevant one for the present paper. This corresponds to the moduli map $C \rightarrow \overline{\mathcal{M}}_{1,1}$ extending to a morphism on all of $\mathcal{C}$ such that the j -map $C \rightarrow \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$ has degree $12d$, and is unramified over ∞ .

Proposition 3.23 *Suppose $(f: X \rightarrow C, S + F)$ is a twisted elliptic surface [7] over a rational curve which is the limit of a degenerating family of smooth elliptic surfaces with $12dI_1$ and arbitrary marked fibers. Then:*

- (i) *If X is reducible, its irreducible components are either attached along nodal fibers, or in the pairs of twisted fibers $I_a^*/I_b^*/N_1$, II/II^* , III/III^* or IV/IV^* .*
- (ii) *The total degree of the j -map $C \rightarrow \overline{\mathcal{M}}_{1,1}$ is $12d$.*
- (iii) *Away from the singular locus of C , the fibers of f are at worst nodal. In particular, every marked fiber in $F = \sum_{i=1}^n F_i$ is an I_a fiber for some $a \geq 0$.*

The surfaces of [Proposition 3.23](#) correspond to genus-zero balanced twisted stable maps to $\overline{\mathcal{M}}_{1,1}$ of degree $12d$ which are parametrized by the space $\mathcal{K}_{0,n}(\overline{\mathcal{M}}_{1,1}, 12d)(\underline{0})$. Here $\underline{0}$ is the tuple of n zeroes, denoting the fact that the marked points have trivial stabilizer.

Theorem 3.24 [[9](#), Theorem 5.5] *Each point*

$$[(f: \mathcal{C} \rightarrow \overline{\mathcal{M}}_{1,1}, p_1, \dots, p_n)] \in \mathcal{K}_{0,n}(\overline{\mathcal{M}}_{1,1}, 12d)(\underline{0})$$

admits a smoothing to a map from a nonsingular n -pointed schematic rational curve.

Corollary 3.25 *A twisted elliptic surface admits a smoothing to a generic $12dI_1$ elliptic surface if and only if it satisfies the conditions of [Proposition 3.23](#).*

3.2.2 Relative twisted stable maps One of the primary moduli spaces of interest from the perspective of stable pairs is the closure of the locus where the marked fibers are exactly the $12dI_1$ fibers. These fibers lie above the preimages of $\infty \in \overline{\mathcal{M}}_{1,1}$ under the j -invariant map $C \rightarrow \overline{\mathcal{M}}_{1,1}$, and thus we are concerned with the closure $\mathcal{K}_\infty \subset \mathcal{K}_{0,24}(\overline{\mathcal{M}}_{1,1}, 24)$ of the locus parametrizing maps from a smooth rational curve which are unramified over ∞ and such that all marked fibers map to ∞ . Equivalently, this locus is the space of maps *relative to the divisor* $[\infty]$ with multiplicities $(1, \dots, 1)$. The closure of such loci has been studied in the Gromov–Witten literature under the name of *relative stable maps*; see eg [[13](#); [17](#); [45](#)]. In [[9](#)], we considered the question of determining the points of this locus for twisted stable maps to stacky curves. The conditions characterizing this locus [[9](#), Conditions $(*)$] can be phrased in the context of elliptic fibrations:

Proposition 3.26 *Suppose $(f: X \rightarrow C, S + F)$ is a twisted elliptic surface over a rational curve which is the limit of a degenerating family of $12dI_1$ elliptic surfaces with marked singular fibers. Then the following hold in addition to the conditions of [Proposition 3.23](#):*

- (i) F consists of $12d$ nodal singular fibers.
- (ii) Every fiber with $j = \infty$ which is not on an isotrivial component is marked.
- (iii) For each maximal connected tree T of isotrivial $j = \infty$ components X , the number of marked fibers contained on T is equal to the sum of the multiplicities of the twisted fibers of the nonisotrivial components along which T is attached.

Remark 3.27 The last condition says that if an isotrivial j -invariant ∞ component is attached to an I_n fiber, there must be n markings on that component, since an I_n fiber is produced when n marked I_1 fibers collide.

Theorem 3.28 [9, Theorems 1.7 and 1.8] *The conditions of Proposition 3.26 characterize the boundary of $\mathcal{K}_\infty$. In particular, any twisted surface satisfying these conditions is the limit of a family of smooth $12dI_1$ elliptically fibered surface with marked singular fibers.*

Remark 3.29 After determining the shape of a twisted stable maps limit, we will use wall-crossing to compute the limits as one reduces weights.

4 Moduli of weighted stable elliptic K3 surfaces

In this section we specialize the discussion of Section 3 to the case of elliptic K3 surfaces and define the various compactifications of the stack $\mathcal{W}$ of elliptic K3 surfaces and its coarse space W which we need. The goal is to obtain an *explicit* description of the compactifications for various choices of weights $\mathcal{A}$. In particular, we will explicitly describe the surfaces parametrized by the boundary of $\mathcal{E}_{\mathcal{A}}$ in this case, as well as understand the wall-crossing morphisms.

From now on we assume that $g(C) = 0$ and $\deg \mathcal{L} = 2$ so that $C \cong \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$, and $(f: X \rightarrow C, S)$ is an elliptic K3 surface with section.

Definition 4.1 Let $\overline{\mathcal{W}}(\mathcal{A})$ be the closure in $\mathcal{E}_{\mathcal{A}}$ of the locus of pairs $(f: X \rightarrow C, S + F_{\mathcal{A}})$ where X is an elliptic K3 surface and $\text{Supp}(F_{\mathcal{A}})$ consists of 24 I_1 singular fibers.

Definition 4.2 If $\mathcal{A} = (a, \dots, a)$ is the constant weight vector, then S_{24} acts on $\overline{\mathcal{W}}(\mathcal{A})$ by permuting the marked fibers, and we denote the quotient by $\overline{\mathcal{W}}_\sigma(a)$.

Proposition 4.3 $\overline{\mathcal{W}}(\mathcal{A})$ and $\overline{\mathcal{W}}_\sigma(a)$ are proper Deligne–Mumford stacks. Moreover, the coarse space $\overline{W}_\sigma(a)$ of $\overline{\mathcal{W}}_\sigma(a)$ is a modular compactifications of W for each $0 < a \leq 1$.

Proof The fact that they are proper Deligne–Mumford stacks follows from [8]. By construction, $\overline{\mathcal{W}}_\sigma(a)$ has an open set parametrizing elliptic K3s with $24I_1$ fibers. Recall that W parametrizes lattice polarized K3 surfaces, and such a lattice polarization is equivalent to the structure of an elliptic fibration with chosen section. The result follows by the observation that a generic elliptically fibered K3 surface has $24I_1$ fibers. $\square$

Brunyate constructs a compactification $\overline{\mathcal{B}}$ of the space of elliptic K3 surfaces by studying degenerations of pairs $(X, \epsilon_1 S + F_B)$ where $B = (\epsilon, \dots, \epsilon)$, ie with small weights on both the section and the fibers (in particular, Brunyate requires $\epsilon_1 \ll \epsilon$), so that $\text{Supp}(F_B)$ is the closure of the rational curves on X [12]; see also [4, Section 7]. In fact there is a morphism $\overline{\mathcal{B}}^\nu \rightarrow \overline{\mathcal{W}}_\sigma(\epsilon)$, given by increasing the weight on the section to 1.

Proposition 4.4 *There is a morphism $\overline{\mathcal{B}}^\nu \rightarrow \overline{\mathcal{W}}_\sigma(\epsilon)$ for $\epsilon \ll 1$.*

Proof Consider a 1-parameter degeneration of pairs $(X, \epsilon S + F_B)$ inside $\overline{\mathcal{B}}$. We may generically choose smooth fibers $G = \bigcup_{i \in I} G_i$ to mark so that the pair $(X, S + F_B + G)$ is stable, where the section has coefficient 1. By the results of [8], there is a sequence of flips and contractions as one reduces the coefficients of G from 1 to 0. The resulting stable limit in $\overline{\mathcal{W}}_\sigma(\epsilon)$ only depends on the point $(X_0, \epsilon S_0 + (F_B)_0)$ in $\overline{\mathcal{B}}$ and not on the family or choice of auxiliary markings. Therefore we obtain the desired morphism by [18, Theorem 7.3]. $\square$

Remark 4.5 Comparing Theorem 6.13 with [12, Theorem 9.1.4] (see also [4, Section 7]), we see that there is a bijection between the boundary strata of $\overline{\mathcal{B}}$ and $\overline{\mathcal{W}}_\sigma(\epsilon) = \mathcal{W}(B)/S_{24}$. For example, the third case in [12, Theorem 9.1.4] maps to case (E) of Theorem 6.13 if there are no $\mathbb{F}_0$ components, and to either case (D) or (F) depending on the parity of the number of components if there are $\mathbb{F}_0$ components.

Corollary 4.6 *The morphism from Proposition 4.4 is an isomorphism.*

Proof It is a proper birational set-theoretic bijection between normal spaces. $\square$

Remark 4.7 It follows from Corollary 4.6 that there is in fact a morphism $\overline{\mathcal{W}}_\sigma(\epsilon) \rightarrow \overline{\mathcal{B}}$ which can be thought of as induced by decreasing weights on the section.

Definition 4.8 Let $\overline{\mathcal{K}}_\epsilon$ denote stable pairs compactification of the space parametrizing pairs with only one singular fiber marked with weight $0 < \epsilon \ll 1$, and let $\overline{\mathcal{K}}_\epsilon$ be its coarse moduli space.

Next we define the moduli space $\overline{\mathcal{F}}_\epsilon$, which is like $\overline{\mathcal{K}}_\epsilon$, only we allow *any* fiber to be marked.

Definition 4.9 Let $\overline{\mathcal{F}}_\epsilon$ be the closure in $\mathcal{E}_A$ of the locus of pairs $(f: X \rightarrow C, S + \epsilon F)$, where f has precisely 24 I_1 fibers, $0 < \epsilon \ll 1$, and F is any fiber.

Remark 4.10 At this point we have introduced many compactifications (see [Figure 1](#) and the list on page 1894):

- $\overline{\mathcal{W}}(\mathcal{A})$ The stable pair compactification with $\mathcal{A}$ -weighted singular fibers.
- $\overline{\mathcal{W}}_\sigma(a)$ The quotient by S_{24} when $\mathcal{A} = (a, \dots, a)$.
- $\overline{\mathcal{K}}_\epsilon$ The stable pairs compactification with a single ϵ -marked singular fiber.
- $\overline{\mathcal{F}}_\epsilon$ The stable pairs compactification with *any fiber* marked by ϵ .
- $\overline{\mathcal{W}}^*$ The SBB compactification of the period domain moduli space W .

We now give a brief overview of how they are related (again, see [Figure 1](#)).

- (i) There are 24 generically finite morphisms $\overline{\mathcal{W}}(\mathcal{A}) \rightarrow \overline{\mathcal{K}}_\epsilon$ of degree $23!$, corresponding to forgetting all but one marked singular fiber.
- (ii) There is a degree 24 generically finite rational map $\overline{\mathcal{K}}_\epsilon \dashrightarrow \overline{\mathcal{W}}_\sigma(\epsilon)$, corresponding to choosing a singular fiber.
- (iii) There are morphisms $\overline{\mathcal{W}}_\sigma(\epsilon) \rightarrow \overline{\mathcal{W}}^*$ and $\overline{\mathcal{K}}_\epsilon \rightarrow \overline{\mathcal{W}}^*$; see [Theorems 6.15](#) and [8.2](#), respectively.
- (iv) We will see in [Section 8.3](#) that the moduli space $\overline{\mathcal{F}}_\epsilon$ is a smooth Deligne–Mumford stack whose coarse space is an (explicit) GIT quotient. Furthermore, there is a morphism $\overline{\mathcal{F}}_\epsilon \rightarrow \overline{\mathcal{W}}^*$ (see [Theorem 8.8](#)) which is generically a $\mathbb{P}^1$ bundle.

We end this section with an important proposition.

Proposition 4.11 *For any surface X parametrized by $\overline{\mathcal{W}}(\mathcal{A})$ (for any $\mathcal{A}$) or $\overline{\mathcal{F}}_\epsilon$ (in particular $\overline{\mathcal{K}}_\epsilon$), we have $H^1(X, \mathcal{O}_X) = 0$.*

Proof Since slc singularities are Du Bois [[26](#), Corollary 6.32; [29](#)], X has Du Bois singularities. Then $H^1(X, \mathcal{O}_X) = 0$ since $H^i(X_b, \mathcal{O}_{X_b})$ is constant in any flat family of varieties with Du Bois singularities [[29](#), Corollary 1.2], and any X arises as the special fiber of a flat family whose general fiber is a surface X_η with $H^1(X_\eta, \mathcal{O}_{X_\eta}) = 0$. $\square$

Remark 4.12 We will see in [Theorem 8.1](#) that the surfaces on the boundary of $\overline{\mathcal{F}}_\epsilon$ (and thus also $\overline{\mathcal{K}}_\epsilon$) satisfy $\omega_X \cong \mathcal{O}_X$. Moreover, if F is the marked fiber, then $2F$ is an ample Cartier divisor such that $(2F)^2 = 2$. Then following [[5](#), Definition 3.4, Proposition 3.8, and Theorem 3.11], we see that $\overline{\mathcal{F}}_\epsilon$ and $\overline{\mathcal{K}}_\epsilon$ are proper Deligne–Mumford stacks representing a functor over arbitrary base schemes. Due to subtleties with defining moduli spaces in higher dimensions, the remaining spaces follow the formalism developed in [[8](#)] and thus correspond to Deligne–Mumford stacks representing functors only over *normal* base schemes; see [[8](#), Section 2.2.2] for more details.

