J. reine angew. Math. **797** (2023), 79–116
DOI 10.1515/crelle-2022-0095

Journal für die reine und angewandte Mathematik

Kähler–Einstein metrics near an isolated log-canonical singularity

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Abstract. We construct Kähler–Einstein metrics with negative scalar curvature near an isolated log canonical (non-log terminal) singularity. Such metrics are complete near the singularity if the underlying space has complex dimension 2. We also establish a stability result for Kähler–Einstein metrics near certain types of isolated log canonical singularity. As application, for complex dimension 2 log canonical singularity, we show that any complete Kähler–Einstein metric of negative scalar curvature near an isolated log canonical (non-log terminal) singularity is smoothly asymptotically close to model Kähler–Einstein metrics from hyperbolic geometry.

1. Introduction

The existence of Kähler–Einstein metrics on complex manifold has been the central topic in complex geometry for decades. In [51], Yau established the existence of Ricci-flat metrics on complex manifolds with zero Chern class by solving the Calabi conjecture, while Aubin and Yau [2,51] proved the existence of Kähler–Einstein metrics independently on canonically polarized compact manifolds. Recent results of Chen, Donaldson and Sun [11–13], Tian [47] and many interesting subsequent papers proved equivalence between existence of Kähler–Einstein metrics and K-stability for Fano manifolds, confirming the Yau–Tian–Donaldson conjecture. Also there has been intensive study of degenerate complex Monge–Ampère equations and construction of singular Kähler–Einstein metrics on singular varieties with Kawamata log terminal (klt) singularities, for example in [22, 44, 54], based on Kolodziej's fundamental result in [34, 35]. Kähler–Einstein metrics on canonical polarized variety with log canonical singularity are constructed in [4] by the variational approach. When such varieties appear on the boundary of the KSB compactification of smooth canonical models of general type, such Kähler–Einstein metrics turn out to be genuinely geometric as degeneration of Kähler–Einstein metrics [41–43]. However, little is known about the geometric behavior of these singular Kähler–Einstein metrics near the log canonical singularities. For smoothable klt singularities, the fundamental work of Donaldson and Sun [20, 21] and Li and Xu [37] show that the tangent cone at any given

Research supported in part by National Science Foundation grant DMS-1711439, and by the Infosys Young Investigator award.

singular point is unique and admits a Ricci-flat cone metric as the Gromov–Hausdorff limit of smooth Kähler–Einstein metrics (see also general structure results for noncollapsed limit in [10]). Such noncollapsed local models for smoothable klt singularities are always built on Sasakian geometry and an interesting metric stability result is obtained in [16, 28]. In this paper, we will study Kähler–Einstein metrics near isolated log canonical (non-log terminal) singularity and their asymptotic collapsing behavior near the complete end.

Let (X, p) be a germ of an isolated log canonical algebraic singularity p embedded in $(\mathbb{C}^N, 0)$. In this paper, we will assume p is not a log terminal singularity. We would like to construct local Kähler–Einstein metrics with negative scalar curvature in an open neighborhood of the singular point p. Let ρ be a nonnegative smooth plurisubharmonic (PSH) function on $\mathbb{C}^N$ with $\rho(0) = 0$ and $\sqrt{-1}\partial\bar\partial\rho > 0$. We let $\mathcal{U} \subset\subset X$ be the open domain defined by

$$\mathcal{U} = \{\rho < a\} \cap X \tag{1.1}$$

for some sufficiently small constant $a > 0$. For example, we can always choose

$$\rho = \sum_{j=1}^{N} |z_j|^2.$$

By choosing a generic sufficiently small $a > 0$, we can assume that $\partial\mathcal{U}$ is smooth and strongly pseudoconvex. We will fix such a domain $\mathcal{U}$ for (X, p) for the rest of the paper. We further require that the canonical divisor $K_{\mathcal{U}}$ is Cartier and fix a local volume measure Ω_X on $\mathcal{U}$ by

$$\Omega_X = (\sqrt{-1})^n \nu \wedge \bar\nu, \tag{1.2}$$

where ν is a local generator of $K_{\mathcal{U}}$. Similarly, we can define Ω_X when $K_{\mathcal{U}}$ is $\mathbb{Q}$-Cartier.

We will consider the following Dirichlet problem of a complex Monge–Ampère equation related to the Kähler–Einstein metric on $\mathcal{U}$:

$$\begin{cases} (\sqrt{-1}\partial\bar\partial\varphi)^n = e^{\varphi}\Omega_X & \text{in } \mathcal{U} \setminus \{p\}, \\ \varphi = \psi & \text{on } \partial\mathcal{U}, \end{cases} \tag{1.3}$$

where $\psi \in C^\infty(\partial\mathcal{U})$. Our first result is on existence of finite volume Kähler–Einstein metrics near isolated log-canonical singularities. To state the theorem, fix a log resolution $\pi : Y \to X$ and let D be the simple normal crossing exceptional divisor. We also fix a defining section σ_D and a hermitian metric h_D for the line bundle corresponding to D (cf. Section 2 for more precise definition).

Theorem 1.1. *Let $(\mathcal{U}, p)$ be a germ of an isolated log canonical algebraic (non-log terminal) singularity defined as above. For any smooth function ψ on $\partial\mathcal{U}$, there exists a function φ_{KE} satisfying the following:*

(1) $\varphi_{\mathrm{KE}} \in C^\infty(\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)) \cap \mathrm{PSH}(\pi^{-1}(\mathcal{U}))$,

(2) *for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$, such that on $\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)$,*

$$C \geq \varphi_{\mathrm{KE}} \geq -(2n + \varepsilon)\log(-\log|\sigma_D|^2_{h_D}) - C_\varepsilon,$$

(3) *φ_{KE} solves equation (1.3) on $\mathcal{U} \setminus \{p\}$ if we identify $\mathcal{U} \setminus \{p\}$ with $\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)$, and the Kähler metric $\omega_{\mathrm{KE}} = \sqrt{-1}\partial\bar\partial\varphi_{\mathrm{KE}}$ is Kähler–Einstein with finite volume*

$$\int_{\mathcal{U}\setminus\{p\}} \omega_{\mathrm{KE}}^n < \infty,$$

(4) φ_{KE} *has vanishing Lelong number at any point in* $\pi^{-1}(\mathcal{U})$ *and for* $x \in \pi^{-1}(\mathcal{U})$,

$$\lim_{x \to \mathrm{Supp}(D)} \varphi_{\mathrm{KE}}(x) = -\infty.$$

Moreover, any solution to (1.3) *(when pulling back to* $\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)$*) satisfying* (1)–(2) *above is unique.*

One naturally would ask how the Kähler–Einstein metric g_{KE} in Theorem 1.1 behaves near the singularity p geometrically. We propose the following conjecture.

Conjecture 1.1. Let g_{KE} be a Kähler–Einstein metric constructed in Theorem 1.1 on $\mathcal{U} \setminus \{p\}$. Then g_{KE} is complete near p. More precisely, for any fixed point $q \in \mathcal{U} \setminus \{p\}$ and $R > 0$, there exists an open neighborhood $V_{q,R}$ of p in $\mathcal{U}$ such that for any point $x \in V_{q,R} \setminus \{p\}$ and any smooth path γ joining q and x in $\mathcal{U} \setminus \{p\}$, we have

$$|\gamma|_{g_{\mathrm{KE}}} > R,$$

where $|\gamma|_{g_{\mathrm{KE}}}$ is the arc length of γ with respect to g_{KE}.

We are able to confirm the above conjecture in the following two special cases.

Theorem 1.2. *Conjecture* 1.1 *holds when* $\dim \mathcal{U} = 2$.

The construction from Theorem 1.1 gives infinitely many Kähler–Einstein metrics near the log canonical singularity p by assigning different boundary conditions. However, certain asymptotic stability should hold for such complete Kähler–Einstein metrics. More precisely, the asymptotic geometric behavior of complete Kähler–Einstein metrics near a log canonical singularity p should be completely determined by the analytic structure of p.

Our next result is to establish a volume stability for such complete Kähler–Einstein metrics. We can always shrink $\mathcal{U}$ slightly because we are interested in the behavior of complete Kähler–Einstein metrics near the isolated log canonical singularity p.

Theorem 1.3. *Let* g_{KE} *and* g'_{KE} *be two Kähler–Einstein metrics on* $\mathcal{U} \setminus \{p\}$ *for an isolated log canonical (non-log terminal) singularity* p. *Let* $R(x)$ *and* $R'(x)$ *be the distance functions from any point* $x \in \mathcal{U} \setminus \{p\}$ *to* $\partial\mathcal{U}$ *with respect to* g_{KE} *and* g'_{KE}. *If both* g_{KE} *and* g'_{KE} *are complete near* p, *then there exists* $c = c(n) > 0$ *such that*

$$1 - \frac{c(n)}{R'(x)} \le \frac{\det(g'_{\mathrm{KE}})}{\det(g_{\mathrm{KE}})}(x) \le 1 + \frac{c(n)}{R(x)}$$

for any $x \in \mathcal{U} \setminus \{p\}$ *with* $R(x) > 1$ *and* $R'(x) > 1$. *In particular,*

$$\lim_{x \to p} \frac{\det(g'_{\mathrm{KE}})}{\det(g_{\mathrm{KE}})}(x) = 1,$$

where $x \to p$ *with respect to the Euclidean metric from any local embedding of* p *in* $\mathbb{C}^N$.

Theorem 1.3 implies that the potentials of two complete Kähler–Einstein metrics near the isolated log canonical singularity p must be asymptotically close to each other at infinity.

There is a large class of log canonical singularities that admits complete Kähler–Einstein metrics with bounded geometry at the complete end. For example, a metric uniformization is obtained in [31, 32] for isolated log canonical singularity in complex dimension 2. Another interesting example of cones over an abelian variety is constructed in [23]. We apply the method of bounded geometry by [14] (see also [31, 32, 48]) to prove the following stability result.

Theorem 1.4. *Let $(\mathcal{U}, p)$ be a germ of an isolated log canonical singularity p. If there exists a complete Kähler–Einstein metric θ near p with bounded geometry of order $k \geq 2$, then for any complete Kähler–Einstein metric g_{KE} on $\mathcal{U} \setminus \{p\}$ and $k \geq 0$, there exists $C = C(n, k, \theta, g_{\mathrm{KE}}) > 0$ such that for any $x \in \mathcal{U}$ with $R_\theta(x) > 1$, we have*

$$\|\nabla_\theta^k (\omega_{\mathrm{KE}} - \theta)\|_\theta(x) \leq \frac{C}{(R_\theta(x))^{\frac{1}{2}}},$$

where $R_\theta(x)$ is the distance from x to ∂U and ∇_θ is the covariant derivative with respect to θ.

Remark 1.1. We remark that the completeness assumption of ω_{KE} is necessary, because there are examples of germ of log canonical singularity $(\mathcal{U} \setminus p)$ which admit both complete and incomplete Kähler–Einstein metrics (cf. [25, Example 2.7]). We also want to compare our stability result with the results of [16, 28]. There in the Ricci-flat case, if we could relate two different Kähler–Einstein metrics $\omega_{\mathrm{KE}}^1, \omega_{\mathrm{KE}}^2$ as

$$\omega_{\mathrm{KE}}^1 = \omega_{\mathrm{KE}}^2 + \sqrt{-1}\partial\bar{\partial}u,$$

the difficulty lies in the higher order estimate of u. While in our case, one new ingredient is that we can show the boundedness of volume ratio u unconditionally in Theorem 1.3 and then use it to show that any local complete ω_{KE} can be obtained from Cheng–Yau-type construction once we have a good model metric. Then higher order regularity of volume ratio is a byproduct of Cheng–Yau-type construction. And to our knowledge, the asymptotics result improves Cheng–Yau global construction and hence is new.

We also have the following immediate corollary.

Corollary 1.1. *Let $(\mathcal{U}, p)$ be the germ of a log canonical singularity p with* $\dim \mathcal{U} = 2$. *Then for any complete Kähler–Einstein metric g_{KE} near p, $(\mathcal{U} \setminus \{p\}, g_{\mathrm{KE}})$ must be asymptotically isometric to one of the following two local models:*

(1) $(\mathbb{B}^2/\Gamma, g_{\mathbb{B}^2/\Gamma})$, *where $\mathbb{B}^2$ is the unit ball in $\mathbb{C}^2$, Γ is a parabolic discrete subgroup of* $\mathrm{Aut}(\mathbb{B}^2)$ *and $g_{\mathbb{B}^2/\Gamma}$ is the Kähler metric induced by the group action invariant hyperbolic metric on $\mathbb{B}^2$,*

(2) $((\mathbb{H} \times \mathbb{H})/\Gamma, g_{(\mathbb{H}\times\mathbb{H})/\Gamma})$, *where $\mathbb{H}$ is the hyperbolic upper half plane, Γ is a parabolic discrete subgroup of* $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ *and $g_{(\mathbb{H}\times\mathbb{H})/\Gamma}$ is the Kähler metric induced by the group action invariant hyperbolic metric $g_{\mathbb{H}\times\mathbb{H}}$.*

Furthermore, if we take a sequence of points $p_j \in (\mathcal{U} \setminus p)$ with $p_j \to p$ (in the Euclidean topology), then $(\mathcal{U}, p_j, g_{\mathrm{KE}})$ converges to line $\mathbb{R}$ as in case (1) *and converges to flat cylinder $S^1 \times \mathbb{R}$ as in case* (2)*, in pointed Gromov–Hausdorff topology.*

Note that Theorem 1.4 can also be applied to higher dimensional complex hyperbolic cusps. (cf. [23]).

Remark 1.2. For application, our Theorem 1.4 and Corollary 1.1 can be applied to understand the geometry of Kähler–Einstein metric with negative scalar curvature constructed on certain stable canonical polarized variety by Berman-Guenancia [4] and Song [42] (cf. Example 5.1).

We further conjecture that for any complete Kähler–Einstein metric g_{KE} on $(\mathcal{U}, p)$ an isolated log canonical singularity p, any sequence of points $x_j \to p$ and positive $\lambda_j \geq 1$, then $(\mathcal{U} \setminus \{p\}, x_j, \lambda_j g_{\mathrm{KE}})$ converges in pointed Gromov–Hausdorff topology to a product of $\mathbb{R}$, $\mathbb{C}$, compact Calabi–Yau varieties and complete Calabi–Yau varieties with cylindrical end.

We briefly outline our paper. In Section 2, we prove Theorem 1.1 by solving the Dirichlet problem of singular complex Monge-Ampere equations. We prove the volume stability of Theorem 1.3 for complete Kähler–Einstein metrics near isolated log canonical singularities in Section 3 and a metric stability Theorem 1.4 if one of the Kähler–Einstein metric satisfies the bounded geometry condition in Section 4. In Section 6, we prove Theorem 1.2 for surfaces and Corollary 1.1. In Section 6, we give a short discussion of conjecture 1.1 for smoothable isolated log singularities.

Acknowledgement. The second author would like to thank Professor H-J. Hein for his illuminating lectures on his joint work with Professor S. Sun [28] during the Summer School at Notre Dame University in 2017 and F. Tong for many interesting discussions. We also thank the anonymous referee for many useful suggestions.

2. Kähler–Einstein metrics near log canonical singularities

2.1. The set-up. In this section, we will prove Theorem 1.1. We first recall the definition for log canonical singularities.

