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The quantum pencil revisited

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The Quantum Pencil Revisited

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Abstract

A quantum description is used to find the time limit for balancing an idealized pencil-like object on end. In agreement with some prior studies, analysis for this system shows that quantum effects are not observable in any foreseeable circumstances, either by casual experimentation or by extensive laboratory preparation. A direct solution of Schrodinger's equation avoids an *a priori* use of the Heisenberg uncertainty relation in the classical equations of motion, which, although very much a proper usage, may lead to misunderstanding of the results. The classical equations of motion are adequate for the pencil in all circumstances.

Introduction

In classical mechanics there is no bound on the time an isolated object might be balanced in a metastable position. Apparently the first mention of the application of quantum mechanics to a pencil balancing on its point is in the text on quantum mechanics by Dicke and Wittke [1]. From the context and the problem statement, the correct answer for the time limit would utilize the Heisenberg uncertainty principle (henceforth denoted HUP). There have been a few publications and many discussions analyzing this problem in chemistry and physics courses because it appears to present the possibility of a practical and interesting macroscopic observation of the quantum uncertainty principle. We analyze a model problem [2] and solve it fully in a small angle approximation to the time dependent Schrödinger equation (henceforth denoted TDSE). We conclude with Easton [3] and with Lynch [4] that quantum effects due to balance stability are not observable in this situation, and also in fact, we show that the application of the HUP can be easily misinterpreted. The double slit experiment

[1] seems to remain as the nearest contact with an observable, "hands-on", quantum phenomenon. This article is of primary interest to graduate and advanced undergraduate students in physics and chemistry.

The model "pencil" problem will be that of a point mass maintained at constant distance L from a fixed pivot point in space. Initially the mass is balanced vertically and is unstable to tipping over due to earth gravity. The question is: How long can one balance this pencil (or more accurately, inverted pendulum) on end? What limit does quantum mechanics play, if any, in the answer? We start with the TDSE written in spherical polar coordinates with the polar angle measured from the vertical axis and the origin at the fixed space point. We make the small angle approximation to the polar angle θ, namely $\sin(\theta) \approx \theta$ in radians, and assume azimuthal symmetry of the initial state. The polar angle θ is scaled with the fixed distance L into a new variable r, $r = L\theta$, and this remains as the only part of the kinetic energy operator once the azimuthal angle dependence is dropped. It is seen that r is the small-angle projection of the pencil vector onto the surface plane which is normal to the vertical gravity direction. The tangential component of the downward gravitational force which accelerates the constrained mass is approximately in the cylindrical radial direction for small angles, allowing us to write the TDSE:

$$(1) \qquad i\hbar\dot{\psi}(r,t) = -\frac{\hbar^2}{2M}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\psi(r,t) - \frac{1}{2}\frac{gM}{L}r^2\psi(r,t) \; ,$$

where g is the acceleration of gravity and M is the mass of the idealized pencil or of an inverted pendulum bob. The Laplacian is now in the form of the radial variable of a cylindrical coordinate system. The small angle approximation has allowed us to convert the two-dimensional (2D) motion of a particle constrained to the surface of a sphere into 2D motion in a plane. The assumption of azimuthal symmetry then leaves an equation solely in the spatial variable r. Eq.(1) will be solved shortly.

Note that the idealized model contains no real physical or chemical effects such as internal structure or temperature, gas collisions, pivot support binding, surface flatness, pencil point geometry, pivot strength, etc. It is certain that Dicke and Wittke did not intend their quantum mechanics exercise [1] to explore these perturbations. We must mention that the analysis of Easton [3] is based on an exploration of some of these real effects and obtains a result with a balance time depending on point geometry and quantum tunneling. Lynch's analysis [4] follows the spirit of the original problem of Dicke and Wittke and parallels ours. These realistic aspects are forced into importance as a consequence of the solution to the idealized model. Nevertheless, they will become irrelevant in our conclusion, which is that the degree of stability of this macroscopic object cannot be attributed to quantum mechanics.