4.1 Isotrivial j -invariant ∞ fibrations

Here we prove some preliminary results on isotrivial j -invariant ∞ elliptic fibrations of K3 type which appear in the boundary of the various moduli spaces described above. We begin by bounding the number of N_i fibers (see [Definition 3.16](#)) which can appear on an slc elliptic K3.

Proposition 4.13 *Let $(f: X \rightarrow \mathbb{P}^1, S)$ be an isotrivial $j = \infty$ slc Weierstrass fibration of K3 type. Then X has one of the following configurations of cuspidal fibers:*

- (i) $4N_1$,
- (ii) $2N_1N_2$, or
- (iii) $2N_2$.

Proof We must have only N_0 , N_1 and N_2 by the slc assumption, so, by [Proposition 3.20](#), $2 = \frac{1}{2}a_1 + a_2$, which only admits the nonnegative integer solutions $(4, 0)$, $(2, 1)$ and $(0, 2)$ for (a_1, a_2) . $\square$

Remark 4.14 Up to automorphisms of $\mathbb{P}^1$, the global Weierstrass equation for the surfaces in [Proposition 4.13](#) can be written as follows:

- (i) $y^2 = x^3 - \frac{1}{3}t^2s^2(t-s)^2(t-\lambda s)^2x + \frac{2}{27}t^3s^3(t-s)^3(t-\lambda s)^3$ for $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- (ii) $y^2 = x^3 - \frac{1}{3}t^2s^2(t-s)^4x + \frac{2}{27}t^3s^3(t-s)^6$.
- (iii) $y^2 = x^3 - \frac{1}{3}t^4s^4x + \frac{2}{27}t^6s^6$.

In particular, up to isomorphism there are unique surfaces with configurations (ii) and (iii).

Finally, we need the following key proposition.

Proposition 4.15 *Suppose $(f_0: X \rightarrow \mathbb{P}^1, S)$ is an isotrivial $j = \infty$ slc Weierstrass fibration of K3 type and $F \subset X$ is an N_k fiber. If f_0 is the central fiber of a 1-parameter family of Weierstrass models $(f: \mathcal{X} \rightarrow \mathcal{C}, \mathcal{S}) \rightarrow B$ with generic fiber $(f_\eta: \mathcal{X}_\eta \rightarrow \mathcal{C}_\eta, \mathcal{S}_\eta)$ a $24I_1$ elliptic fibration, then there are at least $k + 1$ type I_1 fibers of f_η that limit to the N_k fiber F for $k = 1, 2, 3, 4$.*

Proof Consider the twisted stable maps limit of f_η . By [Proposition 3.23\(i\)](#), the Weierstrass N_1 fiber F must be replaced by a surface component Y attached along the twisted model of F by a twisted fiber of type I^* (resp. I) if k is odd (resp. even).

By Proposition 4.13, the possibilities for X are $4N_1$, $2N_1N_2$ and $2N_2$, as well as the non-slc cases N_1N_3 and N_4 . Since the degree of the j -map is constant for a family of twisted stable maps, the sum of degrees of the j -map of the components of the twisted model is 24. This means that Y is rational when $k = 1, 2$ and K3 when $k = 3, 4$. The number of I_1 fibers of f_η limiting to the N_1 fiber F of f_0 is the same as the number of I_1 fibers limiting to the component Y in the twisted model. By Proposition 3.23(ii)–(iii), the component Y cannot be isotrivial and $\deg(\mathcal{L}) \geq 1$. By Persson's classification [41], a rational elliptic surface Y with an I^* fiber has at least $2I_1$ fibers, and one with an I_n fiber has at least three other I_1 fibers counted with multiplicity. Similarly, by [44, Theorems 1.1 and 1.2], an elliptic K3 surface with an I^* fiber has at least $4I_1$ fibers, and one with an I_n fiber has at least five other I_1 fibers counted with multiplicity. $\square$

5 Wall crossings inside $\overline{\mathcal{W}}_\sigma(a)$ for $a > \frac{1}{12}$

Recall that $\overline{\mathcal{W}}_\sigma(a)$ denotes the space where all singular fibers are marked with weight a and we have taken the S_{24} quotient. The main goal of this section is to describe the surfaces parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ for $0 < \epsilon \ll 1$. In particular, we explicitly describe the wall crossings that happen as we vary the weight vector from $a = 1$ to $a = \frac{1}{12} + \epsilon$.

By Corollary 5.6 we see that surfaces parametrized by $\overline{\mathcal{W}}_\sigma(a)$ have *at most* two elliptically fibered components, but possibly with trees of pseudoelliptic surfaces attached to them. In Proposition 5.15 we classify the possible surfaces parametrized by $\overline{\mathcal{W}}_\sigma(a)$ with a *single normal elliptically fibered component*. In Theorem 5.16 we classify the possible surfaces parametrized by $\overline{\mathcal{W}}_\sigma(a)$ with a *single nonnormal elliptically fibered component*. In Theorem 5.19, we classify the possible surfaces parametrized by $\overline{\mathcal{W}}_\sigma(a)$ with *two elliptically fibered components*. Finally, in Propositions 5.18 and 5.20, we show that surfaces of each type appearing in the aforementioned results do exist on the boundary of $\overline{\mathcal{W}}_\sigma(a)$.

Lemma 5.1 *There are type W_{II} walls where type I pseudoelliptic surfaces form at $a = 1/k$ for $k = 1, \dots, 11$.*

Proof Recall that type I pseudoelliptic surfaces form when a component of the underlying weighted curve is contracted — this occurs when $ka = 1$. Finally, note that $24a > 2$ for each of these values of k , so the moduli space is nontrivial. $\square$

Lemma 5.2 *There are type W_{III} walls at $a = \frac{5}{12}$, $\frac{3}{12}$ and $\frac{2}{12}$ where rational pseudo-elliptic surfaces attached along intermediate type II, III and IV fibers, respectively, contract to a point.*

Proof This follows from [8, Theorem 6.3] as well as the observation that a rational elliptic surface attached to a type II, III or IV fiber must have a II^* , III^* or IV^* fiber, respectively, and so it has 2, 3 or 4 other marked fibers counted with multiplicity. $\square$

Since these walls are all above $\frac{1}{12}$, we obtain:

Corollary 5.3 *Any type II, III or IV fiber on a surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ is a Weierstrass fiber. In particular, there are no pseudoelliptic trees sprouting off of it.*

In a similar vein we have the following two lemmas:

Lemma 5.4 *There are type W_{III} walls at $a = \frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{10}$, where:*

- (i) *Rational pseudoelliptic surfaces attached along intermediate type N_1 fibers contract onto a point.*
- (ii) *Isotrivial j -invariant ∞ surfaces with $\deg \mathcal{L} = 1$ attached along intermediate type N_1 fibers contract onto a point.*

Proof A rational elliptic surface attached along an N_1 fiber must have an I_k^* fiber in the double locus. Since an I_k^* has discriminant $6 + k$, there are $6 - k$ markings counted with multiplicity on the rational pseudoelliptic. By the classification in [41], there exist rational elliptic surfaces with I_k^* for $0 \leq k \leq 4$. Since the log canonical threshold of an intermediate N_1 fiber is $\frac{1}{2}$, the surfaces with an N_1/I_k^* double locus contract at $1/(2(6 - k))$. These give walls above $\frac{1}{12}$ for $1 \leq k \leq 4$. Similarly, isotrivial j -invariant ∞ surfaces with an N_1 fiber and $\deg \mathcal{L} = 1$ must be attached along another N_1 fiber and so contract at $1/(2k)$, where they support k fibers. $\square$

Next we consider the base curve at $\frac{1}{12} + \epsilon$:

Lemma 5.5 *Let $\mathcal{A} = (a, \dots, a)$ for $a = \frac{1}{12} + \epsilon$. Then curves C parametrized by $\overline{\mathcal{M}}_{0,\mathcal{A}}$ are either*

- (i) *a smooth $\mathbb{P}^1$ with 24 marked points, with at most 11 markings coinciding, or*
- (ii) *the union of two rational curves, each with 12 marked points and at most 11 markings coinciding.*

Proof If C is a smooth $\mathbb{P}^1$, since the total weight for any marking is at most 1, we see that at most 11 points can coincide. If C is the union of two rational curves, since each point is weighted by $\frac{1}{12} + \epsilon$ and since each curve needs total weight greater than 2 (including the node), each curve must have (exactly) 12 points, and again at most 11 can coincide. Finally, suppose C is the union of three components $C = \bigcup_{i=1}^3 C_i$ with C_1 and C_3 the end components. Since the C_2 component needs at least one marking to be stable, at least one of C_1 and C_3 will not have enough marked points to be stable. $\square$

Corollary 5.6 *Let $(f : X \rightarrow C, S + F_a)$ be a surface pair parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$. Then $f : X \rightarrow C$ has at most two elliptically fibered components.*

Remark 5.7 X can have many type I pseudoelliptic components mapping by f onto marked points of C .

Definition 5.8 If $(f : X \rightarrow C, S + F_a)$ is a surface pair parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$, the *main component* of X , denoted by X_m , is the union of all elliptically fibered components of $f : X \rightarrow C$.

Remark 5.9 By [Corollary 5.6](#), for all surfaces pairs parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$, either X_m and C are irreducible or $X_m = X_1 \cup X_2$ and $C = C_1 \cup C_2$, where X_i and C_i are irreducible and $f|_{X_i} : X_i \rightarrow C_i$ is an elliptic fibration.

5.1 Explicit classification of surfaces inside $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$

We conclude that every surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ consists of a main component (see [Definition 5.8](#)) possibly with trees of pseudoelliptics sprouting off. In order to do understand the possible main components X_m parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$, we will use the following construction of a Weierstrass model for X_m .

5.1.1 Construction of a family of Weierstrass models

Let

$$(f_0 : X_0 \rightarrow C_0, S_0 + (F_a)_0)$$

be an elliptic surface pair parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$, which by [Corollary 5.6](#) has at most two elliptic components. Consider a 1-parameter family $(f : \mathcal{X} \rightarrow \mathcal{C}, \mathcal{S} + \mathcal{F}_a) \rightarrow T$ with generic fiber $(f : X_\eta \rightarrow C_\eta, S_\eta + (F_a)_\eta)$, a $24I_1$ elliptic K3 surface, and special fiber X_0 . Let $\mathcal{G}_\eta$ be a generic smooth fiber of the elliptic fibration $f : \mathcal{X} \rightarrow \mathcal{C}$ such that

the closure $\mathcal{G}$ is a generic smooth fiber of $f_0: X_0 \rightarrow C_0$. In particular, $G_0 = \mathcal{G}_0$ avoids any pseudoelliptic trees of X_0 .

Let Y_0 denote the irreducible component of X_0 on which G_0 lies. The component Y_0 is necessarily elliptically fibered, and so either $Y_0 = X_m$ is the main component or $X_m = Y_0 \cup_{H_0} Y_1$ glued along a twisted fiber H_0 . To classify the possible elliptically fibered components of X_0 , we will take the relative log canonical model of the pair $(\mathcal{X}, \mathcal{S} + \mathcal{G}) \rightarrow T$ using the main results of [8].

First, if $X_m = Y_0 \cup Y_1$, there is a type W_{II} crossing causing a flip of the section of Y_1 such that Y_1 becomes a type I pseudoelliptic. Then in either case we have a new family where Y_0 is the unique elliptically fibered component with trees of type I pseudoelliptic surfaces sprouting off of it. We make the following assumption, and revisit it when we see it holds in Lemmas 5.13 and 5.14:

Assumption 5.10 Suppose every type I pseudoelliptic tree attached to Y_0 is attached along the intermediate model of a log canonical Weierstrass cusp.

There exists a sequence of type W_{III} extremal contractions followed by a type W_{III} relative log canonical morphism of the family that contract the trees of type I pseudoelliptic components to a point, resulting in a Weierstrass model Y' of Y_0 . Denote the resulting family of surfaces by $\mathcal{X}' \rightarrow T$.

Since type W_{III} contractions preserve the generic fiber of the family $\mathcal{X} \rightarrow T$, we must only check type W_{II} contractions of the section S . By [25, Proposition 5.9], we may blow up the point to which the section has contracted to preserve the generic fiber of the family, and so we have that $\mathcal{X}'_{\eta} = \mathcal{X}$. The resulting family of fibrations $(\mathcal{X}' \rightarrow \mathcal{C}) \rightarrow T$ is a family of slc Weierstrass models over $\mathbb{P}^1$ with $\deg(\mathcal{L}) = 2$, generic fiber a $24I_1$ elliptic K3, and special fiber Y' . By Remark 3.15, we can conclude that Y' is one of the following Weierstrass limits:

- (i) a minimal Weierstrass elliptic K3 surface ($\deg \mathcal{L} = 2$),
- (ii) a rational elliptic surface with a single type L cusp, or
- (iii) an isotrivial elliptic surface with two type L cusps and all other fibers stable.

By considering the discriminant of $\mathcal{X}' \rightarrow \mathcal{C}$ as a flat family of divisors on $\mathcal{C}$, we have the following key observation:

Remark 5.11 Suppose $Y' \rightarrow C_0$ is normal. The number of I_1 fibers of the generic fibration $X_\eta \rightarrow C_\eta$ that collide onto a singular fiber F of $Y' \rightarrow C_0$ is the multiplicity of F in the discriminant of the Weierstrass model $Y' \rightarrow C_0$.