Definition 2.1. Let X be a normal variety such that K_X is a $\mathbb{Q}$-Cartier divisor. Let $\pi : Y \to X$ be a log resolution and $\{E_i\}_{i=1}^p$ the irreducible components of the exceptional locus $Exc(\pi)$ of π. There there exists a unique collection $a_i \in \mathbb{Q}$ such that

$$K_Y = \pi^* K_X + \sum_{i=1}^{p} a_i E_i.$$

Then X is said to have log-canonical (resp. klt) singularities if

$$a_i \geq -1 \quad (\text{resp. } a_i > -1) \quad \text{for all } i.$$

In this paper, since we are considering isolated singularity, we have $\pi(Exc(\pi)) = p$. We have to prescribe singularities of the solution φ to obtain a canonical and unique Kähler–Einstein current on X. To do so, we lift all the data to a log resolution $\pi : Y \to X$. It is more convenient to write the adjunction formula in the following form:

$$K_Y = \pi^* K_X + \sum_{i}^{n} a_i E_i - \sum_{j}^{m} b_j F_j, \quad a_i \geq 0,\ 0 < b_j \leq 1.$$

Let σ_{E_i}, $1 \leq i \leq n$, be the defining section for line bundle associated to E_i and let σ_{F_i}, $1 \leq j \leq m$, be the defining section for line bundle associated to F_j. We also equip the line bundles associated to E_i and F_i with smooth hermitian metric h_{E_i}, h_{F_j} on Y. Then we define

$$|\sigma_E|^2_{h_E} := \prod_i |\sigma_{E_i}|^{2a_i}_{h_i}, \quad |\sigma_F|^2_{h_F} := \prod_j |\sigma_{F_j}|^{2b_j}_{h_j}.$$

Recall that Ω_X is a local volume form defined on $\mathcal{U}$ in formula (1.2), now let Ω_Y be a smooth strictly positive volume form on $\pi^{-1}(\mathcal{U})$, defined by

$$\Omega_Y = (|\sigma_E|^2_{h_E})^{-1}|\sigma_F|^2_{h_F}\pi^*\Omega_X.$$

Then lifting equation (1.3) to Y, we have

$$\begin{cases} (\pi^*\sqrt{-1}\partial\bar\partial\varphi)^n = e^{\pi^*\varphi}(|\sigma_F|^2_{h_F})^{-1}|\sigma_E|^2_{h_E}\Omega_Y, \\ \pi^*\varphi_{|\partial\mathcal{U}} = \psi. \end{cases}$$

Abusing notation, we still denote the domain $\pi^{-1}(\mathcal{U})$ by $\mathcal{U}$. Let θ be a fixed smooth Kähler form on Y and we consider the following perturbed family of complex Monge–Ampère equations on $\pi^{-1}(\mathcal{U})$ for $s \in (0,1)$:

$$\begin{cases} (s\theta + \sqrt{-1}\partial\bar\partial\varphi_s)^n = e^{\varphi_s}\dfrac{\prod_i(|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j(|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}}\Omega_Y, \\ \varphi_{s|\partial\mathcal{U}} = \psi. \end{cases} \tag{2.1}$$

For each $s > 0$, we shall first get solution φ_s for equation (2.1). When $s = 0$, equation (2.1) coincides with equation (1.3). Next we want to use pluripotential theory to get a uniform C^0 estimate with barrier of φ_s. Using pluripotential theory [3], a similar C^0 estimate of degenerate Monge–Ampère equations has been obtained in different settings such as on unit ball in [34], on singular Stein domain in [26] and on singular variety with klt singularity in [18, 22, 54]. The main difference of our geometric domain with previous setting is the following: we consider the log canonical singularity, which means we do not have $L^p (p > 1)$ integrability of the right-hand side of equations (2.1), hence we do not have uniform boundedness control of φ_s, which will also cause extra trouble for high order derivative estimates.

Firstly, let us recall that our domain $\mathcal{U}$ is defined by $\mathcal{U} := \{\rho < a\}$ in formula (1.1), where ρ is smooth on $\mathbb{C}^N$. Hence $\pi^*\rho$ is smooth on $\pi^{-1}\mathcal{U}$, where π is a resolution of singularity. Now choose an arbitrary smooth extension ψ_1 of ψ, which is supported on a neighborhood of $\partial\mathcal{U}$. Hence $\pi^*\rho$ is smooth on $\pi^{-1}\mathcal{U}$. By choosing A large enough, we define a semipositive $(1,1)$-form ω on $\mathcal{U}$:

$$\omega := A\sqrt{-1}\partial\bar\partial(\rho - a) + \sqrt{-1}\partial\bar\partial\psi_1. \tag{2.2}$$

Now

$$s\theta + \sqrt{-1}\partial\bar\partial\varphi_s = \omega + s\theta + \sqrt{-1}\partial\bar\partial(\varphi_s - A(\rho - a) - \psi_1).$$

Let $\phi_s := \varphi_s - A(\rho - a) - \psi_1$ and $M := A(\rho - a) + \psi_1$. Then we can rewrite equations (2.1) as a new family of equations with ϕ_s as unknown functions and with zero Dirichlet boundary:

$$\begin{cases} (\omega + s\theta + \sqrt{-1}\partial\bar\partial\varphi_s)^n = \dfrac{e^{\varphi_s + M}\prod_i(|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j(|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}}\Omega_Y, \\ \varphi_{s|\partial\mathcal{U}} = 0. \end{cases} \tag{2.3}$$

The above equations hold for ϕ_s, we abuse notation and still denote the unknown functions by φ_s. Notice all the estimate we get for ϕ_s also holds for φ_s due to the fact that M is independent of s. From now on, we will focus on equations (2.3).

Lemma 2.1. *For each fixed $s > 0$, there exists a unique smooth solution φ_s to equation* (2.3).

Proof. Recall that $\partial\mathcal{U} := \{\rho = a\}$, hence $\phi := \rho - a$ is equal to 0 on $\partial\mathcal{U}$. Now one subsolution of equation (2.3) will be $A\phi$ by choosing A sufficient large. it is well known that existence of subsolution implies existence of solution, see for example [7, Theorem A]. □

Next, we state the Kodaira Lemma which is very useful in the estimates that follows.

Lemma 2.2 ([33, Lemma 2.62]). *Let $\pi : Y \to X$ be log resolution of singularity, where X is $\mathbb{Q}$ factorial. Also let ω be a Kähler form on X and θ be a Kähler form on Y. Then there exist a simple normal crossing divisor $D = \sum D_i$ supporting on the exceptional locus, hermitian metrics h_{D_i} on the line bundle associated to D_i and a sequence of constants $\alpha_i > 0$ and $s_0 > 0$ such that*

$$\pi^*\omega + s\sum_i \alpha_i\sqrt{-1}\partial\bar{\partial}\log h_{D_i} > 0$$

for all $0 < s \le s_0$. By adjusting the coefficients α_i of D_i, we may assume that $s_0 = 1$ and that there exists a constant $\beta > 0$ such that

$$\pi^*\omega + \sum_i \sqrt{-1}\partial\bar{\partial}\alpha_i \log h_{D_i} > \beta\theta.$$

For simplicity, we also define

$$|\sigma_D|^2_{h_D} := \prod_i |D_i|^{2\alpha_i}_{h_{D_i}}, \quad \sqrt{-1}\partial\bar{\partial}\log h_D := \sum_i \alpha_i\sqrt{-1}\partial\bar{\partial}\log h_{D_i}. \tag{2.4}$$

In the next two subsections we obtain uniform estimates for φ_s, completing the proof of Theorem 1.1.

2.2. The C^0 estimate.

Proposition 2.1. *Let $|\sigma_D|^2_{h_D}$ be defined as in* (2.4). *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$C \ge \varphi_s \ge -(2n+\varepsilon)\log(-\log|\sigma_D|^2_{h_D}) - C_\varepsilon.$$

Proof. First note that all φ_s are $C\theta$-PSH for some fixed large constant C independent of s and $\varphi_s = 0$ on $\partial\mathcal{U}$. This implies that

$$\Delta_\theta\varphi_s \ge -C.$$

The upper bound then follows easily by comparing φ_s to the solution to the following Dirichlet problem:

$$\begin{cases} \Delta_\theta u = -C, \\ \quad u = 0 \quad \text{on } \partial\mathcal{U}. \end{cases}$$

For the lower bound we will use a barrier function inspired by [43, Proposition 3.1] (see also [19] for an interesting C^0 estimate for singular Kähler–Einstein metrics in families). From now on, denote by $\varepsilon > 0$ a small but fixed constant and by C_ε a constant depending on ε which might change from line to line. Let D be as in (2.4). We scale h_D so that $|\sigma_D|^2_{h_D} < e^{\frac{-3n}{\varepsilon}}$. Note that this has no effect on $\sqrt{-1}\partial\bar\partial \log h_D$. Now let H be a function defined on $(-\infty, 0]$ satisfying $H' > 0$, $H'' > 0$ to be determined. For each fixed divisor F_j and some constant $0 < \delta < 1$, consider the barrier function $H(u^j_\varepsilon)$, where

$$u^j_\varepsilon := \varepsilon \log|\sigma_D|^2 - \varepsilon^2(-\log|F_j|^2_{h_{F_j}})^{1-\delta}.$$

For $\varepsilon > 0$ sufficiently small, direct calculation shows that

$$\begin{aligned}\omega + \sqrt{-1}\partial\bar\partial u^j_\varepsilon = \omega &- \varepsilon \operatorname{Ric}(D) - \varepsilon^2(1-\delta)(-\log|F_j|^2_{h_{F_j}})^{-\delta}\sqrt{-1}\partial\bar\partial(-\log|F_j|^2_{h_{F_j}}) \\ &+ \varepsilon^2\delta(1-\delta)(-\log|F_j|^2_{h_{F_j}})^{-1-\delta}\sqrt{-1}\partial\log|F_j|^2_{h_{F_j}} \wedge \bar\partial\log|F_j|^2_{h_{F_j}}.\end{aligned}$$

Notice that $-\log|F_j|^2_{h_{F_j}} > 1$. Therefore (possibly change θ)

$$\varepsilon^2(1-\delta)(-\log|F_j|^2_{h_{F_j}})^{-\delta}\sqrt{-1}\partial\bar\partial(-\log|F_j|^2_{h_{F_j}}) \le \varepsilon^2\theta.$$

It follows from Kodaira's lemma that for ε sufficiently small, one has

$$\omega - \varepsilon\operatorname{Ric}(D) - \varepsilon^2(1-\delta)(-\log|F_j|^2_{h_{F_j}})^{-\delta}\sqrt{-1}\partial\bar\partial(-\log|F_j|^2_{h_{F_j}}) \ge \frac{\varepsilon}{2}\theta. \tag{2.5}$$

If we write $|F_j|^2 = |z_j|^2 e^{\varphi_j}$ locally near divisor F_j, we have

$$\sqrt{-1}\partial\log|F_j|^2_{h_{F_j}} \wedge \bar\partial\log|F_j|^2_{h_{F_j}} \ge \frac{\sqrt{-1}dz_j\wedge d\bar z_j}{2|z_j|^2} - \theta. \tag{2.6}$$

Combining inequalities (2.5) and (2.6) and further assuming that H can be chosen such that $H'(u^j_\varepsilon) < 1$, we have

$$\begin{aligned}\omega + s\theta + \sqrt{-1}\partial\bar\partial H(u^j_\varepsilon) &\ge \omega + H'\sqrt{-1}\partial\bar\partial u^j_\varepsilon \\ &\ge H'(\omega + \sqrt{-1}\partial\bar\partial u^j_\varepsilon) \\ &\ge H'\Big(\frac{\varepsilon}{4}\theta + \varepsilon^2\delta(1-\delta)(-\log|F_j|^2_{h_{F_j}})^{-1-\delta}\frac{\sqrt{-1}dz_j\wedge d\bar z_j}{|z_j|^2}\Big).\end{aligned}$$

Now define

$$\hat H = \frac{\sum_j H(u^j_\varepsilon)}{m}$$

(recall that m is the number of components of $F = \sum F_j$) and $\chi_s = \omega + s\theta + \sqrt{-1}\partial\bar\partial\hat H$. Direct calculations show that

$$\chi^n_s \ge C_{\delta,\varepsilon,n}(H')^n\frac{dz_1\wedge d\bar z_1\cdots dz_n\wedge d\bar z_n}{\Pi_j(|z_j|^2(-\log|z_j|)^{1+\delta})}. \tag{2.7}$$

We define $H(x) = -B\log(-x)$ for some $3n > B > 0$ to be chosen later, and consider

$$\phi_s := \varphi_s - \hat H.$$

Then ϕ_s solves the following equation:

$$(\chi_s + \sqrt{-1}\partial\bar{\partial}\phi_s)^n = e^{\phi_s + \widehat{H} + M} \frac{\prod_i (|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \Omega_Y.$$

Let q_s be the minimum point of ϕ_s. Without loss of generality we may assume that q_s lies in the interior. Since $\phi_s(x) \to +\infty$ as $x \to \mathrm{Supp}(D)$, we may also assume that $q_s \notin \mathrm{Supp}(D)$. Then by the maximum principle, at q_s we have the following estimate:

$$\chi_s^n \le e^{\phi_s + \widehat{H} + M} \frac{\prod_i (|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \Omega_Y \le C \frac{e^{\phi_s + \widehat{H}}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \theta^n \tag{2.8}$$

for some constant C independent of s. Locally near q_s, we have $\sigma_D = z_1^{\alpha_1} \cdots z_k^{\alpha_n}$, where $(z_1, \ldots, z_n)$ are complex coordinates and $\alpha_i > 0$ for $i = 1, \ldots, k$. Combining inequalities (2.7) and (2.8), we have

$$e^{\phi_s} \ge C_{\delta,\varepsilon,n} e^{-\widehat{H}} \frac{(H')^n}{\Pi_j (-\log |z_j|)^{1+\delta}} \ge C_{\delta,\varepsilon,B} \frac{(-\log |\sigma_D|^2)^{B-n}}{\Pi_j (-\log |z_j|)^{1+\delta}},$$

where we have used $b_j \le 1$ and the two inequalities

$$e^{-\widehat{H}} \ge B \log(-\log |\sigma_D|^2_{h_D}), \quad (H')^n \ge (-2B\varepsilon \log |\sigma_D|^2_{h_D})^{-n}.$$

Choosing $\delta \ll \varepsilon$ and $B = 2n + \varepsilon$, we have $e^{\phi_s} \ge C_\varepsilon$. Therefore

$$\varphi_s \ge \widehat{H} - C_\varepsilon = -\sum_j \frac{(2n + \varepsilon)}{m} \log(-u_\varepsilon^j) - C_\varepsilon \ge -(2n + \varepsilon) \log(-\log |\sigma_D|^2) - C_\varepsilon,$$

which ends the proof. □

Corollary 2.1. *There exists a constant C independent of s such that*

$$\int_{\mathcal{U} \setminus \mathrm{Supp}(D)} (\omega + s\theta + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n \le C.$$

Proof. Recall that h_D is chosen such that $|\sigma_D|^2_{h_D} < e^{-4n}$. Let

$$f = -3n \log(-\log |\sigma_D|^2_{h_D}) + A$$

for some sufficiently large $A > 0$ so that

$$f \ge \varphi_s$$

on $\partial\mathcal{U}$ for all $s \in (0, 1)$. From the calculations above one can see that $f \in \mathrm{PSH}(\omega + s\theta)$. For $\varepsilon > 0$, we let

$$\varphi_{s,\varepsilon} = \varphi_s + \varepsilon \log |\sigma_D|^2_{h_D}.$$

By Proposition 2.1 we have $\varphi_{s,\varepsilon} < f$ near $\mathrm{Supp}(D)$ and on $\partial\mathcal{U}$. In particular, this implies that the set $\{\varphi_{s,\varepsilon} > f\}$ will be supported on a relatively compact set contained in $(\mathcal{U} \setminus \mathrm{Supp}(D))$.

