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SCHAUDER ESTIMATES FOR EQUATIONS WITH CONE METRICS, II

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We continue our work on the linear theory for equations with conical singularities. We derive interior Schauder estimates for linear elliptic and parabolic equations with a background Kähler metric of conical singularities along a divisor of simple normal crossings. As an application, we prove the short-time existence of the conical Kähler–Ricci flow with conical singularities along a divisor with simple normal crossings.

1. Introduction

This is a continuation of our paper [20]. Regularity of solutions of complex Monge–Ampère equations is a central problem in complex geometry. Complex Monge–Ampère equations with singular and degenerate data can be applied to study compactness and moduli problems of canonical Kähler metrics in Kähler geometry. In [43], Yau considered special cases of complex singular Monge–Ampère equations as generalizations of his solution to the Calabi conjecture. Conical singularities along complex hypersurfaces of a Kähler manifold are among the mildest singularities in Kähler geometry, and they have been extensively studied, especially in the case of Riemann surfaces [28; 41]. The study of such Kähler metrics with conical singularities has many geometric applications, for example, the Chern number inequality in various settings [38; 39]. Recently, Donaldson [14] initiated the program of studying analytic and geometric properties of Kähler metrics with conical singularities along a smooth complex hypersurface on a Kähler manifold. This is an essential step in the solution of the Yau–Tian–Donaldson conjecture relating existence of Kähler–Einstein metrics and algebraic K-stability on Fano manifolds [7; 8; 9; 40]. In [14], the Schauder estimate for linear Laplace equations with the conical background metric is established using classical potential theory. This is crucial for the openness of the continuity method to find a desirable

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(conical) Kähler–Einstein metric. Donaldson's Schauder estimate is generalized to the parabolic case [5] with a similar classical approach. There is also an alternative approach for the conical Schauder estimates using microlocal analysis [23]. Various global and local estimates and regularity are also derived in the conical setting [1; 6; 11; 12; 13; 15; 19; 24; 29; 32; 44; 45].

The Schauder estimates play an important role in linear PDE theory. Apart from the classical potential theory, various proofs have been established by different analytic techniques. In fact, the blow-up or perturbation techniques developed in [36; 42] (also see [2; 3; 33; 34]) are much more flexible and sharper than the classical method. The authors combined the perturbation method in [20] and geometric gradient estimates to establish sharp Schauder estimates for Laplace equations and heat equations on $\mathbb{C}^n$ with a background flat Kähler metric of conical singularities along the smooth hyperplane $\{z_1 = 0\}$ and derived explicit and optimal dependence on conical parameters.

In algebraic geometry, one often has to consider pairs (X, D) with X an algebraic variety of complex dimension n and the boundary divisor D a complex hypersurface of X. After possible log resolution, one can always assume the divisor D is a union of smooth hypersurfaces with simple normal crossings. The suitable category of Kähler metrics associated to (X, D) is the family of Kähler metrics on X with conical singularities along D. In order to study canonical Kähler metrics on pairs and related moduli problems, we are obliged to study regularity and asymptotics for complex Monge–Ampère equation with prescribed conical singularities of normal crossings. However, the linear theory is still missing and has been open for a while. The goal of this paper is to extend our result [20] and establish the sharp Schauder estimates for linear equations with background Kähler metric of conical singularities along divisors of simple normal crossings. We can apply and extend many techniques developed in [20]; however, new estimates and techniques have to be developed because, in the case of conical singularities along a single smooth divisor, the difficult estimate in the conical direction can sometimes be bypassed and reduced to estimates in the regular directions, while such treatment does not work in the case of simple normal crossings. One is forced to treat regions near high codimensional singularities directly with new and more delicate estimates beyond the scope of [20]. More crucially, the estimates in the mixed normal directions (see Section 3D) relies on those in Lemma 3.3, which is new compared to the case of a smooth divisor [20]. This enables us to compare the difference of mixed normal derivatives at two different points. Readers who are interested only in the case of smooth divisors are advised to omit Section 3D.

The standard local models for such conical Kähler metrics are described below.

Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p) \in (0, 1)^p$, $p \le n$, and let $\omega_{\boldsymbol{\beta}}$ (or $g_{\boldsymbol{\beta}}$) be the standard cone metric on $\mathbb{C}^p \times \mathbb{C}^{n-p}$ with cone singularity along $\mathcal{S} = \bigcup_{i=1}^p \mathcal{S}_i$, where $\mathcal{S}_i = \{z_i = 0\}$, that is,

$$\omega_{\boldsymbol{\beta}} = \sum_{j=1}^{p} \beta_j^2 \frac{\sqrt{-1}\, dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \sum_{j=p+1}^{n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j. \tag{1-1}$$

We shall use $s_{2p+1}, \dots, s_{2n}$ to denote the real coordinates of $\mathbb{C}^{n-p} = \mathbb{R}^{2n-2p}$ such that, for $j = p+1, \dots, n$,

$$z_j = s_{2j-1} + \sqrt{-1} s_{2j}.$$

In this paper we will study the conical Laplacian equation with the background metric $g_{\boldsymbol{\beta}}$ on $\mathbb{C}^n$

$$\Delta_{\boldsymbol{\beta}} u = f \quad \text{in } B_{\boldsymbol{\beta}}(0,1)\backslash\mathcal{S}, \tag{1-2}$$

where $B_{\boldsymbol{\beta}}(0,1)$ is the unit ball with respect to $g_{\boldsymbol{\beta}}$ centered at 0. The Laplacian $\Delta_{\boldsymbol{\beta}}$ is defined as

$$\Delta_{\boldsymbol{\beta}} u = \sum_{j,k} g_{\boldsymbol{\beta}}^{j\bar{k}} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = \sum_{j=1}^{p} \beta_j^{-1} |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} + \sum_{j=p+1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}.$$

We always assume

$$f \in C^0(B_{\boldsymbol{\beta}}(0,1)) \quad \text{and} \quad u \in C^0(\overline{B_{\boldsymbol{\beta}}(0,1)}) \cap C^2(B_{\boldsymbol{\beta}}(0,1)\backslash\mathcal{S}).$$

Throughout this paper, given a continuous function f, we write

$$\omega(r) := \omega_f(r) = \sup_{\substack{z,w \in B_{\boldsymbol{\beta}}(0,1) \\ d_{\boldsymbol{\beta}}(z,w) < r}} |f(z) - f(w)|$$

for the oscillation of f with respect to $g_{\boldsymbol{\beta}}$ in the ball $B_{\boldsymbol{\beta}}(0,1)$. It is clear that $\omega(2r) \le 2\omega(r)$ for any $r < \frac{1}{2}$. We say a continuous function f is *Dini continuous* if $\int_0^1 \omega(r)/r \, dr < \infty$.

Definition 1.1. We will write the (weighted) polar coordinates of z_j for $1 \le j \le p$ as

$$r_j = |z_j|^{\beta_j}, \quad \theta_j = \arg z_j.$$

We define D' to be one of the first-order operators $\{\partial/\partial s_{2p+1}, \dots, \partial/\partial s_{2n}\}$, and N_j to be one of the operators $\{\partial/\partial r_j, (\beta_j r_j)^{-1}(\partial/\partial\theta_j)\}$ which as vector fields are transversal to $\mathcal{S}_j$.

Our first main result is the Hölder estimates of the solution u to (1-2).

Theorem 1.2. *Suppose $\boldsymbol{\beta} \in \left(\frac{1}{2}, 1\right)^p$ and $f \in C^0(B_{\boldsymbol{\beta}}(0,1))$ is Dini continuous with respect to $g_{\boldsymbol{\beta}}$. Let $u \in C^0(\overline{B_{\boldsymbol{\beta}}(0,1)}) \cap C^2(B_{\boldsymbol{\beta}}(0,1)\backslash\mathcal{S})$ be the solution to* (1-2). *Then there exists $C = C(n, \boldsymbol{\beta}) > 0$ such that, for any two points $p, q \in B_{\boldsymbol{\beta}}\left(0, \frac{1}{2}\right)\backslash\mathcal{S}$,*

$$\begin{aligned} |(D')^2 u(p) - (D')^2 u(q)| + \sum_{j=1}^{p} \left| |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(p) - |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(q) \right| & \\ \le C\left(d\|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \int_0^d \frac{\omega(r)}{r} \, dr + d \int_d^1 \frac{\omega(r)}{r^2} \, dr \right), & \end{aligned} \tag{1-3}$$

for any $1 \le j \le p$,

$$|N_j D' u(p) - N_j D' u(q)| \le C\left(d^{1/\beta_j - 1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \int_0^d \frac{\omega(r)}{r} \, dr + d^{1/\beta_j - 1} \int_d^1 \frac{\omega(r)}{r^{1/\beta_j}} \, dr \right), \tag{1-4}$$

and, for any $1 \le j, k \le p$ with $j \ne k$,

$$|N_j N_k u(p) - N_j N_k u(q)| \le C\left(d^{1/\beta_{\max} - 1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \int_0^d \frac{\omega(r)}{r} \, dr + d^{1/\beta_{\max} - 1} \int_d^1 \frac{\omega(r)}{r^{1/\beta_{\max}}} \, dr \right), \tag{1-5}$$

where $d = d_{\boldsymbol{\beta}}(p,q) > 0$ is the $g_{\boldsymbol{\beta}}$-distance of p and q and $\beta_{\max} = \max\{\beta_1, \dots, \beta_p\} \in \left(\frac{1}{2}, 1\right)$.

Remarks 1.3. (1) The number $\beta_{\max}$ on the right-hand side of (1-5) can be replaced by $\max\{\beta_j, \beta_k\}$.

(2) We assume $\boldsymbol{\beta} \in \left(\frac{1}{2}, 1\right)^p$ for the purposes of exposition and simplification of the statements of Theorems 1.2 and 1.7. When some of the angles β_j lie in $\left(0, \frac{1}{2}\right]$, the pointwise Hölder estimates in Theorem 1.2 are adjusted as follows: in (1-4), if $\beta_j \in \left(0, \frac{1}{2}\right]$, we replace the right-hand side by the right-hand side of (1-3); in (1-5), if both β_j and $\beta_k \in \left(0, \frac{1}{2}\right]$, we also replace the right-hand side by that of (1-3); if at least one of the β_j, β_k is bigger than $\frac{1}{2}$, (1-5) remains unchanged. The inequalities in Theorem 1.7 can be adjusted similarly. The proofs of these estimates are contained in the proof of the case when $\beta_j \in \left(\frac{1}{2}, 1\right)$ by using the corresponding estimates in (2-3).

An immediate corollary of Theorem 1.2 is a precise form of Schauder estimates for (1-2).

Corollary 1.4. *Given $\boldsymbol{\beta} \in (0, 1)^p$ and $f \in C^{0,\alpha}_{\boldsymbol{\beta}}(\overline{B_{\boldsymbol{\beta}}(0, 1)})$ for some $0 < \alpha < \min\{1, 1/\beta_{\max} - 1\}$, if $u \in C^0(B_{\boldsymbol{\beta}}(0, 1)) \cap C^2(B_{\boldsymbol{\beta}}(0, 1)\backslash \mathcal{S})$ solves (1-2), then $u \in C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0, 1))$. Moreover, for any compact subset $K \Subset B_{\boldsymbol{\beta}}(0, 1)$, there exists a constant $C = C(n, \boldsymbol{\beta}, K) > 0$ such that the following estimate holds (see Definition 2.1 for the notations):*

$$\|u\|_{C^{2,\alpha}_{\boldsymbol{\beta}}(K)} \le C\left(\|u\|_{C^0(B_{\boldsymbol{\beta}}(0,1))} + \frac{\|f\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,1))}}{\alpha\left(\min\left\{\frac{1}{\beta_{\max}} - 1, 1\right\} - \alpha\right)}\right). \tag{1-6}$$

Remark 1.5. A scaling-invariant version of the Schauder estimate (1-6) is that, for any $0 < r < 1$, there exists a constant $C = C(n, \boldsymbol{\beta}, \alpha) > 0$ such that (see Definition 2.4 for the notations)

$$\|u\|^*_{C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,r))} \le C(\|u\|_{C^0(B_{\boldsymbol{\beta}}(0,r))} + \|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,r))}), \tag{1-7}$$

which follows from a standard rescaling argument by scaling r to 1.

Let g be a $C^{0,\alpha}_{\boldsymbol{\beta}}$-conical Kähler metric on $B_{\boldsymbol{\beta}}(0, 1)$ (see Definition 3.31). By definition g is equivalent to $g_{\boldsymbol{\beta}}$. We consider the equation

$$\Delta_g u = f \quad \text{in } B_{\boldsymbol{\beta}}(0, 1) \qquad \text{and} \qquad u = \varphi \quad \text{on } \partial B_{\boldsymbol{\beta}}(0, 1) \tag{1-8}$$

for some $\varphi \in C^0(\partial B_{\boldsymbol{\beta}}(0, 1))$. The following theorem is the generalization of Corollary 1.4 for nonflat background conical Kähler metrics, which is useful for applications of global geometric complex Monge–Ampère equations.

Theorem 1.6. *For any given $\boldsymbol{\beta} \in (0, 1)^p$, $f \in C^{0,\alpha}_{\boldsymbol{\beta}}(\overline{B_{\boldsymbol{\beta}}(0, 1)})$ and $\varphi \in C^0(\partial B_{\boldsymbol{\beta}}(0, 1))$, there is a unique solution $u \in C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0, 1)) \cap C^0(\overline{B_{\boldsymbol{\beta}}(0, 1)})$ to (1-8). Moreover, for any compact subset $K \Subset B_{\boldsymbol{\beta}}(0, 1)$, there exists $C = C(n, \boldsymbol{\beta}, \alpha, g, K) > 0$ such that*

$$\|u\|_{C^{2,\alpha}_{\boldsymbol{\beta}}(K)} \le C(\|u\|_{C^0(B_{\boldsymbol{\beta}}(0,1))} + \|f\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,1))}).$$

Theorem 1.6 can immediately be applied to study complex Monge–Ampère equations with prescribed conical singularities along divisors of simple normal crossings, and most of the geometric and analytic results for canonical Kähler metrics with conical singularities along a smooth divisor can be generalized to the case of simple normal crossings.

We now turn to the parabolic Schauder estimates for the solution $u \in \mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^2(\mathcal{Q}^{\#}_{\boldsymbol\beta})$ to the equation

$$\frac{\partial u}{\partial t} = \Delta_{g_{\boldsymbol\beta}} u + f \tag{1-9}$$

for a Dini continuous function f in $\mathcal{Q}_{\boldsymbol\beta}$, where for convenience of notation we write

$$\mathcal{Q}_{\boldsymbol\beta} := B_{\boldsymbol\beta}(0,1) \times (0,1] \quad \text{and} \quad \mathcal{Q}^{\#}_{\boldsymbol\beta} := B_{\boldsymbol\beta}(0,1) \backslash \mathcal{S} \times (0,1].$$

Our second main theorem is the following pointwise estimate.

Theorem 1.7. *Suppose $\boldsymbol\beta \in \left(\frac{1}{2}, 1\right)^p$ and u is the solution to* (1-9). *Then there exists a computable constant $C = C(n, \boldsymbol\beta) > 0$ such that, for any $Q_p = (p, t_p)$, $Q_q = (q, t_q) \in B_{\boldsymbol\beta}\left(0, \frac{1}{2}\right) \backslash \mathcal{S} \times (\hat t, 1]$ (for some $\hat t \in (0,1)$),*

$$\begin{aligned} &|(D')^2 u(Q_p) - (D')^2 u(Q_q)| \\ &\qquad + \sum_{j=1}^p \left| |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar z_j}(Q_p) - |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar z_j}(Q_q) \right| + \left| \frac{\partial u}{\partial t}(Q_p) - \frac{\partial u}{\partial t}(Q_q) \right| \\ &\qquad\qquad \le C \left(\frac{d}{\hat t^{3/2}} \|u\|_{L^\infty(B_{\boldsymbol\beta}(0,1))} + \hat t^{-1} \int_0^d \frac{\omega(r)}{r}\, dr + \frac{d}{\hat t^{3/2}} \int_d^1 \frac{\omega(r)}{r^2}\, dr \right), \end{aligned}$$

and, for any $1 \le j \le p$,

$$|N_j D' u(Q_p) - N_j D' u(Q_q)| \le C \left(\frac{d^{1/\beta_j - 1}}{\hat t^{3/2}} \|u\|_{L^\infty(B_{\boldsymbol\beta}(0,1))} + \hat t^{-1} \int_0^d \frac{\omega(r)}{r}\, dr + \frac{d^{1/\beta_j - 1}}{\hat t^{3/2}} \int_d^1 \frac{\omega(r)}{r^{1/\beta_j}}\, dr \right),$$

and, for any $1 \le j, k \le p$ with $j \ne k$,

$$|N_j N_k u(Q_p) - N_j N_k u(Q_q)| \le C \left(\frac{d^{1/\beta_{\max} - 1}}{\hat t^{3/2}} \|u\|_{L^\infty(B_{\boldsymbol\beta}(0,1))} + \hat t^{-1} \int_0^d \frac{\omega(r)}{r}\, dr + \frac{d^{1/\beta_{\max} - 1}}{\hat t^{3/2}} \int_d^1 \frac{\omega(r)}{r^{1/\beta_{\max}}}\, dr \right),$$

where $d = d_{\mathcal{P},\boldsymbol\beta}(Q_p, Q_q) > 0$ is the parabolic $g_{\boldsymbol\beta}$-distance of Q_p and Q_q, $\beta_{\max} = \max\{\beta_1, \dots, \beta_p\}$, and $\omega(r)$ is the oscillation of f in $\mathcal{Q}_{\boldsymbol\beta}$ under the parabolic distance $d_{\mathcal{P},\boldsymbol\beta}$ (see Section 2A2).

If $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$ for some $\alpha \in (0, \min(1/\beta_{\max} - 1, 1))$, then we have the following precise estimates as the parabolic analogue of Corollary 1.4.

Corollary 1.8. *Suppose $\boldsymbol\beta \in (0,1)^p$ and $u \in \mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^2(\mathcal{Q}^{\#}_{\boldsymbol\beta})$ satisfies* (1-9). *Then there exists a constant $C = C(n, \boldsymbol\beta) > 0$ such that (see Definition 2.6 for the notations)*

$$\|u\|_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1/2)\times(1/2,1])} \le C \left(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \frac{\|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})}}{\alpha\left(\min\left\{\frac{1}{\beta_{\max}} - 1, 1\right\} - \alpha\right)} \right).$$

For general nonflat $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}$-conical Kähler metrics g, we consider the linear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta}, \qquad u = \varphi \quad \text{on } \partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol\beta}. \tag{1-10}$$

We then have the following parabolic Schauder estimates as an analogue of Theorem 1.6.

Theorem 1.9. *Given $\boldsymbol{\beta} \in (0,1)^p$, $f \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$ and $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol{\beta}})$, there exists a unique solution $u \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(\alpha+2)/2}(B_{\boldsymbol{\beta}}(0,1)\times(0,1]) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$ to the Dirichlet boundary value problem* (1-10). *For any compact subset $K \Subset B_{\boldsymbol{\beta}}(0,1)$ and $\varepsilon_0 > 0$, there exists $C = C(n, \boldsymbol{\beta}, \alpha, K, \varepsilon_0, g) > 0$ such that the following interior Schauder estimate holds*:

$$\|u\|_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(K\times[\varepsilon_0,1])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + \|f\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol{\beta}})}).$$

Furthermore, if we assume $u|_{t=0} = u_0 \in C_{\boldsymbol{\beta}}^{2,\alpha}(B_{\boldsymbol{\beta}}(0,1))$, then $u \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(\alpha+2)/2}(B_{\boldsymbol{\beta}}(0,1)\times[0,1])$, and there exists a constant $C = C(n, \boldsymbol{\beta}, \alpha, g, K) > 0$ such that

$$\|u\|_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(\alpha+2)/2}(K\times[0,1])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + \|f\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol{\beta}})} + \|u_0\|_{C_{\boldsymbol{\beta}}^{2,\alpha}(B_{\boldsymbol{\beta}}(0,1))}).$$

As an application of Theorem 1.9, we derive the short-time existence of the conical Kähler–Ricci flow with background metric being conical along divisors with simple normal crossings.

Let (X, D) be a compact Kähler manifold, where $D = \sum_j D_j$ is a finite union of smooth divisors $\{D_j\}$ and D has only simple normal crossings. Let ω_0 be a $C_{\boldsymbol{\beta}}^{0,\alpha'}(X)$-conical Kähler metric with cone angle $2\pi\boldsymbol{\beta}$ along D (see Definition 2.8), let $\hat{\omega}_t$ be a family of conical metrics with bounded norm $\|\hat{\omega}\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}}$, and let $\hat{\omega}_0 = \omega_0$. We consider the complex Monge–Ampère flow

$$\frac{\partial\varphi}{\partial t} = \log\left(\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n}\right) + f \quad \text{and} \quad \varphi|_{t=0} = 0 \tag{1-11}$$

for some $f \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}(X\times[0,1])$.

Theorem 1.10. *Given $\alpha \in (0,\alpha')$, there exists $T = T(n, \hat{\omega}, f, \alpha', \alpha) > 0$ such that* (1-11) *admits a unique solution $\varphi \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X\times[0,T])$.*

An immediate corollary of Theorem 1.10 is the short-time existence for the conical Kähler–Ricci flow

$$\frac{\partial\omega}{\partial t} = -\operatorname{Ric}(\omega) + \sum_j (1-\beta_j)[D_j], \quad \omega|_{t=0} = \omega_0, \tag{1-12}$$

where $\operatorname{Ric}(\omega)$ is the unique extension of the Ricci curvature of ω from $X \setminus D$ to X, and $[D_j]$ denotes the current of integration over the component D_j. In addition we assume ω_0 is a $C_{\boldsymbol{\beta}}^{0,\alpha'}(X, D)$-conical Kähler metric such that

$$\omega_0^n = \frac{\Omega}{\prod_j (|s_j|_{h_j}^2)^{1-\beta_j}}, \tag{1-13}$$

where s_j and h_j are holomorphic sections and hermitian metrics, respectively, of the line bundle associated to D_j, and Ω is a smooth volume form.

Corollary 1.11. *For any given $\alpha \in (0,\alpha')$, there exists a constant $T = T(n, \omega_0, \alpha, \alpha') > 0$ such that the conical Kähler–Ricci flow* (1-12) *admits a unique solution $\omega = \omega_t$, where $\omega \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X\times[0,T])$ and, for each $t \in [0,T]$, ω_t is still a conical metric with cone angle $2\pi\boldsymbol{\beta}$ along D.*

Furthermore, ω is smooth in $X\backslash D \times (0,T]$ and the (*normalized*) *Ricci potentials of ω, by which we mean $\log(\omega^n/\omega_0^n)$, are still in $\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X\times[0,T])$.*

The short-time existence of the conical Kähler–Ricci flow with singularities along a smooth divisor is derived in [5] by adapting the elliptic potential techniques of Donaldson [14]. Corollary 1.11 treats the general case of conical singularities with simple normal crossings. There have been many results in the analytic aspects of the conical Ricci flow [5; 6; 15; 16; 24; 30; 43]. In [31], the conical Ricci flow on Riemann surfaces is completely classified with jumping conical structure in the limit. Such phenomena is also expected in higher dimension, but it requires much deeper and delicate technical advances both in analysis and geometry.

2. Preliminaries

We explain the notations and give some preliminary tools which will be used later in this section.

2A. ***Notations.*** To distinguish the elliptic from parabolic norms, we will use C to denote the norms in the elliptic case and $\mathscr{C}$ to denote the norms in the parabolic case.

We always assume the Hölder component α appearing in $C^{0,\alpha}_{\boldsymbol{\beta}}$ or $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}$ (or other Hölder norms) is in $(0, \min\{\beta_{\max}^{-1}-1, 1\})$.

2A1. *Elliptic case.* We will denote $d_{\boldsymbol{\beta}}(x, y)$ to be the distance between two points $x, y \in \mathbb{C}^n$ under the metric $g_{\boldsymbol{\beta}}$. $B_{\boldsymbol{\beta}}(x, r)$ will be the metric ball under the metric induced by $g_{\boldsymbol{\beta}}$ with radius r and center x. It is well known that $(\mathbb{C}^n\backslash\mathcal{S}, g_{\boldsymbol{\beta}})$ is geodesically convex, i.e., any two points $x, y \in \mathbb{C}^n\backslash\mathcal{S}$ can be joined by a $g_{\boldsymbol{\beta}}$-minimal geodesic γ which is disjoint from $\mathcal{S}$.

Definition 2.1. We define the $g_{\boldsymbol{\beta}}$-Hölder norm of functions $u \in C^0(B_{\boldsymbol{\beta}}(0, r))$ for some $\alpha \in (0, 1)$ as

$$\|u\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,r))} := \|u\|_{C^0(B_{\boldsymbol{\beta}}(0,r))} + [u]_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,r))},$$

where the seminorm is defined as

$$[u]_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,r))} := \sup_{x\neq y\in B_{\boldsymbol{\beta}}(0,r)} \frac{|u(x)-u(y)|}{d_{\boldsymbol{\beta}}(x, y)^\alpha}.$$

We denote by $C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0, r))$ the subspace of all continuous functions u such that $\|u\|_{C^{0,\alpha}_{\boldsymbol{\beta}}} < \infty$.

Definition 2.2. The $C^{2,\alpha}_{\boldsymbol{\beta}}$-norm of a function u on $B_{\boldsymbol{\beta}}(0, r) =: B_{\boldsymbol{\beta}}$ is defined as

$$\begin{aligned}\|u\|_{C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} := \|u\|_{C^0(B_{\boldsymbol{\beta}})} + \|\nabla_{g_{\boldsymbol{\beta}}} u\|_{C^0(B_{\boldsymbol{\beta}}, g_{\boldsymbol{\beta}})} + \sum_{j=1}^{p} \|N_j D' u\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} \\ + \|(D')^2 u\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} + \sum_{1\leq j\neq k\leq p} \|N_j N_k u\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} + \sum_{j=1}^{p} \left\| |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} \right\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})}.\end{aligned}$$

We denote by $C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0, r))$ the subspace of all continuous functions u such that $\|u\|_{C^{2,\alpha}_{\boldsymbol{\beta}}} < \infty$.

Remark 2.3. These spaces are generalizations of those defined in [20] and are slight variations of those introduced in [14; 23].

Let us compare the Schauder estimates in [14; 23; 20] in the special case when $p=1$, i.e., the conical singularities are supported on $\mathbb{C}^{n-1}$. The Hölder space of [14] is defined using a collection of differential operators as components of $\sqrt{-1}\partial\bar\partial$. The collection of differential operators in our definition for $C^{0,\alpha}_{\boldsymbol\beta}$ (see [20]) is given by

$$\left\{\frac{\partial}{\partial r}, r^{-1}\frac{\partial}{\partial\theta}, D', \frac{\partial^2}{\partial r^2}+r^{-1}\frac{\partial}{\partial r}+(\beta r)^{-2}\frac{\partial^2}{\partial\theta^2}, \frac{\partial}{\partial r}D', r^{-1}\frac{\partial}{\partial\theta}D'\right\},$$

while those defined in [23] for the Hölder space $\mathcal{D}^{0,\alpha}_\omega$ gives the collection

$$\left\{\frac{\partial}{\partial r}, r^{-1}\frac{\partial}{\partial\theta}, D', \frac{\partial^2}{\partial r^2}+r^{-1}\frac{\partial}{\partial r}+(\beta r)^{-2}\frac{\partial^2}{\partial\theta^2}, \frac{\partial}{\partial r}D', r^{-1}\frac{\partial}{\partial\theta}D', \frac{\partial^2}{\partial r\partial\theta}\right\}. \tag{2-1}$$

Here the operators D' are given in Definition 1.1. There seems to a typo in the original definition of (2-1) in [23, p. 104, (16)], where the factor r^{-1} is missing in the operator $r^{-1}\partial/\partial\theta D'$ (see [32, p. 57]). It was pointed out by the referee that this typo does not affect Proposition 3.3 in [23] since the correct operator was used in the proof. The space $\mathcal{D}^{0,\alpha}_\omega$ is introduced in [23] as an alternative definition for the Hölder space of [14] as a consequence of the Schauder estimates in [23, Proposition 3.3]. The Schauder estimates in [23] are stronger than those established in [14] by Donaldson and later in [20] by the authors because of the additional operator $\partial^2/\partial r\partial\theta$ in (2-1). This also implies that the two Hölder spaces from [14] and [20] must coincide. For interested readers, we refer to the survey paper [32] for more details on the characterization of the Hölder space of [14] in terms of the operators in (2-1).

For a given set $\Omega\subset B_{\boldsymbol\beta}(0,1)$, we define the following weighted (semi)norms.

Definition 2.4. Suppose $\sigma\in\mathbb{R}$ is a given real number and u is a $C^{2,\alpha}_{\boldsymbol\beta}$-function in Ω. We will write $d_x=d_{\boldsymbol\beta}(x,\partial\Omega)$ for any $x\in\Omega$. We define the weighted (semi)norms

$$[u]^{(\sigma)}_{C^{0,\alpha}_{\boldsymbol\beta}(\Omega)}=\sup_{x\neq y\in\Omega}\min(d_x,d_y)^{\sigma+\alpha}\frac{|u(x)-u(y)|}{d_{\boldsymbol\beta}(x,y)^\alpha},$$

$$\|u\|^{(\sigma)}_{C^0(\Omega)}=\sup_{x\in\Omega}d_x^\sigma|u(x)|,\quad [u]^{(\sigma)}_{C^1_{\boldsymbol\beta}(\Omega)}=\sup_{x\in\Omega\setminus\mathcal{S}}d_x^{\sigma+1}\left(\sum_j|N_ju|(x)+|D'u|(x)\right),$$

$$[u]^{(\sigma)}_{C^2_{\boldsymbol\beta}(\Omega)}=\sup_{x\in\Omega\setminus\mathcal{S}}d_x^{\sigma+2}|Tu(x)|,\quad [u]^{(\sigma)}_{C^{2,\alpha}_{\boldsymbol\beta}(\Omega)}=\sup_{x\neq y\in\Omega\setminus\mathcal{S}}\min(d_x,d_y)^{\sigma+2+\alpha}\frac{|Tu(x)-Tu(y)|}{d_{\boldsymbol\beta}(x,y)^\alpha},$$

$$\|u\|^{(\sigma)}_{C^{2,\alpha}_{\boldsymbol\beta}(\Omega)}=\|u\|^{(\sigma)}_{C^0(\Omega)}+[u]^{(\sigma)}_{C^1_{\boldsymbol\beta}(\Omega)}+[u]^{(\sigma)}_{C^2_{\boldsymbol\beta}(\Omega)}+[u]^{(\sigma)}_{C^{2,\alpha}_{\boldsymbol\beta}(\Omega)},$$

where T is the collection of operators of second-order

$$\left\{|z_j|^{2(1-\beta_j)}\frac{\partial^2}{\partial z_j\partial\bar z_j},\ N_jN_k\ (j\neq k),\ N_jD',\ (D')^2\right\}. \tag{2-2}$$

When $\sigma=0$, we write the norms above as $[\,\cdot\,]^*$ or $\|\cdot\|^*$ for simplicity of notation.

2A2. *Parabolic case.* We define $\mathcal{Q}_{\boldsymbol\beta}=\mathcal{Q}_{\boldsymbol\beta}(0,1)=B_{\boldsymbol\beta}(0,1)\times(0,1]$ to be a parabolic cylinder and

$$\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}=(\overline{B_{\boldsymbol\beta}(0,1)}\times\{0\})\cup(\partial B_{\boldsymbol\beta}(0,1)\times(0,1])$$

to be the parabolic boundary of the cylinder $\mathcal{Q}_{\boldsymbol\beta}$. We write $\mathcal{S}_{\mathcal{P}} = \mathcal{S} \times [0, 1]$ for the singular set and $\mathcal{Q}^{\#}_{\boldsymbol\beta} = \mathcal{Q}_{\boldsymbol\beta} \backslash \mathcal{S}_{\mathcal{P}}$ for the complement of $\mathcal{S}_{\mathcal{P}}$. For any two space-time points $Q_i = (p_i, t_i)$, we define their parabolic distance $d_{\mathcal{P},\boldsymbol\beta}(Q_1, Q_2)$ as

$$d_{\mathcal{P},\boldsymbol\beta}(Q_1, Q_2) = \max\{\sqrt{|t_1 - t_2|},\ d_{\boldsymbol\beta}(p_1, p_2)\}.$$

Definition 2.5. We define the $g_{\boldsymbol\beta}$-Hölder norm of functions $u \in \mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})$ for some $\alpha \in (0, 1)$ as

$$\|u\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} := \|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + [u]_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})},$$

where the seminorm is

$$[u]_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} := \sup_{Q_1 \neq Q_2 \in \mathcal{Q}_{\boldsymbol\beta}} \frac{|u(Q_1) - u(Q_2)|}{d_{\mathcal{P},\boldsymbol\beta}(Q_1, Q_2)^\alpha}.$$

We denote by $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$ the subspace of all continuous functions u such that $\|u\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} < \infty$.

Definition 2.6. The $\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}$-norm of a function u on $\mathcal{Q}_{\boldsymbol\beta}$ is defined as

$$\|u\|_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} := \|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|\nabla_{g_{\boldsymbol\beta}} u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta}, g_{\boldsymbol\beta})} + \|\mathcal{T} u\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})},$$

where $\mathcal{T}$ is the collection of all the second-order operators in (2-2) with the first-order operator $\partial/\partial t$.

For a given set $\Omega \subset \mathcal{Q}_{\boldsymbol\beta}$ we define the following weighted (semi)norms.

Definition 2.7. Suppose $\sigma \in \mathbb{R}$ is a real number and u is a $\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}$-function in Ω. We will write $d_{\mathcal{P},Q} = d_{\mathcal{P},\boldsymbol\beta}(Q, \partial_{\mathcal{P}}\Omega)$ for any $Q \in \Omega$. We define the weighted (semi)norms

$$[u]^{(\sigma)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\Omega)} = \sup_{Q_1 \neq Q_2 \in \Omega} \min(d_{\mathcal{P},Q_1}, d_{\mathcal{P},Q_2})^{\sigma+\alpha} \frac{|u(Q_1) - u(Q_2)|}{d_{\mathcal{P},\boldsymbol\beta}(Q_1, Q_2)^\alpha}, \quad \|u\|^{(\sigma)}_{\mathscr{C}^0(\Omega)} = \sup_{Q \in \Omega} d^{\sigma}_{\mathcal{P},Q} |u(Q)|,$$

$$[u]^{(\sigma)}_{\mathscr{C}^1_{\boldsymbol\beta}(\Omega)} = \sup_{Q \in \Omega \backslash \mathcal{S}_{\mathcal{P}}} d^{\sigma+1}_{\mathcal{P},Q} \Big(\sum_j |N_j u|(Q) + |D' u|(Q) \Big), \quad [u]^{(\sigma)}_{\mathscr{C}^{2,1}_{\boldsymbol\beta}(\Omega)} = \sup_{Q \in \Omega \backslash \mathcal{S}_{\mathcal{P}}} d^{\sigma+2}_{\mathcal{P},Q} |\mathcal{T} u(Q)|,$$

$$[u]^{(\sigma)}_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\Omega)} = \sup_{Q_1 \neq Q_2 \in \Omega \backslash \mathcal{S}_{\mathcal{P}}} \min(d_{\mathcal{P},Q_1}, d_{\mathcal{P},Q_2})^{\sigma+2+\alpha} \frac{|\mathcal{T} u(Q_1) - \mathcal{T} u(Q_2)|}{d_{\mathcal{P},\boldsymbol\beta}(Q_1, Q_2)^\alpha},$$

$$\|u\|^{(\sigma)}_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\Omega)} = \|u\|^{(\sigma)}_{\mathscr{C}^0(\Omega)} + [u]^{(\sigma)}_{\mathscr{C}^1_{\boldsymbol\beta}(\Omega)} + [u]^{(\sigma)}_{\mathscr{C}^{2,1}_{\boldsymbol\beta}(\Omega)} + [u]^{(\sigma)}_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\Omega)}.$$

When $\sigma = 0$, we write the norms above as $[\,\cdot\,]^*$ or $\|\cdot\|^*$ for simplicity of notation.

2A3. *Compact Kähler manifolds.* Let (X, D) be a compact Kähler manifold with a divisor $D = \sum_j D_j$ with simple normal crossings, i.e., on an open coordinate chart (U, z_j) of any $x \in D$, $D \cap U$ is given by $\{z_1 \cdots z_p = 0\}$ and $D_j \cap U = \{z_j = 0\}$ for any component D_j of D. We fix a finite cover $\{U_a, z_{a,j}\}$ of D.

Definition 2.8. A (singular) Kähler metric ω is called a conical metric with cone angle $2\pi\boldsymbol\beta$ along D if ω is equivalent to $\omega_{\boldsymbol\beta}$ locally on any coordinate chart U_a under the coordinates $\{z_{a,j}\}$, where $\omega_{\boldsymbol\beta}$ is the standard cone metric (1-1) with cone angle $2\pi\beta_j$ along $\{z_{a,j} = 0\}$, and ω is a smooth Kähler metric in the usual sense on $X \backslash \bigcup_a U_a$.

A conical metric ω is in $C^{0,\alpha}_{\boldsymbol\beta}(X, D)$ if ω is in $C^{0,\alpha}_{\boldsymbol\beta}(U_a)$ for each a and ω is smooth in the usual sense on $X \backslash \bigcup_a U_a$. Similarly we can define the $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}$-conical Kähler metrics on $X \times [0, 1]$.

Definition 2.9. A continuous function $u \in C^0(X)$ is said to be in $C^{0,\alpha}_{\boldsymbol\beta}(X, D)$ if u is in $C^{0,\alpha}_{\boldsymbol\beta}(U_a)$ locally on each U_a and u is $C^{0,\alpha}$-continuous in the usual sense on $X \setminus \bigcup_a U_a$. We define the $C^{0,\alpha}_{\boldsymbol\beta}(X, D)$-norm of u

$$\|u\|_{C^{0,\alpha}_{\boldsymbol\beta}(X,D)} := \|u\|_{C^{0,\alpha}(X\setminus\bigcup_a U_a,\omega)} + \sum_a \|u\|_{C^{0,\alpha}_{\boldsymbol\beta}(U_a)}.$$

The $C^{0,\alpha}_{\boldsymbol\beta}(X, D)$-norm depends on the choice of finite covers, and another cover yields a different but equivalent norm. The space $C^{0,\alpha}_{\boldsymbol\beta}(X, D)$ is clearly independent of the choice of finite covers.

The other spaces and norms like $C^{2,\alpha}_{\boldsymbol\beta}(X, D)$, $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(X \times [0, 1], D)$, etc., can be defined similarly.

2B. ***A useful lemma.*** We will frequently use the following elementary estimates from [20]. We write $B_{\mathbb{C}}(0, r)$ for the Euclidean ball in $\mathbb{C}$ with center 0 and radius $r > 0$.

Lemma 2.10 (Lemma 2.2 in [20]). *Given $r \in (0, 1]$, suppose $v \in C^0(B_{\mathbb{C}}(0, r)) \cap C^2(B_{\mathbb{C}}(0, r)\setminus\{0\})$ satisfies*

$$|z|^{2(1-\beta_1)} \frac{\partial^2 v}{\partial z \partial \bar z} = F \quad \textit{in } B_{\mathbb{C}}(0, r)\setminus\{0\}$$

for some $F \in L^\infty(B_{\mathbb{C}}(0, r))$. Then we have the following pointwise estimate for any $z \in B_{\mathbb{C}}\big(0, \frac{9}{10}r\big)\setminus\{0\}$:

$$\left|\frac{\partial v}{\partial z}(z)\right| \le C\frac{\|v\|_{L^\infty}}{r} + C\|F\|_{L^\infty} \cdot \begin{cases} r^{2\beta_1-1} & \text{if } \beta_1 \in \big(\frac12, 1\big), \\ |z|^{2\beta_1-1} & \text{if } \beta_1 \in \big(0, \frac12\big), \\ \left|\log\Big(\frac{|z|}{2r}\Big)\right| & \text{if } \beta_1 = \frac12, \end{cases} \tag{2-3}$$

where the L^∞-norms are taken in $B_{\mathbb{C}}(0, r)$ and $C > 0$ is a uniform constant depending only on the angle β_1.

Finally we remark that the idea of the proof of the estimates in Theorems 1.2 and 1.7 is the same for general $2 \le p \le n$. To explain the argument more clearly, we prove the theorems assuming $p = 2$, i.e., the cone metric of $\omega_{\boldsymbol\beta}$ is singular along the two components $\mathcal{S}_1$ and $\mathcal{S}_2$.

3. Elliptic estimates

In this section, we will prove Theorems 1.2 and 1.6, the Schauder estimates for the Laplace equation (1-2). To begin with, we first observe the simple C^0-estimate based on the maximum principle.

Suppose $u \in C^2(B_{\boldsymbol\beta}(0, 1)\setminus\mathcal{S}) \cap C^0(\overline{B_{\boldsymbol\beta}(0, 1)})$ satisfies the equation

$$\begin{cases} \Delta_{\boldsymbol\beta} u = 0 & \text{in } B_{\boldsymbol\beta}(0, 1)\setminus\mathcal{S}, \\ u = \varphi & \text{on } \partial B_{\boldsymbol\beta}(0, 1) \end{cases} \tag{3-1}$$

for some $\varphi \in C^0(\partial B_{\boldsymbol\beta}(0, 1))$, then we have the following lemma.

