# MEAN CURVATURE FLOW WITH GENERIC LOW-ENTROPY INITIAL DATA

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**Abstract**

*We prove that sufficiently low-entropy closed hypersurfaces can be perturbed so that their mean curvature flow encounters only spherical and cylindrical singularities. Our theorem applies to all closed surfaces in $\mathbb{R}^3$ with entropy at most 2 and to all closed hypersurfaces in $\mathbb{R}^4$ with entropy at most $\lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. When combined with recent work of Daniels and Holgate, this strengthens Bernstein and Wang's low-entropy Schoenflies-type theorem by relaxing the entropy bound to $\lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. Our techniques, based on a novel density drop argument, also lead to a new proof of generic regularity result for area-minimizing hypersurfaces in eight dimensions (due to Hardt, Simon, and Smale).*

## 1. Introduction

Mean curvature flow is the natural heat equation for submanifolds. A family of hypersurfaces $M(t) \subset \mathbb{R}^{n+1}$ flows by mean curvature flow if

$$\left(\frac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \mathbf{H}_{M(t)}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{H}_{M(t)}(\mathbf{x})$ denotes the mean curvature vector of $M(t)$ at $\mathbf{x}$. When $M(0)$ is compact, mean curvature flow is guaranteed to become singular in finite time. Understanding the potential singularities is thus a fundamental problem. One approach to this issue is to study the flow in the generic case: a well-known conjecture of Huisken suggests that the singularities of a generic mean curvature flow should be as simple as possible, namely, spherical and cylindrical (see [49, #8]).

The main results of this note completely resolve Huisken's conjecture in three and four dimensions for low-entropy initial data (see (1.2) for the definition of entropy). Informally stated (see Corollaries 1.8 and 1.9 for precise statements), we prove the following results.

DUKE MATHEMATICAL JOURNAL
Advance publication—final volume, issue, and page numbers to be assigned.
© 2023 DOI 10.1215/00127094-2023-0034
Received 15 April 2022. Revision received 21 May 2023.
2020 *Mathematics Subject Classification*. Primary 53E10; Secondary 35B35.

THEOREM 1.1 (Low-entropy generic flow in $\mathbb{R}^3$, informal)
*If $M^2 \subset \mathbb{R}^3$ is a closed embedded surface with entropy $\lambda(M) \leq 2$, then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that the mean curvature flow starting from $M'$ has only multiplicity-one spherical and cylindrical singularities.*

THEOREM 1.2 (Low-entropy generic flow in $\mathbb{R}^4$, informal)
*If $M^3 \subset \mathbb{R}^4$ is a closed embedded hypersurface with entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that the mean curvature flow starting from $M'$ has only multiplicity-one spherical and cylindrical singularities.*

In an earlier version of this paper, we conjectured that Theorem 1.2 could be combined with a surgery construction to yield a strengthened version of Bernstein and Wang's low-entropy Schoenflies theorem (see [13]; cf. Theorem 1.4 below). This surgery construction has been recently carried out by Daniels and Holgate [35] who showed that if a mean curvature flow has only spherical and neckpinch singularities, then one can construct a mean curvature flow with surgery. As such, combining these results leads to the following.

COROLLARY 1.3 (Strengthened low-entropy Schoenflies-type theorem)
*If $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, then $M$ is smoothly isotopic to the round $\mathbb{S}^3$.*

See Sections 1.2 and 1.4 for an expanded discussion of this result.

### *1.1. Previous work on generic mean curvature flow*

Trailblazing work of Colding and Minicozzi [32] demonstrated that spheres and cylinders are the only *linearly stable* singularity models for mean curvature flow. In particular, the remaining singularity models are unstable so they should not generically occur (as conjectured by Huisken). In a previous paper [25], the authors introduced new methods to the study of generic mean curvature flow, proving that a large class of singularity models (specifically, singularities with tangent flows modeled on multiplicity-one compact or asymptotically conical self-shrinkers) can be indeed avoided by a slight perturbation of the initial conditions.

In particular, our previous work shows that for a generic initial surface in $\mathbb{R}^3$, either the mean curvature flow has only spherical and cylindrical singularities or at the first singular time it has a tangent flow with a cylindrical end or higher multiplicity (both possibilities are conjectured not to happen). We refer the reader to the introduction to our previous article [25] for further discussion of generic mean curvature flows and related work.

*1.1.1. Relationship between this paper and our previous work*
In [25], we proved a classification of ancient one-sided flows (analogous to the minimal surface results of Hardt and Simon [37]; see Appendix D for further discussion) which led to a complete understanding of flows on either side of a neighborhood of a nongeneric (compact or asymptotically conical) singularity. In particular, we showed that nearby flows to either side do not have such singularities nearby.

In $\mathbb{R}^3$, to understand generic mean curvature flow *without* a low-entropy condition (in contrast with this note), one must work at the first nongeneric time rather than globally in space-time. However, two serious issues arise when working this way. First, there is no partial regularity known for tangent flows past the first singular time without a low-entropy bound.[1] Second, there is the possibility that a small perturbation of the initial data may increase the first singular time slightly without improving the flow in an effective way. To that end, in [25] we had to additionally prove that the nearby flows strictly decrease genus as they avoid the nongeneric singularity. This genus-loss property is crucial for tackling Huisken's conjecture in $\mathbb{R}^3$ without a low-entropy condition and is a consequence of the classification of ancient one-sided flows, as obtained in [25].

On the other hand, by including a low-entropy condition, here we are able to work *globally* in space-time. This allows for significantly simplified arguments. In fact, the key observation of this paper is that in this setting one can completely avoid the classification of one-sided ancient flows and instead rely on a soft argument based on compactness and a new geometric property of nongeneric shrinkers (see Proposition 2.2). We emphasize that a drawback of the methods used in this note as compared to our previous work is that the arguments used here give no indication as to the local dynamics near a nongeneric singularity (such information was obtained in [25] near asymptotically conical and compact shrinkers; see also [29], [34]).

*Remark*
After the first version of this paper (as well as our previous paper [25]) was posted, another approach to the generic perturbation of the initial data was pursued by Sun and Xue (see [62], [63]). This approach is in the spirit of local ODE dynamics, as suggested by the Colding–Minicozzi program (cf. [34]). The analytic framework in [62] and [63] has the interesting feature that non-one-sided perturbations are analyzed, but the applications are currently limited to locally perturbing away singularities that

[1] At the first singular time, work of Ilmanen [48] and Wang [65] show that the support of any tangent flow is a smooth self-shrinker with only conical/cylindrical ends.

arise at the first singular time. Conversely, our geometric approach (first developed in [25]) is motivated by global results such as the ones stated in Theorems 1.1 and 1.2. Of course, our approach also admits localizations (see Appendix C).