So by the comparison principle (cf. [9, Lemma 3.4]) and the fact that $s\theta + \varepsilon\sqrt{-1}\partial\bar\partial \log h_D \geq 0$ for sufficiently small ε, we have

$$\int_{\varphi_{s,\varepsilon}>f} (\omega + s\theta + \sqrt{-1}\partial\bar\partial\varphi_s)^n \leq \int_{\varphi_{s,\varepsilon}>f} (\omega + 2s\theta + \sqrt{-1}\partial\bar\partial\varphi_{s,\varepsilon})^n \leq \int_{\varphi_{s,\varepsilon}>f} (\omega + 2s\theta + \sqrt{-1}\partial\bar\partial f)^n.$$

If we note that as $\varepsilon \searrow 0$, the open sets $W_\varepsilon = \{\varphi_{s,\varepsilon} \geq f\}$ increase to $W = \{\varphi_s > f\}$. Note that $\mathrm{Supp}(D) \subset W$ by Proposition 2.1. We shall see in the next subsection that on compact subsets of $\mathcal{U} \setminus \mathrm{Supp}(D)$ (for instance on $\mathcal{U} \setminus W$), one has uniform second derivative bounds on φ_s (cf. Lemma 2.6). So the corollary immediately follows as long as we can prove that

$$\int_{\mathcal{U}\setminus\mathrm{Supp}(D)} (\omega + s\theta + \sqrt{-1}\partial\bar\partial f)^n < C$$

for some uniform constant C. Similar to the calculations above, we have

$$\omega + s\theta + \sqrt{-1}\partial\bar\partial f \leq C\theta + \frac{2n}{(-\log|\sigma_D|^2_{h_D})^2}\sum_{j=1}^k \alpha_j \frac{\sqrt{-1}dz_j \wedge d\bar z_j}{|z_j|^2} \leq C\theta + C\sum_{j=1}^k \alpha_j \frac{\sqrt{-1}dz_j \wedge d\bar z_j}{(-\log|z_j|^2)^2|z_j|^2}$$

for some uniform constant C. In the second line we also used the fact that

$$-\log|\sigma_D|^2_{h_D} \geq -c\log|z_j|^2$$

for all j, where $c = \min\alpha_j$. The required estimate then follows from the binomial theorem and the elementary observation that

$$\int_{|z|<\frac12} \frac{\sqrt{-1}dz \wedge d\bar z}{(-\log|z|)^2|z|^2} = \int_0^{\frac12} \frac{1}{(-\log r)^2}\cdot\frac{dr}{r} = \int_{\log 2}^\infty \frac{ds}{s^2} < \infty. \qquad \square$$

2.3. Further estimates and proof of Theorem 1.1. We first prove the boundary C^1 estimate. We denote the covariant derivative of θ by ∇. We also let D be the effective divisor from Lemma 2.2 such that $\mathrm{Supp}(D) = \mathrm{Supp}(F) \cup \mathrm{Supp}(E)$ and

$$\omega + \sqrt{-1}\partial\bar\partial \log h_D > \beta\theta \tag{2.9}$$

for some $\beta > 0$. Note that by the support condition, if σ_D is a defining section of D, then there exists a uniform constant C such that

$$|\sigma_F|^{-2}, |\sigma_E|^{-2}_{h_E} \leq C|\sigma_D|^{-2l} \tag{2.10}$$

for some $l \in \mathbb{N}$.

Lemma 2.3. *There exists a constant $C > 0$ such that for all $0 < s < 1$,*

$$|\nabla\varphi_s|_{\partial\mathcal{U}} \leq C.$$

Proof. As in the proof of Proposition 2.1, let u solve the Dirichlet problem

$$\begin{cases} \Delta u = -A, \\ u = 0 \quad \text{on } \partial\mathcal{U}. \end{cases}$$

Then if we choose A to be large enough, we have $\varphi_s \leq u$. Next, again by Proposition 2.1,

$$\varphi_s \quad \text{and} \quad \frac{e^{\varphi_s} \prod_i (|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \frac{\Omega_Y}{(\omega + s\theta)^n}$$

are uniformly bounded in the neighborhood of the boundary $\partial\mathcal{U}$. Let $U_\varepsilon := \{\rho > a - \varepsilon\}$. We fix a small ε such that U_ε has smooth boundary consisting of two components $\partial\mathcal{U}$ and $\{\rho = a - \varepsilon\}$. Then we can choose $b \gg 1$ such that

$$\begin{aligned}(\omega + s\theta + \sqrt{-1}\partial\bar{\partial}[b(\rho - a)])^n &> \frac{e^{\varphi_s + M} \prod_i (|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \Omega_Y \\ &= (\omega + s\theta + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n.\end{aligned}$$

On the other hand we also have

$$\begin{aligned} b(\rho - a)|_{\partial\mathcal{U}} &= 0 = \varphi_s|_{\partial\mathcal{U}}, \\ b(\rho - a)|_{\rho = a-\varepsilon} &= -b\varepsilon < \varphi_s|_{\rho = a - \varepsilon}, \end{aligned}$$

if we pick $b \gg 1$. Then by the maximum principle, and the upper bound above,

$$b(\rho - a) \leq \varphi_s \leq u.$$

But then it is easy to see that for any $x \in \partial\mathcal{U}$,

$$|\nabla\varphi_s|(x) \leq \max(|\nabla b(\rho - a)|(x), |\nabla u|(x)),$$

and the boundary C^1 estimate follows. □

Next, we prove the global C^1 estimate with suitable barrier function. Such gradient estimate without barrier is firstly studied in [5] for standard non-degenerate Monge–Ampère equation where φ is bounded and later improved in [40] when the potential only has an upper bound in a different geometric setting.

Proposition 2.2. *Let φ_s be the solution of equation* (2.3). *There exist $N, C > 0$ such that for all* $0 < s < 1$,

$$|\nabla\varphi_s|^2 |\sigma_D|^{2N}_{h_D} \leq C.$$

Proof. Once again we fix a constant $\beta > 0$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\log h_D > \beta\theta$ and rewrite equation (2.3) as

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}\log h_D + s\theta + \sqrt{-1}\partial\bar{\partial}\phi_s\right)^n = \frac{e^{\varphi_s + M} \prod_i (|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j (|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \Omega_Y,$$

where

$$\phi_s = \varphi_s - \log|\sigma_D|^2_{h_D}.$$

It follows from Proposition 2.1 that $\phi_s > -C$ for some uniform constant $C > 0$. Note that our reference metrics $\omega + \sqrt{-1}\partial\bar\partial \log h_D + s\theta$ in the above equation are uniformly non-degenerate as $s \to 0$. By abusing notation, in the rest of proof, we will use θ to denote the Kähler form $\omega + \sqrt{-1}\partial\bar\partial \log h_D + s\theta$. Define

$$F := \frac{e^{\varphi_s + M}|\sigma_D|^2_{h_D}\prod_i(|\sigma_{E_i}|^2_{h_{E_i}} + s)^{a_i}}{\prod_j(|\sigma_{F_j}|^2_{h_{F_j}} + s)^{b_j}} \frac{\Omega_Y}{(\omega + \sqrt{-1}\partial\bar\partial \log h_D + s\theta)^n}.$$

For the rest of the proof, for convenience, we drop the sub-script s from the notation, i.e., we denote ϕ_s by simply ϕ. We define

$$H = \log|\nabla\phi|^2_\theta + \log|\sigma_D|^{2k}_{h_D} - \gamma(\phi),$$

where k is a constant, γ is a one-variable monotone increasing function to be determined. Since the leading term of our function $\gamma(x)$ will be chosen as Bx, and ϕ blows up in the rate of $-\log|\sigma_D|^2_{h_D}$, it follows that $-\gamma(\phi)$ has upper bound. Now $\log|\nabla\phi|_\theta < C - 2\log|\sigma_D|_{h_D}$, this implies that when $k > 2$, H has a maximum in $\mathcal{U} \setminus \mathrm{Supp}(D)$. Direct computation (cf. [39, p. 21]) shows that

$$\Delta' \log|\nabla\phi|^2_\theta \geq \frac{2\Re\langle\nabla \log F, \nabla\phi\rangle_\theta}{|\nabla\phi|^2_\theta} - \Lambda \operatorname{tr}_{\omega'}\theta + \frac{|\nabla\nabla\phi|^2_{\theta,\omega'} + |\bar\nabla\nabla\phi|^2_{\theta,\omega'}}{|\nabla\phi|^2_\theta} - \frac{|\nabla|\nabla\phi|^2_\theta|^2_{\omega'}}{|\nabla\phi|^4_\theta}, \tag{2.11}$$

where Δ' is taken with respect to the metric $\omega' = \omega + \sqrt{-1}\partial\bar\partial \log h_D + s\theta + \sqrt{-1}\partial\bar\partial\phi_s$, Λ is the bound of bisectional curvature of metric θ and $|\cdot|_{\theta,\omega'}$ means that the norm of the two tensor is taken with respect to θ on the first entry and ω' on the second entry. When $|\nabla\phi|_\theta > 1$, by the Cauchy–Schwarz inequality,

$$\left|\frac{2\Re\langle\nabla \log F, \nabla\phi\rangle_\theta}{|\nabla\phi|^2_\theta}\right| \leq C + C\sum_i |\sigma_{E_i}|^{-2}_{h_{E_i}} + \sum_j |\sigma_{F_j}|^{-2}_{h_{F_j}} + |\sigma_D|^{-2}_{h_D} \leq C|\sigma_D|^{-2l}_{h_D} \tag{2.12}$$

for some $l > 0$. As observed before, the final equality follows from the fact that

$$\mathrm{Supp}(D) = \mathrm{Supp}(E) \cup \mathrm{Supp}(F).$$

Next, by [39, Lemma 13], we have

$$\frac{|\nabla\nabla\phi|^2_{\theta,\omega'} + |\nabla\bar\nabla\phi|^2_{\theta,\omega'}}{|\nabla\phi|^2_\theta} - \frac{|\nabla|\nabla\phi|^2_\theta|^2_{\omega'}}{|\nabla\phi|^4_\theta} \geq 2\Re\left\langle\frac{\nabla|\nabla\phi|^2_\theta}{|\nabla\phi|^2_\theta}, \frac{\nabla\phi}{|\nabla\phi|^2_\theta}\right\rangle_{\omega'} - 2\Re\left\langle\frac{\nabla|\nabla\phi|^2_\theta}{|\nabla\phi|^2_\theta}, \frac{\nabla\phi}{|\nabla\phi|^2_\theta}\right\rangle_\theta. \tag{2.13}$$

At the maximum of H, we have

$$\nabla \log|\nabla\phi|^2 + \nabla \log|\sigma_D|^{2k}_{h_D} - \gamma'\nabla\phi = 0.$$

Hence we have

$$2\Re\left\langle \frac{\nabla|\nabla\phi|_\theta^2}{|\nabla\phi|_\theta^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\omega'} - 2\Re\left\langle \frac{\nabla|\nabla\phi|_\theta^2}{|\nabla\phi|_\theta^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\theta} \tag{2.14}$$
$$= 2k\Re\left\langle -\nabla\log|\sigma_D|_{h_D}^2 + \gamma'\nabla\phi, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\omega'} - 2k\Re\left\langle -\nabla\log|\sigma_D|_{h_D}^2 + \gamma'\nabla\phi, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\theta}$$
$$\geq 2k\Re\left\langle -\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\omega'} + 2k\gamma'\frac{|\nabla\phi|_{\omega'}^2}{|\nabla\phi|_\theta^2} + 2k\Re\left\langle \frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\theta} - 2k\gamma'.$$

We choose γ as a monotone increasing function, so we can drop the term $2\gamma'|\nabla\phi|_{\omega'}^2|\nabla\phi|_\theta^{-2}$. At the maximum of H, we can assume $|\sigma_D|_{h_D}^{2k}|\nabla\phi|_\theta^2 \geq 1$ otherwise we are done. Choosing $k \geq 2$, we have the following two estimates:

$$\left|2\Re\left\langle -\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\omega'}\right| \leq 2\left|\Re\left\langle \nabla|\sigma_D|_{h_D}^2, \frac{\nabla\phi}{|\nabla\phi|_\theta}\right\rangle_{\omega'}\right| \tag{2.15}$$
$$\leq |\nabla|\sigma_D|_{h_D}^2|_{\omega'}^2 + \frac{|\sigma_D|_{h_D}^2|\nabla\phi|_{\omega'}^2}{|\sigma_D|_{h_D}^2|\nabla\phi|_\theta^2}$$
$$\leq C|\nabla|\sigma_D|_{h_D}^2|_\theta^2 \operatorname{tr}_{\omega'}\theta + |\sigma_D|_{h_D}^2|\nabla\phi|_{\omega'}^2,$$
$$\left|2\Re\left\langle -\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2}, \frac{\nabla\phi}{|\nabla\phi|_\theta^2}\right\rangle_{\theta}\right| \leq 2\left|\Re\left\langle \nabla|\sigma_D|_{h_D}^2, \frac{\nabla\phi}{|\nabla\phi|_\theta}\right\rangle_{\theta}\right| \leq C.$$

On the other hand, one has

$$-\Delta'\gamma(\phi) = -\gamma'\Delta'\phi - \gamma''|\nabla\phi|_{\omega'}^2 \tag{2.16}$$
$$= \gamma'\operatorname{tr}_{\omega'}\theta - n\gamma' - \gamma''|\nabla\phi|_{\omega'}^2, \ \Delta'\log|\sigma_D|_{h_D}^k \leq C\operatorname{tr}_{\omega'}\theta.$$

Combining estimate (2.16) with preceding estimates (2.12)–(2.15), we have

$$\Delta' H \geq (\gamma' - \Lambda - C)\operatorname{tr}_{\omega'}\theta - (n+2)\gamma' - (\gamma'' + |\sigma_D|_{h_D}^2)|\nabla\phi|_{\omega'}^2 - C|\sigma_D|_{h_D}^{-2l}. \tag{2.17}$$

Recalling that $\phi > -C'$, now we construct our function γ as

$$\gamma(x) = (\Lambda + C + 1)x - \frac{E}{x + C' + 1},$$

where E is a constant to be determined. Then by (2.17) we have

$$\Delta' H \geq \operatorname{tr}_{\omega'}\theta - (n+2)(C+1+\Lambda) - C|\sigma_D|_{h_D}^{-2l} + \left(\frac{2E}{(\phi + C' + 1)^3} - |\sigma_D|_{h_D}^2\right)|\nabla\phi|_{\omega'}^2.$$

Noticing that $\phi \leq C - 2\log|\sigma_D|_{h_D}^2$, we can assume that

$$\left(\frac{2E}{(\phi + C' + 1)^3} - |\sigma_D|_{h_D}^2\right) \geq |\sigma_D|_{h_D}^2$$

by choosing suitable large constant E depending on C and C'. Now we conclude that at the maximum of H, we have

$$\operatorname{tr}_{\omega'}\theta \leq (|\sigma_D|_{h_D})^{-2l}, \quad |\nabla\phi|^2_{\omega'} \leq C(|\sigma_D|_{h_D})^{-2(l+1)}.$$

Hence

$$|\nabla\phi|^2_\theta \leq C(|\sigma_D|_{h_D})^{-2(2l+1)}.$$

Choosing $k = 2l+1$, we have $H_{\max} \leq C$, and letting $N := k + (C + 1 + \Lambda)$, it follows from the definition of $\gamma(\phi)$ that

$$|\nabla\phi|^2_\theta |\sigma_D|^{2N}_{h_D} \leq C.$$

□

Lemma 2.4. *Let φ_s be the solution of equation* (2.3). *Then there exist constant $C > 0$ such that for all $0 < s < 1$,*

$$|\nabla^2_\theta \varphi_s|_{\partial\mathcal{U}} \leq C.$$

Proof. Notice that our boundary is strictly pseudoconvex, and all data in equation (2.3) is uniformly bounded near the boundary, so the second order estimates on the boundary follow directly from the arguments in [8, Section 1.3]. □

Next, we will prove second order estimates with bounds from suitable barrier functions.