We can use this observation to constrain the possible components of the twisted stable maps limit of $(f: \mathcal{X}_\eta \rightarrow \mathcal{C}_\eta, \mathcal{I}_\eta + \mathcal{F})$. In this limit, the singular fibers $(f: \mathcal{X}_\eta \rightarrow \mathcal{C}_\eta)$ cannot collide since they are marked with coefficient one. Let Y'' be the unique component of a twisted model that maps birationally to the component Y' in the above family of Weierstrass models. Then each connected component of the complement of Y'' is a tree of twisted surfaces that gets collapsed onto a fiber of Y'' by the sequence of flips and contractions that produce the Weierstrass model above. In particular the number of marked fibers on each tree of elliptic components sprouting off a fiber of Y'' is exactly the multiplicity of the discriminant of the resulting singular fiber on the Weierstrass model Y' .

Remark 5.12 The type L cusps are the Weierstrass model of an intermediate fiber of type I_m for $m \geq 0$. Such fibers are not contracted until they have coefficient 0, and so any pseudoelliptic tree glued along a type I_m fiber will remain when lowering coefficients to any $\epsilon > 0$.

Finally we revisit [Assumption 5.10](#). We first need the following characterization of intermediate models of non-log canonical Weierstrass cusps:

Lemma 5.13 Suppose $X = X_0 \cup_G X_1$ is a smoothable broken elliptic surface that is the union of broken elliptic surfaces $X_i \rightarrow C_i$, where $C_i \cong \mathbb{P}^1$ and each X_i has a unique main component. Let X' be the result of the type II pseudoelliptic flip of the section of X_0 , so that the strict transform X'_0 is attached to X'_1 by an intermediate fiber $A \cup G$. Then $A \cup G$ is the intermediate fiber of an slc cusp if and only if $-S_0^2 \leq 1$, where S_0 is the section of $X_0 \rightarrow C_0$.

Proof The question is local around a neighborhood of the flip. Therefore, we may assume that X_0 and X_1 are irreducible, so that there are no pseudoelliptic trees sprouting off either of them. On the component X'_1 we have the divisor $S_1 + aA + G$. Note that G has coefficient 1 since it is in the double locus, and the coefficient a is given by the sum of coefficients of marked fibers on X'_1 . Then the Weierstrass model of $A \cup G$ inside X'_1 has log canonical singularities if and only if G contracts onto the Weierstrass model in the log canonical model of the pair $(X_1, S + G)$, ie when all the coefficients

on X'_0 are 0. Since the pair is smoothable, this occurs if and only if X'_0 contracts to a point in the log canonical model of X , where all the coefficients on X'_0 are set to 0. Since G is marked with coefficient 1 on X'_0 , this occurs if and only if X'_0 is a minimal rational elliptic surface by [6, Proposition 7.4], which holds if and only if $-S_0^2 \leq 1$ (where the strictly less than 1 case happens if G is a twisted fiber rather than a stable fiber of X_0). $\square$

Lemma 5.14 *Let X be a surface parametrized by $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$ and suppose $Y \subset X_m$ is a normal main component. Then Assumption 5.10 is satisfied for every pseudoelliptic tree attached to Y . Moreover, the fibers these pseudoelliptic trees are attached to are minimal intermediate fibers.*

Proof Let $X' \rightarrow C'$ denote the twisted stable maps model of $X \rightarrow C$, and let X'_m and Y' denote the strict transform of X_m and Y in X' . Let Z be a pseudoelliptic glued to an intermediate fiber F of Y , and let Z' be the components of X' that map to Z . By Remark 5.11, the number of markings on Z is equal to the contribution of F to the discriminant of the Weierstrass model of Y . Since X_m is the main component, there are less than 12 markings on Z , and so the order of vanishing of the discriminant of F in Y is less than 12. It follows that the order of vanishing of the Weierstrass data in a neighborhood of this fiber satisfies $\min\{3v(a), 2v(b)\} < 12$, so these are minimal Kodaira types by the standard classification. $\square$

5.1.2 X_m is irreducible We first deal with the case where the main component X_m of a surface parametrized by $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$ is irreducible.

Proposition 5.15 *Let X be a surface parametrized by $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$ such that the main component X_m is irreducible and normal. Then X_m is a minimal elliptic K3 surface with trees of pseudoelliptic surfaces of type I attached along intermediate models of I_n^* , II^* , III^* and IV^* fibers.*

Proof By Lemma 5.14, Assumption 5.10 is satisfied. Following Section 5.1.1, we saw that there are three possibilities for the Weierstrass stable replacement of the main component X_m of a surface in $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$. In case (i) we have a minimal Weierstrass elliptic K3 surface. Then since all fibers are minimal Weierstrass fibers, any pseudoelliptic surface has to be attached by the intermediate model of a minimal Weierstrass fiber. These are exactly the intermediate models of type I_n^* , II , III , IV , II^* ,

III* and IV*, since type I_n Weierstrass fibers do not have intermediate models. By [Corollary 5.3](#), pseudoelliptics sprouting off of II, III and IV fibers have contracted onto the Weierstrass model. We now rule out cases (ii) and (iii) of [Section 5.1.1](#).

In case (ii), the Weierstrass model of the main component is a rational elliptic surface with exactly one type L cusp. In this case, there must be a type I pseudoelliptic tree Z in X attached to X_m along an intermediate model of an L cusp, and by [Remark 5.11](#), there are 12 marked pseudofibers on Z . Let $X_1 \rightarrow C_1$ be a twisted stable maps model that maps to X in $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$. We may write $X_1 = Y_1 \sqcup_{I_n} Z_1$, where

- (i) Z_1 is a broken elliptic fibration that dominates the pseudoelliptic tree Z ,
- (ii) Y_1 is a broken elliptic fibration that dominates $X \setminus Z$,
- (iii) the component of Y_1 supporting the fiber $Y_1 \cap Z_1 = I_n$ is birational to X_m , and
- (iv) the $Y_1 \cap Z_1 = I_n$ fiber becomes the intermediate fiber on X_m after Z_1 undergoes a type II transformation into the pseudoelliptic tree Z .

Then 12 of the marked fibers of $X_1 \rightarrow C_1$ must lie on Z_1 and the other 12 on Y_1 . In particular there is a node of C_1 , such that if we separate C_1 along that node we obtain two trees of rational curves each with 12 marked points. However, this means the stable replacement of C_1 inside the Hassett space $\overline{\mathcal{M}}_{0,\mathcal{A}}$, for $\mathcal{A} = (a, \dots, a)$ with $a = \frac{1}{12} + \epsilon$, is a nodal union of two components, contradicting that X has only one main component.

In case (iii), the Weierstrass model of X_m is a trivial surface with exactly two type L cusps and all other fibers stable. There must be type I pseudoelliptic trees attached along each of these L cusp fibers in X_m , and no other pseudoelliptic trees attached to X_m , as every other fiber of its Weierstrass model is stable. As in the previous analysis, let $X_1 \rightarrow C_1$ be a twisted stable maps surface whose image in $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ is X , and let X' be the component of X_1 that dominates X_m . Then X' is attached to exactly two other components of X_1 , so by stability it must have at least one marked point on it. Since $X_1 \rightarrow C_1$ is the twisted stable maps model, all the marked fibers have j -invariant ∞ , and so since X' is isotrivial, it must be nonnormal, a contradiction. $\square$

Next we consider the irreducible, but nonnormal main component case:

Theorem 5.16 *Let X be a surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ with an irreducible nonnormal main component X_m . Then one of the following holds:*

- (a) X_m is an isotrivial $j = \infty$ fibration with $4N_1$ minimal Weierstrass fibers.

- (b) X_m is an isotrivial $j = \infty$ fibration with $2N_1$ minimal Weierstrass fibers, as well as an intermediate N_2 fiber which must have a tree of pseudoelliptic surfaces attached to it along a type I_n pseudofiber.
- (c) X_m is an isotrivial $j = \infty$ fibration with $2N_2$ intermediate fibers, each of which has a tree of pseudoelliptic surfaces attached to it by an I_n fiber.
- (d) X_m is an isotrivial $j = \infty$ fibration with a minimal Weierstrass N_1 fiber, as well as an intermediate N_3 fiber which has a tree of pseudoelliptic surfaces attached to it by an I_n^* fiber.
- (e) X_m is an isotrivial $j = \infty$ fibration with a single intermediate N_4 fiber which has a tree of pseudoelliptic surfaces attached to it by an I_n fiber.

Moreover, if we denote by l the number of marked N_0 fibers on X_m , then

(a)	(b)	(c)	(d)	(e)
$4 \leq l \leq 16$	$3 \leq l \leq 17$	$2 \leq l \leq 18$	$8 \leq l \leq 18$	$13 \leq l \leq 19$

Proof Suppose that [Assumption 5.10](#) is satisfied. By [Section 5.1.1](#), the Weierstrass model of the main component must be an slc isotrivial $j = \infty$ Weierstrass fibration with $\deg \mathcal{L} = 2$, which are classified by [Proposition 4.13](#). The lct of a type N_2 fiber is 0, so these do not contract to Weierstrass models, and any attached pseudoelliptic trees do not contract for nonzero weight.

In case (c), the stability condition on the twisted stable maps limit implies that there must be at least one marked N_0 fiber to give that rational component of the base curve at least three special points.

The types of pseudofibers that are attached to intermediate N_1 and N_2 fibers must have j -invariant ∞ , so they are either type I_n or I_n^* , respectively. The twisted model of an N_1 fiber is a nonreduced rational curve, and so must have a stabilizer at the corresponding point of the twisted stable map. Therefore, it must be attached to an I_n^* fiber, which also has a nontrivial stabilizer at the corresponding point of the twisted stable map. Similarly, the twisted model of an N_2 fiber is a nodal curve so it has no stabilizer, and therefore must be attached to an I_n fiber.

If [Assumption 5.10](#) is not satisfied, then by [Lemma 5.13](#) we must have a K3 component Y attached to X_m along a fiber F such that Y is not the main component. This only happens if Y has less than 12 singular fibers counted with multiplicity away from the fiber along which Y is attached to X_m . In that case F is a fiber of Y with discriminant

at least 13, so F is either an I_n fiber for $n \geq 13$ or an I_n^* for $n \geq 7$. Consider a generic family of $24I_1$ surfaces degenerating to this surface as in [Section 5.1.1](#).

In the first case, we have that n type I_1 fibers collide to sprout out a trivial component with n markings, which becomes the main component when Y flips into a pseudoelliptic. Since X_m has only N_0 fibers away from where Y is attached and the degree of $\mathcal{L}$ must be 2, the attaching fiber is N_4 by [Proposition 3.20](#). This gives us (e). In the second case, let us denote by Y' and X'_m the strict transforms of Y and X_m in the twisted stable maps replacement of the limit of the family. Then Y' and X'_m are glued along twisted I_n^*/N_1 fibers since the order of the stabilizer is 2. Then the base curve of the X'_m component must have at least one more point with a stabilizer since any finite cover of $\mathbb{P}^1$ is ramified in at least two points. On the other hand, the stabilizer of any j -invariant ∞ -curve is μ_2 so these other points have to have stabilizers of order 2. Now when the component Y' flips into the pseudoelliptic surface Y , the twisted fiber on X'_m to which it is attached must flip into a *non-semi-log canonical* intermediate fiber since [Assumption 5.10](#) fails. Thus it must be an N_k fiber for $k \geq 3$. The other twisted fibers on X'_m must flip into intermediate models of N_k fibers for $k \geq 1$ since the N_0 fiber has no stabilizers. Since the degree of $\mathcal{L}$ for the main component X_m must be 2, by [Proposition 3.20](#), the fiber along which Y is attached must be N_3 , and the only other nonstable fiber is a single N_1 . This gives us case (d).

To obtain the number of markings, we may apply [Proposition 4.15](#) to see that each N_k fiber is marked with multiplicity at least $k + 1$. This gives an upper bound on n . For the lower bound, we look at the largest number of marked I_1 fibers that can appear on a component attached to the N_k fiber. For an N_1 fiber this is five markings on a $5I_1I_1^*$ rational, for N_2 this is 11 markings on a $12I_1$ (attached along one of the I_1 fibers), for N_3 this is 11 markings on an $11I_1I_7^*$ elliptic K3, and for N_4 this is 11 markings on a $12I_1I_{13}$ elliptic K3. Here we have used that X_m is the main component so all the other components must have undergone pseudoelliptic flips at a wall above $\frac{1}{12} + \epsilon$. Finally, each N_1 fiber is Weierstrass since there are at most five markings on the component attached to it, and so by [Lemma 5.4](#), these components contract to a point at a W_{III} wall above $\frac{1}{12} + \epsilon$. $\square$

Remark 5.17 Each of the main components in [Theorem 5.16](#) that have only intermediate models of semi-log canonical cusps (cases (a), (b) and (c)) are $j = \infty$ limits of normal isotrivial elliptic surfaces. The $4N_1$ surfaces are limits of $4I_0^*$ isotrivial fibrations. Indeed, the locus in the moduli space of such surfaces is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, where the

first coordinate parametrizes the j -invariant of the fibration and the second coordinate parametrizes the configuration of the $4I_0^*$ (or $4N_1$) singular fibers. Similarly the $2N_1N_2$ surface is the limit of the isotrivial $2I_0^*L$ surface and there is a rational curve of these in the moduli space. Finally the $2N_2$ surface is the limit of isotrivial $2L$ Weierstrass fibrations, but this family of $2L$ surfaces does *not* actually appear on this component of the moduli space as we describe below.

