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VOLUME GROWTH OF 3-MANIFOLDS WITH SCALAR CURVATURE LOWER BOUNDS

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Abstract. We give a new proof of a recent result of Munteanu–Wang relating scalar curvature to volume growth on a 3-manifold with non-negative Ricci curvature. Our proof relies on the theory of μ-bubbles introduced by Gromov [Geom. Funct. Anal. 28 (2018), pp. 645–726] as well as the almost splitting theorem due to Cheeger–Colding [Ann. of Math. (2) 144 (1996), pp. 189–237].

1. Introduction

In this note we give a new proof of (and slightly generalize) the following volume growth estimate recently proven by Munteanu–Wang [MW22, Theorem 5.6].

Theorem 1.1. *Let (M^3, g) be a complete non-compact 3-manifold with $\mathrm{Ric}_g \geq 0$. Then*

$$\liminf_{d(x_0,x)\to\infty} R_g(x) \leq C(x_0, M, g) < +\infty \tag{1.1}$$

for all $x_0 \in M$. Moreover, if

$$R_g(x) \geq d(x, x_0)^{-\alpha} \tag{1.2}$$

outside a compact set K for some $x_0 \in M$ and $0 \leq \alpha < 2$, then

$$\mathrm{Vol}(B_r(x_0)) \leq C(x_0, M, g) r^{1+\alpha} \tag{1.3}$$

for all $r > 0$.

(We note that Theorem 1.1 here considers the optimal range[1] of α while [MW22, Theorem 5.6] only proves Theorem 1.1 for $0 \leq \alpha \leq 1$.[2]) The proof of Theorem 1.1 given by Munteanu–Wang is based on their analysis of certain harmonic functions on such manifolds (see also their earlier work [MW23] as well as [CL21]). Our proof is rather different and instead relies on the theory of μ-bubbles introduced by Gromov [Gro18].

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[1] When $\alpha \geq 2$, any estimate from the scalar curvature inequality (1.2) would be weaker than the Bishop–Gromov cubic volume growth estimate from non-negative Ricci; moreover, by (1.1) it is not possible that (1.2) holds with $\alpha < 0$.

[2] On the other hand, we point out that [MW22, Theorem 5.6] does consider a certain notion of negativity of the Ricci curvature at infinity, so in that respect [MW22, Theorem 5.6] is more general than Theorem 1.1. Moreover, [MW22, Theorem 5.6] yields a more explicit estimate for the constant $C(x_0, M, g)$ than we do here.

The techniques used here are inspired by our recent article on (non-compact) stable minimal hypersurfaces in 4-manifolds [CLS22] (see also [CL23]). We note that in this paper it is necessary to handle the possibility that $\partial B_r(x_0) \subset M$ may have many connected components (even when (M, g) has only one end). (In [CLS22] this issue was avoided since all that was needed was an efficient cutoff function.) Here we use the almost splitting theorem of Cheeger–Colding to show that even if there are many components of $B_r(x_0)$ they do not contribute too much to the volume growth.

We note that Theorem 1.1 is related to well-known conjectures of Yau [Yau92] and Gromov [Gro86]. Yau has conjectured that if (M^n, g) has $\mathrm{Ric}_g \geq 0$, then $\int_{B_r(x_0)} R_g \leq Cr^{n-2}$ for all $r > 0$ while Gromov has conjectured that if (M, g) satisfies $\mathrm{Ric}_g \geq 0$ and $R_g(x) \geq 1$ then $\mathrm{Vol}(B_r(x_0)) \leq Cr^{n-2}$. For some works related to these conjectures we refer to [Pet08, Nab20, Xu20, Zhu22].[3]

2. Proof of main result

By using the splitting theorem [CG72], it is easy to see that Theorem 1.1 holds for complete 3-manifolds with $\mathrm{Ric}_g \geq 0$ and two (or *a priori* more) ends, so it suffices to handle the case when M has one end.

A key tool is the following result which is a consequence of the theory of μ-bubbles due to Gromov [Gro18]. See, for example, [CLS22, Lemma 5.3] (with references to [CL20]) for a proof.

Lemma 2.1 (μ-bubble diameter bound)**.** *Let (N^3, g) be a 3-manifold with boundary satisfying $R_g \geq 1$. Then there are universal constants $L > 0$ and $c > 0$ such that if there is a $p \in N$ with $d_N(p, \partial N) > L/2$, then there is an open set $\Omega \subset B_{L/2}(\partial N) \cap N$ and a smooth surface Σ^2 such that $\partial\Omega = \Sigma \sqcup \partial N$ and each component of Σ has diameter at most c.*

Let L and c denote the constants from Lemma 2.1. Note that we are free to make L larger, so we will assume $L \gg c$. The following is the main geometric result used in the proof of Theorem 1.1.