Lemma 2.5. *There exist constants $N, C > 0$ such that for all $0 < s < 1$,*

$$\sup_{\mathcal{U}} (|\sigma_D|^N_{h_D}) |\Delta_\theta \varphi_s| \leq C,$$

where Δ_θ is the Laplace operator with respect to the Kähler metric θ.

Proof. We remark that the constant C in the proof might change from line to line and it depends on $\theta, \mathcal{U}$ but does not depend on s. Let $\omega' = \omega + s\theta + \sqrt{-1}\partial\bar\partial\varphi_s$. Then we consider the quantity

$$H = \log \operatorname{tr}_\theta(\omega') - B\varphi_s + B\log|\sigma_D|^2_{h_D}$$

for some large constant $B > 2$. By the C^0 estimate of φ_s, H is bounded above in $\mathcal{U}$. Standard calculations (cf. [45, Lemma 3.7]) show that

$$\Delta' \log \operatorname{tr}_\theta \omega' \geq -C \operatorname{tr}_{\omega'}\theta - \frac{\operatorname{tr}_\theta \operatorname{Ric}(\omega')}{\operatorname{tr}_\theta(\omega')}, \tag{2.18}$$

where C depends on bisectional curvature of θ. From equations (2.3), (2.10) and the elementary observation that

$$\sqrt{-1}\partial\bar\partial\log(f+s) = \frac{f}{f+s}\sqrt{-1}\partial\bar\partial\log f + s\frac{\sqrt{-1}\partial f\wedge\bar\partial f}{f(f+s)}$$

holds for any smooth nonnegative function f, it is easy to see that

$$-\operatorname{tr}_\theta \operatorname{Ric}(\omega') \geq \frac{-C}{|\sigma_D|^{2l}_{h_D}}$$

for some constant $C > 0$ independent of s. Together with (2.18) and our choice of D (cf. (2.9)) we see that

$$\begin{aligned}
\Delta' H &\geq -C \operatorname{tr}_{\omega'} \theta - \frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} + B \operatorname{tr}_{\omega'}(\sqrt{-1}\partial\bar\partial \log h_D - \sqrt{-1}\partial\bar\partial\varphi_s) \\
&\geq (B\beta - C) \operatorname{tr}_{\omega'} \theta - \frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} - Bn \\
&\geq \operatorname{tr}_{\omega'} \theta - \frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} - Bn \quad (\text{if } B \gg 1) \\
&\geq (\operatorname{tr}_\theta(\omega'))^{\frac{1}{n-1}} \left(\frac{\theta^n}{\omega'^n}\right)^{\frac{1}{n-1}} - \frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} - Bn \\
&\geq (\operatorname{tr}_\theta(\omega'))^{\frac{1}{n-1}} |\sigma_D|^{2\alpha}_{h_D} - \frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} - Bn
\end{aligned}$$

for some constant α independent of s and $B \gg 1$ so that $B\beta > C + 1$ in line three.

By Lemma 2.4, it suffices to assume that H obtains maximum at a point $p \in \mathcal{U}$. Moreover, since H goes to $-\infty$ on $\operatorname{Supp}(D)$, clearly $p \notin \operatorname{Supp}(D)$. From the maximum principle it follows that at point p,

$$\frac{C}{|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega')} + Bn \geq (\operatorname{tr}_\theta(\omega'))^{\frac{1}{n-1}} |\sigma_D|^{2\alpha}_{h_D}.$$

We first assume that $|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega') \geq 1$ at p. Then

$$(\operatorname{tr}_\theta(\omega'))^{\frac{1}{n-1}} |\sigma_D|^{2\alpha}_{h_D} \leq C + Bn.$$

By noticing that the other case is $|\sigma_D|^{2l}_{h_D} \operatorname{tr}_\theta(\omega') \leq 1$ at p, it follows that in both cases there is an integer k (depending on α, l and n) such that

$$|\sigma_D|^{2k}_{h_D} (\operatorname{tr}_\theta \omega')(p) \leq C + Bn.$$

Notice that $\varphi_s \geq \varepsilon \log |\sigma_D|^2_{h_D} - C_\varepsilon$ for any $\varepsilon > 0$, so it follows by choosing a $B \gg k$ that $H(p) \leq C + Bn$. Now fixing this B, we have $H(x) \leq C$ for any $x \in \mathcal{U} \setminus \operatorname{Supp}(D)$. By the definition of H, we have

$$|\sigma_D|^{2B}_{h_D} (\operatorname{tr}_\theta \omega')(x) \leq C.$$

Choosing $N = 2B$ will finish proof. □

The following lemma on local higher order regularity of φ_s is established by the standard linear elliptic theory after applying Lemma 2.5 and linearizing the complex Monge–Ampère equation (2.3).

Lemma 2.6. *For any compact $K \subset\subset (\mathcal{U} \setminus p)$, and any natural number $k \in \mathbb{N}$, there exists a constant $C = C(k, K) > 0$ such that for any $0 < s < 1$,*

$$\|\varphi_s\|_{C^k(K)} \leq C.$$

In our context, we need a lemma for compactness of quasi-PSH function in L^1 topology. This kind of lemma is standard and we include it for convenience.

Lemma 2.7 ([27]). *Let $\mathcal{U}, \theta$ and φ_s be as above. Denote $A(\rho - a) + \psi_1 + \varphi_s$ by ψ_s (cf. (2.2) for the definition of $A(\rho - a) + \psi_1$). Then ψ_s converge to a PSH function on $\pi^{-1}(\mathcal{U})$ by taking a subsequence.*

Proof. Fix a measure θ^n and a finite open covering V_i for $\pi^{-1}(\mathcal{U})$ such that θ is $\sqrt{-1}\partial\bar{\partial}$ exact on each chart. On each V_i, ψ_s is a $s\theta$-PSH function satisfying $\|\psi_s\|_{L^1} < C$ and $\psi_s < C$, for some C independent of s (cf. Proposition 2.1). By passing to a subsequence, one has

$$\psi_s \xrightarrow{L^1(V_i)} \varphi_i,$$

where φ_i is a $s\theta$-PSH function for any $s > 0$ (cf. [17]). Passing $s \to 0$, φ_i is PSH on V_i. For different charts V_i, V_j with $V_i \cap V_j \neq \emptyset$, we aim to show that $\varphi_i = \varphi_j$. Then we can patch all φ_i to get a PSH function on $\pi^{-1}(\mathcal{U})$. By taking a further subsequence, we may assume that $\varphi_i = \varphi_j$ on a full measure subset W of $V_i \cap V_j$. Hence it suffices to show that for any point $q \in V_i \cap V_j$,

$$\varphi_i(q) = \limsup_{x \to q, x \in W} \varphi_i(x).$$

It follows from the upper semicontinuity of φ_i that

$$\varphi_i(q) \geq \limsup_{x \to q, x \in W} \varphi_i(x).$$

To get the reversed inequality, we argue by contradiction. If not, then

$$\varphi_i(q) > \limsup_{x \to q, x \in W} \varphi_i(x).$$

By the mean value inequality for PSH function, one has that, for some sufficiently small $r > 0$,

$$c_r \int_{B(q,r) \cap W} \varphi_i \, dV < \varphi_i(q) \leq c_r \int_{B(q,r)} \varphi_i \, dV,$$

where $B(q,r)$ is a radius r ball in $V_i \cap V_j$ and c_r is a constant depending on r. This is a contradiction by noticing that $\varphi_i < C$ and $B(q,r) \setminus W$ has measure zero. □

Now we proceed to prove our first main theorem.

Proof of Theorem 1.1. By Lemma 2.7, for any sequence $s_j \to 0$, one has

$$\varphi_{s_j} \xrightarrow{L^1(\pi^{-1}(\mathcal{U}))} \varphi,$$

by taking a subsequence and moreover if we define

$$\varphi_{\mathrm{KE}} := A(\rho - a) + \psi_1 + \varphi, \tag{2.19}$$

then φ_{KE} is a PSH function on $\pi^{-1}(\mathcal{U})$. It follows from the uniform estimates for φ_s away from $\mathrm{Supp}(D)$ that by taking a further subsequence, one has

$$\varphi_{s_j} \xrightarrow{C^\infty_{\mathrm{loc}}(\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D))} \varphi.$$

Clearly, φ solves the desired Monge–Ampère equation on $(\bar{\mathcal{U}} \setminus \{p\})$ by design. Moreover, by Proposition 2.1 there exists $C > 0$ such that

$$-(2n+\varepsilon)\log(-\log|\sigma_D|^2_{h_D}) - C_\varepsilon \le \varphi \le C.$$

With the above sublog C^0 estimate of φ and the uniqueness Lemma 2.8 below, we have

$$\varphi_s \xrightarrow{C^\infty_{\mathrm{loc}}(\pi^{-1}(\mathcal{U})\setminus \mathrm{Supp}(D))} \varphi$$

when $s \to 0$. The PSH function φ_{KE} satisfies properties (1)–(2) in the statement of Theorem 1.1 and $\omega_{\mathrm{KE}} := \sqrt{-1}\partial\bar{\partial}\varphi_{\mathrm{KE}}$ is a Kähler–Einstein metric on $\mathcal{U} \setminus \{p\} \cong \pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)$. From Corollary 2.1 and local smooth convergence on $\pi^{-1}(\mathcal{U}) \setminus \mathrm{Supp}(D)$, we also get by the Fatou Lemma that

$$\int_{\mathcal{U}\setminus\{p\}} \omega^n_{\mathrm{KE}} \le \lim_{s\to 0} \int_{\mathcal{U}\setminus\{p\}} (\omega + s\theta + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n < \infty,$$

and this proves (3).

For part (4) we remark that in (2.19), our function φ_{KE} is PSH function on $\pi^{-1}(\mathcal{U})$. Notice that $\pi^{-1}(\mathcal{U})$ is smooth, by the sublog pole estimate of φ_{KE} in Proposition 2.1, it follows that φ_{KE} has vanishing Lelong number at any point of $\pi^{-1}(\mathcal{U})$.

At last, we show that $\lim_{x\to \mathrm{Supp}(D)} \varphi_{\mathrm{KE}}(x) = -\infty$ for $x \in \pi^{-1}(\mathcal{U})$. We argue by contradiction. If not, noticing that φ_{KE} is bounded from above, then there is a smooth irreducible component E of D and a sequence of point $y_i \to y \in E$ such that

$$C \le \lim_{i\to+\infty} \varphi_{\mathrm{KE}}(y_i) \le \varphi_{\mathrm{KE}}(y) \tag{2.20}$$

for some finite constant C. The second inequality is due to the upper semicontinuity of φ_{KE}. Now by the definition of PSH function (cf. [17, Definition 1.4]), φ_{KE} is still a PSH function when restricted to a smooth component E of D. Indeed, a function with value in $[-\infty, +\infty)$ is PSH if and only if it is upper semicontinuous and it satisfies the mean value inequality when restricted to any complex line. One can check easily that the mean value inequality on complex line and the upper semicontinuity are preserved when restricted to complex submanifold.

Now we have showed that $\varphi_{\mathrm{KE}}|_E$ is a PSH function and moreover it is finite at a point $y \in E$ (cf (2.20)). We claim that φ_{KE} is constant on E. Indeed, since $\varphi_{\mathrm{KE}}|_E$ is upper semicontinuous, we may assume that $\varphi_{\mathrm{KE}}|_E$ achieves its maximum at some point $x_{\max}$. Then by the mean value inequality for PSH function, it is clear that φ_{KE} is locally constant. Now E is a smooth manifold without boundary, hence $\varphi_{\mathrm{KE}}|_E$ is constant on E. The claim is proved. Next, using the connectedness of $\mathrm{Supp}(D)$ (using p is a normal singularity), φ_{KE} must be constant on $\mathrm{Supp}(D)$. Then we need the following general fact from [4, Lemma 2.7]: Suppose ϕ is a plurisubharmonic function on the unit ball $B \subset \mathbb{C}^n$ such that

$$\int_B |z_1|^{-2} e^{\phi} (\sqrt{-1})^n dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n} < \infty.$$

Then ϕ tends to $-\infty$ near $B \cap \{z_1 = 0\}$. Now pick a smooth exceptional divisor F with discrepancy -1 from D and also pick a generic point $q \in F$ (q is not included in other components of D). Then applying the fact just recalled, one has $\varphi_{\mathrm{KE}}(x_i)$ tends to $-\infty$ when $x_i \to q$. This is a contradiction since φ_{KE} is a finite constant on $\mathrm{Supp}(D)$.

Finally, the uniqueness is a consequence of the slightly stronger uniqueness theorem below.

Lemma 2.8. *There exists a unique smooth PSH function $\varphi \in C^\infty(\mathcal{U} \setminus p)$ satisfying the following:*

(1) $(\sqrt{-1}\partial\bar{\partial}\varphi)^n = e^\varphi \Omega_X$ *on* $(\mathcal{U} \setminus p)$, $\varphi|_{\partial\mathcal{U}} = \psi$,

(2) *for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\varepsilon \log |\sigma_D|^2_{h_D} - C_\varepsilon \le \varphi \le C,$$

where D is a SNC divisor supported on the exceptional locus of a log resolution π (cf. Lemma 2.2 for the definition).