Note that in each of these cases, when the surface is isotrivial with $j \neq \infty$, all the markings must be concentrated on the special fibers. Indeed by [Remark 5.11](#), there must be six markings concentrated at an I_0^* fiber and 12 concentrated at a type L fiber. Therefore the isotrivial $j = \infty$ surface pairs that are limits of Weierstrass models as in the above paragraph must have six markings concentrated at each N_1 fiber and 12 markings concentrated at each N_2 fiber. In particular, they *cannot* have any marked N_0 fibers. Therefore, not all surface pairs with isotrivial $j = \infty$ main components are in the limit of the above locus of normal Weierstrass fibrations. In particular, since the type $2N_2$ fibrations must have at least one marked N_0 fiber by stability for twisted stable maps, we see that the $2L$ family limiting to $2N_2$ does not appear.

Finally we address the question of existence of each of the limits described above.

Proposition 5.18 *Each of the cases described by [Proposition 5.15](#) and [Theorem 5.16](#) occurs in $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$.*

Proof We may take the Weierstrass model of the described main component. In each case it has a Weierstrass equation with A and B of degree 8 and 12, respectively. Since the space of Weierstrass equations is irreducible, there exists a family of $24I_1$ elliptic K3 surfaces with this Weierstrass limit. By taking the stable replacement in $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ we must obtain stable limits as described. $\square$

5.1.3 X_m is reducible Now we classify the broken elliptic surfaces in $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ where X_m is the union of two irreducible surfaces.

Theorem 5.19 *Let X be a surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ with reducible main component $X_m = Y_0 \cup Y_1$. Then one of the following holds:*

- (i) *The Y_i are rational elliptic surfaces glued along an I_0 fiber. They are minimal Weierstrass surfaces away from possible intermediate type II^* , III^* and IV^* fibers along which type I pseudoelliptic trees are attached.*

- (ii) Y_0 is an elliptic K3 surface, Y_1 is a trivial j -invariant ∞ surface, and they are glued along I_{12}/N_0 fibers. There are 12 marked N_0 fibers on Y_1 , and Y_0 has minimal Weierstrass fibers or minimal intermediate type II^* , III^* or IV^* fibers where type I pseudoelliptic trees are attached.
- (iii) Y_0 is an elliptic K3 with an I_6^* fiber, Y_1 is a $2N_1$ isotrivial j -invariant ∞ surface, and they are glued along twisted I_6^*/N_1 fibers. Away from the I_6^* fiber, Y_0 has minimal Weierstrass fibers or minimal intermediate type II^* , III^* and IV^* fibers where type I pseudoelliptic trees are attached. There are $7 \leq l \leq 10$ marked N_0 fibers on Y_1 .
- (iv) The Y_i are isotrivial j -invariant ∞ surfaces glued along N_0 fibers. Each surface has a single intermediate N_2 fiber with a type I pseudoelliptic tree attached. There are $1 \leq l_i \leq 9$ marked N_0 fibers on Y_i .
- (v) The Y_i are isotrivial j -invariant ∞ surfaces glued along N_0 fibers. Each surface has two minimal Weierstrass N_1 fibers. There are $2 \leq l_i \leq 8$ marked N_0 fibers on Y_i .
- (vi) The Y_i are isotrivial j -invariant ∞ surfaces glued along N_0 fibers. Y_0 has two minimal Weierstrass N_1 fibers and Y_1 has one intermediate N_2 fiber with a type I pseudoelliptic tree attached. There are $2 \leq l_0 \leq 8$ marked N_0 fibers on Y_0 and $1 \leq l_1 \leq 9$ marked N_0 fibers on Y_1 .

Proof We will proceed by taking the Weierstrass limit of the main component and using the classification in [Section 5.1.1](#) to determine what can be attached as the other main component.

First suppose that [Assumption 5.10](#) does not hold for the fiber along which the Y_i are glued, so that after performing a pseudoelliptic flip of Y_0 , the fiber on Y_1 is not the intermediate model of a semi-log canonical Weierstrass cusp. Then as in the proof of [Theorem 5.16](#), Y_0 is a K3 component and Y_1 is an isotrivial j -invariant ∞ surface. Furthermore, they are either glued along twisted I_n/N_0 or I_n^*/N_1 fibers. Since they are the two main components, they must each have 12 markings, so we conclude that $n = 12$ in the first case and $n = 6$ in the second case. Furthermore, as in the proof of [Theorem 5.16](#), in the I_n^*/N_1 case Y_1 must have another N_1 fiber. This gives us cases (ii) and (iii), respectively.

From now on we can suppose that [Assumption 5.10](#) holds. Let us fix some notation. Denote the Weierstrass limit of the Y_i by Y_i^0 , which must be one of the surfaces listed

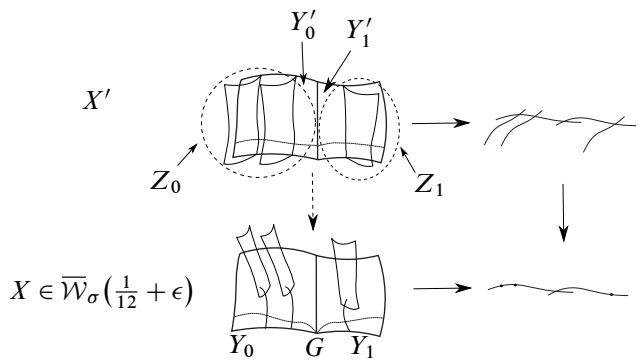


Figure 3: The circled component Z_i represents the union of Y'_i along with the pseudoelliptic trees emanating from Y'_i . The entire Z_i component dominates Y_i , and the Y'_i component contains the pseudoelliptics.

in [Section 5.1.1](#) if it is normal, or [Proposition 4.13](#) if it is isotrivial j -invariant ∞ . We will denote by $X^1 \rightarrow C^1$ a twisted stable maps model of the surface $X \rightarrow C$ in $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$ and we will denote by Y_i^1 the unique component of X^1 dominating Y_i . Let $Z_i^1 \subset X^1$ be the maximal connected union of connected components of X^1 that contains Y_i^1 . Finally we will denote by G the fiber along which Y_0 and Y_1 are glued, and by G_i its model in the Weierstrass limit, which is obtained by flipping one of the Y_i and contracting the transform on G on the other; see [Figure 3](#).

Now, since Y_0 and Y_1 satisfy [Assumption 5.10](#) for the fiber along which they are glued, by [Lemma 5.13](#) we must have $0 < -S_i^2 \leq 1$, where S_i is the section of Y_i . Note that $S_0^2 \neq 0$, otherwise Y_0 would be *trivial* and so the degree of the j -map on Z_0 would be 0 and the degree of the j -map on Z_1 would be 24, which would put us in situation (ii).

Suppose that Y_0 is normal. Then, by [Section 5.1.1](#), Y_0 is a rational elliptic surface and G_0 is a type L cusp. Since the twisted model of a type L cusp is a stable curve, G is an I_n fiber. On the other hand, there must be 12 markings on Y_0 away from G , and so $n = 0$ and G is in fact a smooth fiber. Since G is smooth, Y_1 cannot be isotrivial j -invariant ∞ so it is normal, and the same analysis applies to Y_1 . Thus we obtain (i).

Next, if Y_0 is not normal, then as above Y_1 is also nonnormal. Now the Y_i satisfy [Assumption 5.10](#) for the fiber G . We claim that they must also satisfy it for any pseudoelliptic trees away from G . Indeed suppose that Y_0 has an intermediate fiber F not satisfying [Assumption 5.10](#). Then by [Lemma 5.13](#), there must be an elliptic K3 attached to it. Every fiber of Y_i is N_k for $k \leq 2$, and we get cases (iv), (v) and (vi) by considering the various possible N_k fibers on a surface with $-S^2 \leq 1$.

Since N_2 fibers have 0 lct, they must be intermediate with pseudoelliptic trees attached, while pseudoelliptic trees attached to an N_1 fiber undergo type W_{III} contractions at walls above $\frac{1}{12} + \epsilon$ by [Lemma 5.4](#) so N_1 fibers are minimal Weierstrass. Finally, the number of markings is constrained by [Proposition 4.15](#), stability, and the fact that there are two main components so there must be 12 total markings on each. $\square$

Proposition 5.20 *Each of the cases described in [Theorem 5.19](#) occurs in the boundary of $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$.*

Proof Case (i) is the stable replacement in $\overline{W}_\sigma(\frac{1}{12} + \epsilon)$ of a Kulikov degeneration of type II. Case (ii) occurs when $12I_1$ fibers collide to give an I_{12} fiber. Similarly, case (iii) occurs when $12I_1$ fibers collide to form an I_6^* fiber. Case (iv) occurs when one starts with a degeneration of type (i) and takes the limit as the I_1 fibers approach the double locus G . Since marked I_1 fibers from both Y_0 and Y_1 must fall into G as the j -invariant of G must match on both sides, two isotrivial components appear such that each rational surface is attached to one of them along an N_0 fiber which leads to N_2 fibers when the rational surfaces undergo a flip. Similarly, case (v) occurs when you start with a surface of type (i) and degenerate the two rational components into $2N_1$ isotrivial j -invariant ∞ surfaces. Finally, for case (vi), take a degeneration as in case (i) and then further degenerate Y_0 so that it is an isotrivial $2I_0^*$ surface. Then the stable replacement of the limit as the j -invariant of the $2I_0^*$ surface approaches ∞ is case (vi). $\square$

6 Surfaces in $\overline{W}_\sigma(\epsilon)$, the 24-marked space at $a = \epsilon$

In the previous section, we studied the wall crossings that occur in $\overline{W}_\sigma(a)$ as we let the weight vary from 1 to $\frac{1}{12} + \epsilon$, and we used this to classify the surfaces parametrized by the boundary of $\overline{W}_\sigma(a)$ for $a = \frac{1}{12} + \epsilon$. The goal of this section is to explicitly study the wall crossings that occur as we reduce the weight further, from $a = \frac{1}{12} + \epsilon$ to $a = \epsilon$ for $0 < \epsilon \ll 1$. As a result, we determine the surfaces parametrized by the boundary of $\overline{W}_\sigma(\epsilon)$. The main results in this direction are [Theorems 6.13](#) and [6.14](#). In [Theorem 6.13](#) we describe the possible surfaces on the boundary, and in [Theorem 6.14](#) we use the theory of twisted stable maps (see [Section 3.2](#)) to show that all such surfaces appear on the boundary. Finally, in [Theorem 6.15](#), we describe a morphism from the coarse space of $\overline{W}_\sigma(\epsilon)$ to the GIT quotient $\overline{W}^G$. These three theorems together give a proof of [Theorem 1.1](#).

We begin with the wall at $\frac{1}{12}$:

Lemma 6.1 *At $a = \frac{1}{12}$, there are type III contractions of rational pseudoelliptic components attached by an I_0^* fiber.*

Proof An I_0^* must be attached along another I_0^* by the stabilizer condition. Furthermore, an I_0^* rational surface has six other markings with multiplicity. Putting this together with the description of the walls, we get a wall at $1/(2k) = \frac{1}{12}$ since $\frac{1}{2}$ is the lct of I_0^* ; see (2) in Section 3. $\square$

Lemma 6.2 *At $a = \frac{1}{12}$ the trivial component Y_1 in case (ii) of Theorem 5.19 contracts onto the I_{12} fiber it is attached to.*

Proof The component of the base curve lying under Y_1 contracts to a point, but since Y_1 is trivial, it contracts onto a fiber. $\square$

Lemma 6.3 *Let X be a surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ from Theorem 5.19(iii). Then the stable replacement for coefficients $\frac{1}{12} - \epsilon$ is an irreducible pseudoelliptic K3 surface with an I_6^* fiber.*

Proof X has main component $X_m = Y_0 \cup Y_1$ consisting of an elliptic K3 with a twisted I_6^* fiber glued to an isotrivial j -invariant ∞ surface along a twisted N_1 fiber. Each surface has 12 markings. At coefficient $\frac{1}{12} - \epsilon$, both section components are contracted by an extremal contraction. We first perform the extremal contraction of the section of Y_1 which results in a flip of Y_1 to a pseudoelliptic surface. Then the section of Y_0 contracts to form a pseudoelliptic with the pseudoelliptic model of Y_1 glued along an I_6^* pseudofiber. Finally, Y_1 contracts onto a point as in Lemma 6.1. $\square$

Putting the above together with the observation that the Hassett space becomes a point at $\frac{1}{12}$ so the base curves all contract to a point, we get:

Theorem 6.4 *Let X be a surface parametrized by $\overline{\mathcal{W}}_\sigma(\frac{1}{12} - \epsilon)$.*

- (i) *If X has a single main component, then X_m is the pseudoelliptic surface associated to an elliptic surface, as in Proposition 5.15 and Theorem 5.16, with an A_1 singularity where the section contracted. Any type II, III, IV, N_1 and I_k^* for $k \leq 5$ pseudofibers of X_m are Weierstrass and any I_n fibers satisfy $n \leq 12$. There are pseudoelliptic trees sprouting off of intermediate type II^* , III^* , IV^* and N_k for $k \geq 2$ fibers as before.*

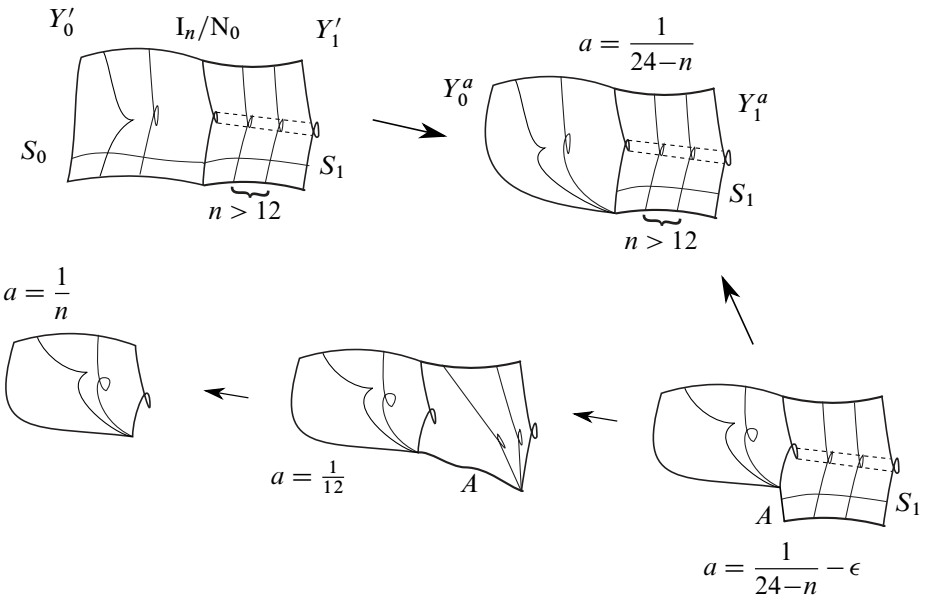


Figure 4: Illustration of Example 6.6.

