



Algebraicity of the Bergman kernel

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Abstract

Our main result introduces a new way to characterize two-dimensional finite ball quotients by algebraicity of their Bergman kernels. This characterization is particular to dimension two and fails in higher dimensions, as is illustrated by a counterexample in dimension three constructed in this paper. As a corollary of our main theorem, we prove, e.g., that a smoothly bounded strictly pseudoconvex domain G in \mathbb{C}^2 has rational Bergman kernel if and only if there is a rational biholomorphism from G to \mathbb{B}^2 .

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1 Introduction

The Bergman kernel, introduced by Bergman in [1, 2] for domains in \mathbb{C}^n and later cast in differential geometric terms by Kobayashi [3], plays a fundamental role in several complex variables and complex geometry. Its biholomorphic invariance properties and intimate connection with the CR geometry of the boundary make it an important tool in the study of open complex manifolds. The use of the Bergman kernel, e.g., in the study of biholomorphic mappings and the geometry of bounded strictly pseudoconvex domains in \mathbb{C}^n was pioneered by C. Fefferman [4–6], who developed a theory of

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Bergman kernels in such domains and initiated a now famous program to describe the boundary singularity in terms of the local invariant CR geometry; see also [7, 8] for further progress on Fefferman's program.

A broad and general problem of foundational importance is that of classifying complex manifolds, or more generally analytic spaces, in terms of their Bergman kernels or Bergman metrics. For example, a well-known result of Lu [9] implies that if a relatively compact domain in an n -dimensional Kähler manifold has a complete Bergman metric with constant holomorphic sectional curvature, then the domain is biholomorphic to the unit ball \mathbb{B}^n in \mathbb{C}^n . Another example is the conjecture of Cheng [10], which states that the Bergman metric of a smoothly bounded strongly pseudoconvex domain in \mathbb{C}^n is Kähler–Einstein (i.e., has Ricci curvature equal to a constant multiple of the metric tensor) if and only if it is biholomorphic to the unit ball \mathbb{B}^n . This conjecture was confirmed by Fu–Wong [11] and Nemirovski–Shafikov [12] in the two dimensional case, and in the higher dimensional case by X. Huang and the second author [13].

In this paper, we introduce a new characterization of the two-dimensional unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ and, more generally, two-dimensional finite ball quotients \mathbb{B}^2/Γ in terms of algebraicity of the Bergman kernel. It is interesting, and perhaps surprising then, to note that such a characterization fails in the higher dimensional case. Indeed, in Sect. 6 below we construct a relatively compact domain G with smooth strongly pseudoconvex boundary in a three-dimensional algebraic variety $V \subset \mathbb{C}^4$, with an isolated normal singularity in the interior of G , such that its Bergman kernel is algebraic, but the boundary ∂G is not spherical and, furthermore, G is not biholomorphic to any finite ball quotient; recall that a CR hypersurface M of dimension $2n - 1$ is said to be *spherical* if near each point $p \in M$, it is locally CR diffeomorphic to an open piece of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Nevertheless, in two dimensions it turns out that algebraicity of the Bergman kernel does characterize finite ball quotients:

Theorem 1.1 *Let V be a 2-dimensional algebraic variety in \mathbb{C}^N , and G a relatively compact domain in V . Assume that every point in \overline{G} is a smooth point of V except for finitely many isolated normal singularities in the interior of G , and that G has a smooth strongly pseudoconvex boundary. Then the Bergman kernel form of G is algebraic if and only if there is an algebraic branched covering map F from \mathbb{B}^2 onto G , which realizes G as a ball quotient \mathbb{B}^2/Γ where Γ is a finite unitary group with no fixed points on $\partial\mathbb{B}^2$.*

Remark 1.2 We note that in addition to showing that Theorem 1.1 fails in dimension at least 3, our example in Sect. 6 also shows that the Ramanadov Conjecture for the Bergman kernel fails for higher dimensional normal Stein spaces. Recall that the Ramanadov Conjecture (c.f., [14], [15, Question 3]) proposes that if the logarithmic term in Fefferman's asymptotic expansion [5] of the Bergman kernel vanishes to infinite order at the boundary of a normal reduced Stein space with compact, smooth strongly pseudoconvex boundary, then the boundary is spherical. The Ramanadov Conjecture has been established in two dimensions by the work of D. Burns and R. C. Graham (see [16]). The normal reduced Stein space constructed in Sect. 6 gives a 3-dimensional counterexample with one isolated singularity. The counterexamples in [15] are smooth, but not Stein.

Theorem 1.1 has two immediate consequences in the non-singular case:

Corollary 1.3 *Let V be a 2-dimensional algebraic variety in \mathbb{C}^N , and let G be a relatively compact domain in V with smooth strongly pseudoconvex boundary. Assume that every point in \overline{G} is a smooth point of V . Then the Bergman kernel form of G is algebraic if and only if G is biholomorphic to \mathbb{B}^2 by an algebraic map.*

Corollary 1.4 *Let G be a bounded domain in \mathbb{C}^2 with smooth strongly pseudoconvex boundary. Then the Bergman kernel of G is rational (respectively, algebraic) if and only if there is a rational (respectively, algebraic) biholomorphic map from G to \mathbb{B}^2 .*

We remark that although Theorem 1.1 fails in higher dimension, Corollary 1.3 and 1.4 might still be true. For instance, it is clear from the proof below of Theorem 1.1 (see Remark 5.3) that if the Ramadanov Conjecture is proved to hold for, e.g., strongly pseudoconvex bounded domains in \mathbb{C}^n , which is still a possibility despite Remark 1.2 above, then Corollary 1.4 also holds in \mathbb{C}^n .

We also remark that the rationality of the biholomorphic map $G \rightarrow \mathbb{B}^2$ in Corollary 1.4, once its existence has been established, follows from the work of S. Bell [17]. For the reader's convenience, a self-contained proof of the rationality is given in Sect. 5.

As a final remark in this introduction, we note that, by Lempert's algebraic approximation theorem [18], if G is a relatively compact domain in a reduced Stein space X with only isolated singularities, then there exist an affine algebraic variety V , a domain $\Omega \subset V$, and a biholomorphism F from a neighborhood of \overline{G} to a neighborhood of $\overline{\Omega}$ with $F(\Omega) = G$. We shall say such a domain Ω is an *algebraic realization* of G . Theorem 1.1 implies the following corollary.

Corollary 1.5 *Let G be a relatively compact domain in a 2-dimensional reduced Stein space X with smooth strongly pseudoconvex boundary and only isolated normal singularities. If G has an algebraic realization with an algebraic Bergman kernel, then G is biholomorphic to a ball quotient \mathbb{B}^2/Γ , where Γ is a finite unitary group with no fixed point on $\partial\mathbb{B}^2$.*

To prove the "only if" implication in Theorem 1.1, we use the asymptotic boundary behavior of the Bergman kernel to establish algebraicity and sphericity of the boundary of G . Fefferman's asymptotic expansion [5] and the Riemann mapping type theorems due to Huang–Ji [19] and Huang [20] play important roles in the proof. To prove the converse ("if") implication in the theorem, we will need to compute the Bergman kernel forms of finite ball quotients. In order to do so, we shall establish a transformation formula for (possibly branched) covering maps of complex analytic spaces. This formula generalizes a classical theorem of Bell ([21], [22], cf. also [23]):

Theorem 1.6 *Let M_1 and M_2 be two complex analytic sets. Let $V_1 \subset M_1$ and $V_2 \subset M_2$ be proper analytic subvarieties such that $M_1 - V_1$, $M_2 - V_2$ are complex manifolds of the same dimension. Assume that $f : M_1 - V_1 \rightarrow M_2 - V_2$ is a finite (m -sheeted) holomorphic covering map. Let Γ be the deck transformation group for the covering*

map (with $|\Gamma| = m$), and denote by $K_i(z, \bar{w})$ the Bergman kernels of M_i for $i = 1, 2$. Then the Bergman kernel forms transform according to

$$\sum_{\gamma \in \Gamma} (\gamma, \text{id})^* K_1 = \sum_{\gamma \in \Gamma} (\text{id}, \gamma)^* K_1 = (f, f)^* K_2 \quad \text{on } (M_1 - V_1) \times (M_1 - V_1), \quad (1.1)$$

where $\text{id} : M_1 \rightarrow M_1$ is the identity map.

See Sect. 2 for the notation used in the formula in Theorem 1.6. We expect that this formula will be useful in other applications as well. In a recent paper [24], the authors apply it to study the question of when the Bergman metric of a finite ball quotient is Kähler–Einstein. (This is always the case for finite disk quotients, i.e., one-dimensional ball quotients, by recent work of Huang and Li [25].)

The paper is organized as follows. Section 2 gives some preliminaries on algebraic functions and Bergman kernels of complex analytic spaces. Section 3 is devoted to establishing the transformation formula in Theorem 1.6. Then in Sect. 4 we apply it to show that every standard algebraic realization (in particular, Cartan’s canonical realization) of a finite ball quotient must have algebraic Bergman kernel, and thus prove the "if" implication in Theorem 1.1. Section 5 gives the proof of the "only if" implication in Theorem 1.1, as well as those of Corollaries 1.3 and 1.4. In Sect. 6 and the Appendix, we construct the counterexample mentioned above to the corresponding statement of Theorem 1.1 in higher dimensions.

2 Preliminaries

2.1 Algebraic functions

In this subsection, we will review some basic facts about algebraic functions. For more details, we refer the readers to [26, Chapter 5.4] and [27].

Definition 2.1 (Algebraic functions and maps) Let \mathbb{K} be the field \mathbb{R} or \mathbb{C} . Let $U \subset \mathbb{K}^n$ be a domain. A \mathbb{K} -analytic function $f : U \rightarrow \mathbb{K}$ is said to be \mathbb{K} -algebraic (i.e., real/complex-algebraic) on U if there is a non-trivial polynomial $P(x, y) \in \mathbb{K}[x, y]$, with $(x, y) \in \mathbb{K}^n \times \mathbb{K}$, such that $P(x, f(x)) = 0$ for all $x \in U$. We say that a \mathbb{K} -analytic map $F : U \rightarrow \mathbb{C}^N$ is \mathbb{K} -algebraic if each of its components is so on U .

Remark 2.2 We make two remarks:

- (i) If $f(x)$ is an \mathbb{K} -analytic function in a domain $U \subset \mathbb{K}^n$, then f is \mathbb{K} -algebraic if and only if it is \mathbb{K} -algebraic in some neighborhood of any point $x_0 \in U$.
- (ii) If $f(x)$ is an \mathbb{R} -analytic function in a domain $U \subset \mathbb{R}^n$, then there is domain $\hat{U} \subset \mathbb{C}^n$ containing $U \subset \mathbb{R}^n \subset \mathbb{C}^n$ and a \mathbb{C} -analytic (i.e., holomorphic) function $g(x + iy)$ in \hat{U} such that $f = g|_U$; i.e., $f(x) = g(x)$ for $x \in U$. Moreover, f is \mathbb{R} -algebraic if and only if g is \mathbb{C} -algebraic.