*1.2. Entropy*

To state our main results, we first recall Colding and Minicozzi's definition in [32] of entropy of $M^n \subset \mathbb{R}^{n+1}$:

$$\lambda(M) := \sup_{\substack{\mathbf{x}_0 \in \mathbb{R}^{n+1} \\ t_0 > 0}} \int_M (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{1}{4t_0}|\mathbf{x}-\mathbf{x}_0|^2}. \tag{1.2}$$

By Huisken's monotonicity of Gaussian area, we see that $t \mapsto \lambda(M(t))$ is nonincreasing when $M(t)$ is flowing by mean curvature flow. A computation of Stone [61] shows that the entropies of the self-shrinking cylinders $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ satisfy[2]

$$2 > \lambda(\mathbb{S}^1) = \sqrt{\frac{2\pi}{e}} \approx 1.52 > \frac{3}{2} > \lambda(\mathbb{S}^2) = \frac{4}{e} \approx 1.47 > \cdots > \lambda(\mathbb{S}^n).$$

Several fundamental results have been obtained about hypersurfaces with sufficiently small entropy, starting with work of Colding, Ilmanen, Minicozzi, and White [31], who proved that the round sphere $\mathbb{S}^n(\sqrt{2n})$ has minimal entropy among all closed self-shrinkers. This was extended by Bernstein and Wang [9] who showed that the round sphere minimizes entropy among all closed hypersurfaces (see also [40], [70]). Moreover, Bernstein and Wang [10] have also proved that the cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R} \subset \mathbb{R}^3$ has the second least entropy among all self-shrinkers in $\mathbb{R}^3$ (their result crucially relies on Brendle's classification of genus-0 self-shrinkers in [18]).

Subsequent work of Bernstein and Wang provides a robust picture of hypersurfaces with sufficiently small entropy (see [11], [12], [14]; see also [15]). In particular, they obtained the following low-entropy Schoenflies result.

THEOREM 1.4 (Bernstein and Wang [13])
*If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, then $M$ is smoothly isotopic to the round $\mathbb{S}^3$.*

In [13], this is proved by flowing $M$ by mean curvature flow and then smoothing out any potential nongeneric singularities to construct the desired isotopy. Our previous work [25] on generic mean curvature flow gave an alternative approach to this result by showing that if one perturbs $M$ slightly, the mean curvature flow directly *provides* the isotopy.

[2]Note that $\lambda(\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}) = \lambda(\mathbb{S}^k)$.

THEOREM 1.5 ([25])
*If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \le \lambda(\mathbb{S}^2 \times \mathbb{R})$, then after a small $C^\infty$-perturbation to a nearby hypersurface $M'$, the mean curvature flow $M'(t)$ is completely smooth until it disappears in a round point.*

One of the consequences of this paper is a simplified proof of Theorem 1.5 (see also the stronger version stated in Corollary 1.3).

### 1.3. Main results

We now describe our main results in full generality. We construct generic mean curvature flows of sufficiently low-entropy hypersurfaces in all dimensions. To quantify the low-entropy condition, we provide several definitions.[3] Let $\mathcal{S}_n$ denote the set of smooth self-shrinkers in $\mathbb{R}^{n+1}$ with $\lambda(\Sigma) < \infty$, that is, properly embedded hypersurfaces $\Sigma$ satisfying $\mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0$ with finite Gaussian area. Let $\mathcal{S}_n^*$ denote the nonflat elements of $\mathcal{S}_n$. For $\Lambda > 0$, let

$$\mathcal{S}_n(\Lambda) := \left\{\Sigma \in \mathcal{S}_n : \lambda(\Sigma) < \Lambda\right\}, \qquad \mathcal{S}_n^*(\Lambda) := \mathcal{S}_n(\Lambda) \cap \mathcal{S}_n^*.$$

We also define

$$\mathcal{S}_n^{\mathrm{gen}} := \left\{O\left(\mathbb{S}^j(\sqrt{2j}) \times \mathbb{R}^{n-j}\right) \in \mathcal{S}_n : j = 1, \ldots, k,\ O \in O(n+1)\right\}$$

to be the set of (round) self-shrinking spheres and cylinders in $\mathbb{R}^{n+1}$.

Similarly, we let $\mathcal{RMC}_n$ denote the space of regular minimal cones in $\mathbb{R}^{n+1}$, that is, the set of $\mathcal{C} \subset \mathbb{R}^{n+1}$ with $\mathcal{C} \setminus \{\mathbf{0}\}$ a smooth properly embedded hypersurface invariant under dilations and having vanishing mean curvature. Let $\mathcal{RMC}_n^*$ denote the nonflat elements of $\mathcal{RMC}_n$. Define

$$\mathcal{RMC}_n(\Lambda) := \left\{\mathcal{C} \in \mathcal{RMC}_n : \lambda(\mathcal{C}) < \Lambda\right\}, \quad \mathcal{RMC}_n^*(\Lambda) := \mathcal{RMC}_n(\Lambda) \cap \mathcal{RMC}_n^*.$$

For a dimension $n \ge 2$ and entropy bound $\Lambda \in (\lambda(\mathbb{S}^n), 2]$, our first hypothesis is

$$\text{For } 3 \le k \le n,\ \mathcal{RMC}_k^*(\Lambda) = \emptyset \tag{$\dagger_{n,\Lambda}$}$$

while our second hypothesis is

$$\mathcal{S}_{n-1}^*(\Lambda) \subset \mathcal{S}_{n-1}^{\mathrm{gen}}. \tag{$\dagger\dagger_{n,\Lambda}$}$$

Finally, we define certain notation that will be used throughout.

[3] The definitions here are closely related to the hypotheses $(\star_{n,\Lambda})$, $(\star\star_{n,\Lambda})$ introduced by Bernstein and Wang (cf. [12], [13]), but our second hypothesis is less restrictive.