- (ii) If X has two main components, then X_m is a union along a twisted pseudofiber of the surfaces appearing in Theorem 5.19, parts (i), (iv), (v) and (vi). Any type II, III, IV, N_1 and I_k^* for $k \leq 5$ pseudofibers are Weierstrass. There are pseudoelliptic trees sprouting off of intermediate II^* , III^* , IV^* and N_2 fibers as before.

Lemma 6.5 There are type III walls at $a = \frac{1}{60}, \frac{1}{36}$ and $\frac{1}{24}$ where rational pseudoelliptic surfaces attached along intermediate type II^* , III^* and IV^* fibers, respectively, contract to a point.

Proof This follows from [8, Theorem 6.3] as well as the observation that a rational elliptic surface attached to a type II^* , III^* or IV^* fiber must have a II, III or IV fiber, respectively, and so it has 10, 9 or 8 other marked fibers counted with multiplicity. $\square$

Next we study some examples of the transformations that occur for small coefficients.

Example 6.6 (Figure 4) Suppose X_η is a smooth elliptic K3 surface with 24 (I_1) fibers, and suppose it appears as the general fiber of a family $(f: \mathcal{X} \rightarrow B, \mathcal{S} + \mathcal{F}_a)$ with limit as in Theorem 5.16, case (d). In particular, this is a stable limit for $a = \frac{1}{12} + \epsilon$ and $\mathcal{F}$ consisting of the 24 I_1 fibers on the generic surface X_η . We will compute the stable limit of this family for $a < \frac{1}{12}$. We will denote by X^a the a -stable special fiber of $\mathcal{X} \rightarrow B$.

We begin with the twisted stable maps limit $X^1 \rightarrow C^1$. It consists of a union $Y_0^1 \cup Y_1^1$ where Y_0^1 is an elliptic K3 and Y_1^1 is a trivial j -invariant ∞ surface with n marked fibers glued along an I_n fiber of Y_0^1 where $n > 12$. At $a = 1/(24 - n)$, the component Y_0^1 undergoes a pseudoelliptic flip to obtain the model in [Theorem 5.16\(d\)](#), ie Y_0^a is a pseudoelliptic K3 glued along an intermediate N_4 fiber $A^a \cup G^a$ of Y_1^a . Next, for $a \leq \frac{1}{12}$, the section of Y_1^a contracts onto an A_1 singularity so that X^a consists of a pseudoelliptic isotrivial j -invariant ∞ surface with an intermediate N_4 pseudofiber and a pseudoelliptic K3 sprouting off it. To continue the MMP on this 1-parameter family and compute the stable limit for smaller a , we need to compute $(K_{\mathcal{X}^a} + \mathcal{F}^a).A^a$ and $(K_{\mathcal{X}^a} + \mathcal{F}^a).G^a$. We can restrict the log canonical divisor to the component Y_1^a to obtain

$$K_{Y_1^a} + G + (24 - n)aA^a + naf,$$

where f is a pseudofiber class. Pulling back to the blowup of the section $\mu: Y_1^b \rightarrow Y_1^a$ where $b = \frac{1}{12} + \epsilon$,

$$\mu^*(K_{Y_1^a} + G + (24 - n)aA^a + naf^a) = K_{Y_1^b} + G^b + (24 - n)A^b + naf^b + 12aS_1^b.$$

Here S_1^b is the section which is a (-2) -curve and f^b is a fiber class. Now A^b is the curve obtained by flipping the section S_0 of Y_0^1 . Using the local structure of the flip (see eg [\[33, Section 7.1\]](#)), we compute that $(A^b)^2 = -\frac{1}{2}$, $A^b.G^b = \frac{1}{2}$ and $(G^b)^2 = -\frac{1}{2}$. Similarly, using push-pull for the contraction $\rho: Y_1^b \rightarrow Y_1^1$ onto the twisted model of Y_1^1 , we get that $K_{Y_1^b} = -2f^b + 2A^b$. Putting all these together and using push-pull for μ ,

$$\begin{aligned} (K_{Y_1^a} + G + (24 - n)aA^a + naf).A^a &= (K_{Y_1^b} + G^b + (24 - n)A^b + naf^b + 12aS_1^b).A^b \\ &= \frac{1}{2}na - \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} (K_{Y_1^a} + G + (24 - n)aA^a + naf).B^a &= (K_{Y_1^b} + G^b + (24 - n)A^b + naf^b + 12aS_1^b).G^b \\ &= \frac{1}{2} + (24 - n) \cdot \frac{1}{2}a. \end{aligned}$$

In particular, for $a < 1/n$, there is an extremal contraction of the curve class of A^a in $\mathcal{X}^a$. On the other hand, since $(A^b)^2 = -\frac{1}{2}$ and μ is the contraction of a (-2) -curve which intersects A^b transversely, we have $(A^a)^2 = 0$, so this curve class rules Y_1^b over G^b and the extremal contraction for $a < 1/n$ contracts X^a onto Y_0^a , the pseudoelliptic K3.

Remark 6.7 In the above example, $n \leq 19$, by eg [\[44\]](#).

Example 6.8 ([Figure 5](#)) Suppose X_η as above is a smooth elliptic K3 surface with 24 (I_1) fibers, which appears as the general fiber of a family $(f: \mathcal{X} \rightarrow B, \mathcal{S} + \mathcal{F}_a)$ with limit as in [Theorem 5.16\(e\)](#). We compute the stable limit for small a as above and we keep the same notation.

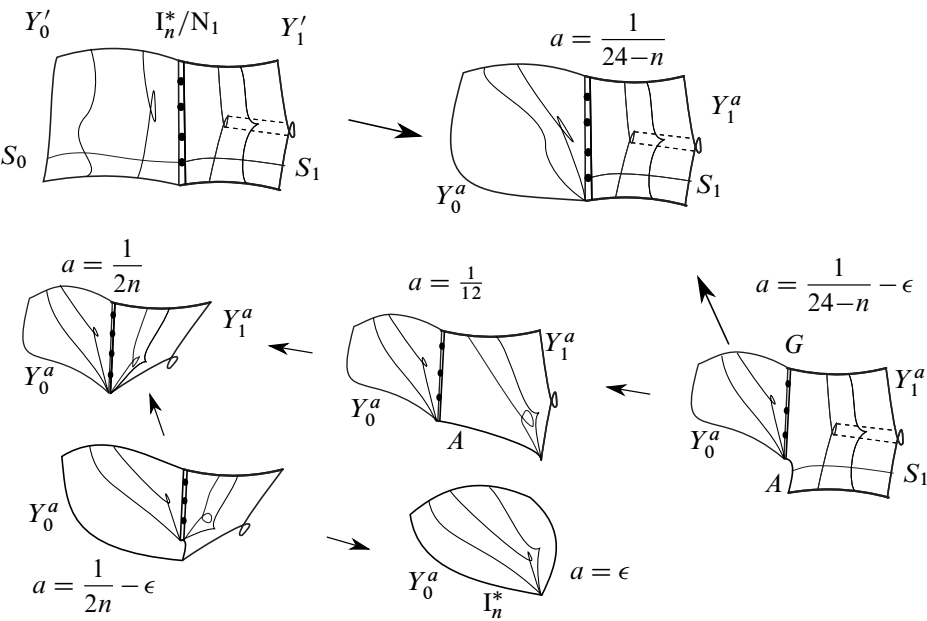


Figure 5: Illustration of Example 6.8.

The twisted stable maps limit $X^1 \rightarrow C^1$ consists of a union $Y_0^1 \cup Y_1^1$ where Y_0^1 is an elliptic K3 and Y_1^1 is a $2N_1$ isotrivial j -invariant ∞ surface. They are glued along twisted I_n^*/N_1 fibers with $n > 6$. At $a = 1/(18 - n)$, the component Y_0^1 undergoes a pseudoelliptic flip to obtain the model in Theorem 5.16, case (e), ie Y_0^a is a pseudoelliptic K3 with a twisted I_n^* pseudofiber glued along an intermediate N_3 fiber $A^a \cup G^a$ of Y_1^a . As above, the section of Y_1^a contracts onto an A_1 singularity for $a \leq \frac{1}{12}$ so that X^a consists of a pseudoelliptic isotrivial j -invariant ∞ surface with an intermediate N_3 pseudofiber and a pseudoelliptic K3 sprouting off it. The N_1 pseudofiber of Y_1^a may have a pseudoelliptic tree sprouting off of it, but it exhibits a type W_{III} contraction onto the Weierstrass model of the N_1 fiber by Lemma 5.4.

Restricting the log canonical divisor to the component Y_1^a , we obtain

$$K_{Y_1^a} + G + (18 - n)aA^a + (6 + n)af$$

where f is a pseudofiber class. Pulling back to the blowup of the section $\mu: Y_1^b \rightarrow Y_1^a$ where $b = \frac{1}{12} + \epsilon$,

$$\begin{aligned} \mu^*(K_{Y_1^a} + G + (18 - n)aA^a + (6 + n)af^a) \\ = K_{Y_1^b} + G^b + (18 - n)aA^b + (6 + n)af^b + 12aS_1^b. \end{aligned}$$

As above, A^b is the curve obtained by flipping the section S_0 of Y_0^1 which is a rational curve with self intersection $-\frac{3}{2}$ since Y_0^1 has a twisted I_n^* fiber. Thus we can compute that $(A^b)^2 = -\frac{2}{3}$, $A^b \cdot G^b = \frac{1}{3}$ and $(G^b)^2 = -\frac{1}{6}$. Using push-pull for the contraction $\rho: Y_1^b \rightarrow Y_1^1$ onto the model of Y_1^1 with a twisted N_1 fiber for the double locus and a Weierstrass N_1 fiber for the other N_1 , we get that $K_{Y_1^b} = -f^b + A^b$. Putting all these together and using push-pull for μ ,

$$\begin{aligned} (K_{Y_1^a} + G + (18-n)aA^a + (6+n)af).A^a \\ &= (K_{Y_1^b} + G^b + (18-n)aA^b + (6+n)af^b + 12aS_1^b).A^b \\ &= \frac{2}{3}an - \frac{1}{3}, \\ (K_{Y_1^a} + G + (18-n)aA^a + (6+n)af).B^a \\ &= (K_{Y_1^b} + G^b + (18-n)aA^b + (6+n)af^b + 12aS_1^b).G^b \\ &= \frac{1}{6} + (18-n) \cdot \frac{1}{3}a. \end{aligned}$$

For $a < 1/(2n)$, there is an extremal contraction of the curve class of A^a in $\mathcal{X}^a$. On the other hand, since $(A^b)^2 = -\frac{2}{3}$ and μ is the contraction of a (-2) -curve which intersects A^b transversely, we have $(A^a)^2 = -\frac{1}{6}$ so this curve class is rigid and therefore undergoes a flip. After the flip, the strict transform Y_1^a for $a < 1/(2n)$ is now a pseudoelliptic attached along an intermediate pseudofiber of Y_0^a . By Lemma 5.13, the flipped pseudoelliptic contracts and goes through a type W_{III} pseudoelliptic flip for some small $a = \epsilon > 0$, giving the stable limit as the minimal Weierstrass pseudoelliptic of Y_0^a .

Remark 6.9 By eg [44], the maximum n such that there exists an elliptic K3 with an I_n^* is 14 and so the above phenomena occur for $6 < n \leq 14$.

Combining the above examples gives:

- Proposition 6.10** (i) *There are type III walls at $1/k$ for $13 \leq k \leq 19$ where the isotrivial j -invariant ∞ main component of the surfaces from Theorem 5.16, case (d), contract as a ruled surface onto the I_n fiber of the pseudoelliptic K3 sprouting off of it.*
- (ii) *There are type III walls at $1/(2n)$ for $6 < n \leq 14$, where the isotrivial j -invariant ∞ main component as in Theorem 5.16, case (e), goes through a flip to become a pseudoelliptic attached to an intermediate model of the I_n^* on the K3 component. At some smaller $a = \epsilon > 0$, this pseudoelliptic contracts onto the Weierstrass model of the I_n^* fiber.*

Corollary 6.11 *The stable replacements in $\overline{\mathcal{W}}_\sigma(\epsilon)$ of the two main component surfaces of $\overline{\mathcal{W}}_\sigma(\frac{1}{12} + \epsilon)$ from [Theorem 5.19\(d\)–\(e\)](#) are pseudoelliptic K3s with Weierstrass I_n and I_n^* fibers, respectively.*

Proposition 6.12 *If X is a surface parametrized by $\overline{\mathcal{W}}_\sigma(\epsilon)$ then $\omega_X \cong \mathcal{O}_X$.*

Proof If X is irreducible then the result is clear, since X is the contraction of the section, a (-2) –curve, on a K3 type Weierstrass fibration.