We say a differential form on $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ is real-algebraic if each of its coefficient functions is so. We can also define real-algebraicity of a differential form on an affine (algebraic) variety.

Definition 2.3 Let $V \subset \mathbb{C}^N$ be an affine variety and write $\text{Reg } V$ for the set of its regular points. Let ϕ be a real analytic differential form on $\text{Reg } V$. We say ϕ is real-algebraic on V if for every point $z_0 \in \text{Reg } V$, there exists a real-algebraic differential form ψ in a neighborhood U of z_0 in $\mathbb{C}^N \cong \mathbb{R}^{2N}$ such that

$$\psi|_V = \phi, \quad \text{on } U \cap V.$$

Let $T_{z_0} V \cong T_{z_0}^{1,0} V$ be the complex tangent space of V at a smooth point $z_0 \in V$ considered as an affine complex subspace in \mathbb{C}^n through z_0 , and let $\xi = (\xi_1, \dots, \xi_n)$ be affine coordinates for $T_{z_0} V$. Since V can be realized locally as a graph over $T_{z_0} V$, the real and imaginary parts of ξ also serve as local real coordinates for V near z_0 . We call such coordinates the *canonical extrinsic coordinates at z_0* . Then the following statements are equivalent.

- (a) ϕ is real-algebraic on $\text{Reg } V$ (in the sense of Definition 2.3).
- (b) For any $z_0 \in \text{Reg } V$, ϕ is real-algebraic in canonical extrinsic coordinates at z_0 .

If in addition, there is a domain $G \subset \mathbb{C}^n$ and a \mathbb{C} -algebraic (i.e., holomorphic algebraic) immersion $f : G \rightarrow \mathbb{C}^N$ such that $f(G) = \text{Reg } V$, then (a) and (b) are further equivalent to

- (c) $f^* \phi$ is real-algebraic on G .

Remark 2.4 We can define complex-algebraicity of $(p, 0)$ -forms, $p > 0$, on an complex affine (algebraic) variety in a similar manner as in Definition 2.3.

2.2 The Bergman Kernel

In this section, we will briefly review some properties of the Bergman kernel on a complex manifold. More details can be found in [28].

Let M be an n -dimensional complex manifold. Write $L^2_{(n,0)}(M)$ for the space of L^2 -integrable $(n, 0)$ forms on M , which is equipped with the following inner product:

$$(\varphi, \psi)_{L^2(M)} := i^{n^2} \int_M \varphi \wedge \bar{\psi}, \quad \varphi, \psi \in L^2_{(n,0)}(M). \quad (2.1)$$

Define the *Bergman space* of M to be

$$A^2_{(n,0)}(M) := \{\varphi \in L^2_{(n,0)}(M) : \varphi \text{ is a holomorphic } (n, 0) \text{ form on } M\}. \quad (2.2)$$

Assume $A^2_{(n,0)}(M) \neq \{0\}$. Then $A^2_{(n,0)}(M)$ is a separable Hilbert space. Taking any orthonormal basis $\{\varphi_k\}_{k=1}^q$ of $A^2_{(n,0)}(M)$ (here $1 \leq q \leq \infty$), we define the *Bergman*

kernel (form) of M to be

$$K_M(x, \bar{y}) = i^{n^2} \sum_{k=1}^q \varphi_k(x) \wedge \overline{\varphi_k(y)}.$$

Then, $K_M(x, \bar{x})$ is a real-valued, real analytic form of degree (n, n) on M and is independent of the choice of orthonormal basis. When M is also (the set of regular points on) an affine variety, we say that the Bergman kernel of M is *algebraic* if $K_M(x, \bar{x})$ is real-algebraic in the sense of Definition 2.3. The following definitions and facts are standard in literature.

Definition 2.5 (*Bergman projection*) Given $g \in L^2_{(n,0)}(M)$, we define for $x \in M$

$$Pg(x) = \int_M g(\zeta) \wedge K_M(x, \bar{\zeta}) := i^{n^2} \sum_{k=1}^q \left(\int_M g(\zeta) \wedge \overline{\varphi_k(\zeta)} \right) \varphi_k(x).$$

$P: L^2_{(n,0)} \rightarrow A^2_{(n,0)}(M)$ is called the *Bergman projection*, and is the orthogonal projection to the Bergman space $A^2_{(n,0)}(M)$.

The Bergman kernel form remains unchanged if we remove a proper complex analytic subvariety. The following theorem is from [3].

Theorem 2.6 ([3]) *If M' is a domain in an n -dimensional complex manifold M and if $M - M'$ is a complex analytic subvariety of M of complex dimension $\leq n - 1$, then*

$$K_M(x, \bar{y}) = K_{M'}(x, \bar{y}) \quad \text{for any } x, y \in M'.$$

This theorem suggests the following generalization of the Bergman kernel form to complex analytic spaces.

Definition 2.7 Let M be a reduced complex analytic space, and let $V \subset M$ denote its set of singular points. The Bergman kernel form of M is defined as

$$K_M(x, \bar{y}) = K_{M-V}(x, \bar{y}) \quad \text{for any } x, y \in M - V,$$

where K_{M-V} denotes the Bergman kernel form of the complex manifold consisting of regular points of M .

Let N_1, N_2 be two complex manifolds of dimension n . Let $\gamma: N_1 \rightarrow M$ and $\tau: N_2 \rightarrow M$ be holomorphic maps. The pullback of the Bergman kernel $K_M(x, \bar{y})$ of M to $N_1 \times N_2$ is defined in the standard way. That is, for any $z \in N_1, w \in N_2$,

$$((\gamma, \tau)^* K)(z, \bar{w}) = \sum_{k=1}^q \gamma^* \varphi_k(z) \wedge \overline{\tau^* \varphi_k(w)}.$$

In terms of local coordinates, writing the Bergman kernel form of M as

$$K_M(x, \bar{y}) = \tilde{K}(x, \bar{y}) dx_1 \wedge \cdots \wedge dx_n \wedge d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_n,$$

we have

$$((\gamma, \tau)^* K_M)(z, \bar{w}) = \tilde{K}(\gamma(z), \overline{\tau(w)}) J_\gamma(z) \overline{J_\tau(w)} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n,$$

where J_γ and J_τ are the Jacobian determinants of the maps γ and τ , respectively.

3 The transformation law for the Bergman kernel

In this section, we shall prove Theorem 1.6. For this, we shall adapt the ideas in [22] to our situation. More precisely, we shall first prove the following transformation law for the Bergman projections. Then (1.1) will follow readily by comparing the associated distributional kernels for the projection operators.

Proposition 3.1 *Under the assumptions and notation in Theorem 1.6, we denote by n the complex dimension of $M_1 - V_1$ and $M_2 - V_2$. Let $P_i : L^2_{(n,0)}(M_i - V_i) \rightarrow A^2_{(n,0)}(M_i - V_i)$ be the Bergman projection for $i = 1, 2$. Then the Bergman projections transform according to*

$$P_1(f^* \phi) = f^*(P_2 \phi) \quad \text{for any } \phi \in L^2_{(n,0)}(M_2 - V_2). \quad (3.1)$$

We first check that $f^* \phi \in L^2_{(n,0)}(M_1 - V_1)$ if $\phi \in L^2_{(n,0)}(M_2 - V_2)$ in the next lemma. Recall that f is an m -sheeted covering map $M_1 - V_1 \rightarrow M_2 - V_2$.

Lemma 3.2

$$\|f^* \phi\|_{L^2(M_1 - V_1)} = m^{\frac{1}{2}} \|\phi\|_{L^2(M_2 - V_2)} \quad \text{for any } \phi \in L^2_{(n,0)}(M_2 - V_2).$$

Proof Let $\{U_j\}$ be a countable, locally finite open cover of $M_2 - V_2$ such that

- each U_j is relatively compact;
- $f^{-1}(U_j) = \bigcup_{k=1}^m V_{j,k}$ for some pairwise disjoint open sets $\{V_{j,k}\}_{k=1}^m$ on $M_1 - V_1$;
- $f : V_{j,k} \rightarrow U_j$ is a biholomorphism for each $j = 1, 2, \dots, m$.

Let $\{\rho_j\}$ be a partition of unity subordinate to the cover $\{U_j\}$. Then

$$i^{n^2} \int_{M_2 - V_2} \phi \wedge \bar{\phi} = \sum_j i^{n^2} \int_{U_j} \rho_j \phi \wedge \bar{\phi} = \frac{1}{m} \sum_j \sum_{k=1}^m i^{n^2} \int_{V_{j,k}} (f^* \rho_j) f^* \phi \wedge \overline{f^* \phi}.$$

Note that $\{f^* \rho_j\}$ is a partition of unity subordinate to the countable, locally finite open cover $\{\cup_{k=1}^m V_{j,k}\}$ of $M_1 - V_1$. Thus,

$$\begin{aligned} \frac{1}{m} \sum_j \sum_{k=1}^m i^{n^2} \int_{V_{j,k}} (f^* \rho_j) f^* \phi \wedge \overline{f^* \phi} &= \frac{1}{m} \sum_j i^{n^2} \int_{\cup_{k=1}^m V_{j,k}} (f^* \rho_j) f^* \phi \wedge \overline{f^* \phi} \\ &= \frac{1}{m} i^{n^2} \int_{M_1 - V_1} f^* \phi \wedge \overline{f^* \phi}. \end{aligned}$$

The result therefore follows immediately. \square

Let F_1, F_2, \dots, F_m be the m local inverses to f defined locally on $M_2 - V_2$. Note that $\sum_{k=1}^m F_k^*$ is a well-defined operator on $L^2_{(n,0)}(M_1 - V_1)$, though each individual F_k is only locally defined.

