



Mixed Matrix Completion in Complex Survey Sampling under Heterogeneous Missingness

Xiaojun Mao, Hengfang Wang, Zhonglei Wang & Shu Yang

To cite this article: Xiaojun Mao, Hengfang Wang, Zhonglei Wang & Shu Yang (29 Mar 2024): Mixed Matrix Completion in Complex Survey Sampling under Heterogeneous Missingness, Journal of Computational and Graphical Statistics, DOI: [10.1080/10618600.2024.2319154](https://doi.org/10.1080/10618600.2024.2319154)

To link to this article: <https://doi.org/10.1080/10618600.2024.2319154>



[View supplementary material](#)



Published online: 29 Mar 2024.



[Submit your article to this journal](#)



Article views: 142



[View related articles](#)



[View Crossmark data](#)

Mixed Matrix Completion in Complex Survey Sampling under Heterogeneous Missingness

Xiaojun Mao^a , Hengfang Wang^b , Zhonglei Wang^c , and Shu Yang^d

^aSchool of Mathematical Sciences, Ministry of Education Key Laboratory of Scientific and Engineering Computing, Shanghai Jiao Tong University, Shanghai, China; ^bSchool of Mathematics and Statistics & Fujian Provincial Key Laboratory of Statistics and Artificial Intelligence, Fujian Normal University, Fuzhou, Fujian, China; ^cWang Yanan Institute for Studies in Economics and School of Economics, Xiamen University, Xiamen, Fujian, China; ^dDepartment of Statistics, North Carolina State University, Raleigh, NC

ABSTRACT

Modern surveys with large sample sizes and growing mixed-type questionnaires require robust and scalable analysis methods. In this work, we consider recovering a mixed data frame matrix, obtained by complex survey sampling, with entries following different canonical exponential distributions and subject to heterogeneous missingness. To tackle this challenging task, we propose a two-stage procedure: in the first stage, we model the entry-wise missing mechanism by logistic regression, and in the second stage, we complete the target parameter matrix by maximizing a weighted log-likelihood with a low-rank constraint. We propose a fast and scalable estimation algorithm that achieves sublinear convergence, and the upper bound for the estimation error of the proposed method is rigorously derived. Experimental results support our theoretical claims, and the proposed estimator shows its merits compared to other existing methods. The proposed method is applied to analyze the National Health and Nutrition Examination Survey data. Supplementary materials for this article are available online.

ARTICLE HISTORY

Received August 2023
Accepted February 2024

KEYWORDS

Covariate matrix; Exponential family; Fast iterative shrinkage-thresholding algorithm

1. Introduction

Survey sampling is a touchstone for social science (Elliott and Valliant 2017; Haziza and Beaumont 2017). Modern technologies have accelerated the sampling speed with more mixed-type questionnaires, for example, National Health and Nutrition Examination Survey and web-based surveys (Rivers 2007). However, in practice, nonresponse is ubiquitous in survey data with arbitrary missingness patterns. If the missing mechanism is informative, ignoring the missing values leads to biased estimation. More importantly, a large survey with many mixed-type questionnaires also requires more scalable and robust methods.

We envision the survey data frame as a matrix, where rows correspond to subjects and columns to responses to different questions whose entries suffer from missingness. Imputation methods are commonly used to address such missingness. Generally, imputation methods fall into two categories: row-wise imputation and column-wise imputation. Multiple imputation (Rubin 1976) is a popular row-wise imputation method and it leverages a posterior predictive distribution given the observations to impute the missing ones for each row. Multiple imputation requires positing a joint distribution, which is stringent in practice, and it is computationally heavy or even infeasible with a growing number of questions. On the other hand, hot-deck imputation (Chen and Shao 2000; Kim and Fuller 2004) is a typical column-wise imputation method and imputes missing

values by observations in the same column with a predetermined distance metric. However, it is unrealistic to use the same metric to impute all the missing values, especially for large surveys with many mixed-type questions. Furthermore, due to the arbitrarily missing pattern, some donor pools may be limited.

As their names manifest, row-wise and column-wise imputation methods use only partial information, so do not harvest the full information of the data matrix. In contrast, matrix completion methods (Candès and Recht 2009; Keshavan, Montanari, and Oh 2010; Mazumder, Hastie, and Tibshirani 2010; Koltchinskii, Lounici, and Tsybakov 2011; Negahban and Wainwright 2012; Fan, Gong, and Zhu 2019) leverage matrix structures, such as low-rankness, to impute missing values simultaneously. Additionally, low-rankness is naturally present for survey data with block-wise questionnaires and common sampling designs such as stratified and cluster sampling. Mixed-type responses in large surveys require matrix completion for mixed data frames (Kiers 1991; Pages 2014; Udell et al. 2016). In this vein, Robin et al. (2020) studied the main effects and interactions with low-rankness and sparse matrix completion. Alaya and Klopp (2019) studied collective matrix completion whose entries come from exponential family distributions. Some survey variables are fully observed and can serve as side information to improve estimation efficiency. Inductive matrix completion (Xu, Jin, and Zhou 2013; Jain and Dhillon 2013; Wang et al.

2023) modeled side information by the matrix factorization method. Fithian and Mazumder (2018) leveraged row-wise and column-wise side information for reduced-rank modeling of matrix-valued data. Chiang, Dhillon, and Hsieh (2018) studied matrix completion with missing and corrupted side information. Mao, Chen, and Wong (2019) and Mao, Wang, and Yang (2023) proposed matrix completion with covariates using the column-space decomposition method. Meanwhile, most of the matrix completion literature treats the missing scheme as uniform missing, that is, missing completely at random. In the vein of missing data literature, heterogeneous missingness is more realistic and has been well studied (Little and Rubin 2019; Kim and Shao 2021), and the missing mechanism is usually modeled by logistic regression on the covariates. Without covariate information, it is difficult to model heterogeneous missingness for matrix completion (Mao, Wong, and Chen 2021).

This article considers recovering a mixed survey data matrix with entries following canonical exponential distributions given auxiliary information and subject to heterogeneous missingness. To tackle this challenging problem, we propose a two-stage procedure: in the first stage, we employ logistic regression to model the entry-wise missing mechanism with auxiliary variables, and in the second stage, we leverage an inverse probability weighted pseudolikelihood with a low-rankness constraint for matrix recovery, where the estimated response rates serve as surrogate probabilities. We establish the statistical guarantee of the proposed method and present the upper bound of the estimation error. Computationally, we adopt a fast iterative shrinkage-thresholding algorithm (FISTA) (Beck and Teboulle 2009) for estimation and show that it enjoys a sublinear convergence rate. To support our theoretical analysis, a synthetic experiment is conducted and the proposed estimator is shown to have its merits compared to other competitors. We apply the proposed method to recover mixed-type missing values in the 2015–2016 National Health and Nutrition Examination Survey data.

Notation. For a matrix $\mathbf{S} = (s_{ij}) \in \mathbb{R}^{n_1 \times n_2}$, its singular values are $\sigma_1(\mathbf{S}), \dots, \sigma_r(\mathbf{S})$ in descending order. We denote the Frobenius norm of \mathbf{S} by $\|\mathbf{S}\|_F = (\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} s_{ij}^2)^{1/2}$, the operator norm by $\|\mathbf{S}\| = \sigma_{\max}(\mathbf{S}) = \sigma_1(\mathbf{S})$, the nuclear norm $\|\mathbf{S}\|_* = \sum_{i=1}^{n_1} \sigma_i(\mathbf{S})$ and the sup norm $\|\mathbf{S}\|_\infty = \max_{i,j} |s_{ij}|$. For a positive integer n , define $\llbracket n \rrbracket = \{1, \dots, n\}$.

2. Methodology

2.1. Problem Formulation

We consider a finite population containing N subjects represented as $\mathcal{F} = \{(\mathbf{x}_i, \mathbf{y}_i) : i \in \llbracket N \rrbracket\}$, where $\mathbf{x}_i \in \mathbb{R}^D$ is a D -dimensional covariate, and $\mathbf{y}_i = (y_{i1}, \dots, y_{iL})^T$ is the response of interest to L questions. The dimension of \mathbf{y}_i is denoted by L . We consider a sampling design without replacement. Denote I_i to be the sampling indicator of the subject i , where $I_i = 1$ if the subject i is sampled and 0 otherwise. The first-order inclusion probability for the i th subject is denoted by $\pi_i = \mathbb{E}(I_i)$, where the expectation corresponds to the sampling process. Denote $\Pi = \{\pi_1, \dots, \pi_N\}$ to be the known first-order inclusion probabilities. For simplicity, we assume that the first n subjects are included in the sample. The covariate matrix $\mathbf{X} = (x_{ij}) \in \mathbb{R}^{n \times D}$ is then

formed, where x_{ij} represents the j th covariate for the i th subject. This covariate matrix contains demographic information and typically has fixed D columns. Furthermore, for a questionnaire with L questions, we construct an $n \times L$ matrix denoted as $\mathbf{Y} = (y_{ij}) \in \mathbb{R}^{n \times L}$. Each entry y_{ij} in this matrix corresponds to the answer provided by the subject i to the j th question. Due to nonresponses, the response probability of \mathbf{Y} is often low. To this end, we use the corresponding missing indicator matrix $\mathbf{R} = (r_{ij}) \in \mathbb{R}^{n \times L}$, where for any $(i, j) \in \llbracket n \rrbracket \times \llbracket L \rrbracket$, the value of r_{ij} is 1 if y_{ij} is observed (not missing), and 0 otherwise.