*Definition 1.6*
For a closed embedded hypersurface $M^n \subset \mathbb{R}^{n+1}$ we denote by $\mathfrak{F}(M)$ the set of cyclic[4] unit-regular integral Brakke flows $\mathcal{M}$ with $\mathcal{M}(0) = \mathcal{H}^n \lfloor M$, and for each $\mathcal{M} \in \mathfrak{F}(M)$, we define $\operatorname{sing}_{\text{gen}} \mathcal{M} \subset \operatorname{sing} \mathcal{M}$ to be the set of singular points $(\mathbf{x}, t)$ so that some[5] tangent flow to $\mathcal{M}$ at $(\mathbf{x}, t)$ is a multiplicity-one flow associated to elements of $\mathcal{S}_n^{\text{gen}}$.

Having given these definitions, we can now state our main technical result. By convention, we take $\lambda(\mathbb{S}^0) = 2$. Everywhere below, $M$ is taken to be closed and embedded.

THEOREM 1.7
*Assume that $n \geq 2$ and that $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2})]$ satisfy hypotheses $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$. If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \Lambda$, then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that $\lambda(M') < \Lambda$ and all $\mathcal{M}' \in \mathfrak{F}(M')$ have* $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\text{gen}} \mathcal{M}'$. *In particular, the level set flow of $M'$ does not fatten.*

See [25, Section 1.2] for a discussion of results related to the regularity of flows satisfying $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\text{gen}} \mathcal{M}'$.

In low dimensions, the hypotheses $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ can be understood more concretely. This leads to the following results.

COROLLARY 1.8
*If $M^2 \subset \mathbb{R}^3$ has $\lambda(M) \leq 2$, then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that the level-set flow of $M'$ is nonfattening and the associated Brakke flow $\mathcal{M}' \in \mathfrak{F}(M')$ has* $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\text{gen}} \mathcal{M}'$.

*Proof*
Condition $(\dagger_{2,2})$ is vacuous while $(\dagger\dagger_{2,2})$ holds by the classification of self-shrinking curves (see [1]). ☐

COROLLARY 1.9
*If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that the level-set flow of $M'$ is nonfattening and the associated Brakke flow $\mathcal{M}' \in \mathfrak{F}(M')$ has* $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\text{gen}} \mathcal{M}'$.

[4] Recall that an integral varifold $V$ is *cyclic* if the unique mod 2 flat chain $[V]$ has $\partial[V] = 0$. Work of White [69] shows that this property is preserved under varifold (and Brakke flow) convergence.
[5] Note that if some tangent flow is a multiplicity-one element of $\mathcal{S}_n^{\text{gen}}$, then all are by [30] and [33] (cf. [8]).

*Proof*

By the resolution of the Willmore conjecture (see [53]), $\mathcal{RMC}_3^*(\Lambda_{\mathcal{C}}) = \emptyset$ for

$$\Lambda_{\mathcal{C}} = \frac{2\pi^2}{4\pi} \approx 1.57 > \lambda(\mathbb{S}^1) \approx 1.52.$$

Thus, $(\dagger_{3,\Lambda})$ holds for all $\Lambda \leq \Lambda_{\mathcal{C}}$. Furthermore, by the classification of low-entropy shrinkers in $\mathbb{R}^3$ from [10], it holds that $\mathcal{S}_2^*(\lambda(\mathbb{S}^1)) = \mathcal{S}_2^{\mathrm{gen}}$. Thus, $(\dagger\dagger_{3,\lambda(\mathbb{S}^1)})$ holds. □

*1.4. Generic mean curvature flow with surgery*

As already observed in [25], we can apply Corollary 1.9 to give direct proofs of Theorems 1.4 and 1.5. Moreover, Daniels and Holgate recently proved that if an initial hypersurface admits a (cyclic, unit-regular, integral) Brakke flow with only[6] spherical- and neckpinch-type singularities[7] then it is possible to construct a smooth mean curvature flow with surgery starting from this initial condition (see [35] for the precise definition of mean curvature flow with surgery).

As such, Corollaries 1.8 and 1.9 combined with [35, Theorem 1.2] yield the following generic surgery construction.

COROLLARY 1.10 (Generic mean curvature flow with surgery)
*Assume that $n \geq 2$ and that $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2})]$ satisfy $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$. If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \Lambda$, then there is an arbitrarily small $C^\infty$ graph $M'$ over $M$ and a smooth mean curvature flow with surgery starting from $M'$.*

In particular, when $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, the mean curvature flow with surgery can be used (see [35, Theorem 6.4]) to construct an isotopy to the round 3-sphere. This yields the strengthened version of the low-entropy Schoenflies theorem stated in Corollary 1.3.

*Remark*

In the setting of 2-convex mean curvature flow with surgery (see [3], [4], [17], [19]–[21], [38], [39], [44]), the surgery to isotopy construction has been studied in several works (see [22], [23], [45], [54], [55]). (We also mention related work using Ricci flow with surgery in [24] and [52] and singular Ricci flow in [5]–[7].)

[6]The spherical and neckpinch singularities are the tangent flows for which a canonical neighborhood theorem is proved, thanks to [27] and [28].

[7]Note that if $\mathcal{M}'$ is such a Brakke flow in $\mathbb{R}^{n+1}$ and $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\mathrm{gen}} \mathcal{M}'$, then the condition "$\mathcal{M}'$ has only spherical and neckpinch singularities" is a consequence of $\lambda(\mathcal{M}') < \lambda(\mathbb{S}^{n-2})$.

*1.5. Generic regularity of area-minimizing hypersurfaces in eight dimensions*

We remark that the study of generic mean curvature flow in our previous work [25] can be viewed as the parabolic analogue of the work of Hardt and Simon [37] and Smale [60] concerning the generic regularity of area-minimizing hypersurfaces in eight dimensions. In particular, the existence and uniqueness of the ancient one-sided mean curvature flow (see [25]) is a direct analogue of the existence and uniqueness of the foliation on either side of a regular area-minimizing cone, as proved in [37] (see also [67]).

In this paper, we develop a new technique based on density drop that avoids the classification of the ancient one-sided flow. As one might expect, this also yields a new proof of the generic regularity results of Hardt and Simon [37] and Smale [60] that avoids the need to classify the foliation. This is discussed further in Appendix D.

*1.6. Organization*

See [25, Section 2] for the conventions used in this paper. In Section 2, we prove entropy drop near nongeneric singularities and we use this to prove Theorem 1.7 in Section 3. Appendices A and B recall some standard stability results. Appendix C contains a localized perturbative result. In Appendix D, we discuss how the arguments here relate to generic regularity of area-minimizing hypersurfaces in eight dimensions.