Proof. Let φ be the limit of φ_s as $s \to 0$ as above. Suppose there exists another φ' satisfying condition (2) in the lemma solving equation 2.3, and for any $\varepsilon > 0$, there exist $C_1 > 0$ and $C_2 = C_2(\varepsilon) > 0$ such that

$$\varepsilon \log |\sigma_D|^2_{h_D} - C_2 \le \varphi' \le C_1.$$

We consider the quantity

$$\phi = \varphi - \varphi' + \delta^3 \log |\sigma_D|^2_{h_D} + \delta M,$$

where recall that $\omega = \sqrt{-1}\partial\bar{\partial} M$ in equation (2.3). Then ϕ satisfies the following equation on the log resolution Y:

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi' - \delta\omega + \delta^3 \operatorname{Ric}(h_D) + \sqrt{-1}\partial\bar{\partial}\phi)^n}{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi')^n} = e^{\phi - \delta M} |\sigma_D|^{-2\delta^3}.$$

Noticing that φ, M are bounded from above on $(\mathcal{U} \setminus p)$ and $(\delta^3 \log |\sigma_D|^2_{h_D} - \varphi')(x) \to -\infty$ when $x \to \operatorname{supp}(D)$, ϕ obtains its maximum in $(\mathcal{U} \setminus p)$. It follows from the maximum principle that there exists $C > 0$ such that

$$\sup_{(\mathcal{U} \setminus p)} \phi \le \delta C.$$

Therefore for any point $x \in (\mathcal{U} \setminus p)$, one has

$$\varphi - \varphi' \le \delta C - \delta M - \delta^3 \log |\sigma_D|^2_{h_D}.$$

Let $\delta \to 0$. On $(\mathcal{U} \setminus p)$, one has

$$\varphi \le \varphi'.$$

Similarly, one has $\varphi \ge \varphi'$ on $(\mathcal{U} \setminus p)$. Therefore $\varphi = \varphi'$. □

3. Volume stability

The goal of this section is to prove Theorem 1.3 and some consequences by a localized version of Yau's Schwarz Lemma [24, 50]. Suppose we have two complete Kähler–Einstein metric ω_{KE} and ω'_{KE} on $(\mathcal{U} \setminus p)$. By the Kähler–Einstein condition, we can write

$$\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where

$$\varphi := \log \frac{\omega'^n_{\mathrm{KE}}}{\omega^n_{\mathrm{KE}}}.$$

Proof of Theorem 1.3. Fix a point q such that $\operatorname{dist}_{\omega_{\mathrm{KE}}}(q, \partial\mathcal{U}) =: 2R(q)$. We construct a cut-off function

$$\zeta(x) = \eta\left(\frac{r(x)}{R}\right) \geq 0,$$

where $r(x)$ is some smoothening of $d_{\omega_{\mathrm{KE}}}(x, q)$ (obtained for instance by running the heat flow) and

$$\eta \in [0,1], \quad \eta^{-1}(\eta')^2 \leq C(n), \quad |\eta''| \leq C(n),$$

and let $H = \zeta\varphi$. Note that ζ satisfies

$$\zeta = \begin{cases} 1 & \text{on } B_{\omega_{\mathrm{KE}}}(q, R), \\ 0 & \text{outside } B_{\omega_{\mathrm{KE}}}(q, 2R). \end{cases}$$

Moreover, by Laplace comparison, we also have

$$\Delta_{\omega_{\mathrm{KE}}}\zeta \geq \frac{-C}{R^2}(1+R)$$

for some dimensional constant C. Since both ω_{KE} and ω'_{KE} are KE metrics, we have

$$\operatorname{tr}_{\omega_{\mathrm{KE}}} \sqrt{-1}\partial\bar{\partial}\varphi = -n + \operatorname{tr}_{\omega_{\mathrm{KE}}} \omega'_{\mathrm{KE}} \geq n(e^{\frac{\varphi}{n}} - 1).$$

Assume H attains a positive maximum at a point Q (otherwise $H(q) \leq 0$). Then at the point Q, we have

$$\begin{aligned}
\Delta H &\geq (\Delta_{\omega_{\mathrm{KE}}}\zeta)\left(\frac{H}{\zeta}\right) + \zeta\Delta_{\omega_{\mathrm{KE}}}\varphi + 2\Re\left\langle\nabla\zeta, \nabla\frac{H}{\zeta}\right\rangle \\
&\geq \frac{-c(n)H}{\zeta}(R^{-2}(1+R)) + \zeta n(e^{\frac{\varphi}{n}} - 1) + 2\Re\left\langle\nabla\zeta, \frac{1}{\zeta}\nabla H\right\rangle - 2H\left\langle\nabla\zeta, \frac{1}{\zeta^2}\nabla\zeta\right\rangle \\
&\geq \frac{-c(n)H}{\zeta}(R^{-2}(1+R)) + n\zeta\frac{\varphi^2}{n^2} + 2\Re\left\langle\nabla\zeta, \frac{1}{\zeta}\nabla H\right\rangle - 2\frac{H}{\zeta R^2} \\
&\geq \frac{-c(n)H}{\zeta}\left(R^{-2}(1+R) - \frac{H}{c(n)n}\right) + 2\Re\left\langle\nabla\zeta, \frac{1}{\zeta}\nabla H\right\rangle - 2\frac{H}{\zeta R^2} \\
&= \frac{c(n)H}{\zeta}\left(-R^{-1} - R^{-2} - \frac{2}{c(n)}R^{-2} + \frac{H}{nc(n)}\right) + 2\Re\left\langle\nabla\zeta, \frac{1}{\zeta}\nabla H\right\rangle,
\end{aligned}$$

where $c(n)$ is a dimensional constant which might change from line to line. By maximum principle on the ball of radius R, noticing that $\nabla H(Q) = 0$, we get

$$H(Q) \leq c(n)\left(\frac{1}{R} + \frac{1}{R^2}\right).$$

Hence $\varphi(q) = H(q) \leq 2c(n)\frac{1}{R(q)}$ when $R \geq 1$. □

Remark 3.1. The above theorem is true as long as ω_{KE} and ω'_{KE} are complete. No other metric properties of $\omega_{\mathrm{KE}}, \omega'_{\mathrm{KE}}$ are required.

We prove a corollary of Theorem 1.3.

Corollary 3.1. *Suppose we are in the setting of Theorem* 1.3, *i.e., $\mathcal{U}$ admits a complete Kähler–Einstein metric ω_{KE} with negative scalar curvature and* $\operatorname{Vol}_{\omega_{\mathrm{KE}}}(\mathcal{U}) < \infty$. *Then for any other complete Kähler–Einstein metric ω'_{KE} with negative scalar curvature,* $\operatorname{Vol}_{\omega'_{\mathrm{KE}}}(\mathcal{U}) < \infty$.

Proof. It is obvious since $\varphi = \log\frac{\omega'^n_{\mathrm{KE}}}{\omega^n_{\mathrm{KE}}}$ is bounded by Theorem 1.3. □

By applying Yau's Schwarz Lemma, we have the following theorem concerning the comparison of two different Kähler–Einstein metrics.

Corollary 3.2. *Let ω_{KE} and ω'_{KE} be two complete Kähler Einstein metrics with negative scalar curvature. If moreover the bisectional curvature of ω_{KE} is smaller than $-K_2$, where K_2 is a positive constant, then there is a constant c such that*

$$\frac{1}{C}\omega'_{\mathrm{KE}} \le \omega_{\mathrm{KE}} \le C\omega'_{\mathrm{KE}}.$$

Proof. Let $u = \mathrm{tr}_{\omega'_{\mathrm{KE}}}\omega_{\mathrm{KE}}$. By Chern–Lu's inequality we have

$$\Delta_{\omega'_{\mathrm{KE}}} u \ge -K_1 u + K_2 u^2,$$

where $-K_1$ is the Ricci curvature of ω'. We still use the cut-off function ζ as in Theorem 1.3. Let $G = \zeta u$. Then by the Chern–Lu inequality and the same argument as in the proof of Theorem 1.3, we get the following inequality:

$$G \le \frac{K_1}{K_2} + c(K_1, K_2)R^{-\frac{1}{2}}.$$

When R is larger, the estimate is better, hence we have $\omega_{\mathrm{KE}} \le c\omega'_{\mathrm{KE}}$ for some constant which depending on the metric ω'. The volume ratio estimate in Theorem 1.3 also gives us the reverse inequality. □

To end this section, we prove a uniqueness theorem for Dirichlet problem which will be used later in Section 4.

Lemma 3.1. *Suppose there exists a complete Kähler metric $\omega = \sqrt{-1}\partial\bar\partial\rho$ on $(\mathcal{U} \setminus p)$. Further assume that the Kähler potential ρ is bounded from above and $\rho(x) \to -\infty$ when $x \to p$. Then any smooth bounded solution φ of the Dirichlet problem*

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{\varphi}\omega^n & \text{on } \mathcal{U} \setminus p, \\ \varphi|_{\partial\mathcal{U}} = 0, \end{cases}$$

is unique, i.e., $\varphi = 0$.

Proof. Let $\varphi_\varepsilon = \varphi - \varepsilon\rho$ and $\omega_\varepsilon = (1+\varepsilon)\omega$. Then φ_ε satisfies the equation

$$\begin{cases} (\omega_\varepsilon + \sqrt{-1}\partial\bar\partial\varphi_\varepsilon)^n = \dfrac{e^{\varphi}}{(1+\varepsilon)^n}\omega_\varepsilon^n & \text{on } \mathcal{U}, \\ \varphi_\varepsilon|_{\partial\mathcal{U}} = -\varepsilon\rho. \end{cases}$$

Since ρ goes to $-\infty$, it follows that φ_ε admits a minimum in $\mathcal{U}$. If the minimum is on the boundary, we have

$$\varphi_\varepsilon \ge -\varepsilon \inf_{\partial\mathcal{U}} \rho.$$

If the minimum is in the interior point Q, then

$$\varphi_\varepsilon(Q) = \varphi(Q) - \varepsilon\rho(Q) \ge \log(1+\varepsilon) - \varepsilon \max \rho.$$

In both cases, letting $\varepsilon \to 0$, we get $\varphi \ge 0$. Similar arguments show that $\varphi \le 0$. Hence 0 is the unique solution. □

Remark 3.2. We remark that Lemma 3.1 can also be proved by using maximum principle when ω has bounded geometry property.

4. Kähler–Einstein metrics with bounded geometry

4.1. Preliminaries of the bounded geometry method. In this section, we want to use bounded geometry methods of [15, 31, 48] to construct complete Kähler–Einstein metric on $\mathcal{U} \setminus \{p\}$. So our geometric domain of interest will be $\mathcal{U} \setminus \{p\}$ with boundary $\partial\mathcal{U}$ and also one complete end of infinite distance, which is punctured neighborhood of p in Euclidean topology.

We recall definitions of quasi-coordinate which are used by [14, 31, 48] to deal with complete Riemannian manifolds with bounded curvature but with shrinking injectivity radius.

Definition 4.1. Let V be an open set in $\mathbb{C}^n$ with coordinates $(v^1, v^2, \dots, v^n)$. Let X be an n-dimensional complex manifold and ϕ a holomorphic map of V into X. We call ϕ a quasi-coordinate map if it is of maximal rank everywhere. In this case, $(V, \phi, (v^1, v^2, \dots, v^n))$ is called a quasi-coordinate of X.

Definition 4.2. Let $\mathcal{W}$ be a neighborhood of p compactly contained in $\mathcal{U}$, and let ω be a Kähler metric on $(\mathcal{U} \setminus p)$ which is complete towards p. A system of quasi-coordinates on $(\widehat{\mathcal{W}} := \mathcal{W} \setminus \{p\}, \omega)$ is a set of quasi-coordinates $\Gamma = \{(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2, \dots, v_\alpha^n))\}$ of $\widehat{\mathcal{W}}$ with the following properties:

(a) $\widehat{\mathcal{W}} \subset \bigcup_\alpha (\text{Image of } V_\alpha) \subset (\mathcal{U} \setminus p)$,

(b) for each point $x \in \widehat{\mathcal{W}}$, there is a quasi-coordinate V_β and $\tilde{x} \in V_\beta$ such that $\phi_\beta(\tilde{x}) = x$ and $\operatorname{dist}(\tilde{x}, \partial V_\beta) \geq \varepsilon_1$ in the Euclidean sense, where ε_1 is constant independent of β,

(c) there are positive constant c and A_k, $k = 1, 2, \dots$, independent of α, such that for each quasi-coordinate $(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2, \dots, v_\alpha^n))$, the following inequalities hold:

$$c^{-1}(\delta_{i\bar{j}}) \leq (g_{\alpha i\bar{j}}) \leq c(\delta_{i\bar{j}}), \qquad \left| \frac{\partial^{p+q}}{\partial v_\alpha^p \bar{v}_\alpha^q} g_{\alpha i\bar{j}} \right| < A_{p+q} \quad \text{for all } p, q,$$

where $(g_{\alpha i\bar{j}})$ denote the metric tensor of the Riemannian metric associated to ω on the chart $(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2, \dots, v_\alpha^n))$.

Roughly speaking, a set of quasi-coordinates of metric domain $(\widehat{\mathcal{W}}, \omega)$ is a set of coverings of $\mathcal{W}$ by charts such that the pull back of ω satisfies uniform bounded metric properties. Now we define the Cheng–Yau function space.

Definition 4.3. We define the Hölder space of $C^{k,\alpha}$ function on $\widehat{\mathcal{U}} := \mathcal{U} \setminus p$ by exploiting the quasi-coordinate system. For any nonnegative integer $k, \alpha \in (0, 1)$, we define

$$\|u\|_{k,\alpha}(\widehat{\mathcal{W}}) = \sup_{V_\beta \in \Gamma} \Bigg(\sup_{z \in V_\beta} \sum_{p+q \leq k} \left| \frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z) \right| + \sup_{z, z' \in V_\beta} \sum_{p+q=k} |z - z'|^{-\alpha} \left| \frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z) - \frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z') \right| \Bigg).$$

Let us introduce one more compact set V with $\widehat{\mathcal{U}} \setminus \widehat{\mathcal{W}} \subset V \subset \widehat{\mathcal{U}}$ to cover the whole $\widehat{\mathcal{U}}$. Now define

$$\|u\|_{k,\alpha}(\widehat{\mathcal{U}}) = \|u\|_{k,\alpha}(\widehat{\mathcal{W}}) + \|u\|_{k,\alpha}(V).$$

The function space $C^{k,\alpha}(\widehat{\mathcal{U}})$ is the completion of $\{u \in C^k(\widehat{\mathcal{U}}) : \|u\|_{k,\alpha}(\widehat{\mathcal{U}}) < \infty\}$.

Remark 4.1. The existence of quasi-coordinate is crucially used in our proof. The classical interior Schauder estimate for a linear elliptic operator L is as follows:

$$\|u\|_{C^{k,\alpha}(V_1)} \le C(\sup |u|_{V_2} + \|Lu\|_{C^{k-2,\alpha}(V_2)}),$$

where $V_1 \subset\subset V_2 \subset R^m$. Notice that the constant C depends on the ellipticity of L, the $C^{k-2,\alpha}$ norms of the coefficients of L and the distance between V_1 and ∂V_2. If we have a quasi-coordinate system defined above, the Schauder estimate on $\widehat{\mathcal{U}}$ is reduced to that on a fixed bounded domain in Euclidean space.

We also introduce the following assumption for our geometric domain $(\mathcal{U}, p)$.

Definition 4.4 (Bounded geometry of order k). Let (X, p) be a germ of isolated log canonical singularity embedded in $(\mathbb{C}^N, 0)$. We say that (X, p) has bounded geometry of order k if

(1) there is a complete metric $\omega = \sqrt{-1}\partial\bar{\partial}\rho$ defined on $(X \setminus p)$ which has a system of quasi-coordinates up to kth derivative of metric g, see Definition 4.2 (d),

(2) there is a function M on $(\mathcal{U} \setminus p)$ satisfying $\mathrm{Ric}(\omega) + \omega = \sqrt{-1}\partial\bar{\partial}M$ and for any $i \le k$, $\|\nabla^i_\omega M\| < C_i$. (Here the potential function M is not unique, we only require one of them satisfy the boundedness property, and in this note, the most interesting case is $M = 0$.)

Before we proceed, we state and prove the following modified version of Yau's generalized maximum principle on noncompact manifold.