In a survey questionnaire, the questions are usually grouped into different categories based on the types of possible answers they can have. For instance, questions that elicit responses like “yes” or “no,” or “like” and “dislike” fall under the category of binary responses. Questions that require answers in the form of nonnegative integers, such as the number of household members or pets, belong to the category of nonnegative integer responses. Meanwhile, questions that prompt continuous values like household income or odometer readings are classified as continuous responses. Assume that there are a total of S categories of questions in a specific questionnaire, and denote the number of questions within the s th category as m_s . Naturally, the total number of questions in the survey is given by $L = \sum_{s=1}^S m_s$. For the sake of clarity and with a slight abuse of notation, we arrange the responses for each subject i in a concatenated manner as $\mathbf{y}_i = ((y_i^{(1)})^T, \dots, (y_i^{(S)})^T)^T$. Here, $y_i^{(s)} \in \mathbb{R}^{m_s}$ represents the responses to the questions of the i th subject in the s th category, for $s \in \llbracket S \rrbracket$. Consequently, the full response matrix \mathbf{Y} can be formulated as $\mathbf{Y} = [\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(S)}]$, with $\mathbf{Y}^{(s)} = (y_{ij}^{(s)}) \in \mathbb{R}^{n \times m_s}$ containing the responses to the questions in the s th category. Following a similar concatenation approach for \mathbf{Y} , we denote the missing indicator matrix as \mathbf{R} to account for missing responses in the data. For each $s \in \llbracket S \rrbracket$, we define $\mathbf{R} = [\mathbf{R}^{(1)}, \dots, \mathbf{R}^{(S)}]$, where $\mathbf{R}^{(s)} = (r_{ij}^{(s)}) \in \mathbb{R}^{n \times m_s}$. Each entry $r_{ij}^{(s)}$ in $\mathbf{R}^{(s)}$ indicates whether the corresponding response $y_{ij}^{(s)}$ in $\mathbf{Y}^{(s)}$ is observed (1) or missing (0). Traditional matrix completion methods assume a specific distribution for the entire response matrix, limiting their ability to handle mixed-type responses. To overcome this limitation, we introduce an exponential family approach, which can flexibly deal with diverse types of responses (Alaya and Klopp 2019; Robin et al. 2020; Wang et al. 2023). Specifically, we assume that all entries within the same response category come from the same generic exponential family. Mathematically, we consider the conditional density function within the s th category as introduce the exponential family to handle different types of responses,

$$f^{(s)}(y_{ij}^{(s)} | z_{ij}^{*(s)}) = h^{(s)}(y_{ij}^{(s)}) \exp \left\{ y_{ij}^{(s)} z_{ij}^{*(s)} - g^{(s)}(z_{ij}^{*(s)}) \right\},$$

for $(s, i, j) \in \llbracket S \rrbracket \times \llbracket n \rrbracket \times \llbracket m_s \rrbracket$, where $\{(g^{(s)}, h^{(s)}) : s \in \llbracket S \rrbracket\}$ contains S doublets of functions and $\mathbf{Z}^* = [\mathbf{Z}^{*(1)}, \dots, \mathbf{Z}^{*(S)}]$ is the parameter matrix with $\mathbf{Z}^{*(s)} = (z_{ij}^{*(s)}) \in \mathbb{R}^{n \times m_s}$. In the context of our investigation, we posit that \mathbf{Z}^* exhibits low-rank characteristics. For illustrative purposes, let us consider the example of the exponential family. Some commonly used distributions from the exponential family are listed in Table 1.

Remark 1. Our setup differs from existing ones in the following aspects. Traditional matrix completion problem mainly assumes

Table 1. Commonly used distributions from the exponential family.

Exponential family	$g(z)$	$h(y)$
Bernoulli	$\log(1 + \exp(z))$	1
Poisson	$\exp(z)$	$1/(y!)$
Gaussian	$\sigma^2 z^2/2$	$(2\pi\sigma^2)^{-1/2} \exp\{-y^2/(2\sigma^2)\}$
Exponential	$-\log(-z)$	1

independence among subjects, but they are usually correlated under complex sampling designs. Additionally, under survey sampling, there exists another parameter matrix $\mathbf{Z}_p^* \in \mathbb{R}^{N \times L}$ on the population level, and the parameter matrix \mathbf{Z}^* consists of rows of $\mathbf{Z}_p^* \in \mathbb{R}^{N \times L}$ with respect to the sample $i \in \llbracket n \rrbracket$. Furthermore, if the sampling design is informative, traditional matrix completion techniques, such as Alaya and Klopp (2019), will lead to biased recovery of \mathbf{Z}^* since the sampling information is overlooked; see Pfeffermann (1996) for details.

Our goal is to estimate the parameter matrix \mathbf{Z}^* based on available information. To this end, given the density functions $\{f^{(s)} : s \in \llbracket S \rrbracket\}$, the negative quasi-log-likelihood function for \mathbf{Z}^\dagger is

$$\begin{aligned} \ell_0(\mathbf{Z}^\dagger) &= - \sum_{s=1}^S \sum_{(i,j) \in \llbracket n \rrbracket \times \llbracket m_s \rrbracket} \frac{r_{ij}^{(s)}}{\pi_i} \log \left\{ f^{(s)}(y_{ij}^{(s)} | z_{ij}^{\dagger(s)}) \right\} \\ &= \sum_{s=1}^S \sum_{(i,j) \in \llbracket n \rrbracket \times \llbracket m_s \rrbracket} \frac{r_{ij}^{(s)}}{\pi_i} \left\{ -y_{ij}^{(s)} z_{ij}^{\dagger(s)} + g^{(s)}(z_{ij}^{\dagger(s)}) \right\} \quad (1) \\ &\quad + \sum_{s=1}^S \sum_{(i,j) \in \llbracket n \rrbracket \times \llbracket m_s \rrbracket} \frac{r_{ij}^{(s)}}{\pi_i} \log \left\{ h^{(s)}(y_{ij}^{(s)}) \right\}, \end{aligned}$$

where $\mathbf{Z}^\dagger = [\mathbf{Z}^{\dagger(1)}, \dots, \mathbf{Z}^{\dagger(S)}]$ and $\mathbf{Z}^{\dagger(s)} = (z_{ij}^{\dagger(s)}) \in \mathbb{R}^{n \times m_s}$ for $s \in \llbracket S \rrbracket$. In (1), we have incorporated the inclusion probabilities to obtain the Horvitz-Thompson estimator (Horvitz and Thompson 1952). The Horvitz-Thompson estimator is commonly used to estimate population parameters under complex survey sampling. It adjusts for the unequal probabilities of selection by weighting each unit's study variable by the inverse of its probability of being selected, ensuring that the sample more accurately represents the finite population. Because the second term of (1) is irrelevant to the argument \mathbf{Z}^\dagger , we concentrate on

$$\ell(\mathbf{Z}^\dagger) = \sum_{s=1}^S \sum_{(i,j) \in \llbracket n \rrbracket \times \llbracket m_s \rrbracket} \frac{r_{ij}^{(s)}}{\pi_i} \left\{ -y_{ij}^{(s)} z_{ij}^{\dagger(s)} + g^{(s)}(z_{ij}^{\dagger(s)}) \right\}.$$

Further, let $\mathbb{P}(r_{ij}^{(s)} = 1) = p_{ij}^{(s)}$ and $\mathbf{P}^{(s)} = (p_{ij}^{(s)})$ for $(s, i, j) \in \llbracket S \rrbracket \times \llbracket n \rrbracket \times \llbracket m_s \rrbracket$. Due to complex survey sampling and missingness, we consider the weighted loss function

$$\ell_w(\mathbf{Z}^\dagger) = \sum_{i=1}^n \frac{1}{\pi_i} \left[\sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{p_{ij}^{(s)}} \left\{ -y_{ij}^{(s)} z_{ij}^{\dagger(s)} + g^{(s)}(z_{ij}^{\dagger(s)}) \right\} \right].$$

Different from existing works (Alaya and Klopp 2019; Robin et al. 2020), in this article, we target on estimating the population-level parameters but not the sample-level ones. For example, Fang et al. (2018) considered an unweighted loss function with max-norm penalization for robust estimation.

Unfortunately, under informative sampling, such unweighted loss functions may lead to biased estimation; see Pfeffermann (1996) and Mao, Wang, and Yang (2023) for more details.