## 2. Entropy drop near nongeneric singularities

LEMMA 2.1

*Assume that* $(\dagger_{n,\Lambda})$ *holds for some* $\Lambda \le 2$. *Suppose that* $V$ *is an* $F$*-stationary cyclic integral* $n$*-varifold in* $\mathbb{R}^{n+1}$ *satisfying* $F(V) < \Lambda$. *Then there is* $\Sigma \in \mathcal{S}_n(\Lambda)$ *so that* $V = \mathcal{H}^n \lfloor \Sigma$.

*Proof*

This follows from the proofs of [12, Lemma 3.1 and Proposition 3.2] except that the cyclic property of $V$ is used to rule out three half-spaces as a potential iterated tangent cone (cf. [69, Corollary 4.5]). □

Recall that Huisken (see [42], [43]) has classified the cylinders $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ as the unique smooth embedded self-shrinkers with nonnegative mean curvature $H \ge 0$ (the technical assumption of bounded curvature was later removed by Colding and Minicozzi [32]). The following result can be viewed as a geometric consequence of Huisken's result. It will serve as our key mechanism for perturbing away "nongeneric" singularities.

PROPOSITION 2.2
*For* $\Sigma \in \mathcal{S}_n^*$, *fix an open set* $\Omega \subset \mathbb{R}^{n+1}$ *with* $\Sigma = \partial\Omega$. *Assume that there is a space-time point* $(\mathbf{x}_0, t_0) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$ *so that*

$$\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0 \subset \sqrt{-t}\,\bar{\Omega} \tag{2.1}$$

*for all* $t < \min\{0, t_0\}$. *Then one of the following holds:*
(1) $\Sigma = \mathbb{S}^n(\sqrt{2n})$, *or*
(2) $\Sigma = O(\hat{\Sigma} \times \mathbb{R})$ *for* $\hat{\Sigma} \in \mathcal{S}_{n-1}^*$ *and* $O \in O(n+1)$.

Note that if we replaced condition (2.1) with

$$\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0 \subset \sqrt{-t}\,\Omega \tag{2.2}$$

(i.e., if we replaced the closure of $\Omega$ with the interior of $\Omega$), we could use an inductive argument to conclude that $\Sigma \in \mathcal{S}_n^{\mathrm{gen}}$.

Let us give the geometric intuition underlying our proof strategy. Let $\mathcal{M}_0$ denote the space-time track of $t \mapsto \sqrt{-t}\,\Sigma$, and let $\mathcal{M}$ denote the space-time track of $t \mapsto \sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0$. For $\lambda \in (0, 1]$, let $\mathcal{M}_\lambda$ be the parabolic rescaling of $\mathcal{M}$ by a factor of $\lambda$; thus, $\mathcal{M}_1 = \mathcal{M}$ and, as $\lambda \to 0$, $\mathcal{M}_\lambda \to \mathcal{M}_0$ smoothly locally away from $(\mathbf{0}, 0)$. Note that $\mathcal{M}_0$ is invariant under parabolic dilations, so $\mathcal{M}_\lambda$ always lies weakly to one side of $\mathcal{M}_0$.

If $\mathcal{M}_\lambda$ touches $\mathcal{M}_0$ for some $\lambda > 0$ (equivalently, for all $\lambda > 0$ due to $\mathcal{M}_0$'s parabolic dilation invariance), it is then a simple consequence of the strong maximum principle and monotonicity that $\Sigma$ splits a line.

Otherwise, $\mathcal{M}_\lambda$ was disjoint from $\mathcal{M}_0$ for all $\lambda \in (0, 1]$. It is then standard to use the height of $\mathcal{M}_\lambda$ over $\mathcal{M}_0$ at time $t = -1$, for $\lambda > 0$ small, to produce a kernel element of the linearized operator that is everywhere nonnegative ($\mathcal{M}_\lambda$ always lies weakly to one side of $\mathcal{M}_0$). By studying the geometry of parabolic dilations, the kernel element produced is $\mathbf{x}_0 \cdot \nu_\Sigma$ if $\mathbf{x}_0 \neq \mathbf{0}$ or $\mathbf{x} \cdot \nu_\Sigma$ if $\mathbf{x}_0 = \mathbf{0}$ ($\implies t_0 \neq 0$). It turns out that the former case implies splitting once again, while the latter implies the mean-convexity of $\Sigma$.

The proof we give below is a more succinct version of the argument above: it handles both cases in a unified way.

*Proof of Proposition 2.2*
Observe that the set $\cup_{t<0} \sqrt{-t}\,\bar{\Omega} \times \{t\}$ is invariant under parabolic dilation around the space-time origin. We thus conclude that for all $\lambda \in [0, \infty)$ and $t < \min\{0, \lambda^2 t_0\}$,

$$\sqrt{\lambda^2 t_0 - t}\,\Sigma + \lambda \mathbf{x}_0 \subset \sqrt{-t}\,\bar{\Omega}.$$

In particular, taking $t = -1$ and $\lambda \geq 0$ small, we have that

$$\lambda \mapsto \Sigma_\lambda := \sqrt{1+\lambda^2 t_0}\,\Sigma + \lambda \mathbf{x}_0 \subset \bar{\Omega}$$

is a 1-parameter family of hypersurfaces with $\Sigma_0 = \Sigma = \partial\Omega$. The normal speed at $\lambda = 0$ is $\mathbf{x}_0 \cdot \nu_\Sigma \geq 0$ (where $\nu_\Sigma$ is the unit normal pointing into $\Omega$). Because (cf. [32, Theorem 5.2])

$$\Delta_\Sigma(\mathbf{x}_0 \cdot \nu_\Sigma) - \frac{1}{2}\mathbf{x} \cdot \nabla_\Sigma(\mathbf{x}_0 \cdot \nu_\Sigma) + |A_\Sigma|^2(\mathbf{x}_0 \cdot \nu_\Sigma) = 0,$$

the maximum principle implies that either $\mathbf{x}_0 \cdot \nu_\Sigma > 0$ along $\Sigma$ or $\mathbf{x}_0 \cdot \nu_\Sigma = 0$ along $\Sigma$. (Note that $\Sigma$ is connected thanks to the Frankel property of shrinkers; cf. [25, Corollary C.4].)