Lemma 4.1 ([14]). *Suppose $(\widehat{\mathcal{U}}, \omega)$ is of bounded geometry of order k with $k \ge 2$. Let f be a smooth function on $\widehat{\mathcal{U}}$, which is bounded from above, and $\sup f > \sup_{\partial\mathcal{U}} f$. Then there is a sequence $\{y_i\}$ in $\widehat{\mathcal{U}}$ such that $\lim_{i\to\infty} f(y_i) = \sup f$, $\lim_{i\to\infty} |\nabla_g f|(y_i) = 0$ and $\lim_{i\to\infty} |\Delta_g f|(y_i) \le 0$, where the derivatives are taken with respect to metric g associated to the Käher form ω.*

Proof. Without loss of generality, let us assume that $\sup f = 0$. If $\sup f$ is attained, the lemma is obvious. Otherwise we choose a sequence x_i with $\lim f(x_i) = 0$. It is easy to see that $\{x_i\}$ must go to infinity. Now at each point we take a quasi-coordinate chart V_i covering x_i. On each V_i, define a nonnegative function $\beta^i : V_i \to \mathbb{R}$ such that

$$\beta^i(x_i) = 1, \quad \beta^i = 0 \text{ on } \partial V_i, \quad \beta^i \le C, \quad |\nabla\beta^i| \le C, \quad (\beta^i_{p\bar{q}}) \ge -C(\delta_{p\bar{q}}),$$

where C is positive number independent of i, and all norms are taken with respect to the Euclidean norm. Now consider

$$\frac{-f}{\beta^i}$$

as a function on V_i. Notice that $\frac{-f}{\beta^i}$ blows up on the boundary of V_i, so it admits minimum at a point y_i which is in the interior of V_i. Now let

$$\frac{-f}{\beta^i}(y_i) = \inf_{V_i} \frac{-f}{\beta^i}.$$

Then

$$\begin{aligned} \frac{-f}{\beta^i}(y_i) &\leq \frac{-f}{\beta^i}(x_i) = -f(x_i), \\ \frac{df}{f}(y_i) &= \frac{d\beta^i}{\beta^i}(y_i), \\ \frac{f_{p\bar{q}}}{f}(y_i) &\geq \frac{\beta^i_{p\bar{q}}}{\beta^i}(y_i). \end{aligned}$$

Using these inequalities and our choice of β^i, we have

$$\begin{aligned} 0 < -f(y_i) &\leq -Cf(x_i), \\ |df(y_i)| &\leq -Cf(x_i), \\ (f_{p\bar{q}})(y_i) &\leq -Cf(x_i)(\delta_{p\bar{q}}). \end{aligned}$$

By the bounded geometry of quasi-coordinates, the above norms can also be take with respect to the metric ω. Hence sequence $\{y_i\}$ satisfies all the properties required in the lemma. □

4.2. Construction of Kähler–Einstein metrics with bounded geometry. We use the function ρ in item (1) of Definition 4.4 to define a domain $(\mathcal{U} \setminus p) := \{\rho < a\}$. We point out that the function ρ used here is different from the one used in definition (1.1). The main goal of this subsection is to prove the following theorem concerning the solvability of Kähler–Einstein equation on $(\mathcal{U} \setminus p)$ by a perturbative method.

Theorem 4.1. *Suppose (X, p) is a germ of log canonical singularity and a punctured neighborhood $\mathcal{U} \setminus p$ of p admits a complete Kähler metric ω with bounded geometry of order k. Then for any smooth function ψ on the boundary $\partial\mathcal{U}$, the Dirichlet problem*

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\varphi+M}\omega^n & \text{on } \mathcal{U}, \\ \varphi|_{\partial\mathcal{U}} = \psi, \end{cases} \tag{4.1}$$

admits a solution in Cheng–Yau function space defined in (4.3) *with* $\|\varphi\|_{k,\alpha} < C(\psi, \rho, M)$.

We first take the function space U to be an open set of $C^{k,\alpha}(\mathcal{U})$, which is defined in (4.3), as follows:

$$U = \left\{\phi \in C^{k,\alpha}(\mathcal{U}) : \frac{1}{c}(g_{\alpha i\bar{j}}) \leq (g_{\alpha i\bar{j}} + \phi_{i\bar{j}}) \leq c(g_{\alpha i\bar{j}}) \text{ in each quasi-coordinate } V_\alpha\right\}$$

for some constant c, which however is not fixed.

Proof of Theorem 4.1. The proof consists of several steps.

Step 1: Find an ω-PSH extension of function ψ to the domain $\mathcal{U}$. Choose an arbitrary smooth extension ψ_1 of ψ, which is supported on $\{x : a \geq \rho(x) \geq c\}$. Also choose a convex monotone increasing function $H : [-\infty, a] \to \mathbb{R}$ which is zero on $[-\infty, b]$ for some constant $b < c$. Now define

$$P = AH(\rho) - AH(a).$$

By choosing A large, $\omega + \sqrt{-1}\partial\bar{\partial}P + \sqrt{-1}\partial\bar{\partial}\psi_1$ is still a Kähler form. If we choose $P + \psi_1$ as our new extension of ψ, then by construction $\sqrt{-1}\partial\bar{\partial}(P + \psi_1)$ is supported on a neighborhood of $\partial\mathcal{U}$. Therefore ω coincides with $\omega + \sqrt{-1}\partial\bar{\partial}P + \sqrt{-1}\partial\bar{\partial}\psi_1$ in a punctured neighborhood of p. In sum, we have

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}(\psi_1 + P))^n = e^{-F}\omega^n & \text{on } \mathcal{U}, \\ \psi_1 + P|_{\partial\mathcal{U}} = \psi, \end{cases}$$

where the function F is still in Cheng–Yau's function space. Hence if we define $\tilde{\omega}$ by

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}(\psi_1 + P), \quad \tilde{\varphi} = \varphi - (\psi_1 + P) \quad \text{and} \quad \tilde{F} = F + \psi_1 + P.$$

Simple calculations show that equation (4.1) is equivalent to

$$\begin{cases} (\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi})^n = e^{\tilde{\varphi}+\tilde{F}}\tilde{\omega}^n & \text{on } \mathcal{U}, \\ \tilde{\varphi}|_{\partial\mathcal{U}} = 0. \end{cases}$$

So from now on, we will focus on zero boundary value problem. The rest of the proof is by continuity method, which is based on a combination of estimates from [5, 8, 14, 31]. We set up the continuity method as follows:

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{\varphi_t + tM}\omega^n, \\ \varphi_t|_{\partial\mathcal{U}} = 0, \end{cases}$$

where M belongs to Cheng–Yau function space defined in (4.3).

Step 2: Openness part in the continuity method. As usual, the openness will follow from the inverse mapping theorem. We need to show the linearized equation at ω_t

$$\begin{cases} \Delta_{\omega_t}h - h = v & \text{on } \mathcal{U}, \\ h|_{\partial\mathcal{U}} = 0, \end{cases}$$

has a unique solution in $C^{k,\alpha}(\mathcal{U})$ with the estimate

$$\|h\|_{k,\alpha}(\mathcal{U}) \leq c\|v\|_{k-2,\alpha}(\mathcal{U})$$

for some constant c independent of the function v. We first remark that $\omega_t := \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ is a complete metric of bounded geometry up to $k - 2$ covariant derivatives by the function choice of function space U at the beginning of the proof. Next take an exhaustion $\{\mathcal{U}_i\}$ of the domain $\mathcal{U}$ towards infinity. (Here the boundary of our compact domain $\mathcal{U}_i$ has two components and one of them coincide with $\partial\mathcal{U}$.) The equation

$$\begin{cases} \Delta_{\omega_t}h_i - h_i = v & \text{on } \mathcal{U}_i, \\ h|_{\partial\mathcal{U}_i} = 0, \end{cases}$$

has a unique solution h_i for each i. The maximum principle implies that $\sup_{\mathcal{U}_i}|h_i| \leq \sup|v|$. The interior Schauder estimate of our function space implies that

$$\|h_i\|_{k,\alpha}(K) \leq c\|v\|_{k-2,\alpha}(\mathcal{U})$$

for any compact set K strict away from $\partial\mathcal{U}$. This inequality combined with the standard global Schauder estimate for a fixed compact set V containing $\partial\mathcal{U}$ imply that $h_i \to h$ pointwise with

$$\Delta_{\omega_t} h - h = v, \quad h|_{\partial\mathcal{U}} = 0.$$

Moreover, we have

$$\|h\|_{k,\alpha}(\mathcal{U}) \leq c\|v\|_{k-2,\alpha}(\mathcal{U}).$$

Hence we establish the openness part.

Step 3: C^0 estimate. We have the following equality:

$$\begin{aligned}\varphi + M &= \log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det g_{i\bar{j}} \\ &= \int_0^1 \frac{\partial}{\partial t}\log\det(g_{i\bar{j}} + t\varphi_{i\bar{j}})\,dt \\ &= \int_0^1 B(t)_{i\bar{j}}\varphi_{i\bar{j}}\,dt,\end{aligned}$$

where $B(t)_{i\bar{j}}$ is the cofactor matrix of the matrix $(g_{i\bar{j}} + t\varphi_{i\bar{j}})$. At a point $x \in \Omega$ we may assume $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\varphi_{i\bar{j}} = \delta_{i\bar{j}}\varphi_{i\bar{i}}$. If $\varphi_{i\bar{i}} \geq 0$, then

$$\frac{\varphi_{i\bar{i}}}{1 + t\varphi_{i\bar{i}}} \leq \varphi_{i\bar{i}}.$$

If $\varphi_{i\bar{i}} \leq 0$, then

$$\frac{\varphi_{i\bar{i}}}{1 + t\varphi_{i\bar{i}}} \leq \varphi_{i\bar{i}}.$$

Hence we have the two inequalities

$$\varphi + M \leq \Delta_\omega\varphi, \quad \varphi + M \geq \Delta_{\omega_1}\varphi,$$

where $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. By Lemma 4.1, we get the C^0 estimate.

Step 4: C^1 boundary estimate. On the one hand, since $\varphi + M \leq \Delta_\omega\varphi$, we can construct a barrier function h from above as follows. Take a domain $V \subset \mathcal{U}$ satisfying $p \notin V$. We also require that $\partial V = \partial\mathcal{U} \cup C$, where C is a smooth connected manifold disjoint with $\partial\mathcal{U}$. Then derive h by solving the following Dirichlet problem in V:

$$\begin{cases}\Delta_\omega h = c, \\ h|_{\partial\mathcal{U}} = 0 \quad \text{and} \quad h|_C = d,\end{cases}$$

where $d := \sup_{\mathcal{U}}|\varphi|$ and $c := \inf_{\mathcal{U}}(\varphi + M)$. Then the maximum principle implies that $h \geq \varphi$ in V. On the other hand, we construct a barrier function h_1 from below as follows. Take the global strict ω-PSH function P we constructed in Step 1 and choose a constant B large enough such that

$$\begin{cases}(\omega + \sqrt{-1}\partial\bar{\partial}BP)^n \geq e^{\sup\varphi + M}\omega^n & \text{on } V, \\ BP \leq \varphi & \text{on } C, \\ BP = 0 & \text{on } \partial\mathcal{U}.\end{cases}$$

Then the maximum principle of the Monge–Ampère equation implies $h_1 := BP \leq \varphi$ on V. Noticing that h and h_1 coincide with φ on $\partial\Omega$, we get the boundary gradient estimate of φ.

Step 5: Global C^1 estimate. On a noncompact manifold, we do not necessarily have a maximum point with vanishing gradient, etc. We will follow [5] to obtain the C^1 estimate. Define $\phi = \log|\nabla\varphi|^2 - \gamma(\varphi)$, where γ is a monotone increasing function to be determined. Assume that $\sup_\Omega \phi$ is not obtained on $\partial\mathcal{U}$. Then by the generalized maximum principle, we can find a point $q \in \Omega$ with

$$\phi(q) + \varepsilon > \sup_\Omega \phi,$$

$$|V|_\omega(q) < \varepsilon,$$

$$\Delta_\omega \phi(q) < \varepsilon,$$

where $V := \nabla_\omega \phi$. By (2.11), at q, we have

$$\Delta' \log|\nabla\varphi|_g^2 \geq \frac{2\mathrm{Re}\nabla_m \log F \nabla^m \varphi}{|\nabla\varphi|_g^2} - \Lambda \operatorname{tr}_{g'} g + 2\Re\left\langle \frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g'} - 2\Re\left\langle \frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g}.$$

Now using $V = \nabla \log|\nabla\varphi|^2 - \gamma'\nabla\varphi$, we get

$$\begin{aligned} &2\Re\left\langle \frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g'} - 2\Re\left\langle \frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g} \\ &= 2\Re\left\langle V + \gamma'\nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g'} - 2\Re\left\langle V + \gamma'\nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\right\rangle_{g} \\ &> -\varepsilon \operatorname{tr}_{g'} g - \varepsilon - \gamma'. \end{aligned}$$

The inequality above together with

$$\Delta'\gamma(\varphi) = \gamma'\Delta'\varphi + (\gamma'')|\nabla\varphi|^2$$

imply that

$$\varepsilon \operatorname{tr}_{g'} g > \Delta'\phi > (\gamma' - \varepsilon - \Lambda)\operatorname{tr}_{g'} g + (\gamma'')|\nabla\varphi|^2 - \gamma' - \varepsilon - n - C,$$

where C is the bound of the gradient of the function $\log F$. Now we construct our function γ as

$$\gamma(x) = (\Lambda + 2)x - \frac{1}{x + C' + 1},$$

where C' is the lower bound of φ. Then by a standard argument we get a global C^1 estimate.

Step 6: Boundary C^2 estimate. We notice the argument of [8] that the boundary C^2 estimate is purely local around the boundary and our equation can be written as

$$\begin{cases} \det \varphi_{i\bar{j}} = e^{\varphi + f} & \text{on } \Omega, \\ \varphi|_{\partial\mathcal{U}} = \psi. \end{cases}$$

Locally, this is exactly the equation considered in [8], hence the estimate follows by the fact that $\sqrt{-1}\partial\bar{\partial}\rho$ is strictly positive in a neighborhood of $\partial\mathcal{U}$.

Step 7: Global C^2 estimate and completion of the proof. We have the well-known inequality

$$\Delta' \log \operatorname{tr}_g g' \geq -B \operatorname{tr}_{g'} g - C,$$

where B, C depends on the geometry of the good background metric g and the Ricci curvature of the volume form on the right-hand side of the equation. Notice that

$$\Delta' \varphi = n - \operatorname{tr}_{g'} g.$$

By setting $A = B + C + 1$, we have the differential inequality

$$\Delta'(\log \operatorname{tr}_g g' - A\varphi) = \operatorname{tr}_{g'} g - An.$$

This inequality and the boundary C^2 estimate imply the global C^2 estimate. Then by the Evans–Krylov Theorem, we have the interior $C^{2,\alpha}$ estimate. Since the metric is uniformly bounded in a neighborhood of the boundary, the local argument of [8] will give the global $C^{2,\alpha}$ estimate. □

Remark 4.2. The $C^{2,\alpha}$ estimate of φ depends on the extension ψ_1 of our boundary function ψ in Step 1. It is not clear if we can have $\|\psi_1\|_{2,\alpha} < C\|\psi\|_{2,\alpha}$ for some constant C independent of the boundary value.

In the following lemma, we will prove that any complete local Kähler–Einstein metric is indeed obtained from solving the Monge–Ampère equations (4.1) with different Dirichlet boundary conditions.

Lemma 4.2. *Let ω_{KE} be a complete Kähler–Einstein metric with bounded geometry property on $\mathcal{U} \setminus p$ and let ω'_{KE} be another complete Kähler–Einstein metric on $\mathcal{U} \setminus p$. Then there exist a function ψ on $\partial\mathcal{U}$ and a solution φ of equation* (4.1) *(with $\omega = \omega_{\mathrm{KE}}$, $M = 0$, Dirichlet boundary value ψ) obtained by the bounded geometry method as in Theorem* 4.1. *Moreover, $\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\varphi$.*

Proof. From the fact that ω'_{KE} is a complete Kähler–Einstein metric and Theorem 1.3, we know that

$$\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi}$$

and $\widetilde{\varphi}$ is a bounded smooth solution of equation (4.1). On the other hand, by Theorem 4.1, we can find another function φ solving equation (4.1) with Dirichlet boundary value $\psi := \widetilde{\varphi}|_{\partial\mathcal{U}}$. By the uniqueness Lemma 3.1, we conclude that $\varphi = \widetilde{\varphi}$, and hence

$$\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\varphi.$$

This completes the proof. □

We shall now prove Theorem 1.4 on the asymptotic behavior of KE metrics we constructed in Theorem 4.1. With the help of estimates of higher order derivatives of φ and quasi-coordinates, we prove the stability of local complete Kähler–Einstein metrics with negative scalar curvature near isolated log canonical singularity.