For concreteness, we focus on stratified sampling. In this scenario, we assume the existence of H strata, and a total of n subjects are sampled, with n_h subjects selected from the h th stratum for $h \in \llbracket H \rrbracket$. In addition, we assume logistic models within each stratum,

$$p_{ij}^{(s)} = \frac{\exp\{(1, \mathbf{x}_i^T) \boldsymbol{\xi}_{j,h}^{(s)}\}}{1 + \exp\{(1, \mathbf{x}_i^T) \boldsymbol{\xi}_{j,h}^{(s)}\}}, \quad (2)$$

for $\sum_{l=1}^{h-1} n_l + 1 \leq i \leq \sum_{l=1}^h n_l$, where $\boldsymbol{\xi}_{j,h}^{(s)} \in \mathbb{R}^{D+1}$ are the coefficients for the j th question within the s th category and the h th stratum. We can fit the logistic model within each stratum and each question to estimate $\boldsymbol{\xi}_{j,h}^{(s)}$ and further obtain $\widehat{p}_{ij}^{(s)}$ in a plug-in fashion of (2). Therefore, the surrogate-weighted loss is

$$\widehat{\ell}_w(\mathbf{Z}^\dagger) = \sum_{i=1}^n \frac{1}{\pi_i} \left[\sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\widehat{p}_{ij}^{(s)}} \left\{ -y_{ij}^{(s)} z_{ij}^{\dagger(s)} + g^{(s)}(z_{ij}^{\dagger(s)}) \right\} \right].$$

The low-rankness of \mathbf{Z}^* and \mathbf{X} naturally allows us to formulate the estimation procedure as

$$\widehat{\mathbf{Z}} = \arg \min_{\mathbf{Z}^\dagger \in \mathbb{R}^{n \times L}} \frac{1}{NL} \widehat{\ell}_w(\mathbf{Z}^\dagger) + \tau \|\llbracket \mathbf{X}, \mathbf{Z}^\dagger \rrbracket\|_* =: \arg \min_{\mathbf{Z}^\dagger \in \mathbb{R}^{n \times L}} \mathcal{L}_\tau(\mathbf{Z}^\dagger), \quad (3)$$

where $\tau > 0$ is a tuning parameter. The concatenation $\llbracket \mathbf{X}, \mathbf{Z}^\dagger \rrbracket$ helps us capture the potential nonlinear relationship between \mathbf{Z} and \mathbf{X} . Similar ideas can be found in the literature of multi-task learning and multi-view data learning; see Goldberg et al. (2010), Zhang and Huan (2012), Chen et al. (2018), and Ashraphijuo, Wang, and Aggarwal (2020) for more details. Even though we focus on stratified sampling, the derivation of our proof can be extended to general sampling designs. Further, in our setting, we consider different missing mechanisms across strata. In a nutshell, our methodology is summarized in Figure 1.

2.2. Estimation Algorithm

We propose a mixed matrix completion method in survey with heterogeneous missing (MMCSHM) algorithm to obtain the estimator in (3) summarized in Algorithm 1. For ease of presentation, denote

$$\widehat{\ell}_w(\mathbf{Z}) = \sum_{i=1}^n \frac{1}{\pi_i} \left[\sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\widehat{p}_{ij}^{(s)}} \left\{ -y_{ij}^{(s)} + (g^{(s)})'(z_{ij}^{(s)}) \right\} \right] \mathbf{E}_{ij}^{(s)}, \quad (4)$$

where $\mathbf{E}_{ij}^{(s)}$ is the indicator matrix for the i th subject and the j th variable within the s th category. Furthermore, for a matrix $\mathbf{S} \in \mathbb{R}^{n \times L}$, a singular value decomposition is $\mathbf{S} = \mathbf{U}\mathbf{W}\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{n \times r}$, $\mathbf{V} \in \mathbb{R}^{L \times r}$, $\mathbf{W} = \text{diag}(\sigma_1, \dots, \sigma_r)$ and $\sigma_1 \geq \dots \geq \sigma_r > 0$. Let $\mathbf{W}_1 = \text{diag}(\sigma_1, 0, \dots, 0)$. Denote the rank-1 approximation of \mathbf{S} as rank-1 SVD(\mathbf{S}) = $\mathbf{U}\mathbf{W}_1\mathbf{V}^T$, and it is used as the initialization for our proposed estimation procedure. Define

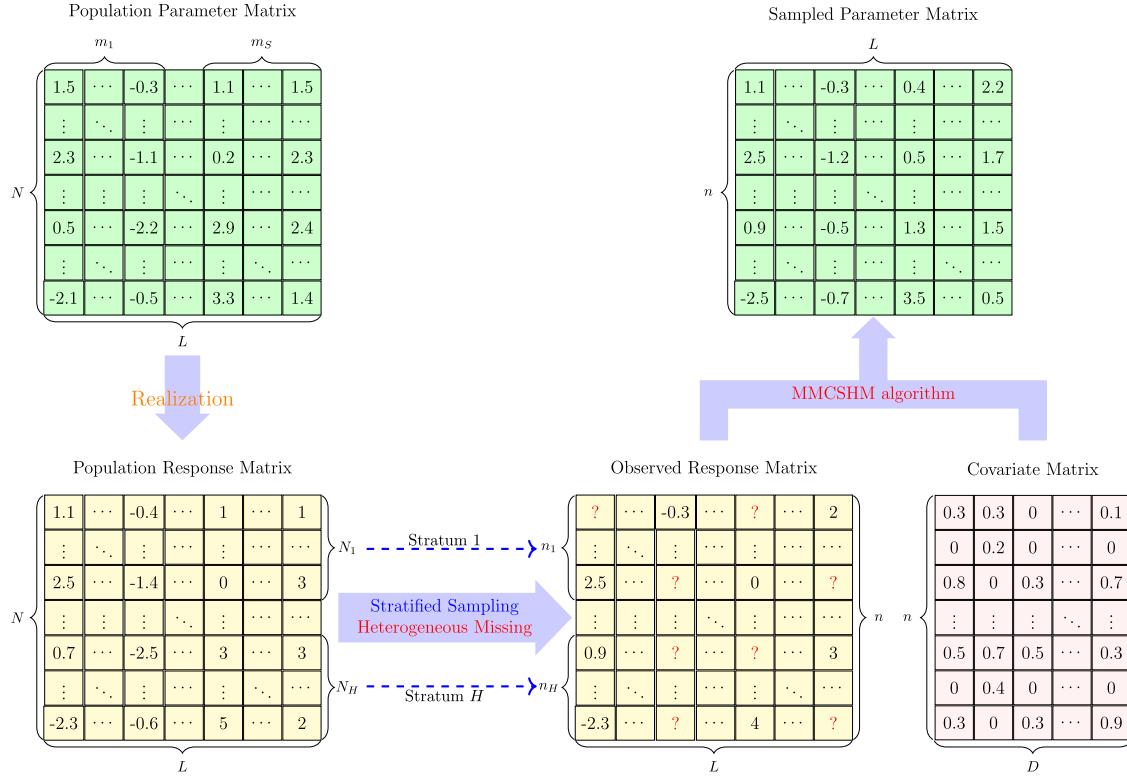


Figure 1. Algorithm illustration, where a question mark represents a missing value.

the singular value thresholding operator (Cai, Candès, and Shen 2010) for \mathbf{S} with the parameter τ as $\text{SVT}_\tau(\mathbf{S}) = \mathbf{U}\text{diag}((\sigma_1 - \tau)_+ \dots (\sigma_r - \tau)_+) \mathbf{V}^T$, where $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. Specifically, in the first stage, we opt for logistic regression to model the entry-wise missing mechanism with auxiliary information. In the second stage (Step 2 to Step 12 of Algorithm 1), with the fitted response probability, we leverage a decent version of FISTA (Beck 2017) to solve (3). The computational complexity for a single iteration is mainly determined by the singular value decomposition, which is $\mathcal{O}(n(D + L) \min\{n, D + L\})$. Consequently, for K iterations, the overall computational complexity becomes $\mathcal{O}(n(D+L)K \min\{n, D+L\})$. The convergence analysis of the proposed algorithm will be presented in the following section.

3. Theoretical Guarantee

In this section, we present the statistical guarantee and the convergence analysis of the proposed algorithm. Before we dive into the theoretical results, we introduce the following technical assumptions.

Assumption 1. There exist positive constants α_L and α_U such that $\alpha_L \leq (N\pi_i)^{-1}n_h \leq \alpha_U$, for $\sum_{l=1}^{h-1} n_l + 1 \leq i \leq \sum_{l=1}^h n_l$ and $h \in [H]$.

Assumption 2. There exists a positive constant $p_{\min} > 0$ such that $p_{\min} \leq p_{ij}^{(s)}$ for $(s, i, j) \in [\![S]\!] \times [\![n]\!] \times [\![m_s]\!]$.

Assumption 3. There exists a positive constant β , such that $\|\mathbf{Z}^*\|_\infty \leq \beta$.

Algorithm 1: MMCSHM algorithm

Input: The observed matrix \mathbf{Y} , missingness matrix \mathbf{R} , covariate matrix \mathbf{X} , sample size $\{n_h\}_{h=1}^H$, population size N , sampling probability $\{\pi_i\}_{i=1}^n$, tuning parameter τ , and learning depth K .

Initialize: $[\mathbf{U}_0, \mathbf{W}_0, \mathbf{V}_0] = \text{rank} - 1 \text{SVD}(\mathbf{R} \circ \mathbf{Y})$.
 $\mathbf{Z}_1^{(0)} = \mathbf{Z}_2^{(0)} = \mathbf{U}_0 \mathbf{W}_0 \mathbf{V}_0^T$.