There are many additional studies of the pendulum and its quantum properties and balance phenomena. None address the question as to whether a limit is imposed by quantum mechanics in the case of a real pencil. E. U. Condon [5] gave the general quantum solution in 1928. However the time independent solution is not easily used to investigate the stability of the inverted quantum pendulum. An interesting application of quantum pendulum stability is in the study of hindered molecular rotation [6]. An investigation of classical and quantum stability is presented by Cook and Zaidins [7]. Their results differ only by a small amount from ours, but the question of observation of a quantum limit is not addressed.

Occasionally the stability of a balanced pencil is investigated from the view of a “rod”, which is a pencil with a flattened point. This is realistic, but not the case implied by Dicke and Wittke[1] . Studies here include Shegelski, Lundeberg, and Goodvin [8] and the application to the pencil case by Easton [3]. The quantum solution can also be obtained by path integral procedures [9], which give corresponding results.

The Classical Motion of the Idealized Pencil

One should be familiar with the classical behavior of this object before seeking a reason for the introduction of quantum mechanics. In this section we quickly develop a classical model for the motion. The approximation leading to the TDSE for the metastable pencil given in Eq.(1) leads in a similar fashion to the approximate classical Newtonian equations for the radial motion:

(2) $$\ddot{r} = \omega^2 r\,, \quad \omega^2 = g/L \quad .$$

This has the general solution:

(3) $$r(t) = r(0)\cosh(\omega t) + (\dot{r}(0)/\omega)\sinh(\omega t) \quad .$$

For arguments much greater than unity, the hyperbolic functions, $\sinh(x) = (\exp(x) - \exp(-x))/2$ and $\cosh(x) = (\exp(x) + \exp(-x))/2$, are approximately exponential and equal in size. These are the equations for the 2D harmonic oscillator with imaginary frequency. The neglect of angular momentum about the vertical axis is expected to play very little part in the overall stability. The complete solution for motion of the unrestricted pendulum is expressed in terms of elliptic integrals [4]. The small-angle approximation is the result of examining the singular region near the vertical orientation.

The observable quantity of interest is the “fall time” or how long the model remains upright. The exact classical time of fall could be found from the pendulum solutions [4]. However, experience with any given real pencil reveals that the fall time is but a fraction of a second from the time that the pencil begins noticeable motion from vertical. Thus, calculations of stability will depend on the analysis near the vertical point where the small angle approximation is quite appropriate. Without loss of generality, we define the fall time for the ideal pencil model to occur when the radial position evolves to be nearly the pencil length,

(4) $$r(t_{fall}) = L\,.$$

We set $g = 10\,m/s^2$, $L = 0.1m$, $M = 0.01kg$, giving $\omega = 10\,s^{-1}$ for our "pencil". From Eq.(3) we solve for the fall time noting that ωt_{fall} is considerably larger than unity allowing us to approximate the hyperbolic functions as simpler exponentials.

Assume that the initial verticality, or width as measured from vertical, $r(0) = w_o$, is fixed with zero velocity at 1mm. Then we find the time for fall is

$$t_{fall} = \frac{1}{\omega}\ln\left(2L/w_o\right) = 0.1\ \ln(200) = 0.53\,s\,.$$

If the verticality is considerably more precise, say 0.01 mm, the result is increased slightly,

$$t_{fall} = 0.1\ \ln(20000) = 0.99\,s\quad.$$

Each decrease by a factor of ten in w_o adds to t_{fall} by 0.23 s, the approximate value of $0.1\ln(10)$. One ***nanometer*** of precision would only approach two seconds of balance time! This shows how difficult it is to stand a pencil-like object on end according to classical mechanics. In possession of this simple fact, one begins to suspect that quantum effects might be observable, but only if it were possible to prepare the initial alignment. At least two researchers have cautioned of this situation [3], [4]. The real chemical and physical effects mentioned earlier that are neglected in the ideal model are certain to become important as dimensions approach and fall below atomic size, which can either increase or decrease the stability.

Now the posited problematic question [1] is: how long can one expect a real, but idealized, pencil or pendulum to balance? Clearly this must be a quantum effect because the classical limit is ideally unbounded even though practicality is seriously in question because of the exceedingly precise conditions necessary for a fall time greater than the order of seconds.