Lemma 3.1. *We have the maximum principle*

$$\inf_{\partial B_{\boldsymbol\beta}(0,1)} \varphi \le \inf_{B_{\boldsymbol\beta}(0,1)} u \le \sup_{B_{\boldsymbol\beta}(0,1)} u \le \sup_{\partial B_{\boldsymbol\beta}(0,1)} \varphi. \tag{3-2}$$

Proof. Consider the functions $\tilde u_\epsilon = u \pm \epsilon(\log|z_1|^2 + \log|z_2|^2)$ for any $\epsilon > 0$. By the proof of Lemma 2.1 in [20], (3-2) is established. □

The next step is to show (3-1) is solvable for suitable boundary values.

3A. *Conical harmonic functions.*

3A1. *Smooth approximating metrics.* Let $\epsilon \in (0, 1)$ be a given small positive number and define a smooth approximating Kähler metric g_ϵ on $B_{\boldsymbol\beta}(0, 1)$ as

$$g_\epsilon = \beta_1^2 \frac{\sqrt{-1}\, dz_1 \wedge d\bar z_1}{(|z_1|^2+\epsilon)^{1-\beta_1}} + \beta_2^2 \frac{\sqrt{-1}\, dz_2 \wedge d\bar z_2}{(|z_2|^2+\epsilon)^{1-\beta_2}} + \sum_{j=3}^n \sqrt{-1}\, dz_j \wedge d\bar z_j. \tag{3-3}$$

The g_ϵ are product metrics on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^{n-2}$. It is clear that their Ricci curvatures satisfy

$$\mathrm{Ric}(g_\epsilon) = \sqrt{-1}\partial\bar\partial \log((|z_1|^2+\epsilon)^{1-\beta_1}(|z_2|^2+\epsilon)^{1-\beta_2}) \ge 0.$$

Let $u_\epsilon \in C^2(B_{\boldsymbol\beta}(0, 1))$ be the solution to the equation

$$\Delta_{g_\epsilon} u_\epsilon = 0 \quad \text{in } B_{\boldsymbol\beta}(0,1) \qquad \text{and} \qquad u_\epsilon = \varphi \quad \text{on } \partial B_{\boldsymbol\beta}(0,1) \tag{3-4}$$

with a given $\varphi \in C^0(\partial B_{\boldsymbol\beta}(0, 1))$. Note that the metric balls $B_{\boldsymbol\beta}(0, 1)$ and $B_{g_\epsilon}(0, 1)$ are uniformly close when ϵ is sufficiently small, so for the following estimates we will work on $B_{\boldsymbol\beta}(0, 1)$.

Let u_ϵ be the harmonic function for $\Delta_\epsilon = \Delta_{g_\epsilon}$ as in (3-4), which we may assume without loss of generality is positive by replacing u_ϵ by $u_\epsilon - \inf u_\epsilon$ if necessary. We will study the Cheng–Yau-type gradient estimate of u_ϵ and the estimate of $\Delta_{1,\epsilon} u_\epsilon := (|z_1|^2+\epsilon)^{1-\beta_1}(\partial^2 u_\epsilon/\partial z_1 \partial \bar z_1)$. Let us recall Cheng–Yau's gradient estimate first.

In Sections 3A2–3A5, for convenience of notation, we will omit the subscript ϵ in g_ϵ and u_ϵ in the proofs of the lemmas.

3A2. *Cheng–Yau gradient estimate revisited.* We assume $u_\epsilon > 0$, as otherwise we could consider the function $u_\epsilon + \delta$ for some $\delta > 0$ and then let $\delta \to 0$. We fix a metric ball $B_{g_\epsilon}(p, R) \subset B_{\boldsymbol\beta}(0, 1)$ centered at some point $p \in B_{\boldsymbol\beta}(0, 1)$. Since $\mathrm{Ric}(g_\epsilon) \ge 0$, the Cheng–Yau gradient estimate holds for Δ_{g_ϵ}-harmonic functions.

Lemma 3.2 [10]. *Let $u_\epsilon \in C^2(B(p, R))$ be a positive Δ_{g_ϵ}-harmonic function. There exists a uniform constant $C = C(n) > 0$ such that* (*the metric balls are taken under the metric g_ϵ*)

$$\sup_{x\in B(p,3R/4)} |\nabla u_\epsilon|_{g_\epsilon}(x) \le C(n) \frac{\mathrm{osc}_{B(p,R)}\, u_\epsilon}{R}. \tag{3-5}$$

As we mentioned above, we will omit the ϵ in the subscript of u_ϵ and g_ϵ. The proof of the lemma is standard [10]. For completeness and to motivate the proofs of Lemmas 3.3 and 3.4, we sketch a proof. Defining $f = \log u$, it can be calculated that

$$\Delta f = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -|\nabla f|^2. \tag{3-6}$$

Then by Bochner's formula we have

$$\begin{aligned} \Delta |\nabla f|^2 &= |\nabla\nabla f|^2 + |\nabla\overline\nabla f|^2 + 2\,\mathrm{Re}\langle \nabla f, \overline\nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \overline\nabla f) \\ &\ge |\nabla\overline\nabla f|^2 - 2\,\mathrm{Re}\langle \nabla f, \overline\nabla |\nabla f|^2\rangle. \end{aligned} \tag{3-7}$$

Let $\phi : [0,1] \to [0,1]$ be a standard cut-off function such that $\phi|_{[0,3/4]} = 1$, $\phi_{[5/6,1]} = 0$ and $0 < \phi < 1$ otherwise. Let $r(x) = d_{g_\epsilon}(p, x)$ be the distance function to p under the metric $g = g_\epsilon$. By abusing notation, we also write $\phi(x) = \phi(r(x)/R)$. It can be calculated by Laplacian comparison and the Bochner formula (3-7) that, at the (positive) maximum point $p_{\max}$ of $H := \phi^2|\nabla f|^2$,

$$\frac{2}{n}H^2 - \frac{4|\phi'|}{R}H^{3/2} - \frac{8(\phi')^2}{R^2}H + \frac{2H}{R^2}((2n-1)\phi\phi' + \phi\phi'' + (\phi')^2) \le 0.$$

Therefore, for any $x \in B\big(p, \frac{3}{4}R\big)$,

$$\frac{|\nabla u|^2}{u^2}(x) = |\nabla f(x)|^2 = H(x) \le H(p_{\max}) \le \frac{C(n)}{R^2}. \tag{3-8}$$

3A3. *Laplacian estimate in singular directions.* We will prove the estimates of

$$\Delta_{j,\epsilon} u_\epsilon := (|z_j|^2 + \epsilon)^{1-\beta_j} \frac{\partial u_\epsilon}{\partial z_j \partial \bar z_j}$$

for a Δ_{g_ϵ}-harmonic function u_ϵ.

Lemma 3.3. *Under the same assumptions as in Lemma 3.2, along the "bad" directions z_1 and z_2, we have that $\Delta_{1,\epsilon} u_\epsilon$ and $\Delta_{2,\epsilon} u_\epsilon$ satisfy the estimates*

$$\sup_{x \in B(p,R/2)} (|\Delta_{1,\epsilon} u_\epsilon|(x) + |\Delta_{2,\epsilon} u_\epsilon|(x)) \le C(n) \frac{\operatorname{osc}_{B(p,R)} u_\epsilon}{R^2}. \tag{3-9}$$

As in the proof of Cheng–Yau gradient estimates, we will work on the function $f = f_\epsilon = \log u$, and we only need to prove the estimate for $\Delta_{1,\epsilon} u_\epsilon$. We write $\Delta_{1,\epsilon} f := (|z_1|^2 + \epsilon)^{1-\beta_1}(\partial^2 f / \partial z_1 \partial \bar z_1)$.

As above, we will omit the subscript ϵ in $\Delta_{1,\epsilon} f$. We first observe that

$$\Delta_1 \Delta_{g_\epsilon} f = \Delta_{g_\epsilon} \Delta_1 f. \tag{3-10}$$

Equation (3-10) can be checked from the definitions using the property that g_ϵ is a product metric. Indeed

$$\begin{aligned}\Delta_1 \Delta_{g_\epsilon} f &= (|z_1|^2+\epsilon)^{1-\beta_1} \frac{\partial^2}{\partial z_1 \partial \bar z_1}\left((|z_1|^2+\epsilon)^{1-\beta_1}\frac{\partial^2 f}{\partial z_1 \partial \bar z_1} + (|z_2|^2+\epsilon)^{1-\beta_2}\frac{\partial^2 f}{\partial z_2 \partial \bar z_2} + \sum_j \frac{\partial^2 f}{\partial z_j \partial \bar z_j}\right) \\ &= (|z_1|^2+\epsilon)^{1-\beta_1}\frac{\partial^2}{\partial z_1 \partial \bar z_1}\Delta_1 f + (|z_2|^2+\epsilon)^{1-\beta_2}\frac{\partial^2}{\partial z_2 \partial \bar z_2}\Delta_1 f + \sum_{j=3}^n \frac{\partial^2}{\partial z_j \partial \bar z_j}\Delta_1 f = \Delta_{g_\epsilon}\Delta_1 f.\end{aligned}$$

On the other hand, note that $\Delta_{g_\epsilon} f = \Delta_g f = -|\nabla f|^2$ by (3-6). Choosing a normal frame $\{e_1, \dots, e_n\}$ at some point x such that $dg(x) = 0$ and $\Delta_1 f = f_{1\bar 1}$, we calculate

$$\begin{aligned}\Delta_1 |\nabla f|^2 = (f_j f_{\bar j})_{1\bar 1} &= f_{j1} f_{\bar j \bar 1} + f_{j\bar 1} f_{\bar j 1} + f_{j1\bar 1} f_{\bar j} + f_j f_{\bar j 1 \bar 1} \\ &= f_{j1} f_{\bar j \bar 1} + f_{j\bar 1} f_{\bar j 1} + f_j f_{\bar 1 1 \bar j} + f_{\bar j}(f_{1\bar 1 j} + f_m R_{1\bar m j \bar 1}) \\ &= |\nabla_1 \nabla f|^2 + |\nabla_1 \overline{\nabla} f|^2 + 2\operatorname{Re}\langle \nabla f, \overline{\nabla}\Delta_1 f\rangle + f_m f_{\bar j} R_{1\bar 1 j \bar m} \\ &\ge (\Delta_1 f)^2 + 2\operatorname{Re}\langle \nabla f, \overline{\nabla}\Delta_1 f\rangle.\end{aligned} \tag{3-11}$$

Then

$$\Delta(-\Delta_1 f) = -\Delta_1 \Delta f = \Delta_1 |\nabla f|^2 \geq (\Delta_1 f)^2 + 2\,\mathrm{Re}\langle \nabla f, \bar{\nabla}\Delta_1 f\rangle.$$

Let $\varphi : [0, 1] \to [0, 1]$ be a standard cut-off function such that $\varphi|_{[0,1/2]} = 1$ and $\varphi|_{[2/3,1]} = 0$. We also define $\varphi(x) = \varphi(r(x)/R)$. Then consider the function $G := \varphi^2 \cdot (-\Delta_1 f)$. We calculate

$$\begin{aligned}\Delta G &= \Delta(\varphi^2(-\Delta_1 f))\\ &= \varphi^2 \Delta(-\Delta_1 f) + 2\,\mathrm{Re}\langle \nabla\varphi^2, \bar{\nabla}(-\Delta_1 f)\rangle + (-\Delta_1 f)\Delta\varphi^2\\ &\geq \varphi^2((\Delta_1 f)^2 + 2\,\mathrm{Re}\langle \nabla f, \bar{\nabla}\Delta_1 f\rangle) + 2\,\mathrm{Re}\langle \nabla\varphi^2, \bar{\nabla}(-\Delta_1 f)\rangle + (-\Delta_1 f)\Delta\varphi^2. \end{aligned} \tag{3-12}$$

We want to estimate the upper bound of G. If the maximum value of $G = \varphi^2(-\Delta_1 f)$ is negative, we are done. So we assume the maximum of G on $B(p, R)$ is *positive*, which is achieved at some point $p_{\max} \in B\big(p, \tfrac{2}{3}R\big)$. Hence, at $p_{\max}$ we have $(-\Delta_1 f) > 0$. By Laplacian comparison, $\Delta r \leq (2n-1)/r$, and we get, at $p_{\max}$,

$$\Delta\varphi^2 \geq \frac{2}{R^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2). \tag{3-13}$$

Thus, at $p_{\max}$, the last term on the right-hand side of (3-12) is greater than or equal to

$$(-\Delta_1 f)\frac{2}{R^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2).$$

Substituting this into (3-12), it follows that, at $p_{\max}$, we have $\Delta G \leq 0$ and $\nabla\Delta_1 f = -2\varphi^{-1}\Delta_1 f \nabla\varphi$, and hence

$$\begin{aligned} 0 &\geq \Delta G\\ &\geq \varphi^2(\Delta_1 f)^2 + 2\varphi^2\,\mathrm{Re}\langle\nabla f, \bar{\nabla}\Delta_1 f\rangle + 4\varphi\,\mathrm{Re}\langle\nabla\varphi, \bar{\nabla}(-\Delta_1 f)\rangle + (-\Delta_1 f)\frac{2}{R^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2)\\ &\geq \varphi^2(\Delta_1 f)^2 - 4\varphi|\Delta_1 f||\nabla f||\nabla\varphi| + 8\Delta_1 f|\nabla\varphi|^2 + (-\Delta_1 f)\frac{2}{R^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2)\\ &= \frac{G^2}{\varphi^2} - 4\varphi^{-1}G|\nabla f||\nabla\varphi| - 8G\frac{|\nabla\varphi|^2}{\varphi^2} + \frac{2G}{R^2\varphi^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2)\\ &\geq \frac{G^2}{\varphi^2} - 4\frac{|\varphi'||\nabla f|}{R\varphi}G - 8\frac{|\varphi'|^2}{R^2\varphi^2}G + \frac{2G}{R^2\varphi^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2). \end{aligned} \tag{3-14}$$

Therefore, at $p_{\max} \in B\big(p, \tfrac{2}{3}R\big)$,

$$G^2 - 4\frac{\varphi|\varphi'\nabla f|}{R}G - 8\frac{|\varphi'|^2}{R^2}G + \frac{2G}{R^2}((2n-1)\varphi\varphi' + \varphi\varphi'' + (\varphi')^2) \leq 0,$$

and combining (3-8) and the fact that φ, φ', φ'' are all uniformly bounded, we can get, at $p_{\max}$,

$$G^2 \leq C(n)R^{-2}G \quad \Longrightarrow \quad G(p_{\max}) \leq \frac{C(n)}{R^2}.$$

Then, for any $x \in B\big(p, \tfrac{1}{2}R\big)$, where $\varphi = 1$, we have

$$-\Delta_1 f(x) = G(x) \leq G(p_{\max}) \leq \frac{C(n)}{R^2}.$$

Moreover, recall that $f = \log u$ and $-\Delta_1 f = -\Delta_1 u/u + |\nabla_1 f|^2$, therefore it follows that

$$\sup_{x\in B(p,R/2)} \Big(-\frac{\Delta_1 u}{u}(x)\Big) \le \frac{C(n)}{R^2}. \tag{3-15}$$

This in particular implies that

$$\sup_{x\in B(p,R/2)} (-\Delta_1 u(x)) \le C(n)\frac{\operatorname{osc}_{B(p,R/2)} u}{R^2} \le C(n)\frac{\operatorname{osc}_{B(p,R)} u}{R^2}. \tag{3-16}$$

On the other hand, consider the function $\hat{u} = \max_{B(p,R)} u - u$, which is still a positive g_ϵ-harmonic function with $\Delta_g \hat{u} = \Delta_{g_\epsilon}\hat{u} = 0$. Applying (3-15) to the function $\hat{u}$, we get

$$\sup_{x\in B(p,R/2)} \Big(\frac{\Delta_1 u(x)}{\max_{B(p,R)} u - u(x)}\Big) = \sup_{x\in B(p,R/2)} \Big(-\frac{\Delta_1 \hat{u}}{\hat{u}}(x)\Big) \le \frac{C(n)}{R^2}, \tag{3-17}$$

which yields

$$\sup_{x\in B(p,R/2)} \Delta_1 u(x) \le C(n)\frac{\operatorname{osc}_{B(p,R)} u}{R^2}. \tag{3-18}$$

Combining (3-18) and (3-16), we get

$$\sup_{x\in B(p,R/2)} |\Delta_1 u|(x) \le C(n)\frac{\operatorname{osc}_{B(p,R)} u}{R^2}. \tag{3-19}$$

3A4. *Mixed derivatives estimates.* In this subsection we will estimate the mixed derivatives

$$|\nabla_1\nabla_2 f|^2 = \frac{\partial^2 f}{\partial z_1\partial z_2}\overline{\frac{\partial^2 f}{\partial z_1\partial z_2}} g^{1\bar{1}}g^{2\bar{2}} \quad\text{and}\quad |\nabla_1\nabla_{\bar{2}} f|^2 = \frac{\partial^2 f}{\partial z_1\partial \bar{z}_2}\overline{\frac{\partial^2 f}{\partial z_1\partial \bar{z}_2}} g^{1\bar{1}}g^{2\bar{2}},$$

where as before $f = \log u$ and u is a positive harmonic function of Δ_{g_ϵ}. Here for simplicity, we omit the subscript ϵ in u_ϵ, f_ϵ and g_ϵ. Observing that since $g_\epsilon = g$ is a product metric with the nonzero components $g_{k\bar{k}}$ depending only on z_k, it follows that the curvature tensor

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l} + g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}$$

vanishes unless $i = j = k = l \in \{1, 2\}$ and also $R_{i\bar{i}i\bar{i}} \ge 0$ for all $i = 1, \dots, n$.

We fix some notation: we will write $f_{12} = \nabla_1\nabla_2 f$ (in fact this is just the ordinary derivative of f with respect to g, since g is a product metric), $|f_{12}|_g^2 = |\nabla_1\nabla_2 f|_g^2$, etc.

Let us first recall that (3-11) implies

$$\begin{aligned}
&\Delta(-\Delta_1 f - \Delta_2 f)\\
&= \sum_{k=1}^n (g^{1\bar{1}}g^{k\bar{k}} f_{1k} f_{\bar{1}\bar{k}} + g^{1\bar{1}}g^{k\bar{k}} f_{1\bar{k}} f_{\bar{1}k} + g^{2\bar{2}}g^{k\bar{k}} f_{2k} f_{\bar{2}\bar{k}} + g^{2\bar{2}}g^{k\bar{k}} f_{2\bar{k}} f_{k\bar{2}})\\
&\qquad - 2\operatorname{Re}\langle\nabla f, \overline{\nabla}(-\Delta_1 f - \Delta_2 f)\rangle + f_1 f_{\bar{1}} g^{1\bar{1}}g^{1\bar{1}}R_{1\bar{1}1\bar{1}} + f_2 f_{\bar{2}} g^{2\bar{2}}g^{2\bar{2}}R_{2\bar{2}2\bar{2}}\\
&\ge \sum_{k=1}^n (|\nabla_1\nabla_k f|^2 + |\nabla_1\nabla_{\bar{k}} f|^2 + |\nabla_2\nabla_k f|^2 + |\nabla_2\nabla_{\bar{k}} f|^2) - 2\operatorname{Re}\langle\nabla f, \overline{\nabla}(-\Delta_1 f - \Delta_2 f)\rangle.
\end{aligned} \tag{3-20}$$

Next we calculate $\Delta|\nabla_1\nabla_2 f|^2$. For convenience of notation we will write $f^{12} = f_{\bar 1\bar 2}g^{1\bar 1}g^{2\bar 2}$, and hence $|\nabla_1\nabla_2 f|^2 = f_{12}f^{12}$. We calculate

$$\begin{aligned}\Delta|\nabla_1\nabla_2 f|^2 &= g^{k\bar l}(f_{12}f^{12})_{k\bar l} = g^{k\bar k}(f_{12}f^{12})_{k\bar k} \qquad \text{(since } g \text{ is a product metric)}\\ &= g^{k\bar k}(f_{12k\bar k}f^{12} + f_{12k}f^{12}{}_{,\bar k} + f_{12\bar k}f^{12}{}_{,k} + f_{12}f^{12}{}_{,k\bar k}).\end{aligned} \tag{3-21}$$

The first term on the right-hand side of (3-21) is (by Ricci identities and switching the indices)

$$\begin{aligned}&g^{k\bar k}f^{12}(f_{k1\bar k2} + g^{m\bar m}f_{m1}R_{k\bar m2\bar k} + g^{m\bar m}f_{km}R_{1\bar m2\bar k})\\ &\quad= g^{k\bar k}f^{12}(f_{k\bar k12} + g^{m\bar m}f_{m2}R_{k\bar m1\bar k} + g^{m\bar m}f_{m}R_{k\bar m1\bar k} + g^{m\bar m}f_{m1}R_{k\bar m2\bar k} + g^{m\bar m}f_{km}R_{1\bar m2\bar k})\\ &\quad= g^{k\bar k}f^{12}(f_{k\bar k12} + g^{m\bar m}f_{m2}R_{k\bar m1\bar k} + g^{m\bar m}f_{m1}R_{k\bar m2\bar k})\\ &\quad= g^{k\bar k}f^{12}f_{k\bar k12} + g^{1\bar 1}g^{1\bar 1}f^{12}f_{21}R_{1\bar 11\bar 1} + g^{2\bar 2}g^{2\bar 2}f^{12}f_{12}R_{2\bar 22\bar 2},\end{aligned} \tag{3-22}$$

and the last term on the right-hand side of (3-21) is the conjugate of the first term; hence we get

$$\begin{aligned}\Delta|\nabla_1\nabla_2 f|^2 = 2\,\mathrm{Re}(f^{12}(\Delta f)_{12}) &+ 2f^{12}f_{12}(g^{1\bar 1}g^{1\bar 1}R_{1\bar 11\bar 1} + g^{2\bar 2}g^{2\bar 2}R_{2\bar 22\bar 2})\\ &+ g^{k\bar k}f_{12k}f^{12}{}_{,\bar k} + g^{k\bar k}f_{12\bar k}f^{12}{}_{,k}.\end{aligned} \tag{3-23}$$

Recall from (3-6) that $\Delta f = -|\nabla f|^2$; hence the first term on the right-hand side of (3-23) is

$$\begin{aligned}2\,\mathrm{Re}(f^{12}(\Delta f)_{12}) &= 2\,\mathrm{Re}(f^{12}(-|\nabla f|^2)_{12})\\ &= -2\,\mathrm{Re}(f^{12}g^{k\bar k}(f_{k12}f_{\bar k} + f_{k1}f_{\bar k2} + f_{k2}f_{\bar k1} + f_k f_{\bar k12}))\\ &= -2\,\mathrm{Re}(f^{12}g^{k\bar k}(f_{12k}f_{\bar k} + f_{k1}f_{\bar k2} + f_{k2}f_{\bar k1} + f_k f_{12\bar k} - f_k f_m R_{1\bar m2k}g^{m\bar m}))\\ &= -4\,\mathrm{Re}\langle\nabla f, \bar\nabla|\nabla_1\nabla_2 f|^2\rangle - 2\,\mathrm{Re}(f^{12}g^{k\bar k}f_{k1}f_{2\bar k} + f^{12}g^{k\bar k}f_{k2}f_{\bar k1}).\end{aligned} \tag{3-24}$$

Combining (3-24) and (3-23), we get

$$\begin{aligned}\Delta|\nabla_1\nabla_2 f|^2 \ge -4\,\mathrm{Re}\langle\nabla f, \bar\nabla|\nabla_1\nabla_2 f|^2\rangle &+ \sum_k (f_{12k}f^{12k} + f_{12\bar k}f^{12\bar k})\\ &- 2\sum_k (|\nabla_1\nabla_2 f||\nabla_1\nabla_k f||\nabla_2\nabla_{\bar k} f| + |\nabla_1\nabla_2 f||\nabla_2\nabla_k f||\nabla_1\nabla_{\bar k} f|).\end{aligned} \tag{3-25}$$

On the other hand, by Kato's inequality we have

$$\begin{aligned}\Delta|\nabla_1\nabla_2 f|^2 &= 2|\nabla_1\nabla_2 f|\Delta|\nabla_1\nabla_2 f| + 2\big|\nabla|\nabla_1\nabla_2 f|\big|^2\\ &\le 2|\nabla_1\nabla_2 f|\Delta|\nabla_1\nabla_2 f| + \sum_k |\nabla_k\nabla_1\nabla_2 f|^2 + |\nabla_{\bar k}\nabla_1\nabla_2 f|^2\\ &= 2|\nabla_1\nabla_2 f|\Delta|\nabla_1\nabla_2 f| + \sum_k f_{12k}f^{12k} + f_{12\bar k}f^{12\bar k}.\end{aligned} \tag{3-26}$$

Combining (3-25) and (3-26), it follows that

$$\Delta|\nabla_1\nabla_2 f| \ge -2\,\mathrm{Re}\langle\nabla f, \bar\nabla|\nabla_1\nabla_2 f|\rangle - \sum_k (|\nabla_1\nabla_k f||\nabla_2\nabla_{\bar k} f| + |\nabla_2\nabla_k f||\nabla_1\nabla_{\bar k} f|). \tag{3-27}$$

Combining (3-20) and (3-27) and applying the Cauchy–Schwarz inequality, we have

$$\Delta(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) \geq -2\,\mathrm{Re}\langle \nabla f, \overline{\nabla}(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))\rangle + \sum_{k=1}^{n}(|\nabla_1\nabla_k f|^2 + |\nabla_1\nabla_{\bar{k}} f|^2 + |\nabla_2\nabla_k f|^2 + |\nabla_2\nabla_{\bar{k}} f|^2). \quad (3\text{-}28)$$

Note that the sum on the right-hand side of (3-27) is (recall under our notation $|\nabla_1\nabla_{\bar{1}} f|^2 = (\Delta_1 f)^2$) greater than or equal to

$$|\nabla_1\nabla_2 f|^2 + |-\Delta_1 f|^2 + |-\Delta_2 f|^2 \geq \tfrac{1}{12}(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))^2,$$

so we get the equation

$$\Delta(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) \geq -2\,\mathrm{Re}\langle \nabla f, \overline{\nabla}(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))\rangle + \tfrac{1}{12}(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))^2. \quad (3\text{-}29)$$

Write

$$Q = \eta^2(|\nabla_1\nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) =: \eta^2 Q_1,$$

where $\eta(x) = \tilde{\eta}(r(x)/R)$ and $\tilde{\eta}$ is a cut-off function such that $\tilde{\eta}|_{[0,1/3]} = 1$ and $\tilde{\eta}|_{[1/2,1]} = 0$. The following arguments are similar to the previous two cases. We calculate

$$\begin{aligned}\Delta Q &= \eta^2\Delta Q_1 + 2\,\mathrm{Re}\langle \nabla\eta^2, \nabla Q_1\rangle + Q_1\Delta\eta^2\\ &\geq -2\eta^2\,\mathrm{Re}\langle \nabla f, \overline{\nabla} Q_1\rangle + 2\,\mathrm{Re}\langle \nabla\eta^2, \nabla Q_1\rangle + \tfrac{1}{12}\eta^2 Q_1^2 + Q_1\Delta\eta^2. \end{aligned} \quad (3\text{-}30)$$

Apply the maximum principle to Q, and if $\max Q \leq 0$, we are done. So we may assume that $\max Q > 0$ and that it is attained at $p_{\max}$; thus at $p_{\max}$, we have $Q_1 > 0$, $\Delta Q \leq 0$, $\nabla Q_1 = -2\eta^{-1} Q_1 \nabla\eta$ and

$$Q_1\Delta\eta^2 \geq Q_1\frac{2}{R^2}((2n-1)\eta\eta' + \eta\eta'' + (\eta')^2).$$

So, at $p_{\max}$,

$$\begin{aligned}0 \geq \Delta Q &\geq 4\eta Q_1\,\mathrm{Re}\langle \nabla f, \overline{\nabla}\eta\rangle - 8Q_1|\nabla\eta|^2 + \eta^2\frac{Q_1^2}{12} + Q_1\frac{2}{R^2}((2n-1)\eta\eta' + \eta\eta'' + (\eta')^2)\\ &= \frac{Q^2}{12\eta^2} + 4Q\eta^{-1}\,\mathrm{Re}\langle \nabla f, \overline{\nabla}\eta\rangle - \frac{8Q}{\eta^2}\frac{(\eta')^2}{R^2} + \frac{2Q}{R^2\eta^2}((2n-1)\eta\eta' + \eta\eta'' + (\eta')^2)\\ &\geq \frac{1}{\eta^2}\left(\frac{Q^2}{12} - \frac{40|\nabla f|}{R}Q - \frac{800}{R^2}Q - \frac{100n}{R^2}Q\right), \end{aligned} \quad (3\text{-}31)$$

where we choose η such that $|\eta'|, |\eta''| \leq 10$, for example. Therefore, at $p_{\max} \in B\big(p, \tfrac{1}{2}R\big)$, we have

$$\frac{Q^2}{12} - Q\left(\frac{40|\nabla f|}{R} + \frac{800}{R^2} + \frac{100n}{R^2}\right) \leq 0 \quad\Longrightarrow\quad Q(p_{\max}) \leq \frac{C(n)}{R^2},$$

since $\sup_{B(p,R/2)} |\nabla f| \leq C(n)R^{-1}$ from the previous estimates. Then, for any $x \in B\big(p, \tfrac{1}{3}R\big)$, we have

$$Q_1(x) = \eta^2(x) Q_1(x) = Q(x) \leq Q(p_{\max}) \leq \frac{C(n)}{R^2}.$$

Thus it follows that

$$|\nabla_1\nabla_2 f|(x) \le Q_1(x) + 2(\Delta_1 f(x) + \Delta_2 f(x)) \le \frac{C(n)}{R^2} + 2(\Delta_1 f(x) + \Delta_2 f(x)).$$

On the other hand, from $|\nabla_1\nabla_2 f| = |(\nabla_1\nabla_2 u/u) - (\nabla_1 u/u)(\nabla_2 u/u)|$, we get

$$\begin{aligned}|\nabla_1\nabla_2 u|(x) &\le |\nabla_1\nabla_2 f(x)|u(x) + u(x)\frac{|\nabla_1 u(x)|}{u}\frac{|\nabla_2 u(x)|}{u}\\ &\le C(n)\frac{u(x)}{R^2} + 2\Delta_1 u(x) + 2\Delta_2 u(x) + u(x)\frac{|\nabla_1 u(x)|}{u}\frac{|\nabla_2 u(x)|}{u}\\ &\le C(n)\frac{\operatorname{osc}_{B(p,R)} u}{R^2}.\end{aligned} \tag{3-32}$$

Therefore we obtain

$$\sup_{B(p,R/3)} |\nabla_1\nabla_2 u| \le C(n)\frac{\operatorname{osc}_{B(p,R)} u}{R^2}. \tag{3-33}$$

By exactly the same argument we get similar estimates for $|\nabla_1\nabla_{\bar 2} u|$ and $|\nabla_1\nabla_k u| + |\nabla_1\nabla_{\bar k} u|$ for $k \ne 1$.

Hence we have proved the following lemma.

Lemma 3.4. *There exists a constant $C(n) > 0$ such that, for the solution u_ϵ to* (3-4),

$$\sup_{B_{g_\epsilon}(0,R/2)} (|\nabla_i\nabla_j u_\epsilon|_{g_\epsilon} + |\nabla_i\nabla_{\bar j} u_\epsilon|_{g_\epsilon}) \le C(n)\frac{\operatorname{osc}_{B_{g_\epsilon}(0,R)} u_\epsilon}{R^2}$$

for all $i, j = 1, 2, \dots, n$.

3A5. *Convergence of u_ϵ.* In this subsection, we will show that the Dirichlet problem (3-1) admits a unique solution for any $\varphi \in C^0(\partial B_{\boldsymbol\beta}(0, 1))$. Here we will write $B_{\boldsymbol\beta} = B_{\boldsymbol\beta}(0, 1)$ for simplicity of notation.

Proposition 3.5. *For any $\varphi \in C^0(\partial B_{\boldsymbol\beta})$, the Dirichlet boundary value problem* (3-1) *admits a unique solution $u \in C^2(B_{\boldsymbol\beta}\setminus\mathcal{S}) \cap C^0(\overline{B_{\boldsymbol\beta}})$. Moreover, u satisfies the estimates in Lemmas 3.2–3.4 with u_ϵ replaced by u and the metric balls replaced by those under the metric $g_{\boldsymbol\beta}$, which we will refer to as "derivatives estimates" throughout this section.*

Proof. Given the estimates of u_ϵ as in Lemmas 3.2–3.4, we can derive the uniform local $C^{2,\alpha}$ estimates of u_ϵ on any compact subsets of $B_{\boldsymbol\beta}(0, 1)\setminus\mathcal{S}$.

The C^0 estimates of u_ϵ follow immediately from the maximum principle (see Lemma 3.1).

Take any compact subsets $K \Subset K' \Subset B_{\boldsymbol\beta}(0, 1)$. By Lemmas 3.2 and 3.3, we have

$$\sup_{K'}\left(|z_1|^{1-\beta_1}\left|\frac{\partial u_\epsilon}{\partial z_1}\right| + |z_2|^{1-\beta_2}\left|\frac{\partial u_\epsilon}{\partial z_2}\right| + \left|\frac{\partial u_\epsilon}{\partial s_j}\right|\right) \le C(n)\frac{\|u_\epsilon\|_\infty}{d(K', \partial B_{\boldsymbol\beta})}, \tag{3-34}$$

$$\sup_{K'}\left(|z_1|^{1-\beta_1}\left|\frac{\partial^2 u_\epsilon}{\partial s_k\partial z_1}\right| + |z_2|^{1-\beta_2}\left|\frac{\partial^2 u_\epsilon}{\partial s_k\partial z_2}\right| + \left|\frac{\partial^2 u_\epsilon}{\partial s_k\partial s_j}\right|\right) \le C(n)\frac{\|u_\epsilon\|_\infty}{d(K', \partial B_{\boldsymbol\beta})^2}, \tag{3-35}$$

and the third-order estimates

$$\sup_{K'}\left(|z_1|^{1-\beta_1}\left|\frac{\partial^3 u_\epsilon}{\partial z_1\partial s_k\partial s_l}\right| + |z_2|^{1-\beta_2}\left|\frac{\partial^3 u_\epsilon}{\partial z_2\partial s_k\partial s_l}\right| + \left|\frac{\partial^3 u_\epsilon}{\partial s_j\partial s_k\partial s_l}\right|\right) \le C(n)\frac{\|u_\epsilon\|_\infty}{d(K', \partial B_{\boldsymbol\beta})^3}. \tag{3-36}$$

Moreover, applying the gradient estimate to the Δ_{g_ϵ}-harmonic function $\Delta_{1,\epsilon}u_\epsilon$, we get

$$\sup_{K'}\left(|z_1|^{1-\beta_1}\left|\frac{\partial}{\partial z_1}\Delta_{1,\epsilon}u_\epsilon\right|+|z_2|^{1-\beta_2}\left|\frac{\partial}{\partial z_2}\Delta_{1,\epsilon}u_\epsilon\right|+\left|\frac{\partial}{\partial s_j}\Delta_{1,\epsilon}u_\epsilon\right|\right)\le C(n)\frac{\|u_\epsilon\|_\infty}{d(K',\partial B_{\boldsymbol\beta})^3}.$$

From (3-34)–(3-36), we see that the functions u_ϵ have uniform C^3-estimates in the "tangential directions" on any compact subset of $B_{\boldsymbol\beta}(0,1)$. Moreover, for any fixed small constant $\delta>0$, let $T_\delta(\mathcal{S})$ be the tubular neighborhood of $\mathcal{S}$. We consider the equation

$$\Delta_\epsilon u_\epsilon=(|z_1|^2+\epsilon)^{1-\beta_1}\frac{\partial^2 u_\epsilon}{\partial z_1\partial\bar z_1}+(|z_2|^2+\epsilon)^{1-\beta_2}\frac{\partial^2 u_\epsilon}{\partial z_2\partial\bar z_2}+\sum_{j=5}^{2n}\frac{\partial^2 u_\epsilon}{\partial s_j^2}=0\quad\text{on } K'\backslash T_{\delta/2}(\mathcal{S}),$$

which is strictly elliptic (with ellipticity depending only on $\delta>0$). Hence by standard elliptic Schauder theory, we also have $C^{2,\alpha}$-estimates of u_ϵ in the "transversal directions" (i.e., normal to $\mathcal{S}$) and the mixed directions on the compact subset $K\backslash T_\delta(\mathcal{S})$. By taking $\delta\to 0$ and $K\to B_{\boldsymbol\beta}$, and using a diagonal argument, up to a subsequence, the u_ϵ converge in $C^{2,\alpha}_{\text{loc}}(B_{\boldsymbol\beta}\backslash\mathcal{S})$ to a function $u\in C^{2,\alpha}(B_{\boldsymbol\beta}\backslash\mathcal{S})$. Clearly, u satisfies the equation $\Delta_{\boldsymbol\beta}u=0$ on $B_{\boldsymbol\beta}\backslash\mathcal{S}$, and the estimates (3-34)–(3-36) hold for u outside $\mathcal{S}$, which implies that u can be continuously extended through $\mathcal{S}$ and defines a continuous function in $B_{\boldsymbol\beta}(0,1)$. It remains to check the boundary value of u.

Claim: $u=\varphi$ on $\partial B_{\boldsymbol\beta}(0,1)$. It remains to show the limit function u of u_ϵ satisfies the boundary condition $u=\varphi$ on $\partial B_{\boldsymbol\beta}(0,1)$, which will be proved by constructing suitable barriers as we did in [20].

The metric ball $B_{\boldsymbol\beta}(0,1)$ is given by

$$B_{\boldsymbol\beta}(0,1)=\left\{z\in\mathbb{C}^n\;\middle|\;d_{\boldsymbol\beta}(0,z)^2:=|z_1|^{2\beta_1}+|z_2|^{2\beta_2}+\sum_{j=5}^{2n}s_j^2<1\right\}.$$

$B_{\boldsymbol\beta}(0,1)\subset B_{\mathbb{C}^n}(0,1)$, and their boundaries only intersect at $\mathcal{S}_1\cap\mathcal{S}_2$, where $z_1=z_2=0$. Fix any point $q\in\partial B_{\boldsymbol\beta}(0,1)$ and consider the cases $q\in\mathcal{S}_1\cap\mathcal{S}_2$ and $q\notin\mathcal{S}_1\cap\mathcal{S}_2$.

Case 1: $q\in\mathcal{S}_1\cap\mathcal{S}_2$, i.e., $z_1(q)=z_2(q)=0$. Consider the point

$$q'=-q\in\partial B_{\boldsymbol\beta}(0,1)\cap\partial B_{\mathbb{C}^n}(0,1).$$

Since q is the *unique* farthest point from q' on $\partial B_{\boldsymbol\beta}(0,1)$ under the Euclidean distance, the function $\Psi_q(z):=d_{\mathbb{C}^n}(z,q')^2-4$ satisfies $\Psi_q(q)=0$ and $\Psi_q(z)<0$ for all $z\in\partial B_{\boldsymbol\beta}(0,1)\backslash\{q\}$. By the continuity of φ for any $\delta>0$, there is a small neighborhood V of q such that $\varphi(q)-\delta<\varphi(z)<\varphi(q)+\delta$ for all $z\in\partial B_{\boldsymbol\beta}(0,1)\cap V$, and, on $\partial B_{\boldsymbol\beta}(0,1)\backslash V$, we have that Ψ_q is bounded above by a negative constant. Hence we can define

$$\varphi_q(z):=\varphi(q)-\delta+A\Psi_q(z)<\varphi(z)$$

for all $z\in\partial B_{\boldsymbol\beta}(0,1)$ if A is chosen large enough. The function φ_q is Δ_{g_ϵ}-subharmonic; hence by the maximum principle we have $u_\epsilon(z)\ge\varphi_q(z)$ for all $z\in B_{\boldsymbol\beta}(0,1)$. Letting $\epsilon\to 0$ we get $u(z)\ge\varphi_q(z)$, taking $z\to q$ we have $\liminf_{z\to q}u(z)\ge\varphi(q)-\delta$, and since $\delta>0$ is arbitrary we have $\liminf_{z\to q}u(z)\ge\varphi(q)$.

By considering the barrier function $\varphi(q)+\delta-A\Psi_q(z)$ and using a similar argument it is not hard to see that $\limsup_{z\to q}u(z)\le\varphi(q)$; hence $\lim_{z\to q}u(z)=\varphi(q)$ and u is continuous up to $q\in\partial B_{\boldsymbol\beta}(0,1)$.