In the first case (i.e., $\mathbf{x}_0 \cdot \nu_\Sigma > 0$), each component of $\Sigma$ is a graph over the $\mathbf{x}_0^\perp$-hyperplane. By [64] (cf. [36]), each component of $\Sigma$ must be a hyperplane, so there is only one component and $\Sigma$ is a flat hyperplane. This contradicts the assumption that $\Sigma \in \mathcal{S}_n^*$ (the set of nonflat shrinkers).

In the second case (i.e., $\mathbf{x}_0 \cdot \nu_\Sigma = 0$), we see that $\mathbf{x}_0 \in T_p\Sigma$ for all $p \in \Sigma$. In particular, if $\mathbf{x}_0 \neq \mathbf{0}$, then $\Sigma$ splits a line in the $\mathbf{x}_0$-direction. It thus remains to consider the situation in which $\mathbf{x}_0 = \mathbf{0}$. If this is the case, then it must hold that $t_0 \neq 0$ and we have

$$\tilde{\Sigma}_\mu := (1+\mu t_0)\Sigma \subset \bar{\Omega}$$

for $\mu \geq 0$ sufficiently small. The normal speed at $\mu = 0$ is $t_0 \mathbf{x} \cdot \nu_\Sigma \geq 0$. Using the shrinker equation, we thus find that $t_0 H_\Sigma \geq 0$. Since $t_0 \neq 0$, we can assume that $H_\Sigma \geq 0$. Thus, up to a rotation, $\Sigma = \mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for $k = 1, \ldots, n$ by [32, Theorem 10.1]. This completes the proof. □

Recall the definition of smoothly crossing Brakke flows in Definition B.1.

PROPOSITION 2.3

*Fix $n \geq 2$, $\varepsilon > 0$, and $\Lambda \in (\lambda(\mathbb{S}^n), 2]$ so that $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ hold. There is $\delta = \delta(n, \varepsilon, \Lambda) > 0$ with the following property.*

*Consider $\Sigma \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\mathrm{gen}}$ and $\tilde{\mathcal{M}}$ an ancient cyclic unit-regular integral $n$-dimensional Brakke flow in $\mathbb{R}^{n+1}$ with $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$ so that $\tilde{\mathcal{M}}$ does not smoothly cross the flow $(-\infty, 0) \ni t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\,\Sigma$. Then $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}, t) \leq F(\Sigma) - \delta$ for all $(\mathbf{x}, t) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$.*

*Proof*

We argue by contradiction. Consider a sequence of $\Sigma_i \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\mathrm{gen}}$ and $\mathcal{M}_i$ ancient cyclic unit-regular integral Brakke flows in $\mathbb{R}^{n+1}$ with $\lambda(\tilde{\mathcal{M}}_i) \leq F(\Sigma)$ so that

$\tilde{\mathcal{M}}_i$ does not smoothly cross the flow $(-\infty, 0) \ni t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma_i$ and so that there are points $(\mathbf{x}_i, t_i) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$ with

$$\Theta_{\tilde{\mathcal{M}}_i}(\mathbf{x}_i, t_i) \geq F(\Sigma_i) - o(1) \tag{2.3}$$

as $i \to \infty$. We can assume that $|(\mathbf{x}_i, t_i)| = 1$.

By Lemma 2.1 and Allard's theorem (see [2], [57]), we can pass to a subsequence so that $\Sigma_i$ converges in $C^\infty_{\text{loc}}$ to $\Sigma \in \mathcal{S}_n(\Lambda)$. By Brakke's theorem (see [16], [68]), $\Sigma$ is nonflat. Because cylinders are isolated in $C^\infty_{\text{loc}}$ by [30], we thus see that $\Sigma \in \mathcal{S}^*_n(\Lambda) \setminus \mathcal{S}^{\text{gen}}_n$. Note that $F(\Sigma_i) \to F(\Sigma)$.

We now pass to a further subsequence so that $(\mathbf{x}_i, t_i) \to (\mathbf{x}_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ with $|(\mathbf{x}_0, t_0)| = 1$ and the Brakke flows $\tilde{\mathcal{M}}_i$ converge to an ancient cyclic unit-regular integral Brakke flow $\tilde{\mathcal{M}}$ with $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$. By upper semicontinuity of Gaussian density, (2.3) implies that $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}_0, t_0) \geq F(\Sigma)$. Because $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$, $\tilde{\mathcal{M}}$ is a self-similar flow around $(\mathbf{x}_0, t_0)$. By stability of smoothly crossing flows, Lemma B.2, $\tilde{\mathcal{M}}$ does not smoothly cross $(-\infty, 0) \ni t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma$.

Consider any tangent flow to $\tilde{\mathcal{M}}$ at $t = -\infty$. By Huisken's monotonicity formula and Lemma 2.1, there is a smooth shrinker $\tilde{\Sigma}$ so that this tangent flow at $t = -\infty$ corresponds to some $\tilde{\Sigma} \in \mathcal{S}_n(\Lambda)$ with multiplicity one. By the Frankel property for self-shrinkers (cf. [25, Corollary C.4]) and the strong maximum principle, if $\tilde{\Sigma} \neq \Sigma$, then the flows $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma$ and $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \tilde{\Sigma}$ smoothly cross each other at some point. This contradicts the stability of smooth crossings.

We conclude that any tangent flow to $\tilde{\mathcal{M}}$ at $t = -\infty$ is the flow associated to $\Sigma$. Since $\tilde{\mathcal{M}}$ is self-similar around $(\mathbf{x}_0, t_0)$, we find that

$$\tilde{\mathcal{M}}(t) = \mathcal{H}^n \lfloor (\sqrt{t_0 - t}\, \Sigma + \mathbf{x}_0)$$

for $t < t_0$. Since $\tilde{\mathcal{M}}$ does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma$, we see that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial\Omega = \Sigma$ so that

$$\sqrt{t_0 - t}\, \Sigma + \mathbf{x}_0 \subset \sqrt{-t}\, \bar{\Omega}$$

for $t < \min\{0, t_0\}$. We can thus apply Proposition 2.2 to conclude that (up to a rotation) $\Sigma = \hat{\Sigma} \times \mathbb{R}$ for $\hat{\Sigma} \in \mathcal{S}^*_{n-1}(\Lambda)$. By hypothesis $(\dagger\dagger_{n,\Lambda})$, $\hat{\Sigma} \in \mathcal{S}^{\text{gen}}_{n-1}$ so $\Sigma = \hat{\Sigma} \times \mathbb{R} \in \mathcal{S}^{\text{gen}}_n$. This is a contradiction. □

## 3. Proof of Theorem 1.7

For $M' \subset \mathbb{R}^{n+1}$ a smooth closed hypersurface, recall that $\mathfrak{F}(M')$ is the set of cyclic unit-regular integral Brakke flows $\mathcal{M}'$ with $\mathcal{M}'(0) = \mathcal{H}^n \lfloor M'$. Note that [46] and [69] imply that $\mathfrak{F}(M') \neq \emptyset$ (see also [41, Appendix B]).