Use of the Heisenberg Uncertainty Principle for Quantum Behavior of the Pencil

The simplest way to introduce quantum mechanics into pencil stability is to utilize the HUP, as has been done in all previous analyses and implied in the original exercise by Dicke and Wittke [1]. As mentioned, it is especially interesting in that it presents the possibility of observing true quantum behavior in an everyday situation. Its one defect, however, is that it allows the casual analyst to neglect the spatial implications. The HUP for a 1D Cartesian coordinate is $\Delta x\,\Delta p > \hbar/2$ where $\hbar \approx 10^{-34}\,J\,s$. Looking ahead, we rewrite this for our pendulum bob (ideal pencil model) in polar coordinates, as

$$\Delta r\,\Delta p > \hbar\ . \tag{5}$$

The initial position and momentum preparations, $\Delta r = r(0) = w_o$ and $\Delta p = \dot{r}(0)M = \dot{w}_o\,M$, are set in Eq.(5). The solution for radial motion, Eq.(3), is now used to find the fall time for these initial conditions. The hyperbolic functions can be replaced by the single dominant exponential because of the large argument. We now have the two relations from which to find the maximum of t_{fall}:

$$\begin{aligned} 2L &= \exp(\omega t_{fall})\,(w_0 + (\dot{w}_0/\omega)) \\ w_0\,\dot{w}_o &> \hbar/M \end{aligned}\ . \tag{6}$$

Examining the conditions for the maximum of t_{fall} subject to variation in w_0 and $\dot{w}_0$ with the HUP constraint gives the result:

$$\begin{aligned} w_o &= \sqrt{\hbar/M\omega}\ , \\ t_{fall} &= \frac{1}{2\omega}\ln\left(L^2 M\omega/\hbar\right) \end{aligned}\ . \tag{7}$$

Our specifications, i.e. $\omega = 10\,s^{-1}$, $L = 0.1\,m$, $M = 0.01\,kg$, will be used to evaluate the fall time. The fall time is a very reasonable 3.57 s. This is very encouraging goal for the inquisitive student hoping to observe a real sign of quantum mechanics. However, there is a serious flaw: the initial alignment value of $r(0) = w_o$ computes to be $w_o \approx \sqrt{\hbar/M\omega} = \sqrt{10}\,10^{-17}\,m = 3.16\times10^{-7}$ Å, less than a millionth of an Ångstrom! If one evaluates the limit on the time of balance without checking the initial alignment required to achieve this result, one has been negligent indeed. We explore this curious result by means of a full solution to the TDSE and then discuss the answer.

Full Solution of the Quantum Problem

The application of the HUP to the classical solution is an adequate treatment of quantum effects. However, there is the sense that something may be missing. For that reason, we do a full quantum solution to the small-angle ideal model. An appropriate particular solution of the TDSE can be found by making an exponential substitution for the quantum mechanical amplitude using the exponential form of the ground state of a harmonic oscillator:

$$\psi(r,t) = A(t)\exp(-r^2\,B(t))\ . \tag{8}$$

This solution allows the Gaussian form of a minimum uncertainty wave packet [1] for the prepared initial state of width w_o. The choice of this form for the initial state is optimal for stabilizing the pencil as it minimizes quantum spread. Substituting Eq.(8) into Eq.(1), we develop the solution to be:

(9)
$$A(t)=\sqrt{2/\pi w_o^2}\Big/D(t) \quad , \quad B(t)=-\frac{iM}{2\hbar}\frac{\dot{D}(t)}{D(t)} \quad ,$$
$$\ddot{D}(t)=\omega^2 D(t)\,, \quad D(t)=\cosh(\omega t)+i\,\gamma\sinh(\omega t) \quad ,$$
$$\omega=\sqrt{g/L} \quad , \quad \gamma=2\hbar/M\,\omega\,w_o^2 \quad .$$

It is instructional to know that *B(t)* can be expressed as

(10)
$$B(t)=\frac{1}{w_o^2}\left(1-i\,\gamma^{-1}\tanh(\omega t)\right)/\left(1+i\gamma\tanh(\omega t)\right) \quad ,$$

for evaluation of the $\hbar\to 0$ limit, which implies $\gamma\to 0$ also. Performing the mathematical limit allows us to obtain the classical equations of motion of the quantum distribution but of course does not change the proper description of the system. It should be noted that the approximation of $\hbar\to 0$ is precisely the same in the solution as $M\to\infty$, which is physically more reasonable because $\hbar$ is a constant of nature whereas *M* is truly large.

