ARAKELOV INEQUALITIES IN HIGHER DIMENSIONS

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Abstract. We develop a Hodge theoretic invariant for families of projective manifolds that measures the potential failure of an Arakelov-type inequality in higher dimensions, one that naturally generalizes the classical Arakelov inequality over regular quasi-projective curves. We show that for families of manifolds with ample canonical bundle this invariant is uniformly bounded. As a consequence we establish that such families over a base of arbitrary dimension satisfy the aforementioned Arakelov inequality, answering a question of Viehweg.

Contents

1. Introduction

While numerical invariants play a central role in classification in all fields of mathematics, it is often very difficult to compute their exact value. As a result we opt for the next best thing: try to give estimates by finding upper or lower bounds. In algebraic geometry, and in particular in the construction of moduli spaces, giving bounds for certain invariants provides a fundamnatal tool. Without such bounds it would be extremely difficult to find reasonably-behaved moduli spaces; for example, we could not even hope for such spaces to be of finite type.

One of the early examples of such bounds, with an eye towards the construction of moduli spaces of higher dimensional varieties, is Matsusaka's Big Theorem [Mat72]. Boundedness questions are present in many other more or less related questions, such as Mordell's Conjecture, Lang's Conjecture, or Shafarevich's Conjecture. The latter, and its more modern generalizations, are the most relevant to the present work.

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Shafarevich [Sha63] conjectured that there are only finitely many non-isotrivial families of smooth projective curves of fixed genus (≥ 2) over a fixed curve. Parshin [Par68] and Arakelov [Ara71] proved this conjecture in two steps: *boundedness*, that is, there are only finitely many deformation types of such families, and *rigidity*; those families are actually rigid, so each one is the *only one* in its deformation type.

Boundedness can be roughly translated to some associated parameter scheme being of finite type. These parameter spaces are often constructed via an appropriate Hilbert scheme and hence being of finite type is closely related to bounding the degree of an ample line bundle. In fact, already Arakelov used this idea to prove boundedness in order to prove Shafarevich's conjecture in the curve case.

More generally, we consider a smooth projective family of canonically polarized varieties $\pi : U \to V$. Then V maps to a moduli space parametrizing the fibers. This target moduli space is equipped with an ample line bundle cf. [Kol90, Fuj18, KP17]. The pullback of this line bundle to V is $\det \pi_* \omega^m_{U/V}$ (for some well-chosen $m > 0$ and up to a suitable power). Therefore, in order to carry out the above sketched plan for the boundedness problem, one would need to uniformly bound the degree of this line bundle.

This is exactly what Arakelov did. He established such a universal bound for all families of curves of genus at least 2 over base spaces of dimension one [Ara71]. More precisely, he showed that, for every sufficiently large $m \in \mathbb{N}$, there is a polynomial function $b_{m,g} \in \mathbb{Z}_{>0}[x_1, x_2]$, depending only on m and a fixed integer $g \in \mathbb{N}$, $g \geq 2$, such that the inequality

$$(\star) \qquad \deg(\det f_* \omega^m_{X/B}) \leq b_{m,g}\big(g(B), \deg(D)\big)$$

holds for any smooth compactification $f : X \to B$ of *any* non-isotrivial smooth projective family $f_U : U \to V$ of curves of genus g over a one dimensional base V, where $D := B \setminus V$. In fact Arakelov showed that the coefficients of $b_{m,g}$ are themselves purely g-dependent functions of m and $r_m := \operatorname{rank}(f_* \omega^m_{X/B})$.

This result was partially generalized to the case of higher dimensional fibers in [Kov96, Kov97, Kov00]. Subsequently, Bedulev and Viehweg [BV00] proved a further generalization of Arakelov's inequality for families of canonically polarized manifolds, *still* over curves. Other, more Hodge theoretic analogues of ($\star$) were also established by Deligne [Del87] and Peters [Pet00] (see Subsection 1.C below or [Vie08] for a more detailed account).