Therefore, suppose X consists of multiple components. Let $p: \mathcal{X} \rightarrow D$ be a 1–parameter family over the spectrum of a DVR with generic fiber a $24I_1$ elliptic K3 and central fiber X . Now there is a sequence of pseudoelliptic flips producing a model $p': \mathcal{X}' \rightarrow D$, where the sections of X are blown back up so that the components of central fiber X' of p' are all elliptically fibered and glued along twisted fibers (for example, these flips occur as part of the MMP when decreasing the coefficient on the section of the twisted model, or equivalently, X' is the model parametrized by the Brunyate/Inchiostro moduli space). Then $X' = X_0 \cup_{F_0} X_1 \cup_{F_1} \cdots \cup_{F_{n-1}} X_n \cup_{F_n} X_{n+1}$, where X_0 and X_{n+1} are rational elliptic surfaces and $X_1, \dots, X_n$ are trivial j –invariant ∞ fibrations.

Then $K_{X'}|_{X_0} = K_{X_0} + F_0$, $K_{X'}|_{X_{n+1}} = K_{X_{n+1}} + F_n$ and $K_X|_{X_i} = K_{X_i} + F_{i-1} + F_i$ for $i = 1, \dots, n$, which are all 0 by the canonical bundle formula since X_0 and X_{n+1} (resp. $X_1, \dots, X_n$) satisfy $\deg \mathcal{L} = 1$ (resp. $\deg \mathcal{L} = 0$). Thus $K_{X'}$ is numerically trivial, that is, $K_{X'} \equiv 0$.

We proceed in two steps. First we show that X' is Gorenstein and then we show that the pullback

(4)

$$\mathrm{Pic}(X') \rightarrow \bigoplus_{i=0}^{n+1} \mathrm{Pic}(X_i)$$

is injective. For the first claim, note that away from the gluing fibers F_i , the surface X' is a minimal Weierstrass fibration. From the classification of surfaces (see [Corollary 6.11](#)), the components X_i are glued along I_n type fibers, and so in a neighborhood of F_i the surface corresponds to a map from a nonstacky nodal curve into $\overline{\mathcal{M}}_{1,1}$. In particular, in a neighborhood of F_i , the elliptic fibration $X' \rightarrow C$ is a flat family of nodal curves over a nodal curve. In either case, X' is Gorenstein.

Next, denote by $\pi: \bigsqcup X_i \rightarrow X'$ the natural morphism. By [\[22, Proposition 2.6 and Remark 2.7\]](#) there is a diagram of short exact sequences of sheaves of abelian groups

on X'

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_{X'}^* & \xrightarrow{\alpha} & \prod_{i=0}^{n+1} \pi_* \mathcal{O}_{X_i}^* & \longrightarrow & \mathcal{N} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mathcal{O}_{F'}^* & \xrightarrow{\beta} & \pi_* \mathcal{O}_F^* & \longrightarrow & \mathcal{N} \longrightarrow 0 \end{array}$$

where F' is the double locus on X' and F is the double locus on X_i . As an abstract variety, F is the disjoint union of two copies of F' . By [22, Proposition 4.2], (4) is injective if and only if $\gamma : \text{Pic}(F') \rightarrow \text{Pic}(F)$ is injective and $\text{coker} H^0(\alpha) = \text{coker} H^0(\beta)$. The map γ is simply the diagonal, so it is injective. Moreover, since X' , X_i and F_i are all connected projective varieties, applying H^0 to the above diagram gives

$$\begin{array}{ccc} 1 & \longrightarrow & k^* \xrightarrow{H^0(\alpha)} \prod_{i=0}^{n+1} k^* \\ & & \downarrow f_1 \qquad \qquad \downarrow f_2 \\ 1 & \longrightarrow & \prod_{i=0}^n k^* \xrightarrow{H^0(\beta)} \prod_{i=0}^n k^* \times k^* \end{array}$$

Here f_1 and $H^0(\alpha)$ are the diagonal maps, $H^0(\beta)$ is the product of diagonal maps for each i , and f_2 is given by $(x_0, \dots, x_{n+1}) \mapsto (x_0, x_1, x_1, x_2, \dots, x_n, x_{n+1})$. The cokernel of $H^0(\alpha)$ can be identified with $\prod_{i=1}^{n+1} k^*$ by the map

$$(x_0, \dots, x_{n+1}) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_{n+1}}{x_0} \right).$$

Similarly, the cokernel of $H^0(\beta)$ can be identified with $\prod_{i=0}^n k^*$ by the map

$$(a_0, b_0, a_1, b_1, \dots, a_n, b_n) \mapsto \left(\frac{b_0}{a_0}, \frac{b_1}{a_1}, \dots, \frac{b_n}{a_n} \right).$$

Therefore the induced map on cokernels is given by

$$(x_1, \dots, x_{n+1}) \mapsto \left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_{n+1}}{x_n} \right),$$

which is an isomorphism. Thus we conclude that (4) is an injection.

This means that X' is Gorenstein and $\omega_{X'}$ pulls back to the trivial line bundle under (4), so $\omega_{X'} \cong \mathcal{O}_{X'}$. It follows that $\omega_{\mathcal{X}'/D} \cong \mathcal{O}_{\mathcal{X}'}$. Now $\mathcal{X}'$ is related to $\mathcal{X}$ by a sequence of log flips. Since these flips always contract K -trivial curves, we conclude from the cone theorem (see eg [30, Theorem 3.7(4)]) that the canonical line bundle is preserved, so $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ and so $\omega_X \cong \mathcal{O}_X$. □

Putting all of this together, we have a classification of the boundary components of $\overline{\mathcal{W}}_{\sigma}(\epsilon)$; see Section 7 for an alternative description.

Theorem 6.13 *The surfaces in $\overline{\mathcal{W}}_\sigma(\epsilon)$ are*

- (A) *an irreducible pseudoelliptic K3 with the section contracted to an A_1 singularity and minimal Weierstrass pseudofibers,*
- (B) *an irreducible isotrivial $j = \infty$ pseudoelliptic with $4N_1$ Weierstrass fibers,*
- (C) *an isotrivial $j = \infty$ fibration with $2N_1$ Weierstrass fibers and an N_2 intermediate fiber with a tree of pseudoelliptics sprouting off of it,*
- (D) *an isotrivial $j = \infty$ fibration with $2N_2$ intermediate fibers each sprouting a tree of pseudoelliptics,*
- (E) *a union of irreducible pseudoelliptic rational surfaces along an I_0 fiber,*
- (F) *a union of isotrivial $j = \infty$ pseudoelliptic surfaces with a single intermediate N_2 fiber sprouting a pseudoelliptic tree on each, glued along an N_0 fiber,*
- (G) *a union of irreducible isotrivial $j = \infty$ surfaces each with $2N_1$ Weierstrass fibers glued along an N_0 fiber,*
- (H) *a union of an irreducible isotrivial $j = \infty$ surface with $2N_1$ Weierstrass fibers and an isotrivial $j = \infty$ surface with a single N_2 fiber sprouting a pseudoelliptic tree, glued along an N_0 fiber.*

Furthermore, every surface X satisfies $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Finally, the number of marked N_0 fibers are as in Theorems 5.16 and 5.19.

Now we show that each surface actually appears on the boundary, using the full smoothability results of Section 3.2.

Theorem 6.14 *Every slc surface pair in Theorem 6.13 appears in the boundary of $\overline{\mathcal{W}}_\sigma(\epsilon)$.*

Proof Given any surface satisfying the conditions of Theorem 6.13, we can construct a twisted surface whose stable replacement is the surface obtained by flipping the pseudoelliptic components into elliptically fibered ones as in the previous section, replacing each cuspidal fiber by a twisted fiber, and attaching a component with dual monodromy satisfying the conditions of Propositions 3.23 and 3.26 to each of these twisted fibers. By full smoothability (Theorems 3.24 and 3.28), this twisted model is the limit of a family of $24I_1$ elliptic K3 surfaces with singular fibers marked, and its stable replacement must be the initial surface as computed in the previous two sections. $\square$

We conclude this section by discussing the connection between $\overline{\mathcal{W}}_\sigma(\epsilon)$ and the GIT quotient $\overline{\mathcal{W}}^G$.

Theorem 6.15 (connection with GIT/SBB) *If $\overline{\mathcal{W}}_\sigma(\epsilon)$ denotes the coarse space of $\overline{\mathcal{W}}_\sigma(\epsilon)$ then there is a morphism $\overline{\mathcal{W}}_\sigma(\epsilon) \rightarrow \overline{\mathcal{W}}^G \cong \overline{\mathcal{W}}^*$ with the following structure:*

- (i) *The locus of surfaces of type (A) maps isomorphically onto $\mathcal{W}_s^G$.*
- (ii) *The locus of surfaces of type (B) maps as a generic $\mathbb{P}^{12}$ -bundle onto $\overline{\mathcal{W}}_{\text{slc},o}^G$ by forgetting the marked fibers. The closure of this locus in $\overline{\mathcal{W}}_\sigma(\epsilon)$ parametrizes the unique surface of type (G) along with a choice of marked fibers, and this locus all maps onto $\overline{\mathcal{W}}_{\text{slc}}^G \cap \overline{\mathcal{W}}_L^G$.*
- (iii) *The locus of surfaces of type (E) maps onto $\overline{\mathcal{W}}_L^G$ by taking the j -invariant of the I_0 fiber along which the two components are glued.*
- (iv) *The surfaces of type (C), (D), (F) and (H) all get mapped onto the point $\overline{\mathcal{W}}_{\text{slc}}^G \cap \overline{\mathcal{W}}_L^G$.*

Proof By Theorem 6.13, we have a classification of surfaces in $\overline{\mathcal{W}}_\sigma(\epsilon)$. Each of the irreducible surfaces mentioned in the theorem is also parametrized by $\overline{\mathcal{W}}^*$, yielding a rational map $\overline{\mathcal{W}}_\sigma(\epsilon) \rightarrow \overline{\mathcal{W}}^G$ defined on a dense open subset. Now one can easily check that the limit in $\overline{\mathcal{W}}^G$ of a Weierstrass family limiting to a surface of type (B) (resp. type (C), (D), (G), (F) or (H)) is the j -invariant of the L (resp. N_2) fiber in $\overline{\mathcal{W}}_L^G$. This depends only the central fiber of the family, not the family itself, so the morphism extends uniquely by normality after applying [18, Theorem 7.3]. $\square$

7 Explicit description of the boundary of $\overline{\mathcal{W}}_\sigma(\epsilon)$

In the previous section, specifically Theorems 6.13 and 6.14, we gave an explicit description of the surfaces parametrized by the boundary of $\overline{\mathcal{W}}_\sigma(\epsilon)$. The goal of this section is to enumerate the resulting boundary strata of $\overline{\mathcal{W}}_\sigma(\epsilon)$ in a combinatorial way, akin to Kulikov models; see Proposition 7.2 for the analogue of *type II* degenerations, and Theorems 7.5, 7.7 and 7.9 for the analogues of the *type III* degenerations.

Before starting, we define R_n to be the space parametrizing pairs $(X, S + F)$, where X is a minimal Weierstrass rational elliptic surface, S is a section, and F is a fiber of type I_n . Note that $n \leq 9$. The following is well known:

Lemma 7.1 [24, Section 3.3] R_n is a $(9-n)$ -dimensional affine variety which is irreducible for $n \neq 8$, while R_8 has two components.

Using these spaces, we will explicitly describe the boundary of $\overline{\mathcal{W}}_\sigma(\epsilon)$. To do so, we use the notation of Kulikov models (models of type II and III).

7.1 Type II degenerations

Proposition 7.2 *There are two type II strata:*

- (i) *The first is a dimension-17 stratum W_{II} isomorphic to a quotient of the fiber product $R_0 \times_j R_0$, namely the self fiber product of the j -map $j: R_0 \rightarrow \mathbb{A}^1$. A point parametrizes two rational elliptic surfaces with a marked I_0 fiber of the same j -invariant glued along this fiber, and the quotient comes from swapping the two surfaces; see [Theorem 6.13\(E\)](#).*
- (ii) *The second is a dimension-17 stratum $W_{II}^\infty \cong \text{Sym}^{16}(\mathbb{P}^1) \times \mathbb{A}^1$ where $\mathbb{A}^1$ is the j -line. The j -line parametrizes the $4N_1$ isotrivial j -invariant ∞ component, and $\text{Sym}^{16}(\mathbb{P}^1)$ parametrizes the m markings on this surface other than the N_1 fibers counted with multiplicity; see [Theorem 6.13\(B\)](#).*

7.2 Type III degenerations

The first step is to “unflip” the pseudoelliptic components in [Theorem 6.13](#). After, we can describe each surface as a chain $X_0 \cup \cdots \cup X_{n+1}$, where both X_0 and X_{n+1} are Weierstrass fibrations of rational type ($\deg \mathcal{L} = 1$), and $X_1, \dots, X_n$ are all isomorphic to trivial j -invariant ∞ fibrations $C \times \mathbb{P}^1$, with C being a nodal cubic. These surfaces are all glued along nodal cubic fibers (ie either I_n or N_0 fibers). Further, each X_i for $i = 1, \dots, n$ must have *at least* one marked fiber by stability. We call the surfaces X_0 and X_{n+1} the *end components* and $X_1, \dots, X_n$ the *intermediate components*.