Lemma 2.2. *Let (M^3, g) be a complete 3-manifold with $\mathrm{Ric}_g \geq 0$ and one end. Let $x \in M$. There is an $r_0(x, M, g) > 0$ and a universal constant $C > 0$ so that if*

$$R_g \,|_{\overline{B}_{r+a_1+a_2}(x) \setminus B_r(x)} \geq 1$$

for some $r \geq r_0$ and some $a_1, a_2 \in [L, 2L]$, then

$$\mathrm{Vol}(\overline{E}_1 \setminus E_2) \leq C,$$

where E_1 and E_2 are the unique unbounded components of $M \setminus \overline{B}_{r+a_1}(x)$ and $M \setminus \overline{B}_{r+a_1+a_2}(x)$ respectively.

Proof. Let E_0 be the unique unbounded component of $M \setminus \overline{B}_r(x)$, and E_1, E_2 as in the statement. By [And90, Corollary 1.5] (or [SY82, Liu13]), we have $b_1(M) < \infty$. Then by [CLS22, Proposition 3.2], ∂E_k are connected so long as $r \geq r_0$ for $r_0(x, M, g) > 0$ fixed.

Let $\gamma : \mathbb{R}_+ \to M$ be a geodesic ray associated with the unique end of M. For all $r > r_0$, $\gamma \cap \partial B_r(x)$ lies on the boundary of the unique unbounded component

[3] Added in proof: there has recently been related work in these directions, see [WXZZ22] and [ZZ23].

of $M \setminus B_r(x)$. We apply Lemma 2.1 to E_k for $k = 0, 1$. As in the proof of [CLS22, Lemma 5.4], we obtain a connected surface Σ_k in $B_{L/2}(\partial E_k) \cap E_k$ with $\operatorname{diam}(\Sigma_k) \le c$ that separates ∂E_k from the end of M. Choose $t_k \in \mathbb{R}_+$ with $\gamma(t_k) \in \Sigma_k$, for $k = 0, 1$. Note that $d_g(\gamma(t_0), \gamma(t_1)) \le a_1 + a_2 \le 4L$ so $d_g(x_0, x_1) \le 4L + 2c =: D$ for any $x_0 \in \Sigma_0$ and $x_1 \in \Sigma_1$.

We now define some constants. Let $b = 3c, A = 2\sqrt{b^2 + D^2}$. Fix $R \in \mathbb{R}$ such that

$$R \ge A + 4L, \quad \sqrt{b^2 + (2R + D)^2} + 1 < 2D + 2R. \tag{2.1}$$

Then take $\delta \in (0, 1)$ such that

$$\delta < \sqrt{b^2 + D^2}, \quad 14\delta + 6\sqrt{\delta(D + R)} < c, \quad 2c + 22\delta + 6\sqrt{\delta(D + R)} < \frac{L}{2}. \tag{2.2}$$

Note that all constants here are numerical (i.e., independent of (M, g) and r_0).

By the Cheeger–Colding almost splitting theorem [CC96, Theorem 6.62], assuming r_0 is sufficiently large depending on D, R, δ, there is a length space (Y, d) with

$$d_{GH}(B_{D+R}(\gamma(t_1)) \subset (M, d_g),\ B_{D+R}(y, 0) \subset (Y \times \mathbb{R}, d \times d_{\mathrm{Euc}})) < \delta.$$

(See Definition A.2 for the definition of the Gromov–Hausdorff distance.) Below we fix (Y, d) with this property.

Claim 1. $\operatorname{diam}(Y, d) \le b$.

Proof. By the definition of D, we have $\Sigma_0 \cup \Sigma_1 \subset B_{D+R}(\gamma(t_1))$. Since Σ_1 is connected and separating, and $B_{D+R}(\gamma(t_1)) \supset \Sigma_1$ is connected, $B_{D+R}(\gamma(t_1)) \setminus \Sigma_1$ has two components. Let Ω_i, $i = 1, 2$, denote the two components. Let

$$f : B_{D+R}(\gamma(t_1)) \to B_{D+R}(y, 0)$$

be a δ-Gromov–Hausdorff approximation (cf. Definition A.1) given by the almost splitting theorem. Then,