We define

$$\mathcal{D}(M') := \sup\{\Theta_{\mathcal{M}'}(\mathbf{x},t) : \mathcal{M}' \in \mathfrak{F}(M'), (\mathbf{x},t) \in \operatorname{sing}\mathcal{M}' \setminus \operatorname{sing}_{\text{gen}}\mathcal{M}'\}.$$

Recall that by convention $\sup \emptyset = -\infty$.

Assume that hypotheses $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ hold for $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2}]$ fixed. Consider a smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with $\lambda(M) \leq \Lambda$. Flowing $M$ by mean curvature flow for a short time strictly decreases the entropy unless $M$ is homothetic to a self-shrinker. If $M$ is homothetic to a self-shrinker other than $\mathbb{S}^n(\sqrt{2n})$, then by [32], a small $C^\infty$-perturbation of $M$ has strictly smaller entropy.

As such, either $M = \mathbb{S}^n(r)$, in which case the Theorem 1.7 trivially holds, or we can perform an initial perturbation and assume that $\lambda(M) \leq \Lambda - 2\varepsilon$ for some $\varepsilon > 0$. Choose a foliation $\{M_s\}_{s\in(-1,1)}$ of a tubular neighborhood of $M$ so that $M_0 = M$ and so that $\lambda(M_s) \leq \Lambda - \varepsilon$. Fix $\delta = \delta(n, \varepsilon, \Lambda) > 0$ from Proposition 2.3.

LEMMA 3.1
*We have*

$$\limsup_{s\to s_0} \mathcal{D}(M_s) \leq \mathcal{D}(M_{s_0}) - \delta$$

*for all* $s_0 \in (-1, 1)$.

Lemma 3.1 implies Theorem 1.7 by a straightforward iteration argument since by Brakke's regularity theorem (see [16], [68]), if $\mathcal{D}(M') \leq 1$, then $\mathcal{D}(M') = -\infty$ implying that $\operatorname{sing}\mathcal{M}' = \operatorname{sing}_{\text{gen}}\mathcal{M}'$ for all $\mathcal{M}' \in \mathfrak{F}(M')$. Since $\lambda(M') < \Lambda \leq \lambda(\mathbb{S}^{n-2} \times \mathbb{R}^2)$, any $\mathcal{M}' \in \mathfrak{F}(M')$ has only (multiplicity-one) $\mathbb{S}^n$- and $(\mathbb{S}^{n-1} \times \mathbb{R})$-type singularities. Thus, the resolution of the mean convex neighborhood conjecture for $\mathbb{S}^{n-1} \times \mathbb{R}$ singularities (see [27], [28]; cf. [41]) implies nonfattening of the flow of $M'$.

*Proof of Lemma 3.1*
Assume that there is $s_i \to s_0 \in (-1, 1)$ with $s_i \neq s_0$ but

$$\lim_{i\to\infty} \mathcal{D}(M_{s_i}) > \mathcal{D}(M_{s_0}) - \delta.$$

Fix $\mathcal{M}_i \in \mathfrak{F}(M_{s_i})$ and $(\mathbf{x}_i, t_i) \in \operatorname{sing}\mathcal{M}_i \setminus \operatorname{sing}_{\text{gen}}\mathcal{M}_i$ with

$$\lim_{i\to\infty} \Theta_{\mathcal{M}_i}(\mathbf{x}_i, t_i) > \mathcal{D}(M_{s_0}) - \delta.$$

Pass to a subsequence $\mathcal{M}_i$ converging to $\mathcal{M} \in \mathfrak{F}(M_{s_0})$ and $(\mathbf{x}_i, t_i) \to (\mathbf{x}_0, t_0) \in \operatorname{sing}\mathcal{M}$. Since $s_i \neq s_0$ for all $i$, we have that $M_{s_i}$ is disjoint from $M_{s_0}$ for all $i$. In particular, $\operatorname{supp}\mathcal{M}_i \cap \operatorname{supp}\mathcal{M} = \emptyset$ (by the avoidance principle for Brakke flows; see [47, Section 10.6]). Thus, $(\mathbf{x}_i, t_i) \neq (\mathbf{x}_0, t_0)$.

Observe that that if $(\mathbf{x}_0, t_0) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}$, then since $\lambda(M) < \Lambda \leq \lambda(\mathbb{S}^{n-2})$, we see that $(\mathbf{x}_0, t_0)$ must be an $\mathbb{S}^n$- or $(\mathbb{S}^{n-1} \times \mathbb{R})$-type singularity. Proposition A.1 then implies that $(\mathbf{x}_i, t_i) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}_i$, which is a contradiction. Thus, it must hold that $(\mathbf{x}_0, t_0) \in \operatorname{sing} \mathcal{M} \setminus \operatorname{sing}_{\mathrm{gen}} \mathcal{M}$.

Translate $(\mathbf{x}_0, t_0)$ to the space-time origin and parabolically dilate to yield $\tilde{\mathcal{M}}_i$ and $(\tilde{\mathbf{x}}_i, \tilde{t}_i)$ with $|(\tilde{\mathbf{x}}_i, \tilde{t}_i)| = 1$ and

$$\lim_{i \to \infty} \Theta_{\tilde{\mathcal{M}}_i}(\tilde{\mathbf{x}}_i, \tilde{t}_i) > \mathcal{D}(M_{s_0}) - \delta.$$

Pass to a subsequence so that $\tilde{\mathcal{M}}_i \rightharpoonup \tilde{\mathcal{M}}$ and $(\tilde{\mathbf{x}}_i, \tilde{t}_i) \to (\tilde{\mathbf{x}}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$. By upper semicontinuity of density,

$$\Theta_{\tilde{\mathcal{M}}}(\tilde{\mathbf{x}}, \tilde{t}) > \mathcal{D}(M_{s_0}) - \delta. \tag{3.1}$$