Lemma 7.3 *An end component must have at least three marked fibers if it is normal, or at least four marked fibers if it is isotrivial j -invariant ∞ , counted with multiplicity.*

Proof If an end component is an isotrivial j -invariant ∞ surface, then it must be a $2N_1$ fibration glued along an N_0 fiber. Each N_1 must carry at least two markings counted with multiplicity so the surface carries at least four. If it is a normal rational elliptic surface, then the number of markings is given by $12 - n$, where the surface is glued along an I_n fiber. Since $n \leq 9$ for I_n fibers on a rational elliptic surface, then there are at most three markings on such a component. $\square$

Corollary 7.4 *For the chains $X_0 \cup \cdots \cup X_{n+1}$ in the type III locus, n is at most 18.*

Proof As there is at least one marking on each of the intermediate components, the number of components is bounded by the number of markings not on X_0 and X_{n+1} . By Lemma 7.3, there are at least six combined on these components so there are at least 18 markings to be distributed among the intermediate components. $\square$

Now we will describe an explicit parametrization of each of the type III strata. There are three cases, depending on whether none, one or both of the end components X_0 and X_{n+1} are isotrivial j -invariant ∞ . We call these strata type III_0 , III_1 and III_2 , respectively. The type III_0 strata are further indexed by the fiber types I_r and I_s along which X_0 and X_{n+1} are glued. In this case, there are $12 - r$ and $12 - s$ fibers marked on X_0 and X_{n+1} , respectively, which gives us $r + s$ markings remaining for the middle components $X_1, \dots, X_n$. Thus, n must satisfy $1 \leq n \leq r + s$.

Finally, for each n , we can fix a single marking on each component $X_1, \dots, X_n$ and fix coordinates so that the components are glued along fibers at 0 and ∞ , and the chosen marking is at 1. That gives us freedom to parametrize $r + s - n$ additional markings among $X_1, \dots, X_n$. For each choice of partition $\sum_{i=1}^n a_i = r + s - n$ we can consider the stratum where there are a_i markings on X_i .

Theorem 7.5 (type III_0 locus) *Fix data*

$$1 \leq r, s \leq 9, \quad 1 \leq n \leq r + s, \quad \sum_{i=1}^n a_i = r + s - n.$$

There is a type III_0 stratum $\text{III}_{0,a_1,\dots,a_n}^{r,s,n}$ of dimension $\dim(\text{III}_{0,a_1,\dots,a_n}^{r,s,n}) = 18 - n$ with a finite parametrization by $R_s \times \mathbb{G}_m^{a_1} \times \cdots \times \mathbb{G}_m^{a_n} \times R_r$. Here a point of the above product determines the surface pairs X_0, X_{n+1} as well as the configuration of a_i marked fibers on $X_1, \dots, X_n$ avoiding the double locus.

Remark 7.6 Just to reiterate, R_s and R_r parametrize the surfaces X_0 and X_{n+1} , respectively, and the $\mathbb{G}_m^{a_i}$ parametrize the marked fibers on the X_i avoiding the double locus.

Next, we consider type III_1 strata where exactly one of the end surfaces, without loss of generality X_0 , is an isotrivial j -invariant ∞ surface of rational type. Then X_0 must be the $2N_1$ surface glued along an N_0 fiber. There are two markings each on the N_1 fibers

for a total of four. Then for each $0 \leq s \leq 17$, there is a stratum with $17 - s$ marked N_0 fibers on X_0 ; see [Theorem 5.16](#). After picking coordinates so that the N_1 fibers are at 0 and 1 and the double locus is at ∞ , these $17 - s$ markings must avoid ∞ and so give a factor of $\mathbb{A}^{17-s}$ parametrizing X_0 . The other end component X_{n+1} is a rational elliptic surface glued along an I_r fiber for some r and with $12 - r$ marked fibers.

This gives $33 - s - r$ total markings on X_0 and X_{n+1} . On the other hand, there are at most 24 markings, so $33 - s - r \leq 24$. In the case of equality, there are no intermediate components and we have a stratum parametrized by $\mathbb{A}^{17-s} \times R_r$. Otherwise, we have $1 \leq n \leq s + r - 9$ intermediate components with $s + r - 9$ markings distributed on them. After fixing one marking on each intermediate component at coordinate 1, there are $r + s - 9 - n$ marked fibers partitioned into $\sum_{i=1}^n a_i = r + s - 9 - n$. This gives a finite parametrization by $\mathbb{A}^{17-s} \times \mathbb{G}_m^{a_1} \times \cdots \times \mathbb{G}_m^{a_n} \times R_r$.

Theorem 7.7 (type III₁ locus) (i) *Fix the data*

$$1 \leq r \leq 9, \quad 0 \leq s \leq 17, \quad s + r = 9.$$

There is a type III₁ stratum $\text{III}_1^{r,s}$ of dimension $\dim(\text{III}_1^{r,s}) = 17$ with a finite parametrization by $\mathbb{A}^{17-s} \times R_r$.

(ii) *Fix the data*

$$1 \leq r \leq 9, \quad 1 \leq s \leq 17, \quad 1 \leq n \leq s + r - 9, \quad \sum_{i=1}^n a_i = r + s - 9 - n.$$

There is a type III₁ stratum $\text{III}_1^{r,s,n}$ of dimension $\dim(\text{III}_1^{r,s,n}) = 17 - n$ with a finite parametrization by $\mathbb{A}^{17-s} \times \mathbb{G}_m^{a_1} \times \cdots \times \mathbb{G}_m^{a_n} \times R_r$.

Remark 7.8 Again, here $\mathbb{A}^{8-s}$ parametrizes the $8 - s$ marked N_0 fibers on X_0 , the $\mathbb{G}_m^{a_i}$ parametrize the marked N_0 fibers on the X_i , and R_r parametrizes the surface X_{n+1} .

Finally, we have the type III₂ stratum where both X_0 and X_{n+1} are isotrivial j -invariant ∞ . In this case, X_0 and X_{n+1} are described by affine spaces of dimension $17 - s$ and $17 - r$, respectively, where there are $17 - s$ and $17 - r$ marked N_0 fibers on X_0 and X_{n+1} in addition to the $2N_1$ which each appear with multiplicity two. This gives $42 - r - s$ total marked fibers among the end components, so $42 - r - s \leq 24$, and we again have two cases: this is an equality and there are no intermediate components, or this inequality is strict and there are intermediate components with $r + s - 18$ marked fibers. Thus, as before:

Theorem 7.9 (type III₂ locus) (i) *Fix the data*

$$0 \leq s, r \leq 17, \quad s + r = 18.$$

There is a type III₂ stratum $\text{III}_2^{r,s}$ of dimension $\dim(\text{III}_2^{r,s}) = 16$ with a finite parametrization by $\mathbb{A}^{17-s} \times \mathbb{A}^{17-s} = \mathbb{A}^{16}$.

(ii) *Fix the data*

$$1 \leq s, r \leq 17, \quad 1 \leq n \leq s + r - 18, \quad \sum_{i=1}^n a_i = r + s - n - 18.$$

There is a type III₂ stratum $\text{III}_{2,a_1,\dots,a_n}^{r,s,n}$ of dimension $\dim(\text{III}_{2,a_1,\dots,a_n}^{r,s,n}) = 16 - n$ with a finite parametrization by $\mathbb{A}^{17-s} \times \mathbb{G}_m^{a_1} \times \cdots \times \mathbb{G}_m^{a_n} \times \mathbb{A}^{17-r}$.

Remark 7.10 In the above theorem, the $\mathbb{A}^{17-s}$ (resp. $\mathbb{A}^{17-r}$) parametrize the markings on X_0 (resp. X_{n+1}), and the $\mathbb{G}_m^{a_i}$ parametrize the markings on the X_i .

8 Spaces with one marked fiber

The goal of this section is to describe the surfaces parametrized by the boundary of the moduli spaces $\bar{\mathcal{K}}_\epsilon$ (resp. $\bar{\mathcal{F}}_\epsilon$), ie the moduli spaces parametrizing one ϵ -marked singular fiber (resp. any fiber). In [Section 8.1](#) we describe the boundary of the two moduli spaces; see [Theorem 8.1](#). In [Section 8.2](#) we prove [Theorem 8.2](#), which describes a morphism from $\bar{\mathcal{K}}_\epsilon$ to $\bar{\mathcal{W}}^G$. Finally, in [Section 8.3](#) we extend Miranda's GIT construction to produce a moduli space of Weierstrass surfaces with a choice of marked fiber. The main result in this direction is [Theorem 8.8](#), which shows that $\bar{\mathcal{F}}_\epsilon$ is a smooth Deligne–Mumford stack with coarse space map $\bar{\mathcal{F}}_\epsilon \rightarrow \widetilde{\mathcal{W}}^G$ given by the extended GIT compactification we discuss in [Section 8.3](#).

8.1 Spaces with one marked fiber

In this section we first consider the moduli space $\bar{\mathcal{F}}_\epsilon$ (see [Definition 4.9](#)), which corresponds to marking only one (possibly singular) fiber with ϵ weight. In particular, we give a description of the surfaces parametrized by the boundary. Note that since $\bar{\mathcal{K}}_\epsilon$ is a slice of $\bar{\mathcal{F}}_\epsilon$, this description also applies to the surfaces parametrized by $\bar{\mathcal{K}}_\epsilon$.

Theorem 8.1 (characterization of the boundary) *The surfaces parametrized by $\bar{\mathcal{F}}_\epsilon$ are single-component pseudoelliptic K3 surfaces whose corresponding elliptic surfaces*

are semi-log canonical Weierstrass elliptic K3s, and the marked fiber F can be any fiber other than an L type cusp. Moreover, all surfaces parametrized by $\overline{\mathcal{F}}_\epsilon$ satisfy $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$.

Proof We follow the explicit stable reduction process explained in eg [8, Section 6]. Let

$$(f: \mathcal{X} \rightarrow \mathcal{C}, \mathcal{S} + \mathcal{F}) \rightarrow T$$

be a 1-parameter family whose generic fiber $(f: X_\eta \rightarrow C_\eta, S_\eta + F_\eta)$ is a Weierstrass elliptic K3 surface with 24 I_1 fibers, and a single (possibly singular) marked fiber F_η . Denote by $(f_0: X_0 \rightarrow C_0, S_0 + F_0)$ the special fiber, and consider the limit obtained via twisted stable maps; see eg [7]. The limit $(f_0: X'_0 \rightarrow C'_0, S'_0 + F'_0)$ will be a tree of elliptic fibrations glued along twisted fibers, and the closure of the fiber F will be contained in precisely one such surface component. While this surface will be stable as a map to $\overline{\mathcal{M}}_{1,1}$, it will not necessarily be stable as a surface pair. To resolve this, choose some generic markings $G = \bigcup_{i \in I} G_i$ to make the above limit stable as a surface pair. In this case, G will consist of generic smooth fibers.

As we (uniformly) lower the coefficients marking G towards 0, there will be some choice of coefficients such that the weighted stable base curve is an irreducible rational curve. Indeed, the components of the base curve will contract precisely when there is not enough weight being supported on the marked fibers. As we only lowered the coefficients marking G , and the fiber F'_0 remained marked with coefficient 1, the (unique) main component, call it Y_0 , fibered over the rational curve will contain the original marked fiber.

Now we have a single main component with marked fiber F'_0 with type I pseudoelliptic trees attached to it. When the coefficients of G are set to 0, the type I trees will undergo type W_{III} contractions to a point to produce the Weierstrass model of Y_0 , away from the fiber F'_0 . When the coefficient of F'_0 is reduced to $0 < \epsilon \ll 1$, it will cross W_I walls to become a Weierstrass fiber.

We saw in Proposition 4.11 that $H^1(X, \mathcal{O}_X) = 0$, so it suffices to show that $\omega_X \cong \mathcal{O}_X$. This holds on any Weierstrass elliptic K3 surface (see [37, Proposition III.1.1]), and since X is obtained from a Weierstrass elliptic K3 by contracting a (-2) -curve (the section), we have $\omega_X \cong \mathcal{O}_X$. $\square$

8.2 Stable pairs to GIT/SBB

The goal of this section is to describe the morphism from $\overline{W}_\sigma(\epsilon)$ to $\overline{W}^G$ (and thus to $\overline{W}^*$).

Theorem 8.2 (connection with GIT/SBB) *Let $\overline{\mathcal{K}}_\epsilon$ be the coarse moduli space of $\overline{\mathcal{K}}_\epsilon$ and let $\Delta \subset \overline{\mathcal{K}}_\epsilon$ be the boundary locus parametrizing surfaces with an L type cusp, with $U = \overline{\mathcal{K}}_\epsilon \setminus \Delta$. There is a morphism $\overline{\mathcal{K}}_\epsilon \rightarrow \overline{\mathcal{W}}^G \cong \overline{\mathcal{W}}^*$ such that the diagram*

$$\begin{array}{ccccc} \Delta & \hookrightarrow & \overline{\mathcal{K}}_\epsilon & \hookleftarrow & U \\ j \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \overline{\mathcal{W}}^G & \hookleftarrow & \overline{\mathcal{W}}_s^G \end{array}$$

commutes, where $j : \Delta \rightarrow \mathbb{P}^1$ sends a surface with an L cusp to its j -invariant, the morphism $U \rightarrow \overline{\mathcal{W}}_s^G$ is proper and finite of degree 24, and $\mathbb{P}^1 \rightarrow \overline{\mathcal{W}}_L^G \subset \overline{\mathcal{W}}^G$ maps bijectively onto the strictly GIT semistable locus.