Proof of Theorem 1.4. First of all, since ω'_{KE} is complete, by Lemma 4.2,

$$\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where φ is the solution from Theorem 4.1 with suitable Dirichlet boundary value. For any point $q \in (\mathcal{U} \setminus p)$, we can choose a quasi-coordinate $(\widehat{V}, \phi)$ covering q such that there is a point $\widehat{q} \in V \subset \widehat{V}$, $\phi(\widehat{q}) = q$ and $\operatorname{dist}(\widehat{q}, \partial\widehat{V}) \geq \operatorname{dist}(V, \partial\widehat{V}) \geq \varepsilon_1$. Let β be the cut-off function we constructed in the proof of Lemma 4.1. Then we have the inequality

$$\sum_{i=1}^{k} |\beta^{(k)}|_{\mathrm{Euc}} \leq B_k,$$

where B_k are constants independent of p and V by the existence of quasi-coordinates (Definition 4.2). This is true because under the construction of the system of quasi-coordinates, we have $\operatorname{dist}(V, \partial\widehat{V}) \geq \varepsilon_1 > 0$, hence controlling the derivatives of cut-off function uniformly. Actually, we can even assume the covering domains we chose are $B_{\frac{1}{4}\varepsilon_1}, B_{\frac{1}{2}\varepsilon_1}$ by subdividing the original coverings. By our previous proof of the a priori estimates of φ from equation (4.1), we also have the following inequality for any point $q \in (\mathcal{U} \setminus p)$ and any nonnegative integer k:

$$\sum_{i=1}^{k} \|\nabla^{(k)}\varphi\|_{\omega_{\mathrm{KE}}}(q) \leq C_k.$$

When $k = 0$, the C^0 decay of φ is Theorem 1.3. For $k \geq 1$, we do computations in the quasi-coordinate as follows:

$$-\int_{\widehat{V}} \beta\varphi\Delta\varphi = \int_{\widehat{V}} \beta|\nabla\varphi|^2 + \int_{\widehat{V}} \varphi\langle\nabla\varphi, \nabla\beta\rangle,$$
$$\int_V |\nabla\varphi|^2 \leq \int_{\widehat{V}} \beta|\nabla\varphi|^2 \leq C \cdot (C_2 + B_1) \cdot \sup_{\widehat{V}} |\varphi|.$$

Similarly, using integration by part,

$$\int_V \beta|\nabla^k\varphi|^2 = \int_V \Big(\sum_{i+j=2k-1} \nabla^i\beta\nabla^j\varphi\Big)\varphi \leq (B_k + C_{2k-1}) \cdot \sup_{\widehat{V}} |\varphi|.$$

Note that by Theorem 4.1, there is a constant $C(\psi, \omega_{\mathrm{KE}}, n)$ such that

$$\frac{\omega_{\mathrm{KE}}}{C(\psi, \omega, n)} \leq \omega'_{\mathrm{KE}} \leq C(\psi, \omega_{\mathrm{KE}}, n)\omega_{\mathrm{KE}}.$$

Suppose $\operatorname{dist}_{\omega_{\mathrm{KE}}}(q, \partial\mathcal{U}) \geq R$. Then

$$\operatorname{dist}_{\omega'_{\mathrm{KE}}}(q, \partial\mathcal{U}) \geq \frac{1}{C(\omega_{\mathrm{KE}}, \psi, n)} R,$$

and by the triangle inequality, we have

$$\operatorname{dist}_{\omega_{\mathrm{KE}}}(\partial(\phi(\widehat{V})), \partial\mathcal{U}) \geq \operatorname{dist}_{\omega_{\mathrm{KE}}}(q, \partial\mathcal{U}) - \operatorname{dist}_{\omega_{\mathrm{KE}}}(q, \partial(\phi(\widehat{V}))) \geq R - C'\varepsilon_1 \geq \frac{R}{2},$$

where $\widehat{V}$ is a covering of point q and C' is the metric equivalence constant in the definition of quasi-coordinates (Definition 4.2), which depends on the geometry of ω_{KE}. Similarly, we have

$$\operatorname{dist}_{\omega'_{\mathrm{KE}}}(\partial(\phi(\widehat{V})), \partial\mathcal{U}) \geq \frac{R}{2C(\omega_{\mathrm{KE}}, \psi, n)}.$$

Hence by the C^0 estimate of φ in Theorem 1.3,

$$\int_V |\nabla^k\varphi|^2 \leq C(\psi, \omega_{\mathrm{KE}}, n) \cdot (C_{2k-1} + B_k) \cdot \frac{c(n)}{R}.$$

Now that we have L^2 norm control of all higher order derivatives, by Sobolev embedding on Euclidean space and property (d) of quasi-coordinates (Definition 4.2), we can conclude that

$$\sum_{i=1}^{k} \|\nabla^i \varphi\|_{\omega_{\mathrm{KE}}}(q) \le \frac{1}{R(q)^{\frac{1}{2}}} \cdot C(k, \omega_{\mathrm{KE}}, \psi). \qquad \square$$

Remark 4.3. We remark that the decay rate $R^{\frac{-1}{2}}$ of the function φ obtained in the above theorem is far from optimal. The optimal decay rate might need a case-by-case treatment depending on the type of the singularity. It is pointed out to us by Professor Hein that if $|\varphi| \le R^{-1}$, and if all its higher order derivatives are bounded, then by using quasi-coordinates and the regularity theory of Monge–Ampère equations, the decay rate can be improved to R^{-1}.

We use the metric stability result to show that to construct a geometric domain with bounded geometry property in a punctured neighborhood of an isolated log canonical singularity, we only need to show that there exists a Kähler metric with bounded geometry property on its finite cover. Therefore, once for some singularity it has bounded geometry property, so it does for its finite quotient, hence more examples of singularity with bounded geometry property will be obtained. This will be useful when the metric on the finite cover is not necessary invariant under the finite group action.

Corollary 4.1. *Let (X, p) be an isolated log canonical singularity embedded in $\mathbb{C}^N$ and let $(\mathcal{U} \setminus p, \omega = \sqrt{-1}\partial\bar{\partial}\rho)$ be a Kähler–Einstein domain with bounded geometry property of infinite order defined as in Definition* 4.4. *Let (Y, p') be the quotient space $(X, p)/G$, where G is a finite group acting freely on $(X \setminus p)$ and fixing point p. Then for any $k \ge 1$, there is a punctured neighborhood $\mathcal{U}' \setminus p'$ of p' admitting a Kähler metric ω' with bounded geometry property of order k. Hence Theorem* 1.4 *can be applied to the finite quotient domain $(\mathcal{U}' \setminus p', \omega')$.*

Proof. For simplicity, assume $G = \mathbb{Z}_2$. Let f be the nontrivial element of G. Define

$$\rho' := \rho + f^*\rho, \quad \omega' := \sqrt{-1}\partial\bar{\partial}\rho'.$$

Then ρ' is invariant under the $\mathbb{Z}_2$ action. Note that $\hat{\omega} = \sqrt{-1}\partial\bar{\partial} f^*\rho$ is also a complete Kähler–Einstein metric with bounded geometry property near p. Hence by Theorem 1.4, we have

$$\hat{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi, \quad \sum_{i=1}^{k} \|\nabla^i \varphi\|_{\omega}(q) \le \frac{C(k, \omega, \psi)}{R^{\frac{1}{2}}(q)}.$$

In particular, this shows that $\omega' = \omega + \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\varphi$ has bounded geometry property of order k by possibly shrinking the domain. This completes the proof. $\square$

5. Log canonical singularities on surfaces

5.1. Model metrics with bounded geometry property. In this short subsection, we provide some explicit examples of (X, p) with bounded geometry property of infinite order. We mainly focus on complex dimension 2 and expect that there are more examples in higher dimension by using the arithmetic quotient of symmetric domains (cf. [23, 52]). We should remark that log canonical surface singularities are classified in [1,29,36] and they admit a model metric with bounded geometry property. However, in complex dimension higher than two, there are examples of log canonical singularity which admit a local Kähler–Einstein metrics but these metrics do not have bounded geometry property. For such an example, we can take a negative line bundle over a non-flat Calabi–Yau manifold and then contract the zero section.

Lemma 5.1 ([32,38,53]). *Any isolated normal log canonical (non-log terminal) surface singularity can be uniformized by bounded symmetric domains with invariant Kähler–Einstein metric* $\omega = \sqrt{-1}\partial\bar{\partial}\rho$ *and classified as follows:*

(1) *One point partial compactification of* $\mathbb{B}^2/\Gamma$, Γ *a parabolic discrete subgroup of* $\mathrm{Aut}(\mathbb{B}^2)$. *Invariant metric is defined by* $\sqrt{-1}\partial\bar{\partial}\rho$, *where*

$$\rho = \log \frac{1}{1 - |z_1|^2 - |z_2|^2}$$

and z_1, z_2 *are complex coordinates of* $\mathbb{B}^2$.

(2) *One point compactification of* $\mathbb{H} \times \mathbb{H}/\Gamma$, Γ *a parabolic discrete subgroup of* $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ *corresponding to a boundary point. Invariant metric is defined by* $\sqrt{-1}\partial\bar{\partial}\rho$, *where*

$$\rho = \log\left(\frac{1}{y_1 y_2}\right)$$

and y_1, y_2 *are the imaginary parts of the complex coordinates of* $\mathbb{H} \times \mathbb{H}$.

Remark 5.1. Type 1 singularities, up to a finite quotient, have a minimal resolution whose exceptional divisor is an elliptic curve. Type 2 singularities, once again up to finite quotients, have minimal resolutions whose exceptional divisor is a cycle of smooth rational curves.

Remark 5.2. The invariant Kähler–Einstein metrics in Lemma 5.1 have a system of quasi-coordinates in a punctured neighborhood of the isolated log canonical singularities. This is the main property we will use in the following proof of Corollary 1.1.

Proof of Corollary 1.1. By Lemma 5.1, for any isolated log canonical surface singularity, there is a reference metric with bounded geometry of order k for any k. Hence Corollary 1.1 is a direct consequence of Theorem 1.4 except for the statement about the Gromov–Hausdorff convergence. We deal with each of the cases separately.

(1) By [23], the uniformization metric in the upper half plane model is

$$\omega = -\sqrt{-1}\partial\bar{\partial}(\operatorname{Im} u - |v|^2) = -\sqrt{-1}\partial\bar{\partial}\log(-\log|\sigma_D|_{h_D}),$$

where D is an elliptic curve. The total space of resolution is a negative line bundle over the elliptic curve. Direct calculations show that the metric ω degenerates along the tangential direction of D and the S^1 circle direction contained in the fiber of the negative line bundle. Hence it is clear that, when we choose the base point $p_j \to p$, $(\mathcal{U}, p_j, g_{\mathrm{KE}}) \to \mathbb{R}$ in the pointed Gromov–Hausdorff topology.

(2) In this case, the model is $\mathbb{H} \times \mathbb{H}/\Gamma$ with invariant metric given by $\sqrt{-1}\partial\bar{\partial}\log(\frac{1}{y_1 y_2})$, where y_i is the imaginary part z_i, $i = 1, 2$. To make things clear, we describe what is the group action (see [31, p. 55] or [32, p. 344] for more details).

Let $G(M,V) = \{\left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & 1 \end{smallmatrix}\right) : \varepsilon \in V,\ \mu \in M\}$ act on $\mathbb{C}^2$ properly discontinuous and without fixed points as follows:

$$\left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & 1 \end{smallmatrix}\right) \bullet (z_1, z_2) := (\varepsilon z_1 + \mu, \varepsilon' z_2 + \mu'), \tag{5.1}$$

where $M \cong \mathbb{Z} \times \mathbb{Z}\omega_0$ is a rank 2 free module over $\mathbb{Z}$ generated by 1 and another real quadratic irrational number ω_0 and $V \cong \mathbb{Z}$ generated by a quadratic irrational number $\varepsilon > 0$. Here ε' (resp. μ') is the Galois conjugation over $\mathbb{Q}$ of ε (resp. μ') and ε also satisfies $\varepsilon' = \frac{1}{\varepsilon}$. The action of $G(M,V)$ can be restricted onto $\mathbb{H}^2$, where $\mathbb{H}$ is the upper half plane.

Let x_1, y_1 and x_2, y_2 be the real coordinates of z_1, z_2 separately and define

$$L_c := \{y_1 y_2 = c\} \subset \mathbb{H} \times \mathbb{H},$$

where c is a positive constant. By $\varepsilon\varepsilon' = 1$, we know that L_c is invariant under the action of Γ. Now show that the set L_c/Γ is a torus bundle over a circle. Let x_1, x_2, y_1 be the coordinates on the set L_c. By (5.1), the action of $\left(\begin{smallmatrix} \varepsilon^n & \mu \\ 0 & 1 \end{smallmatrix}\right)$ when restricted on L_c is given by

$$(x_1, x_2, y_1) \to (\varepsilon^n x_1 + \mu, \varepsilon^{-n} x_2 + \mu', \varepsilon^n y_1).$$

The action of M via the embedding

$$\mu \mapsto \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

on $\mathbb{R}^2(x_1, x_2)$ realizes M as a lattice of $\mathbb{R}^2$, so that the projection of L_c/Γ onto y_1 identifies L_c/Γ with a $\mathbb{T}^2$ bundle over $\mathbb{S}_1$.

The metric tensor is given by

$$\begin{aligned} g &= \frac{dx_1 \otimes dx_1}{y_1^2} + \frac{dy_1 \otimes dy_1}{y_1^2} + \frac{dx_2 \otimes dx_2}{y_2^2} + \frac{dy_2 \otimes dy_2}{y_2^2} \\ &= \frac{dx_1 \otimes dx_1}{y_1^2} + \frac{y_1^2 dx_2 \otimes dx_2}{c^2} + 2\frac{dy_1 \otimes dy_1}{y_1^2} + \frac{dc \otimes dc}{c^2} - 2\frac{dc \otimes dy_1}{c y_1}, \end{aligned}$$

where $c = y_1 y_2$. Letting $\widehat{c} = \log c$, we have

$$g = \frac{dx_1 \otimes dx_1}{y_1^2} + \frac{y_1^2 dx_2 \otimes dx_2}{e^{2\widehat{c}}} + 2\frac{dy_1 \otimes dy_1}{y_1^2} + d\widehat{c} \otimes d\widehat{c} - 2\frac{d\widehat{c} \otimes dy_1}{y_1}.$$

Let $\mathbb{T}^2(y_1)$ be the family of torus defined by the lattice action on $\mathbb{R}^2(x_1, x_2)$, which depends on y_1. We claim that the diameter of $(\mathbb{T}^2, g_{\widehat{c}}) \to 0$ when $\widehat{c} \to \infty$, where

$$g_{\widehat{c}} = \frac{dx_1 \otimes dx_1}{y_1^2} + \frac{y_1^2 dx_2 \otimes dx_2}{e^{2\widehat{c}}}.$$

Since y_1 is bounded, without loss of generality, we may assume

$$g_{\widehat{c}} = dx_1 \otimes dx_1 + \frac{dx_2 \otimes dx_2}{e^{2\widehat{c}}}.$$

We prove the claim by contradiction. If not, we can assume that $\operatorname{diam}(\mathbb{T}^2, g_{\widehat{c}})$ decreases to some constant $\eta \neq 0$. Hence for each $i \in \mathbb{Z}$, we have a point $q_i \in \mathbb{T}^2$ such that $\operatorname{dist}_{g_i}(o, q_i) \geq \frac{\eta}{2}$.