On the other hand, we can perform the same translation and parabolic dilation to $\mathcal{M}$ and by extracting a further subsequence, the resulting flows converge to a tangent flow to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$. By Lemma 2.1, the tangent flow is the multiplicity-one flow associated to a smooth shrinker $\Sigma$. Note that

$$F(\Sigma) \leq \lambda(\mathcal{M}) \leq \limsup_{s \to s_0} \lambda(M_s) \leq \Lambda - \varepsilon.$$

Since $(\mathbf{x}_0, t_0) \in \operatorname{sing} \mathcal{M} \setminus \operatorname{sing}_{\mathrm{gen}} \mathcal{M}$, it must hold that $\Sigma \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\mathrm{gen}}$. Huisken's monotonicity formula implies that $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma) = \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0)$ (cf. the proof of Proposition 10.6 in [25]). Finally, since the supports of $\mathcal{M}$ and $\mathcal{M}_i$ are disjoint, $\mathcal{M}_i$ does not smoothly cross $\mathcal{M}$. As such (using Lemma B.2), $\tilde{\mathcal{M}}$ does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\,\Sigma$. We can now apply Proposition 2.3 to conclude that

$$\Theta_{\tilde{\mathcal{M}}}(\tilde{\mathbf{x}}, \tilde{t}) \leq F(\Sigma) - \delta = \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - \delta \leq \mathcal{D}(M_{s_0}) - \delta.$$

This contradicts (3.1), completing the proof. ☐

## Appendix A. Stability of generic singularities

Based on [28], the following stability of generic singularities was proved in [56, Proposition 2.3] (see [25, Lemma 10.4] for the simple argument when the singularity is modeled on $\mathbb{S}^n$). When $n = 2$, this also follows via density considerations using [10].

PROPOSITION A.1

*Suppose that $\mathcal{M}_i \rightharpoonup \mathcal{M}$ are unit-regular integral Brakke flows in $\mathbb{R}^{n+1}$ and that $(\mathbf{x}_i, t_i) \in \operatorname{sing} \mathcal{M}_i$ converge to $(\mathbf{0}, 0) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}$. If the singularity at $(\mathbf{0}, 0)$ is modeled on $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$, then for $i$ sufficiently large $(\mathbf{x}_i, t_i) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}_i$.*

## Appendix B. Stability of crossing points

*Definition B.1*
Given two integral unit Brakke flows $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, we say that $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ *smoothly cross* at $(\mathbf{x}, t)$ if there is $r > 0$ with

$$\mathcal{M}^{(j)}(s) \lfloor B_r(\mathbf{x}) = \mathcal{H}^n \lfloor \Gamma^{(j)}(s)$$

for $s \in (t - r^2, t + r^2)$, where $\Gamma^{(j)}(s)$ are smooth connected mean curvature flows so that in any small neighborhood of $\mathbf{x}$ there are points of $\Gamma^{(1)}(0)$ on both sides of $\Gamma^{(2)}(0)$.

The following is a straightforward consequence of Brakke's regularity theorem (see [16], [68]).

LEMMA B.2
*For $j = 1, 2$, suppose that $\mathcal{M}_i^{(j)} \rightharpoonup \mathcal{M}^{(j)}$ are integral unit-regular $n$-dimensional Brakke flows in $\mathbb{R}^{n+1}$. Assume that $\mathcal{M}^{(1)}$ smoothly crosses $\mathcal{M}^{(2)}$ at $(\mathbf{x}, t)$. Then, for $i$ sufficiently large, there is $(\mathbf{x}_i, t_i) \to (\mathbf{x}, t)$ so that $\mathcal{M}_i^{(1)}$ smoothly crosses $\mathcal{M}_i^{(2)}$ at $(\mathbf{x}_i, t_i)$.*

## Appendix C. Local results

In this appendix, we prove the following local perturbative result.

PROPOSITION C.1
*Suppose that $M^n \subset \mathbb{R}^{n+1}$ is a closed embedded hypersurface, and suppose that $\mathcal{M} \in \mathcal{F}(M)$ is a cyclic unit-regular integral Brakke flow starting at $M$. Assume that for $(\mathbf{x}_0, t_0) \in \operatorname{sing} \mathcal{M}$, the following hold:*

- *$\operatorname{reg} \mathcal{M} \cap \{t < t_0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is connected, and*
- *any tangent flow $\mathcal{N}$ to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$ has $\mathcal{N}(-1) = \mathcal{H}^n \lfloor \Sigma$, for $\Sigma \in \mathcal{S}_n^* \setminus \mathcal{S}_n^{\mathrm{gen}}$ that does not split a line.*

*Then there is $r = r(\mathcal{M}, \mathbf{x}_0, t_0) > 0$ so that for $M_j = \operatorname{graph}_M(u_j)$, $u_j > 0$ with $u_j \to 0$ in $C^\infty$, it holds that any*

$$(\mathbf{x}, t) \in B_r(\mathbf{x}_0) \times (t_0 - r^2, t_0 + r^2)$$

*has $\Theta_{\mathcal{M}_j}(\mathbf{x}, t) \leq \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - r$ for $j$ sufficiently large.*

In particular, no tangent flow to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$ can arise as the tangent flow to $\mathcal{M}_j$ at some point in $B_r(\mathbf{x}_0) \times (t_0 - r^2, t_0 + r^2)$, for $j$ large.

*Proof*
If this failed, there is $(\mathbf{x}_j, t_j) \to (\mathbf{x}_0, t_0)$ with

$$\Theta_{\mathcal{M}_j}(\mathbf{x}, t) \geq \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - o(1).$$

The assumption on the connectedness of the regular set implies that $\mathcal{M}_j \lfloor \{t < t_0\} \rightharpoonup \mathcal{M} \lfloor \{t < t_0\}$. Thus, by rescaling around $(\mathbf{x}_0, t_0)$ so that $(\mathbf{x}_j, t_j)$ is scaled to a unit distance from $(\mathbf{0}, 0)$, we obtain $\Sigma \in \mathcal{S}_n^* \setminus \mathcal{S}_n^{\mathrm{gen}}$ that does not split a line and an ancient Brakke flow $\tilde{\mathcal{M}}$ that does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma$, so that $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$, but for some $(\tilde{\mathbf{x}}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus \{(\mathbf{0}, 0)\}$ it holds that $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}, t) \geq F(\Sigma)$.