Lemma 3.3 *Let $v \in L^2_{(n,0)}(M_1 - V_1)$ and $\phi \in L^2_{(n,0)}(M_2 - V_2)$. Then $\sum_{k=1}^m F_k^*(v) \in L^2_{(n,0)}(M_2 - V_2)$ and*

$$(v, f^* \phi)_{L^2(M_1 - V_1)} = \left(\sum_{k=1}^m F_k^*(v), \phi \right)_{L^2(M_2 - V_2)}. \quad (3.2)$$

Proof We first verify $\sum_{k=1}^m F_k^*(v) \in L^2_{(n,0)}(M_2 - V_2)$. For that we note

$$f^* \sum_{k=1}^m F_k^*(v) = \sum_{\gamma \in \Gamma} \gamma^* v.$$

By the same argument as in Lemma 3.2, we have

$$\left\| \sum_{\gamma \in \Gamma} \gamma^* v \right\|_{L^2(M_1 - V_1)} = m^{\frac{1}{2}} \left\| \sum_{k=1}^m F_k^*(v) \right\|_{L^2(M_2 - V_2)}. \quad (3.3)$$

Since each deck transformation $\gamma : M_1 - V_1 \rightarrow M_1 - V_1$ is biholomorphic, it follows that

$$\left\| \sum_{\gamma \in \Gamma} \gamma^* v \right\|_{L^2(M_1 - V_1)} \leq \sum_{\gamma \in \Gamma} \|\gamma^* v\|_{L^2(M_1 - V_1)} = m \|v\|_{L^2(M_1 - V_1)}.$$

Therefore by (3.3), $\sum_{k=1}^m F_k^*(v) \in L^2_{(n,0)}(M_2 - V_2)$.

Now we are ready to prove (3.2). Let $\{U_j\}$, $\{V_{j,k}\}$ and $\{\rho_j\}$ be the open covers and partition of unity as in Lemma 3.2. Then

$$\left(\sum_{k=1}^m F_k^*(v), \phi \right)_{L^2(M_2 - V_2)} = \sum_j i^{n^2} \int_{U_j} \rho_j \sum_{k=1}^m F_k^*(v) \wedge \overline{\phi}.$$

Note that every $F_k : U_j \rightarrow V_{j,k}$ is biholomorphic and the inverse of $f : V_{j,k} \rightarrow U_j$. Thus,

$$\sum_j i^{n^2} \int_{U_j} \rho_j \sum_{k=1}^m F_k^*(v) \wedge \bar{\phi} = \sum_j \sum_{k=1}^m i^{n^2} \int_{V_{j,k}} (f^* \rho_j) v \wedge \bar{f^* \phi} = (v, f^* \phi)_{L^2(M_1 - V_1)}.$$

The last equality follows from the fact that $\{f^* \rho_j\}$ is a partition of unity subordinate to the countable, locally finite open cover $\{\cup_{k=1}^m V_{j,k}\}$ of $M_1 - V_1$. This proves (3.2). \square

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1 If $\phi \in A_{(n,0)}^2(M_2 - V_2)$, then $f^* \phi \in A_{(n,0)}^2(M_1 - V_1)$ by Lemma 3.2, whence (3.1) holds trivially. It thus suffices to prove (3.1) for $\phi \in A_{(n,0)}^2(M_2 - V_2)^\perp$. In this case, (3.1) reduces to

$$P_1(f^* \phi) = 0 \quad \text{for any } \phi \in A_{(n,0)}^2(M_2 - V_2)^\perp;$$

i.e., $\phi \in A_{(n,0)}^2(M_2 - V_2)^\perp$ implies that $f^* \phi \in A_{(n,0)}^2(M_1 - V_1)^\perp$. To prove this, we note that for any $v \in A_{(n,0)}^2(M_1 - V_1)$, we have by Lemma 3.3

$$(v, f^* \phi)_{L^2(M_1 - V_1)} = \left(\sum_{k=1}^m F_k^*(v), \phi \right)_{L^2(M_2 - V_2)} = 0.$$

The last equality follows from the fact $\phi \in A_{(n,0)}^2(M_2 - V_2)^\perp$. Thus, $f^* \phi \in A_{(n,0)}^2(M_1 - V_1)^\perp$ and the proof is completed. \square

We are now in a position to prove Theorem 1.6.

Proof of Theorem 1.6 Let id_{M_i} be the identity map on M_i for $i = 1, 2$. Recall that $\{F_k\}_{k=1}^m$ are local inverses of f . Note that $\sum_{k=1}^m (\text{id}_{M_1}, F_k)^* K_1$ is a well-defined (n, n) form on $(M_1 - V_1) \times (M_2 - V_2)$ though each $(\text{id}_{M_1}, F_k)^* K_1$ is only locally defined.

We shall write out the Bergman projection transformation law (3.1) in terms of integrals of the Bergman kernel forms. For any $\phi \in L_{(n,0)}^2(M_2 - V_2)$, by Lemma 3.3 we have for any $z \in M_1 - V_1$,

$$P_1(f^* \phi)(z) = \int_{M_1 - V_1} f^* \phi(\eta) \wedge K_1(z, \eta) = \int_{M_2 - V_2} \phi(\eta) \wedge \sum_{k=1}^m (\text{id}_{M_1}, F_k)^* K_1(z, \eta).$$

On the other hand,

$$P_2(\phi)(\xi) = \int_{M_2 - V_2} \phi(\eta) \wedge K_2(\xi, \eta) \quad \text{for any } \xi \in M_2 - V_2.$$

If we pull back the forms on both sides by f , then

$$f^* P_2(\phi)(z) = \int_{M_2 - V_2} \phi(\eta) \wedge (f, \text{id}_{M_2})^* K_2(z, \eta) \quad \text{for any } z \in M_1 - V_1.$$

Therefore, the Bergman projection transformation law (3.1) translates to

$$\int_{M_2 - V_2} \phi(\eta) \wedge \sum_{k=1}^m (\text{id}_{M_1}, F_k)^* K_1(z, \eta) = \int_{M_2 - V_2} \phi(\eta) \wedge (f, \text{id}_{M_2})^* K_2(z, \eta).$$

As this equality holds for any $\phi \in L^2_{(n,0)}(M_2 - V_2)$, it follows that for any $z \in M_1 - V_1$ and $\eta \in M_2 - V_2$,

$$\sum_{k=1}^m (\text{id}_{M_1}, F_k)^* K_1(z, \eta) = (f, \text{id}_{M_2})^* K_2(z, \eta). \quad (3.4)$$

If we further pull back the forms on both sides by $(\text{id}_{M_1}, f) : (M_1 - V_1) \times (M_1 - V_1) \rightarrow (M_1 - V_1) \times (M_2 - V_2)$, then we obtain for $z, w \in M_1 - V_1$,

$$\sum_{k=1}^m (\text{id}_{M_1}, F_k \circ f)^* K_1(z, w) = (f, f)^* K_2(z, w). \quad (3.5)$$

By using the notation γ_k for the deck transformation $F_k \circ f$, we may write this as

$$\sum_{k=1}^m (\text{id}_{M_1}, \gamma_k)^* K_1(z, w) = (f, f)^* K_2(z, w). \quad (3.6)$$

Note that

$$\begin{aligned} \sum_{k=1}^m (\text{id}_{M_1}, \gamma_k)^* K_1(z, w) &= \sum_{k=1}^m (\gamma_k \circ \gamma_k^{-1}, \gamma_k \circ \text{id}_{M_1})^* K_1(z, w) \\ &= \sum_{k=1}^m (\gamma_k^{-1}, \text{id}_{M_1})^* (\gamma_k, \gamma_k)^* K_1(z, w). \end{aligned}$$

Since γ_k is a biholomorphism on $M_1 - V_1$, we have

$$(\gamma_k, \gamma_k)^* K_1(z, w) = K_1(z, w),$$

and hence

$$\sum_{k=1}^m (\text{id}_{M_1}, \gamma_k)^* K_1(z, w) = \sum_{k=1}^m (\gamma_k^{-1}, \text{id}_{M_1})^* K_1(z, w) = \sum_{k=1}^m (\gamma_k, \text{id}_{M_1})^* K_1(z, w).$$

Theorem 1.6 now follows by combining the above identity with (3.6). \square

4 Proof of Theorem 1.1, part I: Bergman kernels of ball quotients

In this section, we will apply the transformation law in Theorem 1.6 to study the Bergman kernel form of a finite ball quotient and prove the "if" implication in Theorem 1.1. For this part, the restriction of the dimension of the algebraic variety to two is not needed, and we shall therefore consider the situation in an arbitrary dimension n .

Let \mathbb{B}^n denote the unit ball in \mathbb{C}^n and $\text{Aut}(\mathbb{B}^n)$ its (biholomorphic) automorphism group.

Let Γ be a finite subgroup of $\text{Aut}(\mathbb{B}^n)$. As the unitary group $U(n)$ is a maximal compact subgroup of $\text{Aut}(\mathbb{B}^n)$, by basic Lie group theory, there exists some $\psi \in \text{Aut}(\mathbb{B}^n)$ such that $\Gamma \subset \psi^{-1} \cdot U(n) \cdot \psi$. Thus without loss of generality, we can assume $\Gamma \subset U(n)$, i.e., Γ is a finite unitary group. Note that the origin $0 \in \mathbb{C}^n$ is always a fixed point of every element in Γ . We say Γ is *fixed point free* if every $\gamma \in \Gamma - \{\text{id}\}$ has no other fixed point, or equivalently, if every $\gamma \in \Gamma - \{\text{id}\}$ has no fixed point on $\partial\mathbb{B}^n$. In this case, the action of Γ on $\partial\mathbb{B}^n$ is properly discontinuous and $\partial\mathbb{B}^n / \Gamma$ is a smooth manifold.

By a theorem of Cartan [29], the quotient \mathbb{C}^n / Γ can be realized as a normal algebraic subvariety V in some \mathbb{C}^N . To be more precise, we write \mathcal{A} for the algebra of Γ invariant holomorphic polynomials, that is,

$$\mathcal{A} := \{p \in \mathbb{C}[z_1, \dots, z_n] : p \circ \gamma = p \text{ for all } \gamma \in \Gamma\}.$$

By Hilbert's basis theorem, \mathcal{A} is finitely generated. Moreover, we can find a minimal set of homogeneous polynomials $\{p_1, \dots, p_N\}$ such that every $p \in \mathcal{A}$ can be expressed in the form

$$p(z) = q(p_1(z), \dots, p_N(z)) \quad \text{for } z \in \mathbb{C}^n,$$

where q is some holomorphic polynomial in \mathbb{C}^N . The map $Q := (p_1, \dots, p_N) : \mathbb{C}^n \rightarrow \mathbb{C}^N$ is proper and induces a homeomorphism of \mathbb{C}^n / Γ onto $V := Q(\mathbb{C}^n)$. By Remmert's proper mapping theorem (see [30]), V is an analytic variety. As Q is a polynomial holomorphic map, V is furthermore an algebraic variety. The restriction of Q to the unit ball \mathbb{B}^n maps \mathbb{B}^n properly onto a relatively compact domain $G \subset V$. In this way, \mathbb{B}^n / Γ is realized as G by Q . Following [31], we call such Q the *basic map* associated to Γ . The ball quotient $G = \mathbb{B}^n / \Gamma$ is nonsingular if and only if the group Γ is generated by *reflections*, i.e., elements of finite order in $U(n)$ that fix a complex subspace of dimension $n - 1$ in \mathbb{C}^n (see [31]); thus, if Γ is fixed point free and nontrivial, then $G = \mathbb{B}^n / \Gamma$ must have singularities. Moreover, G has smooth boundary if and only if Γ is fixed point free (see [32] for more results along this line).