$$B_\delta(f(\Omega_1)) \cup B_\delta(f(\Sigma_1)) \cup B_\delta(f(\Omega_2))$$

covers $B_{D+R}(y, 0)$. Let

$$S := B_\delta(f(B_{2\delta}(\Sigma_1))) \subset B_{D+R}(y, 0),$$

and let

$$\Lambda_i := B_{D+R}(y, 0) \cap B_\delta(f(\Omega_i)) \setminus S.$$

Then

$$B_{D+R}(y, 0) = \Lambda_1 \cup S \cup \Lambda_2.$$

We first show an upper bound for the (extrinsic) diameter of S. Suppose $p,\ q \in S$. Then there are $p',\ q' \in B_{2\delta}(\Sigma_1)$ and $p'',\ q'' \in \Sigma_1$ so that

$$d(p, q) \le d(f(p'), f(q')) + 2\delta \le d(p', q') + 3\delta \le d(p'', q'') + 7\delta \le c + 7\delta.$$

Hence,

$$\operatorname{diam}(S) \le c + 7\delta.$$

Second, we show a lower bound for the diameter of Λ_i. Note that the length of a component of γ in Ω_i is at least $L/2$. Then we can take $p,\ q \in \Omega_i \setminus B_{5\delta}(\Sigma_1)$ with $d(p, q) \ge L/2 - 5\delta$. For any $x \in B_{2\delta}(\Sigma_1)$, we have

$$d(f(p), f(x)) \ge d(p, x) - \delta \ge d(p, \Sigma_1) - 3\delta \ge 2\delta$$

(and similarly for q). Thus there are $p', q' \in \Lambda_i$ satisfying

$$d(p', q') \geq d(f(p), f(q)) - 2\delta \geq d(p, q) - 3\delta \geq L/2 - 8\delta.$$

Hence,

$$\operatorname{diam}(\Lambda_i) \geq L/2 - 8\delta.$$

Third, we show that Λ_1 and Λ_2 are separated by a positive distance. Let $p \in \Lambda_1$ and $q \in \Lambda_2$. Then there are $p' \in \Omega_1 \setminus B_{2\delta}(\Sigma_1)$ and $q' \in \Omega_2 \setminus B_{2\delta}(\Sigma_1)$ satisfying (by the fact that Σ_1 is separating Ω_1 and Ω_2)

$$d(p, q) \geq d(f(p'), f(q')) - 2\delta \geq d(p', q') - 3\delta \geq \delta.$$

Hence,

$$d(\Lambda_1, \Lambda_2) \geq \delta.$$

Finally, we show that

$$\operatorname{diam}(Y, d) \leq 2c + 14\delta + 6\sqrt{\delta(D + R)}.$$

Suppose otherwise for contradiction. Then $c + 7\delta + 3\sqrt{\delta(D+R)} < \frac{1}{2}\operatorname{diam}(Y, d)$. By Proposition A.3 (using $9\delta \leq D + R$) and the diameter bound for S, we have

$$S \subset B_{c+7\delta+3\sqrt{\delta(D+R)}}(y, 0).$$

By Proposition A.4,

$$B_{D+R}(y, 0) \setminus B_{c+7\delta+3\sqrt{\delta(D+R)}}(y, 0)$$

is path connected and does not contain S. Hence, (without loss of generality)

$$\Lambda_2 \subset B_{c+7\delta+3\sqrt{\delta(D+R)}}(y, 0).$$

But then

$$\operatorname{diam}(\Lambda_2) \leq \operatorname{diam}(B_{c+7\delta+3\sqrt{\delta(D+R)}}(y, 0)) \leq 2c + 14\delta + 6\sqrt{\delta(D + R)}.$$

Thus (2.2) implies that

$$\operatorname{diam}(\Lambda_2) < L/2 - 8\delta,$$

which contradicts the diameter lower bound. □

Claim 2. $B_{D+R}(\gamma(t_1)) \subset B_{2\sqrt{b^2+D^2}}(\gamma\,|_{|t-t_1|\leq D+R})$.

Proof. Let σ denote the segment of γ in $B_{D+R}(\gamma(t_1))$. Note that $\operatorname{diam}(\sigma) = 2D + 2R$.