The equation ($\star$) became known as Arakelov's inequality. To see its usefulness the reader is invited to consult [Vie08] for a survey of related results available at the time and §8 of that paper for several open questions. Based on [KL10], Viehweg and others speculated that the inequality ($\star$) should have analogues over higher dimensional base spaces. In fact, at the end of his survey [Vie08] Viehweg explains how a higher dimensional Arakelov inequality would be useful, and goes on to say that none of the known methods (at the time) give any hope of obtaining it [Vie08, §8:III,IV].

Definition 1.2 provides a natural higher dimensional analogue of ($\star$) and the main result of the present paper is that under natural assumptions this inequality holds for canonically polarized families.

Remark 1.1. It used to be customary to fix a Hilbert polynomial when one is discussing moduli functors in order to have a finite type moduli space. Recently Kollár showed that for families of stable varieties it is actually enough to fix the volume of the canonical divisor (which appears as a coefficient of the canonical Hilbert polynomial) [Kol22, 5.1,6.19]. As this is now the standard, we will follow this approach and refer to the volume of the canonical divisor as the *canonical volume*. More generally, we will follow the terminology

of the [Kol22] on everything related to moduli spaces of stable varieties. In order to keep the introduction manageable, we only address some of the details on moduli spaces in Section 4.

Definition 1.2 (Higher dimensional Arakelov type inequalities)**.** Let V be a smooth quasi-projective variety of dimension d and B a smooth compactification of V such that $B \setminus D \simeq V$, with D being a reduced divisor on B having simple normal crossing support. Further let $f_U : U \to V$ be a smooth family of projective varieties and let X be a smooth compactification of U such that there exists a projective morphism $f : X \to B$ with $f|_U = f_U$. We will refer to these by saying that (the pair) (B, D) *is a smooth compactification of* V and that $f : X \to B$ is a *smooth compactification of* $f_U : U \to V$.

Still working with the above notations, let H be an ample Cartier divisor on B and set $\mathcal{Sm}_{n,\nu}(V)$ to denote the class of smooth projective families, $f_U : U \to V$, of canonically polarized varieties of relative dimension n and canonical volume $\nu = \mathrm{vol}(K_{U_t}) := K^n_{U_t}$ over V. Members of a subclass of $\mathcal{S} \subseteq \mathcal{Sm}_{n,\nu}(V)$ will be said to satisfy an *Arakelov inequality*, if for all sufficiently large and divisible $m \in \mathbb{N}$, there exists a function $b_{m,n,\nu} \in \mathbb{Z}_{>0}[x_1, x_2]$, depending only on m, n, and ν, for which the inequality

$$\deg_H \left(\det f_* \omega^m_{X/B} \right) \leq b_{m,n,\nu} \left(\deg_H(K_B + D), \deg_H(D) \right) \tag{1.2.1}$$

holds for any smooth compactification $f : X \to B$ of any family $(f_U : U \to V) \in \mathcal{S}$. Here for any divisor Δ and line bundle $\mathscr{L}$ on B, we define $\deg_H(\Delta) := \Delta \cdot H^{d-1}$ and $\deg_H(\mathscr{L}) := c_1(\mathscr{L}) \cdot H^{d-1}$.

Theorem 1.3 (for a more precise version see Theorem 5.2)**.** *In the setting of Definition 1.2, if $K_B + D$ is pseudo-effective, then all members of $\mathcal{Sm}_{n,\nu}(V)$ satisfy an Arakelov inequality.*

Note that when $d = 1$, our Arakelov-type inequality in Theorem 5.2 fully recovers the original one for curves in $(\star)$. In fact, in Theorem 5.2 we prove a more sophisticated and precise version. In particular, we prove that there exist $b'_{m,n,\nu}$, a polynomial whose coefficients are given by explicit functions of m and $\mathrm{rank}(f_* \omega^m_{X/B})$ (see Theorem 5.2 and (5.3.4)), and an integer γ_m such that the polynomial $b_{m,n,\nu}$ in (1.2.1) can be written as

$$b_{m,n,\nu} = (\dim V)^{\gamma_m} b'_{m,n,\nu}.$$

The dependence of $b_{m,n,\nu}$ on γ_m disappears in the case $d = 1$. Further note that the integer γ_m is an upperbound for an invariant for members of $\mathcal{Sm}_{n,\nu}(V)$, which we will refer to as *Viehweg number* (cf. Definition 3.8).