We are now in a position to state the following theorem, which implies the "if" implication in Theorem 1.1.

Theorem 4.1 *Let G be a domain in an algebraic variety V in \mathbb{C}^N and $\Gamma \subset U(n)$ a finite unitary subgroup with $|\Gamma| = m$. Suppose there exist proper complex analytic*

varieties $V_1 \subset \mathbb{B}^n$, $V_2 \subset G$ and $F : \mathbb{B}^n - V_1 \rightarrow G - V_2$ such that F is an m -sheeted covering map with deck transformation group Γ . If F is algebraic, then the Bergman kernel form of G is algebraic.

Proof Note that the Bergman kernel form of G coincides with that of $\tilde{G} := G - V_2$ by Theorem 2.6, and likewise the Bergman kernel form $K_{\mathbb{B}^n}$ of \mathbb{B}^n coincides with that of $\tilde{B} := \mathbb{B}^n - V_1$. By the transformation law in Theorem 1.6, we have

$$\sum_{\gamma \in \Gamma} (\text{id}_{\mathbb{B}^n}, \gamma)^* K_{\mathbb{B}^n} = (F, F)^* K_G \quad \text{on } \tilde{B} \times \tilde{B}.$$

Since all $\gamma \in \Gamma$ and $K_{\mathbb{B}^n}$ are rational, so is the right hand side of the equation. This implies that K_G is algebraic (see the equivalent condition (c) of algebraicity in Sect. 2.1). \square

Theorem 4.1 applies in particular to Cartan's canonical realization of ball quotients.

Corollary 4.2 *Let $\Gamma \subset U(n)$ be a finite unitary group. Suppose $Q : \mathbb{C}^n \rightarrow \mathbb{C}^N$ is the basic map associated to Γ . Let $G = Q(\mathbb{B}^n)$, which is a relatively compact domain in the algebraic variety $V = Q(\mathbb{C}^n)$. Then the Bergman kernel form of G is algebraic.*

Proof We let

$$Z = \{z \in \mathbb{C}^n : \text{the Jacobian of } Q \text{ at } z \text{ is not full rank}\}.$$

Clearly, Z is a proper complex analytic variety in \mathbb{C}^n . By Remmert's proper mapping theorem, $Q(Z) \subset V$ is a proper complex analytic variety. Moreover, $Q : \mathbb{B}^n - Z \rightarrow G - Q(Z)$ is a covering map with m sheets, where $m = |\Gamma|$, and Γ is its deck transformation group (Note that $Q^{-1}(Q(Z)) = Z$; see [29]). The conclusion now follows from Theorem 4.1. \square

Remark 4.3 Note that the "if" implication in Theorem 1.1 in fact holds under a much weaker assumption than that stipulated in the theorem. In Theorem 4.1 we do not assume $n = 2$ nor that the group Γ is fixed point free. We remark that the formula for the Bergman kernel of the finite ball quotient is also obtained by Huang-Li [25].

5 Proof of Theorem 1.1, part II

In this section, we prove one of the main results of the paper—the "only if" implication in Theorem 1.1. We also prove Corollary 1.3 and 1.4.

Proof of the "only if" implication in Theorem 1.1 Let V and G be as in Theorem 1.1 and assume that G has algebraic Bergman kernel. We shall prove that G is a finite ball quotient. We proceed in several steps.

Step 1. In this step, we prove ∂G is real analytic, and furthermore, real algebraic. For this step, we do not need to assume that the dimension of V is two.

Proposition 5.1 *Let G be a relatively compact domain with smooth strongly pseudoconvex boundary in an n -dimensional ($n \geq 2$) algebraic variety $V \subset \mathbb{C}^N$. If the Bergman kernel K_G of G is algebraic, then the boundary ∂G of G is Nash algebraic, i.e., ∂G is locally defined by a real algebraic function.*

Proof Fix a point $p \in \partial G$. Then there exists a neighborhood U of p in V with canonical extrinsic coordinates $z = (z_1, \dots, z_n)$ on U (see Sect. 2). Write the Bergman kernel form K_G of G as

$$K_G(z, \bar{z}) = K(z, \bar{z})dz \wedge d\bar{z} \quad \text{on } U \cap G,$$

where $dz = dz_1 \wedge \dots \wedge dz_n$, $d\bar{z} = d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$ and $K(z, \bar{z})$ is a real algebraic function on $U \cap G$.

As K is real algebraic, there exist real-valued polynomials $a_1(z, \bar{z}), \dots, a_q(z, \bar{z})$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with $a_q \neq 0$ such that

$$a_q(z, \bar{z})K(z, \bar{z})^q + \dots + a_1(z, \bar{z})K(z, \bar{z}) + a_0(z, \bar{z}) = 0, \quad \text{on } U \cap G. \quad (5.1)$$

Note that when $z \rightarrow \partial G$, we have $K(z, \bar{z}) \rightarrow \infty$ as ∂G is strictly pseudoconvex. We divide both sides of (5.1) by $K(z, \bar{z})^q$ and let $z \rightarrow \partial G$ to obtain

$$a_q(z, \bar{z}) = 0, \quad \text{on } U \cap \partial G.$$

Write $z_k = x_k + iy_k$ for $1 \leq k \leq n$, $z' = (z_1, \dots, z_{n-1})$ and $x' = (x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n)$. By rotation, we can assume that ∂G near p is locally defined by

$$y_n = \varphi(x'),$$

where φ is a smooth function. We then have

$$a_q(z', x_n + i\varphi(x'), \bar{z}', x_n - i\varphi(x')) = 0.$$

By Malgrange's theorem (see [33] and references therein), φ is real analytic and thus, since a_q is a polynomial, also real algebraic. Hence, ∂G is Nash algebraic. \square

Step 2. We now return to the case where V is two-dimensional. We shall prove that ∂G is spherical, where G is as in Theorem 1.1. Fix $p \in \partial G$, and a canonical extrinsic coordinates chart (U, z) of V at p , where $z = (z_1, z_2)$. We again write

$$K_G(z, \bar{z}) = K(z, \bar{z})dz \wedge d\bar{z} \quad \text{on } U \cap G,$$

where $dz = dz_1 \wedge dz_2$ and $d\bar{z} = d\bar{z}_1 \wedge d\bar{z}_2$. Choose a strongly pseudoconvex domain $D \Subset U \cap G$ such that

$$B(p, \delta) \cap D = B(p, \delta) \cap G \quad \text{for some small } \delta > 0.$$

Here $B(p, \delta) = \{z \in U : \|z - p\| < \delta\}$ is the ball centered at p with radius δ with respect to the coordinates (U, z) . Write K_D for the Bergman kernel of D , which is now considered as a function. Then $K_D - K$ extends smoothly across $B(p, \delta) \cap \partial D$ (see [4, 34], see also [25] for a nice and detailed proof of this fact). Consequently,

$$K_D(z, \bar{z}) = K(z, \bar{z}) + h(z, \bar{z}) \quad \text{on } D,$$

where $h(z, \bar{z})$ is real analytic in D and extends smoothly across $B(p, \delta) \cap \partial D$. Let r be a Fefferman defining function of D and express the Fefferman asymptotic expansion of K_D as

$$K_D(z, \bar{z}) = \frac{\phi(z, \bar{z})}{r(z)^3} + \psi(z, \bar{z}) \log r(z) \quad \text{on } D,$$

where ϕ and ψ are smooth functions on D that extend smoothly across $B(p, \delta) \cap \partial D$; see [5]. Thus,

$$K(z, \bar{z}) = \frac{\phi(z, \bar{z}) - h(z, \bar{z})r(z)^3}{r(z)^3} + \psi(z, \bar{z}) \log r(z) \quad \text{on } D. \quad (5.2)$$

As in Step 1, there exist real-valued polynomials $a_1(z, \bar{z}), \dots, a_q(z, \bar{z})$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ with $a_q \neq 0$ for some $q \geq 1$, such that

$$a_q K^q + \dots + a_1 K + a_0 = 0 \quad \text{on } D.$$

If we substitute (5.2) into the above equation and multiply both sides by r^{3q} , then

$$a_q \psi^q r^{3q} (\log r)^q + \sum_{j=0}^{q-1} b_j (\log r)^j = 0 \quad \text{on } D, \quad (5.3)$$

where all b_j for $0 \leq j \leq q-1$ are smooth on D and extend smoothly across $B(p, \delta) \cap \partial D$. We recall the following lemma from [11].

Lemma 5.2 ([11]) *Let $f_0(t), \dots, f_q(t) \in C^\infty(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$. If*

$$f_0(t) + f_1(t) \log t + \dots + f_q(t) (\log t)^q = 0$$

for all $t \in (0, \varepsilon)$, then each $f_j(t)$ for $0 \leq j \leq q$, vanishes to infinite order at 0.

It follows from the above lemma and (5.3) that the coefficient ψ of the logarithmic term vanishes to infinite order at ∂D near p . Since G is two-dimensional, it follows that G is locally spherical near p by [16] (see page 129 where the result is credited to Burns) and [35] (see page 23).

Remark 5.3 Recall from the introduction that the sphericity near p above follows from the affirmation of the Ramadanov Conjecture in two dimensions. This is also the only place where the fact that G is two dimensional is essentially used.

Step 3. In this step, we will prove there is an algebraic branched covering map $F : \mathbb{B}^2 \rightarrow G$ with finitely many sheets. Since we have already shown that ∂G is a Nash algebraic and spherical CR submanifold in \mathbb{C}^N , by a theorem of Huang (see Corollary 3.3 in [20]), it follows that ∂G is CR equivalent to a CR spherical space form $\partial\mathbb{B}^2/\Gamma$ with $\Gamma \subset U(n)$ a finite group with no fixed points on $\partial\mathbb{B}^2$. In particular, there is a CR covering map $f : \partial\mathbb{B}^2 \rightarrow \partial G$ (see the proof of Theorem 3.1 in [20] and also [19]). By Hartogs's extension theorem, f extends as a smooth map $F : \overline{\mathbb{B}^2} \rightarrow V$, holomorphic in \mathbb{B}^2 and sending $\partial\mathbb{B}^2$ onto ∂G . The latter implies that F is moreover algebraic by X. Huang's algebraicity theorem [36]. It is not difficult to see that F sends \mathbb{B}^2 into G . Since F maps $\partial\mathbb{B}^2$ to ∂G , we conclude that F is a proper algebraic mapping $\mathbb{B}^2 \rightarrow G$.