- 1 Compute $\{\hat{p}_{ij}^{(s)}\}$ by fitting the logistic model with \mathbf{X} and \mathbf{R} within the corresponding strata.
- 2 **for** $k = 1$ **to** K , **do**
- 3 Compute $\theta_k = 2/(k + 1)$.
- 4 Compute $\mathbf{Q} = (1 - \theta_k)\mathbf{Z}_1^{(k-1)} + \theta_k\mathbf{Z}_2^{(k-1)}$.
- 5 Compute $\mathbf{T} = \mathbf{Q} - \tau^{-1}\text{SVT}_\tau(\nabla \hat{\ell}_w(\mathbf{Q}))$.
- 6 Compute $\tilde{\mathbf{T}} = \text{SVT}_\tau[\mathbf{X}, \mathbf{T}]$.
- 7 Compute $\tilde{\mathbf{Z}}_1^{(k)} = \tilde{\mathbf{T}}[\mathbf{0}_{n \times D}, \mathbf{I}_{n \times L}]^T$.
- 8 **if** $\mathcal{L}_\tau(\tilde{\mathbf{Z}}_1^{(k)}) < \mathcal{L}_\tau(\mathbf{Z}_1^{(k)})$ **then**
- 9 $\mathbf{Z}_1^{(k)} = \tilde{\mathbf{Z}}_1^{(k)}$;
- 10 **else**
- 11 $\mathbf{Z}_1^{(k)} = \mathbf{Z}_1^{(k-1)}$;
- 12 $\mathbf{Z}_2^{(k)} = \mathbf{Z}_1^{(k-1)} + \theta_k^{-1}(\tilde{\mathbf{Z}}_1^{(k)} - \mathbf{Z}_1^{(k-1)})$.

Output: $\mathbf{Z}_1^{(K)}$.

Assumption 4. Denote $C(z_{ij}^{*(s)}) = \inf_{0 \leq r \leq r(\mathbf{Z})} r^{-1} \mathbb{E}_{z_{ij}^{*(s)}} [\exp\{r|Y_{ij}^{(s)} - (g^{(s)})'(z_{ij}^{*(s)})|\}]$, where $r(\mathbf{Z})$ is the natural parameter space for the exponential family. Assume that there exists a

constant C_Z such that $C(z_{ij}^{*(s)}) \leq C_Z < \infty$ for $(s, i, j) \in \llbracket S \rrbracket \times \llbracket n \rrbracket \times \llbracket m_s \rrbracket$.

Assumption 5. Let $\mathcal{D} = [-\beta - \epsilon, \beta + \epsilon]$, for some $\epsilon > 0$. For any $z \in \mathcal{D}$, there exist positive constants L_β and U_β , such that $L_\beta \leq (g^{(s)})''(z) \leq U_\beta$, where $g^{(s)}$ is the link function of the exponential family for the s th category of the questionnaire, for $s \in \llbracket S \rrbracket$.

Assumption 1 is commonly used to control the sampling weights in survey sampling; see sec. 1.3 in Fuller (2009). **Assumption 2** controls the response probabilities for the entries. **Assumption 3** implies that the parameters are bounded. **Assumption 4** requires all $C(z_{ij}^{*(s)})$'s bounded from above by a constant C_Z . Under the framework of the canonical exponential family, **Assumption 5** indicates that $\text{var}(Y_{ij}^{(s)} | z_{ij}^{*(s)}) = (g^{(s)})''(z_{ij}^{*(s)}) > 0$ with extended support.

Denoting $\mathbf{P}^{(s)\dagger} = ((p_{ij}^{(s)})^{-1})_{ij}$, the inverse probability matrix is $\mathbf{P}^\dagger = [\mathbf{P}^{(1)\dagger}, \dots, \mathbf{P}^{(S)\dagger}]$. Similarly, let $\hat{\mathbf{P}}^\dagger = [\hat{\mathbf{P}}^{(1)\dagger}, \dots, \hat{\mathbf{P}}^{(S)\dagger}]$, where $\hat{\mathbf{P}}^{(s)\dagger} = ((\hat{p}_{ij}^{(s)})^{-1})_{ij}$. Further, denote

$$\zeta_R = \max \left\{ \|\mathbf{R} \circ (\hat{\mathbf{P}}^\dagger - \mathbf{P}^\dagger)\|_{\infty,2}, \|\mathbf{R}^T \circ (\hat{\mathbf{P}}^\dagger - \mathbf{P}^\dagger)^T\|_{\infty,2} \right\},$$

and

$$\tilde{\Delta} = \max \left\{ \frac{\{C_Z(n \vee L) \log(n + L)\}^{1/2}}{\sqrt{p_{\min} n L}}, \frac{\alpha_U C_Z \zeta_R (\log(n + L))^{1+\delta}}{n L}, \sqrt{\frac{\alpha_U \beta L_\beta^2 |1/p_{\min}| (n \vee L) \log(n + L)}{n L}} \right\}$$

for some $\delta > 0$. The upper bound of $\|\hat{\mathbf{Z}} - \mathbf{Z}^*\|_F^2$ can be obtained by the following theorem, which provides a statistical guarantee of our method.

Theorem 1. Suppose Assumptions 1–5 hold. Then there exist positive constants C_1, C_2, C_3 such that for $\tau \geq C_3 \tilde{\Delta}$,

$$\frac{1}{nL} \|\hat{\mathbf{Z}} - \mathbf{Z}^*\|_F^2 \leq \max \left\{ \frac{C_1 \alpha_L}{\alpha_U \beta^2} \sqrt{\frac{\log(n + L)}{nL}}, \frac{C_2 \text{rank}(\mathbf{M}^*)}{\alpha_L p_{\min}} \right. \\ \left. \times \left\{ \tau^2 + \frac{\alpha_U^2 \log(n + L)}{n \wedge L} \right\} \right\}$$

holds with probability at least $1 - 3/(n + L)$, where $\mathbf{M}^* = [\mathbf{X}, \mathbf{Z}^*]$.

Proof. First of all, by $R^*(\hat{\mathbf{Z}}) \leq R^*(\mathbf{Z})$, we have

$$\frac{1}{NL} \hat{\ell}_w(\hat{\mathbf{Z}}) + \tau \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_* \leq \frac{1}{NL} \hat{\ell}_w(\mathbf{Z}^*) + \tau \|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_*,$$

which yields

$$\frac{1}{NL} \sum_{i=1}^n \{\hat{\ell}_{w,i}(\hat{\mathbf{Z}}) - \hat{\ell}_{w,i}(\mathbf{Z}^*)\} \leq \tau (\|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_* - \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_*).$$

Expand the terms $\ell_{w,i}(\hat{\mathbf{Z}})$ and $\ell_{w,i}(\mathbf{Z}^*)$, we obtain

$$\begin{aligned} & \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} \left\{ g^{(s)}(\hat{z}_{ij}^{(s)}) - g^{(s)}(z_{ij}^{*(s)}) \right\} \\ & \leq \tau (\|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_* - \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_*) \\ & + \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} y_{ij}^{(s)} \left(\hat{z}_{ij}^{(s)} - z_{ij}^{*(s)} \right). \end{aligned}$$

By definition of Bregman divergence, it follows that

$$\begin{aligned} & \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} d_{g^{(s)}}(\hat{z}_{ij}^{(s)}, z_{ij}^{*(s)}) \\ & \leq \tau (\|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_* - \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_*) \\ & - \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} \left\{ \left(g^{(s)} \right)'(z_{ij}^{*(s)}) - y_{ij}^{(s)} \right\} \left(\hat{z}_{ij}^{(s)} - z_{ij}^{*(s)} \right). \end{aligned} \quad (5)$$

To handle the left-hand-side of (5), by $L_\beta(x - y)^2 \leq 2d_{g^{(s)}}(x - y) \leq U_\beta(x - y)^2$, we have

$$\begin{aligned} & \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} \left(\hat{z}_{ij}^{(s)} - z_{ij}^{*(s)} \right)^2 \\ & \leq \frac{2}{nL\beta} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} d_{g^{(s)}}(\hat{z}_{ij}^{(s)}, z_{ij}^{*(s)}). \end{aligned} \quad (6)$$

Plug (6) into (5), it turns out that

$$\begin{aligned} & \frac{L_\beta}{2nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} \left(\hat{z}_{ij}^{(s)} - z_{ij}^{*(s)} \right)^2 \\ & \leq \tau (\|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_* - \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_*) \\ & - \frac{1}{nL} \sum_{i=1}^n \frac{n}{N\pi_i} \sum_{s=1}^S \sum_{j=1}^{m_s} \frac{r_{ij}^{(s)}}{\hat{p}_{ij}^{(s)}} \left\{ \left(g^{(s)} \right)'(z_{ij}^{*(s)}) - y_{ij}^{(s)} \right\} \left(\hat{z}_{ij}^{(s)} - z_{ij}^{*(s)} \right). \end{aligned} \quad (7)$$