For a non-isotrivial smooth family $f_U : U \to V$ of curves of genus at least 2 over a quasi-projective curve V, by Arakelov and Parshin's resolution of the Shafarevich hyperbolicity conjecture, we know that $K_B + D$ is effective (same is true when fibers are canonically polarized manifolds by [Kov00], [Kov02]). Therefore, the pesudo-effectivity of $K_B + D$ in Theorem 1.3 is a natural assumption. In fact, for families in $\mathcal{Sm}_{n,\nu}(V)$ with maximal variation (or equivalently those with a generically finite moduli morphism $\mu : U \to M_{n.\nu}$ to the coarse moduli scheme $M_{n,\nu}$, see Section 4 for more details), by the culmination of the works of [VZ02], [KK08], [KK10], [Pat12], [CP19] on Viehweg's hyperbolicity conjecture, we know that $\kappa(B, D) = \dim B$. In particular, we have the following direct consequence of Theorem 1.3:

Corollary 1.4. *Let (B, D) be a smooth compactification of a smooth quasi-projective variety V (as in Definition 1.2). Then each member f_U of the subclass of $\mathcal{Sm}_{n,\nu}(V)$ consisting of the families of maximal variation satisfies an Arakelov inequality as in (1.2.1).*

Finally note that when f is semistable, the inequality in Theorem 1.3 can be sharpened by replacing $\deg_H(D)$ by zero, but because this case is not the focus of the current paper, we omit additional references and details.

1.A. **Bounding heights for substacks of stable varieties.** By using fundamental properties of the moduli stack of stable curves Arakelov and Bedulev-Viehweg introduced a notion of a height function on canonically polarized varieties X of dimension n, with fixed canonical volume $\nu := K_X^n \in \mathbb{Q}$. That is, given any morphism $\mu : V \to M_{n,\nu}$ arising from a smooth family over the curve V, let $\Phi : B' \to M_{n,\nu}$ be its KSB-stable closure via a finite surjective morphism $\eta : B' \to B$. For sufficiently large m, there is an ample line bundle λ_m on $M_{n,\nu}$ and a positive integer p_m such that $\deg(\Phi^*\lambda_m)$—which we think of as a height function associated to μ—has an upperbound by $p_m \cdot \deg \eta \cdot b_{m,n,\nu}\big(g(B), \deg(D)\big)$.

Similarly, in higher dimensions we can think of $\mathrm{vol}(\Phi^*\lambda_m)$ as a height function which can be uniformly bounded using Theorem 1.3 by the numerical properties of (B, D). In fact, following Arakelov, one may go further and divide $\deg \eta$ by $\deg \Phi$ to get a bound on $\mathrm{vol}(\lambda_m)$ on the image of μ, which, as $\deg \Phi \geq \deg \eta$, is independent of $\deg \eta$.