Case 2: $q \in \partial B_{\boldsymbol{\beta}}(0,1)\backslash \mathcal{S}_1 \cap \mathcal{S}_2$. We first consider the case when $z_1(q) \neq 0$ and $z_2(q) \neq 0$. The boundary $\partial B_{\boldsymbol{\beta}}(0,1)$ is smooth near q, and hence satisfies the exterior sphere condition. We choose an exterior Euclidean ball $B_{\mathbb{C}^n}(\tilde{q}, r_q)$ which is tangential to $\partial B_{\boldsymbol{\beta}}(0,1)$ (only) at q, i.e., under the Euclidean distance, q is the unique closest point to $\tilde{q}$ on $\partial B_{\boldsymbol{\beta}}(0,1)$. So the function

$$G(z) = \frac{1}{|z-\tilde{q}|^{2n-2}} - \frac{1}{r_q^{2n-2}} \tag{3-37}$$

satisfies $G(q) = 0$ and $G(z) < 0$ for all $z \in \partial B_{\boldsymbol{\beta}}(0,1)\backslash\{q\}$. We calculate

$$\begin{aligned}
\Delta_{g_\epsilon} G &= (|z_1|^2+\epsilon)^{-\beta_1+1}\frac{\partial^2 G}{\partial z_1 \partial \bar{z}_1} + (|z_2|^2+\epsilon)^{-\beta_2+1}\frac{\partial^2 G}{\partial z_2 \partial \bar{z}_2} + \sum_{k=3}^{n}\frac{\partial^2 G}{\partial z_k \partial \bar{z}_k} \\
&= ((|z_1|^2+\epsilon)^{-\beta_1+1}-1)\frac{\partial^2 G}{\partial z_1 \partial \bar{z}_1} + ((|z_2|^2+\epsilon)^{-\beta_2+1}-1)\frac{\partial^2 G}{\partial z_2 \partial \bar{z}_2} \\
&= \sum_{k=1}^{2}(-n+1)\frac{(|z_k|^2+\epsilon)^{-\beta_k+1}-1}{|z-\tilde{q}|^{2n}}\left(-\frac{n|z_k-\tilde{q}_k|^2}{|z-\tilde{q}|^2}+1\right) \geq -C(q, r_q).
\end{aligned}$$

The function

$$\Psi_q(z) = A(d_{\boldsymbol{\beta}}(z,0)^2-1) + G(z)$$

is Δ_{g_ϵ}-subharmonic for $A \gg 1$, and $\Psi_q(q) = 0$ and $\Psi_q(z) < 0$ for all $z \in \partial B_{\boldsymbol{\beta}}(0,1)\backslash\{q\}$. We are in the same situation as Case 1, so by the same argument as above, we can show the continuity of u at such a boundary point q.

In the case when $z_1(q) \neq 0$ and $z_2(q) = 0$, the boundary $\partial B_{\boldsymbol{\beta}}(0,1)$ is not smooth at q and we cannot apply the exterior sphere condition to construct the barrier. Instead we use the geometry of the metric ball $B_{\boldsymbol{\beta}}(0,1)$. Consider the standard cone metric

$$g_{\beta_1} = \beta_1^2 \frac{dz_1 \otimes d\bar{z}_1}{|z_1|^{2(1-\beta_1)}} + \sum_{k=2}^{n} dz_k \otimes d\bar{z}_k$$

with cone singularity only along $\mathcal{S}_1 = \{z_1 = 0\}$. We observe that the metric ball $B_{\boldsymbol{\beta}}(0,1)$ is strictly contained in $B_{g_{\beta_1}}(0,1)$, and the boundaries of these balls are tangential at the points with vanishing z_2-coordinate. Thus $q \in \partial B_{\boldsymbol{\beta}}(0,1) \cap \partial B_{g_{\beta_1}}(0,1)$ and $\partial B_{g_{\beta_1}}(0,1)$ is smooth at q, so there exists an exterior sphere for $\partial B_{g_{\beta_1}}(0,1)$ at q. We define a similar function $G(z)$ as in (3-37), and, by the strict inclusion of the metric balls $B_{\boldsymbol{\beta}}(0,1) \subset B_{g_{\beta_1}}(0,1)$, it follows that $G(q) = 0$ and $G(z) < 0$ for all $z \in \partial B_{\boldsymbol{\beta}}(0,1)\backslash\{q\}$. The remaining argument is the same as before. □

Remark 3.6. For any constant $c \in \mathbb{R}$, the Dirichlet boundary value problem

$$\Delta_{g_{\boldsymbol{\beta}}} u = c \quad \text{in } B_{\boldsymbol{\beta}}(0,1)\backslash\mathcal{S} \qquad \text{and} \qquad u = \varphi \quad \text{on } \partial B_{\boldsymbol{\beta}}(0,1)$$

admits a solution $u \in C^2(B_{\boldsymbol{\beta}}\backslash\mathcal{S}) \cap C^0(\overline{B_{\boldsymbol{\beta}}})$ for any given $\varphi \in C^0(\partial B_{\boldsymbol{\beta}})$. This follows from the solution $\tilde{u}$ of (3-1) with boundary value $\tilde{\varphi} = \varphi - \frac{1}{2}c(n-2)^{-1}\sum_{j=5}^{2n} s_j^2$. Then the function $u = \tilde{u} + \frac{1}{2}c(n-2)^{-1}\sum_j s_j^2$ solves the equation above.

For later application, we prove the existence of solutions for a more general right-hand side of the Laplace equation with the standard background metric. This result is not needed to prove Theorem 1.2.

Proposition 3.7. *For any given $\varphi \in C^0(\partial B_{\boldsymbol\beta}(0,1))$ and $f \in C^{0,\alpha}_{\boldsymbol\beta}(\overline{B_{\boldsymbol\beta}(0,1)})$, the Dirichlet boundary value problem*

$$\begin{cases} \Delta_{g_{\boldsymbol\beta}} v = f & \text{in } B_{\boldsymbol\beta}(0,1)\backslash\mathcal{S}, \\ v = \varphi & \text{on } \partial B_{\boldsymbol\beta}(0,1) \end{cases} \tag{3-38}$$

admits a unique solution $v \in C^2(B_{\boldsymbol\beta}(0,1)\backslash\mathcal{S}) \cap C^0(\overline{B_{\boldsymbol\beta}(0,1)})$.

By Theorem 1.2, the solution v to (3-38) belongs to $C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1)) \cap C^0(\overline{B_{\boldsymbol\beta}(0,1)})$.

Proof. The proof is similar to that of Proposition 3.5. As before, let g_ϵ be the approximating metrics (3-3) of $g_{\boldsymbol\beta}$ which are smooth metrics on $B_{\boldsymbol\beta}(0,1)$. By standard elliptic theory we can solve the equations

$$\begin{cases} \Delta_{g_\epsilon} v_\epsilon = f & \text{in } B_{\boldsymbol\beta}(0,1), \\ v_\epsilon = \varphi & \text{on } \partial B_{\boldsymbol\beta}(0,1). \end{cases} \tag{3-39}$$

For any compact subset $K \Subset B_{\boldsymbol\beta}(0,1)$ and small $\delta > 0$, we have a uniform $C^{2,\alpha'}$-bound of v_ϵ on $K\backslash T_\delta(\mathcal{S})$ for some $\alpha' < \alpha$. Thus v_ϵ converges in the $C^{2,\alpha'}$-norm to a function v on $K\backslash T_\delta(\mathcal{S})$ as $\epsilon \to 0$. By a standard diagonal argument, letting $K \to B_{\boldsymbol\beta}(0,1)$ and $\delta \to 0$, we can achieve

$$v_\epsilon \xrightarrow{C^{2,\alpha'}_{\mathrm{loc}}(B_{\boldsymbol\beta}(0,1)\backslash\mathcal{S})} v \in C^{2,\alpha'}_{\mathrm{loc}}(B_{\boldsymbol\beta}(0,1)\backslash\mathcal{S}) \quad \text{as } \epsilon \to 0.$$

Clearly v satisfies (3-38) in $B_{\boldsymbol\beta}(0,1)\backslash\mathcal{S}$. It only remains to show the boundary value of v coincides with φ and v is globally continuous in $B_{\boldsymbol\beta}(0,1)$.

Global continuity: $v \in C^0(B_{\boldsymbol\beta}(0,1))$. It suffices to show v is continuous at any $p \in \mathcal{S} \cap B_{\boldsymbol\beta}(0,1)$. Fix such a point p and take $R_0 > 0$ small enough that $B_{\mathbb{C}^n}(p, 10R_0) \cap \partial B_{\boldsymbol\beta}(0,1) = \varnothing$. We observe that $\frac{1}{2} g_{\mathbb{C}^n} \le g_\epsilon \le g_{\boldsymbol\beta}$, so for any $r \in \left(0, \frac{1}{2}\right)$,

$$B_{g_{\boldsymbol\beta}}(p,r) \subset B_{g_\epsilon}(p,r) \subset B_{\mathbb{C}^n}(p,2r). \tag{3-40}$$

In particular, the balls $B_{g_\epsilon}(p, 5R_0)$ are also disjoint with $\partial B_{\boldsymbol\beta}(0,1)$.

Since $\mathrm{Ric}(g_\epsilon) \ge 0$, we have the following Sobolev inequality [25]: there exists a constant $C = C(n) > 0$ such that, for any $h \in C^1_0(B_{g_\epsilon}(p,r))$,

$$\left(\int_{B_{g_\epsilon}(p,r)} h^{2n/(n-1)} \omega_\epsilon^n \right)^{(n-1)/n} \le C \left(\frac{r^{2n}}{\mathrm{Vol}_{g_\epsilon}(B_{g_\epsilon}(p,r))} \right)^{1/n} \int_{B_{g_\epsilon}(p,r)} |\nabla h|^2_{g_\epsilon} \omega_\epsilon^n. \tag{3-41}$$

It can be checked by straightforward calculations that $\mathrm{Vol}_{g_\epsilon}(B_{g_\epsilon}(p,1)) \ge c_0(n) > 0$ for some constant c_0 independent of ϵ. Then Bishop's volume comparison yields, for any $r \in (0,1)$,

$$C_1(n) r^{2n} \ge \mathrm{Vol}_{g_\epsilon}(B_{g_\epsilon}(p,r)) \ge c_1(n) r^{2n}.$$

Thus the Sobolev inequality (3-41) is reduced to

$$\left(\int_{B_{g_\epsilon}(p,r)} h^{2n/(n-1)} \omega_\epsilon^n \right)^{(n-1)/n} \le C \int_{B_{g_\epsilon}(p,r)} |\nabla h|^2_{g_\epsilon} \omega_\epsilon^n \quad \text{for all } h \in C^1_0(B_{g_\epsilon}(p,r)). \tag{3-42}$$

With (3-42) at hand, we can apply the same proof of the standard De Giorgi–Nash–Moser theory (see the proof of Corollary 4.18 in [22]) to derive the uniform Hölder continuity of v_ϵ at p, i.e., there exists a constant $C = C(n, \boldsymbol{\beta}, R_0) > 0$ such that

$$\operatorname{osc}_{B_{\boldsymbol{\beta}}(p,r)} v_\epsilon \le \operatorname{osc}_{B_{g_\epsilon}(p,r)} v_\epsilon \le C r^{\alpha''} \quad \text{for all } r \in (0, R_0)$$

for some $\alpha'' = \alpha''(n, \boldsymbol{\beta}, R_0) \in (0, 1)$, where in the first inequality we use the relation (3-40). Letting $\epsilon \to 0$ we see the continuity of v at p.

Boundary value: $v = \varphi$ on $\partial B_{\boldsymbol{\beta}}(0, 1)$. The proof is almost identical to that of Proposition 3.5. For example, the function $\varphi_q(z) = \varphi(q) - \delta + A\Psi_q(z)$ defined in Case 1 in the proof of Proposition 3.5 satisfies $\Delta_{g_\epsilon}\varphi_q(z) \ge \max_X f$ if $A > 0$ is taken large enough. Then from $\Delta_{g_\epsilon}(\varphi_q - v_\epsilon) \ge 0$ in $B_{\boldsymbol{\beta}}$ and $\varphi_q - \varphi \le 0$ on $\partial B_{\boldsymbol{\beta}}$, applying the maximum principle we get $\varphi_q \le v_\epsilon$ in $B_{\boldsymbol{\beta}}(0, 1)$. The remaining arguments are the same as in Proposition 3.5. Case 2 can be dealt with similarly. □

Remark 3.8. Let $H_0^1(B_{\boldsymbol{\beta}}(0, 1), g_{\boldsymbol{\beta}})$ be the completion of the space of $C_0^1(B_{\boldsymbol{\beta}}(0, 1))$-functions under the norm

$$\|\nabla u\|_{L^2(g_{\boldsymbol{\beta}})} = \left(\int_{B_{\boldsymbol{\beta}}(0,1)} |\nabla u|^2_{g_{\boldsymbol{\beta}}} \omega^n_{\boldsymbol{\beta}}\right)^{1/2}.$$

For any $h \in C_0^1(B_{\boldsymbol{\beta}}(0, 1))$, letting $\epsilon \to 0$ in (3-42), we get

$$\left(\int_{B_{\boldsymbol{\beta}}(p,r)} |h|^{2n/n-1} \omega^n_{\boldsymbol{\beta}}\right)^{(n-1)/n} \le C \int_{B_{\boldsymbol{\beta}}(p,r)} |\nabla h|^2_{g_{\boldsymbol{\beta}}} \omega^n_{\boldsymbol{\beta}} \tag{3-43}$$

for the same constant C in (3-42). That is, the Sobolev inequality also holds for the conical metric $\omega_{\boldsymbol{\beta}}$.

3B. *Tangential and Laplacian estimates.* In this section, we will prove the Hölder continuity of $\Delta_k u$ for $k = 1, 2$ and $(D')^2 u$ for the solution u to (1-2). The arguments of [20] can be adopted here. We recall that we assume $\beta_1, \beta_2 \in \left(\frac{1}{2}, 1\right)$. We fix some notations first.

For a given point $p \notin \mathcal{S}$, we define $r_p = d_{g_{\boldsymbol{\beta}}}(p, \mathcal{S})$, the $g_{\boldsymbol{\beta}}$-distance of p to the singular set $\mathcal{S}$. For simplicity of notation we will fix $\tau = \frac{1}{2}$ and an integer $k_p \in \mathbb{Z}_+$ to be the smallest integer such that $\tau^{k_p} < r_p$, and $k_{i,p} \in \mathbb{Z}_+$ the smallest integer $k_{i,p}$ such that $\tau^{k_{i,p}} < d_{\boldsymbol{\beta}}(p, \mathcal{S}_i)$ for $i = 1, 2$. So $k_p = \max\{k_{1,p}, k_{2,p}\}$. We write $p_1 \in \mathcal{S}_1$ and $p_2 \in \mathcal{S}_2$ for the projections of p to $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

For $j = 1, 2$, we will write

$$\Delta_j u := |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}.$$

We will consider a family of conical Laplace equations with different choices of $k \in \mathbb{Z}^+$.

(i) If $k \ge k_p$, the geodesic balls $B_{\boldsymbol{\beta}}(p, \tau^k)$ are disjoint from $\mathcal{S}$ and have smooth boundaries. We note that $g_{\boldsymbol{\beta}}$ is smooth on such balls. By standard theory we can solve the following Dirichlet problem for $u_k \in C^\infty(B_{\boldsymbol{\beta}}(p, \tau^k)) \cap C^0(\overline{B_{\boldsymbol{\beta}}(p, \tau^k)})$:

$$\begin{cases} \Delta_{\boldsymbol{\beta}} u_k = f(p) & \text{in } B_{\boldsymbol{\beta}}(p, \tau^k), \\ u_k = u & \text{on } \partial B_{\boldsymbol{\beta}}(p, \tau^k). \end{cases} \tag{3-44}$$

(ii) Without loss of generality, we assume $d_{\boldsymbol{\beta}}(p, \mathcal{S}_1) \le d_{\boldsymbol{\beta}}(p, \mathcal{S}_2)$, i.e., $k_{1,p} \ge k_{2,p}$. We now solve the following Dirichlet problem for $u_k \in C^2(B_{\boldsymbol{\beta}}(p_1, 2\tau^k)\backslash\mathcal{S}_1) \cap C^0(\overline{B_{\boldsymbol{\beta}}(p_1, 2\tau^k)})$ for $k_{2,p} + 2 \le k < k_{1,p}$:

$$\begin{cases} \Delta_{\boldsymbol{\beta}} u_k = f(p) & \text{in } B_{\boldsymbol{\beta}}(p_1, 2\tau^k)\backslash\mathcal{S}_1, \\ u_k = u & \text{on } \partial B_{\boldsymbol{\beta}}(p_1, 2\tau^k). \end{cases} \tag{3-45}$$

By similar arguments to those in the proof of Proposition 3.5 and Remark 3.6, such u_k exists.

(iii) For $2 \le k \le k_{2,p} + 1$, let $u_k \in C^2(B_{\boldsymbol{\beta}}(p_{1,2}, 2\tau^k)\backslash\mathcal{S}) \cap C^0(\overline{B_{\boldsymbol{\beta}}(p_{1,2}, 2\tau^k)})$ solve the equation

$$\begin{cases} \Delta_{\boldsymbol{\beta}} u_k = f(p) & \text{in } B_{\boldsymbol{\beta}}(p_{1,2}, 2\tau^k), \\ u_k = u & \text{on } \partial B_{\boldsymbol{\beta}}(p_{1,2}, 2\tau^k), \end{cases} \tag{3-46}$$

whose existence follows from Remark 3.6. Here $p_{1,2} = (0; 0; s(p)) \in \mathcal{S}_1 \cap \mathcal{S}_2$ is the projection of p_1 to $\mathcal{S}_2$.

We remark that we may take $f(p) = 0$ by considering $\tilde{u} = u - \frac{1}{2} f(p)(n-2)^{-1}|s - s(p)|^2$. If the estimate holds for $\tilde{u}$, it also holds for u. So from now on we assume $f(p) = 0$.

Lemma 3.9. *Let u_k be the solutions to* (3-44)–(3-46). *There exists a constant $C = C(n) > 0$ such that, for all $k \in \mathbb{Z}_+$, we have the estimates*

$$\|u_k - u\|_{L^\infty(\hat{B}_k(p))} \le C(n)\tau^{2k}\omega(\tau^k), \tag{3-47}$$

where we define $\hat{B}_k(p)$ as

$$\hat{B}_k(p) := \begin{cases} B_{\boldsymbol{\beta}}(p, \tau^k) & \text{if } k \ge k_p, \\ B_{\boldsymbol{\beta}}(p_1, 2\tau^k) & \text{if } k_{2,p} + 2 \le k < k_{1,p}, \\ B_{\boldsymbol{\beta}}(p_{1,2}, 2\tau^k) & \text{if } 2 \le k \le k_{2,p} + 1 \end{cases} \tag{3-48}$$

for different choices of $k \in \mathbb{Z}_+$.

We will also define $\lambda\hat{B}_k(p)$ to be the ball concentric with $\hat{B}_k(p)$ with radius scaled by $\lambda \in (0, 1)$.

This lemma follows straightforwardly from Lemma 3.1 and the definition of $\omega(r)$, so we omit the proof. By the triangle inequality, we get the estimates

$$\|u_k - u_{k+1}\|_{L^\infty(\hat{B}_k/2)} \le C(n)\tau^{2k}\omega(\tau^k). \tag{3-49}$$

Since $u_k - u_{k+1}$ are $g_{\boldsymbol{\beta}}$-harmonic functions on $\frac{1}{2}\hat{B}_k$, applying the gradient and Laplacian estimates (3-5) and (3-9) for harmonic functions, we get the following lemma.

Lemma 3.10. *There exists a constant $C(n) > 0$ such that, for all $k \in \mathbb{Z}_+$,*

$$\|D'u_k - D'u_{k+1}\|_{L^\infty(\hat{B}_k/3)} \le C(n)\tau^k\omega(\tau^k) \tag{3-50}$$

and

$$\sup_{(\hat{B}_k/3)\backslash\mathcal{S}} \left(\sum_{i=1}^{2} |\Delta_i(u_k - u_{k+1})| + |(D')^2 u_k - (D')^2 u_{k+1}| \right) \le C(n)\omega(\tau^k), \tag{3-51}$$

where we recall that D' denotes the first-order operators $\partial/\partial s_i$ for $i = 5, \dots, 2n$.

The following lemma can be proved by looking at the Taylor expansion of u_k at p for $k \gg 1$ as in Lemma 2.8 of [20].

Lemma 3.11. *For $i = 1, 2$, we have the limits*

$$\lim_{k\to\infty} D'u_k(p) = D'u(p), \quad \lim_{k\to\infty} (D')^2u_k(p) = (D')^2u(p), \quad \lim_{k\to\infty} \Delta_i u_k(p) = \Delta_i u(p). \tag{3-52}$$

Combining Lemmas 3.10 and 3.11, we obtain estimates on the second-order (tangential) derivatives.

Proposition 3.12. *There exists a constant $C = C(n, \beta) > 0$ such that*

$$\sup_{B_\beta(0,1/2)\setminus\mathcal{S}} |(D')^2u| + |\Delta_i u| \le C\biggl(\|u\|_{L^\infty(B_\beta(0,1))} + \int_0^1 \frac{\omega(r)}{r}\,dr + |f(0)|\biggr). \tag{3-53}$$

Proof. From the triangle inequality we have, for any given $z \in B_{\boldsymbol\beta}\bigl(0, \tfrac{1}{2}\bigr)\setminus\mathcal{S}$,

$$\begin{aligned}|(D')^2u(z)| &\le \sum_{k=2}^{\infty} |(D')^2u_k(z) - (D')^2u_{k+1}(z)| + |(D')^2u_2(z)| \\ &\le C(n)\sum_{k=2}^{\infty}\omega(\tau^k) + C(n)\operatorname{osc}_{B_\beta(0,1)} u_0 \le C(n, \boldsymbol\beta)\biggl(\|u\|_{L^\infty} + \int_0^1 \frac{\omega(r)}{r}\,dr + |f(0)|\biggr).\end{aligned}$$

The estimates for $\Delta_i u$ can be proved similarly. □

For any other given point $q \in B_{\boldsymbol\beta}\bigl(0, \tfrac{1}{2}\bigr)\setminus\mathcal{S}$, we can solve a similar Dirichlet boundary problems as u_k with the metric balls centered at q, and we obtain a family of functions v_k such that

$$\Delta_{\boldsymbol\beta} v_k = f(q) \quad \text{in } \widetilde{B}_k(q), \qquad v_k = u \quad \text{on } \partial\widetilde{B}_k(q), \tag{3-54}$$

where $\widetilde{B}_k(q)$ are metrics balls centered at q given by

$$\widetilde{B}_k(q) = \widetilde{B}_k := \begin{cases} B_{\boldsymbol\beta}(q, \tau^k) & \text{if } k \ge k_q, \\ B_{\boldsymbol\beta}(q_i, 2\tau^k) & \text{if } k_{j,q} + 2 \le k < k_q \;\; (\text{here } k_{i,q} = \max(k_{1,q}, k_{2,q}) \text{ and } j \ne i), \\ B_{\boldsymbol\beta}(q_{i,j}, 2\tau^k) & \text{if } k \le k_{j,q} + 1. \end{cases}$$

Similar estimates as in Lemmas 3.9–3.11 also hold for v_k within the balls $\widetilde{B}_k(q)$.

We are now ready to state the main result in this subsection on the continuity of second-order derivatives.

Proposition 3.13. *Let $d = d_{\boldsymbol\beta}(p, q) < \frac{1}{16}$. There exists a constant $C = C(n) > 0$ such that if u solves the conical Laplace equation* (1-2), *then the following holds for $i = 1, 2$:*

$$|\Delta_i u(p) - \Delta_i u(q)| + |(D')^2u(p) - (D')^2u(q)| \le C\biggl(d\|u\|_{L^\infty(B_\beta(0,1))} + \int_0^d \frac{\omega(r)}{r}\,dr + d\int_d^1 \frac{\omega(r)}{r^2}\,dr\biggr).$$

Proof. We only prove the estimate for $(D')^2u$; the estimates for $\Delta_i u$ can be dealt with in the same way.

We may assume $r_p = \min(r_p, r_q)$. We fix $\ell \in \mathbb{Z}$ such that τ^ℓ is comparable to d; more precisely, take

$$\tau^{\ell+4} \le d < \tau^{\ell+3} \quad \text{or} \quad \tau^{\ell+1} \le 8d \le \tau^\ell.$$

We calculate by the triangle inequality

$$\begin{aligned}|(D')^2u(p) - (D')^2u(q)| &\le |(D')^2u(p) - (D')^2u_\ell(p)| + |(D')^2u_\ell(p) - (D')^2u_\ell(q)| \\ &\qquad + |(D')^2u_\ell(q) - (D')^2v_\ell(q)| + |(D')^2v_\ell(q) - (D')^2u(q)| \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We will estimate I_1–I_4 one by one.

$\underline{I_1 \text{ and } I_4}$: By (3-51) and (3-52), we have

$$I_1 = |(D')^2u(p) - (D')^2u_\ell(p)| \le C(n)\sum_{k=\ell}^{\infty}\omega(\tau^k),$$

and a similar estimate holds for I_4 as well:

$$I_4 = |(D')^2u(q) - (D')^2v_\ell(q)| \le C(n)\sum_{k=\ell}^{\infty}\omega(\tau^k).$$

$\underline{I_3}$: By the choice of ℓ, it is not hard to see that $\frac{2}{3}\widetilde{B}_\ell(q) \subset \hat{B}_\ell(p)$. In particular, u_ℓ and v_ℓ are both defined on $\frac{2}{3}\widetilde{B}_\ell(q)$ and satisfy the equations

$$\Delta_{\boldsymbol{\beta}}u_\ell = f(p) \quad \text{and} \quad \Delta_{\boldsymbol{\beta}}v_\ell = f(q),$$

respectively, on this ball. From (3-47) for u_ℓ and from a similar estimate for v_ℓ, we get

$$\|u_\ell - v_\ell\|_{L^\infty(2\widetilde{B}_\ell(q)/3)} \le C\tau^{2\ell}\omega(\tau^\ell).$$

Consider the function

$$U := u_\ell - v_\ell - \frac{f(p)-f(q)}{2(n-2)}|s - s(\tilde{q})|^2, \tag{3-55}$$

where $\tilde{q}$ is the center of the ball $\widetilde{B}_\ell(q)$. U is $g_{\boldsymbol{\beta}}$-harmonic in $\frac{2}{3}\widetilde{B}_\ell(q)$ and satisfies the estimate

$$\|U\|_{L^\infty(2\widetilde{B}/3_\ell(q))} \le C\tau^{2\ell}\omega(\tau^\ell) + C\tau^{2\ell}\omega(d) \le C(n)\tau^{2\ell}\omega(\tau^\ell).$$

The derivatives estimates imply that

$$|(D')^2U(q)| \le C\tau^{-2\ell}\|U\|_{L^\infty(2\widetilde{B}/3_\ell(q))} \le C(n)\omega(\tau^\ell).$$

Hence

$$I_3 = |(D')^2u_\ell(q) - (D')^2v_\ell(q)| \le C(n)\omega(\tau^\ell).$$

$\underline{I_2}$: This is a little more complicated than the previous estimates. We define $h_k = u_{k-1} - u_k$ for $k \le \ell$. We observe that h_k is $g_{\boldsymbol{\beta}}$-harmonic on $\hat{B}_k(p)$ and by (3-47) satisfies the L^∞-estimate $\|h_k\|_{\hat{B}_k(p)} \le C\tau^{2k}\omega(\tau^k)$ and the derivatives estimates $\|(D')^2h_k\|_{L^\infty(2\hat{B}_k(p)/3)}\omega(\tau^k)$. On the other hand, the function $(D')^2h_k$ is also $g_{\boldsymbol{\beta}}$-harmonic on $\frac{2}{3}\hat{B}_k(p)$, so the gradient estimate implies that

$$\|\nabla_{g_{\boldsymbol{\beta}}}(D')^2h_k\|_{L^\infty((\hat{B}_k(p)/2)\setminus\mathcal{S})} \le C\tau^{-k}\omega(\tau^k). \tag{3-56}$$

Integrating this along the minimal $g_{\boldsymbol{\beta}}$-geodesic γ connecting p and q and noting that γ avoids $\mathcal{S}$ since $(\mathbb{C}^n\setminus\mathcal{S}, g_{\boldsymbol{\beta}})$ is strictly geodesically convex, we get

$$|(D')^2h_k(p) - (D')^2h_k(q)| \le d\cdot\|\nabla_{g_{\boldsymbol{\beta}}}(D')^2h_k\|_{L^\infty((\hat{B}_k(p)/2)\setminus\mathcal{S})} \le dC\tau^{-k}\omega(\tau^k).$$

By the triangle inequality, for each $k \le \ell$,

$$I_2 = |(D')^2u_\ell(p) - (D')^2u_\ell(q)| \le |(D')^2u_2(p) - (D')^2u_2(q)| + dC\sum_{k=2}^{\ell}\tau^{-k}\omega(\tau^k). \tag{3-57}$$

Observe that $p, q \in \hat{B}_2(p)$ and the function $(D')^2 u_2$ is $g_{\boldsymbol{\beta}}$-harmonic on $\hat{B}_2(p)$. From (3-47) and derivatives estimates we have

$$\|(D')^2 u_2\|_{L^\infty(2\hat{B}_2(p)/3)} \le C\|u_2\|_{L^\infty(\hat{B}_2(p))} \le C(\|u\|_{L^\infty} + \omega(\tau^2)).$$

Again by the gradient estimate we have

$$\|\nabla_{g_{\boldsymbol{\beta}}} (D')^2 u_2\|_{L^\infty(\hat{B}_2(p)/2)} \le C(\|u\|_{L^\infty} + \omega(\tau^2)).$$

Integrating along the minimal geodesic γ we arrive at

$$|(D')^2 u_2(p) - (D')^2 u_2(q)| \le dC(\|u\|_{L^\infty} + \omega(\tau^2)).$$

Combining this with (3-57), we obtain

$$I_2 \le Cd\biggl(\|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \sum_{k=2}^{\ell} \tau^{-k}\omega(\tau^k)\biggr).$$

Combing the estimates for I_1–I_4, we get

$$|(D')^2 u(p) - (D')^2 u(q)| \le C\biggl(d\biggl(\|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \sum_{k=2}^{\ell} \tau^{-k}\omega(\tau^k)\biggr) + \sum_{k=\ell}^{\infty} \omega(\tau^k)\biggr).$$

Proposition 3.13 now follows from this and the fact that $\omega(r)$ is monotonically increasing. □

3C. ***Mixed normal-tangential estimates along the directions $\mathcal{S}$.*** Throughout this section, we fix two points $p, q \in B_{\boldsymbol{\beta}}\bigl(0, \tfrac{1}{2}\bigr)\backslash\mathcal{S}$ and assume $r_p \le r_q$. Recall that we defined the weighted "polar coordinates" (r_i, θ_i) for (z_1, z_2):

$$\rho_i = |z_i|, \quad r_i = \rho_i^{\beta_i}, \quad \theta_i = \arg z_i, \qquad i = 1, 2.$$

Under these coordinates,

$$\Delta_i u = |z_i|^{2(1-\beta_i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_i} = \frac{\partial^2 u}{\partial r_i^2} + \frac{1}{r_i}\frac{\partial u}{\partial r_i} + \frac{1}{\beta_i^2 r_i^2}\frac{\partial^2 u}{\partial \theta_i^2}. \tag{3-58}$$

Let u_k and v_k be the solutions to (3-44)–(3-46) on $\hat{B}_k(p)$ and $\widetilde{B}_k(q)$, respectively. Recalling that $u_k - u_{k+1}$ satisfies (3-49) and applying gradient estimates to the $g_{\boldsymbol{\beta}}$-harmonic function $u_k - u_{k+1}$, we get the bound of $\|\nabla_{g_{\boldsymbol{\beta}}}(u_k - u_{k+1})\|_{L^\infty(\hat{B}_k(p)/3)}$, which in particular implies that, for $i = 1, 2$,

$$\left\| |z_i|^{1-\beta_i} \left(\frac{\partial u_k}{\partial z_i} - \frac{\partial u_{k+1}}{\partial z_i} \right) \right\|_{L^\infty(\hat{B}_k(p)/3)} \le C\tau^k \omega(\tau^k). \tag{3-59}$$

Similarly, $D'u_k - D'u_{k+1}$ is also $g_{\boldsymbol{\beta}}$-harmonic on $\frac{1}{2}\hat{B}_k(p)$, and applying gradient estimates to this function we get, for $i = 1, 2$,

$$\left\| |z_i|^{1-\beta_i} \left(\frac{\partial D'u_k}{\partial z_i} - \frac{\partial D'u_{k+1}}{\partial z_i} \right) \right\|_{L^\infty(\hat{B}_k(p)/3)} \le C\omega(\tau^k). \tag{3-60}$$

The next lemma can be proved in the same way as Lemma 2.10 of [20] since $p \notin \mathcal{S}$; we omit the proof.

Lemma 3.14. *For $i = 1, 2$, we have the limits*

$$\lim_{k\to\infty} \frac{\partial u_k}{\partial r_i}(p) = \frac{\partial u}{\partial r_i}(p), \quad \lim_{k\to\infty} \frac{\partial u_k}{r_i \partial\theta_i}(p) = \frac{\partial u}{r_i\partial\theta_i}(p)$$

and

$$\lim_{k\to\infty} \frac{\partial D' u_k}{\partial r_i}(p) = \frac{\partial D' u}{\partial r_i}(p), \quad \lim_{k\to\infty} \frac{\partial D' u_k}{r_i \partial\theta_i}(p) = \frac{\partial D' u}{r_i\partial\theta_i}(p). \tag{3-61}$$

Similar formulas also hold for v_k at the point q.

We are going to estimate the quantities

$$J := \left| \frac{\partial D' u}{\partial r_i}(p) - \frac{\partial D' u}{\partial r_i}(q) \right| \quad \text{and} \quad K := \left| \frac{\partial D' u}{r_i\partial\theta_i}(p) - \frac{\partial D' u}{r_i\partial\theta_i}(q) \right|, \qquad i = 1, 2.$$

Note that J, K correspond to $|N_j D'u(p) - N_j D'u(q)|$ in Theorem 1.2. We will estimate the case for $i = 1$ and J, since the other cases are completely the same. By the triangle inequality we have

$$\begin{aligned} J \le \left| \frac{\partial D' u}{\partial r_i}(p) - \frac{\partial D' u_\ell}{\partial r_i}(p) \right| + \left| \frac{\partial D' u_\ell}{\partial r_i}(p) - \frac{\partial D' u_\ell}{\partial r_i}(q) \right| \\ + \left| \frac{\partial D' u_\ell}{\partial r_i}(q) - \frac{\partial D' v_\ell}{\partial r_i}(q) \right| + \left| \frac{\partial D' v_\ell}{\partial r_i}(q) - \frac{\partial D' u}{\partial r_i}(q) \right| \\ =: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Lemma 3.15. *There exists a constant $C(n) > 0$ such that J_1, J_3 and J_4 satisfy*

$$J_1 + J_4 \le C \sum_{k=\ell}^{\infty} \omega(\tau^k), \quad J_3 \le C\omega(\tau^\ell).$$

Proof. The estimates for J_1 and J_4 can be proved similarly to those of I_1 and I_4 in Section 3B, using (3-60) and (3-61). J_3 can be estimated in a similar way to I_3 in Section 3B, using (3-60). We omit the details. □

To estimate J_2, as in Section 3B we define $h_k := u_{k-1} - u_k$ for $2 \le k \le \ell$ which is $g_{\boldsymbol\beta}$-harmonic on $\hat B_k(p)$ and satisfies the L^∞-estimate $\|h_k\|_{L^\infty(\hat B_k(p))} \le C\tau^{2k}\omega(\tau^k)$ by (3-60). We rewrite (3-56) as

$$\|(D')^3 h_k\|_{L^\infty((\hat B_k(p)/2)\setminus\mathcal S)} + \sum_{i=1}^{2} \left\| |z_i|^{1-\beta_i} \frac{\partial}{\partial z_i} (D')^2 h_k \right\|_{L^\infty((\hat B_k(p)/2)\setminus\mathcal S)} \le C\tau^{-k}\omega(\tau^k). \tag{3-62}$$

Lemma 3.16. *There exists a constant $C = C(n, \boldsymbol\beta) > 0$ such that, for any $z \in \frac14 \hat B_k(p)\setminus\mathcal S$, the following pointwise estimate holds for all $k \le \min(\ell, k_p)$:*

$$\left| \frac{\partial D' h_k}{\partial r_1}(z) \right| + \left| \frac{\partial D' h_k}{r_1\partial\theta_1}(z) \right| \le C r_1(z)^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k).$$

Proof. We define a function F as

$$|z_1|^{2(1-\beta_1)} \frac{\partial^2 D' h_k}{\partial z_1 \partial \bar z_1} = -|z_2|^{2(1-\beta_2)} \frac{\partial^2 D' h_k}{\partial z_2 \partial \bar z_2} - \sum_{j=5}^{2n} \frac{\partial^2 D' h_k}{\partial s_j^2} =: F. \tag{3-63}$$

The Laplacian estimates (3-9) and derivatives estimates applied to the $g_{\boldsymbol\beta}$-harmonic function $D'h_k$ imply that F satisfies

$$\|F\|_{L^\infty(\hat B_k(p)/2)} \le C(n)\tau^{-k}\omega(\tau^k). \tag{3-64}$$

For any $k \le \min(\ell, k_p)$ and $x \in \mathcal{S}_1 \cap \frac14 \hat B_k(p)$, we have that $B_{\boldsymbol\beta}(x, \tau^k) \subset \frac13 \hat B_k(p)$. The intersection of $B_{\boldsymbol\beta}(x, \tau^k)$ with the complex plane $\mathbb{C}$ passing through x and orthogonal to the hyperplane $\mathcal{S}_1$ lies in a metric ball of radius τ^k under the standard cone metric $\hat g_{\beta_1}$ on $\mathbb{C}$. We view (3-63) as defined on the ball $\hat B := B_{\mathbb{C}}(x, (\tau^k)^{1/\beta_1}) \subset \mathbb{C}$. The estimate (2-3) applied to the function $D'h_k$ gives rise to

$$\sup_{B_{\mathbb{C}}(x,(\tau^k)^{1/\beta_1}/2)\setminus\{x\}} \left|\frac{\partial D'h_k}{\partial z_1}\right| \le C\frac{\|D'h_k\|_{L^\infty(\hat B)}}{(\tau^k)^{1/\beta_1}} + C\|F\|_{L^\infty(\hat B)}(\tau^k)^{2-1/\beta_1}.$$

Therefore, on $B_{\mathbb{C}}\big(x, \frac12(\tau^k)^{1/\beta_1}\big)\setminus\{x\}$,

$$\left|\frac{\partial D'h_k}{\partial r_1}(z)\right| + \left|\frac{\partial D'h_k}{r_1\partial\theta_1}(z)\right| \le \frac{1}{\beta_1} r_1^{1/\beta_1-1}\left|\frac{\partial D'h_k}{\partial z_1}(z)\right| \le C r_1^{1/\beta_1-1}\tau^{k(1-1/\beta_1)}\omega(\tau^k). \tag{3-65}$$

On other hand, since $B_{\mathbb{C}}\big(x, \frac12(\tau^k)^{1/\beta_1}\big) = B_{\hat g_{\beta_1}}(x, 2^{-\beta_1}\tau^k)$,

$$\tfrac14 \hat B_k(p) \subset \bigcup_{x\in\mathcal{S}_1\cap\hat B_k/4} B_{\mathbb{C}}\big(x, \tfrac12(\tau^k)^{1/\beta_1}\big). \tag{3-66}$$

Equation (3-65) implies the desired estimate on the balls $\frac14 \hat B_k(p)$. ☐

Remark 3.17. By similar arguments we also get the following estimates for any $k \le \min(\ell, k_p)$ and $z \in \frac14 \hat B_k(p)\setminus\mathcal{S}_1$:

$$\left|\frac{\partial (D')^2 h_k}{\partial r_1}(z)\right| + \left|\frac{\partial (D')^2 h_k}{r_1\partial\theta_1}(z)\right| \le C r_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}\omega(\tau^k). \tag{3-67}$$

Lemma 3.18. *There exists a constant $C = C(n, \boldsymbol\beta) > 0$ such that, for all $k \le \min(k_p, \ell)$ and $z \in \frac14 \hat B_k(p)\setminus\mathcal{S}$, the following pointwise estimates hold*:

$$\left|\frac{\partial^2 D'h_k}{r_1^2\partial\theta_1^2}(z)\right| + \left|\frac{\partial^2 D'h_k}{r_1\partial r_1\partial\theta_1}(z)\right| \le C r_1(z)^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)}\omega(\tau^k), \tag{3-68}$$

$$\left|\frac{\partial^2 D'h_k}{\partial r_1^2}(z)\right| \le C r_1(z)^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)}\omega(\tau^k). \tag{3-69}$$

Proof. Applying the gradient estimate to the $g_{\boldsymbol\beta}$-harmonic function $D'h_k$, we get

$$\left\|\frac{\partial D'h_k}{r_1\partial\theta_1}\right\|_{L^\infty(\hat B_k(p)/2)} \le \|\nabla_{g_{\boldsymbol\beta}} D'h_k\|_{L^\infty(\hat B_k(p)/2)} \le C\omega(\tau^k).$$

The function $\partial_{\theta_1} D'h_k$ is also a continuous $g_{\boldsymbol\beta}$-harmonic function, so the derivatives estimates implies, on $\frac13 \hat B_k(p)\setminus\mathcal{S}$,

$$|F_1| \le \left||z_2|^{2(1-\beta_2)}\frac{\partial^2(\partial_{\theta_1} D'h_k)}{\partial z_2\partial\bar z_2}\right| + \left|\frac{\partial^2(\partial_{\theta_1} D'h_k)}{\partial s_j^2}\right| \le C\tau^{-k}\omega(\tau^k),$$

where F_1 is defined as

$$|z_1|^{2(1-\beta_1)}\frac{\partial^2(\partial_{\theta_1}D'h_k)}{\partial z_1\partial\bar{z}_1}=-|z_2|^{2(1-\beta_2)}\frac{\partial^2(\partial_{\theta_1}D'h_k)}{\partial z_2\partial\bar{z}_2}-\sum_{j=5}^{2n}\frac{\partial^2(\partial_{\theta_1}D'h_k)}{\partial s_j^2}=:F_1. \tag{3-70}$$