The argument in the second half of the proof of Proposition 2.3 carries over without change to show that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial \Omega = \Sigma$ and

$$\sqrt{\tilde{t} - t}\, \Sigma + \tilde{\mathbf{x}} \subset \sqrt{-t}\, \overline{\Omega}$$

for $t < \min\{0, t_0\}$. By Proposition 2.2, we have that either $\Sigma = \mathbb{S}^n(\sqrt{2n}) \in \mathcal{S}_n^{\mathrm{gen}}$ or $\Sigma$ splits a line. Either case contradicts the assumption that $\mathcal{M}$ has no such tangent flow at $(\mathbf{x}_0, t_0)$. This completes the proof. $\square$

Note that Proposition C.1 does not give any indication as to *how* the perturbation avoids the singularity (the trade-off is that the proof is very short). On the other hand, the results in [25] give a rather *complete* description of how the perturbed flow avoids a compact/asymptotically conical singularity. The works [62] and [63] also obtain some information along these lines, but only as long as the perturbed flow remains graphical over the original flow.

## Appendix D. The setting of area-minimizing hypersurfaces

We recall the following fundamental result.

THEOREM D.1 (Hardt and Simon [37, Theorem 2.1])
*If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a regular area-minimizing cone, then there exist smooth area-minimizing hypersurfaces $S_\pm$ in each component of $\mathbb{R}^{n+1} \setminus \mathcal{C} = U_+ \cup U_-$ so that if $S'$ is area-minimizing and contained in $U_\pm$, then $S' = \lambda S_\pm$.*

The uniqueness statement in Theorem D.1 implies smoothness of solution to the Plateau problem for seven-dimensional currents in $\mathbb{R}^8$ with generic boundary data (see [37, Theorem 5.6]). Later, Smale [60] used Theorem D.1 to prove that for $(M^8, g)$ a closed Riemannian manifold and $\alpha \in H_7(M; \mathbb{Z})$, there is a $C^k$-close metric $g'$ so that the least area representative of $\alpha$ is smooth.

*Remark*

Besides their role in generic regularity of area-minimizing hypersurfaces in eight-dimensional manifolds, the surfaces $S_\pm$ are important objects in their own right (cf. [26], [50], [51], [58], [59], [66]). In our previous paper [25], we proved the parabolic analogue of Theorem D.1 (for compact/asymptotically conical self-shrinkers) by constructing and classifying ancient one-sided flows analogous to the surfaces $S_\pm$.

We explain here how the main idea of this note can be used to prove the generic regularity results from [37] and [60] using the following result in lieu of Theorem D.1 (cf. Proposition 2.3).

PROPOSITION D.2

*There is $\delta > 0$ with the following property. Suppose that $\mathcal{C}^7 \subset \mathbb{R}^8$ is a nonflat area-minimizing cone. If $S'$ is area-minimizing with support contained in $\bar{U}_\pm$, where $\mathbb{R}^8 \setminus \mathcal{C} = U_+ \cup U_-$, then*

$$\Theta_{S'}(\mathbf{x}) \leq \Theta_{\mathcal{C}}(0) - \delta.$$

*Proof*

Using smooth compactness of the links of area-minimizing cones in $\mathbb{R}^8$, it suffices to rule out the case where $S' \subset \bar{U}_\pm$ is area-minimizing and there is $|\mathbf{x}_0| = 1$ so that

$$\Theta_{S'}(\mathbf{x}_0) = \Theta_{\mathcal{C}}(\mathbf{0}).$$

Because $S'$ is contained in $\bar{U}_\pm$, its tangent cone at $\infty$ must be $\mathcal{C}$ (e.g., using the Frankel property of minimal hypersurfaces in $\mathbb{S}^n$). Thus, $S' = \mathcal{C} + \mathbf{x}_0$. This implies that $\mathcal{C} + \lambda \mathbf{x}_0 \subset \bar{U}_\pm$ as $\lambda \to 0$, so $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} \geq 0$. It cannot hold that $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} = 0$ since $\mathcal{C}$ does not split a line, so $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} > 0$. This would imply that $\mathcal{C}$ is a graph, which is impossible since $\mathcal{C}$ is nonflat. □

Using this, we obtain the following density drop result (cf. Lemma 3.1).

COROLLARY D.3

*There is $\delta > 0$ with the following property. Suppose that $\Sigma = \partial[\Omega] \subset B_2 \subset \mathbb{R}^8$ is an area-minimizing boundary with $\operatorname{sing}\Sigma = \{\mathbf{0}\}$. Suppose that $\Omega_1, \Omega_2, \dots \supset \Omega$ is a sequence of sets of finite perimeter in $B_2$ with $\Sigma_i := \partial[\Omega_i]$ area-minimizing, $\Sigma_i \cap \Sigma = \emptyset$, and $\Omega_i \to \Omega$. Then, for $\mathbf{x}_i \in \Sigma_i \cap B_1$, we have*

$$\limsup_{i \to \infty} \Theta_{\Sigma_i}(\mathbf{x}_i) \leq \Theta_\Sigma(\mathbf{0}) - \delta.$$

Note that this result can be iterated exactly in the proof of Theorem 1.7 to obtain generic regularity of area-minimizing hypersurfaces in eight dimensions.

COROLLARY D.4 (cf. [37, Theorem 5.6])
*For $\Gamma^6 \subset \mathbb{R}^8$ a smooth compact oriented submanifold without boundary, there is an arbitrarily small $C^\infty$-perturbation of $\Gamma$ to $\Gamma'$ so that any area-minimizing integral current bounded by $\Gamma'$ is completely smooth.*

COROLLARY D.5 (cf. [60])
*For $(M^8, g)$ a closed oriented Riemannian manifold and $\alpha \in H_7(M;\mathbb{Z})$ a codimension-one integral homology class, there is an arbitrarily small $C^k$-perturbation of $g$ to $g'$ so that there is a unique $g'$-area-minimizing representative $\Sigma$ of $\alpha$ and $\Sigma$ is completely smooth.*

*Acknowledgments.* We would like to thank Richard Bamler for some discussions related to weak flows and surgery constructions. We are also grateful to the referees for many helpful suggestions.

Chodosh's work was partially supported by a Sloan Fellowship, a Terman Fellowship, and National Science Foundation (NSF) grants DMS-1811059 and DMS-2016403. Choi's work was partially supported by Korea Institute for Advanced Study (KIAS) Individual Grant MG078901. Mantoulidis's work was partially supported by NSF grants DMS-2050120 and DMS-2147521. Schulze's work was partially supported by Leverhulme Trust Research Project Grant RPG-2016-174.

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