Theorem 1.5. *Using the notation introduced in Definition 1.2, we have that for a sufficiently large $m \in \mathbb{N}$ there exists a function $c_m = c_{m,n,\nu} \in \mathbb{Z}[x_1, x_2, x_3]$, depending only on m, n, and ν, such that, for every $\mu : V \to M_{n,\nu}$, arising from some $(f_U : U \to V) \in \mathcal{Sm}_{n,\nu}(V)$, and the associated compactification $\Phi : B' \to M_{n,\nu}$, we have*

$$\mathrm{vol}(\Phi^*\lambda_m) \leq (\deg \eta)^d c_m\big((K_B + D) \cdot H^{d-1}, D \cdot H^{d-1}, H^d\big) \in \mathbb{Q}_{>0}.$$

1.B. **Outline of the proof.** As we will see in Section 5, using results from [CP19], [Taj21] and others, one can establish a naive, f_U-dependent, upperbound for a smooth projective family f_U in the form of (1.2.1) (see (5.2.1)), as soon as $K_B + D$ is pseudo-effective. However, in order to obtain an inequality where this upperbound is independent of the choice of the family—the aim of an Arakelov-type inequality—one needs a more careful approach. We start by defining a local system via a prescribed global section s of line bundle $\mathscr{M}$, which can be naturally defined on any compactification of a smooth (or stable) projective family f. It turns out that to obtain the desired Arakelov-type inequality, it is sufficient to find a uniform bound for the rank of this local system. We denote this rank by $\alpha_s(f)$ and refer to it as the *Viehweg number* of f (cf. §§3.D).

The problem of proving an Arakelov inequality over higher dimensional base spaces is then reduced to establishing the existence of a suitable global section s for which $\alpha_s(f)$ has an upperbound that does not depend on f, but only on fixed invariants. To achieve this, we introduce what we call the *twisted direct image sheaf*. This is defined on B and simultaneously encodes information about $f_*\omega_{X/B}^m$ (for an appropriate $m \in \mathbb{N}$), its determinant, and the semistable locus of f. This sheaf is closely related to the above $\mathscr{M}$ and, as we will show in Section 3, it is weakly positive (see Proposition 3.3). The latter property is of particular importance for the construction of s in Theorem 4.11.

1.B.1. *Deformation spaces of families of canonically polarized manifolds.* A key component of our argument is based on the results of [KL10] on finiteness of deformation classes for members of $\mathcal{Sm}_{n,\nu}(V)$. That is, by [KL10], there is a finite subset $\{f_i\}_{1 \leq i \leq k} \subset \mathcal{Sm}_{n,\nu}(V)$ such that for any $f_U \in \mathcal{Sm}_{n,\nu}(V)$, there is some $1 \leq i \leq k$, a connected scheme W and a projective family $f_W : U_W \to W \times V$ of canonically polarized manifolds such that $(U_W)_{\{w\}\times V} \cong_V U_i$ and $(U_W)_{\{w'\}\times V} \cong_V U$, for some closed points $w, w' \in W$, i.e., up to an isomorphism U_W pulls back to U_i and U.

In fact, [KL10] goes further by showing that there is a finite type substack of the stack of canonically polarized manifolds over a finite type scheme of the form $W \times V$ whose connected components *parametrize* members of $\mathcal{Sm}_{n,\nu}(V)$ (see Subsection 4.A for more details). We use this latter result in two ways; to find a suitable section s_m for all members of $\mathcal{Sm}_{n,\nu}(V)$, with m being sufficiently large and divisible, and to use the (generic) deformation-invariance property of $\alpha_{s_m}(f)$ to establish an upperbound for $\alpha_{s_m}(f)$ that is independent of the choice of f, proving the existence of γ_m as stated in Theorem 5.2 (see also Theorem 4.11 and Corollary 4.12).

1.B.2. *The role of stable reductions.* As mentioned earlier, Viehweg numbers are closely related to the twisted direct image sheaf (cf. §§1.B) The connection is through weak positivity of this sheaf. That is, as we will show, up to a twist by an ample line bundle, a large enough symmetric power of this sheaf is generically globally generated. It turns out that the problem of bounding Viehweg numbers can be traced to bounding the necessary exponent for this symmetric power. Although the finiteness of deformation classes for $\mathcal{Sm}_{n,\nu}(V)$ proved in [KL10] is key to uniformly bounding it, as we will see in §§4.B, it is not enough for finding such bounds for compactifications of members of $\mathcal{Sm}_{n,\nu}(V)$. To accomplish the latter, strict base change properties (in the sense of Proposition 3.2) are needed for the twisted direct image sheaf that generally only holds for KSB-stable closures via stable reductions. The required compactifications exist by [Kol22, Thm. 4.59] (cf. [KSB88, Kar00]) and play a significant role in our strategy to establish Arakelov inequalities over higher dimensional base spaces.