Claim 1. $F : \mathbb{B}^2 \rightarrow G$ is surjective.

Proof of Claim 1 By the properness of F , $F(\mathbb{B}^2)$ is closed in G . Let us denote by

$$Z := \{z \in \mathbb{B}^2 : F \text{ is not full rank at } z\}.$$

Since F is a local biholomorphic map at every point of $\partial\mathbb{B}^2$, Z is a finite set. We also note that if $p \in \mathbb{B}^2 - Z$, then $F(p)$ is a smooth point of V , and $F(p)$ is an interior point of $F(\mathbb{B}^2)$. Assume, in order to reach a contradiction, that $F(\mathbb{B}^2) \neq G$. Since $F(\mathbb{B}^2)$ is closed in G , its complement $G \setminus F(\mathbb{B}^2)$ is then a non-empty open subset of G . Note that any boundary point of $F(\mathbb{B}^2)$ in G can only be in $F(Z)$. But $F(Z)$ is a finite set, which cannot separate the (non-empty) interior of $F(\mathbb{B}^2)$ and the (non-empty) open complement $G \setminus F(\mathbb{B}^2)$ in the domain G . This is the desired contradiction and, hence, $F(\mathbb{B}^2) = G$. \square

Now, we let $T := F^{-1}(F(Z)) \supset Z$. Then T is a compact analytic subvariety of \mathbb{B}^2 and thus is a finite set. Consider the restriction of F :

$$F|_{\mathbb{B}^2 - T} : \mathbb{B}^2 - T \rightarrow G - F(Z),$$

still denoted by F . Clearly, F is a proper surjective map. Since F is also a local biholomorphism, F is a finite covering map.

Note that $\mathbb{B}^2 - T$ is simply connected. It follows that the deck transformation group $\tilde{\Gamma} = \{\tilde{\gamma}_k\}_{k=1}^m$ of the covering map $F : \mathbb{B}^2 - T \rightarrow G - f(Z)$ acts transitively on each fiber. Since each $\tilde{\gamma}_k$ is a biholomorphism from $\mathbb{B}^2 - T$ to $\mathbb{B}^2 - T$, it extends to an automorphism of \mathbb{B}^2 . Consequently,

$$\tilde{\Gamma} = \{\tilde{\gamma} \in \text{Aut}(\mathbb{B}^2) : F \circ \tilde{\gamma} = F \text{ on } \mathbb{B}^2\}.$$

Recall that Γ is the deck transformation group of the original covering map $f : \partial\mathbb{B}^2 \rightarrow \partial G$. From this, it is clear that we can identify Γ with $\tilde{\Gamma}$. From now on, we will simply use the notation Γ for either group.

Note that Z and T are both closed under the action of Γ , and $(\mathbb{B}^2 - T)/\Gamma$ is biholomorphic to $G - f(Z)$.

Claim 2: If $z, w \in \mathbb{B}^2$ satisfy $F(z) = F(w)$, then $w = \gamma(z)$ for some $\gamma \in \Gamma$. Consequently, $T = Z$.

Proof of Claim 2 We only need to prove the first assertion. If both z, w are in $\mathbb{B}^2 - T$, then the conclusion is clear as Γ acts transitively on each fiber of the covering map $F : \mathbb{B}^2 - T \rightarrow G - f(Z)$. Next we assume one of z and w is in T . Seeking a contradiction, suppose $w \neq \gamma(z)$ for every $\gamma \in \Gamma$. Writing $q := F(z) = F(w)$, there are then points in distinct orbits of Γ that are mapped to q . Writing t for the number of orbits of Γ that are mapped to q , we must then have $t \geq 2$. Pick p_1, \dots, p_t from these t distinct orbits of Γ . Since T is a finite set, we can choose, for each $1 \leq i \leq t$, some disjoint neighborhoods U_i of p_i such that $U_i \cap T \subseteq \{p_i\}$. Moreover, we can make $\gamma(U_i) \cap U_j = \emptyset$ for all $\gamma \in \Gamma$ if $i \neq j$. Consequently, $F(U_i - \{p_i\}) \cap F(U_j - \{p_j\}) = \emptyset$. Note there is a small open subset W containing q such that $W \subseteq \bigcup_{i=1}^t F(\bigcup_{\gamma \in \Gamma} \gamma(U_i)) = \bigcup_{i=1}^t F(U_i)$. Thus $W - \{q\} \subseteq \bigcup_{i=1}^t F(U_i - \{p_i\})$. But the sets $F(U_i - p_i) \cap (W - \{q\})$ are open and disjoint, and we can choose $W - \{q\}$ to be connected. This is a contradiction. Thus we must have $t = 1$ and $w = \gamma(z)$ for some $\gamma \in \Gamma$. \square

Recall that 0 is assumed to be the only fixed point for elements in Γ . We write $q_0 := F(0)$ and prove that q_0 is the only possible singularity in G . Also, recall that all singularities of G are assumed to be isolated and normal.

Claim 3. G can only have a singularity at q_0 .

Proof of Claim 3 Suppose q_1 is a (normal) singular point in G and $q_1 \neq q_0$. Since F is onto, there exists some $p_1 \in \mathbb{B}^2$ such that $f(p_1) = q_1$.

First, note that we can find a small neighborhood U_0 of p_1 , and a small neighborhood W of q_1 in G such that

- (i) $U_0 \cap T = \{p_1\}$;
- (ii) F is injective on U_0 ;
- (iii) $W \subseteq F(U_0)$ and $W \cap F(Z) = \{q_1\}$.

It is easy to see that we can make (i) and (ii) hold. It is guaranteed by Claim 2 (see its proof) that we can find $W \subseteq F(U_0)$; the second condition in (iii) is then easy to satisfy, since $F(Z)$ is a finite set. Now, we let $U := U_0 \cap F^{-1}(W)$, which is an open subset of \mathbb{B}^2 containing p_1 . Then $F : U - \{p_1\} \rightarrow W - \{q_1\}$ is a biholomorphism. We let $g : W - \{q_1\} \rightarrow U - \{p_1\}$ denote its inverse. By the normality of q_1 , we can assume that g is the restriction of some holomorphic map \widehat{g} defined on some open set $\widehat{W} \subset \mathbb{C}^N$, where \widehat{W} contains W . Since $g \circ F|_{U - \{p_1\}}$ equals the identity map, $\widehat{g} \circ F$ equals identity on U by continuity. Similarly $F \circ (\widehat{g}|_W)$ equals the identity on W . Therefore, q_1 cannot be a singular point. \square

By Claims 2 and 3, we also see that $T = Z = \{0\}$ or \emptyset . Therefore, F gives a holomorphic algebraic branched covering map from \mathbb{B}^2 to G with a possible branch point at 0. This completes the proof of the "only if" implication in Theorem 1.1. \square

Remark 5.4 We reiterate (see Remark 5.3 above) that in the above proof, the condition that $\dim V = 2$ is only used in the second step where we apply the affirmative solution of the Ramadanov conjecture in \mathbb{C}^2 [16, 35].

We shall now prove Corollaries 1.3 and 1.4.

Proof of Corollary 1.3 By Theorem 1.1, it follows that G can be realized as a finite ball quotient \mathbb{B}^2 / Γ by an algebraic map for some finite unitary group Γ with no fixed point on $\partial\mathbb{B}^2$. We must prove that $\Gamma = \{\text{id}\}$. Suppose not. But, then G must have a singular point (see [31]), which is a contradiction. \square

Proof of Corollary 1.4 The algebraic case follows immediately from Corollary 1.3. Thus, we only need to consider the rational case. First, as a consequence of the algebraic case, there exists an algebraic biholomorphic map $f : G \rightarrow \mathbb{B}^2$. It remains to establish that f is in fact rational. This follows immediately from a result by Bell [17]. For the convenience of the readers, however, we sketch an independent proof here.

Denote by K_G and $K_{\mathbb{B}^2}$ the Bergman kernels (now considered as functions) of G and \mathbb{B}^2 , respectively. By the transformation law, they are related as

$$\begin{aligned} K_G(z, w) &= \det(Jf(z)) \cdot K_B(f(z), f(w)) \cdot \overline{\det(Jf(w))} \\ &= \frac{2!}{\pi^2} \det(Jf(z)) \cdot \overline{\det(Jf(w))} \cdot \frac{1}{(1 - f(z) \cdot \overline{f(w)})^3}. \end{aligned} \quad (5.4)$$

We may assume $0 \in G$ by translating G if necessary and, by composing f with an automorphism of \mathbb{B}^2 , we may also assume $f(0) = 0$. Thus, at $w = 0$, we have

$$K_G(z, 0) = \frac{2!}{\pi^2} \det(Jf(z)) \cdot \overline{\det(Jf(0))}.$$

It follows that

$$\det(Jf(z)) = \det(Jf(0)) \frac{K_G(z, 0)}{K_G(0, 0)}. \quad (5.5)$$

In particular, this implies that $K_G(z, 0) \neq 0$ for any $z \in D$. We evaluate (5.4) on the diagonal $w = z$ and use (5.5) to obtain

$$K_G(z, z) = \frac{2!}{\pi^2} |\det(Jf(0))|^2 \frac{|K_G(z, 0)|^2}{|K_G(0, 0)|^2} \frac{1}{(1 - \|f(z)\|^2)^3}.$$

Taking the logarithm of both sides yields

$$\begin{aligned} &\log K_G(z, z) + 3 \log(1 - \|f(z)\|^2) \\ &= \log \frac{2!}{\pi^2} + \log |\det(Jf(0))|^2 + \log |K_G(z, 0)|^2 - \log |K_G(0, 0)|^2. \end{aligned}$$

For $j = 1, 2$, we apply the derivative $\frac{\partial}{\partial \bar{z}_j}$ to both sides and obtain

$$\frac{1}{K_G(z, z)} \frac{\partial K_G(z, z)}{\partial \bar{z}_j} - \frac{3}{1 - \|f(z)\|^2} \sum_{i=1}^2 \frac{\partial \overline{f_i(z)}}{\partial \bar{z}_j} f_i(z) = \frac{1}{K_G(z, 0)} \frac{\partial \overline{K_G(z, 0)}}{\partial \bar{z}_j}.$$

Complexifying the above equation and evaluating it at $w = 0$, after rearrangement, we obtain

$$\sum_{i=1}^2 \frac{\partial \overline{f_i}}{\partial \overline{z_j}}(0) f_i(z) = \frac{1}{3} \left(\frac{1}{K_G(z, 0)} \frac{\partial K_G}{\partial \overline{z_j}}(z, 0) - \frac{1}{\overline{K_G(0, 0)}} \frac{\partial \overline{K_G}}{\partial \overline{z_j}}(0, 0) \right).$$

Note this is a linear system for $f(z) = (f_1(z), f_2(z))$ and the coefficient matrix $\overline{Jf(0)}$ is non-singular. By solving this linear system for f , it is immediately clear that the rationality of K_G implies that of f . \square

Remark 5.5 Corollary 1.3 implies, in particular, that the Burns–Shnider domains in \mathbb{C}^2 (see page 244 in [37]) cannot have algebraic Bergman kernels. In fact, this holds for any Burns–Shnider domain in \mathbb{C}^n for $n \geq 2$, which can be seen as follows. By Proposition 5.1, if the Bergman kernel were algebraic, then the boundary would be Nash algebraic. While this can be seen to not be so by inspection, a contradiction would also be reached by the Huang–Ji Riemann mapping theorem [27] since the boundary of a Burns–Shnider domain is spherical while the domain itself is not biholomorphic to the unit ball.