We first show that in $Y \times \mathbb{R}$, $B_{D+R}(y, 0) \subset B_{\sqrt{b^2+D^2}}(f(\sigma))$. Suppose for contradiction that there is a point $p \in B_{D+R}(y, 0)$ satisfying

$$d(p, f(\sigma)) > \sqrt{b^2 + D^2}.$$

Let π denote the projection to $\mathbb{R}$ in $Y \times \mathbb{R}$. Then

$$d_{\mathbb{R}}(\pi(p), \pi(f(\sigma))) > D.$$

Since $D > \delta$, f is a δ-approximation, and σ is connected, we have

$$\operatorname{diam}_{\mathbb{R}}(\pi(f(\sigma))) \leq 2R + D.$$

Then

$$\operatorname{diam}(f(\sigma)) \leq \sqrt{b^2 + (2R + D)^2}.$$

By (2.1) and $\delta < 1$, we have

$$\operatorname{diam}(\sigma) \le \sqrt{b^2 + (2R+D)^2} + \delta < 2D + 2R,$$

which yields a contradiction.

Take $z \in B_{D+R}(\gamma(t_1))$. By the above, there is an $z' \in \sigma$ with $d(f(z), f(z')) \le \sqrt{b^2 + D^2}$. Then

$$d(z,\sigma) \le d(z,z') \le d(f(z), f(z')) + \delta \le \sqrt{b^2 + D^2} + \delta.$$

The claim follows. □

Claim 3. $\overline{E}_1 \setminus E_2 \subset B_A(\gamma\,|_{|t-t_1|<D+R})$.

Proof. Let $x' \in \overline{E}_1 \setminus E_2$.

If $d(x', x) \le R + c + r$, then

$$\begin{aligned} d(x', \gamma(t_1)) &\le d(x', \Sigma_0) + \operatorname{diam}(\Sigma_0) + d(\gamma(t_0), \gamma(t_1)) \\ &\le (R + c) + c + 4L \\ &= R + D. \end{aligned}$$

Then by Claim 2, we have $d(x', \gamma\,|_{|t-t_1|<D+R}) \le A$.

Now, suppose for contradiction that $d(x', x) > R + c + r$. Take the radial geodesic μ from x to x', and let x'' be the point on μ with

$$d(x'', x) = R + c + r.$$

By the above observation (since $R > L$, we still have $x'' \in \overline{E}_1 \setminus E_2$), we have

$$d(x'', \gamma\,|_{|t-t_1|<D+R}) \le A.$$

However, since $\partial B_{r+a_1+a_2}(x)$ separates x'' from γ (by the definition of E_2), we have

$$d(x'', \gamma\,|_{|t-t_1|<D+R}) \ge R + c - a_1 - a_2 \ge R + c - 4L.$$

Since $R \ge A + 4L$, we reach a contradiction. □

By Claim 3, the diameter of $\overline{E}_1 \setminus E_2$ is bounded from above by $2A + 2D + 2R$ (which is bounded by a universal constant). Since $\operatorname{Ric}_g \ge 0$, the Bishop–Gromov inequality yields a universal constant C so that

$$\operatorname{Vol}(\overline{E}_1 \setminus E_2) \le C,$$

as desired. □

We can now prove the main result.

Proof of Theorem 1.1. We assume M has one end. Let $r_0(x_0, M, g)$ and C_1 be the constants in Lemma 2.2, where we assume $K \subset B_{r_0}(x_0)$.

First, assume (1.2). Take $r > 0$ very large (so that $r^{1-\alpha/2} > r_0$). Set $\tilde{g} := r^{-\alpha} g$. Then

$$R_{\tilde{g}} \ge 1 \quad \text{on} \quad B^{\tilde{g}}_{r^{1-\alpha/2}}(x_0).$$

Let $k \in \mathbb{N}$ so that $r_0 \le r^{1-\alpha/2} - kL < r_0 + L$. Set

$$r_i := r^{1-\alpha/2} - kL + iL.$$

Let E_i be the unique unbounded component of $M \setminus \overline{B}^{\tilde{g}}_{r_i}(x_0)$. By Lemma 2.2 (with $a_1 = a_2 = L$), we have

$$\operatorname{Vol}_{\tilde{g}}(\overline{E}_i \setminus E_{i+1}) \le C_1$$

for $1 \leq i \leq k-1$, so

$$\mathrm{Vol}_g(\overline{E}_i \setminus E_{i+1}) \leq C_1 r^{3\alpha/2}.$$

Note that

$$B_r^g(x_0) = B_{r^{1-\alpha/2}}^{\tilde{g}}(x_0) \subset (M \setminus E_1) \cup \bigcup_{i=1}^{k-1} (\overline{E}_i \setminus E_{i+1}).$$