1.C. **Related results.** As mentioned above, Arakelov type inequalities generally fall into two related categories. The geometric one ($\star$) goes back to Arakelov. This was later generalized in [BV00]. Further refinements and generalizations over curves was established in Viehweg-Zuo [VZ02]. There are also more Hodge theoretic Arakelov-type inequalities, which are concerned with establishing universal bounds for the degree of direct summands of Hodge bundles underlying (canonical extensions of) variation of Hodge structures (or VHS for short) of geometric origin. When the fibers are of dimension 1, then one can interpret this type of Arakelov inequalities for VHSs of weight one as essentially the same as ($\star$), with $m = 1$. Such inequalities were initiated by Deligne [Del87] and later extended by Peters [Pet00] and Jost-Zuo [JZ02]. All of these results are restricted to the case when the base of the family is 1-dimensional. Using Simpson's nonabelian Hodge theory, under rather strong positivity assumptions for $\Omega^1_X(\log B)$, some Arakelov-type inequalities for VHSs of weight one were generalized to higher dimensional base spaces in [VZ07]. Topological counterparts of these inequalities were also studied by Bradlow, García-Prada, and Gothen [BGPG06] and by Koziarz and Maubon [KM08], [KM10]. A detailed review of Arakelov inequalities can be found in [Vie08]. We also refer to the paper of Brotbek and Brunebarbe [BB20] for some other recent developments in this area.

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2. Preliminaries and Background

Definition 2.1. In this article a *variety* means a reduced, finite type scheme, and every scheme will be assumed to be defined over $\mathbb{C}$. A *pair* (X, Δ) consists of a variety X and an effective $\mathbb{Q}$-Weil divisor $\Delta = \sum a_i \Delta_i$, $a_i \leq 1$. We say (X, Δ) is a *reduced pair* if $a_i = 1$, for each i. An *snc pair* is a reduced pair (X, Δ) such that X is regular and Δ is an snc divisor. A *morphism of reduced pairs* $f : (X, \Delta) \to (B, D)$ is a dominant morphism $f : X \to B$

of schemes with connected fibers such that $f^{-1}(\operatorname{supp} D) \subseteq \operatorname{supp} \Delta$. Assuming that D is $\mathbb{Q}$-Cartier, we will use the notation $f^{-1}D := (f^*D)_{\mathrm{red}}$ to denote the *reduced preimage* of D. Using this notation, the above criterion can be replaced by $\Delta \geq f^{-1}D$. A *morphism of snc pairs* is a morphism of reduced pairs $f : (X, \Delta) \to (B, D)$ such that both (X, Δ) and (B, D) are snc pairs.

Definition 2.2. [Kol22, 8.34] Let X be a proper scheme and $\mathscr{L}$ a line bundle on X. $\mathscr{L}$ is said to be *strongly ample* if it is very ample and $H^i(X, \mathscr{L}^q) = 0$ for $i, q > 0$. Note that by [Laz04, I.8.3], if this holds for all $q \leq \dim X + 1$ then it holds for all $q > 0$. In particular, strong ampleness is an open condition in flat families.