We apply similar arguments as in the proof of Lemma 3.16. For any $x\in\mathcal{S}_1\cap\frac{1}{4}\hat{B}_k(p)$, we view (3-70) as defined on the $\mathbb{C}$-ball $B_{\mathbb{C}}(x,(\tau^k)^{1/\beta_1})$, and by the estimate (2-3) we have, on $B_{\mathbb{C}}(x,(\tau^k)^{1/\beta_1}/2)\backslash\{x\}$,

$$\left|\frac{\partial(\partial_{\theta_1}D'h_k)}{\partial z_1}\right|\le C\frac{\|\partial_{\theta_1}D'h_k\|_{L^\infty(\hat{B}_{\mathbb{C}})}}{(\tau^k)^{1/\beta_1}}+C\|F_1\|_{L^\infty(\hat{B}_{\mathbb{C}})}(\tau^k)^{2-1/\beta_1}.$$

Equivalently, this means that, on $B_{\mathbb{C}}(x,(\tau^k)^{1/\beta_1}/2)\backslash\{x\}$,

$$\left|\frac{\partial^2D'h_k}{\partial r_1\partial\theta_1}\right|+\left|\frac{\partial^2D'h_k}{r_1\partial\theta_1^2}\right|\le r_1^{1/\beta_1-1}\left|\frac{\partial(\partial_{\theta_1}D'h_k)}{\partial z_1}\right|\le Cr_1^{1/\beta_1-1}\tau^{k(1-1/\beta_1)}\omega(\tau^k).$$

Again by the inclusion (3-66), we get (3-68). The estimate (3-69) follows from Lemma 3.16, (3-68), (3-64) and the equation (from (3-63))

$$\frac{\partial^2D'h_k}{\partial r_1^2}=\frac{1}{r_1}\frac{\partial D'h_k}{\partial r_1}-\frac{1}{\beta_1^2r_1^2}\frac{\partial^2D'h_k}{\partial\theta_1^2}+F. \qquad\square$$

Lemma 3.19. *There exists a constant $C(n,\boldsymbol{\beta})>0$ such that, for $k\le\min(k_{2,p},\ell)$, the following pointwise estimates hold for any $z\in\frac{1}{4}\hat{B}_k(p)\backslash\mathcal{S}$:*

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)(z)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)(z)\right|\le C(n,\boldsymbol{\beta})r_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(-1+1/\beta_1+1/\beta_2)}\omega(\tau^k). \tag{3-71}$$

Proof. By the Laplacian estimate in (3-9) for the harmonic function $D'h_k$ on $\frac{1}{2}\hat{B}_k(p)$, we have

$$\sup_{\hat{B}_k(p)/2.2}(|\Delta_1D'h_k|+|\Delta_2D'h_k|)\le C(n)\tau^{-2k}\operatorname{osc}_{\hat{B}_k(p)/2}(D'h_k)\le C(n)\tau^{-k}\omega(\tau^k). \tag{3-72}$$

Since $\Delta_1(D'h_k)$ is also $g_{\boldsymbol{\beta}}$-harmonic, the Laplacian estimates (3-9) imply

$$\sup_{\hat{B}_k(p)/2.4}(|\Delta_1\Delta_1D'h_k|+|\Delta_2\Delta_1D'h_k|)\le C(n)\tau^{-2k}\operatorname{osc}_{\hat{B}_k(p)/2.2}\Delta_1D'h_k\le C\tau^{-3k}\omega(\tau^k). \tag{3-73}$$

Now from the equation $\Delta_{\boldsymbol{\beta}}(\Delta_1D'h_k)=0$, we get

$$|z_1|^{2(1-\beta_1)}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}\Delta_1D'h_k=-\Delta_2\Delta_1D'h_k-\sum_j\frac{\partial^2}{\partial s_j^2}\Delta_1D'h_k=:F_2. \tag{3-74}$$

From (3-73) and the Laplacian estimates (3-9), we see that $\sup_{\hat{B}_k(p)/2.4}|F_2|\le C\tau^{-3k}\omega(\tau^k)$. Using similar arguments, by considering $x\in\frac{1}{3}\hat{B}_k(p)\cap\mathcal{S}_1$, we obtain from (3-74) that, on $\hat{B}:=B_{\mathbb{C}}(x,(\tau^k)^{1/\beta_1}/2)\backslash\{x\}$,

$$\left|\frac{\partial}{\partial z_1}\Delta_1D'h_k\right|\le C\frac{\|\Delta_1D'h_k\|_{L^\infty(\hat{B})}}{(\tau^k)^{1/\beta_1}}+C\|F_2\|_{L^\infty(\hat{B})}(\tau^k)^{2-1/\beta_1}\le C\tau^{-k(1+1/\beta_1)}\omega(\tau^k).$$

This implies that, for any $z \in \frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$,

$$\left|\frac{\partial}{\partial r_1}\Delta_1 D'h_k(z)\right| + \left|\frac{\partial}{r_1\partial\theta_1}\Delta_1 D'h_k(z)\right| \le Cr_1^{1/\beta_1-1}\tau^{-k(1+1/\beta_1)}\omega(\tau^k). \tag{3-75}$$

Now taking $\partial/\partial r_1$ on both sides of $\Delta_{\boldsymbol\beta} D'h_k = 0$, we get

$$|z_2|^{2(1-\beta_2)}\frac{\partial^2}{\partial z_2\partial\bar z_2}\left(\frac{\partial D'h_k}{\partial r_1}\right) = -\frac{\partial}{\partial r_1}(\Delta_1 D'h_k) - \sum_j \frac{\partial^2}{\partial s_j^2}\left(\frac{\partial D'h_k}{\partial r_1}\right) =: F_3. \tag{3-76}$$

From (3-75), for any $z \in \frac{1}{3}\hat{B}_k\backslash\mathcal{S}$, we have $|F_3|(z) \le Cr_1^{1/\beta_1-1}\tau^{-k(1+1/\beta_1)}\omega(\tau^k)$. By a similar argument, for any $y \in \frac{1}{3.2}\hat{B}_k(p)\cap\mathcal{S}_2$, we apply estimate (2-3) to $\partial D'h_k/\partial r_1$ and get, on $A_1 := B_{\mathbb{C}}\big(y, \frac{1}{2}(\tau^k)^{1/\beta_2}\big)\backslash\{y\}$ — the punctured ball in the complex plane $\mathbb{C}$ of (Euclidean) radius $\frac{1}{2}(\tau^k)^{1/\beta_2}$ and orthogonal to $\mathcal{S}_2$ passing through y — that

$$\begin{aligned}\left|\frac{\partial}{\partial z_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)(z)\right| &\le C\frac{\left\|\frac{\partial D'h_k}{\partial r_1}\right\|_{L^\infty(A_1)}}{(\tau^k)^{1/\beta_2}} + C\|F_3\|_{L^\infty(A_1)}(\tau^k)^{2-1/\beta_2}\\ &\le Cr_1^{1/\beta_1-1}\tau^{-k(1/\beta_1+1/\beta_2-1)}\omega(\tau^k).\end{aligned} \tag{3-77}$$

Varying $y \in \frac{1}{3.2}\hat{B}_k(p)\cap\mathcal{S}_2$ we get, for any $z \in \frac{1}{4}\hat{B}_k\backslash\mathcal{S}$, that the following pointwise estimate holds:

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)(z)\right| + \left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)(z)\right| \le Cr_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(1/\beta_1+1/\beta_2-1)}\omega(\tau^k). \qquad \square$$

Lemma 3.20. *Let $d = d_{\boldsymbol\beta}(p,q)$. There exists a constant $C(n,\boldsymbol\beta)$ such that, for all $k \le \ell$,*

$$\left|\frac{\partial D'h_k}{\partial r_1}(p) - \frac{\partial D'h_k}{\partial r_1}(q)\right| \le Cd^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}\omega(\tau^k), \tag{3-78}$$

$$\left|\frac{\partial D'h_k}{r_1\partial\theta_1}(p) - \frac{\partial D'h_k}{r_1\partial\theta_1}(q)\right| \le Cd^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}\omega(\tau^k). \tag{3-79}$$

Proof. We will consider the different cases $r_p = \min(r_p, r_q) \le 2d$ and $r_p = \min(r_p, r_q) > 2d$.

Case 1: $r_p \le 2d$. In this case, it is clear by the choice of ℓ that $r_p \approx \tau^{k_p} \le 2d \le \tau^{\ell+2}$, so $k_p \ge \ell+2$.

From our assumption when solving (3-45), $r_p = d_{\boldsymbol\beta}(p,\mathcal{S}_1)$, i.e., $r_1(p) = r_p \le 2d$. By the triangle inequality we have $r_1(q) \le 3d$. We also remark that, for $k \le \ell$, we have $\tau^k \ge \tau^\ell > 8d$. In particular, the geodesics considered below all lie inside the balls $\frac{1}{4}\hat{B}_k(p)$, and the estimates in Lemmas 3.16–3.19 hold for points on these geodesics.

Let the coordinates of the points p and q be given by

$$p = (r_1(p), \theta_1(p); r_2(p), \theta_2(p); s(p)) \quad\text{and}\quad q = (r_1(q), \theta_1(q); r_2(q), \theta_2(q); s(q)).$$

Let $\gamma : [0,d] \to B_{\boldsymbol\beta}(0,q)\backslash\mathcal{S}$ be the unique $g_{\boldsymbol\beta}$-geodesic connecting p and q. We know the curve γ is disjoint from $\mathcal{S}$, and we write

$$\gamma(t) = (r_1(t), \theta_1(t); r_2(t), \theta_2(t); s(t))$$

for the coordinates of $\gamma(t)$ for $t \in [0,d]$. By definition we have, for all $t \in [0,d]$,

$$|\gamma'(t)|^2_{g_{\boldsymbol\beta}} = (r_1'(t))^2 + \beta_1^2 r_1(t)^2(\theta_1'(t))^2 + (r_2'(t))^2 + \beta_2^2 r_2(t)^2(\theta_2'(t))^2 + |s'(t)|^2 = 1.$$

So $|s(p)-s(q)|\le d$ and $|r_i(p)-r_i(q)|\le d$ for $i=1,2$. We define

$$q' := (r_1(q),\theta_1(q);r_2(p),\theta_2(p);s(p)) \quad\text{and}\quad p' := (r_1(p),\theta_1(q);r_2(p),\theta_2(p);s(p)), \tag{3-80}$$

the points with coordinates related to p and q. Let γ_1 be the $g_{\boldsymbol\beta}$-geodesic connecting q and q', γ_2 be the $g_{\boldsymbol\beta}$-geodesic joining q' to p', and γ_3 be the $g_{\boldsymbol\beta}$-geodesic joining p' to p.

By the triangle inequality, we have

$$\begin{aligned}\left|\frac{\partial D'h_k}{\partial r_1}(p)-\frac{\partial D'h_k}{\partial r_1}(q)\right| &\le \left|\frac{\partial D'h_k}{\partial r_1}(p)-\frac{\partial D'h_k}{\partial r_1}(p')\right|+\left|\frac{\partial D'h_k}{\partial r_1}(p')-\frac{\partial D'h_k}{\partial r_1}(q')\right|+\left|\frac{\partial D'h_k}{\partial r_1}(q')-\frac{\partial D'h_k}{\partial r_1}(q)\right|\\ &=: J_1'+J_2'+J_3'.\end{aligned}$$

Integrating along γ_3, on which the points have fixed r_1-coordinate $r_1(p)$, we get (by (3-68))

$$J_1' = \left|\int_{\gamma_3}\frac{\partial}{\partial\theta_1}\left(\frac{\partial D'h_k}{\partial r_1}\right)d\theta_1\right| \le C(n,\boldsymbol\beta)r_1(p)^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}\omega(\tau^k). \tag{3-81}$$

Integrating along γ_2, we get (by (3-69))

$$\begin{aligned}J_2' &= \left|\int_{\gamma_2}\frac{\partial}{\partial r_1}\left(\frac{\partial D'h_k}{\partial r_1}\right)dr_1\right| \le C(n,\boldsymbol\beta)\tau^{-k(1/\beta_1-1)}\omega(\tau^k)\left|\int_{r_1(p)}^{r_1(q)} t^{1/\beta_1-2}\,dt\right|\\ &= C(n,\boldsymbol\beta)\tau^{-k(1/\beta_1-1)}\omega(\tau^k)|r_1(p)^{1/\beta_1-1}-r_1(q)^{1/\beta_1-1}|\\ &\le C(n,\boldsymbol\beta)\tau^{-k(1/\beta_1-1)}\omega(\tau^k)|r_1(p)-r_1(q)|^{1/\beta_1-1}\\ &\le C(n,\boldsymbol\beta)\tau^{-k(1/\beta_1-1)}\omega(\tau^k)d^{1/\beta_1-1}.\end{aligned} \tag{3-82}$$

To deal with J_3', we need to consider different choices of $k\le\ell$.

Case 1a: $k_{2,p}+1\le k\le\ell$. In this case, the balls $\hat B_k(p)$ are centered at $p_1\in\mathcal S_1$ (recall p_1 is the projection of p to $\mathcal S_1$; hence p and p_1 have the same $(r_2,\theta_2;s)$-coordinates). We have $\tau^{-k}\le 8^{-1}d^{-1}$ by the choice of ℓ. The balls $\hat B_k(p)$ are disjoint from $\mathcal S_2$, so we can introduce the smooth coordinates $w_2=z_2^{\beta_2}$, and under the coordinates $(r_1,\theta_1;w_2,z_3,\dots,z_n)$, the metric $g_{\boldsymbol\beta}$ becomes the smooth cone metric with conical singularity *only* along $\mathcal S_1$ with angle $2\pi\beta_1$. Therefore we can derive the following estimate as in (3-62):

$$\sup_{(\hat B_k(p)/2)\setminus\mathcal S_1}\left|\frac{\partial(D')^2h_k}{\partial r_1}\right|+\left|\frac{\partial}{\partial r_1}\left(\frac{\partial D'h_k}{\partial w_2}\right)\right| \le C\tau^{-k}\omega(\tau^k). \tag{3-83}$$

Since q and q' have the same (r_1,θ_1)-coordinates and $g_{\boldsymbol\beta}$ is a product metric, γ_1 is in fact a straight line segment (under the coordinates $(w_2,z_3,\dots,z_n)$) in the hyperplane with fixed (r_1,θ_1)-coordinates. Integrating over γ_1, we get

$$\begin{aligned}J_3' \le \int_{\gamma_1}\left|\frac{\partial}{\partial w_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|+\sum_j\left|\frac{\partial}{\partial s_j}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right| &\le C\tau^{-k}\omega(\tau^k)d_{\boldsymbol\beta}(q,q')\le C\tau^{-k}\omega(\tau^k)d\\ &\le C(n,\boldsymbol\beta)\tau^{-k(1/\beta_1-1)}\omega(\tau^k)d^{1/\beta_1-1}.\end{aligned}$$

Case 1b: $k \le k_{2,p}$. In this case, the balls $\hat{B}_k(p)$ are centered at $p_{1,2} \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $\tau^k \ge r_2(p)$. By the triangle inequality, $r_2(q) \le d + r_2(p) \le \frac{9}{8}\tau^k$. We choose the points

$$\tilde{q} = (r_1(q), \theta_1(q); r_2(p), \theta_2(p); s(q)) \quad \text{and} \quad \hat{q} = (r_1(q), \theta_1(q); r_2(q), \theta_2(p); s(q)). \tag{3-84}$$

Let $\tilde{\gamma}_1$ be the $g_{\boldsymbol{\beta}}$-geodesic joining q' to $\tilde{q}$, $\tilde{\gamma}$ the $g_{\boldsymbol{\beta}}$-geodesic joining $\tilde{q}$ to $\hat{q}$, and $\hat{\gamma}$ the $g_{\boldsymbol{\beta}}$-geodesic joining $\hat{q}$ to q. The curves $\tilde{\gamma}_1$, $\tilde{\gamma}$ and $\hat{\gamma}$ all lie in the hyperplane with constant (r_1, θ_1)-coordinates $(r_1(q), \theta_1(q))$. Then by the triangle inequality we have

$$J_3' \le \left|\frac{\partial D'h_k}{\partial r_1}(q') - \frac{\partial D'h_k}{\partial r_1}(\tilde{q})\right| + \left|\frac{\partial D'h_k}{\partial r_1}(\tilde{q}) - \frac{\partial D'h_k}{\partial r_1}(\hat{q})\right| + \left|\frac{\partial D'h_k}{\partial r_1}(\hat{q}) - \frac{\partial D'h_k}{\partial r_1}(q)\right| =: J_1'' + J_2'' + J_3''.$$

We will use frequently the inequalities $r_1(q) \le 3d$ and $\max(r_2(q), r_2(p)) \le 2\tau^k$ in the estimates below. Integrating along $\hat{\gamma}$ we get (by (3-71))

$$\begin{aligned} J_3'' \le \left|\int_{\hat{\gamma}} \frac{\partial}{\partial \theta_2}\left(\frac{\partial D'h_k}{\partial r_1}\right) d\theta_2\right| &\le C r_1(q)^{1/\beta_1 - 1} r_2(q)^{1/\beta_2} \tau^{-k(-1+1/\beta_1+1/\beta_2)} \omega(\tau^k) \\ &\le C d^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k). \end{aligned}$$

Integrating along $\tilde{\gamma}$ we get (again by (3-71))

$$\begin{aligned} J_2'' \le \left|\int_{\tilde{\gamma}} \frac{\partial}{\partial r_2}\left(\frac{\partial D'h_k}{\partial r_1}\right) dr_2\right| &\le C r_1(q)^{1/\beta_1 - 1} \tau^{-k(-1+1/\beta_1+1/\beta_2)} \omega(\tau^k) \left|\int_{r_2(q)}^{r_2(p)} t^{1/\beta_2 - 1}\, dt\right| \\ &\le C r_1(q)^{1/\beta_1 - 1} \tau^{-k(-1+1/\beta_1+1/\beta_2)} \omega(\tau^k) \max(r_2(q), r_2(p))^{1/\beta_2 - 1} d \\ &\le C d^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k). \end{aligned}$$

Integrating along $\tilde{\gamma}_1$ we get (by (3-67))

$$J_1'' \le \left|\int_{\tilde{\gamma}_1} \frac{\partial}{\partial s_j}\left(\frac{\partial D'h_k}{\partial r_1}\right) dt\right| \le C r_1(q)^{1/\beta_1 - 1} \tau^{-k/\beta_1} \omega(\tau^k) d \le C d^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k).$$

Combining the three inequalities above, we get, in the case $k \le k_{2,p}$,

$$J_3' \le C d^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k).$$

Combining the estimates on J_1', J_2' and J_3', we finish the proof of (3-78) in the case $r_p \le 2d$.

Case 2a: $r_p > 2d$ and $\ell \le k_p$. In this case $\tau^{k_p} \approx r_p > 2d \ge \tau^{\ell+3}$. From the triangle inequality we get $d_{\boldsymbol{\beta}}(\gamma(t), \mathcal{S}) \ge d$. In particular, the r_1 and r_2 coordinates of $\gamma(t)$ are both bigger than d. In this case $k \le \ell \le k_p$, and Lemmas 3.16–3.19 hold for the points in γ. So $r_1(\gamma(t)) \le r_1(p) + d \le 2\tau^k$. We calculate the gradient of $\partial D'h_k / \partial r_1$ along γ:

$$\begin{aligned} \left|\nabla_{g_{\beta}} \frac{\partial D'h_k}{\partial r_1}\right|^2 = \left|\frac{\partial}{\partial r_1}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|^2 + \left|\frac{\partial}{\beta_1 r_1 \partial \theta_1}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|^2 + \left|\frac{\partial}{\partial r_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|^2 \\ + \left|\frac{\partial}{\beta_2 r_2 \partial \theta_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|^2 + \sum_j \left|\frac{\partial}{\partial s_j}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|^2. \end{aligned}$$

(1) When $k_{2,p}+1\le k\le \ell$ we have by (3-83) that

$$\sup_{(\hat{B}_k/2)\setminus\mathcal{S}_1}\left|\frac{\partial}{\partial r_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial D'h_k}{\partial r_1}\right)\right|\le C\tau^{-k}\omega(\tau^k). \tag{3-85}$$

Thus by Lemma 3.18, (3-67) and (3-85), along γ we have

$$\left|\nabla_{g_{\beta}}\frac{\partial D'h_k}{\partial r_1}\right|\le C\omega(\tau^k)(d^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)}+\tau^{-k})$$

Integrating along γ we get

$$\begin{aligned}\left|\frac{\partial D'h_k}{\partial r_1}(p)-\frac{\partial D'h_k}{\partial r_1}(q)\right|\le\int_\gamma\left|\nabla_{g_\beta}\frac{\partial D'h_k}{\partial r_1}\right|&\le C\omega(\tau^k)(d^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}+d\tau^{-k})\\&\le C\omega(\tau^k)d^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}.\end{aligned}$$

(2) When $k\le k_{2,p}$, we have $r_2(\gamma(t))\le r_2(p)+d\le\tau^k+d\le\frac{9}{8}\tau^k$ and similar estimates hold for $r_1(\gamma(t))$ too. Then by Lemma 3.18, Lemma 3.19 and (3-67) along γ the following estimate holds

$$\left|\nabla_{g_{\beta}}\frac{\partial D'h_k}{\partial r_1}\right|(\gamma(t))\le C\omega(\tau^k)(d^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)}+\tau^{-k})$$

Integrating along γ we get

$$\begin{aligned}\left|\frac{\partial D'h_k}{\partial r_1}(p)-\frac{\partial D'h_k}{\partial r_1}(q)\right|\le\int_\gamma\left|\nabla_{g_\beta}\frac{\partial D'h_k}{\partial r_1}\right|&\le C\omega(\tau^k)(d^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}+d\tau^{-k})\\&\le C\omega(\tau^k)d^{1/\beta_1-1}\tau^{-k(1/\beta_1-1)}.\end{aligned}$$

This finishes the proof of the lemma in this case.

Case 2b: $r_p>2d$ but $\ell\ge k_p+1$. When $k\le k_p$, the estimate (3-78) follows in the same way as the case above. Hence it suffices to consider the case when $k_p+1\le k\le\ell$. In this case the balls $\hat{B}_k(p)=B_{\boldsymbol\beta}(p,\tau^k)$ and it can be seen by triangle inequality that the geodesic $\gamma\subset\frac{1}{3}\hat{B}_k(p)\setminus\mathcal{S}$. Since the metric balls $\hat{B}_k(p)$ are disjoint with $\mathcal{S}$ we can use the smooth coordinates $w_1=z_1^{\beta_1}$ and $w_2=z_2^{\beta_2}$ as before, and everything becomes smooth under these coordinates in $\hat{B}_k(p)$.

The estimate (3-79) can be shown by the same argument, so we skip the details. □

Iteratively applying (3-78) for $k\le\ell$, we get

$$\begin{aligned}J_2=\left|\frac{\partial D'u_\ell}{\partial r_1}(p)-\frac{\partial D'u_\ell}{\partial r_1}(q)\right|&\le\left|\frac{\partial D'u_2}{\partial r_1}(p)-\frac{\partial D'u_2}{\partial r_1}(q)\right|+Cd^{1/\beta_1-1}\sum_{k=3}^{\ell}\tau^{-k(1/\beta_1-1)}\omega(\tau^k)\\&\le Cd^{1/\beta_1-1}\biggl(\|u\|_{C^0}+\sum_{k=2}^{\ell}\tau^{-k(1/\beta_1-1)}\omega(\tau^k)\biggr),\end{aligned}$$

where the inequality

$$\left|\frac{\partial D'u_2}{\partial r_1}(p)-\frac{\partial D'u_2}{\partial r_1}(q)\right|\le Cd^{1/\beta_1-1}\|u\|_{C^0}$$

can be proved by the same argument as in proving (3-78).

Combining the estimates for $J_1,\ J_2,\ J_3,\ J_4$ we finish the proof of (1-4).

We remark that in solving (3-45) we assume $r_1(p) \le r_2(p)$, so we need also to deal with the following case, whose proof is more or less parallel to that of Lemma 3.20, so we just point out the differences and sketch the proof.

Lemma 3.21. *Let $d = d_{\boldsymbol\beta}(p, q) > 0$. There exists a constant $C(n, \boldsymbol\beta) > 0$ such that, for all $k \le \ell$,*

$$\left|\frac{\partial D'h_k}{\partial r_2}(p) - \frac{\partial D'h_k}{\partial r_2}(q)\right| \le C d^{1/\beta_2 - 1}\tau^{-k(1/\beta_2-1)}\omega(\tau^k), \tag{3-86}$$

$$\left|\frac{\partial D'h_k}{r_2\partial\theta_2}(p) - \frac{\partial D'h_k}{r_2\partial\theta_2}(q)\right| \le C d^{1/\beta_2 - 1}\tau^{-k(1/\beta_2-1)}\omega(\tau^k). \tag{3-87}$$

Proof. We consider the cases when $k \le k_{1,p}$ and $k_{1,p}+1 \le k \le \ell$.

Case 1: $k_{1,p}+1 \le k \le \ell$. The balls $\hat B_k(p)$ are disjoint with $\mathcal{S}_2$, so we can introduce the complex coordinate $w_2 = z_2^{\beta_2}$ on these balls as before. Let t_1 and t_2 be the real and imaginary parts of w_2, respectively. The derivatives estimates imply that

$$\|\partial_{w_2} D'h_k\|_{L^\infty(\hat B_k(p)/2)} \le C\omega(\tau^k) \quad\text{and}\quad \|\partial^2_{w_2} D'h_k\|_{L^\infty(\hat B_k(p)/2)} \le C\tau^{-k}\omega(\tau^k),$$

where $\partial^2_{w_2}$ denotes the full second-order derivatives in the $\{t_1, t_2\}$-directions. Also

$$\left\|\frac{\partial}{\partial r_1}\left(\frac{\partial D'h_k}{\partial w_2}\right)\right\|_{L^\infty(\hat B_k(p)/2)} + \left\|\frac{\partial}{r_1\partial\theta_1}\left(\frac{\partial D'h_k}{\partial w_2}\right)\right\|_{L^\infty(\hat B_k(p)/2)} \le C\tau^{-k}\omega(\tau^k).$$

Since

$$\frac{\partial}{\partial r_2} = \frac{w_2}{\beta_2 r_2}\frac{\partial}{\partial w_2} + \frac{\bar w_2}{\beta_2 r_2}\frac{\partial}{\partial \bar w_2}, \tag{3-88}$$

we have

$$\frac{\partial}{\partial w_2}\left(\frac{\partial D'h_k}{\partial r_2}\right) = \frac{1}{r_2}\frac{\partial D'h_k}{\partial w_2} - \frac{|w_2|^2}{2r_2^3}\frac{\partial D'h_k}{\partial w_2} - \frac{\bar w_2\cdot\bar w_2}{2r_2^3}\frac{\partial D'h_k}{\partial \bar w_2} + \frac{w_2}{r_2}\partial^2_{w_2}D'h_k,$$

and we have, on $\frac12\hat B_k(p)$,

$$\left|\frac{\partial}{\partial w_2}\left(\frac{\partial D'h_k}{\partial r_2}\right)\right| \le \frac{C}{r_2}\omega(\tau^k) + C\tau^{-k}\omega(\tau^k)$$

and

$$\left\|\frac{\partial}{\partial r_1}\left(\frac{\partial D'h_k}{\partial r_2}\right)\right\|_{L^\infty(\hat B_k(p)/2)} + \left\|\frac{\partial}{r_1\partial\theta_1}\left(\frac{\partial D'h_k}{\partial r_2}\right)\right\|_{L^\infty(\hat B_k(p)/2)} \le C\tau^{-k}\omega(\tau^k).$$

Therefore,

$$\left|\nabla_{g_{\boldsymbol\beta}}\frac{\partial D'h_k}{\partial r_2}\right|^2 = \left|\frac{\partial^2 D'h_k}{\partial r_1\partial r_2}\right|^2 + \left|\frac{\partial^2}{r_1\partial\theta_1\partial r_2}\right|^2 + \left|\frac{\partial^2 D'h_k}{\partial w_2\partial r_2}\right|^2 + \sum_j\left|\frac{\partial^2 D'h_k}{\partial s_j\partial r_2}\right|^2 \le C(\tau^{-k}\omega(\tau^k))^2 + C\frac{1}{r_2^2}\omega(\tau^k)^2.$$

In this case we know that $r_1(p) \approx \tau^{k_p} \ge 2\tau^k \ge \tau^\ell > 8d$, so along γ

$$r_2(\gamma(t)) \ge r_2(p) - d \ge r_1(p) - d \ge \tfrac74\tau^k.$$

Integrating along γ we get

$$\left|\frac{\partial D'h_k}{\partial r_2}(p)-\frac{\partial D'h_k}{\partial r_2}(q)\right|\le\int_\gamma\left|\nabla_{g_\beta}\frac{\partial D'h_k}{\partial r_2}\right|\le C\tau^{-k}\omega(\tau^k)d\le C\tau^{-k(1/\beta_2-1)}\omega(\tau^k)d^{1/\beta_2-1}.$$

Case 2: $k\le k_{1,p}$. This case is the same as in the proof of (3-78), replacing r_1 by r_2 and β_1 by β_2. We omit the details.

We can prove (3-87) similarly. □

3D. *Mixed normal directions.* In this section, we deal with Hölder continuity of the four mixed derivatives

$$\frac{\partial^2 u}{\partial r_1\partial r_2},\quad\frac{\partial^2 u}{r_1\partial\theta_1\partial r_2},\quad\frac{\partial^2 u}{r_2\partial r_1\partial\theta_2},\quad\frac{\partial^2 u}{r_1r_2\partial\theta_1\partial\theta_2},\tag{3-89}$$

which by our previous notation correspond to N_1N_2u. Since the proof for each of them is more or less the same, we will only prove Hölder continuity for $\partial^2u/\partial r_1\partial r_2$. The following holds at p and q by the same reasoning of Lemma 3.11:

$$\lim_{k\to\infty}\frac{\partial^2 u_k}{\partial r_1\partial r_2}(p)=\frac{\partial^2 u}{\partial r_1\partial r_2}(p),\qquad\lim_{k\to\infty}\frac{\partial^2 v_k}{\partial r_1\partial r_2}(q)=\frac{\partial^2 u}{\partial r_1\partial r_2}(q).$$

By the triangle inequality,

$$\begin{aligned}\left|\frac{\partial^2 u}{\partial r_1\partial r_2}(p)-\frac{\partial^2 u}{\partial r_1\partial r_2}(q)\right|\le\left|\frac{\partial^2 u}{\partial r_1\partial r_2}(p)-\frac{\partial^2 u_\ell}{\partial r_1\partial r_2}(p)\right|+\left|\frac{\partial^2 u_\ell}{\partial r_1\partial r_2}(p)-\frac{\partial^2 u_\ell}{\partial r_1\partial r_2}(q)\right|\\+\left|\frac{\partial^2 u_\ell}{\partial r_1\partial r_2}(q)-\frac{\partial^2 v_\ell}{\partial r_1\partial r_2}(q)\right|+\left|\frac{\partial^2 v_\ell}{\partial r_1\partial r_2}(q)-\frac{\partial^2 u}{\partial r_1\partial r_2}(q)\right|\\=:L_1+L_2+L_3+L_4.\end{aligned}$$

Lemma 3.22. *We have the estimate*

$$L_1+L_4\le\sum_{k=\ell}^{\infty}\omega(\tau^k).$$

Proof. We consider the cases when $k\ge k_p+1$ and $\ell\le k\le k_p$.

Case 1: $k\ge k_p+1$. In this case the balls $\hat B_k(p)$ are disjoint from $\mathcal{S}$ and we can introduce the smooth coordinates $w_1=z_1^{\beta_1}$ and $w_2=z_2^{\beta_2}$. Under the coordinates $\{w_1,w_2,z_3,\dots,z_n\}$, the cone metric $g_{\boldsymbol\beta}$ becomes the standard Euclidean metric $g_{\mathbb{C}^n}$ and the metric balls $\hat B_k(p)$ become the standard Euclidean balls with the same radius and center p. Since the $g_{\boldsymbol\beta}$-harmonic functions u_k-u_{k+1} satisfy (3-49), by standard gradient estimates for Euclidean harmonic functions, we get

$$\sup_{\hat B_k(p)/2.1}\left|D_{w_1}D_{w_2}(u_k-u_{k-1})\right|\le C\omega(\tau^k),$$

where we use D_{w_i} to denote either $\partial/\partial w_i$ or $\partial/\partial\bar w_i$ for simplicity. From (3-88) and a similar formula for $\partial/\partial r_1$, we get

$$\sup_{\hat B_k(p)/2.1}\left|\frac{\partial^2}{\partial r_1\partial r_2}(u_k-u_{k-1})\right|\le C\omega(\tau^k).\tag{3-90}$$

Case 2a: $\ell \geq k_{2,p}+1$ and $\ell \leq k_p = k_{1,p}$. For all $\ell \leq k$, the balls $\hat{B}_k(p)$ are disjoint from $\mathcal{S}_2$ and centered at p_1. We can still use $w_2 = z_2^{\beta_2}$ as the smooth coordinate. The cone metric $g_{\boldsymbol{\beta}}$ becomes smooth in the w_2-variable, and we can apply the standard gradient estimate to the $g_{\boldsymbol{\beta}}$-harmonic function $D_{w_2}(u_k - u_{k-1})$ to get

$$\sup_{\hat{B}_k(p)/2.2} \left| \frac{\partial}{\partial r_1} D_{w_2}(u_k - u_{k-1}) \right| + \left| \frac{\partial}{r_1 \partial \theta_1} D_{w_2}(u_k - u_{k-1}) \right| \leq C\omega(\tau^k).$$

Again by (3-88), we get

$$\sup_{\hat{B}_k(p)/2.2} \left| \frac{\partial^2}{\partial r_1 \partial r_2}(u_k - u_{k-1}) \right| + \left| \frac{\partial^2}{r_1 \partial \theta_1 \partial r_2}(u_k - u_{k-1}) \right| \leq C\omega(\tau^k). \tag{3-91}$$

Case 2b: $\ell \leq k_{2,p}$ and $k \geq k_{2,p}+1$. This case can be dealt with similarly as above.

Case 2c: $\ell \leq k \leq k_{2,p}$. In this case $r_2(p) \approx \tau^{k_{2,p}} \leq \tau^k \leq \tau^\ell \approx 8d$. Now the balls $\hat{B}_k(p)$ are centered at $p_{1,2} \in \mathcal{S}_1 \cap \mathcal{S}_2$. We can proceed as in the proof of Lemma 3.19, with the harmonic functions $u_k - u_{k-1}$ replacing the $D' h_k$ in that lemma to prove that, for any $z \in \frac{1}{3}\hat{B}_k(p) \backslash \mathcal{S}$,

$$\begin{aligned} \left| \frac{\partial^2}{\partial r_1 \partial r_2}(u_k - u_{k-1}) \right|(z) + \left| \frac{\partial^2}{r_2 \partial \theta_2 \partial r_1}(u_k - u_{k-1}) \right|(z) \\ \leq C(n, \boldsymbol{\beta}) r_1(z)^{1/\beta_1 - 1} r_2(z)^{1/\beta_2 - 1} \tau^{-k(-2+1/\beta_1+1/\beta_2)} \omega(\tau^k). \end{aligned}$$

In particular, the estimate in each case holds at p, and from $r_1(p) \leq r_2(p) \leq \tau^k$ we obtain

$$\left| \frac{\partial^2}{\partial r_1 \partial r_2}(u_k - u_{k-1}) \right|(p) + \left| \frac{\partial^2}{r_2 \partial \theta_2 \partial r_1}(u_k - u_{k-1}) \right|(p) \leq C\omega(\tau^k). \tag{3-92}$$

Combining each case above, by (3-90)–(3-92), we get, for all $k \geq \ell$,

$$\left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(u_k - u_{k-1}) \right|(p) \leq C(n, \boldsymbol{\beta})\omega(\tau^k).$$

Therefore, by the triangle inequality,

$$L_1 \leq \sum_{k=\ell+1}^{\infty} \left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(u_k - u_{k-1}) \right|(p) \leq C(n, \boldsymbol{\beta}) \sum_{k=\ell+1}^{\infty} \omega(\tau^k).$$

The estimate for L_4 can be dealt with similarly by studying the derivatives of v_k at q. □

Lemma 3.23. $$L_3 \leq C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

Proof. As in the proof of Lemma 3.22, we consider the cases $\ell \geq k_{1,p}+1$, $k_{1,p} \geq \ell \geq k_{2,p}$ and $\ell \leq k_{2,p}-1$.

Case 1: $\ell \geq k_{1,p}+1$. Here the ball $\hat{B}_\ell(p)$ is equal to $B_{\boldsymbol{\beta}}(p, \tau^\ell)$, the function U defined in (3-55) is $g_{\boldsymbol{\beta}}$-harmonic in $\frac{1}{2}\hat{B}_\ell(p)$, and $\sup_{\hat{B}_\ell(p)/2} |U| \leq C\omega^{2\ell}\omega(\tau^\ell)$. Since the ball $\frac{1}{2}\hat{B}_\ell(p)$ is disjoint from $\mathcal{S}$, we have that w_1 and w_2 are well defined on $\frac{1}{2}\hat{B}_\ell(p)$, and thus we have the derivatives estimates

$$\sup_{\hat{B}_\ell(p)/3} \left| \frac{\partial^2 U}{\partial r_1 \partial r_2} \right| \leq \sup_{\hat{B}_\ell(p)/3} \left| D_{w_1} D_{w_2} U \right| \leq C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

In particular, at $q \in \frac{1}{3}\hat{B}_\ell(p)$,

$$L_3 = \left|\frac{\partial^2 u_\ell}{\partial r_1 \partial r_2}(q) - \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2}(q)\right| = \left|\frac{\partial^2 U}{\partial r_1 \partial r_2}(q)\right| \le C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

Case 2: $k_{1,p} \ge \ell \ge k_{2,p}$. Here the ball $\hat{B}_\ell(p)$ is equal to $B_{\boldsymbol{\beta}}(p_1, 2\tau^\ell)$, the function U defined in (3-55) is $g_{\boldsymbol{\beta}}$-harmonic and well defined in a ball

$$B_q := B_{\boldsymbol{\beta}}\big(q, \tfrac{1}{10}\tau^\ell\big) \subset \tfrac{1}{2.2}\hat{B}_\ell(p),$$

and $\sup_{\hat{B}_\ell(p)/2} |U| \le C\omega^{2\ell}\omega(\tau^\ell)$. Since $\frac{1}{2}\hat{B}_\ell(p)$ is disjoint from $\mathcal{S}_2$, we have that w_2 is well defined on $\frac{1}{2.2}\hat{B}_\ell(p)$, and thus we have the derivatives estimates

$$\sup_{B_q/2}\left|\frac{\partial^2 U}{\partial r_1 \partial r_2}\right| \le \sup_{B_q/2}\left|\frac{\partial}{\partial r_1} D_{w_2} U\right| \le C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

In particular, at $q \in \frac{1}{2}B_q$, we have

$$L_3 = \left|\frac{\partial^2 u_\ell}{\partial r_1 \partial r_2}(q) - \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2}(q)\right| = \left|\frac{\partial^2 U}{\partial r_1 \partial r_2}(q)\right| \le C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

Case 3: $\ell \le k_{2,p} - 1$. Here $r_2(p) \approx \tau^{k_{2,p}} \le \tau^{\ell+1} < 8d$, so

$$r_2(q) \le r_2(p) + d \le \tfrac{5}{8}\tau^\ell \quad \text{and} \quad r_1(q) \le d + r_1(p) \le d + r_2(p) \le \tfrac{5}{8}\tau^\ell.$$

Therefore the ball $\widetilde{B}_\ell(q)$ is centered at either $q_1,\ q_2$ or $q_{1,2} \in \mathcal{S}_1 \cap \mathcal{S}_2$, with radius $2\tau^\ell$. It follows that the function U defined in (3-55) is well defined on the ball $\frac{1}{1.8}\hat{B}_\ell(p)$.