Recall that the inverse probability matrix $\mathbf{P}^\dagger = [\mathbf{P}^{(1)\dagger}, \dots, \mathbf{P}^{(S)\dagger}]$ and its estimated surrogate matrix $\hat{\mathbf{P}}^\dagger = [\hat{\mathbf{P}}^{(1)\dagger}, \dots, \hat{\mathbf{P}}^{(S)\dagger}]$, where $\mathbf{P}^{(s)\dagger} = ((p_{ij}^{(s)})^{-1})_{ij}$ and $\hat{\mathbf{P}}^{(s)\dagger} = ((\hat{p}_{ij}^{(s)})^{-1})_{ij}$. Let $\mathbf{1}_m$ be a vector of all ones elements with length m . Denote $\boldsymbol{\Pi}^\dagger = N^{-1}(n_{h(1)}\pi_1^{-1}, \dots, n_{h(n)}\pi_n^{-1})\mathbf{1}_L^T$. Furthermore, let $\mathbf{J} = \mathbf{1}_n\mathbf{1}_L^T$. Inequality (7) can be written into the following matrix representation

$$\begin{aligned} & \frac{L_\beta}{2nL} \langle \mathbf{R} \circ \hat{\mathbf{P}}^\dagger, \boldsymbol{\Pi}^\dagger \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & \leq \tau (\|\llbracket \mathbf{X}, \mathbf{Z}^* \rrbracket\|_* - \|\llbracket \mathbf{X}, \hat{\mathbf{Z}} \rrbracket\|_*) + \frac{1}{nL} \langle \mathbf{R} \circ \hat{\mathbf{P}}^\dagger \circ \mathbf{G}, \boldsymbol{\Pi}^\dagger \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \rangle, \end{aligned} \quad (8)$$

where \circ is the Hadamard product between matrices, $\mathbf{G} = [\mathbf{G}^{(1)}, \dots, \mathbf{G}^{(S)}]$ and $\mathbf{G}^{(s)} = (y_{ij}^{(s)} - (g^{(s)})'(z_{ij}^{*(s)}))_{ij}$. We further expand the terms in (8) as

$$\begin{aligned} & \langle \mathbf{R} \circ \hat{\mathbf{P}}^\dagger, \boldsymbol{\Pi}^\dagger \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & = \langle \mathbf{R} \circ \mathbf{P}^\dagger, \boldsymbol{\Pi}^\dagger \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & + \langle \mathbf{R} \circ \hat{\mathbf{P}}^\dagger - \mathbf{R} \circ \mathbf{P}^\dagger, \boldsymbol{\Pi}^\dagger \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\hat{\mathbf{Z}} - \mathbf{Z}^*) \rangle. \end{aligned} \quad (9)$$

Therefore, plug (9) into (8), we have

$$\begin{aligned} & \frac{L_\beta}{2nL} \langle \mathbf{R} \circ \mathbf{P}^\dagger, \mathbf{\Pi}^\dagger \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & \leq \underbrace{\tau \left(\|\mathbf{[X, Z^*]}\|_* - \|\mathbf{[X, \widehat{Z}]} \|_* \right)}_{T1} + \underbrace{\frac{1}{nL} \langle \mathbf{R} \circ \widehat{\mathbf{P}}^\dagger \circ \mathbf{G}, \mathbf{\Pi}^\dagger (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle}_{T2} \\ & \quad - \underbrace{\frac{L_\beta}{2nL} \langle \mathbf{R} \circ (\widehat{\mathbf{P}}^\dagger - \mathbf{P}^\dagger), \mathbf{\Pi}^\dagger \{ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \} \rangle}_{T3}. \end{aligned} \quad (10)$$

Let $\tilde{\Delta} = 2\Delta^{(1)}/L_\beta + \Delta^{(2)}$ and $\tilde{\tau} = 2\tau/L_\beta$. By (10) and Lemma S1–S3, with probability at least $1 - 3/(n + L)$, there exists a constant $\tilde{C} > 0$ such that

$$\begin{aligned} & \frac{1}{nL} \langle \mathbf{R} \circ \mathbf{P}^\dagger, \mathbf{\Pi}^\dagger \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & \leq \tilde{\tau} \left(\|\mathbf{[X, Z^*]}\|_* - \|\mathbf{[X, \widehat{Z}]} \|_* \right) + \tilde{C}\tilde{\Delta} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_*. \end{aligned}$$

Denote $\mathbf{M}^* = \mathbf{[X, Z^*]}$. Let the singular value decomposition of $\mathbf{M}^* = \sum_{i=1}^{r_{\mathbf{M}^*}} \sigma_{i, \mathbf{M}^*} \mathbf{u}_{i, \mathbf{M}^*} \mathbf{v}_{i, \mathbf{M}^*}^T$. Denote $\mathbf{A}_u = [\mathbf{u}_{1, \mathbf{M}^*}, \dots, \mathbf{u}_{r_{\mathbf{M}^*}, \mathbf{M}^*}]$ and $\mathbf{A}_v = [\mathbf{v}_{1, \mathbf{M}^*}, \dots, \mathbf{v}_{r_{\mathbf{M}^*}, \mathbf{M}^*}]$. The projection operators can be defined as

$$\begin{aligned} \mathcal{P}_{\mathbf{A}^\perp}(\mathbf{B}) &= \mathbf{P}_{\mathbf{A}_u^\perp} \mathbf{B} \mathbf{P}_{\mathbf{A}_v^\perp}, \\ \mathcal{P}_{\mathbf{A}}(\mathbf{B}) &= \mathbf{B} - \mathcal{P}_{\mathbf{A}^\perp}(\mathbf{B}) = \mathbf{P}_{\mathbf{A}_u} \mathbf{B} - \mathbf{P}_{\mathbf{A}_u^\perp} \mathbf{B} \mathbf{P}_{\mathbf{A}_v}, \end{aligned} \quad (11)$$

where $\mathbf{P}_{\mathbf{A}_u^\perp}$ is the projection matrix generated by \mathbf{A}_u^\perp and $\mathbf{A}_u^\perp = \mathbf{A}_u - \mathbf{P}_{\mathbf{A}_u}(\mathbf{A}_u)$. It follows that

$$\begin{aligned} & \|\mathbf{[X, Z^*]}\|_* - \|\mathbf{[X, \widehat{Z}]} \|_* \\ & \leq \|\mathcal{P}_{\mathbf{M}^*}(\mathbf{[X, Z^*]} - \mathbf{[X, \widehat{Z}]})\|_* - \|\mathcal{P}_{(\mathbf{M}^*)^\perp}(\mathbf{[X, Z^*]} - \mathbf{[X, \widehat{Z}]})\|_* \\ & \leq \|\mathcal{P}_{\mathbf{M}^*}(\mathbf{[0, Z^* - \widehat{Z}]})\|_* - \|\mathcal{P}_{(\mathbf{M}^*)^\perp}(\mathbf{[0, Z^* - \widehat{Z}]})\|_*. \end{aligned} \quad (12)$$

Further,

$$\begin{aligned} \|\mathcal{P}_{\mathbf{M}^*}(\mathbf{[0, Z^* - \widehat{Z}]})\|_* &\leq \sqrt{\text{rank}(\mathcal{P}_{\mathbf{M}^*})} \|\mathbf{[0, Z^* - \widehat{Z}]} \|_F \\ &= \sqrt{\text{rank}(\mathcal{P}_{\mathbf{M}^*})} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F. \end{aligned}$$

By the second equation in (11), we have $\text{rank}(\mathcal{P}_{\mathbf{M}^*}) \leq 2\text{rank}(\mathbf{M}^*)$. Take $\tilde{\tau} \geq 2^{-1}3\tilde{C}\tilde{\Delta}$, it follows that

$$\begin{aligned} & \frac{1}{nL} \langle \mathbf{R} \circ \mathbf{P}^\dagger, \mathbf{\Pi}^\dagger \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ & \leq \tilde{\tau} \sqrt{2\text{rank}(\mathbf{M}^*)} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F + \frac{2}{3} \tilde{\tau} \sqrt{2\text{rank}(\mathbf{Z}^*)} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F \\ & \leq \tilde{\tau} \left\{ \sqrt{\text{rank}(\mathbf{M}^*)} + \frac{2}{3} \sqrt{\text{rank}(\mathbf{Z}^*)} \right\} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F \\ & \leq \tilde{\tau} \sqrt{2\text{rank}(\mathbf{M}^*) + \frac{8}{9}\text{rank}(\mathbf{Z}^*)} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F \\ & \leq \tilde{\tau} \sqrt{2D + \frac{26}{9}\text{rank}(\mathbf{Z}^*)} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F. \end{aligned} \quad (13)$$

By Lemma S6 in the supplementary material, we have

$$\begin{aligned} \|\widehat{\mathbf{Z}} - \mathbf{Z}^*\|_* &\leq \|\mathbf{[0, \widehat{Z} - Z^*]}\|_* \\ &\leq \|\mathcal{P}_{\mathbf{M}^*}(\mathbf{[0, Z^* - \widehat{Z}]})\|_* + \|\mathcal{P}_{(\mathbf{M}^*)^\perp}(\mathbf{[0, Z^* - \widehat{Z}]})\|_* \\ &\leq 6 \|\mathcal{P}_{\mathbf{M}^*}(\mathbf{[0, Z^* - \widehat{Z}]})\|_* \\ &\leq \sqrt{72\text{rank}(\mathbf{M}^*)} \|\mathbf{Z}^* - \widehat{\mathbf{Z}}\|_F. \end{aligned}$$

Denote $\tilde{\beta} = \|\widehat{\mathbf{Z}} - \mathbf{Z}^*\|_\infty$. Apparently, $\tilde{\beta} \leq 2\beta$. Further, define

$$\begin{aligned} \mathcal{C}(r) = \left\{ \mathbf{Z} \in \mathbb{R}^{n \times L} : \|\mathbf{Z}\|_\infty = 1, \|\mathbf{Z}\|_* \leq \sqrt{r} \|\mathbf{Z}\|_F, \right. \\ \left. \langle \mathbf{\Pi}^\dagger, (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \geq \alpha_U \sqrt{\frac{64 \log(n+L)}{\log(6/5)nL}} \right\}. \end{aligned}$$

1. If $\langle \mathbf{\Pi}^\dagger, (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle < \alpha_U \beta^2 \sqrt{\frac{64 \log(n+L)}{\log(6/5)nL}}$, it yields

$$\begin{aligned} \frac{1}{nL} \|\widehat{\mathbf{Z}} - \mathbf{Z}^*\|_F^2 &\leq \frac{1}{\alpha_L nL} \langle \mathbf{\Pi}^\dagger, (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ &\leq \frac{\alpha_L}{\alpha_U \beta^2} \sqrt{\frac{64 \log(n+L)}{\log(6/5)nL}}. \end{aligned}$$

2. Otherwise, $\langle \mathbf{\Pi}^\dagger, (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \geq \alpha_U \beta^2 \sqrt{\frac{64 \log(n+L)}{\log(6/5)nL}}$.