Similarly, let $f\colon X \to B$ be a proper, flat morphism and $\mathscr{L}$ a line bundle on X. We say that $\mathscr{L}$ is *strongly f-ample* or *strongly ample over B*, if $\mathscr{L}$ is strongly ample on the fibers. Equivalently, if $\mathscr{L}$ is f-very ample and $\mathscr{R}^i f_* \mathscr{L}^q = 0$ for $i, q > 0$. It follows that in this case $f_*\mathscr{L}$ is locally free and we get an embedding $X \hookrightarrow \mathbb{P}_B(f_*\mathscr{L})$. We will be mainly interested in the case when $f\colon X \to B$ is stable and $\mathscr{L} = \omega_{X/S}^{[q]}$ for some $q > 0$. In this case, if $q > 1$ then $\mathscr{R}^i f_* \mathscr{L}^m = 0$ for $i, m > 0$ by [Kol22, 11.34].

Definition 2.3 (Snc and strongly snc morphisms)**.** Consider a morphism $f : X \to B$ and a decomposition $\Delta = \Delta_v + \Delta_h$ into vertical and horizontal parts, i.e., such that $\operatorname{codim}_B f(\Delta_v) \geq 1$ and that $f|_{\Delta_0}$ dominates B, for any irreducible component $\Delta_0 \subseteq \Delta_h$. Using this decomposition, we call a morphism of snc pairs $f : (X, \Delta) \to (B, D)$ an *snc morphism*, if f is flat, $\Delta_v = f^{-1}D$ and $f|_{X \setminus \Delta_v}$ is smooth.

An snc morphism $f : (X, \Delta) \to (B, D)$ is called *strongly snc*, if f^*D is reduced. Note that this implies that then $\Delta_v = f^*D$. Further note that an snc morphism with reduced fibers is necessarily strongly snc. Semistable (see [AK00, 0.1] for a definition) and stable snc morphisms (Definition 2.7) are the main examples to which this will be applied.

Notation 2.4. Given a reduced scheme X and a coherent sheaf $\mathscr{F}$ of rank r, for any $m \in \mathbb{N}$, we define $\mathscr{F}^{[m]} := \left(\mathscr{F}^{\otimes m}\right)^{**}$, where $(\_)^{**}$ denotes the double dual. We will apply the same notation for all tensor operations. In particular, $\operatorname{Sym}^{[m]} \mathscr{F} := (\operatorname{Sym}^m \mathscr{F})^{**}$ and $\bigwedge^{[m]} \mathscr{F} := (\bigwedge^m \mathscr{F})^{**}$. Furthermore, $\det \mathscr{F}$ will denote $\bigwedge^{[r]} \mathscr{F} = \left(\bigwedge^r \mathscr{F}\right)^{**}$. Notice that if X is regular, then $\det \mathscr{F}$ is a line bundle.

Notation 2.5. Let $f : (X, \Delta) \to (B, D)$ be an snc morphism. Consider the natural morphism $\varrho : f^*\Omega^1_B(\log D) \to \Omega^1_X(\log \Delta)$ and define the *sheaf of relative log differentials* as the cokernel of this morphism: $\Omega^1_{X/B}(\log \Delta) := \operatorname{coker} \varrho$. In other words, there exists a short exact sequence:

$$0 \longrightarrow f^*\Omega^1_B(\log D) \longrightarrow \Omega^1_X(\log \Delta) \longrightarrow \Omega^1_{X/B}(\log \Delta) \longrightarrow 0.$$

The first two sheaves are locally free by definition and a simple local calculation shows that so is the third one. In particular, the exterior powers of this sheaf, denoted by $\Omega^p_{X/B}(\log \Delta) := \bigwedge^p \Omega^1_{X/B}(\log \Delta)$ for $p \in \mathbb{N}$, are also locally free and if $\dim X = d$ and $\dim B = r$, then we have the following isomorphism:

$$\Omega^{d-r}_{X/B}(\log \Delta) \simeq \Omega^d_X(\log \Delta) \otimes \left(f^*\Omega^r_B(\log D)\right)^{-1} \simeq \omega_{X/B}(\Delta - f^*D) \tag{2.5.1}$$

Observe that if f is strongly snc, then (using the notation from Definition 2.3) the last sheaf is isomorphic to $\omega_{X/B}(\Delta_h)$.