We can take an accumulation point q of q_i by the compactness of $\mathbb{T}^2$. By the triangle inequality, we have $\operatorname{dist}_{g_i}(o,q) \geq \frac{\eta}{4}$. This contradicts the fact that $\mathbb{Z}+\mathbb{Z}\omega$ is dense in $\mathbb{R}$, which enables us to show that $\operatorname{dist}_{g_i}(o,q) \to 0$. So if we take a sequence of base point $p_i = (x_{1i}, x_{2i}, y_{1i}, c_i)$ in $\mathcal{U} \setminus p$ with $c_i \to 0$, the Gromov–Hausdorff limit of $(\mathcal{U}, p_j, g_{\mathrm{KE}})$ is a flat cylinder Y. More precisely, Y is defined as $\mathbb{R}^2/\mathbb{Z}$, where

$$n \bullet (\widehat{y_1}, \widehat{c}) = (\widehat{y_1} + n \log \varepsilon, \widehat{c})$$

and the metric is

$$g_Y = 2d\,\widehat{y}_1 \otimes d\,\widehat{y}_1 + d\widehat{c} \otimes d\widehat{c} - d\,\widehat{y}_1 \otimes d\widehat{c} - d\widehat{c} \otimes \widehat{y}_1,$$

where $\widehat{y} = \log y$. □

Proof of Theorem 1.2 (surface case). By Theorem 4.1, when $\omega = \sqrt{-1}\partial\bar{\partial}\rho$ is a Kähler–Einstein metric with bounded geometry property, the following Dirichlet problem on $\mathcal{U} \setminus p$ admits a bounded solution φ:

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\varphi}\omega^n, \\ \varphi|_{\partial\mathcal{U}} = \psi. \end{cases}$$

Now we rewrite the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\varphi}\omega^n$$

as

$$(\sqrt{-1}\partial\bar{\partial}\widetilde{\varphi})^n = e^{\widetilde{\varphi}-\rho}\frac{\omega^n}{(\sqrt{-1})^n \nu \wedge \overline{\nu}}(\sqrt{-1})^n \nu \wedge \overline{\nu},$$

where $\widetilde{\varphi} = \rho + \varphi$ and ν is a local holomorphic volume form (possibly multi valued). Note that $\log \frac{\omega^n}{(\sqrt{-1})^n \nu\wedge\overline{\nu}} - \rho$ is a pluriharmonic function on $\mathcal{U} \setminus p$.

Claim. *The function* $\log \frac{\omega^n}{(\sqrt{-1})^n \nu\wedge\overline{\nu}} - \rho$ *is sublog, i.e.,*

$$\log \frac{\omega^n}{(\sqrt{-1})^n \nu \wedge \overline{\nu}} - \rho \geq \varepsilon \log |\sigma_D|_{h_D} + C_\varepsilon.$$

Assuming this claim, we can complete the proof as follows. By the uniqueness Theorem 2.8, we conclude that

$$\widetilde{\varphi} + \log \frac{\omega^n}{(\sqrt{-1})^n \nu \wedge \overline{\nu}} - \rho$$

coincides with the solution from Theorem 1.1 (when the boundary value is matched). Recall that the Riemannian metric induced by $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ in Theorem 4.1 is complete, in particular, this implies that the Kähler–Einstein metric associated with the solution from Theorem 1.1 is complete towards p. □

Proof of the Claim. We will use the explicit expression of ρ, ω, and ν from Lemma 5.1 to prove the claim case by case. Now we outline the proof. There are essentially four types of isolated log canonical (non-klt) singularities in complex dimension 2.

Type (1.1): The exceptional divisor of the minimal resolution of singularity is an elliptic curve D. Then the model metric is

$$\omega = -\sqrt{-1}\partial\bar{\partial}\rho, \quad \rho = \log(-\log|\sigma_D|_{h_D}).$$

When pulled back to the resolution, $\nu \wedge \bar{\nu}$ has a pole $|\sigma_D|_{h_D}^{-2}$ along the exceptional divisor. Straightforward calculations show that $\log \frac{\omega^2}{\nu\wedge\bar{\nu}}$ and ρ are both sublog functions.

Type (1.2): The singularity is a finite quotient of case (1.1) above. When pulled back to the finite cover,

$$\log \frac{\omega^n}{(\sqrt{-1})^n \nu \wedge \bar{\nu}} - \rho,$$

by the proof of case (1) above, is a bounded function. Here we have used the fact that ω is invariant under the finite quotient, and when pull backed to finite cover, ν is still a generator of holomorphic volume form near the singularity point p.

Type (2.1): The exceptional divisors of the minimal resolution of singularity is circle of rational curves D_i, $0 \le i \le m$. Let us focus on the metric behavior of ω near the intersection of two exceptional divisors. Without loss of generality, assume locally

$$D_k = \{u_k = 0\}, \quad D_{k+1} = \{v_k = 0\},$$

then the metric

$$\omega = \sqrt{-1}\partial\bar{\partial}\rho, \rho = -\log(\log|u_k^{\alpha} v_k^{\beta}| \log|u_k^{\alpha'} v_k^{\beta'}|)$$

can be written as

$$\omega = \frac{\sqrt{-1}\partial \log|u_k^{\alpha} v_k^{\beta}| \wedge \bar{\partial}\log|u_k^{\alpha} v_k^{\beta}|}{(\log|u_k^{\alpha} v_k^{\beta}|)^2} + \frac{\sqrt{-1}\partial \log|u_k^{\alpha'} v_k^{\beta'}| \wedge \bar{\partial}\log|u_k^{\alpha'} v_k^{\beta'}|}{(\log|u_k^{\alpha'} v_k^{\beta'}|)^2}.$$

When pulled back to the resolution, $\nu \wedge \bar{\nu}$ has a pole $|u_k v_k|^{-2}$ along the exceptional divisor. Then direct calculations show that $\log \frac{\omega^2}{(\sqrt{-1})^n \nu\wedge\bar{\nu}}$ and ρ are both sublog functions.

Type (2.2): The singularity is a finite quotient of case (2.1). Then we can argue as case (1.2) to complete the proof. □

We end this section by providing an example of a family of canonical polarized surface with the central fiber equipped with log canonical singularity satisfying bounded geometry property.

Example 5.1 (Degeneration of Godeaux surfaces). A surface X is called a Godeaux surface if $\pi_1(X) = \mathbb{Z}_5$ and the universal cover is a quintic hypersurface. A explicit construction could be as follows: Define $\mathbb{Z}_5$ on $\mathbb{P}^3$ in the following way:

$$\rho \bullet (X_0, X_1, X_2, X_3) = (X_0, \rho X_1, \rho^2 X_2 . \rho^3 X_3).$$

Then there exist quintics (in $\mathbb{P}^3$) invariant and fixed-point free under the $\mathbb{Z}_5$ action with five non-degenerate triple points and no other singularities by a dimension count argument (cf. [49, p. 135]). The $\mathbb{Z}_5$ quotient will give a family of Godeaux surfaces with central fiber a canonical polarized variety coupled with a single simple elliptic singularity (cone over an elliptic curve).

6. Discussion of Conjecture 1.1 for smoothable isolated log canonical singularities

In this section, we give a discussion of Conjecture 1.1 when the isolated log canonical singularity is smoothable. First we set up the question.

Let $\pi : \mathcal{X} \to B$ be a smoothing of an isolated log canonical (non-log terminal) singularity $(\mathcal{X}_0, p)$ over $B \in \mathbb{C}$ such that $\mathcal{X}_t = \pi^{-1}(t)$ is smooth for $t \neq 0$. We assume that the total space $\mathcal{X}$ has at worst canonical singularities and the canonical divisor $K_{\mathcal{X}}$ is ample. For simplicity, we assume $K_{\mathcal{X}/B}$ is Cartier and let ν be a local generator ν of the bundle $K_{\mathcal{X}/B}$ such that $\nu_t = \nu|_{\mathcal{X}_t}$ is a local holomorphic n form on $\mathcal{X}_t$, where $\dim \mathcal{X}_t = n$ for $t \in B$. Then $\mathcal{X}$ can locally be embedded in $\mathbb{C}^{N+1}$ and we apply a pluriharmonic function ρ on $\mathbb{C}^{N+1}$ to define $\mathcal{U} := \{\rho \leq 0\} \cap \mathcal{X}$ so that $p \in \mathcal{U}$. After perturbation, we can always assume $\partial\mathcal{U}$ is smooth and $d\rho \neq 0$ on $\partial\mathcal{U}$. By continuity, we may assume $\partial\mathcal{U}_t$ is smooth for t sufficiently close to 0, where $\mathcal{U}_t = \mathcal{U} \cap \mathcal{X}_t$ and $\rho_t = \rho|_{\mathcal{X}_t}$.

We let ψ be a smooth function in an open neighborhood of $\partial\mathcal{U}$ and let ψ_t be its restriction to $\partial\mathcal{U}_t$. We also let Ω be the real-valued smooth (n,n)-volume measure on $\mathcal{X}$ defined by

$$\Omega = (\sqrt{-1})^n \nu \wedge \bar{\nu}$$

and let

$$\Omega_t = \Omega|_{\mathcal{X}_t} = (\sqrt{-1})^n \nu_t \wedge \bar{\nu}_t$$

be the restriction of Ω on $\mathcal{U}_t$. Immediately we have

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = 0, \quad \sqrt{-1}\partial\bar{\partial}\log\Omega_t = 0$$

for $t \in B$.

By Theorem 1.1, there is a unique solution φ_t to the following Dirichlet problem for Monge–Ampère equation on $\mathcal{U}_t$:

$$\begin{cases} (\sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{\varphi_t}\Omega_t, \\ \varphi_t|_{\partial\mathcal{U}_t} = \psi_t \end{cases}$$

for each $t \in B$. If we let $\omega_t = \sqrt{-1}\partial\bar{\partial}\varphi_t$, then the corresponding Kähler metric g_t satisfies the Kähler–Einstein equation

$$\mathrm{Ric}(g_t) = -g_t$$

on $\mathcal{U}_t$ for $t \in B^*$ and $\mathcal{U}_0 \setminus \{p\}$ for $t = 0$.

In [42], the third author combines semistable reduction [30] and maximum principle to derive uniform estimates with barrier for φ_t for $t \in B$ when $\mathcal{X}$ is a stable family of canonical polarized variety, and then applies the local version of partial C^0 estimate (cf. [20,21]) initiated by [46] for noncollapsed polarized Kähler–Einstein manifolds and some other tools to show that the Kähler–Einstein metric g_0 on $\mathcal{X}_0$ is complete towards the local canonical locus.

In our local situation, now we have a degeneration of polarized Kähler–Einstein manifold

$$(\mathcal{U}_t, g_t, L_t, h_t)$$

with boundary, where L_t is a trivial line bundle and $h_t := e^{-\varphi_t}$ is the hermitian metric on L_t whose curvature is the Kähler form associated to the Kähler–Einstein metric g_t. Notice that $\mathcal{U}_t$, $t \in B^*$, is a strongly pseudoconvex domain, so we can solve the $\bar{\partial}$ equation with L^2 estimate on Ω_t. We believe that the techniques developed in [42] and the local version of the partial C^0 estimate (cf. [21, Theorem 1.1]) can still be applied.

A. Appendix

Lemma A.1. *Let $(\mathcal{U}, \omega)$ be a Kähler manifold with boundary (the boundary can possibly have several components). If φ, ψ are two continuous ω-PSH function on $\bar{\mathcal{U}}$ satisfying $\varphi > \psi$ on $\partial\mathcal{U}$, then for $\Omega := \{\varphi < \psi\}$ we have*

$$\int_\Omega \omega_\psi^n \le \int_\Omega \omega_\varphi^n.$$

Proof. Suppose first that φ, ψ and the boundary Ω are smooth. Set

$$\varphi_t = \max(\varphi + t, \psi), \quad t > 0.$$

Then close to $\partial\Omega$, we have $\varphi_t = \varphi + t$. Define the closed current

$$T_t = \sum_{k=1}^{k=n} \binom{n}{k} (\sqrt{-1}\partial\bar{\partial}\varphi_t)^{k-1} \wedge \omega^{n-k}$$

and set $T = \lim_{t\to 0} T_t$. By Stokes's theorem

$$\begin{aligned}\int_\Omega \omega_{\varphi_t}^n &= \int_\Omega \sqrt{-1}\partial\bar{\partial}\varphi_t \wedge T_t + \omega^n \\ &= \int_{\partial\Omega} d^c\varphi_t \wedge T_t + \int_\Omega \omega^n \\ &= \int_{\partial\Omega} d^c\varphi \wedge T + \int_\Omega \omega^n \\ &= \int_\Omega \omega_\varphi^n.\end{aligned}$$

Since $\varphi_t \to \psi$ in Ω as $t \to 0$, we get by applying the convergence theorem that $\omega_{\varphi_t}^n \to \omega_\psi^n$ in Ω. Hence for a test function χ in Ω with $0 \le \chi \le 1$ we get

$$\int_\Omega \chi\omega_\psi^n = \lim_{t\to 0} \int_\Omega \chi\omega_{\varphi_t}^n \le \liminf_{t\to 0} \int_\Omega \omega_{\varphi_t}^n.$$

So

$$\int_\Omega \omega_\psi^n \le \liminf_{t\to 0} \int_\Omega \omega_{\varphi_t}^n = \int_\Omega \omega_\varphi^n,$$

which completes the proof for smooth functions. Now we suppose that φ and ψ satisfy the extra assumption

$$\sqrt{-1}\partial\bar{\partial}\varphi \ge (\delta - 1)\omega, \quad \sqrt{-1}\partial\bar{\partial}\psi \ge (\delta - 1)\omega \tag{A.1}$$

for some $\delta > 0$. Then by [6, Theorem 2], we can find two sequence of ω-PSH functions φ_j and ψ_j on $\bar{\Omega}$ converging uniformly to φ, ψ respectively. Given a compact set $K \subset \Omega$, we find $t > 0$ and a positive integer j_0 such that

$$K \subset \Omega(t, j) := \{\varphi_j < \psi_j - t\} \subset \Omega$$

for $j > j_0$ and the boundary of $\Omega(t, j)$ is smooth (using Sard's theorem). Now we have

$$\int_K \omega_\psi^n \le \liminf_{j\to\infty} \int_{\Omega(t,j)} \omega_{\psi_j}^n \le \liminf_{j\to\infty} \int_{\Omega(t,j)} \omega_{\varphi_j}^n \le \int_\Omega \omega_\varphi^n,$$

where the second inequality is due to the first part of the proof. We still need to get rid of the assumption of (A.1). Note that for fixed $t \in (0,1)$ and ω-PSH functions φ, ψ, the functions $t\varphi, t\psi$ satisfy (A.1) for some $\delta > 0$. For a fixed compact set $K \subset \Omega$ and constant $t \in (0,1)$, we may choose $\delta > 0$ sufficiently small such that $K \subset \Omega(\delta, t) := \{\varphi < \psi - \frac{\delta}{t}\}$. Now we have

$$\int_K \omega_\psi^n \le \liminf_{t\to 1} \int_{\Omega(t,\delta)} \omega_{t\psi}^n \le \liminf_{t\to 1} \int_{\Omega(t,\delta)} \omega_{t\varphi}^n \le \int_\Omega \omega_\varphi^n,$$

To complete the proof it is enough to consider an exhaustion sequence of compact subsets of Ω. □

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Eingegangen 28. Mai 2022, in revidierter Fassung 9. Dezember 2022