In this case, $\tilde{\beta}^{-1}(\widehat{\mathbf{Z}} - \mathbf{Z}^*) \in \mathcal{C}(72\text{rank}(\mathbf{M}^*))$. By (13), Lemma S4 and Lemma S5 in the supplementary material, we have

$$\begin{aligned} \frac{1}{nL} \|\widehat{\mathbf{Z}} - \mathbf{Z}^*\|_F^2 &\leq \frac{1}{\alpha_L nL} \langle \mathbf{\Pi}^\dagger, (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \circ (\widehat{\mathbf{Z}} - \mathbf{Z}^*) \rangle \\ &\leq \frac{\text{Crank}(\mathbf{M}^*)}{\alpha_L p_{\min}} [\tau^2 + \{\mathbb{E}(\|\Phi^{(4)}\|)\}^2] \\ &\leq \frac{C\tau^2 \text{rank}(\mathbf{M}^*)}{\alpha_L p_{\min}} + \frac{C\alpha_U^2 \text{rank}(\mathbf{M}^*) \log(n+L)}{\alpha_L p_{\min} (n \wedge L)}, \end{aligned}$$

holds with probability at least $1 - 3/(n + L)$, where $\Phi^{(4)}$ is defined in Lemma S4 in the supplementary material, which completes the proof of the theorem. \square

Theorem 1 presents an upper bound with the same order as the methods in Robin et al. (2020) and Alaya and Klopp (2019), but with a different proof technique faithfully incorporating sampling weights, estimated response probabilities, and the covariate matrix. Meanwhile, the following theorem ensures that the proposed algorithm enjoys a sub-linear convergence rate.

Theorem 2. Suppose Assumptions 1–5 hold. Then, there exists constant $C_3 > 0$ such that for $\tau \geq C_3 \tilde{\Delta}$,

$$|\mathcal{L}_\tau(\mathbf{Z}_1^{(k)}) - \mathcal{L}_\tau(\mathbf{Z}^*)| \leq \frac{\tau \|\mathbf{Z}_1^{(0)} - \mathbf{Z}^*\|_F^2}{(k+1)^2} \quad (14)$$

holds with probability at least $1 - 3/(n + L)$.

Proof. The proof of Theorem 2 is similar to the proof of Theorem 4.1 in Ji and Ye (2009). We first claim that h is convex. For $\alpha \in (0, 1)$, $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{n \times L}$, we have

$$\begin{aligned} h(\alpha \mathbf{C} + (1 - \alpha) \mathbf{D}) &= \|\mathbf{[X, \alpha \mathbf{C} + (1 - \alpha) \mathbf{D}]}\|_* \\ &= \|\alpha \mathbf{[X, C]} + (1 - \alpha) \mathbf{[X, D]}\|_* \\ &\leq \alpha \|\mathbf{[X, C]}\|_* + (1 - \alpha) \|\mathbf{[X, D]}\|_* \\ &= \alpha h(\mathbf{C}) + (1 - \alpha) h(\mathbf{D}). \end{aligned}$$

We conclude that $\widehat{\mathcal{L}}_w(\mathbf{Z})$ is convex as $(g^{(s)})''(z) > 0$ from Assumption 4, so $\mathcal{L}_\tau(\mathbf{Z})$ is convex.

Second, we verify the subgradient form of $h(\mathbf{Z}) = \|[\mathbf{X}, \mathbf{Z}]\|_*$. The subgradient \mathbf{G} for h at $\mathbf{Z} = \mathbf{Z}_0$ indicates that, for any $\mathbf{Z} \in \mathbb{R}^{n \times L}$,

$$h(\mathbf{Z}) \geq h(\mathbf{Z}_0) + \langle \mathbf{Z} - \mathbf{Z}_0, \mathbf{G} \rangle, \quad (15)$$

where $\langle \cdot, \cdot \rangle$ is induced by trace norm. Expand (15), it yields $\|[\mathbf{X}, \mathbf{Z}]\|_* - \|\mathbf{X}, \mathbf{Z}_0\|_* \geq \langle \mathbf{Z} - \mathbf{Z}_0, \mathbf{G} \rangle$. On the other hand, for any $\tilde{\mathbf{G}} \in \partial \|\mathbf{X}, \mathbf{Z}_0\|_*$, that is, the subgradient for nuclear norm of the $n \times (D + L)$ matrix at $[\mathbf{X}, \mathbf{Z}_0]$, we have

$$\begin{aligned} \|[\mathbf{X}, \mathbf{Z}]\|_* - \|\mathbf{X}, \mathbf{Z}_0\|_* &\geq \langle [\mathbf{X}, \mathbf{Z}] - [\mathbf{X}, \mathbf{Z}_0], \tilde{\mathbf{G}} \rangle \\ &= \langle [\mathbf{0}, \mathbf{Z} - \mathbf{Z}_0], [\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2] \rangle = \langle \mathbf{Z} - \mathbf{Z}_0, \tilde{\mathbf{G}}_2 \rangle. \end{aligned}$$

We know that the subgradient

$$\begin{aligned} \partial \|\mathbf{X}, \mathbf{Z}_0\|_* = &\{ \mathbf{U} \mathbf{V}^T + \mathbf{W} : \mathbf{W} \in \mathbb{R}^{n \times (d+L)}, \mathbf{U}^T \mathbf{W} = \mathbf{0}, \\ &\mathbf{W} \mathbf{V} = \mathbf{0}, \|\mathbf{W}\| \leq 1 \}, \end{aligned}$$

where $[\mathbf{X}, \mathbf{Z}_0] = \mathbf{U} \Sigma \mathbf{V}$ is an SVD. Therefore, let $\tilde{\mathbf{I}} = [\mathbf{0}_{n \times d}, \mathbf{I}_{n \times L}]^T$, it turns out that

$$\begin{aligned} \partial h(\mathbf{Z}_0) = &\{ (\mathbf{U} \mathbf{V}^T + \mathbf{W}) \tilde{\mathbf{I}} : \mathbf{W} \in \mathbb{R}^{n \times (d+L)}, \mathbf{U}^T \mathbf{W} = \mathbf{0}, \\ &\mathbf{W} \mathbf{V} = \mathbf{0}, \|\mathbf{W}\| \leq 1 \}, \end{aligned}$$

where the subgradient of h is verified. \square

4. Synthetic Experiment

We conduct a simulation study under a stratified two-stage cluster sampling design. Specifically, we assume a finite population consists of H strata, M_h clusters in the h th stratum and M_{hi} elements in the (hi) th cluster. Assume stratum effects $a_h \sim \text{Ex}(1)$ for $h = 1, \dots, H$, where $\text{Ex}(\lambda)$ is an exponential distribution with rate parameter λ . Generate the stratum sizes $M_h \mid a_h \sim 5\text{Po}(a_h) + 20$ for $h = 1, \dots, H$, where $\text{Po}(\lambda)$ is a Poisson distribution with parameter λ . Further, generate cluster effects $b_{hi} \sim \text{Ex}(1)$ for $i = 1, \dots, M_h$ and the cluster sizes $M_{hi} \sim 5\text{Po}(a_h + b_{hi}) + 30$ for $i = 1, \dots, M_h$. We consider the following sampling design. The first-stage sampling design is probability proportional to size sampling with replacement, where the selection probability is proportional to the cluster size, and the sample size is m_1 . That is, for $h = 1, \dots, H$, independently sample m_1 clusters with selection probability proportion to M_{hi} for $i = 1, \dots, M_h$. The second-stage sampling is simple random sampling without replacement, and the sample size is m_2 . Then, for $h = 1, \dots, H$, the inclusion probability for each sampled element is $\pi_{hij} = m_1 m_2 / N_h$ for $i = 1, \dots, M_h$ and $j = 1, \dots, M_{hi}$, where $N_h = \sum_{i=1}^{M_h} M_{hi}$ is the stratum size. We consider $(m_1, m_2) = (5, 20)$ for each setup of H . The above sampling mechanism results in sample size $n = 900$.

Within each cluster, we generate an auxiliary matrix and a response matrix of interest. Specifically, generate $\mathbf{X}_{*,hi}^0 \in \mathbb{R}^{M_{hi} \times D}$ whose entries are independent and identically distributed from $\text{Ex}(1)$. Let $\mathbf{X}_{*,hi} = \mathbf{X}_{*,hi}^0 / \|\mathbf{X}_{*,hi}^0\|_\infty$. Further, generate coefficient matrix $\mathbf{W}_{hi}^{(s)} \in \mathbb{R}^{d \times m_s}$ whose elements are independently generated from a uniform distribution over $(0, 2)$ for $s = 1, 2, 3$. Let $\tilde{\mathbf{Z}}_{*,hi}^{(s)} = \mathbf{X}_{*,hi} \mathbf{W}_{hi}^{(s)}$ and $\mathbf{Z}_{*,hi}^{(s)} = \tilde{\mathbf{Z}}_{*,hi}^{(s)} / \|\tilde{\mathbf{Z}}_{*,hi}^{(s)}\|_\infty$. Each $\mathbf{Y}_{hi}^{(s)}$ are generated by parameters in $\mathbf{Z}_{*,hi}^{(s)}$. Specifically, we assume that the

entries of $\mathbf{Y}_{hi}^{(1)}$ come from the Gaussian distribution, the entries of $\mathbf{Y}_{hi}^{(2)}$ come from the Poisson distribution and the entries of $\mathbf{Y}_{hi}^{(3)}$ come from the Bernoulli distribution.