Moreover, by the choice of k, there is a constant $V(M,g) > 0$ so that

$$\mathrm{Vol}_g(M \setminus E_1) \leq V.$$

Hence, we have (since $k \leq r^{1-\alpha/2}/L$)

$$\mathrm{Vol}_g(B_r(x_0)) \leq V + C_1 k r^{3\alpha/2} \leq V + \frac{C_1}{L} r^{1+\alpha}.$$

Then we have

$$\lim_{r\to\infty} \frac{1}{r^{1+\alpha}} \mathrm{Vol}_g(B_r(x_0)) < +\infty$$

and (since $\alpha < 2$)

$$\lim_{r\to 0} \frac{1}{r^{1+\alpha}} \mathrm{Vol}_g(B_r(x_0)) = 0,$$

so the conclusion holds.

Now, assume for contradiction that (1.1) fails at some $x_0 \in M$. Let

$$f_1(r) := \inf_{M\setminus B_r(x_0)} R_g.$$

By construction, f_1 is non-negative and increasing. By the contradiction assumption, we have

$$\lim_{r\to\infty} f_1(r) = +\infty.$$

Assuming r_0 sufficiently large, we have $f(r_0) \geq 1$. On $[r_0, \infty)$, we define a function f to be the largest nondecreasing, piecewise constant function taking the values $\{4^j f_1(r_0)\} \to \infty$ so that the preimage of each value has length at least 1 and $f \leq f_1$. Then f is non-negative, increasing, has

$$\lim_{r\to\infty} f(r) = +\infty,$$

and satisfies

$$f(r+1) \leq 4f(r) \quad \forall \quad r \geq r_0'.$$

Moreover, $R_g(x) \geq f_1(d(x_0,x)) \geq f(d(x_0,x))$. Starting with r_0, we inductively define

$$r_i := r_{i-1} + 2Lf(r_{i-1})^{-1/2}.$$

Suppose for contradiction that $r_i \leq N < +\infty$ for all i. Then

$$r_i - r_{i-1} = 2Lf(r_{i-1})^{-1/2} \geq 2Lf(N)^{-1/2} > 0,$$

which yields a contradiction. Hence, $r_i \to +\infty$. We also have,

$$r_i - r_{i-1} = 2Lf(r_{i-1})^{-1/2} \leq 1$$

(by assuming r_0 sufficiently large). Let E_i be the unique unbounded component of $M \setminus \overline{B}_{r_i}(x_0)$. Let $g_i := f(r_{i-1})g$. Then

$$R_{g_i} \geq 1 \quad \text{on} \quad M \setminus B_{r_{i-1} f(r_{i-1})^{1/2}}^{g_i}(x).$$

We first note that $r_{i-1} f(r_{i-1})^{1/2} \geq r_0$. Moreover, we have

$$d_{g_i}(\partial E_{i-1}, \partial E_i) = f(r_{i-1})^{1/2}(r_i - r_{i-1}) = 2Lf(r_{i-1})^{-1/2} f(r_{i-1})^{1/2} = 2L$$

and

$$d_{g_i}(\partial E_i, \partial E_{i+1}) = f(r_{i-1})^{1/2}(r_{i+1} - r_i) = 2Lf(r_i)^{-1/2}f(r_{i-1})^{1/2} \in [L, 2L]$$

because

$$\begin{aligned} 2L \geq 2Lf(r_i)^{-1/2}f(r_{i-1})^{1/2} &\geq 2Lf(r_i)^{-1/2}f(r_i - 1)^{1/2} \\ &\geq Lf(r_i)^{-1/2}f(r_i)^{1/2} = L. \end{aligned}$$

Then by Lemma 2.2 (with a_1 and a_2 the distances above), we have

$$\mathrm{Vol}_{g_i}(\overline{E}_i \setminus E_{i+1}) \leq C_1,$$

so

$$\mathrm{Vol}_g(\overline{E}_i \setminus E_{i+1}) \leq C_1 f(r_{i-1})^{-3/2}.$$

Then

$$\mathrm{Vol}_g(B_{r_k}(x_0)) \leq V + C_1 \sum_{i=1}^{k-1} f(r_{i-1})^{-3/2}.$$

Since $r_i - r_{i-1} = 2Lf(r_{i-1})^{-1/2}$, we have

$$\sum_{i=1}^{k-1} f(r_{i-1})^{-3/2} = \frac{1}{2L} \sum_{i=1}^{k-1} (r_i - r_{i-1}) f(r_{i-1})^{-1}.$$