6 Counterexample in higher dimension

In this section, we construct a 3–dimensional reduced Stein space G with only one normal singularity and compact, smooth strongly pseudoconvex boundary, realized as a relatively compact domain in a complex algebraic variety V in \mathbb{C}^4 . We will show that its Bergman kernel is algebraic, while G is not biholomorphic to any finite ball quotient \mathbb{B}^n/Γ , which shows that Theorem 1.1 cannot hold in higher dimensions.

Let G be defined as

$$G = \{w = (w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 < 1, \quad w_1 w_4 = w_2 w_3\}.$$

Then G is a relatively compact domain in the complex algebraic variety

$$V = \{w \in \mathbb{C}^4 : w_1 w_4 = w_2 w_3\}. \quad (6.1)$$

Since G is a closed algebraic subvariety of $\mathbb{B}^4 \subset \mathbb{C}^4$, G is a reduced Stein space. Note that 0 is the only singularity of V . Moreover it is a normal singularity as it is a hypersurface singularity of codimension 3 (> 2 ; see [38]). It is also easy to verify that G has smooth strongly pseudoconvex boundary in V .

Proposition 6.1 *The boundary $M = \partial G$ of G is homogeneous and non-spherical.*

Proof Consider the product complex manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$. For $j = 1, 2$, let $\pi_j : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be the projection map to the j -th component and let $(L_0, h_0) \rightarrow \mathbb{CP}^1$ be the tautological line bundle L_0 with its standard Hermitian metric h_0 . We set the

Hermitian line bundle (L, h) over $\mathbb{CP}^1 \times \mathbb{CP}^1$ to be:

$$(L, h) := \pi_1^*(L_0, h_0) \otimes \pi_2^*(L_0, h_0).$$

We begin the proof with the following claim.

Claim 1. Let $(L, h) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be as above and let $S(L) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be its unit circle bundle. Then M is CR diffeomorphic to $S(L)$ by the restriction of biholomorphic map.

Proof of Claim 1 Note that the circle bundle $S(L) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ can be written as

$$S(L) = \left\{ \left(\lambda(\zeta_1, z_1) \otimes (\zeta_2, z_2), [\zeta_1, z_1], [\zeta_2, z_2] \right) : \begin{array}{l} [\zeta_1, z_1] \in \mathbb{CP}^1, [\zeta_2, z_2] \in \mathbb{CP}^1 \\ |\lambda|^2(|\zeta_1|^2 + |z_1|^2)(|\zeta_2|^2 + |z_2|^2) = 1 \end{array} \right\}. \quad (6.2)$$

Define $F : L \rightarrow \mathbb{C}^4$ as

$$F\left(\lambda(\zeta_1, z_1) \otimes (\zeta_2, z_2), [\zeta_1, z_1], [\zeta_2, z_2]\right) = (\lambda\zeta_1\zeta_2, \lambda z_1\zeta_2, \lambda\zeta_1 z_2, \lambda z_1 z_2). \quad (6.3)$$

Then the map F gives a biholomorphism that sends a neighborhood of $S(L)$ in L to a neighborhood of M in $V \subset \mathbb{C}^4$. This proves the claim. \square

Note that $S(L)$ is homogeneous (see [15]) and non-spherical by Theorem 12 in [39]. Thus, M is homogeneous and non-spherical. \square

Proposition 6.2 *The Bergman kernel form K_G of G is algebraic.*

Proof Set

$$\Omega := \{(\lambda, z) = (\lambda, z_1, z_2) \in \mathbb{C}^3 : |\lambda|^2(1 + |z_1|^2)(1 + |z_2|^2) < 1\}. \quad (6.4)$$

Note that Ω is an unbounded domain with smooth boundary in \mathbb{C}^3 . Moreover, Ω has a rational Bergman kernel form K_Ω (see Appendix 7 for a proof of this fact). Define the map $F : \mathbb{C}^3 \rightarrow \mathbb{C}^4$ as

$$F(\lambda, z_1, z_2) := (\lambda, \lambda z_1, \lambda z_2, \lambda z_1 z_2).$$

We note that $F(\mathbb{C}^3)$ is contained in V as defined by (6.1). And F is a holomorphic embedding on $\mathbb{C}^3 - \{\lambda = 0\}$. Moreover, $F(\Omega) \subset G$ and

$$F : \widetilde{\Omega} := \Omega - \{\lambda = 0\} \rightarrow \widetilde{G} := G - \{w_1 = 0\}$$

is a biholomorphism. By Theorem 2.6, the Bergman kernel form $K_{\widetilde{\Omega}}$ of $\widetilde{\Omega}$ is the restriction (pullback) of K_Ω to $\widetilde{\Omega}$. Thus, $K_{\widetilde{\Omega}}$ is rational. By the transformation law (1.1), we have

$$K_{\widetilde{\Omega}} = (F, F)^* K_{\widetilde{G}}.$$

This implies that $K_{\tilde{G}}$ is algebraic (see the equivalent condition (c) in §2.1), and thus K_G is also algebraic by Theorem 2.6. \square

Before we prove G is not biholomorphic to any finite ball quotient, we pause to study the following bounded domain U in \mathbb{C}^3 :

$$U := \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 < |w_1|^2 \right\}.$$

Proposition 6.3 *The domain U has algebraic Bergman kernel and its boundary is non-spherical at every smooth boundary point.*

Proof Let $\pi : \mathbb{C}^4 \rightarrow \mathbb{C}^3$ be the projection map defined by

$$\pi(w_1, w_2, w_3, w_4) := (w_1, w_2, w_3).$$

Let \overline{G} be the closure of G in \mathbb{C}^4 . Then the image of \overline{G} under the projection π is

$$\begin{aligned} \widehat{U} := \pi(\overline{G}) &= \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 \right. \\ &\quad \leq |w_1|^2, w_1 \neq 0 \} \\ &\quad \cup \left\{ (0, w_2, w_3) \in \mathbb{C}^3 : |w_2|^2 + |w_3|^2 \leq 1, w_2 w_3 = 0 \right\} \\ &= \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 \right. \\ &\quad \leq |w_1|^2, |w_2|^2 + |w_3|^2 \leq 1 \}. \end{aligned}$$

Note that $\widehat{U}^\circ = U$ and $\widehat{U} = \overline{U}$, where \widehat{U}° denotes the interior of \widehat{U} . But $U \neq \pi(G)$. On the other hand if we remove the variety $\{w_1 = 0\}$, then the projection map

$$\pi : G - \{w_1 = 0\} \rightarrow U$$

is an algebraic biholomorphism. Consequently, by Theorem 2.6 the Bergman kernel form K_U of U is algebraic. This proves the first part of the proposition.

To prove the second part of the proposition (i.e., the non-sphericity), we observe that the boundary ∂U of U is given by

$$\begin{aligned} \partial U &= \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 \right. \\ &\quad = |w_1|^2, w_1 \neq 0 \} \\ &\quad \cup \left\{ (0, w_2, w_3) \in \mathbb{C}^3 : |w_2|^2 + |w_3|^2 \leq 1, w_2 w_3 = 0 \right\} \\ &= \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 \right. \\ &\quad = |w_1|^2, |w_2|^2 + |w_3|^2 \leq 1 \}. \end{aligned}$$

Write

$$\partial U = (\partial U \cap \{w_1 \neq 0\}) \cup (\partial U \cap \{w_1 = 0\}).$$

Since the projection map π is a biholomorphism from $\overline{G} - \{w_1 = 0\}$ to $\widehat{U} - \{w_1 = 0\}$, every point $p \in \partial U \cap \{w_1 \neq 0\}$ is a smooth point of ∂U , and, moreover, ∂U is strictly pseudoconvex and non-spherical at p . We note that a defining function for ∂U near p is given by

$$\rho = |w_1|^4 + |w_1|^2(|w_2|^2 + |w_3|^2) + |w_2 w_3|^2 - |w_1|^2.$$

Furthermore, it is easy to verify that every other point $q \in \partial U \cap \{w_1 = 0\}$ is not a smooth boundary point of U . This proves the second part of the assertion. \square

We are now ready to show that G is indeed a counterexample to the conclusion of Theorem 1.1 in three dimensions.

Proposition 6.4 *G is not biholomorphic to any finite ball quotient.*

Proof Seeking a contradiction, we suppose G is biholomorphic to a finite ball quotient \mathbb{B}^3/Γ , where $\Gamma \subset U(n)$ is a finite unitary group. We realize \mathbb{B}^3/Γ as the image $G_0 \subset \mathbb{C}^N$ of \mathbb{B}^3 under the basic map Q associated to Γ , where $Q = (p_1, \dots, p_N) : \mathbb{C}^3 \rightarrow \mathbb{C}^N$ gives a proper map from \mathbb{B}^3 to G_0 . Let F be a biholomorphism from $G_0 \cong \mathbb{B}^3/\Gamma$ to G . Then there is an analytic variety $W_0 \subset G_0$ such that

$$F : G_0 - W_0 \rightarrow G - \{w_1 = 0\} \text{ is a biholomorphism.}$$

There also exists an analytic variety W such that $W = Q^{-1}(W_0)$ and thus $Q : \mathbb{B}^3 - W \rightarrow G - W_0$ is proper and onto. Set

$$f := \pi \circ F \circ Q : \mathbb{B}^3 - W \rightarrow U = \pi(G - \{w_1 = 0\}),$$

where π is the projection defined in the proof of Proposition 6.3 and is a biholomorphism from $G - \{w_1 = 0\}$ to U . Note that f is proper. Since $U \subset \mathbb{C}^3$, we can write f as (f_1, f_2, f_3) .