Proof By Theorem 8.1 every surface parametrized by $\overline{\mathcal{K}}_\epsilon$ is a single-component pseudoelliptic K3 surface. In particular, if we blow up the point to where the section contracted, we obtain an (unstable) slc Weierstrass elliptic K3 surface. Consider the PGL_2 -torsor $\mathcal{P} = \{(X, s, t) \mid (s, t) \in C \cong \mathbb{P}^1\} / \sim$, where X is an slc Weierstrass elliptic K3 surface obtained by blowing up the section of a surface parametrized by $\overline{\mathcal{K}}_\epsilon$, s and t are coordinates on the base $C \cong \mathbb{P}^1$ (or equivalently a basis for the linear series $|F|$ of a fiber F on X), and we quotient by scaling. Note that the Weierstrass coefficients $(A(s, t), B(s, t))$ defining X are unique up to the scaling of the $\mathbb{G}_m$ action $(A, B) \mapsto (\lambda^4 A, \lambda^6 B)$.

Since the semi-log canonical Weierstrass elliptic K3 surfaces are GIT semistable (see [36, Proposition 5.1]), we obtain a PGL_2 -equivariant morphism $\mathcal{P} \rightarrow V$ which induces a morphism $\phi : \overline{\mathcal{K}}_\epsilon \rightarrow \overline{\mathcal{W}}^G$. □

Remark 8.3 (i) The morphism $\overline{\mathcal{K}}_\epsilon \rightarrow \overline{\mathcal{W}}^G$ is generically a 24-to-1 cover, as it requires the choice of some marked fiber and generically there are 24 choices. The morphism is *not* finite — eg families with one L type cusp of fixed j -invariant are all collapsed to the same polystable point.

(ii) All the underlying surfaces of pairs parametrized by $\overline{\mathcal{K}}_\epsilon$ are in fact GIT semistable, even though all pairs with an L type cusp of fixed j -invariant map to the same GIT polystable point. One might wonder if the locus inside the GIT stack $[V_{24}^{\mathrm{ss}} // \mathrm{PGL}_2]$ consisting of those surfaces that appear in $\overline{\mathcal{K}}_\epsilon$ is an open Deligne–Mumford substack with proper coarse moduli space factoring the morphism $\overline{\mathcal{K}}_\epsilon \rightarrow \overline{\mathcal{W}}^G$. Furthermore, it is natural to compare this to a Kirwan desingularization of $\overline{\mathcal{W}}^G$. We will pursue these questions in the future.

(iii) In the morphism from stable pairs to GIT, all surfaces with an L type cusp get collapsed to the polystable orbit corresponding to the KSBA-unstable but GIT semistable (unique) surface with 2L cusps of the same j -invariant.

(iv) The locus of surfaces with an L type cusp is 9-dimensional. Indeed, such surfaces are birational to a rational elliptic surface (which has an 8-dimensional moduli space) with a choice of a fiber to replace by an L type cusp. There is a $\mathbb{P}^1$ worth of choices.

8.3 GIT for Weierstrass surfaces with a marked fiber

We extend Miranda's GIT construction to produce a moduli space of Weierstrass surfaces with a choice of marked fiber. Such data can be represented by triples (A, B, l) , where $(A, B) \in V_{4N} \oplus V_{6N}$ are Weierstrass data as above and $l \in V_1$ is a linear form. Then $\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SL}_2$ acts naturally on $V_{4N} \oplus V_{6N} \oplus V_1$, where the first $\mathbb{G}_m$ acts on $V_{4N} \oplus V_{6N}$ with weights $4N$ and $6N$ and the second acts on V_1 with weight one.

To study GIT (semi)stability, we follow Miranda's strategy. Consider the natural morphism

$$f: V_{4N} \oplus V_{6N} \rightarrow S^3 V_{4N} \oplus S^2 V_{6N},$$

let Z_N be the image of f , and let $\mathfrak{M}_N \subset \mathbb{P}(S^3 V_{4N} \oplus S^2 V_{6N})$ be its projectivization. By [36, Propositions 3.1 and 3.2]:

Proposition 8.4 *The morphism $f \times \mathrm{id}: V_{4N} \oplus V_{6N} \oplus V_1 \rightarrow S^3 V_{4N} \oplus S^2 V_{6N} \oplus V_1$ is finite and SL_2 -equivariant with fibers contained in $\mathbb{G}_m \times \mathbb{G}_m$ orbits. In particular, two triples (A, B, l) and (A', B', l') are in the same $\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SL}_2$ orbit if and only if the corresponding points in $\mathfrak{M}_N \times \mathbb{P}(V_1)$ are in the same SL_2 orbit.*

This lets us compute a GIT compactification of the moduli space of minimal Weierstrass fibrations with a chosen marked fiber as a GIT quotient $(\mathfrak{M}_N \times \mathbb{P}^1) // \mathrm{SL}_2$. We will linearize the moduli problem using the Segre embedding of $\mathbb{P}(S^3 V_{4N} \oplus S^2 V_{6N}) \times \mathbb{P}^1$.

Proposition 8.5 *A triple (A, B, l) is stable if and only if it is semistable. Further, it is not stable if and only if there exists a point $q \in \mathbb{P}^1$ with $v_q(A) > 2N$ and $v_q(B) > 3N$, or with $v_q(A) \geq 2N$, $v_q(B) \geq 3N$ (with at least one equality) and $v_q(l) = 1$.*

Proof Let $(A, B, l) \in \mathfrak{M}_N$, let $\lambda: \mathbb{G}_m \rightarrow \mathrm{SL}_2$ be a 1-parameter subgroup, and pick coordinates $[T_0, T_1]$ so that λ acts by $T_0 \mapsto \lambda^e T_0$ and $T_1 \mapsto \lambda^{-e} T_1$. Then it acts on

(A, B, l) by

$$\begin{aligned} A &= \sum_{i=0}^{4N} a_i T_0^i t_1^{4N-i} \mapsto \sum_{i=0}^{4N} a_i \lambda^{2ei-4eN} T_0^i t_1^{4N-i}, \\ B &= \sum_{i=0}^{6N} b_i T_0^i t_1^{6N-i} \mapsto \sum_{i=0}^{4N} b_i \lambda^{2ei-6eN} T_0^i t_1^{4N-i}, \\ l &= l_0 T_1 + l_1 T^0 \mapsto l_0 \lambda^{-e} T_1 + l_1 \lambda^e T_0. \end{aligned}$$

The coordinates of $\mathbb{P}(S^3 V_{4N} \oplus S^2 V_{6N}) \times \mathbb{P}(V_1)$ are given by $l_0 a_i a_j a_k$, $l_0 b_l b_m$, $l_1 a_i a_j a_k$, and $l_0 b_l b_m$ which respectively have weights

$$\begin{aligned} 2e(i+j+k) - 12eN - e, \quad & 2e(l+m) - 12eN - e, \\ 2e(i+j+k) - 12eN + e, \quad & 2e(l+m) - 12eN + e. \end{aligned}$$

By the Hilbert–Mumford criterion, a point is not stable (resp. semistable) if and only if there exists a 1-parameter subgroup such that all the weights are nonnegative (resp. positive).

Suppose (A, B, l) is not (semi)stable and pick a 1-parameter subgroup and coordinates as above. Then we have, after dividing by $e \neq 0$,

$$\begin{aligned} 2e(i+j+k) - 12eN - e < (\leq) 0 &\implies l_0 a_i a_j a_k = 0, \\ 2e(l+m) - 12eN - e < (\leq) 0 &\implies l_0 b_l b_m = 0, \\ 2e(i+j+k) - 12eN + e < (\leq) 0 &\implies l_1 a_i a_j a_k = 0, \\ 2e(l+m) - 12eN + e < (\leq) 0 &\implies l_1 b_l b_m = 0. \end{aligned}$$

Note that the left-hand side is always odd and so equality is never achieved. From this we can conclude that stability coincides with semistability. Now consider the cases where $i = j = k$ and $l = m$. We see that $l_0 a_i^3 = 0$ for $i \leq 2N$, $l_1 a_i^3 = 0$ for $i \leq 2N - 1$, $l_0 b_l^2 = 0$ for $l \leq 3N$ and $l_1 b_l^2 = 0$ for $l \leq 3N - 1$. Let $q = [0, 1]$ be the point given by $T_0 = 0$. If $l_0 \neq 0$, then we must have that $a_i = 0$ for $i \leq 2N$ and $b_l = 0$ for $l \leq 3N$. Thus the order of vanishing satisfies $v_q(A) > 2N$ and $v_q(B) > 3N$. Otherwise, if $l_0 = 0$ then $l_1 \neq 0$ so we must have that $a_i = 0$ for $i \leq 2N - 1$ and $b_l = 0$ for $l \leq 3N - 1$. In this case, $v_q(l) = 1$, $v_q(A) \geq 2N$ and $v_q(B) \geq 3N$.

Conversely, given a triple (A, B, l) satisfying such order of vanishing conditions, we may pick coordinates such that $q = [0, 1]$. Then clearly the 1-parameter subgroup acting by $(T_0, T_1) \mapsto (\lambda T_0, \lambda^{-1} T_1)$ demonstrates that (A, B, l) is not stable. $\square$

In the case of K3 surfaces where $N = 2$, we obtain an especially pleasant result:

Corollary 8.6 *A point of $\mathfrak{M}_2$ is stable if and only if it represents a 1–marked Weierstrass fibration $(f: X \rightarrow \mathbb{P}^1, S + \epsilon F)$ with at worst semi-log canonical singularities.*

Proof First note that the generic fiber of the fibration $f: X \rightarrow \mathbb{P}^1$ represented by a stable point in $\mathfrak{M}_N$ is at worst nodal, since the Weierstrass data of a stable point cannot be identically 0. Then combining Proposition 8.5 with [33, Lemmas 3.2.1 and 3.2.2 and Corollary 3.2.4], and noting that the log canonical threshold of a type L/N₂ fiber is 0 (see Lemma 3.14), a point is unstable if and only if there exists a point $q \in \mathbb{P}^1$ such that the pair $(X, S + \epsilon F)$ is not semi-log canonical around the singular point of $f^{-1}(q)$. The result then follows since a Weierstrass fibration $(X, S + \epsilon F)$ has semi-log canonical singularities away from the singular points of the fibers. $\square$

Definition 8.7 If $\mathfrak{M}_2^s$ denotes the stable/semistable locus, we define $\widetilde{\mathcal{W}}^G = \mathfrak{M}_2^s // \mathrm{SL}_2$.

Theorem 8.8 $\overline{\mathcal{F}}_\epsilon$ is a smooth Deligne–Mumford stack with a coarse space map $\overline{\mathcal{F}}_\epsilon \rightarrow \widetilde{\mathcal{W}}^G$ given by the GIT compactification. Furthermore, there is a morphism $\overline{\mathcal{F}}_\epsilon \rightarrow \overline{\mathcal{W}}^G$ given by forgetting the marked fiber. A Weierstrass fibration $(f: X \rightarrow \mathbb{P}^1, S)$ is represented by a point in $\overline{\mathcal{W}}^G$ if and only if there exists a fiber F such that $(X, S + \epsilon F)$ is a stable pair.

Proof By the proof of Theorem 8.2 we obtain a birational morphism $\overline{\mathcal{F}}_\epsilon \rightarrow [\mathfrak{M}_2^s / \mathrm{PGL}_2]$. On the other hand, by Corollary 8.6, there is a family of KSBA-stable one ϵ –marked Weierstrass fibrations $(f: X \rightarrow \mathbb{P}^1, S + \epsilon F)$ over $\mathfrak{M}_2^s$. This induces a PGL_2 equivariant map $\mathfrak{M}_2^s \rightarrow \overline{\mathcal{F}}_\epsilon$ which gives an inverse map $[\mathfrak{M}_2^s / \mathrm{PGL}_2] \rightarrow \overline{\mathcal{F}}_\epsilon$ exhibiting these as isomorphisms. Then note that $[\mathfrak{M}_2^s / \mathrm{PGL}_2]$ is a smooth stack, as $\mathfrak{M}_2^s$ is an open subset of a smooth variety, so $\overline{\mathcal{F}}_\epsilon$ is smooth.

The composition $\overline{\mathcal{F}}_\epsilon \rightarrow [\mathfrak{M}_2^s / \mathrm{PGL}_2] \rightarrow \mathfrak{M}_2 // \mathrm{SL}_2$ is the coarse moduli space map. Indeed, $[\mathfrak{M}_2^s / \mathrm{SL}_2]$ and $[\mathfrak{M}_2^s / \mathrm{PGL}_2]$ have the same coarse moduli space; note that $[\mathfrak{M}_2^s / \mathrm{SL}_2] \rightarrow [\mathfrak{M}_2^s / \mathrm{PGL}_2]$ is a μ_2 –gerbe since it is the base change of the map $B\mathrm{SL}_2 \rightarrow B\mathrm{PGL}_2$, so $[\mathfrak{M}_2^s / \mathrm{SL}_2] \rightarrow [\mathfrak{M}_2^s / \mathrm{PGL}_2]$ is a relative coarse space and the coarse map $[\mathfrak{M}_2^s / \mathrm{SL}_2] \rightarrow \mathfrak{M}_2 // \mathrm{SL}_2$ factors through it.

If (A, B, l) is in $\mathfrak{M}_2^s$ then (A, B) is a semistable point for Miranda’s space, and conversely if (A, B) is semistable in Miranda’s space, then for a generic choice of fiber F , the corresponding fibration $(X \rightarrow \mathbb{P}^1, S + \epsilon F)$ is a stable pair and the corresponding GIT data (A, B, l) is GIT stable. $\square$

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