Then

$$\lim_{k\to\infty} \frac{1}{r_k} \mathrm{Vol}_g(B_{r_k}(x_0)) \leq \frac{C_1}{2L} \lim_{k\to\infty} \frac{1}{r_k} \sum_{i=1}^{k} (r_i - r_{i-1}) f(r_{i-1})^{-1}.$$

Let $\varepsilon > 0$. Let $k \in \mathbb{N}$ sufficiently large so that $f(r_k)^{-1} < \varepsilon/2$. Let $l \in \mathbb{N}$ sufficiently large so that

$$f(r_0)^{-1} \frac{r_k - r_0}{r_l} < \varepsilon/2.$$

Then

$$\frac{1}{r_l} \sum_{i=1}^{l} (r_i - r_{i-1}) f(r_{i-1})^{-1} \leq f(r_0)^{-1} \frac{r_k - r_0}{r_l} + f(r_k)^{-1} < \varepsilon.$$

Hence, we have

$$\lim_{k\to\infty} \frac{1}{r_k} \mathrm{Vol}_g(B_{r_k}(x_0)) = 0,$$

which contradicts Yau's linear volume growth (cf. [SY94, Theorem 4.1]) since M is non-compact. □

3. Sharpness of main result

We provide an example to demonstrate the sharpness of the growth upper bounds in Theorem 1.1.

Consider on $[1, \infty) \times S^2$ the metric

$$g = dt^2 + \rho(t)^2 g_{S^2}^{\mathrm{round}}.$$

We glue a compact cap so that positive Ricci curvature is preserved, so it suffices to study the scalar curvature decay and volume growth on this end.

Let $X_i \in T_pS^2$ be an orthonormal basis with respect to g_{S^2}. By [Pet16, §4.2.3], we have

$$R_{(t,p)}(\rho(t)^{-1}X_i, \partial_t, \partial_t, \rho(t)^{-1}X_i) = -\rho''(t)/\rho(t)$$

and

$$R_{(t,p)}(\rho(t)^{-1}X_1, \rho(t)^{-1}X_2, \rho(t)^{-1}X_2, \rho(t)^{-1}X_1) = 1/\rho(t)^2.$$

Hence,

$$\begin{aligned}
&\mathrm{Ric}_g(\partial_t, \partial_t) = -2\rho''(t)/\rho(t)\\
&\mathrm{Ric}_g(\rho(t)^{-1}X_i, \rho(t)^{-1}X_i) = -\rho''(t)/\rho(t) + 1/\rho(t)^2\\
&R_g = -5\rho''(t)/\rho(t) + 1/\rho(t)^2.
\end{aligned}$$

Let $0 < \alpha < 2$. Take $\rho(t) = t^{\alpha/2}$. Then

$$\begin{aligned}
\rho(t) &= t^{\alpha/2}\\
\rho'(t) &= \frac{\alpha}{2} t^{\alpha/2-1}\\
\rho''(t) &= \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) t^{\alpha/2-2}.
\end{aligned}$$

Hence, $\mathrm{Ric}_g > 0$ and

$$R_g = -\frac{5\alpha}{2}\left(\frac{\alpha}{2} - 1\right) t^{-2} + t^{-\alpha} \geq t^{-\alpha}.$$

Take $x_0 = (1, p_0)$. Then

$$d_g((1,p_0),(t,p)) \leq t - 1 + d_{S^2}(p, p_0) \geq t - 1.$$

Then for $x = (t, p)$ with $d(x_0, x) \geq 1$, we have

$$R_g(x) \geq t^{-\alpha} \geq (d_g(x_0, x) + 1)^{-\alpha} \geq 2^{-\alpha} d_g(x_0, x)^{-\alpha}.$$

Finally we compute the volume growth. For $r > 2\pi$, we have

$$\begin{aligned}
\mathrm{Vol}_g(B_r(x_0)) &\geq \mathrm{Vol}_g([1, r/2] \times S^2)\\
&= C \int_1^{r/2} t^\alpha dt\\
&= \frac{C}{2^{1+\alpha}(1+\alpha)} r^{1+\alpha} - \frac{C}{1+\alpha}.
\end{aligned}$$

Then

$$\lim_{r\to\infty} r^{-1-\alpha} \mathrm{Vol}_g(B_r(x_0)) \geq \frac{C}{2^{1+\alpha}(1+\alpha)} > 0.$$

See also the discussion in [MW22] after their statement of Theorem 1.4 for a related example demonstrating that it is possible to have nearly non-negative Ricci curvature $\mathrm{Ric}_g \geq -Cd(x_0, x)^{-2} \log d(x_0, x)$ with $\inf_{M\setminus B_r(x_0)} R_g \to \infty$ as $r \to \infty$.