By the same strategy as in the proof of Lemma 3.19, with the harmonic function $D'h_k$ in that lemma replaced by U on the metric ball $\frac{1}{1.8}\hat{B}_\ell(p)$, we can prove that, for any $z \in \frac{1}{2}\hat{B}_\ell(p)\backslash\mathcal{S}$,

$$\left|\frac{\partial^2 U}{\partial r_1 \partial r_2}(z)\right| \le C(n, \boldsymbol{\beta}) r_1^{1/\beta_1 - 1} r_2^{1/\beta_2 - 1} \tau^{-\ell(-2+1/\beta_1+1/\beta_2)}\omega(\tau^\ell).$$

Applying this inequality at q, we get

$$L_3 = \left|\frac{\partial^2 (u_\ell - v_\ell)}{\partial r_1 \partial r_2}(q)\right| \le C(n, \boldsymbol{\beta}) r_1(q)^{1/\beta_1 - 1} r_2(q)^{1/\beta_2 - 1} \tau^{-\ell(-2+1/\beta_1+1/\beta_2)}\omega(\tau^\ell) \le C(n, \boldsymbol{\beta})\omega(\tau^\ell).$$

In sum, in all cases $L_3 \le C(n, \boldsymbol{\beta})\omega(\tau^\ell)$. □

Lemma 3.24. *There exists a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that, for all $k \le \ell$ and $z \in \frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$,*

$$\left|\frac{\partial}{\partial \theta_1}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)(z)\right| + \left|\left(\frac{\partial^3 h_k}{r_1 \partial \theta_1^2 \partial r_2}\right)\right|$$

$$\le C \cdot \begin{cases} r_1^{1/\beta_1 - 1}\tau^{-k(-1+1/\beta_1)}\omega(\tau^k) & \text{if } k \in [k_{2,p}+1, \min(\ell, k_p)], \\ r_1^{1/\beta_1 - 1} r_2^{1/\beta_2 - 1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}\omega(\tau^k) & \text{if } k \le k_{2,p}. \end{cases} \tag{3-93}$$

Proof. The proof is parallel to that of Lemma 3.19. The function $\partial h_k/\partial\theta_1$ is $g_{\boldsymbol\beta}$-harmonic on $\hat{B}_k(p)$, and by the Laplacian estimates (3-9), we have

$$\sup_{\hat{B}_k(p)/1.2}\left(\left|\Delta_1\frac{\partial h_k}{\partial\theta_1}\right|+\left|\Delta_2\frac{\partial h_k}{\partial\theta_1}\right|\right)\le C(n,\boldsymbol\beta)\omega(\tau^k).$$

The function $\Delta_2(\partial h_k/\partial\theta_1)$ is also $g_{\boldsymbol\beta}$-harmonic, so the Laplacian estimates (3-9) imply

$$\sup_{\hat{B}_k(p)/1.4}\left(\left|\Delta_1\Delta_2\frac{\partial h_k}{\partial\theta_1}\right|+\left|\Delta_2\Delta_2\frac{\partial h_k}{\partial\theta_1}\right|+\left|(D')^2\Delta_2\frac{\partial h_k}{\partial\theta_1}\right|\right)\le C\tau^{-2k}\left(\operatorname{osc}_{\hat{B}_k(p)/1.2}\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)\le C\tau^{-2k}\omega(\tau^k).$$

We consider

$$|z_2|^{2(1-\beta_2)}\frac{\partial^2}{\partial z_2\partial\bar{z}_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)=-\Delta_1\Delta_2\frac{\partial h_k}{\partial\theta_1}-\sum_j\frac{\partial^2}{\partial s_j^2}\Delta_2\frac{\partial h_k}{\partial\theta_1}=:F_5,\tag{3-94}$$

where the function F_5 satisfies $\sup_{\hat{B}_k(p)/1.4}|F_5|\le C\tau^{-2k}\omega(\tau^k)$.

Case 1: $k_{2,p}+1\le k\le\min(\ell,k_p)$. Here we introduce the smooth coordinate $w_2=z_2^{\beta_2}$ in the ball $\frac{1}{1.5}\hat{B}_k(p)$ as before. Since this ball is disjoint from $\mathcal{S}_2$, under the coordinates $(r_1,\theta_1;w_2,z_3,\dots,z_n)$ we can use the usual standard gradient estimate to the $g_{\boldsymbol\beta}$-harmonic function $\Delta_2(\partial h_k/\partial\theta_1)$ to obtain

$$\sup_{\hat{B}_k(p)/2}\left|\frac{\partial}{\partial r_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)\right|\le C\tau^{-k}\omega(\tau^k).\tag{3-95}$$

Case 2: $k\le k_{2,p}$. Here the ball $\hat{B}_k(p)$ is centered at $p_{1,2}$. We apply the usual estimate (2-3) to the function $\Delta_2(\partial h_k/\partial\theta_1)$, the solution to (3-94), on any $\mathbb{C}$-ball $A_2:=B_{\mathbb{C}}(y,(\tau^k)^{1/\beta_2})$ for any $y\in\mathcal{S}_2\cap\frac{1}{1.6}\hat{B}_k(p)$, where A_2 denotes the Euclidean ball in the complex plane orthogonal to $\mathcal{S}_2$ and passing through y. Then, for any $z\in B_{\mathbb{C}}(y,(\tau^k)^{1/\beta_2}/2)\setminus\{y\}$,

$$\left|\frac{\partial}{\partial z_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)(z)\right|\le C\frac{\|\Delta_2\frac{\partial h_k}{\partial\theta_1}\|_{L^\infty(A_2)}}{\tau^{k/\beta_2}}+C\|F_5\|_{L^\infty(A_2)}(\tau^k)^{2-1/\beta_2}\le C\tau^{-k/\beta_2}\omega(\tau^k).$$

This implies that, on $\frac12\hat{B}_k(p)\setminus\mathcal{S}$,

$$\left|\frac{\partial}{\partial r_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)\right|\le Cr_2^{1/\beta_2-1}\tau^{-k/\beta_2}\omega(\tau^k).\tag{3-96}$$

Taking $\partial/\partial r_2$ on both sides of $\Delta_{\boldsymbol\beta}(\partial h_k/\partial\theta_1)=0$, we get

$$|z_1|^{2(1-\beta_1)}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}\left(\frac{\partial^2 h_k}{\partial r_2\partial\theta_1}\right)=-\frac{\partial}{\partial r_2}\left(\Delta_2\frac{\partial h_k}{\partial\theta_1}\right)-\sum_j\frac{\partial^2}{\partial s_j^2}=:F_6.\tag{3-97}$$

It is not hard to see from (3-95), (3-96) and standard derivatives estimates that, on $\frac{1}{1.8}\hat{B}_k(p)\setminus\mathcal{S}$,

- in Case 1 when $k_{2,p}+1\le k\le\min(\ell,k_p)$, we have $|F_6|\le C\tau^{-k}\omega(\tau^k)$,
- in Case 2 when $k\le k_{2,p}$, we have $|F_6|\le Cr_2^{1/\beta_2-1}\tau^{-k/\beta_2}\omega(\tau^k)$.

Then by applying estimate (2-3) to the function $\partial^2 h_k/\partial r_2\partial\theta_1$ on any $\mathbb{C}$-ball $A_3 := B_{\mathbb{C}}(x, (\tau^k)^{1/\beta_1})$ for any $x \in \frac{1}{1.8}\hat{B}_k(p)\cap\mathcal{S}_1$, we get that, on $B_{\mathbb{C}}(x, (\tau^k)^{1/\beta_1}/2)\backslash\{x\}$,

$$\begin{aligned}
&\left|\frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_2\partial\theta_1}\right)\right| + \left|\frac{\partial}{r_1\partial\theta_1}\left(\frac{\partial^2 h_k}{\partial r_2\partial\theta_1}\right)\right| \\
&\qquad\qquad \le C r_1^{1/\beta_1-1}\frac{\left\|\frac{\partial^2 h_k}{\partial r_2\partial\theta_1}\right\|_{L^\infty(A_3)}}{\tau^{k/\beta_1}} + C r_1^{1/\beta_1-1}\|F_6\|_{L^\infty(A_3)}\tau^{k(2-1/\beta_1)} \\
&\qquad\qquad \le C\cdot\begin{cases} r_1^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)}\omega(\tau^k) & \text{if } k\in[k_{2,p}+1, \min(\ell, k_p)], \\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}\omega(\tau^k) & \text{if } k\le k_{2,p}. \end{cases}
\end{aligned}$$

Therefore this estimate holds on $\frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$. ☐

Lemma 3.25. *For any $k \le \ell$ and any point $z \in \frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$,*

$$\begin{aligned}
\left|\frac{\partial^2 h_k}{\partial r_1\partial r_2}(z)\right| &\le C\cdot\begin{cases} r_1^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)}\omega(\tau^k) & \text{if } k\in[k_{2,p}+1, \min(\ell, k_p)], \\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}\omega(\tau^k) & \text{if } k\le k_{2,p}. \end{cases} \\
\left|\frac{\partial^2 D' h_k}{\partial r_1\partial r_2}(z)\right| &\le C\cdot\begin{cases} r_1^{1/\beta_1-1}\tau^{-k/\beta_1}\omega(\tau^k) & \text{if } k\in[k_{2,p}+1, \min(\ell, k_p)], \\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(-1+1/\beta_1+1/\beta_2)}\omega(\tau^k) & \text{if } k\le k_{2,p}. \end{cases}
\end{aligned} \tag{3-98}$$

Proof. This follows from almost the same argument as in the proof of Lemma 3.24, by studying the harmonic functions h_k and $D'h_k$ instead of $\partial h_k/\partial\theta_1$. ☐

Lemma 3.26. *For any $k \le \ell$ and any $z \in \frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$,*

$$\begin{aligned}
&\left|\frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right)\right|(z) \\
&\le C\omega(\tau^k)\cdot\begin{cases} \tau^{-k} + r_1(z)^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)} & \text{if } k\in[k_{2,p}+1, \min(\ell, k_p)], \\ r_2(z)^{1/\beta_2-1}\tau^{-k/\beta_2} + r_1(z)^{1/\beta_1-2}r_2(z)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k\le k_{2,p}. \end{cases}
\end{aligned}$$

Proof. By the Laplacian estimates (3-9) we have

$$\sup_{(\hat{B}_k(p)/1.2)\backslash\mathcal{S}} |\Delta_1 h_k| + |\Delta_2 h_k| \le C(n, \boldsymbol{\beta})\omega(\tau^k). \tag{3-99}$$

Applying again the Laplacian estimates (3-9) to the $g_{\boldsymbol{\beta}}$-harmonic function $\Delta_1 h_k$, we have

$$\sup_{\hat{B}_k(p)/1.4} (|\Delta_1\Delta_1 h_k| + |\Delta_2\Delta_1 h_k| + |(D')^2\Delta_1 h_k|) \le C(n)\tau^{-2k}\omega(\tau^k).$$

We consider the equation

$$|z_2|^{2-2\beta_2}\frac{\partial^2}{\partial z_2\partial\bar{z}_2}\Delta_1 h_k = -\Delta_1\Delta_1 h_k - \sum_j \frac{\partial^2}{\partial s_j^2}\Delta_1 h_k =: F_7. \tag{3-100}$$

From the estimates above, $\|F_7\|_{L^\infty(\hat{B}_k(p)/1.8)} \le C\tau^{-2k}\omega(\tau^k)$.

Case 1: $k_{2,p}+1 \le k \le \min(\ell, k_p)$. Here we directly apply the gradient estimate to $\Delta_1 h_k$ to get

$$\sup_{(\hat{B}_k(p)/1.5)\setminus\mathcal{S}} \left|\frac{\partial}{\partial r_2}\Delta_1 h_k\right| + \left|\frac{\partial}{r_2\partial\theta_2}\Delta_1 h_k\right| \le C\tau^{-k}\omega(\tau^k). \tag{3-101}$$

Case 2: $k \le k_{2,p}$. Here the balls $\hat{B}_k(p)$ are centered at $p_{1,2}$, and we can apply the usual $\mathbb{C}$-ball type estimate to get that, for any $z \in \frac{1}{2}\hat{B}_k(p)\setminus\mathcal{S}$,

$$\begin{aligned}\left|\frac{\partial}{\partial r_2}\Delta_1 h_k\right|(z) + \left|\frac{\partial}{r_2\partial\theta_2}\Delta_1 h_k\right| &\le Cr_2(z)^{1/\beta_2-1}\frac{\|\Delta_1 h_k\|_{L^\infty}}{\tau^{k/\beta_2}} + Cr_2(z)^{1/\beta_2-1}\|F_7\|_{L^\infty}\tau^{k(2-1/\beta_2)}\\ &\le Cr_2(z)^{1/\beta_2-1}\tau^{-k/\beta_2}\omega(\tau^k).\end{aligned}$$

Recall that

$$\frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right) = \frac{\partial}{\partial r_2}\Delta_1 h_k - \frac{1}{r_1}\frac{\partial^2 h_k}{\partial r_1\partial r_2} - \frac{1}{\beta_1^2 r_1^2}\frac{\partial^3 h_k}{\partial\theta_1^2\partial r_2},$$

from which we derive that, for any $z \in \frac{1}{2}\hat{B}_k(p)\setminus\mathcal{S}$,

$$\begin{aligned}&\left|\frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right)\right|(z)\\ &\le C\omega(\tau^k)\begin{cases}\tau^{-k}+r_1(z)^{1/\beta_1-2}\tau^{-k(1/\beta_1-1)} & \text{if } k\in[k_{2,p}+1, \min(\ell,k_p)],\\ r_2(z)^{1/\beta_2-1}\tau^{-k/\beta_2}+r_1(z)^{1/\beta_1-2}r_2(z)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k\le k_{2,p}.\end{cases}\end{aligned}$$

□

Lemma 3.27. *There exists a constant $C = C(n,\boldsymbol{\beta}) > 0$ such that, for all $k \le \ell$ and $z \in \frac{1}{3}\hat{B}_k(p)\setminus\mathcal{S}$,*

$$\begin{aligned}&\left|\frac{\partial}{\partial\theta_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right)(z)\right| + \left|\left(\frac{\partial^3 h_k}{r_2\partial\theta_2^2\partial r_1}\right)(z)\right|\\ &\qquad\le C\omega(\tau^k)\begin{cases} r_1^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)} & \text{if } k\in[k_{2,p}+1,\min(\ell,k_p)],\\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k\le k_{2,p}.\end{cases}\end{aligned} \tag{3-102}$$

Proof. It follows from the Laplacian estimates (3-9) that

$$\sup_{\hat{B}_k(p)/1.2}\left(\left|\Delta_1\frac{\partial h_k}{\partial\theta_2}\right| + \left|\Delta_2\frac{\partial h_k}{\partial\theta_2}\right|\right) \le C(n)\omega(\tau^k).$$

Again by (3-9), we have

$$\sup_{\hat{B}_k(p)/1.4}\left(\left|\Delta_1\Delta_1\frac{\partial h_k}{\partial\theta_2}\right| + \left|\Delta_2\Delta_1\frac{\partial h_k}{\partial\theta_2}\right| + \left|(D')^2\Delta_1\frac{\partial h_k}{\partial\theta_2}\right|\right) \le C\tau^{-2k}\omega(\omega^k).$$

We look at the equation

$$|z_1|^{2(1-\beta_1)}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}\left(\Delta_1\frac{\partial h_k}{\partial\theta_2}\right) = -\Delta_2\Delta_1\frac{\partial h_k}{\partial\theta_2} - \sum_j\frac{\partial^2}{\partial s_j^2}\left(\Delta_1\frac{\partial h_k}{\partial\theta_2}\right) =: F_8$$

and note that

$$\sup_{\hat{B}_k(p)/1.4}|F_8| \le C\tau^{-2k}\omega(\tau^k).$$

As we did before, by estimate (2-3) it follows that, for any $z \in \frac{1}{2}\hat{B}_k(p)\backslash\mathcal{S}$ (remember here $k \le \min(\ell, k_p)$),

$$\left|\frac{\partial}{\partial r_1}\Delta_1\frac{\partial h_k}{\partial\theta_2}\right|(z)+\left|\frac{\partial}{r_1\partial\theta_1}\Delta_1\frac{\partial h_k}{\partial\theta_2}\right|(z) \le Cr_1(z)^{1/\beta_1-1}\frac{\left\|\Delta_1\frac{\partial h_k}{\partial\theta_2}\right\|_{L^\infty}}{\tau^{k/\beta_1}}+Cr_1(z)^{1/\beta_1-1}\|F_8\|_{L^\infty}\tau^{k(2-1/\beta_1)}$$
$$\le Cr_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}\omega(\tau^k).$$

Taking $\partial/\partial r_1$ on both sides of the equation $\Delta_{\boldsymbol\beta}(\partial h_k/\partial\theta_2)=0$, we get

$$|z_2|^{2(1-\beta_2)}\frac{\partial^2}{\partial z_2\partial\bar z_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial\theta_2}\right)=-\frac{\partial}{\partial r_1}\left(\Delta_1\frac{\partial h_k}{\partial\theta_2}\right)-\sum_j\frac{\partial}{\partial r_1}\left(\frac{\partial^2}{\partial s_j^2}\frac{\partial h_k}{\partial\theta_2}\right)=: F_9. \tag{3-103}$$

Here $|F_9(z)|\le Cr_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}\omega(\tau^k)$ for any $z\in\frac{1}{2}\hat{B}_k(p)\backslash\mathcal{S}$. Therefore, by the usual $\mathbb{C}$-ball argument,

- when $k\le k_{2,p}$, for any $z\in\frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$, we have

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial\theta_2}\right)(z)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial\theta_2}\right)(z)\right|\le Cr_2(z)^{1/\beta_2-1}r_1(z)^{1/\beta_1-1}\tau^{k(2-1/\beta_1-1/\beta_2)}\omega(\tau^k),$$

- when $k_{2,p}+1\le k\le\min(\ell,k_p)$, we have

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial\theta_2}\right)(z)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial\theta_2}\right)(z)\right|\le Cr_1(z)^{1/\beta_1-1}\tau^{k(1-1/\beta_1)}\omega(\tau^k).$$ □

Lemma 3.28. *For any $k\le\ell$ and any $z\in\frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$,*

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right)(z)\right|$$
$$\le C\omega(\tau^k)\begin{cases} r_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}+r_1^{1/\beta_1-1}r_2^{-1}\tau^{-k(-1+1/\beta_1)} & \text{if } k\in[k_{2,p}+1,\min(\ell,k_p)],\\ r_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}+r_1^{1/\beta_1-1}r_2^{1/\beta_2-2}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k\le k_{2,p}.\end{cases} \tag{3-104}$$

Proof. We first observe that

$$\frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1\partial r_2}\right)=\frac{\partial}{\partial r_1}\Delta_2 h_k-\frac{1}{r_2}\frac{\partial^2 h_k}{\partial r_1\partial r_2}-\frac{1}{\beta_2^2 r_2^2}\frac{\partial^2}{\partial\theta_2^2}\left(\frac{\partial h_k}{\partial r_1}\right).$$

It can be shown by the $\mathbb{C}$-ball argument that, for any $z\in\frac{1}{2}\hat{B}_k(p)\backslash\mathcal{S}$,

$$\left|\frac{\partial}{\partial r_1}\Delta_2 h_k(z)\right|\le Cr_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1}\omega(\tau^k).$$

From Lemma 3.25, we have, for any $z\in\frac{1}{2}\hat{B}_k(p)\backslash\mathcal{S}$,

$$\left|\frac{1}{r_2}\frac{\partial^2 h_k}{\partial r_1\partial r_2}(z)\right|\le C\cdot\begin{cases} r_1^{1/\beta_1-1}r_2^{-1}\tau^{-k(-1+1/\beta_1)}\omega(\tau^k) & \text{if } k\in[k_{2,p}+1,\min(\ell,k_p)],\\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-2}\tau^{-k(-2+1/\beta_1+1/\beta_2)}\omega(\tau^k) & \text{if } k\le k_{2,p}.\end{cases}$$

From Lemma 3.27, we have, for any $z\in\frac{1}{2}\hat{B}_k(p)\backslash\mathcal{S}$,

$$\left|\frac{1}{r_2^2}\frac{\partial^3 h_k}{\partial r_1\partial\theta_2^2}(z)\right|\le C\omega(\tau^k)\cdot\begin{cases} r_1^{1/\beta_1-1}r_2^{-1}\tau^{-k(-1+1/\beta_1)} & \text{if } k\in[k_{2,p}+1,\min(\ell,k_p)],\\ r_1^{1/\beta_1-1}r_2^{1/\beta_2-2}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k\le k_{2,p}.\end{cases}$$

Therefore, for any $z \in \frac{1}{3}\hat{B}_k(p)\backslash\mathcal{S}$, we have

$$\left|\frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)(z)\right| \le C\omega(\tau^k)\cdot\begin{cases} r_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1} + r_1^{1/\beta_1-1} r_2^{-1}\tau^{-k(-1+1/\beta_1)} & \text{if } k \in [k_{2,p}+1, \min(\ell, k_p)], \\ r_1(z)^{1/\beta_1-1}\tau^{-k/\beta_1} + r_1^{1/\beta_1-1} r_2^{1/\beta_2-2}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k \le k_{2,p}. \end{cases} \qquad \square$$

It remains to estimate L_2. For simplicity, we write $h_k := -u_k + u_{k-1}$ as before, where we take $k \le \ell$. We will define $\beta_{\max} = \max(\beta_1, \beta_2)$.

Lemma 3.29. *Let $d = d_{\boldsymbol\beta}(p, q)$. There exists a constant $C(n, \boldsymbol\beta) > 0$ such that, for all $k \le \ell$,*

$$\left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q)\right| \le C\omega(\tau^k)\tau^{-k(1/\beta_1-1)} d^{1/\beta_1-1} \le C\omega(\tau^k)\tau^{-k(1/\beta_{\max}-1)} d^{1/\beta_{\max}-1}.$$

Proof. Case 1: First we assume that $r_p \le 2d$, so that $r_q \le 3d$ and $\ell + 2 \le k_p$. In particular, the balls $\hat{B}_k(p)$ are centered at either $p_1 \in \mathcal{S}_1$ or 0, depending on whether $k \ge k_{2,p} + 1$ or $k \le k_{2,p}$. As in the proof of Lemma 3.20, let $\gamma : [0, d] \to B_{\boldsymbol\beta}(0, 1)\backslash S$ be the $g_{\boldsymbol\beta}$-geodesic connecting p and q, let the two points q' and p' be defined as in (3-80), and let $\gamma_1, \gamma_2, \gamma_3$ be the $g_{\boldsymbol\beta}$-geodesics defined in that lemma. By the triangle inequality we calculate

$$\left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q)\right| \le \left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p')\right| + \left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p') - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q')\right| + \left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q') - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q)\right| =: L_1' + L_2' + L_3'.$$

Integrating along γ_3, where the coordinates $(r_1; r_2, \theta_2; z_3, \dots, z_n)$ are the same as p, we get (by (3-93))

$$L_1' = \left|\int_{\gamma_3} \frac{\partial}{\partial \theta_1}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right) d\theta_1\right| \le C\omega(\tau^k)\cdot\begin{cases} r_1(p)^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)} & \text{if } k \in [k_{2,p}+1, \ell], \\ r_1(p)^{1/\beta_1-1} r_2(p)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)} & \text{if } k \le k_{2,p}. \end{cases}$$

Integrating along γ_2, where the coordinates $(\theta_1; r_2, \theta_2; z_3, \dots, z_n)$ are the same as p' or q', we get by the estimate in Lemma 3.26 that

$$\begin{aligned} L_2' &= \left|\int_{\gamma_2} \frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right) dr_1\right| \\ &\le C\omega(\tau^k)\cdot\begin{cases} \tau^{-k} d + \tau^{-k(1/\beta_1-1)}|r_1(p) - r_1(q)|^{1/\beta_1-1} & \text{if } k \in [k_{2,p}+1, \ell], \\ r_2(p)^{1/\beta_2-1}\tau^{-k/\beta_2} d \\ \qquad + r_2(p)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}|r_1(p) - r_1(q)|^{1/\beta_1-1} & \text{if } k \le k_{2,p} \end{cases} \\ &\le C\omega(\tau^k)\cdot\begin{cases} \tau^{-k} d + \tau^{-k(1/\beta_1-1)} d^{1/\beta_1-1} & \text{if } k \in [k_{2,p}+1, \ell], \\ r_2(p)^{1/\beta_2-1}\tau^{-k/\beta_2} d + r_2(p)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)} d^{1/\beta_1-1} & \text{if } k \le k_{2,p}. \end{cases} \end{aligned}$$

To deal with the term L_3', we consider two cases for k: $\ell \ge k \ge k_{2,p} + 1$ and $k \le k_{2,p}$.

Case 1a: $k_{2,p}+1\le k\le \ell$. In this case the balls $\hat B_k(p)$ are centered at $p_1\in\mathcal S_1$. Here $\tau^{-k}\le\tau^{-\ell}\le 8^{-1}d^{-1}$ and $\tau^k\le\tau^{k_{2,p}+1}\le\frac12 r_2(p)$, so $r_2(q)\ge -d+r_2(p)\ge\tau^k$. The balls $\hat B_k(p)$ are disjoint from $\mathcal S_2$, and we can use the smooth coordinate $w_2=z_2^{\beta_2}$ as before. The functions $D_{w_2}D'h_k$ are $g_{\boldsymbol\beta}$-harmonic; hence by the gradient estimate we have

$$\sup_{(\hat B_k(p)/1.2)\setminus\mathcal S_1}|\nabla_{g_{\boldsymbol\beta}}(D_{w_2}D'h_k)|\le C(n)\frac{\|D_{w_2}D'h_k\|_{L^\infty(\hat B_k(p)/1.1)}}{\tau^k}\le C\tau^{-k}\omega(\tau^k).$$

From (3-88), we get

$$\sup_{(\hat B_k(p)/1.2)\setminus\mathcal S_1}\left|\frac{\partial^2}{\partial r_1\partial r_2}D'h_k\right|\le C(n)\tau^{-k}\omega(\tau^k). \tag{3-105}$$

Recalling that $r_1(p)=r_p\le 2d\le\frac12\tau^k$, the triangle inequality implies $r_1(q)\le 3d\le\frac12\tau^k$. The points in γ_1 have fixed (r_1,θ_1)-coordinates $(r_1(q),\theta_1(q))$, so integrating along γ_1 we get (by (3-104) and (3-105))

$$\begin{aligned}L_3'&\le\int_{\gamma_1}\left|\frac{\partial}{\partial r_2}\left(\frac{\partial^2h_k}{\partial r_1\partial r_2}\right)\right|+\left|\frac{\partial}{r_2\partial\theta_2}\left(\frac{\partial^2h_k}{\partial r_1\partial r_2}\right)\right|+\left|D'\left(\frac{\partial^2h_k}{\partial r_1\partial r_2}\right)\right|\\&\le Cd\omega(\tau^k)(r_1(q)^{1/\beta_1-1}\tau^{-k/\beta_1}+\tau_1(q)^{1/\beta_1-1}\min(r_2(p),r_2(q))^{-1}\tau^{-k(1/\beta_1-1)}+\tau^{-k})\\&\le C\tau^{-k}\omega(\tau^k)\cdot d\le C\tau^{-k(1/\beta_1-1)}\omega(\tau^k)d^{1/\beta_1-1}.\end{aligned}$$

Case 1b: $k\le k_{2,p}$. In this case $\tau^k\ge\tau^{k_{2,p}}\ge r_2(p)$ and $\tau^k\ge\tau^\ell\ge 8d$. Thus $r_2(q)\le r_2(p)+d\le\frac32\tau^k$. We choose points $\tilde q$ and $\hat q$ as in (3-84), and let $\tilde\gamma_1$, $\tilde\gamma$ and $\hat\gamma$ be $g_{\boldsymbol\beta}$-geodesics defined as in the proof of Lemma 3.20. Then we have

$$L_3'\le\left|\frac{\partial^2h_k}{\partial r_1\partial r_2}(q')-\frac{\partial^2h_k}{\partial r_1\partial r_2}(\tilde q)\right|+\left|\frac{\partial^2h_k}{\partial r_1\partial r_2}(\tilde q)-\frac{\partial^2h_k}{\partial r_1\partial r_2}(\hat q)\right|+\left|\frac{\partial^2h_k}{\partial r_1\partial r_2}(\hat q)-\frac{\partial^2h_k}{\partial r_1\partial r_2}(q)\right|=:L_1''+L_2''+L_3''.$$

We will estimate L_1'', L_2'' and L_3'' term by term by integrating appropriate functions along the geodesics $\tilde\gamma_1$, $\tilde\gamma$ and $\hat\gamma$ as follows: The points in $\hat\gamma$ have fixed $(r_1,\theta_1;r_2;s)$-coordinates $(r_1(q),\theta_1(q);r_2(q);s(q))$, so (by (3-102))

$$\begin{aligned}L_3''=\left|\int_{\hat\gamma}\frac{\partial}{\partial\theta_2}\left(\frac{\partial^2h_k}{\partial r_1\partial r_2}\right)d\theta_2\right|&\le C\omega(\tau^k)r_1(q)^{1/\beta_1-1}r_2(q)^{1/\beta_2-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}\\&\le C\omega(\tau^k)r_1(q)^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)}\le C\tau^{-k(1/\beta_1-1)}\omega(\tau^k)d^{1/\beta_1-1}.\end{aligned}$$

Integrating along $\tilde\gamma$, where the points have constant r_1-coordinate $r_1(q)$, we get (by (3-104))

$$\begin{aligned}L_2''&=\left|\int_{\tilde\gamma}\frac{\partial}{\partial r_2}\left(\frac{\partial^2h_k}{\partial r_1\partial r_2}\right)dr_2\right|\\&\le C\omega(\tau^k)\big(r_1(q)^{1/\beta_1-1}\tau^{-k/\beta_1}|r_2(q)-r_2(p)|\\&\qquad+r_1(q)^{1/\beta_1-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}|r_2(q)^{1/\beta_2-1}-r_2(p)^{1/\beta_2-1}|\big)\\&\le C\omega(\tau^k)(r_1(q)^{1/\beta_1-1}\tau^{-k/\beta_1}d+r_1(q)^{1/\beta_1-1}\tau^{-k(-2+1/\beta_1+1/\beta_2)}d^{1/\beta_2-1})\\&\le C\omega(\tau^k)r_1(q)^{1/\beta_1-1}\tau^{-k(-1+1/\beta_1)}\\&\le C\tau^{-k(1/\beta_1-1)}\omega(\tau^k)d^{1/\beta_1-1}.\end{aligned}$$

Integrating along $\tilde{\gamma}_1$, where the points have constant $(r_1, \theta_1; r_2, \theta_2)$-coordinates, we have (by (3-71))

$$L_1'' \le \int_{\tilde{\gamma}_1} \left| D'\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right| \le C r_1(q)^{1/\beta_1 - 1} r_2(p)^{1/\beta_2 - 1} \tau^{-k(-1+1/\beta_1+1/\beta_2)} d$$
$$\le C d \tau^{-k} \omega(\tau^k) \le C d^{1/\beta_1 - 1} \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k).$$

Combining both cases, we conclude that

$$L_3' \le C \tau^{-k(1/\beta_1 - 1)} \omega(\tau^k) d^{1/\beta_1 - 1}.$$

Then by the estimates above for L_1' and L_2', we finally get, for all $k \le \ell$,

$$\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \le C\omega(\tau^k)\tau^{-k(1/\beta_1 - 1)} d^{1/\beta_1 - 1} \le C\omega(\tau^k)\tau^{-k(1/\beta_{\max} - 1)} d^{1/\beta_{\max} - 1},$$

where in the last inequality we use the fact that $\tau^{-k} d \le \frac{1}{8} < 1$ when $k \le \ell$. Hence we finish the proof of Lemma 3.29 in the case $r_p \le 2d$.

Now we deal with the remaining cases.

Case 2: Here we assume $\min(r_p, r_q) = r_p \ge 2d$ and $\ell \le k_p$. In this case $\tau^{k_p} \approx r_p \ge 2d \ge \tau^{\ell+3}$, so $\ell + 3 \ge k_p$. It follows by the triangle inequality that $d_{\boldsymbol{\beta}}(\gamma(t), \mathcal{S}) \ge d$, where γ is the $g_{\boldsymbol{\beta}}$-geodesic joining p to q as before. In particular, this implies that $\min(r_1(\gamma(t)), r_2(\gamma(t))) \ge d$.

Since $\ell \le k_p$, Lemmas 3.24–3.28 hold for all $k \le \ell$ and $r_1(p) \approx \tau^{k_p} \le \tau^{\ell}$, so

$$r_1(\gamma(t)) \le d + r_1(p) \le \tfrac{9}{8}\tau^{\ell} \le \tfrac{9}{8}\tau^{k}.$$

We calculate the gradient of $\partial^2 h_k / \partial r_1 \partial r_2$ along the geodesic γ as

$$\left| \nabla_{g_{\boldsymbol{\beta}}} \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right|^2 (\gamma(t)) = \left| \frac{\partial}{\partial r_1}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right|^2 + \left| \frac{1}{\beta_1 r_1 \partial \theta_1}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right|^2 + \left| \frac{\partial}{\partial r_2}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right|^2$$
$$+ \left| \frac{\partial}{\beta_2 r_2 \partial \theta_2}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right|^2 + \sum_j \left| \frac{\partial}{\partial s_j}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right|^2.$$

Case 2a: $k_{2,p} + 1 \le k \le \ell$. Here along γ we have

$$r_2(\gamma(t)) \ge r_2(p) - d \ge \tau^k - d \ge \tfrac{7}{8}\tau^k.$$

Then by Lemmas 3.24–3.28, along γ we have

$$\left| \nabla_{g_{\boldsymbol{\beta}}} \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(\gamma(t)) \right| \le C\omega(\tau^k)(\tau^{-k} + d^{1/\beta_1 - 2}\tau^{-k(1/\beta_1 - 1)}).$$

Integrating this inequality along γ we get

$$\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \le \int_{\gamma} \left| \nabla_{g_{\boldsymbol{\beta}}} \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \le C\omega(\tau^k)(d\tau^{-k} + d^{1/\beta_1 - 1}\tau^{-k(1/\beta_1 - 1)})$$
$$\le C d^{1/\beta_1 - 1}\tau^{-k(1/\beta_1 - 1)}\omega(\tau^k).$$

Case 2b: $k \le k_{2,p}$. Here along γ we have

$$r_2(\gamma(t)) \le r_2(p) + d \le \tau^k + d \le \tfrac{9}{8}\tau^k.$$

Then by Lemmas 3.24–3.28, along γ we have

$$\left|\nabla_{g_\beta} \frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right| \le C\omega(\tau^k)(\tau^{-k} + d^{1/\beta_1 - 2}\tau^{-k(1/\beta_1 - 1)}).$$

Integrating this inequality along γ we again get

$$\left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q)\right| \le \int_\gamma \left|\nabla_{g_\beta} \frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right| \le C d^{1/\beta_1 - 1}\tau^{-k(1/\beta_1 - 1)}\omega(\tau^k).$$

Case 3: Here we assume $\min(r_p, r_q) = r_p \ge 2d$ but $\ell \ge k_p + 1$. The case when $k \le k_p$ can be dealt with by the same argument as in Case 2, so we omit it and only consider the case when $k_p + 1 \le k \le \ell$. Here $r_2(p) \ge r_1(p) \ge \tau^k \ge \tau^\ell > 8d$, and hence

$$r_1(\gamma(t)) \ge \tfrac{7}{8}\tau^k \quad \text{and} \quad r_2(\gamma(t)) \ge \tfrac{7}{8}\tau^k$$

for any point $\gamma(t)$ in the geodesic γ. By the triangle inequality it follows that $\gamma \subset \frac{1}{3}\hat{B}_k(p) = B_{\boldsymbol\beta}\big(p, \frac{1}{3}\tau^k\big)$.

As before, we can introduce smooth coordinates $w_1 = z_1^{\beta_1}$ and $w_2 = z_2^{\beta_2}$, and $g_{\boldsymbol\beta}$ becomes the standard smooth Euclidean metric $g_{\mathbb{C}^n}$ under these coordinates. Moreover, h_k is the usual Euclidean harmonic function $\Delta_{g_{\mathbb{C}^n}} h_k = 0$ on $\hat{B}_k(p)$. By the standard derivatives estimates we have

$$\sup_{\hat{B}_k(p)/2} (|D^3_{w_1,w_2} h_k| + |D'(D^2_{w_1,w_2}) h_k|) \le C\tau^{-k}\omega(\tau^k).$$

From the equation

$$\frac{\partial^2 h_k}{\partial r_1 \partial r_2} = \frac{w_1 w_2}{r_1 r_2}\frac{\partial^2 h_k}{\partial w_1 \partial w_2} + \frac{\bar{w}_1 w_2}{r_1 r_2}\frac{\partial^2 h_k}{\partial \bar{w}_1 \partial w_2} + \frac{w_1 \bar{w}_2}{r_1 r_2}\frac{\partial^2 h_k}{\partial w_1 \partial \bar{w}_2} + \frac{\bar{w}_1 \bar{w}_2}{r_1 r_2}\frac{\partial^2 h_k}{\partial \bar{w}_1 \partial \bar{w}_2}$$

we see that, for $i = 1, 2$,

$$\sup_{\hat{B}_k(p)/2}\left|\frac{\partial}{\partial w_i}\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right| \le \frac{C}{r_i}\omega(\tau^k) + C\tau^{-k}\omega(\tau^k) \quad \text{and} \quad \sup_{\hat{B}_k(p)/2}\left|D'\left(\frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right)\right| \le C\tau^{-k}\omega(\tau^k).$$

From this we see that

$$\sup_\gamma \left|\nabla_{g_\beta} \frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right| \le \sup_\gamma \left(C\tau^{-k}\omega(\tau^k) + \frac{C}{r_1}\omega(\tau^k) + \frac{C}{r_2}\omega(\tau^k)\right) \le C\tau^{-k}\omega(\tau^k).$$

Integrating along γ we get

$$\begin{aligned}\left|\frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q)\right| \le \int_\gamma \left|\nabla_{g_\beta} \frac{\partial^2 h_k}{\partial r_1 \partial r_2}\right| &\le C d\tau^{-k}\omega(\tau^k) \\ &\le C d^{1/\beta_1 - 1}\tau^{-k(1/\beta_1 - 1)}\omega(\tau^k).\end{aligned}$$

Combining the estimates in all three cases, we finish the proof of Lemma 3.29. □

By Lemma 3.29,

$$
\begin{aligned}
L_2 &= \left|\frac{\partial^2 u_\ell}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_\ell}{\partial r_1 \partial r_2}(q)\right| \\
&\le \left|\frac{\partial^2 u_2}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_2}{\partial r_1 \partial r_2}(q)\right| + C d^{1/\beta_{\max}-1} \sum_{k=3}^{\ell} \tau^{-k(1/\beta_{\max}-1)} \omega(\tau^k).
\end{aligned}
\tag{3-106}
$$

To finish the proof, it suffices to estimate the first term on the right-hand side of the above equation. Recall that we assume u_2 is a $g_{\boldsymbol{\beta}}$-harmonic function defined on the ball $\hat{B}_2(p)$, which is centered at $p_{1,2} \in \mathcal{S}_1 \cap \mathcal{S}_2$ and has radius $2\tau^2$. We also know u_2 satisfies the L^∞-estimate by the maximum principle: there exists some $C = C(n) > 0$ such that

$$
\|u_2\|_{L^\infty(\hat{B}_k(p))} \le C(\|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \omega(\tau^2)). \tag{3-107}
$$

Recall that the proofs of the estimates in Lemmas 3.24–3.28 in the case when $k \le k_{2,p}$ work for any $g_{\boldsymbol{\beta}}$-harmonic functions defined on suitable balls, and we can repeat the arguments there replacing the L^∞-estimate of h_k, namely $\|h_k\|_{L^\infty} \le C\tau^{2k}\omega(\tau^k)$, by the L^∞-estimate of u_2 given in (3-107) to get similar estimates as in those lemmas. We will omit the details. Given these estimates, we can repeat the proof of Lemma 3.29 to prove the estimates

$$
\left|\frac{\partial^2 u_2}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_2}{\partial r_1 \partial r_2}(q)\right| \le C d^{1/\beta_{\max}-1}(\|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \omega(\tau^2)).
$$

This inequality, combined with (3-106), gives the final estimate of the L_2 term, that is

$$
L_2 \le C d^{1/\beta_{\max}-1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + C d^{1/\beta_{\max}-1} \sum_{k=2}^{\ell} \tau^{-k(1/\beta_{\max}-1)} \omega(\tau^k). \tag{3-108}
$$

By Lemmas 3.22 and 3.23 and the estimate (3-108) for L_2, we are ready to prove the following estimate; see (1-5).

Proposition 3.30. *For given $p, q \in B_{\boldsymbol{\beta}}\big(0, \frac{1}{2}\big) \setminus \mathcal{S}$, there is a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that*

$$
\left|\frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q)\right| \le C\left(d^{1/\beta_{\max}-1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \int_0^d \frac{\omega(r)}{r}\, dr + d^{1/\beta_{\max}-1} \int_d^1 \frac{\omega(r)}{r^{1/\beta_{\max}}}\, dr\right).
$$

Proof. From Lemmas 3.22 and 3.23 and the estimate (3-108) for L_2, we have

$$
\begin{aligned}
\left|\frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q)\right| &\le C\left(d^{1/\beta_{\max}-1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + d^{1/\beta_{\max}-1} \sum_{k=2}^{\ell} \tau^{-k(1/\beta_{\max}-1)} \omega(\tau^k) + \sum_{k=\ell}^{\infty} \omega(\tau^k)\right) \\
&\le C\left(d^{1/\beta_{\max}-1} \|u\|_{L^\infty(B_{\boldsymbol{\beta}}(0,1))} + \int_0^d \frac{\omega(r)}{r}\, dr + d^{1/\beta_{\max}-1} \int_d^1 \frac{\omega(r)}{r^{1/\beta_{\max}}}\, dr\right),
\end{aligned}
$$

where the last inequality follows from the fact that $\omega(r)$ is monotonically increasing. □

Finally, we remark that the estimates for the other operators in (3-89) follow similarly; we omit the proofs and state that the estimates are the same as the estimates for $\partial^2 u / \partial r_1 \partial r_2$ in Proposition 3.30.