For entry-wise missing mechanism, let $\zeta_{j,h}^{(s)} = (\zeta_{1,j,h}^{(s)}, \zeta_{2,j,h}^{(s)}, \zeta_{3,j,h}^{(s)}, \zeta_{4,j,h}^{(s)})^T$. We generate $\zeta_{1,j,h}^{(s)}$ from $N(\xi, 0.1^2)$ and $\zeta_{k,j,h}^{(s)}$ from $N(0.3, 0.1^2)$ for $k = 2, 3, 4$, where ξ is used to adjust the overall missing rate. In this simulation study, we take $\xi = 0.3, -0.1, -0.5$ for response probability 40% and 60%. In addition, take $m_1 = m_2 = m_3 = 300$. We further choose learning depth $K = 200$. We tune the parameter τ using an independently generated validation dataset and apply the same parameter to the 100 Monte Carlo samples. We compare the proposed method with three other popular matrix completion methods:

- **Collective**: penalized likelihood accelerated inexact soft impute method from collective matrix completion Alaya and Klopp (2019).
- **Soft_Impute** : Soft-Impute method Mazumder, Hastie, and Tibshirani (2010).
- **Hot_Deck** : Hot deck imputation method Kim and Fuller (2004).

To measure the performance of each method, we employ the relative error of recovered matrix $\hat{\mathbf{S}}$ compared with true matrix \mathbf{S}^* , that is, $\text{RE}(\hat{\mathbf{S}}, \mathbf{S}^*) = \|\hat{\mathbf{S}} - \mathbf{S}^*\|_F / \|\mathbf{S}^*\|_F$.

In the simulation, we compare different methods in the aspects of $\text{RE}(\hat{\mathbf{Z}}, \mathbf{Z}^*)$ for the overall recovered matrix, $\text{RE}(\hat{\mathbf{Z}}^{(1)}, \mathbf{Z}^{*(1)})$ in Gaussian response part, $\text{RE}(\hat{\mathbf{Z}}^{(2)}, \mathbf{Z}^{*(2)})$ in Poisson response part and $\text{RE}(\hat{\mathbf{Z}}^{(3)}, \mathbf{Z}^{*(3)})$ in Bernoulli response part. The results are presented in Figure 2, where the error bar on top of each colored bar represents one standard error for the corresponding relative error. The proposed method has the lowest overall relative error in all three missing scenarios. More specifically, in each sub-matrix, our proposed method also presents nearly lowest relative errors in $\text{RE}(\hat{\mathbf{Z}}^{(1)}, \mathbf{Z}^{*(1)})$, $\text{RE}(\hat{\mathbf{Z}}^{(2)}, \mathbf{Z}^{*(2)})$ and $\text{RE}(\hat{\mathbf{Z}}^{(3)}, \mathbf{Z}^{*(3)})$. The numerical results partially support the benefit of auxiliary information for heterogeneous missingness for better matrix completion under survey sampling framework.

5. Real Data Application

The National Health and Nutrition Examination Survey (NHANES) is a comprehensive national survey conducted every two years to provide representative data on the health and nutritional status of adults and children in the United States. NHANES employs a multi-faceted approach to data collection, involving interviews, physical examinations, and laboratory tests. This survey covers a wide range of health-related topics, including demographic information, health conditions, nutrition status, and health behaviors, among others. The goal is to recover the values in the 2015–2016 NHANES data.

The dataset comprises $n = 5375$ sampled units, each associated with corresponding sampling weights. The number of strata H in our real data application is 14; see Chen et al. (2020) for more details. The covariate information includes 16 fully

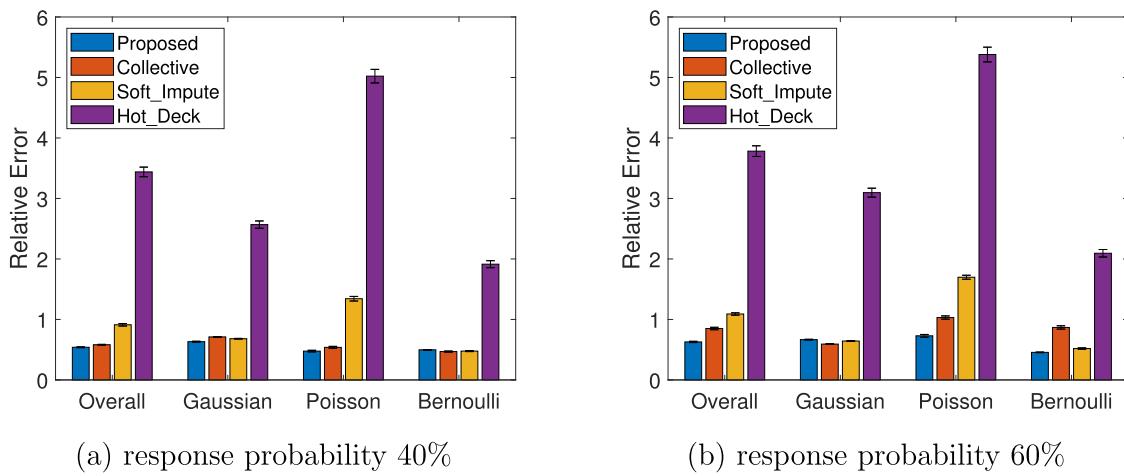


Figure 2. Relative errors for each method with 100 Monte Carlo samples in three response probability scenarios (including hot deck).

observed demographic variables, such as gender, age and marital status. To create the covariate matrix, we standardize these variables, ensuring that their columns have zero mean and unit variance. The response matrix consists of answers to 130 survey questions, and a majority of them have missing values. Among these questions, 57 are binary answers and 73 are continuous-valued. We assume that the binary variables follow Bernoulli distributions and the continuous variables follow Gaussian distributions. Following the convention of Alaya and Klopp (2019) and Robin et al. (2020), we standardize the columns for continuous responses. We revert to the original scale to present the final results. We apply the four methods assessed in the previous section to this dataset. To select the appropriate tuning parameter, we adopt a 5-fold cross-validation approach. The dataset is randomly divided into 5-fold, and within each fold, we employ the training set to complete the matrix and calculate the squared Euclidean norm for the difference between the imputed mean and the observed entries in the validation set. The sum of these errors serves as the cross-validation criterion for a specific tuning parameter. We choose the best tuning parameter with the lowest cross-validation error from $\{2^{-15}, \dots, 2^{-1}, 1, 2\}$.

The weighted mean for each question is computed using the imputed matrix and the survey weights. To present the results, we have randomly selected six questions, categorized by different levels of response probabilities. Specifically, we have two questions with high response rates of 0.95 and 0.93, two questions with moderate response rates of 0.77 and 0.66, and two questions with low response rates of 0.35 and 0.23. The corresponding results along with descriptions of the questions are provided in Table 2. Notably, when the response rates are high, the results obtained from the four methods are relatively similar. However, as the response rates decrease, the four methods begin to diverge from each other. It is important to observe that in all scenarios, the results from our proposed method consistently agree with one of the other three methods, demonstrating its robustness and practical use.

6. Conclusion

In this work, we study mixed matrix completion in survey sampling with heterogeneous missingness. Statistical guarantees

Table 2. Mean estimation for the following six questions.

	Response rate	Collective	Soft_Impute	Hot_Deck	Proposed
I	0.95	3.75	3.75	3.76	3.74
II	0.93	9.88	9.89	9.91	9.89
III	0.77	1.86	1.98	2.08	1.99
IV	0.66	1.57	1.37	1.54	1.30
V	0.35	0.80	0.52	0.80	0.80
VI	0.23	0.56	0.56	0.90	0.55

NOTE: "I" stands for "What is the highest grade or level of school you have/SP has completed or the highest degree you have/s/he has received?", "II" for "Total household income (reported as a range value in dollars)", "III" for "How many of those meals did you/did SP get from a fast-food or pizza place?", "IV" for "Total savings or cash assets at this time for you/NAMES OF OTHER FAMILY/your family", "V" for "Because of your/SP's (high blood pressure/hypertension), have you/has s/he ever been told to . . . take prescribed medicine?", "VI" for "During the past 30 days, how many times did you/SP drink DISPLAY NUMBER or more drinks of any kind of alcohol in about 2 hr?".

of the proposed method and the upper bound of the estimation error are presented. We propose an algorithm for estimation, and its convergence analysis shows that it achieves a sub-linear convergence. Experimental results support our theoretical claims and the proposed estimator shows its merits compared to other existing methods. Some research directions are worthy of investigation. To accommodate longitudinal observations, extension to the tensor completion of the proposed method is a potential research topic. To improve the numerical performance, a non-convex factorization (Zhao, Wang, and Liu 2015) method for Z^\dagger in (3) can be considered.

Supplementary Materials

Supplementary materials contain a demo Matlab code for our proposed method and useful lemmas for Theorem 1.

Acknowledgments

We thank the editor, the AE, and two anonymous reviewers for constructive comments and suggestions.