Appendix A. Gromov-Hausdorff approximations

We recall the definition of a Gromov-Hausdorff approximation.

Definition A.1. A map $f : (X, d_X) \to (Y, d_Y)$ is an *ε-Gromov-Hausdorff approximation* if

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$$

for all $x_1, x_2 \in X$ and

$$Y \subset B_\varepsilon(f(X)).$$

We recall a notion of Gromov-Hausdorff distance between metric spaces using Gromov-Hausdorff approximations.

Definition A.2. We say $d_{\mathrm{GH}}((X,d_X),(Y,d_Y)) < \varepsilon$ if there are ε-Gromov-Hausdorff approximations

$$f : X \to Y \quad \text{and} \quad g : Y \to X.$$

Proposition A.3. *Let*

$$f : B_R(x) \subset (X,d_X) \to B_R(y,0) \subset (Y \times \mathbb{R}, d_Y \times d_{Euc})$$

be an ε-Gromov-Hausdorff approximation. Then

$$d(f(x),(y,0)) < \sqrt{\varepsilon}\sqrt{9\varepsilon + 8R}.$$

Proof. Let $y' \in B_R(y,0)$. By definition, there is an $x' \in B_R(x)$ with $d(f(x'),y') < \varepsilon$. Moreover, $d(x,x') < R$, so $d(f(x),f(x')) < R + \varepsilon$. Hence,

$$d(f(x),y') \le d(f(x),f(x')) + d(f(x'),y') < R + 2\varepsilon,$$

which implies

$$B_R(y,0) \subset B_{R+2\varepsilon}(f(x)).$$

Write $f(x) = (y_0,t_0)$. Without loss of generality (by relabeling plus and minus), we have $t_0 \ge 0$. Since $(y,-R+\varepsilon) \in B_R(y,0)$, we have

$$(R+2\varepsilon)^2 > d(f(x),(y,-R+\varepsilon)) = d(y_0,y)^2 + (t_0 + R - \varepsilon)^2.$$

Since $d(y_0,y)^2 \ge 0$, we have $t_0 \le 3\varepsilon$. Since $t_0 \ge 0 \ge -\varepsilon$, we have

$$d(y_0,y)^2 \le (R+2\varepsilon)^2 - (R-2\varepsilon)^2 = 8R\varepsilon.$$

The conclusion follows. □

Proposition A.4. *Suppose (X,d) is a path connected metric space. Take any $R > 0$ and $x \in X$, and let $(\tilde{X},\tilde{d})$ the ball of radius $R > 0$ centered at $(x,0)$ in the product metric space $(X \times \mathbb{R}, d \times d_{Euc})$. Let $r < \min\{\frac{1}{2}\operatorname{diam}(X), R\}$. Then the subset*

$$\tilde{X} \setminus B_r(x,0) \subset \tilde{X}$$

is path connected.

Proof. We first show that the region $(\tilde{X} \setminus B_r(x,0)) \cap (\tilde{X} \times \mathbb{R}_+)$ is path connected. Let $(x_i,t_i) \in \tilde{X} \setminus B_r(x,0)$ for $i = 1,2$ with $t_i \ge 0$. Let $\gamma(s) = (x(s),t(s))$ be any path in $\tilde{X}$ joining (x_1,t_1) to (x_2,t_2). By replacing $t(s)$ by $\max\{t(s),0\}$, we obtain a continuous path in $(\tilde{X} \setminus B_r(x,0)) \cap (\tilde{X} \times \mathbb{R}_+)$ joining the points, so we can assume $t(s) \ge 0$. Now we take

$$\tilde{t}(s) := \begin{cases} \sqrt{r^2 - d(x(s),x)^2} & d(x(s),x)^2 + t(s)^2 \le r^2 \\ t(s) & \text{otherwise.} \end{cases}$$

Since $t(s) \ge 0$, $\tilde{t}(s)$ is continuous. Moreover, if $\tilde{t}(s) \ne t(s)$, then

$$d(x(s),x)^2 + \tilde{t}(s)^2 = d(x(s),x)^2 + r^2 - d(x(s),x)^2 = r^2,$$

so $\tilde{\gamma}(s) := (x(s),\tilde{t}(s))$ is a path in $(\tilde{X} \setminus B_r(x,0)) \cap (\tilde{X} \times \mathbb{R}_+)$ joining the two points.