Claim 2. There is a sequence $\{\zeta_i\} \subset \mathbb{B}^3 - W$ with $\zeta_i \rightarrow \zeta^* \in \partial \mathbb{B}^3 - \overline{W}$ such that

$$f(\zeta_i) \rightarrow p^* \in \partial U \cap \{w_1 \neq 0\}.$$

Proof of Claim 2 Suppose not. Then for any $\{\zeta_i\} \subset \mathbb{B}^3 - W$ with $\zeta_i \rightarrow \zeta^* \in \partial \mathbb{B}^3 - \overline{W}$, every convergent subsequence of $f(\zeta_i)$ converges to some point in $\partial U \cap \{w_1 = 0\}$. That is to say, if $f(\zeta_{i_k})$ is convergent, then $f_1(\zeta_{i_k}) \rightarrow 0$. Note that U is bounded. Thus, $f_1(\zeta_i) \rightarrow 0$ for any $\{\zeta_i\} \subset \mathbb{B}^3 - W$ with $\zeta_i \rightarrow \zeta^* \in \partial \mathbb{B}^3 - \overline{W}$. By a standard argument using analytic disks attached to $\partial \mathbb{B}^3 - \overline{W}$, we see that

$$f_1 = 0 \quad \text{on } \mathbb{B}^3 - W.$$

This is a contradiction. \square

Let ζ_i, ζ^* and p^* be as in Claim 2. Note that ζ^* is a smooth strictly pseudoconvex boundary point of $\mathbb{B}^3 - W$, and p^* is a smooth strictly pseudoconvex boundary point

of U (see the proof of Proposition 6.3). It follows from [40] (see page 239) that f extends to a Hölder- $\frac{1}{2}$ continuous map on a neighborhood of ζ^* in $\overline{\mathbb{B}^3}$. Since f is proper $\mathbb{B}^3 - W \rightarrow U$, its extension to the boundary is a (Hölder- $\frac{1}{2}$) continuous, nonconstant CR map sending a piece of $\partial\mathbb{B}^3$ containing ζ^* to a piece of ∂U containing p^* . By [41], f extends holomorphically to a neighborhood of ζ^* , since both boundaries are real analytic (in fact, real-algebraic). Now, since f is non-constant and sends a strongly pseudoconvex hypersurface to another, it must be a CR diffeomorphism, which would mean that ∂U is locally spherical near p^* . This contradicts Proposition 6.1. \square

We conclude this section by a couple of remarks.

Remark 6.5 Since the Bergman kernel forms of G and U are algebraic, it follows from the proof of Theorem 1.1 in Section 5 (see Step 2) that the coefficients of the logarithmic term in Fefferman's expansions of K_G and K_V both vanish to infinite order at every smooth boundary point. The reduced normal Stein space G gives the counterexample mentioned in Remark 1.2. The domain $U \subset \mathbb{C}^3$ establishes the following fact, which implies that the Ramadanov conjecture fails for non-smooth domains in higher dimension.

There exists a bounded domain in \mathbb{C}^3 with smooth, real-algebraic boundary away from a 1-dimensional complex curve such that every smooth boundary point is strongly pseudoconvex and non-spherical, while the coefficient of the logarithmic term in Fefferman's asymptotic expansion of the Bergman kernel vanishes to infinite order at every smooth boundary point.

Remark 6.6 Using the same idea as in the above example, we can actually construct significantly more general examples of higher dimensional domains in affine algebraic varieties $V \subset \mathbb{C}^N$ with similar properties. Indeed, let X be a compact Hermitian symmetric space of rank at least 2. Write

$$X = X_1 \times \cdots \times X_t, \quad t \geq 1,$$

where X_1, \dots, X_t are the irreducible factors of X . Fix a Kähler-Einstein metric ω_j on X_j and let $(\widehat{L}_j, \widehat{h}_j)$ be the top exterior product $\Lambda^n T^{1,0}$ of the holomorphic tangent bundle over X_j with the metric induced from ω_j . Then there is a homogeneous line bundle (L_j, h_j) with a Hermitian metric h_j such that its p_j -th tensor power gives $(\widehat{L}_j, \widehat{h}_j)$, where p_j is the genus of X_j . (see [15] for more details).

Let π_j be the projection from X onto the j -th factor X_j for $1 \leq j \leq t$. Define the line bundle L over X with a Hermitian metric h to be:

$$(L, h) := \pi_1^*(L_1, h_1) \otimes \cdots \otimes \pi_t^*(L_t, h_t).$$

Let (L^*, h^*) be the dual line bundle of (L, h) . Write $D(L^*)$ and $S(L^*)$ for the associated unit disc and unit circle bundle. The specific example above is the special case $t = 2$ and $X_1 = X_2 = \mathbb{CP}^1$. Proceeding as in that example, one finds that there is a canonical way to map L^* to \mathbb{C}^N , for some N , induced by the minimal embedding of X into some complex projective space (see [42]). If we denote this map $L^* \rightarrow \mathbb{C}^N$ by F (in the example above, the map F is as given by (6.3)), then F sends the zero section

of L^* to the point 0 and is a holomorphic embedding away from the zero section. It follows that the image of $D(L^*)$ under the map F is a domain G with a singular point at 0. The boundary of G is given by the image of $S(L^*)$. It is not spherical since $S(L^*)$ is not by [39]. Moreover, as the Bergman kernel form of $D(L^*)$ is algebraic by [15], the Bergman kernel form of G is also algebraic by Theorem 2.6.

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Declarations

Conflict of interest The authors state that there is no conflict of interest.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix

In this section, we will prove the claim that the Bergman kernel of the domain Ω in \mathbb{C}^3 as defined in (6.4) is rational. This fact actually follows from a general theorem in [15] (see Theorem 3.3 in [15] and its proof). We include here a proof in this particular example for the convenience of readers and self-containedness of this paper. In fact, we shall compute the Bergman kernel of Ω explicitly (Theorem 7.3 below).

Recall that

$$\Omega := \{(z, \lambda) = (z_1, z_2, \lambda) \in \mathbb{C}^3 : |\lambda|^2(1 + |z_1|^2)(1 + |z_2|^2) < 1\}. \quad (7.1)$$

We let

$$h(z) := (1 + |z_1|^2)(1 + |z_2|^2),$$

and denote the defining function by

$$\rho(z, \lambda) := |\lambda|^2(1 + |z_1|^2)(1 + |z_2|^2) - 1.$$

We recall that the Bergman space on Ω is defined as

$$A^2(\Omega) := \{f(z, \lambda) \text{ is holomorphic in } \Omega : i \int_{\Omega} |f(z, \lambda)|^2 dz \wedge d\lambda \wedge d\bar{z} \wedge d\bar{\lambda} < \infty\}, \quad (7.2)$$

and let

$$A_m^2(\Omega) := \{f(z) \text{ is holomorphic in } \mathbb{C}^2 : \lambda^m f(z) \in A^2(\Omega)\}. \quad (7.3)$$

Note that the L^2 norm of $\lambda^m f(z)$ is given by

$$\begin{aligned}\|\lambda^m f(z)\|^2 &= i \int_{\Omega} |\lambda|^{2m} |f(z)|^2 dz \wedge d\lambda \wedge d\bar{z} \wedge d\bar{\lambda} \\ &= \int_{z \in \mathbb{C}^2} \left(\int_{|\lambda|^2 < h(z)^{-1}} |\lambda|^{2m} i \, d\lambda \wedge d\bar{\lambda} \right) |f(z)|^2 dz \wedge d\bar{z}.\end{aligned}$$

We can rewrite the inner integral as follows:

$$\begin{aligned}\int_{|\lambda|^2 < h(z)^{-1}} |\lambda|^{2m} i \, d\lambda \wedge d\bar{\lambda} &= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{h(z)}}} r^{2m} 2r dr d\theta \\ &= 2\pi \int_0^{\frac{1}{h(z)}} r^m dr = \frac{2\pi}{m+1} h(z)^{-(m+1)}.\end{aligned}$$

Thus,

$$\|\lambda^m f(z)\|^2 = \frac{2\pi}{m+1} \int_{\mathbb{C}^2} |f(z)|^2 h(z)^{-(m+1)} dz \wedge d\bar{z}. \quad (7.4)$$

If we introduce the weighted Bergman space

$$A^2(\mathbb{C}^2, h^{-(m+1)}) = \{f(z) \text{ is holomorphic in } \mathbb{C}^2 : \int_{\mathbb{C}^2} |f(z)|^2 h(z)^{-(m+1)} dz \wedge d\bar{z} < \infty\},$$

then

$$A_m^2(\Omega) = \{\lambda^m f(z) : f(z) \in A^2(\mathbb{C}^2, h^{-(m+1)})\}. \quad (7.5)$$

We note that $A_{m_1}^2(\Omega)$ and $A_{m_2}^2(\Omega)$ are orthogonal to each other if $m_1 \neq m_2$. We can therefore orthogonally decompose $A^2(\Omega)$ into a direct sum as follows.

Lemma 7.1

$$A^2(\Omega) = \bigoplus_{m=0}^{\infty} A_m^2(\Omega).$$

Proof Let $f(z, \lambda) \in A^2(\Omega)$. If we fix $z \in \mathbb{C}^2$, then λ is contained in the disc $\{\lambda \in \mathbb{C}, |\lambda|^2 < h(z)^{-1}\}$. By taking the Taylor expansion at $\lambda = 0$, we obtain

$$f(z, \lambda) = \sum_{j=0}^{\infty} a_j(z) \lambda^j, \quad \text{for } |\lambda|^2 < h(z)^{-1},$$

where each $a_j(z)$ is holomorphic on \mathbb{C}^2 . We shall first write $\|f(z, \lambda)\|^2$ in terms of $\{a_j(z)\}_{j=0}^\infty$. We have

$$\|f(z, \lambda)\|^2 = \int_{z \in \mathbb{C}^2} \left(\int_{|\lambda|^2 < h^{-1}(z)} |f(z, \lambda)|^2 i d\lambda \wedge d\bar{\lambda} \right) dz \wedge d\bar{z}.$$

The inner integral can be computed as

$$\begin{aligned} \int_{|\lambda|^2 < h^{-1}(z)} |f(z, \lambda)|^2 i d\lambda \wedge d\bar{\lambda} &= \lim_{\varepsilon \rightarrow 0^+} \sum_{s=0}^\infty \sum_{t=0}^\infty a_s(z) \overline{a_t(z)} \int_{|\lambda|^2 < h^{-1}(z) - \varepsilon} \lambda^s \bar{\lambda}^t i d\lambda \wedge d\bar{\lambda} \\ &= \lim_{\varepsilon \rightarrow 0^+} \sum_{j=0}^\infty \frac{2\pi}{j+1} |a_j(z)|^2 (h(z)^{-1} - \varepsilon)^{j+1} \\ &= \sum_{j=0}^\infty \frac{2\pi}{j+1} |a_j(z)|^2 h(z)^{-(j+1)}, \end{aligned}$$

where the last equality follows from the Monotone Convergence theorem. Therefore,

$$\|f(z, \lambda)\|^2 = \sum_{j=0}^\infty \frac{2\pi}{j+1} \int_{z \in \mathbb{C}^2} |a_j(z)|^2 h(z)^{-(j+1)} dz \wedge d\bar{z},$$

which immediately implies that each $a_j(z)$ is contained in $A^2(\mathbb{C}^2, h^{-(j+1)})$.