3E. ***Nonflat conical Kähler metrics.*** In this section, we will consider the Schauder estimates for general conical Kähler metrics on $B_{\boldsymbol\beta}(0,2)\subset\mathbb{C}^n$ with cone angle $2\pi\boldsymbol\beta$ along the simple normal crossing hypersurface $\mathcal{S}$. Let ω be such a metric. By definition, there exists a constant $C\ge 1$ such that

$$C^{-1}\omega_{\boldsymbol\beta}\le\omega\le C\omega_{\boldsymbol\beta}\quad\text{in } B_{\boldsymbol\beta}(0,2)\backslash\mathcal{S},\tag{3-109}$$

where $\omega_{\boldsymbol\beta}$ is the standard flat conical metric as before. Since ω is closed and $B_{\boldsymbol\beta}(0,2)$ is simply connected, we can write $\omega=\sqrt{-1}\partial\bar\partial\phi$ for some strictly plurisubharmonic function ϕ. By elliptic regularity, ϕ is Hölder continuous under the Euclidean metric on $B_{\boldsymbol\beta}(0,2)$.

We fix $\alpha\in(0,\min\{1/\beta_{\max}-1,1\})$.

Definition 3.31. We say $\omega=g$ is a $C^{0,\alpha}_{\boldsymbol\beta}$ Kähler metric on $B_{\boldsymbol\beta}(0,2)$ if it satisfies (3-109) and the Kähler potential ϕ of ω belongs to $C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,2))$.

We are interested in studying the Laplacian equation

$$\Delta_g u=f\quad\text{in } B_{\boldsymbol\beta}(0,1),\tag{3-110}$$

where $f\in C^{0,\alpha}_{\boldsymbol\beta}(\overline{B_{\boldsymbol\beta}(0,1)})$ and $u\in C^{2,\alpha}_{\boldsymbol\beta}$. We will prove the following scaling-invariant interior Schauder estimates. The proof closely follows that of Theorem 6.6 in [18], so we mainly focus on the differences.

Proposition 3.32. *There exists a constant* $C=C(n,\boldsymbol\beta,\|g\|^*_{C^{0,\alpha}_{\boldsymbol\beta}})>0$ *such that, if* $u\in C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))$ *satisfies* (3-110), *then*

$$\|u\|^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))}\le C(\|u\|_{C^0(B_{\boldsymbol\beta}(0,1))}+\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))}).\tag{3-111}$$

Proof. Given any points $x_0\ne y_0\in B_{\boldsymbol\beta}(0,1)$, assume $d_{x_0}=\min(d_{x_0},d_{y_0})$; recall $d_x=d_{\boldsymbol\beta}(x,\partial B_{\boldsymbol\beta}(0,1))$. Let $\mu\in\left(0,\tfrac14\right)$ be a small number to be determined later. Write $d=\mu d_{x_0}$, and define $B:=B_{\boldsymbol\beta}(x_0,d)$ and $\tfrac12 B:=B_{\boldsymbol\beta}\left(x_0,\tfrac12 d\right)$.

Case 1: $d_{\boldsymbol\beta}(x_0,y_0)<\tfrac12 d$.

Case 1a: $B_{\boldsymbol\beta}(x_0,d)\cap\mathcal{S}=\varnothing$. We introduce smooth complex coordinates $\{w_1=z_1^{\beta_1},w_2=z_2^{\beta_2},z_3,\dots,z_n\}$ on $B_{\boldsymbol\beta}(x_0,d)$, under which $g_{\boldsymbol\beta}$ becomes the Euclidean metric and the components of g become C^α in the usual sense. Equation (3-110) has C^α leading coefficients, and we can apply Theorem 6.6 in [18] to conclude that (the following inequality is understood in the new coordinates)

$$[u]^*_{C^{2,\alpha}(B)}\le C(\|u\|_{C^0(B)}+\|f\|^{(2)}_{C^{0,\alpha}(B)}).\tag{3-112}$$

Recall that T denotes the second-order operators appearing in (2-2). Let D denote the ordinary first-order operators in $\{w_1,w_2,z_3,\dots,z_n\}$. We calculate

$$\begin{aligned}|Tu(x_0)-Tu(y_0)|&\le|D^2u(x_0)-D^2u(y_0)|+\frac{d_{\boldsymbol\beta}(x_0,y_0)}{d}(|D^2u(x_0)|+|D^2u(y_0)|)\\&\le\frac{4d_{\boldsymbol\beta}(x_0,y_0)^\alpha}{d^{2+\alpha}}[u]^*_{C^{2,\alpha}(B)}+\frac{4d_{\boldsymbol\beta}(x_0,y_0)}{d^3}[u]^*_{C^2(B)}\\&\le\frac{8d_{\boldsymbol\beta}(x_0,y_0)^\alpha}{d^{2+\alpha}}[u]^*_{C^{2,\alpha}(B)}+C\frac{d_{\boldsymbol\beta}(x_0,y_0)^\alpha}{d^{2+\alpha}}\|u\|_{C^0(B)}\ \text{(by the interpolation inequality).}\end{aligned}$$

Then we get

$$d_{x_0}^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha}\le\frac{C}{\mu^{2+\alpha}}\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B)}+\frac{C}{\mu^{2+\alpha}}\|u\|_{C^0(B)}. \tag{3-113}$$

Case 1b: $B_{\boldsymbol\beta}(x_0,d)\cap\mathcal{S}\neq\varnothing$. Let $\hat{x}_0\in\mathcal{S}$ be the nearest possible point x_0 to $\mathcal{S}$. We consider the ball $\hat{B}:=B_{\boldsymbol\beta}(\hat{x}_0,2d)$, which is contained in $B_{\boldsymbol\beta}(0,1)$ by the triangle inequality. As in [14], we introduce a (nonholomorphic) basis of $T^*_{1,0}(\mathbb{C}^n\backslash\mathcal{S})$

$$\{\epsilon_j:=dr_j+\sqrt{-1}\beta_jr_j\,d\theta_j,dz_k\}_{j=1,2;\,k=3,\dots,n},$$

and the dual basis of $T_{1,0}(\mathbb{C}^n\backslash\mathcal{S})$

$$\left\{\gamma_j:=\frac{\partial}{\partial r_j}-\sqrt{-1}\frac{1}{\beta_jr_j}\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial z_k}\right\}_{j=1,2;\,k=3,\dots,n}.$$

We can write the $(1,1)$-form ω in the basis $\{\epsilon_j\wedge\bar\epsilon_k,\epsilon_j\wedge d\bar z_k,dz_k\wedge\bar\epsilon_j,dz_j\wedge d\bar z_k\}$ as

$$\omega=g_{\epsilon_j\bar\epsilon_k}\epsilon_j\wedge\bar\epsilon_k+g_{\epsilon_j\bar k}\epsilon_j\wedge d\bar z_k+g_{k\bar\epsilon_j}\,dz_k\wedge\bar\epsilon_j+g_{j\bar k}\,dz_j\wedge d\bar z_k, \tag{3-114}$$

where

$$\begin{aligned}g_{\epsilon_j\bar\epsilon_k}&=\sqrt{-1}\partial\bar\partial\phi(\gamma_j,\bar\gamma_k), & g_{\epsilon_j\bar k}&=\sqrt{-1}\partial\bar\partial\phi\left(\gamma_j,\frac{\partial}{\partial\bar z_k}\right),\\ g_{k\bar\epsilon_j}&=\sqrt{-1}\partial\bar\partial\phi\left(\frac{\partial}{\partial z_k},\bar\gamma_j\right), & g_{k\bar j}&=\frac{\partial^2}{\partial z_k\partial\bar z_j}\phi.\end{aligned} \tag{3-115}$$

We remark that all the second-order derivatives of ϕ appearing in (3-115) are linear combinations of $|z_j|^{2-2\beta_j}(\partial^2/\partial z_j\partial z_{\bar j})N_jN_k$ $(j\neq k)$, N_jD' and $(D')^2$, which are studied in Theorem 1.2. The standard metric $\omega_{\boldsymbol\beta}$ becomes the identity matrix under the basis above for $(1,1)$-forms. If ω is $C^{0,\alpha}_{\boldsymbol\beta}$, all the coefficients in the expression for ω in (3-114) are $C^{0,\alpha}_{\boldsymbol\beta}$-continuous, and the cross terms $g_{\epsilon_j\bar\epsilon_k}$ $(j\neq k)$ and $g_{\epsilon_j\bar k}$ tend to zero when approaching the corresponding singular sets $\mathcal{S}_j$ or $\mathcal{S}_k$. Moreover, the limit of $g_{j\bar k}\,dz_j\wedge d\bar z_k$ when approaching $\mathcal{S}_1\cap\cdots\cap\mathcal{S}_p$ defines a Kähler metric on it. Rescaling or rotating the coordinates if necessary we may assume at $\hat{x}_0\in\mathcal{S}$ that $g_{\epsilon_j\bar\epsilon_j}(\hat{x}_0)=1$, $g_{j\bar k}(\hat{x}_0)=\delta_{jk}$ and the cross terms vanish at $\hat{x}_0$. Let $\omega_{\boldsymbol\beta}$ be the standard cone metric under these new coordinates near $\hat{x}_0$. We can rewrite (3-110) as

$$\Delta_gu(z)=\Delta_{g_{\boldsymbol\beta}}u(z)+\eta(z)\cdot i\partial\bar\partial u(z)=f(z)\quad\text{for all } z\notin\mathcal{S}$$

for some hermitian matrix $\eta(z)=(\eta^{j\bar k})^n_{j,k=1}$, $\eta^{j\bar k}=g^{j\bar k}(z)-g^{j\bar k}_{\boldsymbol\beta}$. It is not hard to see the term $\eta(z)\cdot i\partial\bar\partial u$ can be written as

$$\sum_{j,k=1}^{2}(g^{\epsilon_j\bar\epsilon_k}(z)-\delta_{jk})u_{\epsilon_j\bar\epsilon_j}+2Re\sum_{\substack{1\le j\le2\\3\le k\le n}}g^{\epsilon_j\bar k}u_{\epsilon_j\bar k}+\sum_{j,k=3}^{n}(g^{j\bar k}(z)-\delta_{jk})u_{j\bar k}, \tag{3-116}$$

where g with the upper indices denotes the inverse matrix of g. We consider the equivalent form of (3-110) on $\hat{B}$:

$$\Delta_{g_{\boldsymbol\beta}}u=f-\eta\cdot\sqrt{-1}\partial\bar\partial u=:\hat f,\quad u\in C^0(\hat B)\cap C^2(\hat B\backslash\mathcal{S}).$$

Observing that $x_0, y_0 \in B_{\boldsymbol\beta}\big(\hat x_0, \frac{3}{2}d\big)$, we can apply the scaled inequality (1-7) of Theorem 1.2 to conclude that

$$d^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha} \le C(\|u\|_{C^0(\hat B)} + \|\hat f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)});$$

thus

$$d_{x_0}^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha} \le \frac{C}{\mu^{2+\alpha}}(\|u\|_{C^0(\hat B)} + \|\hat f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)}). \tag{3-117}$$

<u>Case 2</u>: $d_{\boldsymbol\beta}(x_0,y_0) \ge \frac{1}{2}d$.

$$d_{x_0}^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha} \le 4d_{x_0}^{2+\alpha}\frac{|Tu(x_0)|+|Tu(y_0)|}{d^\alpha} \le \frac{8}{\mu^\alpha}[u]^*_{C^2_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))}. \tag{3-118}$$

Combining (3-113), (3-117) and (3-118) we get

$$\begin{aligned} d_{x_0}^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha} \le \frac{8}{\mu^\alpha}[u]^*_{C^2_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))} &+ \frac{C}{\mu^{2+\alpha}}(\|u\|_{C^0(\hat B)} + \|\hat f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)}) \\ &+ \frac{C}{\mu^{2+\alpha}}\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B)} + \frac{C}{\mu^{2+\alpha}}\|u\|_{C^0(B)}. \end{aligned} \tag{3-119}$$

By definition it is easy to see that (writing $B_{\boldsymbol\beta} = B_{\boldsymbol\beta}(0,1)$)

$$\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B)} \le C\mu^2\|f\|^{(2)}_{C^0(B_{\boldsymbol\beta})} + C\mu^{2+\alpha}[f]^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} \le \mu^2\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}.$$

We calculate

$$\begin{aligned} \|\hat f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)} &\le \|\eta\|^{(0)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)}\|Tu\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)} + \|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(\hat B)} \\ &\le C_0[g]^*_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}\mu^\alpha(\mu^2[u]^*_{C^2_{\boldsymbol\beta}(B_{\boldsymbol\beta})} + \mu^{2+\alpha}[u]^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}) + \mu^2\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} \\ &\le C_0[g]^*_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}\mu^\alpha(C(\mu)\|u\|_{C^0(B_{\boldsymbol\beta})} + 2\mu^{2+\alpha}[u]^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}) + \mu^2\|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}, \\ \frac{8}{\mu^\alpha}[u]^*_{C^2_{\boldsymbol\beta}(B_{\boldsymbol\beta})} &\le \mu^\alpha[u]^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} + C(\mu)\|u\|_{C^0(B_{\boldsymbol\beta})}. \end{aligned}$$

If we choose $\mu > 0$ small enough that $\mu^\alpha(2C_0[g]^*_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} + 1) \le \frac{1}{2}$, then we get from (3-119) and the inequalities above that

$$d_{x_0}^{2+\alpha}\frac{|Tu(x_0)-Tu(y_0)|}{d_{\boldsymbol\beta}(x_0,y_0)^\alpha} \le \tfrac{1}{2}[u]^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} + C(\mu)(\|u\|_{C^0(B_{\boldsymbol\beta})} + \|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}).$$

Taking the supremum over $x_0 \ne y_0 \in B_{\boldsymbol\beta}(0,1)$, we conclude from the inequality above that

$$[u]^*_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})} \le C(\|u\|_{C^0(B_{\boldsymbol\beta})} + \|f\|^{(2)}_{C^{0,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})}).$$

Proposition 3.32 then follows from interpolation inequalities. □

Remark 3.33. It follows easily from the proof of Proposition 3.32 that estimate (3-111) also holds for metric balls $B_{\boldsymbol\beta}(p,R) \subset B_{\boldsymbol\beta}(0,1)$ whose center p may not lie in $\mathcal{S}$.

Remark 3.34. The Schauder estimate was first established by Donaldson [14] for a background cone metric with singularity along a smooth divisor assuming $u \in C^{2,\alpha}_{\beta}$; this latter assumption was removed by Brendle [1] in the case $\beta \in \left(0, \frac{1}{2}\right)$ and by Jeffres, Mazzeo and Rubinstein [23], requiring only a weak solution. Jeffres, Mazzeo and Rubinstein [23] then extend the results to nonflat background metrics using a perturbation argument. This is the first time a Schauder estimate for the linear conic equation in the smooth divisor case appeared in the literature with a full proof.

An immediate corollary to Proposition 3.32 is the following interior Schauder estimate.

Corollary 3.35. *Suppose u satisfies* (3-110). *For any compact subset $K \Subset B_{\boldsymbol{\beta}}(0,1)$, there exists a constant $C = C(n, \boldsymbol{\beta}, K, \|g\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,1))}) > 0$ such that*

$$\|u\|_{C^{2,\alpha}_{\boldsymbol{\beta}}(K)} \le C(\|u\|_{C^0(B_{\boldsymbol{\beta}}(0,1))} + \|f\|_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}(0,1))}).$$

Next we will show that (3-110) admits a unique $C^{2,\alpha}_{\boldsymbol{\beta}}$-solution for any $f \in C^{0,\alpha}_{\boldsymbol{\beta}}(\overline{B_{\boldsymbol{\beta}}(0,1)})$ and boundary value $\varphi \in C^0(\partial B_{\boldsymbol{\beta}}(0,1))$. We will follow the argument in Section 6.5 in [18]. In the following we write $B_{\boldsymbol{\beta}} = B_{\boldsymbol{\beta}}(0,1)$ for simplicity.

Lemma 3.36. *Let $\sigma \in (0,1)$ be a given number. Suppose $u \in C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})$ solves* (3-110), $\|u\|^{(-\sigma)}_{C^0(B_{\boldsymbol{\beta}})} < \infty$ *and $\|f\|^{(2-\sigma)}_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} < \infty$. Then there exists a $C = C(n, \boldsymbol{\beta}, \alpha, g, \sigma) > 0$ such that*

$$\|u\|^{(-\sigma)}_{C^{2,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})} \le C(\|u\|^{(-\sigma)}_{C^0(B_{\boldsymbol{\beta}})} + \|f\|^{(2-\sigma)}_{C^{0,\alpha}_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}})}).$$

Proof. Given the estimates in Proposition 3.32, the proof is identical to that of Lemma 6.20 in [18]. We omit the details. □

Lemma 3.37. *Let $u \in C^2_{\boldsymbol{\beta}}(B_{\boldsymbol{\beta}}) \cap C^0(\overline{B_{\boldsymbol{\beta}}})$ solve the equation $\Delta_g u = f$ and $u \equiv 0$ on $\partial B_{\boldsymbol{\beta}}$. For any $\sigma \in (0,1)$, there exists a constant $C = C(n, \boldsymbol{\beta}, \sigma, g) > 0$ such that*

$$\|u\|^{(-\sigma)}_{C^0(B_{\boldsymbol{\beta}})} = \sup_{x \in B_{\boldsymbol{\beta}}} d_x^{-\sigma} |u(x)| \le C \sup_{x \in B_{\boldsymbol{\beta}}} d_x^{2-\sigma} |f(x)| = C\|f\|^{(2-\sigma)}_{C^0(B_{\boldsymbol{\beta}})},$$

where $d_x = d_{\boldsymbol{\beta}}(x, \partial B_{\boldsymbol{\beta}})$ as before.

Proof. Consider the function $w_1 = (1 - d^2_{\boldsymbol{\beta}})^\sigma$, where $d_{\boldsymbol{\beta}}(x) = d_{\boldsymbol{\beta}}(x, 0)$. We calculate

$$\begin{aligned} \Delta_g w_1 &= \sigma(1-d^2_{\boldsymbol{\beta}})^{\sigma-2}(-(1-d^2_{\boldsymbol{\beta}}) \operatorname{tr}_g g_{\boldsymbol{\beta}} - (1-\sigma)|\nabla d^2_{\boldsymbol{\beta}}|^2_g) \\ &\le \sigma(1-d^2_{\boldsymbol{\beta}})^{\sigma-2}(-C^{-1}(1-d^2_{\boldsymbol{\beta}}) - 4C^{-1} d^2_{\boldsymbol{\beta}}(1-\sigma)) \le -c_0 \sigma(1-d^2_{\boldsymbol{\beta}})^{\sigma-2}. \end{aligned}$$

Take a large constant $A > 1$ such that, for $w = Aw_1$,

$$\Delta_g w \le -(1-d_{\boldsymbol{\beta}})^{\sigma-2} \le -\frac{|f|}{N} \quad \text{in } B_{\boldsymbol{\beta}},$$

where

$$N = \sup_{x \in B_{\boldsymbol{\beta}}} d_x^{2-\sigma} |f(x)| = \sup_{x \in B_{\boldsymbol{\beta}}} (1 - d_{\boldsymbol{\beta}}(x))^{2-\sigma} |f(x)|.$$

Hence $\Delta_g(Nw \pm u) \le 0$, and from the definition of w we also have $w|_{\partial B_{\boldsymbol{\beta}}} \equiv 0$. By the maximum principle we obtain $|u(x)| \le Nw \le CN(1 - d_{\boldsymbol{\beta}}(x))^\sigma = CNd_x^\sigma$, and hence the lemma is proved. □

Proposition 3.38. *Given any function $f \in C^{0,\alpha}_{\beta}(\overline{B_\beta})$, the Dirichlet problem $\Delta_g u = f$ in B_β and $u \equiv 0$ on ∂B_β admits a unique solution $u \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta})$.*

Proof. The proof of this proposition is almost identical to that of Theorem 6.22 in [18]. For completeness, we provide the detailed argument. Fix $\sigma \in (0, 1)$ and define a family of operators $\Delta_t = t\Delta_g + (1-t)\Delta_{g_\beta}$. It is straightforward to see that Δ_t is associated to some cone metric which also satisfies (3-109). We study the Dirichlet problem

$$\Delta_t u_t = f \quad \text{in } B_\beta, \qquad u_t \equiv 0 \quad \text{on } \partial B_\beta. \tag{$*_t$}$$

Equation $(*_0)$ admits a unique solution $u_0 \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta})$ by Proposition 3.7. By Theorem 5.2 in [18], in order to apply the continuity method to solve $(*_1)$, it suffices to show Δ_t^{-1} defines a bounded linear operator between some Banach spaces. More precisely, define

$$\mathcal{B}_1 := \{u \in C^{2,\alpha}_{\beta}(B_\beta) \mid \|u\|^{(-\sigma)}_{C^{2,\alpha}_{\beta}(B_\beta)} < \infty\},$$
$$\mathcal{B}_2 := \{f \in C^{0,\alpha}_{\beta}(B_\beta) \mid \|f\|^{(2-\sigma)}_{C^{0,\alpha}_{\beta}(B_\beta)} < \infty\}.$$

By definition any $u \in \mathcal{B}_1$ is continuous on $\overline{B_\beta}$ and $u = 0$ on ∂B_β. By Lemmas 3.36 and 3.37, we have

$$\|u\|_{\mathcal{B}_1} = \|u\|^{(-\sigma)}_{C^{2,\alpha}_{\beta}(B_\beta)} \le C\|f\|^{(2-\sigma)}_{C^{0,\alpha}_{\beta}(B_\beta)} = C\|\Delta_t u\|_{\mathcal{B}_2},$$

for some constant C independent of $t \in [0, 1]$. Thus $(*_1)$ admits a solution $u \in \mathcal{B}_1$. □

Corollary 3.39. *For any given $\varphi \in C^0(\partial B_\beta)$ and $f \in C^{0,\alpha}_{\beta}(\overline{B_\beta})$, the Dirichlet problem*

$$\Delta_g u = f \quad \textit{in } B_\beta \qquad \textit{and} \qquad u = \varphi \quad \textit{on } \partial B_\beta, \tag{3-120}$$

admits a unique solution $u \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta})$.

Proof. We may extend φ continuously to B_β and assume $\varphi \in C^0(\overline{B_\beta})$. Take a sequence of functions $\varphi_k \in C^{2,\alpha}_{\beta}(\overline{B_\beta}) \cap C^0(\overline{B_\beta})$ which converges uniformly to φ on $\overline{B_\beta}$. The Dirichlet problem

$$\Delta_g v_k = f - \Delta_g \varphi_k \quad \text{in } B_\beta \qquad \text{and} \qquad v_k = 0 \quad \text{on } \partial B_\beta$$

admits a unique solution $v_k \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta})$. Thus the function $u_k := v_k + \varphi_k \in C^{2,\alpha}_{\beta}$ satisfies $\Delta_g u_k = f$ in B_β and $u_k = \varphi_k$ on ∂B_β. By the maximum principle, u_k is uniformly bounded in $C^0(\overline{B_\beta})$. Corollary 3.35 gives uniformly $C^{2,\alpha}_{\beta}(K)$-bounds on any compact subset $K \Subset B_\beta$. Letting $k \to \infty$ and $K \to B_\beta$, by a diagonal argument and up to a subsequence, $u_k \to u \in C^{2,\alpha}_{\beta}(B_\beta)$. On the other hand, from $\Delta_g(u_k - u_l) = 0$, we see that $\{u_k\}$ is a Cauchy sequence in $C^0(\overline{B_\beta})$; thus u_k converges uniformly to u on $\overline{B_\beta}$. Hence $u \in C^0(\overline{B_\beta})$, and u satisfies (3-120). □

Corollary 3.40. *Given $f \in C^{0,\alpha}_{\beta}(B_\beta)$, suppose u is a weak solution to the equation $\Delta_g u = f$ in the sense that*

$$\int_{B_\beta} \langle \nabla u, \nabla \varphi \rangle \omega_g^n = -\int_{B_\beta} f\varphi\omega_g^n \quad \textit{for all } \varphi \in H_0^1(B_\beta),$$

then $u \in C^{2,\alpha}_{\beta}(B_\beta)$.

Proof. We first observe that the Sobolev inequality (3-43) also holds for the metric g, since g is equivalent to $g_{\boldsymbol\beta}$. The metric space $(B_{\boldsymbol\beta}, g)$ also has maximal volume growth/decay, so we can apply the same proof of De Giorgi–Nash–Moser theory [22] to conclude that u is continuous in $B_{\boldsymbol\beta}$. The standard elliptic theory implies that $u \in C^{2,\alpha}_{\mathrm{loc}}(B_{\boldsymbol\beta}\backslash\mathcal{S})$. For any $r \in (0, 1)$, by Corollary 3.39, the Dirichlet problem $\Delta_g \tilde u = f$ in $B_{\boldsymbol\beta}(0, r)$, $\tilde u = u$ on $\partial B_{\boldsymbol\beta}(0, r)$ admits a unique solution $\tilde u \in C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0, r)) \cap C^0(\overline{B_{\boldsymbol\beta}(0, r)})$. Then $\Delta_g(u - \tilde u) = 0$ in $B_{\boldsymbol\beta}(0, r)$ and $u - \tilde u = 0$ on $\partial B_{\boldsymbol\beta}(0, r)$. By the maximum principle, we get $u = \tilde u$ in $B_{\boldsymbol\beta}(0, r)$, so we conclude $u \in C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0, r))$. Since $r \in (0, 1)$ is arbitrary, we get $u \in C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta})$. □

Corollary 3.41. *Let X be a compact Kähler manifold and $D = \sum_j D_j$ be a divisor with simple normal crossings. Let g be a conical Kähler metric with cone angle $2\pi\boldsymbol\beta$ along D. Suppose $u \in H^1(g)$ is a weak solution to the equation $\Delta_g u = f$ in the sense that*

$$\int_X \langle \nabla u, \nabla\varphi\rangle \omega_g^n = -\int_X f\varphi\omega_g^n \quad \text{for all } \varphi \in C^1(X)$$

for some $f \in C^{0,\alpha}_{\boldsymbol\beta}(X)$. Then $u \in C^{2,\alpha}_{\boldsymbol\beta}(X) \cap C^0(X)$ and there exists a constant $C = C(n, \boldsymbol\beta, g, \alpha)$ such that

$$\|u\|_{C^{2,\alpha}_{\boldsymbol\beta}(X)} \le C(\|u\|_{C^0(X)} + \|f\|_{C^{0,\alpha}_{\boldsymbol\beta}(X)}).$$

Proof. We choose finite covers of D, $\{B_a\}$ and $\{B'_a\}$, with $B'_a \Subset B_a$ and centers in D. By assumption u is a weak solution to $\Delta_g u = f$ in each B_a, so by Corollary 3.40 we conclude that $u \in C^{2,\alpha}_{\boldsymbol\beta}(B_a)$ for each B_a. On $X\backslash\mathcal{S}$, the metric g is smooth so standard elliptic theory implies that $u \in C^{2,\alpha}_{\mathrm{loc}}(X\backslash\mathcal{S})$. Since $\{B_a\}$ covers D, we have $u \in C^{2,\alpha}_{\boldsymbol\beta}(X)$.

We can apply Corollary 3.35 to obtain that, for some constant $C > 0$,

$$\|u\|_{C^{2,\alpha}_{\boldsymbol\beta}(B'_a)} \le C(\|u\|_{C^0(B_a)} + \|f\|_{C^{\alpha}_{\boldsymbol\beta}(B_a)}).$$

On $X\backslash\bigcup_a\{B'_a\}$ the metric g is smooth, so the usual Schauder estimates apply. We finish the proof of the corollary using the definition of $C^{2,\alpha}_{\boldsymbol\beta}(X)$; see Definition 2.9. □

Remark 3.42. Let (X, D, g) be as in Corollary 3.41. It is easy to see by the variational method that weak solutions to $\Delta_g u = f$ always exist for any $f \in L^2(X, \omega_g^n)$ satisfying $\int_X f\omega_g^n = 0$.

4. Parabolic estimates

In this section, we will study the heat equation with background metric $\omega_{\boldsymbol\beta}$ and prove the Schauder estimates for solutions $u \in C^0(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^{2,1}(\mathcal{Q}^{\#}_{\boldsymbol\beta})$ to the equation

$$\frac{\partial u}{\partial t} = \Delta_{g_{\boldsymbol\beta}} u + f \tag{4-1}$$

for a function $f \in \mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})$ with some better regularity.

4A. ***Conical heat equations.*** In this section, we will show that, for any $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$, the Dirichlet problem (4-2) admits a unique $\mathscr{C}^{2,1}(\mathcal{Q}^{\#}_{\boldsymbol\beta}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$-solution in $\mathcal{Q}_{\boldsymbol\beta}$. We first observe that a maximum principle argument yields the uniqueness of the solution.

Suppose $u \in \mathscr{C}^{2,1}(\mathcal{Q}^{\#}_{\boldsymbol{\beta}}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$ solves the Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{g_{\boldsymbol{\beta}}} u & \text{in } \mathcal{Q}_{\boldsymbol{\beta}}, \\ u = \varphi & \text{on } \partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}} \end{cases} \tag{4-2}$$

for some given continuous function $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}})$. As in Lemma 3.1, it follows from the maximum principle that

$$\inf_{\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}}} u \le \inf_{\mathcal{Q}_{\boldsymbol{\beta}}} u \le \sup_{\mathcal{Q}_{\boldsymbol{\beta}}} u \le \sup_{\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}}} u. \tag{4-3}$$

So the $\mathscr{C}^{2,1}(\mathcal{Q}^{\#}_{\boldsymbol{\beta}}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$-solution to (4-2) is unique, if it exists.

We prove the existence of solutions to (4-2). As before, we use an approximation argument. Let g_ϵ be the smooth approximation metrics in $B_{\boldsymbol{\beta}}$ from (3-3). Let u_ϵ be the $\mathscr{C}^{2,1}(\mathcal{Q}_{\boldsymbol{\beta}}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$-solution to

$$\frac{\partial u_\epsilon}{\partial t} = \Delta_{g_\epsilon} u_\epsilon \quad \text{in } \mathcal{Q}_{\boldsymbol{\beta}} \qquad \text{and} \qquad u_\epsilon = \varphi \quad \text{on } \partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}}. \tag{4-4}$$

4A1. *Estimates of u_ϵ.* We first recall the Li–Yau gradient estimates [26; 35] for positive solutions to the heat equations.

Lemma 4.1. *Let (M, g) be a complete manifold with $\mathrm{Ric}(g) \ge 0$ and $B(p, R)$ be the geodesic ball with center $p \in M$ and radius $R > 0$. Let u be a positive solution to the heat equation $\partial_t u - \Delta_g u = 0$ on $B(p, R)$. Then there exists $C = C(n) > 0$ such that, for all $t > 0$,*

$$\sup_{B(p,2R/3)} \left(\frac{|\nabla u|^2}{u^2} - \frac{2\dot{u}_t}{u} \right) \le \frac{C}{R^2} + \frac{2n}{t},$$

where $\dot{u}_t = \partial u/\partial t$.

By considering the functions $u_\epsilon - \inf u_\epsilon$ and $\sup u_\epsilon - u_\epsilon$, from Lemma 4.1, we see that there exists a constant $C = C(n) > 0$ such that, for any $R \in (0, 1)$ and $t \in (0, R^2)$,

$$\sup_{B_{g_\epsilon}(0,2R/3)} |\nabla u_\epsilon|^2_{g_\epsilon} \le C \left(\frac{1}{R^2} + \frac{1}{t} \right) (\operatorname{osc}_R u_\epsilon)^2, \tag{4-5}$$

$$\sup_{B_{g_\epsilon}(0,2R/3)} |\Delta_{g_\epsilon} u_\epsilon| = \sup_{B_{g_\epsilon}(0,2R/3)} \left| \frac{\partial u_\epsilon}{\partial t} \right| \le C \left(\frac{1}{R^2} + \frac{1}{t} \right) \operatorname{osc}_R u_\epsilon, \tag{4-6}$$

where $\operatorname{osc}_R u_\epsilon := \operatorname{osc}_{B_{g_\epsilon}(0,R) \times (0,R^2)} u_\epsilon$ is the oscillation of u_ϵ in the cylinder $B_{g_\epsilon}(0, R) \times (0, R^2)$. Replacing u_ϵ by $u_\epsilon - \inf u_\epsilon$, we may assume $u_\epsilon > 0$ and define $f_\epsilon = \log u_\epsilon$. Then we have

$$\frac{\partial f_\epsilon}{\partial t} = \Delta_{g_\epsilon} f_\epsilon + |\nabla f_\epsilon|^2.$$

Let $\varphi(x) = \varphi(r(x)/R)$, where φ is a cut-off function equal to 1 on $\left[0, \frac{3}{5}\right]$ and 0 on $\left[\frac{2}{3}, \infty\right)$ satisfying the inequalities $|\varphi''| \le 10$ and $(\varphi')^2 \le 10\varphi$. Let $r(x)$ be the distance function under g_ϵ to the center 0.

Lemma 4.2. *There exists a constant $C = C(n) > 0$ such that, for any small $\epsilon > 0$,*

$$\sup_{B_{g_\epsilon}(0,3R/5)} |\Delta_i u_\epsilon| \le C \left(\frac{1}{t} + \frac{1}{R^2} \right) \operatorname{osc}_R u_\epsilon \quad \textit{for all } t \in (0, R^2),$$

where we write $\Delta_i u_\epsilon := (|z_i|^2 + \epsilon)^{1-\beta_i} (\partial^2 u_\epsilon / \partial z_i \partial z_{\bar{i}})$ for $i = 1, \dots, p$.

Proof. We only prove the case when $i=1$. We define $F := t\varphi(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon)$, and we calculate

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\Big)(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon) &= -|\nabla_2\nabla f_\epsilon|^2 - |\nabla_1\overline{\nabla} f_\epsilon|^2 - 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon)\rangle - R_{1\bar{1}j\bar{k}} f_{\epsilon,\bar{j}} f_{\epsilon,k}\\ &\le -(-\Delta_1 f_\epsilon)^2 - 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon)\rangle.\end{aligned}$$

F achieves its maximum at a point (p_0, t_0), where we may assume $F(p_0, t_0) > 0$; otherwise we are already done. In particular, $p_0 \in B_{g_\epsilon}\big(0, \frac{2}{3}r\big)$ by the definition of φ and $t_0 > 0$. Then at (p_0, t_0), we have

$$\begin{aligned} 0 &\le \Big(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\Big)F\\ &= \frac{F}{t_0} + t_0\varphi\Big(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\Big)(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon) - \frac{F}{\varphi}\Delta_{g_\epsilon}\varphi - 2t_0\operatorname{Re}\Big\langle\nabla\varphi, \overline{\nabla}\Big(\frac{F}{t_0\varphi}\Big)\Big\rangle\\ &\le \frac{F}{t_0} + t_0\varphi\Big(-(-\Delta_1 f_\epsilon)^2 - 2\frac{F}{t_0\varphi^2}\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}\varphi\rangle\Big) + C\frac{F}{R^2\varphi}(\varphi' + \varphi'') + 2\frac{F}{R^2\varphi^2}(\varphi')^2, \end{aligned} \tag{4-7}$$

where we use the Laplacian comparison and the fact that $\nabla F = 0$ at (p_0, t_0). The second term on the right-hand side satisfies (we write $\widetilde{F} := -\Delta_1 f_\epsilon - 2\dot{f}_\epsilon$ for convenience of notation)

$$\begin{aligned} t_0\varphi\Big(-(-\Delta_1 f_\epsilon)^2 - 2\frac{F}{t_0\varphi^2}\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}\varphi\rangle\Big) &\le t_0\varphi\Big(-\widetilde{F}^2 - 4\widetilde{F}\dot{f}_\epsilon - 4(\dot{f}_\epsilon)^2 + \frac{2\widetilde{F}}{\varphi}\frac{|\nabla f_\epsilon||\varphi'|}{R}\Big)\\ &\le t_0\varphi\Big(-\widetilde{F}^2 - 4\widetilde{F}\dot{f}_\epsilon + 2\widetilde{F}|\nabla f_\epsilon|^2 + \frac{\widetilde{F}|\varphi'|^2}{2R^2\varphi^2}\Big)\\ &\le t_0\varphi\Big(-\widetilde{F}^2 + \frac{\widetilde{F}|\varphi'|^2}{2R^2\varphi^2} + C\frac{\widetilde{F}}{t_0} + C\frac{\widetilde{F}}{R^2}\Big) \qquad \text{(by Lemma 4.1)}\\ &= -\frac{F^2}{t_0\varphi} + C\frac{F}{2R^2\varphi} + C\frac{F}{t_0} + C\frac{F}{R^2}. \end{aligned}$$

Inserting this into (4-7), we get, for some constant $C = C(n) > 0$, at (p_0, t_0),

$$-F^2 + C\varphi F + \frac{t_0\varphi F}{R^2} + Ct_0\frac{F}{R^2} \ge 0,$$

from which we obtain $F(p_0, t_0) \le Ct_0/R^2 + C$. By the choice of (p_0, t_0), we can see that

$$\sup_{B_{g_\epsilon}(0,R/2)}(-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon) \le C\Big(\frac{1}{R^2} + \frac{1}{t}\Big) \quad \text{for all } t \in (0, R^2),$$

which implies that, on $B_{g_\epsilon}\big(0, \frac{3}{5}R\big) \times (0, R^2)$,

$$-\Delta_1 u_\epsilon \le \dot{u}_\epsilon + C\Big(\frac{1}{t} + \frac{1}{R^2}\Big)u_\epsilon. \tag{4-8}$$

Applying (4-8) to the function $\sup u_\epsilon - u_\epsilon$, we obtain, on $B_{g_\epsilon}\big(0, \frac{3}{5}R\big) \times (0, R^2)$,

$$|\Delta_1 u_\epsilon| \le |\dot{u}_\epsilon| + C\Big(\frac{1}{t} + \frac{1}{R^2}\Big)\operatorname{osc}_R u_\epsilon \le C\Big(\frac{1}{t} + \frac{1}{R^2}\Big)\operatorname{osc}_R u_\epsilon$$

by (4-6). Thus we finish the proof of the lemma. □

Lemma 4.3. *There exists a constant $C = C(n) > 0$ such that*

$$\sup_{i\neq j}\ \sup_{B_{g_\epsilon}(0,R/2)} (|\nabla_i\nabla_j u_\epsilon| + |\nabla_i\overline{\nabla}_j u_\epsilon|) \le C\left(\frac{1}{t}+\frac{1}{R^2}\right)\operatorname{osc}_R u_\epsilon$$

for all $t\in(0,R^2)$. Recall, here $|\nabla_i\nabla_j u_\epsilon|^2 = \nabla_i\nabla_j u_\epsilon \nabla_{\bar i}\nabla_{\bar j} u_\epsilon g_\epsilon^{i\bar i} g_\epsilon^{j\bar j}$ (no summation over i, j is taken).

Proof. We only prove the estimate for $|\nabla_1\nabla_2 u_\epsilon|$. The other estimates are similar, so we omit their proofs.

By calculations similar to those used to derive (3-27), we have

$$\left(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\right)|\nabla_1\nabla_2 f_\epsilon| \le 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}|\nabla_1\nabla_2 f_\epsilon|\rangle + \sum_k (|\nabla_1\nabla_k f_\epsilon||\nabla_2\nabla_{\bar k} f_\epsilon| + |\nabla_2\nabla_k f_\epsilon||\nabla_1\nabla_{\bar k} f_\epsilon|), \tag{4-9}$$

and similar to (3-20),

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\right)(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon) &\le 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon)\rangle\\ &\quad -\sum_k (|\nabla_1\nabla_k f_\epsilon|^2+|\nabla_1\nabla_{\bar k} f_\epsilon|^2+|\nabla_2\nabla_k f_\epsilon|^2+|\nabla_2\nabla_{\bar k} f_\epsilon|^2). \end{aligned}\tag{4-10}$$

Combining (4-10), (4-9) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\right)(|\nabla_1\nabla_2 f_\epsilon|+2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon))\\ &\le 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}(|\nabla_1\nabla_2 f_\epsilon|+2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon))\rangle - \sum_k (|\nabla_1\nabla_k f_\epsilon|^2+|\nabla_1\nabla_{\bar k} f_\epsilon|^2+|\nabla_2\nabla_k f_\epsilon|^2+|\nabla_2\nabla_{\bar k} f_\epsilon|^2)\\ &\le 2\operatorname{Re}\langle\nabla f_\epsilon, \overline{\nabla}(|\nabla_1\nabla_2 f_\epsilon|+2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon))\rangle - \tfrac{1}{10}(|\nabla_1\nabla_2 f_\epsilon|+2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon))^2.\end{aligned}$$

We define a cut-off function η similar to φ in the proof of Lemma 4.2 such that $\eta = 1$ on $B_{g_\epsilon}\big(0,\frac12 R\big)$ and η vanishes outside $B_{g_\epsilon}\big(0,\frac35 R\big)$. We write

$$G = t\eta(|\nabla_1\nabla_2 f_\epsilon| + 2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon) - 2\dot f_\epsilon).$$

Like we did for F in the proof of Lemma 4.2, we argue similarly that at the maximum point (p_0,t_0) of G, for which we assume $G(p_0,t_0)>0$,

$$\begin{aligned}0\le\left(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\right)G &\le \frac{G}{t_0}-\frac{G^2}{t_0\eta}+C\frac{G}{R^2\eta}+C\frac{G}{t_0}+C\frac{G}{R^2}+C\frac{G}{R^2}\frac{\eta'+\eta''}{\eta}+\frac{2G}{R^2\eta^2}(\eta')^2\\ &\le \frac{1}{t_0\eta}\left(-G^2+C\eta G+\frac{t_0\eta G}{R^2}+Ct_0\frac{G}{R^2}\right),\end{aligned}$$

so it follows that $G(p_0,t_0)\le C(1+t_0/R^2)$. Therefore by the definition of G, on $B_{g_\epsilon}\big(0,\frac12 R\big)\times(0,R^2)$,

$$|\nabla_1\nabla_2 f_\epsilon| + 2(-\Delta_1 f_\epsilon-\Delta_2 f_\epsilon) - 2\dot f_\epsilon \le C\left(\frac{1}{R^2}+\frac{1}{t}\right),$$

and thus by Lemmas 4.1 and 4.2, we conclude that, on $B_{g_\epsilon}\big(0,\frac12 R\big)\times(0,R^2)$,

$$|\nabla_1\nabla_2 u_\epsilon| \le \dot u_\epsilon + 2|\Delta_1 u_\epsilon| + 2|\Delta_2 u_\epsilon| + \frac{|\nabla u_\epsilon|^2}{u_\epsilon} + Cu_\epsilon\left(\frac{1}{R^2}+\frac{1}{t}\right) \le C\left(\frac{1}{t}+\frac{1}{R^2}\right)\operatorname{osc}_R u_\epsilon,$$

as desired. □

4A2. *Existence of a solution u to* (4-2). We will show the limit function of u_ϵ as $\epsilon \to 0$ solves (4-2).