Disclosure Statement

No potential conflict of interest was reported by the author(s).



Funding

X. Mao was supported by NSFC Grant (No. 12371273), Shanghai Rising Star Program (No. 23QA1404600) and Young Elite Scientists Sponsorship Program by CAST (2023QNRC001). H. Wang was supported by Middle-aged and Young Teachers' Training Program (No. SDPY2023024). Z. Wang was supported in part by National Key R&D Program of China (No. 2022YFA1003800), NSFC (No.: 72033002, 12231011, 71988101), Humanities and Social Sciences Foundation of the Ministry of Education of China Grant (No. 23YJA910005), NSSFC (No. 23CMZ005). S. Yang was partially supported by the USA NIH (No.: 1R01AG066883, 1R01ES03165) and NSF SES (No. 2242776).

ORCID

Xiaojun Mao <http://orcid.org/0000-0002-9362-508X>
 Hengfang Wang <http://orcid.org/0000-0002-0472-2073>
 Zhonglei Wang <http://orcid.org/0000-0001-6712-0750>

References

Alaya, M. Z., and Klopp, O. (2019), "Collective Matrix Completion," *Journal of Machine Learning Research*, 20, 1–43. [\[1,2,3,6,7,8\]](#)

Ashraphijuo, M., Wang, X., and Aggarwal, V. (2020), "Fundamental Sampling Patterns for Low-Rank Multi-View Data Completion," *Pattern Recognition*, 103, 107307. [\[3\]](#)

Beck, A. (2017), *First-Order Methods in Optimization*, Philadelphia: SIAM. [\[4\]](#)

Beck, A., and Teboulle, M. (2009), "A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems," *SIAM Journal on Imaging Sciences*, 2, 183–202. [\[2\]](#)

Cai, J.-F., Candès, E. J., and Shen, Z. (2010), "A Singular Value Thresholding Algorithm for Matrix Completion," *SIAM Journal on Optimization*, 20, 1956–1982. [\[4\]](#)

Candès, E. J., and Recht, B. (2009), "Exact Matrix Completion via Convex Optimization," *Foundations of Computational Mathematics*, 9, 717–772. [\[1\]](#)

Chen, J., and Shao, J. (2000), "Nearest Neighbor Imputation for Survey Data," *Journal of Official Statistics*, 16, 113–132. [\[1\]](#)

Chen, L., Zhang, H., Lu, J., Thung, K., Aibaidula, A., Liu, L., Chen, S., Jin, L., Wu, J., Wang, Q., et al. (2018), "Multi-Label Nonlinear Matrix Completion with Transductive Multi-Task Feature Selection for Joint mgmt and idh1 Status Prediction of Patient with High-Grade Gliomas," *IEEE Transactions on Medical Imaging*, 37, 1775–1787. [\[3\]](#)

Chen, T.-C., Clark, J., Riddles, M. K., Mohadjer, L. K., and Fakhouri, T. H. (2020), "National Health and Nutrition Examination Survey, 2015–2018: Sample Design and Estimation Procedures," in *Vital Health and Statistics*, Series 2, no. 184. [\[7\]](#)

Chiang, K.-Y., Dhillon, I. S., and Hsieh, C.-J. (2018), "Using Side Information to Reliably Learn Low-Rank Matrices from Missing and Corrupted Observations," *Journal of Machine Learning Research*, 19, 3005–3039. [\[2\]](#)

Elliott, M. R., and Valliant, R. (2017), "Inference for Nonprobability Samples," *Statistical Science*, 32, 249–264. [\[1\]](#)

Fan, J., Gong, W., and Zhu, Z. (2019), "Generalized High-Dimensional Trace Regression via Nuclear Norm Regularization," *Journal of Econometrics*, 212, 177–202. [\[1\]](#)

Fang, E. X., Liu, H., Toh, K.-C., and Zhou, W.-X. (2018), "Max-Norm Optimization for Robust Matrix Recovery," *Mathematical Programming*, 167, 5–35. [\[3\]](#)

Fithian, W., and Mazumder, R. (2018), "Flexible Low-Rank Statistical Modeling with Missing Data and Side Information," *Statistical Science*, 33, 238–260. [\[2\]](#)

Fuller, W. A. (2009), *Sampling Statistics*, Hoboken, NJ: Wiley. [\[5\]](#)

Goldberg, A., Recht, B., Xu, J., Nowak, R., and Zhu, J. (2010), "Transduction with Matrix Completion: Three Birds with One Stone," in *Advances in Neural Information Processing Systems* (Vol. 23), pp. 757–765. [\[3\]](#)

Haziza, D., and Beaumont, J.-F. (2017), "Construction of Weights in Surveys: A Review," *Statistical Science*, 32, 206–226. [\[1\]](#)

Horvitz, D. G., and Thompson, D. J. (1952), "A Generalization of Sampling Without Replacement from a Finite Universe," *Journal of the American Statistical Association*, 47, 663–685. [\[3\]](#)

Jain, P., and Dhillon, I. S. (2013), "Provable Inductive Matrix Completion," arXiv preprint arXiv:1306.0626, pp. 1–22. [\[1\]](#)

Ji, S., and J. Ye (2009), "An accelerated gradient method for trace norm minimization. In *Proceedings of the 26th Annual International Conference on Machine Learning*, pp. 457–464. [\[6\]](#)

Keshavan, R. H., Montanari, A., and Oh, S. (2010), "Matrix Completion from Noisy Entries," *Journal of Machine Learning Research*, 11, 2057–2078. [\[1\]](#)

Kiers, H. A. (1991), "Simple Structure in Component Analysis Techniques for Mixtures of Qualitative and Quantitative Variables," *Psychometrika*, 56, 197–212. [\[1\]](#)

Kim, J. K., and Fuller, W. (2004), "Fractional Hot Deck Imputation," *Biometrika*, 91, 559–578. [\[1,7\]](#)

Kim, J. K., and Shao, J. (2021), *Statistical Methods for Handling Incomplete Data*, New York: CRC Press. [\[2\]](#)

Koltchinskii, V., Lounici, K., and Tsybakov, A. B. (2011), "Nuclear-Norm Penalization and Optimal Rates for Noisy Low-Rank Matrix Completion," *The Annals of Statistics*, 39, 2302–2329. [\[1\]](#)

Little, R. J., and Rubin, D. B. (2019), *Statistical Analysis with Missing Data* (3rd ed.), Hoboken: Wiley. [\[2\]](#)

Mao, X., Chen, S. X., and Wong, R. K. (2019), "Matrix Completion with Covariate Information," *Journal of the American Statistical Association*, 114, 198–210. [\[2\]](#)

Mao, X., Wang, Z., and Yang, S. (2023), "Matrix Completion Under Complex Survey Sampling," *Annals of the Institute of Statistical Mathematics*, 75, 463–492. [\[2,3\]](#)

Mao, X., Wong, R. K., and Chen, S. X. (2021), "Matrix Completion Under Low-Rank Missing Mechanism," *Statistica Sinica*, 31, 2005–2030. [\[2\]](#)

Mazumder, R., Hastie, T., and Tibshirani, R. (2010), "Spectral Regularization Algorithms for Learning Large Incomplete Matrices," *Journal of Machine Learning Research*, 11, 2287–2322. [\[1,7\]](#)

Negahban, S., and Wainwright, M. J. (2012), "Restricted Strong Convexity and Weighted Matrix Completion: Optimal Bounds with Noise," *Journal of Machine Learning Research*, 13, 1665–1697. [\[1\]](#)

Pagès, J. (2014), *Multiple Factor Analysis by Example Using R*, New York: CRC Press. [\[1\]](#)

Pfeffermann, D. (1996), "The Use of Sampling Weights for Survey Data Analysis," *Statistical Methods in Medical Research*, 5, 239–261. [\[3\]](#)

Rivers, D. (2007), "Sampling for Web Surveys," in *Joint Statistical Meetings* (Vol. 4). [\[1\]](#)

Robin, G., Klopp, O., Josse, J., Moulines, É., and Tibshirani, R. (2020), "Main Effects and Interactions in Mixed and Incomplete Data Frames," *Journal of the American Statistical Association*, 115, 1292–1303. [\[1,2,3,6,8\]](#)

Rubin, D. B. (1976), "Inference and Missing Data," *Biometrika*, 63, 581–592. [\[1\]](#)

Udell, M., Horn, C., Zadeh, R., Boyd, S., et al. (2016), "Generalized Low Rank Models," *Foundations and Trends® in Machine Learning*, 9, 1–118. [\[1\]](#)

Wang, H., Zhang, Y., Mao, X., and Wang, Z. (2023), "Transductive Matrix Completion with Calibration for Multi-Task Learning," in *ICASSP 2023 - 2023 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 1–5. [\[2\]](#)

Xu, M., Jin, R., and Zhou, Z.-H. (2013), "Speedup Matrix Completion with Side Information: Application to Multi-Label Learning," in *Advances in Neural Information Processing Systems* (Vol. 26), pp. 2301–2309. [\[1\]](#)

Zhang, J., and Huan, J. (2012), "Inductive Multi-Task Learning with Multiple View Data," in *Proceedings of the 18th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pp. 543–551. [\[3\]](#)

Zhao, T., Wang, Z., and Liu, H. (2015), "A Nonconvex Optimization Framework for Low Rank Matrix Estimation," in *Advances in Neural Information Processing Systems* (Vol. 28), pp. 559–567. [\[8\]](#)