By the same argument for the $\mathbb{R}_-$ side, we have $(\tilde{X} \setminus B_r(x,0)) \cap (\tilde{X} \times \mathbb{R}_+)$ is path connected.

Since $r < \frac{1}{2}\operatorname{diam}(X)$ and X is path connected, there is an $x' \in X$ with $r < d(x',x) < R$. Then (x',s) is a path in $\tilde{X} \setminus B_r(x,0)$ for s sufficiently small, which joins the two path connected regions. Hence, the conclusion follows. □

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References

[And90] Michael T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445, DOI 10.1007/BF01233434. MR1074481

[CC96] Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237, DOI 10.2307/2118589. MR1405949

[CG72] Jeff Cheeger and Detlef Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443, DOI 10.2307/1970819. MR309010

[CL20] Otis Chodosh and Chao Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888, 2020.

[CL21] Otis Chodosh and Chao Li, *Stable minimal hypersurfaces in* $\mathbf{R}^4$, Acta Math. (to appear), arXiv:2108.11462, (2021).

[CL23] Otis Chodosh and Chao Li, *Stable anisotropic minimal hypersurfaces in* $\mathbf{R}^4$, Forum Math. Pi **11** (2023), Paper No. e3, 22, DOI 10.1017/fmp.2023.1. MR4546104

[CLS22] Otis Chodosh, Chao Li, and Douglas Stryker, *Complete stable minimal hypersurfaces in positively curved 4-manifolds*, arXiv:2202.07708, (2022).

[Gro86] M. Gromov, *Large Riemannian manifolds*, Curvature and topology of Riemannian manifolds (Katata, 1985), Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, pp. 108–121, DOI 10.1007/BFb0075649. MR859578

[Gro18] Misha Gromov, *Metric inequalities with scalar curvature*, Geom. Funct. Anal. **28** (2018), no. 3, 645–726, DOI 10.1007/s00039-018-0453-z. MR3816521

[Liu13] Gang Liu, *3-manifolds with nonnegative Ricci curvature*, Invent. Math. **193** (2013), no. 2, 367–375, DOI 10.1007/s00222-012-0428-x. MR3090181

[MW22] Ovidiu Munteanu and Jiaping Wang, *Geometry of three-dimensional manifolds with scalar curvature lower bound*, arXiv:2201.05595v2, 2022.

[MW23] Ovidiu Munteanu and Jiaping Wang, *Comparison Theorems for 3D Manifolds With Scalar Curvature Bound*, Int. Math. Res. Not. IMRN **3** (2023), 2215–2242, DOI 10.1093/imrn/rnab307. MR4565611

[Nab20] Aaron Naber, *Conjectures and open questions on the structure and regularity of spaces with lower Ricci curvature bounds*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 104, 8, DOI 10.3842/SIGMA.2020.104. MR4164873

[Pet08] A. M. Petrunin, *An upper bound for the curvature integral* (Russian, with Russian summary), Algebra i Analiz **20** (2008), no. 2, 134–148, DOI 10.1090/S1061-0022-09-01046-2; English transl., St. Petersburg Math. J. **20** (2009), no. 2, 255–265. MR2423998

[Pet16] Peter Petersen, *Riemannian geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016, DOI 10.1007/978-3-319-26654-1. MR3469435

[SY82] Richard Schoen and Shing Tung Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 209–228. MR645740

[SY94] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; Translated from the Chinese by Ding and S. Y. Cheng; With a preface translated from the Chinese by Kaising Tso. MR1333601

[WXZZ22] Jinmin Wang, Zhizhang Xie, Bo Zhu, and Xingyu Zhu, *Positive Scalar Curvature Meets Ricci Limit Spaces*, arXiv:2212.10416, 2022.

[Xu20] Guoyi Xu, *Integral of scalar curvature on non-parabolic manifolds*, J. Geom. Anal. **30** (2020), no. 1, 901–909, DOI 10.1007/s12220-019-00174-7. MR4058542

[Yau92] Shing-Tung Yau, *Open problems in geometry*, Chern—a great geometer of the twentieth century, Int. Press, Hong Kong, 1992, pp. 275–319. MR1201369

[Zhu22] Bo Zhu, *Geometry of positive scalar curvature on complete manifold*, J. Reine Angew. Math. **791** (2022), 225–246, DOI 10.1515/crelle-2022-0049. MR4489630

[ZZ23] Bo Zhu and Xingyu Zhu, *Optimal diameter estimates of three-dimensional Ricci limit spaces*, arXiv:2302.09415, 2023.

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