Proposition 4.4. *Given any $R \in (0, 1)$ and any $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol{\beta}}(0, R))$, there exists a unique function $u \in \mathscr{C}^{2,1}(\mathcal{Q}_{\boldsymbol{\beta}}(0, R)^\#) \cap C^0(\overline{\mathcal{Q}_{\boldsymbol{\beta}}(0, R)})$ solving* (4-2). *Moreover, there exists a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that, for any $t \in (0, R^2)$ we have (defining $B_{\boldsymbol{\beta}}(r)^\# := B_{\boldsymbol{\beta}}(0, r)\backslash\mathcal{S}$),*

$$\sup_{B_{\boldsymbol{\beta}}(R/2)^\#}\left(\sum_{j=1}^{p}|z_j|^{2-2\beta_j}\left|\frac{\partial u}{\partial z_j}\right|^2+|D'u|^2\right)\le C\left(\frac{1}{t}+\frac{1}{R^2}\right)(\mathrm{osc}_R\, u)^2, \tag{4-11}$$

$$\sup_{B_{\boldsymbol{\beta}}(R/2)^\#}\left(\sum_{i\neq j}(|\nabla_i\nabla_j u|_{g_{\boldsymbol{\beta}}}+|\nabla_i\overline{\nabla}_j u|_{g_{\boldsymbol{\beta}}})+\left|\frac{\partial u}{\partial t}\right|\right)\le C\left(\frac{1}{t}+\frac{1}{R^2}\right)\mathrm{osc}_R\, u, \tag{4-12}$$

$$\sup_{B_{\boldsymbol{\beta}}(R/2)^\#}\left(\sum_{j=1}^{p}|\nabla_{g_{\boldsymbol{\beta}}}\Delta_j u|+|\nabla_{g_{\boldsymbol{\beta}}}(D')^2u|+\left|\nabla_{g_{\boldsymbol{\beta}}}\frac{\partial u}{\partial t}\right|\right)\le C\left(\frac{1}{t}+\frac{1}{R^2}\right)^{3/2}\mathrm{osc}_R\, u, \tag{4-13}$$

where by abusing notation we write $\mathrm{osc}_R\, u := \mathrm{osc}_{B_{\boldsymbol{\beta}}(0,R)\times(0,R^2)}\, u$.

Proof. Let u_ϵ be the $\mathscr{C}^{2,1}$-solution to (4-4). The $\mathscr{C}^0$-norm of u_ϵ follows from the maximum principle (4-3).

To prove the higher-order estimates, for any fixed compact subset $K \Subset B_{\boldsymbol{\beta}}(0, R)$ and $\delta > 0$, standard parabolic Schauder theory yields uniform $\mathscr{C}^{4+\alpha,(4+\alpha)/2}$-estimates of u_ϵ on $(K\backslash T_\delta\mathcal{S}) \times (\delta, R^2]$ for any $\alpha \in (0, 1)$. As $\epsilon \to 0$, u_ϵ converges in $\mathscr{C}^{4+\alpha,(4+\alpha)/2}(K\backslash T_\delta\mathcal{S} \cap (\delta, R^2])$ to some function u which is also $\mathscr{C}^{4+\alpha,(4+\alpha)/2}$ in $(K\backslash T_\delta\mathcal{S}) \times (\delta, R^2]$. Letting $\delta \to 0$ and $K \to B_{\boldsymbol{\beta}}(0, R)$ and using a diagonal argument, we can assume that

$$u_\epsilon \xrightarrow{\mathscr{C}^{4+\alpha,(4+\alpha)/2}_{\mathrm{loc}}(B_{\boldsymbol{\beta}}(0,R)^\#\times(0,R^2])} u \quad \text{as } \epsilon \to 0.$$

Letting $\epsilon \to 0$, estimate (4-11) follows from (4-5); (4-12) is a consequence of Lemma 4.3; and (4-13) follows by applying the gradient estimate (4-5) to the Δ_{g_ϵ}-harmonic functions $\Delta_j u_\epsilon$, $(D')^2 u_\epsilon$ and $\partial u_\epsilon/\partial t$, and then letting $\epsilon \to 0$.

The gradient estimate (4-11) implies that, for any compact $K \Subset B_{\boldsymbol{\beta}}(0, R)$,

$$\sup_{K\backslash\mathcal{S}_j}\left|\frac{\partial u}{\partial z_j}\right| \le \frac{C(n, K, \boldsymbol{\beta})(\mathrm{osc}_R\, u)^2}{t}|z_j|^{\beta_j-1} \quad \text{for all } t \in (0, R^2).$$

From this, for any $t \in (0, R^2)$, we see that $u(\,\cdot\,, t)$ can be continuously extended to $\mathcal{S}$, and thus we have $u \in C^0(B_{\boldsymbol{\beta}}(0, R) \times (0, R^2))$.

It only remains to show $u = \varphi$ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol{\beta}}(0, R)$. Fix an arbitrary point $(q_0, t_0) \in \partial_{\mathcal{P}}(\mathcal{Q}_{\boldsymbol{\beta}}(0, R))$.

Case 1: $t_0 = 0$ and $q_0 \in \overline{B_{\boldsymbol{\beta}}(0, R)}$. We define a barrier function $\phi_1(z, t) = e^{-d_{\mathbb{C}^n}(z,q_0)^2-\lambda t} - 1$, where $\lambda > 0$ is to be determined. If $\lambda \ge 4n$, we calculate

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_{g_\epsilon}\right)\phi_1 &= -\lambda e^{-d_{\mathbb{C}^n}(z,q_0)^2-\lambda t}-(-\Delta_{g_\epsilon}d^2_{\mathbb{C}^n}+|\nabla d^2_{\mathbb{C}^n}|^2_{g_\epsilon})e^{-d_{\mathbb{C}^n}(z,q_0)^2-\lambda t}\\ &\le \left(-\lambda+\sum_{j=1}^{p}(|z_j|^2+\epsilon)^{1-\beta_j}+(n-p)\right)e^{-d_{\mathbb{C}^n}(z,q_0)^2-\lambda t}<0.\end{aligned}$$

On the other hand, $\phi_1(q_0,t_0)=0$ and $\phi_1(z,t)<0$ for any $(z,t)\neq(q_0,t_0)$. For any $\varepsilon>0$, we can find a small neighborhood $V\cap\partial_{\mathcal{P}}(\mathcal{Q}_{\boldsymbol\beta}(0,R))$ of (q_0,t_0) such that, on V, we have $\varphi(q_0,t_0)+\varepsilon>\varphi(z,t)>\varphi(q_0,t_0)-\varepsilon$, since φ is continuous. On $\partial_{\mathcal{P}}(\mathcal{Q}_{\boldsymbol\beta}(0,R))\backslash V$, the function ϕ_1 is bounded above by a negative constant. Therefore the function $\phi_1^-:=\varphi(q_0,t_0)-\varepsilon+A\phi_1(z,t)\le\varphi(z,t)$ for any $(z,t)\in\partial_{\mathcal{P}}(\mathcal{Q}_{\boldsymbol\beta}(0,R))$ if $A\gg 1$. Therefore, by the maximum principle, $\phi_1^-(z,t)\le u_\epsilon(z,t)$ for any $(z,t)\in\mathcal{Q}_{\boldsymbol\beta}(0,R)$. Letting $\epsilon\to 0$, we have $\phi_1^-(z,t)\le u(z,t)$. Letting $(z,t)\to(q_0,t_0)$ yields $\varphi(q_0,t_0)-\varepsilon\le\liminf_{(z,t)\to(q_0,t_0)}u(z,t)$. Setting $\varepsilon\to 0$, we conclude that $\varphi(q_0,t_0)\le\liminf_{(z,t)\to(q_0,t_0)}u(z,t)$. Considering $\phi_1^+(z,t)=\varphi(q_0,t_0)+\varepsilon-A\phi_1(z,t)$ and using an argument similar to that above, we can get $\varphi(q_0,t_0)\ge\limsup_{(z,t)\to(q_0,t_0)}u(z,t)$. Thus u coincides with φ at (q_0,t_0).

Case 2: $t_0>0$ and $q_0\in\partial B_{\boldsymbol\beta}(0,R)\cap(\mathcal{S}_1\cap\mathcal{S}_2)$. In this case $z_1(q_0)=z_2(q_0)=0$. We define $q_0'=-q_0\in\partial B_{\boldsymbol\beta}(0,R)$ to be the (Euclidean) opposite point to q_0. For some small $\delta>0$, define

$$\phi_2(z,t)=d_{\mathbb{C}^n}(z,q_0')^2-4R^2-\delta(t-t_0)^2.$$

Then $\phi_2(q_0,z_0)=0$ and $\phi(z,t)<0$ for any $(z,t)\neq(q_0,t_0)$. We calculate $\partial_t\phi_2-\Delta_{g_\epsilon}\phi_2\le 0$. By an argument similar to Case 1, replacing ϕ_1 by ϕ_2 we get $\lim_{(z,t)\to(q_0,t_0)}u(z,t)=\varphi(q_0,t_0)$.

Case 3: $t_0>0$ and $q_0\in\partial B_{\boldsymbol\beta}(0,R)\backslash(\mathcal{S}_1\cap\mathcal{S}_2)$. As in Case 2 in the proof of Proposition 3.5, we define a similar function G. Define $\phi_3(z,t)=A(d_{\boldsymbol\beta}(z,0)^2-R^2)+G(z)-\delta(t-t_0)^2$ for $A\gg 1$ and small $\delta>0$. Then we can calculate that $\partial_t\phi_3\le\Delta_{g_\epsilon}\phi_3$, $\phi_3(q_0,t_0)=0$ and $\phi_3(z,t)<0$ for any other $(z,t)\neq(q_0,t_0)$. Similar arguments to those in Case 1 proves that

$$\lim_{(z,t)\to(q_0,t_0)}u(z,t)=\varphi(q_0,t_0).$$

Combining the three cases above, we obtain that u coincides with φ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}(0,R)$. Thus the Dirichlet problem (4-2) admits a unique solution $u\in\mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}(0,R)})\cap\mathscr{C}^{2,1}(\mathcal{Q}_{\boldsymbol\beta}(0,R)^\#)$. □

Corollary 4.5. *Given any functions $f\in\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ and $\varphi\in\mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$, there exists a unique solution $v\in\mathscr{C}^{2,1}(\mathcal{Q}^\#_{\boldsymbol\beta})\cap\mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ to the Dirichlet problem*

$$\frac{\partial v}{\partial t}=\Delta_{g_{\boldsymbol\beta}}v+f\quad\text{in }\mathcal{Q}_{\boldsymbol\beta}\qquad\text{and}\qquad v=\varphi\quad\text{on }\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}.\tag{4-14}$$

Proof. Let $v_\epsilon\in\mathscr{C}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})\cap\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})$ be the unique solution to the equations

$$\frac{\partial v_\epsilon}{\partial t}=\Delta_{g_\epsilon}v_\epsilon+f\quad\text{in }\mathcal{Q}_{\boldsymbol\beta}\qquad\text{and}\qquad v_\epsilon=\varphi\quad\text{on }\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}.$$

For any compact subset $K\Subset B_{\boldsymbol\beta}(0,1)$ and $\delta\in(0,1)$, the standard Schauder estimates for parabolic equations provide uniform $\mathscr{C}^{2+\alpha,(2+\alpha)/2}$-estimates for v_ϵ on $K\backslash T_\delta\mathcal{S}\times(\delta^2,1)$. Then $v_\epsilon\to v$ for some $v\in\mathscr{C}^{2+\alpha,(2+\alpha)/2}(K\backslash T_\delta\mathcal{S}\times(\delta^2,1))$. Taking $\delta\to 0$ and $K\to B_{\boldsymbol\beta}(0,1)$ and using a diagonal argument, we get that v_ϵ converges in $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\text{loc}}(B_{\boldsymbol\beta}\backslash\mathcal{S}\times(0,1))$ to v and v satisfies the equation $\partial v/\partial t=\Delta_{g_{\boldsymbol\beta}}v+f$ on $B_{\boldsymbol\beta}\backslash\mathcal{S}\times(0,1)$.

It only remains to show $v\in\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})$ and $v=\varphi$ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$. The same proof as in Cases 1, 2 and 3 in Proposition 4.4 yields that v must coincide with φ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$, since we can always choose $A>1$ large enough that (for example in Case 1) $\partial\phi_1^-/\partial t-\Delta_{g_\epsilon}\phi_1^-\le\inf_{\mathcal{Q}_{\boldsymbol\beta}}f\le\partial v_\epsilon/\partial t-\Delta_{g_\epsilon}v_\epsilon$. To see the continuity of v in $\mathcal{Q}_{\boldsymbol\beta}$, because of the Sobolev inequality (3-42) for metric spaces $(B_{\boldsymbol\beta}, g_\epsilon)$ and by the proof of the standard De Giorgi–Nash–Moser theory for parabolic equations, we conclude that, for any $p \in \mathcal{S}$ and $t_0 \in (0,1)$, there exists a small number $R_0 = R_0(p, t_0)$ such that, on the cylinder $\widetilde{\mathcal{Q}}_{R_0} := B_{\boldsymbol\beta}(p, R_0) \times (t_0 - R_0^2, t_0)$, we have $\operatorname{osc}_{\widetilde{\mathcal{Q}}_r} v_\epsilon \le C r^{\alpha'}$ for any $r \in (0, R_0)$ and some $\alpha' \in (0,1)$. Therefore $\operatorname{osc}_{\widetilde{\mathcal{Q}}_r} v \le C r^{\alpha'}$ and v is continuous at (p, t_0), as desired.

The uniqueness of the solution to (4-14) follows from the maximum principle. □

Remark 4.6. Corollary 4.5 is not needed in the proof of Theorem 1.7. So by Theorem 1.7, the solution u to (4-14) is in $\mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$.

4B. ***Sketched proof of Theorem 1.7.*** With Proposition 4.4, we can prove the Schauder estimates for the solution $u \in \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}}) \cap \mathscr{C}^{2,1}(\mathcal{Q}_{\boldsymbol\beta}^\#)$ to (4-1) for a Dini-continuous function f by making use of almost the same arguments as in the proof of Theorem 1.2. We will not provide the full details and only point out the main differences. For any given points $Q_p = (p, t_p)$, $Q_q = (q, t_q) \in \big(B_{\boldsymbol\beta}\big(0, \frac12\big)\backslash\mathcal{S}\big) \times (\hat t, 1)$, to define the approximating functions u_k as in (3-44), we define u_k in this case as the solution to the heat equation

$$\frac{\partial u_k}{\partial t} = \Delta_{g_{\boldsymbol\beta}} u_k + f(Q_p) \quad \text{in } \hat B_k(p) \times (t_p - \hat t \cdot \tau^{2k}, t_p], \qquad u_k = u \quad \text{on } \partial_{\mathcal{P}}(\hat B_k(p) \times (t_p - \hat t \cdot \tau^{2k}, t_p]),$$

where $\hat B_k(p)$ is defined in (3-48). We can now apply the estimates in Proposition 4.4 to the functions u_k or $u_k - u_{k-1}$, instead of those in Lemmas 3.3 and 3.4 as we did in Sections 3B, 3C and 3D, to prove the Schauder estimates for u. Thus we finish the proof of Theorem 1.7. □

4C. ***Interior Schauder estimate for nonflat conical Kähler metrics.*** Let $g = \sqrt{-1} g_{j\bar k}(z,t)\, dz_j \wedge dz_{\bar k}$ be a $\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}$ conical Kähler metric on $\mathcal{Q}_{\boldsymbol\beta}$ with conical singularity along $\mathcal{S}$; that is, $g(\,\cdot\,, t)$ is a $C_{\boldsymbol\beta}^{0,\alpha}$ conical Kähler metric (from Section 3E) for any $t \in [0,1]$, and the coefficients of g in the basis $\{\epsilon_j \wedge \bar\epsilon_k, \dots\}$ are $\frac12\alpha$-Hölder continuous in $t \in [0,1]$. Suppose $u \in \mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})$ satisfies the equation

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta} \tag{4-15}$$

for some $f \in \mathscr{C}^{\alpha,\alpha/2}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$.

Proposition 4.7. *There exists a constant $C = C(n, \boldsymbol\beta, \alpha, g)$ such that*

$$\|u\|^*_{\mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})} \le C\big(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|^{(2)}_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol\beta})}\big).$$

Proof. The proof is parallel to that of Proposition 3.32. Given any two points $P_x = (x, t_x)$, $P_y = (y, t_x) \in \mathcal{Q}_{\boldsymbol\beta}$, we may assume $d_{P_x} = \min\{d_{P_x}, d_{P_y}\} > 0$, where $d_{P_x} := d_{\mathcal{P},\boldsymbol\beta}(P_x, \partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$ is the parabolic distance of P_x to the parabolic boundary $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$. Let $\mu \in \big(0, \frac14\big)$ be a positive number to be determined later. Define $d := \mu d_{P_x}$, $\mathcal{Q} := B_{\boldsymbol\beta}(x, d) \times (t_x - d^2, t_x]$ the "parabolic ball" centered at P_x, and $\frac12\mathcal{Q} := B_{\boldsymbol\beta}\big(x, \frac12 d\big) \times \big(t_x - \frac14 d^2, t_x\big]$.

<u>Case 1</u>: $d_{\mathcal{P},\boldsymbol\beta}(P_x, P_y) < \frac12 d$. In this case we always have $P_y \in \frac12\mathcal{Q}$.

<u>Case 1a</u>: $B_{\boldsymbol\beta}(x, d) \cap \mathcal{S} = \varnothing$. As in the proof of Proposition 3.32, we can introduce smooth complex coordinates $\{w_1, w_2, z_3, \dots, z_n\}$ on $B_{\boldsymbol\beta}(x, d)$ under which $g_{\boldsymbol\beta}$ becomes the standard Euclidean metric and the components of g are $\mathscr{C}^{\alpha,\alpha/2}$ in the usual sense on $\mathcal{Q}$. The leading coefficients and constant term f

in (4-15) are both $\mathscr{C}^{\alpha,\alpha/2}$ in the usual sense, so we apply the standard parabolic Schauder estimates (see Theorem 4.9 in [27]) to get that there exists some constant $C = C(n, \boldsymbol{\beta}, \alpha, g)$ independent of $\mathcal{Q}$ such that

$$[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q})} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q})} + \|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}(\mathcal{Q})}). \tag{4-16}$$

Let D denote the ordinary first-order operators in the coordinates $\{w_1, w_2, z_3, \dots, z_n\}$. We calculate

$$\begin{aligned} |Tu(P_x) - Tu(P_y)| &\le |D^2u(P_x) - D^2u(P_y)| + \frac{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)}{d}(|D^2u(P_x)| + |D^2u(P_y)|) \\ &\le \frac{4d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha}{d^{2+\alpha}}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q})} + \frac{4d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)}{d^3}[u]^*_{\mathscr{C}^{2,1}(\mathcal{Q})} \\ &\le \frac{8d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha}{d^{2+\alpha}}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q})} + C\frac{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha}{d^{2+\alpha}}\|u\|_{\mathscr{C}^0(\mathcal{Q})} \end{aligned}$$

and

$$\left|\frac{\partial u}{\partial t}(P_x) - \frac{\partial u}{\partial t}(P_y)\right| \le \frac{4d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha}{d^{2+\alpha}}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q})}.$$

Recall $\mathcal{T}$ denotes the operators in T and $\partial/\partial t$; then by (4-16) it follows that

$$d_{P_x}^{2+\alpha}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \le \frac{C}{\mu^{2+\alpha}}\|f\|^{(2)}_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\mathcal{Q})} + \frac{C}{\mu^{2+\alpha}}\|u\|_{\mathscr{C}^0(\mathcal{Q})}. \tag{4-17}$$

<u>Case 1b</u>: $B_{\boldsymbol{\beta}}(x, d) \cap \mathcal{S} \ne \varnothing$. Let $\hat{x} \in \mathcal{S}$ be the projection of x onto $\mathcal{S}$ and $\hat{P}_x = (\hat{x}, t_x)$ be the corresponding space-time point. Define $\hat{\mathcal{Q}} := B_{\boldsymbol{\beta}}(\hat{x}, 2d) \times (t_x - 4d^2, t_x]$. As in Case 1b in the proof of Proposition 3.32, we may choose suitable enough complex coordinates that $g_{\epsilon_j\bar{\epsilon}_k}(\hat{P}_x) = \delta_{jk}$ and, for $j, k \ge p+1$, we have $g_{j\bar{k}}(\hat{P}_x) = \delta_{jk}$ and the cross terms in the expansion of g in (3-114) vanish at $\hat{P}_x$. Thus (4-15) can be rewritten as

$$\frac{\partial u}{\partial t} = \Delta_{g_{\boldsymbol{\beta}}} u + \eta \cdot \sqrt{-1}\partial\bar{\partial} u + f =: \Delta_{g_{\boldsymbol{\beta}}} u + \tilde{f}, \quad u \in \mathscr{C}^0(\hat{\mathcal{Q}}) \cap \mathscr{C}^{2,1}(\hat{\mathcal{Q}}^\#),$$

for some (1, 1)-form η as in the proof of Proposition 3.32. From the rescaled version of Theorem 1.7 we conclude that

$$d^{2+\alpha}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \le C(\|u\|_{\mathscr{C}^0(\hat{\mathcal{Q}})} + \|\tilde{f}\|^{(2)}_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\hat{\mathcal{Q}})}).$$

Hence

$$d_{P_x}^{2+\alpha}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \le \frac{C}{\mu^{2+\alpha}}(\|u\|_{\mathscr{C}^0(\hat{\mathcal{Q}})} + \|\tilde{f}\|^{(2)}_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\hat{\mathcal{Q}})}). \tag{4-18}$$

<u>Case 2</u>: $d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y) \ge \frac{1}{2}d$. Here we calculate (recall $\mathcal{Q}_{\boldsymbol{\beta}} := B_{\boldsymbol{\beta}}(0, 1) \times (0, 1]$)

$$d_{P_x}^{2+\alpha}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \le 4d_{P_x}^{2+\alpha}\frac{|\mathcal{T}u|(P_x) + |\mathcal{T}u|(P_y)}{d^\alpha} \le \frac{8}{\mu^\alpha}[u]^*_{\mathscr{C}_{\boldsymbol{\beta}}^{2,1}(\mathcal{Q}_{\boldsymbol{\beta}})}. \tag{4-19}$$

Combining (4-17)–(4-19), we obtain

$$\begin{aligned} d_{P_x}^{2+\alpha}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \le \frac{8}{\mu^\alpha}[u]^*_{\mathscr{C}_{\boldsymbol{\beta}}^{2,1}(\mathcal{Q}_{\boldsymbol{\beta}})} &+ \frac{C}{\mu^{2+\alpha}}(\|u\|_{\mathscr{C}^0(\hat{\mathcal{Q}})} + \|\tilde{f}\|^{(2)}_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\hat{\mathcal{Q}})}) \\ &+ \frac{C}{\mu^{2+\alpha}}\|f\|^{(2)}_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(\mathcal{Q})} + \frac{C}{\mu^{2+\alpha}}\|u\|_{\mathscr{C}^0(\mathcal{Q})}. \end{aligned}$$

Observe that, for any $P \in \mathcal{Q}$ or $P \in \hat{\mathcal{Q}}$, we have $d_{\mathcal{P},\boldsymbol{\beta}}(P, \partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol{\beta}}) \geq (1-2\mu)d_{P_x}$. Then it follows from the definition that

$$\|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q})} \leq C\mu^2\|f\|^{(2)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + C\mu^{2+\alpha}[f]^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} \leq C\mu^2\|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}.$$

We calculate

$$\begin{aligned}\|\tilde{f}\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\hat{\mathcal{Q}})} &\leq \|\eta\|^{(0)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\hat{\mathcal{Q}})}\|Tu\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\hat{\mathcal{Q}})} + \|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\hat{\mathcal{Q}})} \\ &\leq C_1[g]^*_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}\mu^\alpha(\mu^2[u]^*_{\mathscr{C}^{2,1}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} + \mu^{2+\alpha}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}) + \mu^2\|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} \\ &\leq C_1[g]^*_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}\mu^\alpha(C(\mu)[u]^*_{\mathscr{C}^{2,1}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} + 2\mu^{2+\alpha}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}) + \mu^2\|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})},\end{aligned}$$

where in the last inequality we use the interpolation inequality, by which we also have

$$\frac{8}{\mu^\alpha}[u]^*_{\mathscr{C}^{2,1}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} \leq \mu^\alpha[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} + C(\mu)\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})}.$$

If μ is chosen small enough that $\mu^\alpha(2C_1[g]^*_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} + 1) < \frac{1}{2}$, combining the above inequalities yields

$$d^{2+\alpha}_{P_x}\frac{|\mathcal{T}u(P_x) - \mathcal{T}u(P_y)|}{d_{\mathcal{P},\boldsymbol{\beta}}(P_x, P_y)^\alpha} \leq \tfrac{1}{2}[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} + C(\mu)(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + \|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}).$$

Taking the supremum over all $P_x \neq P_y \in \mathcal{Q}_{\boldsymbol{\beta}}$, we obtain

$$[u]^*_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})} \leq C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + \|f\|^{(2)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}).$$

The proposition is proved by invoking the interpolation inequalities. □

Remark 4.8. It follows from the proof that the estimates in Proposition 4.7 also hold on $\mathcal{Q}_{\boldsymbol{\beta}}(p, R) := B_{\boldsymbol{\beta}}(p, R) \times (0, R^2) \subset \mathcal{Q}_{\boldsymbol{\beta}}$, i.e., the cylinder whose spatial center p may not lie in $\mathcal{S}$.

It is easy to derive the following local Schauder estimate for $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}$-solutions to (4-15) from Proposition 4.7.

Corollary 4.9. *Let $K \Subset B_{\boldsymbol{\beta}}(0, 1)$ be a compact subset and $\varepsilon_0 \in (0, 1)$ be a given number. With the same assumptions as in Proposition 4.7, there exists a constant $C = C(n, \boldsymbol{\beta}, \alpha, g, K, \varepsilon_0) > 0$ such that*

$$\|u\|_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}(K\times[\varepsilon_0,1])} \leq C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol{\beta}})} + \|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})}).$$

With the interior Schauder estimates in Proposition 4.7, we can show the existence of $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol{\beta}}(\mathcal{Q}_{\boldsymbol{\beta}})$-solutions to the Dirichlet problem

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \text{in } \mathcal{Q}_{\boldsymbol{\beta}} \qquad \text{and} \qquad u = \varphi \quad \text{on } \partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol{\beta}} \tag{4-20}$$

for any given $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol{\beta}}(\overline{\mathcal{Q}_{\boldsymbol{\beta}}})$ and $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol{\beta}})$. We first show the existence of solutions to (4-20) in the case $\varphi \equiv 0$.

Lemma 4.10. *Let $\sigma \in (0, 1)$ be given and $u \in \mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})$ solve* (4-20), *with $\|u\|^{(-\sigma)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} < \infty$ and $\|f\|^{(2-\sigma)}_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol\beta})} < \infty$. Then there is a constant $C = C(n, \alpha, \boldsymbol\beta, g, \sigma) > 0$ such that*

$$\|u\|^{(-\sigma)}_{\mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})} \le C(\|u\|^{(-\sigma)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|^{(2-\sigma)}_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol\beta})}).$$

Proof. The lemma follows from the definitions of the norms and the estimates in Proposition 4.7. □

Lemma 4.11. *Suppose $u \in \mathscr{C}_{\boldsymbol\beta}^{2,1}(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ satisfies $\partial u/\partial t = \Delta_g u + f$ and $u \equiv 0$ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$. For any $\sigma \in (0, 1)$, there exists a constant $C = C(n, \boldsymbol\beta, g, \sigma) > 0$ such that*

$$\|u\|^{(-\sigma)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} = \sup_{P_x \in \mathcal{Q}_{\boldsymbol\beta}} d_{P_x}^{-\sigma}|u(P_x)| \le C \sup_{P_x \in \mathcal{Q}_{\boldsymbol\beta}} d_{P_x}^{2-\sigma}|f(P_x)| = C\|f\|^{(2-\sigma)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})},$$

where $d_{P_x} = d_{\mathcal{P},\boldsymbol\beta}(P_x, \partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$ is the parabolic distance of P_x to the parabolic boundary $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$.

Proof. We write $N := \|f\|^{(2-\sigma)}_{C^0(\mathcal{Q}_{\boldsymbol\beta})} < \infty$ and $P_x = (x, t_x)$. Define functions

$$w_1(P_x) = (1 - d_{\boldsymbol\beta}(x)^2)^\sigma \quad \text{and} \quad w_2(P_x) = t_x^{\sigma/2},$$

where $d_{\boldsymbol\beta}(x) = d_{\boldsymbol\beta}(x, 0)$ is the $g_{\boldsymbol\beta}$-distance between x and 0. Observe that $d_{P_x} = \min\{1 - d_{\boldsymbol\beta}(x), t_x^{1/2}\}$ by definition. By a straightforward calculation there is a constant $c_0 > 0$ such that

$$\Big(\frac{\partial}{\partial t} - \Delta_g\Big)w_1 \ge c_0(1 - d_{\boldsymbol\beta}(x))^{\sigma-2} \quad \text{and} \quad \Big(\frac{\partial}{\partial t} - \Delta_g\Big)w_2 \ge c_0(t_x^{1/2})^{\sigma-2}.$$

By the maximum principle we get

$$|u(P_x)| \le Nc_0^{-1}(w_1(P_x) + w_2(P_x)) \quad \text{for all } P_x \in \mathcal{Q}_{\boldsymbol\beta}. \tag{4-21}$$

We take the decomposition of $\mathcal{Q}_{\boldsymbol\beta}$ into different regions, $\mathcal{Q}_{\boldsymbol\beta} = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 := \{P_x \in \mathcal{Q}_{\boldsymbol\beta} \mid t_x^{1/2} > 1 - d_{\boldsymbol\beta}(x)\},$$
$$\Omega_2 := \{P_x \in \mathcal{Q}_{\boldsymbol\beta} \mid t_x^{1/2} \le 1 - d_{\boldsymbol\beta}(x)\}.$$

Inequality (4-21) implies that, on the parabolic boundaries $\partial_{\mathcal{P}}\Omega_1$ and $\partial_{\mathcal{P}}\Omega_2$, we have $|u(P_x)| \le 2Nc_0^{-1}d_{P_x}^\sigma$. On Ω_1 we have $(\partial/\partial t - \Delta_g)(2Nc_0^{-1}w_1 \pm u) \ge 0$ and $2Nc_0^{-1}w_1 \pm u \ge 0$ on $\partial_{\mathcal{P}}\Omega_1$, so the maximum principle implies that $2Nc_0^{-1}w_1 \pm u \ge 0$ in Ω_1, i.e., $|u(P_x)| \le 2Nc_0^{-1}d_{P_x}^\sigma$ in Ω_1. Similarly we also have $2Nc_0^{-1}w_2 \pm u \ge 0$ in Ω_2, and thus $|u(P_x)| \le 2Nc_0^{-1}d_{P_x}^\sigma$ in Ω_2. In conclusion, we get

$$|u(P_x)| \le 2c_0^{-1}Nd_{P_x}^\sigma \quad \text{for all } P_x \in \mathcal{Q}_{\boldsymbol\beta}. \qquad \square$$

Proposition 4.12. *If $\varphi \equiv 0$, equation* (4-20) *admits a unique solution $u \in \mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ for any $f \in \mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$.*

Proof. Uniqueness follows from the maximum principle, so it suffices to show existence. We will use the continuity method. Define a continuous family of linear operators as follows: for $s \in [0, 1]$, let $L_s := s(\partial/\partial t - \Delta_g) + (1 - s)(\partial/\partial t - \Delta_{g_{\boldsymbol\beta}})$. It can been seen that $L_s = \partial/\partial t - \Delta_{g_s}$ for some conical

Kähler metric g_s which is uniformly equivalent to $g_{\boldsymbol\beta}$ and has uniform $\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}$-estimate. So the interior Schauder estimates holds also for L_s. Fix $\sigma \in (0,1)$. Define

$$\mathcal{B}_1 := \{u \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta}) \mid \|u\|^{(-\sigma)}_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} < \infty\},$$
$$\mathcal{B}_2 := \{f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta}) \mid \|f\|^{(2-\sigma)}_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} < \infty\}.$$

Observe that any $u \in \mathcal{B}_1$ is continuous in $\overline{\mathcal{Q}_{\boldsymbol\beta}}$ and vanishes on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$. L_s defines a continuous family of linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. By Lemmas 4.10 and 4.11 we have

$$\|u\|_{\mathcal{B}_1} \le C(\|u\|^{(-\sigma)}_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|L_s u\|_{\mathcal{B}_2}) \le C\|L_s u\|_{\mathcal{B}_2} \quad \text{for all } s \in [0,1] \text{ and for all } u \in \mathcal{B}_1.$$

By Corollary 4.5 and Remark 4.6, L_0 is invertible, thus by Theorem 5.2 in [18], L_1 is also invertible. □

Corollary 4.13. *For any $\varphi \in \mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$ and $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$, equation* (4-20) *admits a unique solution $u \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta}) \cap \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$.*

Proof. The proof is identical to that of Corollary 3.39 by an approximation argument. We may assume $\varphi \in \mathscr{C}^0(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ and choose a sequence $\varphi_k \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$ which converges uniformly to φ on $\overline{\mathcal{Q}_{\boldsymbol\beta}}$. The equations

$$\frac{\partial v_k}{\partial t} = \Delta_g v_k + f - \Delta_g \varphi_k \qquad \text{and} \qquad v_k \equiv 0 \quad \text{on } \partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$$

admit a unique $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}$-solution by Proposition 4.12. The interior Schauder estimates in Corollary 4.9 imply that $u_k := v_k + \varphi_k$ converges in $\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta,\mathrm{loc}}$ to some function u in $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$ which solves (4-20). The $\mathscr{C}^0$-convergence $u_k \to u$ is uniform on $\overline{\mathcal{Q}_{\boldsymbol\beta}}$ by the maximum principle, so $u = \varphi$ on $\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta}$, as desired. □

We recall the definition of weak solutions and refer to Section 7.1 in [17] for the notations.

Definition 4.14. We say a function u on $\mathcal{Q}_{\boldsymbol\beta}$ is a weak solution to the equation $\partial u/\partial t = \Delta_g u + f$ if:

(1) $u \in L^2(0,1; H^1(B_{\boldsymbol\beta}))$ and $\partial u/\partial t \in L^2(0,1; H^{-1}(B_{\boldsymbol\beta}))$.

(2) For any $v \in H^1_0(B_{\boldsymbol\beta})$ and $t \in (0,1)$,

$$\int_{B_\beta} \frac{\partial u(x,t)}{\partial t} v(x)\omega_g^n = -\int_{B_\beta} \langle \nabla u(x,t), \nabla v(x)\rangle_g \omega_g^n + \int_{B_\beta} f(x,t)v(x)\omega_g^n.$$

On can use the classical *Galerkin approximations* to construct a weak solution to $\partial u/\partial t = \Delta_g u + f$ (see Section 7.1.2 in [17]). If f has better regularity, so does the weak solution u.

Lemma 4.15. *If $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$, then any weak solution to*

$$\frac{\partial u}{\partial t} = \Delta_g u + f$$

belongs to $\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$.

Proof. The Sobolev inequality holds for the metric g, so by the proof of the standard De Giorgi–Nash–Moser theory for parabolic equations we have that u is in fact continuous on $\mathcal{Q}_{\boldsymbol\beta}$. Since the metric g is smooth on $\mathcal{Q}^\#_{\boldsymbol\beta}$, the weak solution u is also a weak solution in $\mathcal{Q}^\#_{\boldsymbol\beta}$ with the smooth background metric, so we have that $u \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\mathrm{loc}}(\mathcal{Q}^\#_{\boldsymbol\beta})$ in the usual sense by the classical Schauder estimates. Thus it suffices to consider points at $\mathcal{S}$. We choose the worst such point $0 \in \mathcal{S}$ only, since the case when centers are in other components of $\mathcal{S}$ is even simpler. We fix the point $P_0 = (0, t_0) \in \mathcal{Q}_{\boldsymbol\beta}$ with $t_0 > 0$. Fix $r \in (0, \sqrt{t_0})$. By Corollary 4.13,

$$\frac{\partial v}{\partial t} = \Delta v + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta}(P_0, r) := B_{\boldsymbol\beta}(0, r) \times (t_0 - r^2, t_0]$$

with boundary value $v = u$ on $\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol\beta}(P_0, r)$ admits a unique solution $v \in \mathscr{C}^{2+\alpha,(\alpha+2)/2}{}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta}(P_0, r))$. Then by the maximum principle $u = v$ in $\mathcal{Q}_{\boldsymbol\beta}(P_0, r)$. Thus $u \in \mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta}(P_0, r))$ too. Since the argument also works at other space-time points in $\mathcal{S}_{\mathcal{P}}$, we see that $u \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})$, as desired. □

Corollary 4.16. *Let* (X, g, D) *be as in Corollary 3.41, and let* $u_0 \in C^0(X)$ *and* $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(X \times (0, 1])$ *be given functions. The weak solution u to the equation*

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \textit{in } X \times (0, 1], \qquad u|_{t=0} = u_0$$

always exists. Moreover, $u \in \mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(X \times (0, 1])$, *and there exists a constant* $C = C(n, g, \boldsymbol\beta, \alpha) > 0$ *such that*

$$\|u\|_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(X\times(1/2,1])} \le C(\|u_0\|_{C^0(X)} + \|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(X\times(0,1])}).$$

Proof. The weak equation can be constructed using the *Galerkin approximations* [17]. The uniqueness is an easy consequence of the maximum principle. The regularity of u follows from the local results in Lemma 4.15. The estimate follows from the maximum principle, a covering argument as in Corollary 3.41, and the local estimates in Corollary 4.9. □

The *interior* estimate in Corollary 4.16 is not good enough to show the existence of solutions to nonlinear partial differential equations since the estimate becomes worse as t approaches 0. We need some global estimates in the whole time interval $t \in [0, 1]$ if the initial value u_0 has better regularity.

4D. ***Schauder estimate near t = 0.*** In this subsection, we will prove a Schauder estimate in the whole time interval for the solutions to the heat equation when the initial value is 0 or has better regularity. We consider the model case with the background metric $g_{\boldsymbol\beta}$ first, then we generalize the estimate to general nonflat conical Kähler metrics.

4D1. *The model case.* In this subsection, we will assume the background metric is $g_{\boldsymbol\beta}$. Let u be the solution to the equation

$$\frac{\partial u}{\partial t} = \Delta_{g_{\boldsymbol\beta}} u + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta}, \qquad u|_{t=0} \equiv 0, \tag{4-22}$$

and $u = \varphi \in \mathscr{C}^0$ on $\partial B_{\boldsymbol\beta} \times (0, 1]$, where $f \in \mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\overline{\mathcal{Q}_{\boldsymbol\beta}})$. In the calculations below, we should have used the smooth approximating solutions u_ϵ, where $\partial_t u_\epsilon = \Delta_{g_\epsilon} u_\epsilon + f$ and $u_\epsilon = u$ on $\partial_{\mathcal{P}} \mathcal{Q}_{\boldsymbol\beta}$. But by letting $\epsilon \to 0$, the corresponding estimates also hold for u. So for simplicity, we will work directly on u.

We fix $0<\rho<R\le 1$ and define $B_R := B_\beta(0,R)$ and $Q_R := B_R\times[0,R^2]$ in this section. Let u be the solution to (4-22). We first have the following Caccioppoli inequalities.