The probability distribution of the full quantum solution can be found from Eq.(9), introducing the time dependent width *w(t)* :

(11)
$$P(r,t)=\psi^*\psi=\left(2/\pi\,w^2(t)\right)\,\exp\left(-2r^2/w^2(t)\right) \quad ,$$
$$w^2(t)/w_0^2=\cosh^2(\omega t)+\gamma^2\sinh^2(\omega t)=\;D^*(t)\,D(t) \quad ,$$
$$B(t)^*+B(t)=2/w^2(t), \quad w_o=w(0) \quad .$$

The normalization has been incorporated so that $2\pi\int_0^\infty r\,dr\,\psi^*\psi=1$. The probability displays all the necessary information about the stability of the pencil. The width shows that quantum effects are present if $\gamma=2\hbar/M\,\omega\,w_o^2$ is order unity or greater. This requirement will be analyzed after the next section. The dependence of the time-dependent width on the initial width may be deduced from Eq.(11). If the system is classical $\gamma\approx 0$, the time-dependent width $w(t)$ is linear in the initial width w_0. Quantum mechanics produces an inverse dependence of $w(t)$ on w_0 because the initially specified position constraint forces a large momentum subsequent spread.

What we are looking for is a more detailed description of the role that quantum mechanics plays in limiting the stability of the pencil. We have already found that the HUP says quantum effects are not observable. This is the final answer, but a more robust analysis is desired. Schrodinger's equation describes all the information about the system, whether it exhibits quantum or classical behavior. The easiest way to begin this analysis is to re-examine the classical result using the TDSE. We let $\hbar \to 0$ in the TDSE solution and examine the resulting classical predictions. The initial conditions for the probability are obtained from Eq.(11) :

(12) $$P(r,0) = \left(2/\pi w_o^2\right)\exp\left(-2r^2/w_o^2\right) \quad .$$

The classical limit as $\gamma \to 0$ (either viewing it as $\hbar \to 0$ or $M \to \infty$) for all time is obtained from Eq.(9) and Eq.(11) with $P(r,t)$ given by the same expression as in Eq.(11) in terms of the new $w_{CL}(t)$:

(13) $$w_{CL}(t) = w_o \cosh(\omega t), \quad A(t) = \sqrt{2/\pi\, w_{CL}^2(t)} \ ,$$
$$B(t) \to \frac{1}{w_{CL}^2(t)}\left(1 - \frac{i}{\gamma}\sinh(\omega t)\cosh(\omega t)\right) \ .$$

The rate of fall of the classical system is given by the first member of Eq.(13); of course it is independent of mass. The classical probability distribution differs from the exact quantum probability in Eq.(11) by a term of order $\hbar^2$ in the width of the packet. No quantum effects are observable if $\gamma \ll O(1)$. The exponential phase of the wave function, which is the imaginary part of *B(t)*, still contains γ which scales with $\hbar$ as it becomes small. This is an essential singularity that occurs in the classical limit of wave mechanics in the $\psi = A\exp(iS/\hbar)$ decomposition where the classical limit leads to Hamilton-Jacobi equations, not directly to Newtonian equations of motion. This is discussed in Albert Messiah's classic text Quantum Mechanics [10]. Our initial classical state, as given by the probability of Eq.(12), is a distribution, an ensemble of independent, non-interacting particles each obeying the Newtonian equations of motion [10]. It is this distribution that evolves in time. Messiah's real *A* and *S* functions in the classical limit are given in our notation for our problem as:

$$A_{Messiah}(r,t) = \sqrt{2/\pi\, w_{CL}^2(t)}\ \exp\left(-r^2/w_{CL}^2(t)\right) \quad ,$$
$$S_{Messiah}(r,t) = \tfrac{1}{2} r^2\, M\, \omega \tanh(\omega t) \quad .$$

In this approach one could approximate the motion in the small angle limit giving the result for each classical particle moving in the radial direction in the ensemble as given in Eqs.(2) and (3). As we mention in the next section, relating this to the HUP, $\Delta x\, \Delta p \geq \hbar/2$ with $r(0) = \Delta x$ and $\dot{r}(0) = \Delta p / M$, will lead to a conclusion similar to the earlier study [4]. Radial coordinates are distinct in this comparison, so factors of 2 may be different.