Suppose $f(z, \lambda) \perp A_m^2(\Omega)$. Then for any $\lambda^m g(z) \in A_m^2(\Omega)$,

$$\begin{aligned} 0 &= i \int_{\Omega} f(z, \lambda) \overline{\lambda^m g(z)} dz \wedge d\lambda \wedge d\bar{z} \wedge d\bar{\lambda} \\ &= \int_{z \in \mathbb{C}^2} \left(\int_{|\lambda|^2 < h(z)^{-1}} f(z, \lambda) \overline{\lambda^m} i d\lambda \wedge d\bar{\lambda} \right) \overline{g(z)} dz \wedge d\bar{z}. \end{aligned}$$

The inner integral can be computed as follows

$$\begin{aligned} \int_{|\lambda|^2 < h^{-1}(z)} f(z, \lambda) \overline{\lambda^m} i d\lambda \wedge d\bar{\lambda} &= \lim_{\varepsilon \rightarrow 0^+} \sum_{j=0}^\infty a_j(z) \int_{|\lambda|^2 < h(z)^{-1} - \varepsilon} \lambda^j \overline{\lambda^m} i d\lambda \wedge d\bar{\lambda} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{2\pi}{m+1} a_m(z) (h(z)^{-1} - \varepsilon)^{m+1} \\ &= \frac{2\pi}{m+1} a_m(z) h(z)^{-(m+1)}. \end{aligned}$$

Therefore,

$$0 = \frac{2\pi}{m+1} \int_{z \in \mathbb{C}^2} a_m(z) \overline{g(z)} h(z)^{-(m+1)} dz \wedge d\bar{z}, \quad \text{for any } g \in A^2(\mathbb{C}^2, h^{-(m+1)}).$$

Since a_m belongs to the space $A^2(\mathbb{C}^2, h^{-(m+1)})$, we get $a_m = 0$. Therefore, the direct sum of $A_m^2(\Omega)$ for $0 \leq m < \infty$ generates $A^2(\Omega)$. \square

Since $A_m^2(\Omega)$ can be identified with the weighted Bergman space $A^2(\mathbb{C}^2, h^{-(m+1)})$ as in (7.5), we can find an explicit orthonormal basis and compute its reproducing kernel.

Proposition 7.2 *Let $m \geq 1$. The reproducing kernel of $A_m^2(\Omega)$ is*

$$K_m^*(z, \lambda, \bar{w}, \bar{\tau}) = \frac{(m+1)m^2}{(2\pi)^3} \lambda^m \bar{\tau}^m (1 + z_1 \bar{w}_1)^{m-1} (1 + z_2 \bar{w}_2)^{m-1}, \quad (7.6)$$

where $(z, \lambda), (w, \tau)$ are points in Ω .

Proof Denote

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}, \quad \text{for any multi-index } \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_{\geq 0}^2.$$

By (7.5), since Ω is Reinhardt, it is easy to see that

$$\{\lambda^m z^\alpha : z^\alpha \in A^2(\mathbb{C}^2, h^{-(m+1)})\}$$

forms an orthogonal basis of $A_m^2(\Omega)$. We shall compute the norm for each $\lambda^m z^\alpha$. Using (7.4), we have

$$\begin{aligned} \|\lambda^m z^\alpha\|^2 &= \frac{2\pi}{m+1} \int_{\mathbb{C}^2} |z^\alpha|^2 (1 + |z_1|^2)^{-(m+1)} (1 + |z_2|^2)^{-(m+1)} dz \wedge d\bar{z} \\ &= \frac{2\pi}{m+1} \int_{\mathbb{C}} |z_1|^{2\alpha_1} (1 + |z_1|^2)^{-(m+1)} i dz_1 \wedge d\bar{z}_1 \\ &\quad \int_{\mathbb{C}} |z_2|^{2\alpha_2} (1 + |z_2|^2)^{-(m+1)} i dz_2 \wedge d\bar{z}_2 \\ &= \frac{(2\pi)^3}{m+1} \int_0^\infty r_1^{\alpha_1} (1 + r_1)^{-(m+1)} dr_1 \cdot \int_0^\infty r_2^{\alpha_2} (1 + r_2)^{-(m+1)} dr_2. \end{aligned}$$

By the elementary integral identity

$$\int_0^\infty r^p \frac{1}{(1+r)^q} dr = \frac{(q-p-2)! p!}{(q-1)!},$$

for any nonnegative integers p, q with $q \geq p+2$, (7.7)

we get

$$\|\lambda^m z^\alpha\|^2 = \begin{cases} \frac{(2\pi)^3}{m+1} \frac{(m-\alpha_1-1)!(m-\alpha_2-1)!\alpha!}{m!^2} & \text{if } \alpha_1, \alpha_2 \leq m-1, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, $\left\{ \frac{\lambda^m z^\alpha}{\|\lambda^m z^\alpha\|} : 0 \leq \alpha_1, \alpha_2 \leq m-1 \right\}$ is an orthonormal basis of $A_m^2(\Omega)$, and the reproducing kernel of $A_m^2(\Omega)$ is given by

$$\begin{aligned} K_m^*(z, \lambda, \bar{w}, \bar{\tau}) &= \sum_{0 \leq \alpha_1, \alpha_2 \leq m-1} \frac{z^\alpha \lambda^m \bar{w}^\alpha \bar{\tau}^m}{\|z^\alpha \lambda^m\|^2} \\ &= \frac{(m+1)m^2}{(2\pi)^3} \lambda^m \bar{\tau}^m \sum_{\alpha_1=0}^{m-1} \binom{m-1}{\alpha_1} z_1^{\alpha_1} \bar{w}_1^{\alpha_1} \sum_{\alpha_2=0}^{m-1} \binom{m-1}{\alpha_2} z_2^{\alpha_2} \bar{w}_2^{\alpha_2} \\ &= \frac{(m+1)m^2}{(2\pi)^3} \lambda^m \bar{\tau}^m (1 + z_1 \bar{w}_1)^{m-1} (1 + z_2 \bar{w}_2)^{m-1}. \end{aligned}$$

□

Now we are ready to compute the Bergman kernel form of Ω .

Theorem 7.3 *The Bergman kernel form of the domain $\Omega \subset \mathbb{C}^3$ in (7.1) is given by*

$$K_\Omega(z, \lambda, \bar{w}, \bar{\tau}) = i K^*(z, \lambda, \bar{w}, \bar{\tau}) dz \wedge d\lambda \wedge d\bar{w} \wedge d\bar{\tau},$$

where

$$K^*(z, \lambda, \bar{w}, \bar{\tau}) = \sum_{m=1}^{\infty} \frac{(m+1)m^2}{(2\pi)^3} \lambda^m \bar{\tau}^m (1 + z_1 \bar{w}_1)^{m-1} (1 + z_2 \bar{w}_2)^{m-1}.$$

It can be written in terms of the complexified defining function

$$\rho(z, \lambda, \bar{w}, \bar{\tau}) = \lambda \bar{\tau} (1 + z_1 \bar{w}_1) (1 + z_2 \bar{w}_2) - 1$$

as

$$K^*(z, \lambda, \bar{w}, \bar{\tau}) = \frac{1}{(2\pi)^3} \left(\frac{4\lambda \bar{\tau}}{\rho(z, \lambda, \bar{w}, \bar{\tau})^3} + \frac{6\lambda \bar{\tau}}{\rho(z, \lambda, \bar{w}, \bar{\tau})^4} \right).$$

Proof By Lemma 7.1, we immediately get the reproducing kernel of $A^2(\Omega)$ by adding up the reproducing kernels of $A_m^2(\Omega)$ for all m . Since $A_0^2(\Omega) = \{0\}$, we obtain

$$\begin{aligned} K^*(z, \lambda, \bar{w}, \bar{\tau}) &= \sum_{m=1}^{\infty} K_m^*(z, \lambda, \bar{w}, \bar{\tau}) \\ &= \sum_{m=0}^{\infty} \frac{(m+2)(m+1)^2}{(2\pi)^3} \lambda^{m+1} \bar{\tau}^{m+1} (1 + z_1 \bar{w}_1)^m (1 + z_2 \bar{w}_2)^m. \end{aligned}$$

It remains to write $K^*(z, \lambda, \bar{w}, \bar{\tau})$ in terms of the defining function $\rho(z, \lambda, \bar{w}, \bar{\tau})$. We use the Taylor expansion of $1/(1-x)^{j+1}$ for $0 \leq j \leq 3$ to obtain

$$\begin{aligned} \frac{1}{(-\rho(z, \lambda, \bar{w}, \bar{\tau}))^{j+1}} &= \frac{1}{(1 - (1 + z_1 \bar{w}_1)(1 + z_2 \bar{w}_2) \lambda \bar{\tau})^{j+1}} \\ &= \sum_{m=0}^{\infty} \binom{m+j}{j} (1 + z_1 \bar{w}_1)^m (1 + z_2 \bar{w}_2)^m \lambda^m \bar{\tau}^m. \end{aligned}$$

Note that $(m+2)(m+1)^2$ is a polynomial in m of degree 3. Since $\{\binom{m+j}{j}\}_{j=0}^3$ is a basis of polynomials in m with degree ≤ 3 , we can write

$$(m+2)(m+1)^2 = \sum_{j=0}^3 a_j \binom{m+j}{j}.$$

One can check that the coefficients are given by $a_0 = a_1 = 0$, $a_2 = -4$ and $a_3 = 6$. Therefore,

$$\begin{aligned} K^*(z, \lambda, \bar{w}, \bar{\tau}) &= \frac{1}{(2\pi)^3} \sum_{m=0}^{\infty} \sum_{j=0}^3 a_j \binom{m+j}{j} \lambda^{m+1} \bar{\tau}^{m+1} (1 + z_1 \bar{w}_1)^m (1 + z_2 \bar{w}_2)^m \\ &= \frac{1}{(2\pi)^3} \sum_{j=0}^3 a_j \frac{\lambda \bar{\tau}}{(-\rho(z, \lambda, \bar{w}, \bar{\tau}))^{j+1}}, \end{aligned}$$

and the result follows. \square

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