Lemma 4.17. *There exists a constant $C=C(n)>0$ such that*

$$\sup_{t\in[0,\rho^2]}\int_{B_\rho} u^2\omega_\beta^n+\iint_{Q_\rho}|\nabla u|^2_{g_\beta}\omega_\beta^n\,dt\le C\left(\frac{1}{(R-\rho)^2}\iint_{Q_R}u^2\omega_\beta^n\,dt+(R-\rho)^2\iint_{Q_R}f^2\omega_\beta^n\,dt\right) \tag{4-23}$$

and

$$\begin{aligned}\sup_{t\in[0,\rho^2]}\int_{B_\rho}|\nabla u|^2_{g_\beta}\omega_\beta^n&+\iint_{Q_\rho}(|\nabla\nabla u|^2_{g_\beta}+|\nabla\overline{\nabla}u|^2_{g_\beta})\omega_\beta^n\,dt\\&\le C\left(\frac{1}{(R-\rho)^2}\iint_{Q_R}|\nabla u|^2_{g_\beta}\omega_\beta^n\,dt+\iint_{Q_R}(f-f_R)^2\omega_\beta^n\,dt\right),\end{aligned} \tag{4-24}$$

where $f_R:=|Q_R|^{-1}_{\omega_\beta}\iint_{Q_R}f\omega_\beta^n\,dt$ is the average of f over the cylinder Q_R.

Proof. We fix a cut-off function η such that $\operatorname{supp}\eta\subset B_R$, $\eta=1$ on B_ρ, and $|\nabla\eta|_{g_\beta}\le 2/(R-\rho)$. Multiplying both sides of (4-22) by η^2u and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{B_R}\eta^2u^2&=\int_{B_R}2\eta^2u\Delta_{g_\beta}u+2\eta^2uf=\int_{B_R}-2\eta^2|\nabla u|^2_{g_\beta}-4u\eta\langle\nabla u,\nabla\eta\rangle_{g_\beta}+2\eta^2uf\\&\le\int_{B_R}-\eta^2|\nabla u|^2_{g_\beta}+4u^2|\nabla\eta|^2_{g_\beta}+\eta^2\frac{u}{(R-\rho)^2}+\eta^2(R-\rho)^2f^2.\end{aligned}$$

Equation (4-23) follows by integrating this inequality over $t\in[0,s^2]$ for all $s\le\rho$. To see (4-24), observe that the Bochner formula yields

$$\frac{\partial}{\partial t}|\nabla u|^2\le\Delta_{g_\beta}|\nabla u|^2-|\nabla\nabla u|^2_{g_\beta}-|\nabla\overline{\nabla}u|^2_{g_\beta}-2\langle\nabla u,\nabla f\rangle_{g_\beta}.$$

Multiplying both sides of this inequality by η^2 and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{B_R}\eta^2|\nabla u|^2&\le\int_{B_R}-2\eta\langle\nabla\eta,\nabla|\nabla u|^2\rangle_{g_\beta}-\eta^2|\nabla\nabla u|^2-\eta^2|\nabla\overline{\nabla}u|^2-2\eta^2\langle\nabla u,\nabla f\rangle\\&\le\int_{B_R}4\eta|\nabla u||\nabla\eta|\big|\nabla|\nabla u|\big|-\eta^2|\nabla\nabla u|^2-\eta^2|\nabla\overline{\nabla}u|^2\\&\qquad\qquad 4\eta|f-f_R||\nabla\eta||\nabla u|+2\eta^2|f-f_R||\Delta_{g_\beta}u|\\&\le\int_{B_R}-\tfrac12\eta^2(|\nabla\nabla u|^2+|\nabla\overline{\nabla}u|^2)+10\eta^2|\nabla u|^2|\nabla\eta|^2+20\eta^2(f-f_R)^2.\end{aligned}$$

Then (4-24) follows by integrating this inequality over $t\in[0,s^2]$ for any $s\in[0,\rho]$. □

Combining (4-23) and (4-24) we conclude that

$$\begin{aligned}\sup_{t\in[0,R^2/4]}\int_{B_{R/2}}|\nabla u|^2&+\iint_{Q_{R/2}}|\Delta_{g_\beta}u|^2\\&\le\frac{C}{R^4}\iint_{Q_R}u^2+CR^{2n+2}\|f\|^2_{\mathscr{C}^0(Q_R)}+CR^{2n+2+2\alpha}([f]_{\mathscr{C}^{\alpha,\alpha/2}_\beta(Q_R)})^2.\end{aligned} \tag{4-25}$$

By a standard Moser iteration argument we get the following sub-mean-value inequality.

Lemma 4.18. *If in addition $f \equiv 0$, then there exists a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that*

$$\sup_{Q_\rho} |u| \le C\left(\frac{1}{(R-\rho)^{2n+2}} \iint_{Q_R} u^2 \omega_{\boldsymbol{\beta}}^n \, dt\right)^{1/2}.$$

Proof. For any $p \ge 1$, multiplying both sides of the equation by $\eta^2 u_+^p$, where $u_+ = \max\{u, 0\}$, and integrating by parts, we get

$$\frac{d}{dt} \int_{B_R} \frac{\eta^2}{p+1} u_+^{p+1} = \int_{B_R} -p\eta^2 u_+^{p-1} |\nabla u_+|^2 - 2\eta u_+^p \langle \nabla u_+, \nabla \eta\rangle.$$

By the Cauchy–Schwarz inequality and integrating over $t \in [0, R^2]$, we conclude that

$$\sup_{s \in [0, R^2]} \int_{B_R} \eta^2 u_+^{p+1}\Big|_{t=s} + \iint_{Q_R} |\nabla(\eta u_+^{(p+1)/2})|^2 \le \frac{C}{(R-\rho)^2} \iint_{Q_R} u_+^{p+1} \omega_{\boldsymbol{\beta}}^n \, dt =: A.$$

By the Sobolev inequality we get

$$\begin{aligned}\int_0^{R^2}\!\!\int_{B_R} (\eta^2 u_+^{p+1})^{1+1/n} &\le \int_0^{R^2} \left(\int_{B_R} \eta^2 u_+^{p+1}\right)^{1/n} \left(\int_{B_R} (\eta u_+^{(p+1)/2})^{2n/(n-1)}\right)^{(n-1)/n} \\ &\le A^{1/n} C \int_0^{R^2}\!\!\int_{B_R} |\nabla(\eta u_+^{(p+1)/2})|^2 \le C A^{(n+1)/n}.\end{aligned}$$

If we write

$$H(p, \rho) = \left(\int_0^{\rho^2}\!\!\int_{B_\rho} u_+^p\right)^{1/p},$$

the inequality above implies

$$H((p+1)\xi, \rho) \le \frac{C^{1/(p+1)}}{(R-\rho)^{2/(p+1)}} H(p+1, R),$$

where $\xi = (n+1)/n > 1$. Writing $p_k + 1 = 2\xi^k$ and $\rho_k = \rho + (R-\rho)2^{-k}$, we then have $H(p_{k+1}+1, \rho_{k+1}) \le H(p_k+1, \rho_k)$. Iterating this inequality we get

$$H(\infty, \rho) = \sup_{Q_\rho} u_+ \le \frac{C}{(R-\rho)^{n+1}} \left(\iint_{Q_R} u_+^2\right)^{1/2}.$$

Similarly we get the same inequality for $u_- = \max\{-u, 0\}$. □

Corollary 4.19. *If in addition $f \equiv 0$, then there is a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that*

$$\iint_{Q_\rho} u^2 \omega_{\boldsymbol{\beta}}^n \, dt \le C\left(\frac{\rho}{R}\right)^{2+2n} \iint_{Q_R} u^2 \omega_{\boldsymbol{\beta}}^n \, dt. \tag{4-26}$$

Proof. When $\rho \in \left[\frac{1}{2}R, R\right]$, the inequality is trivial; when $\rho \in \left[0, \frac{1}{2}R\right)$, it follows from Lemma 4.18. □

Lemma 4.20. *If in addition $f \equiv 0$, then there is a constant $C = C(n, \boldsymbol{\beta}) > 0$ such that, for any $\rho \in (0, R)$,*

$$\iint_{Q_\rho} u^2 \omega_{\boldsymbol{\beta}}^n \, dt \le C\left(\frac{\rho}{R}\right)^{2n+4} \iint_{Q_R} u^2 \omega_{\boldsymbol{\beta}}^n \, dt.$$

Proof. The inequality is trivial in the case $\rho \in \left[\frac{1}{2}R, R\right]$, so we assume $\rho < \frac{1}{2}R$. First we observe that $\Delta_{\boldsymbol\beta} u$ also satisfies the equations $\partial_t(\Delta_{\boldsymbol\beta} u) = \Delta_{\boldsymbol\beta}(\Delta_{\boldsymbol\beta} u)$ and $(\Delta_{\boldsymbol\beta} u)|_{t=0} \equiv 0$, so (4-26) holds with u^2 replaced by $(\Delta_{\boldsymbol\beta} u)^2$, i.e.,

$$\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \omega_{\boldsymbol\beta}^n \, dt \le C\left(\frac{\rho}{R}\right)^{2+2n} \iint_{Q_R} (\Delta_{\boldsymbol\beta} u)^2 \omega_{\boldsymbol\beta}^n \, dt.$$

Since $u|_{t=0} = 0$, we have $u(x,t) = \int_0^t \partial_s u(x,s)\, ds$, and we calculate

$$\begin{aligned}\iint_{Q_\rho} u^2 \le \rho^4 \iint_{Q_\rho} \left|\frac{\partial u}{\partial t}\right|^2 = \rho^4 \iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \le C\rho^4 \left(\frac{\rho}{R}\right)^{2n+2} \iint_{Q_{R/2}} (\Delta_{\boldsymbol\beta} u)^2 \\ \le C\left(\frac{\rho}{R}\right)^{2n+6} \iint_{Q_R} u^2 \omega_{\boldsymbol\beta}^n \, dt \qquad \text{(by (4-25)).} \qquad \square\end{aligned}$$

Lemma 4.21. *Let u be a solution to* (4-22). *There exists a constant $C = C(n, \boldsymbol\beta, \alpha) > 0$ such that*

$$\frac{1}{\rho^{2n+2+2\alpha}} \iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \le \frac{C}{R^{2n+2+2\alpha}} \iint_{Q_R} (\Delta_{\boldsymbol\beta} u)^2 \omega_{\boldsymbol\beta}^n \, dt + C([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_R)})^2.$$

Proof. Let $u = u_1 + u_2$, where

$$\frac{\partial u_1}{\partial t} = \Delta_{\boldsymbol\beta} u_1 + f_R \quad \text{in } Q_R, \qquad u_1 = u \quad \text{on } \partial_{\mathcal P} Q_R,$$

and

$$\frac{\partial u_2}{\partial t} = \Delta_{\boldsymbol\beta} u_2 + f - f_R \quad \text{in } Q_R, \qquad u_2 = 0 \quad \text{on } \partial_{\mathcal P} Q_R.$$

The function $\Delta_{\boldsymbol\beta} u_1$ satisfies the assumptions of Lemma 4.20. Thus

$$\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u_1)^2 \omega_{\boldsymbol\beta}^n \, dt \le C\left(\frac{\rho}{R}\right)^{2n+4} \iint_{Q_R} (\Delta_{\boldsymbol\beta} u_1)^2 \omega_{\boldsymbol\beta}^n \, dt.$$

Multiplying both sides of the equation for u_2 by $\dot u_2 = \partial u_2/\partial t$ and noting that $\dot u_2 = 0$ on $\partial B_R \times (0, R^2)$, we get

$$\begin{aligned}\int_{B_R} (\dot u_2)^2 &= \int_{B_R} \dot u_2 \Delta_{\boldsymbol\beta} u_2 + \dot u_2 (f - f_R) = \int_{B_R} -2\langle \nabla \dot u_2, \nabla u_2\rangle + \dot u (f - f_R) \\ &\le \int_{B_R} -\frac{\partial}{\partial t} |\nabla u_2|^2 + \tfrac{1}{2}(\dot u_2)^2 + 2(f - f_R)^2.\end{aligned}$$

Integrating over $t \in [0, R^2]$, we obtain

$$\iint_{Q_R} (\dot u_2)^2 \le -2 \int_{B_R} |\nabla u_2|^2 \Big|_{t=R^2} + 4 \iint_{Q_R} (f - f_R)^2,$$

therefore

$$\iint_{Q_R} (\Delta_{\boldsymbol\beta} u_2)^2 \le 2 \iint_{Q_R} (\dot u_2)^2 + 2 \iint_{Q_R} (f - f_R)^2 \le C R^{2n+2+2\alpha} ([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_R)})^2.$$

Then for $\rho < R$ we have

$$\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \le 2\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u_1)^2 + 2\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u_2)^2$$
$$\le C\Big(\frac{\rho}{R}\Big)^{2n+4} \iint_{Q_R} (\Delta_{\boldsymbol\beta} u_1)^2 \omega_{\boldsymbol\beta}^n \, dt + C R^{2n+2+2\alpha} ([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_R)})^2.$$

The estimate is proved by an iteration lemma (see Lemma 3.4 in [22]). □

Lemma 4.22. *Suppose u satisfies* (4-22). *There exists a constant $C = C(n, \boldsymbol\beta, \alpha) > 0$ such that, for any $\rho \in \big(0, \frac{1}{2}R\big)$,*

$$\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u - (\Delta_{\boldsymbol\beta} u)_\rho)^2 \omega_{\boldsymbol\beta}^n \, dt \le C M_R \rho^{2n+2+2\alpha},$$

where

$$M_R := \frac{1}{R^{4+2\alpha}} \|u\|^2_{\mathscr{C}^0(Q_R)} + \frac{1}{R^{2\alpha}} \|f\|^2_{\mathscr{C}^0(Q_R)} + ([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_R)})^2.$$

Proof. From Lemma 4.21, we get

$$\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \le C\rho^{2+2n+2\alpha} \bigg(\frac{1}{R^{2n+2+2\alpha}} \iint_{Q_{2R/3}} (\Delta_{\boldsymbol\beta} u)^2 + ([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_{2R/3})})^2 \bigg)$$
$$\le C\rho^{2+2n+2\alpha} \bigg(\frac{1}{R^{2n+6+2\alpha}} \iint_{Q_R} u^2 + \frac{1}{R^{2\alpha}} \|f\|^2_{C^0(Q_R)} + ([f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(Q_R)})^2 \bigg) \quad \text{(by (4-25))}$$
$$\le C\rho^{2+2n+2\alpha} M_R.$$

On the other hand, by the Hölder inequality,

$$(\Delta_{\boldsymbol\beta} u)^2_\rho = \frac{1}{|Q_\rho|^2_{g_{\boldsymbol\beta}}} \bigg(\iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u) \omega_{\boldsymbol\beta}^n \, dt \bigg)^2 \le \frac{C}{\rho^{2+2n}} \iint_{Q_\rho} (\Delta_{\boldsymbol\beta} u)^2 \le C M_R \rho^{2\alpha}.$$

The lemma is proved by combining the two inequalities above. □

By Campanato's lemma (see Theorem 3.1 in Chapter 3 of [22]), we get the following.

Corollary 4.23. *There is a constant $C = C(n, \boldsymbol\beta, \alpha) > 0$ such that, for any $x \in B_{\boldsymbol\beta}\big(0, \frac{3}{4}\big)$ and $R < \frac{1}{10}$,*

$$[\Delta_{\boldsymbol\beta} u]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(x,R/2)\times[0,R^2/4])}$$
$$\le C\bigg(\frac{1}{R^{2+\alpha}} \|u\|_{\mathscr{C}^0(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} + \frac{1}{R^\alpha} \|f\|_{\mathscr{C}^0(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} + [f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} \bigg). \tag{4-27}$$

Lemma 4.24. *There exists a constant $C = C(n, \boldsymbol\beta, \alpha) > 0$ such that, for any $x \in B_{\boldsymbol\beta}\big(0, \frac{3}{4}\big)$ and $R < \frac{1}{10}$,*

$$[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(x,R/2)\times[0,R^2/4])} + \bigg[\frac{\partial u}{\partial t}\bigg]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(x,R/2)\times[0,R^2/4])}$$
$$\le C\bigg(\frac{1}{R^{2+\alpha}} \|u\|_{\mathscr{C}^0(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} + \frac{1}{R^\alpha} \|f\|_{\mathscr{C}^0(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} + [f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(x,R)\times[0,R^2])} \bigg). \tag{4-28}$$

Proof. It follows from (4-27) and the elliptic Schauder estimates in Theorem 1.2 by adjusting R slightly that, for any $t \in \left[0, \frac{1}{4}R^2\right]$,

$$[Tu(\,\cdot\,,t)]_{C^{0,\alpha}_{\beta}(B_{\beta}(x,R/2))} \le C\left(\frac{1}{R^{2+\alpha}}\|u\|_{\mathscr{C}^0(B_{\beta}(x,R)\times[0,R^2])} + \frac{1}{R^{\alpha}}\|f\|_{\mathscr{C}^0(B_{\beta}(x,R)\times[0,R^2])} + [f]_{\mathscr{C}^{\alpha,\alpha/2}_{\beta}(B_{\beta}(x,R)\times[0,R^2])}\right),$$

that is, in the spatial variables the estimate (4-28) holds. It only remains to show the Hölder continuity of Tu in the time-variable. For this, we fix any two times $0 \le t_1 < t_2 \le \frac{1}{4}R^2$ and denote $r := \frac{1}{2}\sqrt{t_2 - t_1}$. For any $x_0 \in B_{\beta}\left(x, \frac{1}{4}R\right)$, $B_{\beta}(x_0, r) \subset B_{\beta}\left(x, \frac{1}{2}R\right)$. By (4-27) and the equation for u, it is not hard to see that the inequality (4-27) holds when $\Delta_{\beta} u$ on the left-hand side is replaced by $\dot{u} = \partial u/\partial t$. In particular,

$$\frac{|\dot{u}(y,t) - \dot{u}(y,t_1)|}{|t-t_1|^{\alpha/2}} \le A_R \quad \text{for all } y \in B_{\beta}\left(x, \tfrac{1}{2}R\right),$$

where A_R is defined to be the constant on the right-hand side of (4-27). Integrating over $t \in [t_1, t_2]$ we get

$$|u(y,t_2) - u(y,t_1) - \dot{u}(y,t_1)(t_2-t_1)| \le CA_R(t_2-t_1)^{1+\alpha/2}.$$

Thus, for any $y \in B_{\beta}(x_0, r)$,

$$\begin{aligned}|u(y,t_2) - u(y,t_1) - \dot{u}(x_0,t_1)(t_2-t_1)| &\le |u(y,t_2) - u(y,t_1) - \dot{u}(y,t_1)(t_2-t_1)| + |\dot{u}(x_0,t_1) - \dot{u}(y,t_1)|(t_2-t_1)\\ &\le CA_R(t_2-t_1)^{1+\alpha/2} + A_R r^{\alpha}(t_2-t_1).\end{aligned}$$

Write

$$\tilde{u}(y) := u(y,t_2) - u(y,t_1) - \dot{u}(x_0,t_1)(t_2-t_1),$$

which is a function on $B_{\beta}(x_0, r)$. We have that the function $\tilde{f} := \Delta_{\beta}\tilde{u} = \Delta_{\beta}u(\,\cdot\,,t_2) - \Delta_{\beta}u(\,\cdot\,,t_1)$ satisfies the inequalities $\|\tilde{f}\|_{C^0(B_{\beta}(x_0,r))} \le A_R(t_2-t_1)^{\alpha}$ and $[\tilde{f}]_{C^{0,\alpha}_{\beta}(B_{\beta}(x_0,r))} \le A_R$ by (4-27). It follows from the rescaled version of Proposition 3.32 that

$$|T\tilde{u}|_{C^0(B_{\beta}(x_0,r/2))} \le C(n,\boldsymbol{\beta},\alpha)\left(\frac{\|\tilde{u}\|_{C^0(B_{\beta}(x_0,r))}}{r^2} + \|\tilde{f}\|_{C^0(B_{\beta}(x_0,r))} + r^{\alpha}[\tilde{f}]_{C^{0,\alpha}(B_{\beta}(x_0,r))}\right) \le C(t_2-t_1)^{\alpha/2}A_R.$$

Therefore, for any $x_0 \in B_{\beta}\left(x, \frac{1}{4}R\right)$,

$$\frac{|Tu(x_0,t_2) - Tu(x_0,t_1)|}{|t_2-t_1|^{\alpha/2}} \le CA_R.$$

It is then easy to see by the triangle inequality that (adjusting R slightly if necessary)

$$[Tu]_{\mathscr{C}^{\alpha,\alpha/2}_{\beta}(B_{\beta}(x,R/2)\times[0,R^2/4])} \le CA_R,$$

as desired. The estimate for $\dot{u}$ follows from the equation $\dot{u} = \Delta_g u + f$. □

Remark 4.25. By a simple parabolic rescaling of the metric and time, we see from (4-28) that, for any $0 < r < R < \frac{1}{10}$,

$$[Tu]_{\mathscr{C}^{\alpha,\alpha/2}_{\beta}(Q_r)} \le C\left(\frac{\|u\|_{\mathscr{C}^0(Q_R)}}{(R-r)^{2+\alpha}} + \frac{\|f\|_{\mathscr{C}^0(Q_R)}}{(R-r)^{\alpha}} + [f]_{\mathscr{C}^{\alpha,\alpha/2}_{\beta}(Q_R)}\right). \tag{4-29}$$

4D2. *The nonflat metric case.* In this subsection, we will consider the case when the background metrics are general nonflat $\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}$-conical Kähler metrics $g = g(z,t)$. Suppose $u \in \mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(\mathcal{Q}_{\boldsymbol\beta})$ satisfies the equation

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta}, \qquad u|_{t=0} = 0, \tag{4-30}$$

and $u \in \mathscr{C}^0(\partial_{\mathcal{P}}\mathcal{Q}_{\boldsymbol\beta})$.

Proposition 4.26. *There exists a constant $C = C(n, \boldsymbol\beta, \alpha, g) > 0$ such that*

$$\|u\|_{\mathscr{C}_{\boldsymbol\beta}^{2+\alpha,(2+\alpha)/2}(B_{\boldsymbol\beta}(0,1/2)\times[0,1/4])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol\beta})}).$$

Proof. Choosing suitable complex coordinates at the origin $x = 0$, we may assume the components of g in the basis $\{\epsilon_j \wedge \bar\epsilon_k, \dots\}$ satisfy $g_{\epsilon_j\bar\epsilon_k}(\,\cdot\,, 0) = \delta_{jk}$ and $g_{j\bar k}(\,\cdot\,, 0) = \delta_{jk}$ at the origin. As in the proof of Proposition 4.7, we can write (4-30) as

$$\frac{\partial u}{\partial t} = \Delta_{\boldsymbol\beta} u + \eta \cdot \sqrt{-1}\partial\bar\partial u + f =: \Delta_{\boldsymbol\beta} u + \hat f,$$

where η is given in the proof of Proposition 3.32. By (4-29) we get

$$[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_r)} \le C\biggl(\frac{\|u\|_{\mathscr{C}^0(\widetilde Q_R)}}{(R-r)^{2+\alpha}} + \frac{1}{(R-r)^\alpha}\|\hat f\|_{\mathscr{C}^0(\widetilde Q_R)} + [\hat f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}\biggr),$$

where $\widetilde Q_R := B_{\boldsymbol\beta}(0,R) \times [0, R^2]$. Observe that

$$\begin{aligned}\frac{1}{(R-r)^\alpha}\|\hat f\|_{\mathscr{C}^0(\widetilde Q_R)} &\le \frac{1}{(R-r)^\alpha}\|f\|_{\mathscr{C}^0(\widetilde Q_R)} + \frac{1}{(R-r)^\alpha}\|\eta\|_{\mathscr{C}^0(\widetilde Q_R)}\|Tu\|_{C^0(\widetilde Q_R)}\\ &\le \frac{1}{(R-r)^\alpha}\|f\|_{\mathscr{C}^0(\widetilde Q_R)} + \frac{[\eta]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}R^\alpha}{(R-r)^\alpha}(\varepsilon[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + C(\varepsilon)\|u\|_{\mathscr{C}^0(\widetilde Q_R)})\end{aligned}$$

and

$$\begin{aligned}[\hat f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} &\le [f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + \|\eta\|_{\mathscr{C}^0(\widetilde Q_R)}[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + \|Tu\|_{\mathscr{C}^0(\widetilde Q_R)}[\eta]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}\\ &\le [f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + [\eta]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}R^\alpha[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}\\ &\qquad + [\eta]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}(\varepsilon[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + C(\varepsilon)\|u\|_{\mathscr{C}^0(\widetilde Q_R)}).\end{aligned}$$

By choosing $R_0 = R_0(n, \boldsymbol\beta, \alpha, g) > 0$ small enough and suitable $\varepsilon > 0$, for any $0 < r < R < R_0 < \frac{1}{10}$, the combination of the above inequalities yields

$$[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_r)} \le \tfrac12[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)} + C\biggl(\frac{\|u\|_{\mathscr{C}^0(\widetilde Q_R)}}{(R-r)^{2+\alpha}} + \frac{1}{(R-r)^\alpha}\|f\|_{\mathscr{C}^0(\widetilde Q_R)} + [f]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_R)}\biggr).$$

By Lemma 4.27 below (setting $\phi(r) = [Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\widetilde Q_r)}$), we conclude that

$$[Tu]_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(B_{\boldsymbol\beta}(0,R_0/2)\times[0,R_0^2/4])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|_{\mathscr{C}_{\boldsymbol\beta}^{\alpha,\alpha/2}(\mathcal{Q}_{\boldsymbol\beta})}).$$

This is the desired estimate when the center of the ball is the worst possible. For the other balls $B_{\boldsymbol\beta}(x,r)$ with center $x \in B_{\boldsymbol\beta}\bigl(0, \frac12\bigr)$, we can repeat the above procedures and use the smooth coordinates $w_j = z_j^{\beta_j}$

in case the ball is disjoint from $\mathcal{S}_j$. Finitely many such balls cover $B_{\boldsymbol\beta}\big(0, \frac{1}{2}\big)$, so we get

$$[Tu]_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1/2)\times[0,1/100])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})}).$$

The proposition is proved by combining this inequality, the equation for u, interpolation inequalities, and the interior Schauder estimates in Corollary 4.9. □

Lemma 4.27 [22, Lemma 4.3]. *Let $\phi(t) \ge 0$ be bounded in $[0, T]$. Suppose, for any $0 < t < s \le T$, we have*

$$\phi(t) \le \tfrac{1}{2}\phi(s) + \frac{A}{(s-t)^a} + B$$

for some $a > 0$ and $A, B > 0$. Then it holds that, for any $0 < t < s \le T$,

$$\phi(t) \le c(a)\bigg(\frac{A}{(s-t)^a} + B\bigg).$$

Corollary 4.28. *Suppose u satisfies the equations*

$$\frac{\partial u}{\partial t} = \Delta_g u + f \quad \text{in } \mathcal{Q}_{\boldsymbol\beta} \qquad u|_{t=0} = u_0 \in C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1)).$$

Then

$$\|u\|_{\mathscr{C}^{2+\alpha,(\alpha+2)/2}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1/2)\times[0,1])} \le C(\|u\|_{\mathscr{C}^0(\mathcal{Q}_{\boldsymbol\beta})} + \|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(\mathcal{Q}_{\boldsymbol\beta})} + \|u_0\|_{C^{2,\alpha}_{\boldsymbol\beta}(B_{\boldsymbol\beta}(0,1))})$$

for some constant $C = C(n, \boldsymbol\beta, \alpha, g) > 0$.

Proof. We set $\hat u = u - u_0$ and $\hat f = f - \Delta_g u_0$. $\hat u$ satisfies the conditions in Proposition 4.26, so the corollary follows from Proposition 4.26 applied to $\hat u$ and triangle inequalities. □

Corollary 4.29. *In addition to the assumptions in Corollary 4.16, we also assume that $u_0 \in C^{2,\alpha}_{\boldsymbol\beta}(X)$. Then the weak solution to $\partial u/\partial t = \Delta_g u + f$ with $u|_{t=0} = u_0$ exists and is in $\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(X, \times[0,1])$. Moreover, there is a $C = C(n, \boldsymbol\beta, \alpha, g) > 0$ such that*

$$\|u\|_{\mathscr{C}^{2+\alpha,(2+\alpha)/2}_{\boldsymbol\beta}(X\times[0,1])} \le C(\|f\|_{\mathscr{C}^{\alpha,\alpha/2}_{\boldsymbol\beta}(X\times[0,1])} + \|u_0\|_{C^{2,\alpha}_{\boldsymbol\beta}(X)}). \tag{4-31}$$

Proof. Observe that by the maximum principle we have

$$\|u\|_{\mathscr{C}^0(X\times[0,1])} \le \|f\|_{\mathscr{C}^0(X\times[0,1])} + \|u_0\|_{C^0(X)}.$$

Then (4-31) follows from Corollary 4.28 and a covering argument as in the proof of Corollary 3.41. □

5. Conical Kähler–Ricci flow

Let X be a compact Kähler manifold and $D = \sum_j D_j$ be a divisor with simple normal crossings. Let ω_0 be a fixed $C^{0,\alpha'}_{\boldsymbol\beta}(X)$ conical Kähler metric with cone angle $2\pi\boldsymbol\beta$ along D and $\hat\omega_t$ be a family of $\mathscr{C}^{\alpha',\alpha'/2}_{\boldsymbol\beta}$ conical metrics which are uniformly equivalent to ω_0, with $\hat\omega_0 = \omega_0$ and $\|\hat\omega\|_{\mathscr{C}^{\alpha',\alpha'/2}_{\boldsymbol\beta}(X\times[0,1])} \le C_0$. We consider the complex Monge–Ampère equation

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \log\bigg(\dfrac{(\hat\omega_t + \sqrt{-1}\partial\bar\partial\varphi)^n}{\omega_0^n}\bigg) + f, \\ \varphi|_{t=0} = 0, \end{cases} \tag{5-1}$$

where $f \in \mathscr{C}^{\alpha',\alpha'/2}{}_{\beta}(X\times[0,1])$ is a given function. We will use an inverse function theorem argument from [4] which was outlined in [21] to show the short-time existence of the flow (5-1).

Theorem 5.1. *There exists a small* $T = T(n, \boldsymbol{\beta}, \omega_0, f, \alpha, \alpha') > 0$ *such that* (5-1) *admits a unique solution* $\varphi \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0, T])$ *for any* $\alpha < \alpha'$.

Proof. The uniqueness of the solution follows from the maximum principle. We will break the proof of short-time existence into three steps.

Step 1. Let $u \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha',(2+\alpha')/2}(X \times [0, 1])$ be the solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{g_0} u + f & \text{in } X \times [0, 1], \\ u|_{t=0} = 0. \end{cases}$$

Thanks to Corollary 4.29, such a u exists and satisfies the estimate (4-31). We fix $\varepsilon > 0$, so that, as long as $\|\phi\|_{C^{2,\alpha}_{\boldsymbol{\beta}}(X)} \le \varepsilon$, we have that $\hat{\omega}_{t,\phi} := \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi$ is equivalent to ω_0, i.e.,

$$C_0^{-1}\omega_0 \le \omega_{0,\phi} \le C_0\omega_0 \quad \text{and} \quad \|\hat{\omega}_{t,\phi}\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}} \le C_0.$$

We *claim* that, for $T_1 > 0$ small enough, $\|u\|_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X\times[0,T_1])} \le \varepsilon$. We first observe by (4-31) that

$$N := \|u\|_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha',(\alpha'+2)/2}(X\times[0,1])} \le C\|f\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}(X\times[0,1])}.$$

It suffices to show that $[u]_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha',(\alpha'+2)/2}(X\times[0,T_1])}$ is small, since the lower-order derivatives are small because $u|_{t=0} = 0$. We calculate, for any $t_1, t_2 \in [0, T_1]$,

$$\frac{|Tu(x, t_1) - Tu(x, t_2)|}{|t_1 - t_2|^{\alpha/2}} + \frac{|\dot{u}(x, t_1) - \dot{u}(x, t_2)|}{|t_1 - t_2|^{\alpha/2}} \le N|t_1 - t_2|^{(\alpha'-\alpha)/2} \le \tfrac{1}{4}\varepsilon$$

if $NT_1^{(\alpha'-\alpha)/2} < \frac{1}{4}\varepsilon$. For any $x, y \in X$ and $t \in [0, T_1]$,

$$\frac{|Tu(x, t) - Tu(y, t)|}{d_{g_0}(x, y)^\alpha} \le N \min\left\{\frac{2T_1^{\alpha'/2}}{d_{g_0}(x, y)^\alpha}, d_{g_0}(x, y)^{\alpha'-\alpha}\right\} \le \tfrac{1}{2}\varepsilon.$$

The *claim* then follows from the triangle inequality.

We define a function

$$w(x, t) := \frac{\partial u}{\partial t}(x, t) - \log\left(\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_0^n}\right)(x, t) - f(x, t) \quad \text{for all } (x, t) \in X \times [0, T_1].$$

It is clear that $w(x, 0) \equiv 0$.

Step 2: We consider the small ball

$$\mathcal{B} = \{\phi \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0, T_1]) \mid \|\phi\|_{\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(\alpha+2)/2}} \le \varepsilon,\ \phi(\cdot\,, 0) = 0\}$$

in the space $\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0, T_1])$. Then $u|_{t\in[0,T_1]} \in \mathcal{B}$ by the discussion in Step 1.

Define the differential map $\Psi : \mathcal{B} \to \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T_1])$ by

$$\Psi(\phi) = \frac{\partial\phi}{\partial t} - \log\left(\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega_0^n}\right) - f.$$

The map Ψ is well defined and C^1, where the differential $D\Psi_\phi$ at any $\phi \in \mathcal{B}$ is given by

$$D\Psi_\phi(v) = \frac{\partial v}{\partial t} - (\hat{g}_\phi)^{i\bar{j}} v_{i\bar{j}} = \frac{\partial v}{\partial t} - \Delta_{\hat{\omega}_{t,\phi}} v$$

for any

$$v \in T_\phi \mathcal{B} = \{v \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0, T_1]) \mid v(\,\cdot\,, 0) = 0\},$$

where $(\hat{g}_\phi)^{i\bar{j}}$ denotes the inverse of the metric $\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi$. As a linear map,

$$D\Psi_\phi : T_\phi \mathcal{B} \to \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T_1])$$

is injective by the maximum principle and surjective by Corollary 4.29. Thus $D\Psi_\phi$ is invertible at any $\phi \in \mathcal{B}$. In particular, $D\Psi_u$ is invertible, and by the inverse function theorem, $\Psi : \mathcal{B} \to \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T_1])$ defines a local diffeomorphism from a small neighborhood of $u \in \mathcal{B}$ to an open neighborhood of $w = \Psi(u)$ in $\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T_1])$. This implies that, for any $\tilde{w} \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T_1])$ with $\|w - \tilde{w}\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X\times[0,T_1])} < \delta$ for some small $\delta > 0$, there exists a unique $\varphi \in \mathcal{B}$ such that $\Psi(\varphi) = \tilde{w}$.

<u>Step 3</u>. For a small $T_2 < T_1$ to be determined, we define a function

$$\tilde{w}(x, t) = \begin{cases} 0, & t \in [0, T_2], \\ w(x, t - T_2), & t \in [T_2, T_1]. \end{cases}$$

Since $u \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha',(2+\alpha')/2}$, we see that $w \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}(X \times [0, T_1])$ with $M := \|w\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}(X\times[0,T_1])} < \infty$. We *claim* that if T_2 is small enough, then $\|w - \tilde{w}\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X\times[0,T_1])} < \delta$. We write $\eta = w - \tilde{w}$. It is clear from the fact that $w(\,\cdot\,, 0) = 0$ that $\|\eta\|_{\mathscr{C}^0} \le \frac{1}{2}\delta$ if T_2 is small enough.

Spatial directions: If $t < T_2$ then

$$\frac{|\eta(x,t) - \eta(y,t)|}{d_{g_0}(x,y)^\alpha} = \frac{|w(x,t) - w(y,t)|}{d_{g_0}(x,y)^\alpha} \le M \min\left\{ \frac{2T_2^{\alpha'/2}}{d_{g_0}(x,y)^\alpha}, d_{g_0}(x,y)^{\alpha'-\alpha} \right\} \le 2MT_2^{(\alpha'-\alpha)/2}.$$

If $t \in [T_2, T_1]$ then

$$\begin{aligned} \frac{|\eta(x,t) - \eta(y,t)|}{d_{g_0}(x,y)^\alpha} &= \frac{|w(x,t) - w(y,t) - w(x,t-T_2) + w(y,t-T_2)|}{d_{g_0}(x,y)^\alpha} \\ &\le 2M \min\left\{ \frac{T_2^{\alpha'/2}}{d_{g_0}(x,y)^\alpha}, d_{g_0}(x,y)^{\alpha'-\alpha} \right\} \le 2MT_2^{(\alpha'-\alpha)/2}. \end{aligned}$$

Time direction: If $t, t' < T_2$ then

$$\frac{|\eta(x,t) - \eta(x,t')|}{|t-t'|^{\alpha/2}} = \frac{|w(x,t) - w(x,t')|}{|t-t'|^{\alpha/2}} \le M|t-t'|^{(\alpha'-\alpha)/2} \le MT_2^{(\alpha'-\alpha)/2}.$$

If $t, t' \in [T_2, T_1]$ then

$$\frac{|\eta(x,t) - \eta(x,t')|}{|t-t'|^{\alpha/2}} = \frac{|w(x,t) - w(x,t') - w(x,t-T_2) + w(x,t'-T_2)|}{|t-t'|^{\alpha/2}} \le 2MT_2^{(\alpha'-\alpha)/2}.$$

If $t < T_2 \le t' \le T_1$ then

$$\frac{|\eta(x,t)-\eta(x,t')|}{|t-t'|^{\alpha/2}} = \frac{|w(x,t)-w(x,t')+w(x,t'-T_2)|}{|t-t'|^{\alpha/2}} \le 2MT_2^{(\alpha'-\alpha)/2}.$$

Therefore, if we choose $T_2 > 0$ small enough that $2MT_2^{(\alpha'-\alpha)/2} < \frac{1}{4}\delta$, then we have

$$\frac{|\eta(x,t)-\eta(x,t')|}{|t-t'|^{\alpha/2}} + \frac{|\eta(x,t)-\eta(y,t)|}{d_{g_0}(x,y)^\alpha} \le \tfrac{1}{2}\delta \quad \text{for all } x \in X, \quad t, t' \in [0, T_1].$$

It then follows from the triangle inequality that

$$\begin{aligned} |\eta(x,t)-\eta(y,t')| &\le |\eta(x,t)-\eta(y,t)| + |\eta(y,t)-\eta(y,t')| \\ &\le \tfrac{1}{2}\delta(d_{g_0}(x,y)^\alpha + |t-t'|^{\alpha/2}) \le \tfrac{1}{2}\delta d_{\mathcal{P},g_0}((x,t),(y,t'))^\alpha. \end{aligned}$$

In conclusion, $\|\tilde{w}-w\|_{\mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X\times[0,T_1])} < \delta$, so by Step 2 we conclude that there exists a $\varphi \in \mathcal{B}$ such that $\Psi(\varphi) = \tilde{w}$. Since $\tilde{w}|_{t\in[0,T_2]} \equiv 0$ by definition, $\varphi|_{t\in[0,T_2]}$ satisfies (5-1) for $t \in [0, T]$, where $T := T_2$. This shows the short-time existence of the flow (5-1). □

Proof of Corollary 1.11. Recall that in (1-13) we wrote $\omega_0^n = \Omega/\prod_j(|s_j|^2_{h_j})^{1-\beta_j}$, where Ω is a smooth volume form, s_j and h_j are holomorphic sections and hermitian metrics, respectively, of the line bundle associated to the component D_j. Choose a smooth reference form

$$\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega - \sum_j (1-\beta_j)\sqrt{-1}\partial\bar{\partial}\log h_j.$$

Define the reference metrics $\hat{\omega}_t = \omega_0 + t\chi$ which are $\mathscr{C}_{\boldsymbol{\beta}}^{\alpha',\alpha'/2}$-conical and Kähler for small $t > 0$. Let φ be the $\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}$-solution to (1-11) with $f \equiv 0$. Then it is straightforward to check that $\omega_t = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfies the conical Kähler–Ricci flow equation (1-12) and $\omega \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0, T])$ for some small $T > 0$.

The smoothness of ω in $X\backslash D \times (0, T]$ follows from the general smoothing properties of parabolic equations; see [37]. Taking $\partial/\partial t$ on both sides of (1-11) we get

$$\frac{\partial\dot{\varphi}}{\partial t} = \Delta_{\omega_t}\dot{\varphi} + \operatorname{tr}_{\omega_t}\chi \quad \text{and} \quad \dot{\varphi}|_{t=0} = 0.$$

By Corollary 4.29, $\dot{\varphi} \in \mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0,T])$ since $\operatorname{tr}_{\omega_t}\chi \in \mathscr{C}_{\boldsymbol{\beta}}^{\alpha,\alpha/2}(X \times [0,T])$. Therefore the normalized Ricci potential $\log(\omega_t^n/\omega_0^n)$ exists in $\mathscr{C}_{\boldsymbol{\beta}}^{2+\alpha,(2+\alpha)/2}(X \times [0,T])$. □

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