Discussion

We return to the exact solution Eq.(11) for our model and ask for the maximum of t_{fall} subject to variation in w_o, which is the only free parameter that can be adjusted within the context of our model. After introducing $w(t_{fall}) = L$ as a practical definition of the fall time and solving for $dt_{fall} / dw_0 = 0$ to find the maximum, one can show that the maximum occurs precisely at

(14) $$\gamma = 2\hbar / M\omega w_o^2 = \coth(\omega t_{fall}) \quad .$$

This can be evaluated using $\gamma \approx 1$ because $\omega t_{fall} \gg 1$, which is the aforementioned point where quantum effects become important in the Eq.(11) solution. The maximum time may now be determined by introducing Eq.(14) into Eq.(11) and evaluated using the $\omega t_{fall} \gg 1$ approximation:

(15) $$t_{fall} = \frac{1}{2\omega} \operatorname{arcsinh}\left(L^2 M\omega/2\hbar\right) \approx \frac{1}{2\omega} \ln\left(L^2 M\omega/\hbar\right) = 3.57\ s \ .$$

This is exactly the same result as the use of the HUP in the classical solution. It is “exactly” so because we have adjusted the proportionality constant in the Heisenberg uncertainty relation by a bit. From the conditions used to derive Eq.(15) we find the initial alignment from Eq.(14) which produced that maximum time:

(16) $$w_o \approx \sqrt{2\hbar/M\omega} = \sqrt{20}\ 10^{-17} m = 4.47\ 10^{-7}\ \text{Å} \quad .$$

We have used the physical specifications of our ideal pencil. Eq.(16) poses a ***serious*** addendum to the use of the HUP to understanding the balance of a metastable macroscopic object.

The initial spatial precision given in Eq.(16) that is required to achieve the balance time given by Eq.(15) is of a sub-atomic size so extreme that it is barely achieved even in gravitational wave detection experiments [11]. In fact, with this demand on the initial alignment, there is no possibility that a real experimental test of the stability of a free-standing pencil or pencil-like macroscopic object is possible. Any observed stability or instability will be due to limitations in the perfection of the pivot point or in external perturbations. Atomic scale surface roughness, atomic scale point strength, gravity fluctuations due to moving nearby masses, etc. would intervene. These real effects, which themselves may be quantum mechanical because of chemical bonding or fracture, can either increase or decrease the pencil stability, totally obscuring the quantum behavior of the balanced pencil.

Conclusion

Quantum effects are not evident because the prerequisite initial verticality in our model, or in a real pencil, cannot be attained in any practical way. Any or all of neglected real effects will dominate the stability of the balanced pencil before any quantum limit on the fall time is observable. Atomic force and scanning tunneling microscopy experiments routinely achieve sub-atomic spatial control with suspended macroscopic tools with an atomically sharp tip [12]. However, such a tip cannot withstand the free-standing weight of a macroscopic object (~10g). Thus, even an atomic force microscope could not prepare a suspended pencil and release it to satisfy the initial condition for testing quantum effects on stability.

What we add to the exercise is a full quantum mechanical solution of the idealized system illustrating the transition between quantum and classical mechanics. The solution is approximate but the nature of the approximation does not affect the classical-quantum comparison. We have not discussed the interplay of quantum evolution and observation [13], but that is not the issue. The study of Lynch [4] does an approximation to the elliptic integral solution of the classical pendulum equation which itself is valid for the full range of angular motion. His combination of an approximation to this solution with the Heisenberg uncertainty relation gives much the same results as our study.

Solving for the maximum time for balance using the HUP without examining the spatial implications gives one the impression that quantum mechanics could be the limiting factor in the stability of an upright pencil. This is not the case. It would appear to be an oversight in the assigned exercise [1]. Classical mechanics describes the pencil unless the verticality is ideally prepared to be less than one millionth of an Ångstrom. In this case the fall time becomes the order of three or four seconds. Such an experiment is nigh impossible to devise due to surface roughness and fracture, and the precision and chemical strength required of the pivot. Consequently, quantum mechanics plays no part in the stability of a pencil on end. Perhaps an experiment based on atomically smooth fields could be devised to display the HUP with a free-standing macroscopic mass, but that is yet to be seen.

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