Notation 2.6. For a morphism of finite type $f : X \to B$ of relative dimension n we define the relative canonical sheaf by $\omega_{X/B} := h^{-n}\left(f^!\mathscr{O}_B\right)$. For an S_2 and G_1 (Gorenstein in

codimension one) scheme X, a canonical divisor is denoted by K_X (see [Kov13, §5] for more details). If X is S_2 and G_1 and B is Gorenstein, then $\omega_{X/B} \simeq \mathscr{O}_X(K_X - f^*K_B)$ by [Con00, (3.3.6)].

In this paper we only need the following slightly restrictive definition.

Definition 2.7 (Stable families)**.** A projective variety Z is called *stable*, if it has slc singularities [Kol22, 1.41] and ω_Z is an ample $\mathbb{Q}$-line bundle. Let B be a reduced scheme. A projective morphism $f : X \to B$ is called *stable*, if X_b is a stable variety for each $b \in B$ and $\omega_{X/B}^{[m]}$ is invertible, for some $m \in \mathbb{N}$ cf. [Kol22, 3.40].

Remark 2.8. Note that if B is not assumed to be reduced or if one considers families of pairs, then the definition of a stable family is more complicated. The fact that in this case the above definition suffices follows from [Kol22, 4.7].

Notation 2.9 (Pullback). Given morphisms of schemes $f : X \to B$ and $Z \to B$ we denote the pullback of f by $f_Z : X_Z = X \times_B Z \to Z$ and $\mathrm{pr} : X_Z \to X$ denotes the induced natural projection.

Remark 2.10 (Base change properties). Let $f : X \to B$ be a family with slc fibers. Then the relative canonical sheaf of f is flat over B with S_2 fibers and compatible with arbitrary base change by [KK10, KK20], cf. [Kol22, 2.67].

For a stable family $f : X \to B$, the formation of $\omega_{X/B}^{[m]}$ commutes with arbitrary base change for every $m \in \mathbb{N}$, by [Kol22, 4.33], cf. [Kol18, Prop. 16], that is, for any reduced scheme Z and morphism $\psi : Z \to B$, and any $k \in \mathbb{N}$, we have that $\psi_X^*\omega_{X/B}^{[k]} \simeq \omega_{X_Z/Z}^{[k]}$, and that the isomorphism

$$\psi^* f_*\omega_{X/B}^{[m]} \simeq (f_Z)_*\omega_{X_Z/Z}^{[m]} \tag{2.10.1}$$

holds when $\omega_{X/B}^{[m]}$ is strongly f-ample (cf. Definition 2.2), e.g., for all sufficiently large and divisible m.

Note that a stable family as defined in Definition 2.7 is KSB-stable in the sense of [Kol22, Def. 6.16], cf. [Kol22, 4.33].

Notation 2.11 (Discriminant locus). Let $f : X \to B$ be a dominant morphism of regular schemes. Denote the divisorial part of the discriminant locus of f by $\mathrm{disc}(f)$. Setting $D_f = \mathrm{disc}(f)$, we let $\Delta_f := f^{-1}D_f$, a reduced divisor on X. This way the resulting map $f : (X, \Delta_f) \to (B, D_f)$ is a morphism of reduced pairs. If in addition, $f : (X, \Delta_f) \to (B, D_f)$ is strongly snc, then $\Delta_f = f^*D_f$ and if $\dim X/B = n$, then there is an isomorphism (cf. (2.5.1))

$$\Omega^n_{X/B}(\log \Delta_f) \simeq \omega_{X/B}(\Delta_f - f^*D_f) \simeq \omega_{X/B}. \tag{2.11.1}$$

We define a similar notion for morphisms of arbitrary schemes.

Notation 2.12 (Non-reduced locus). Let $f : X \to B$ be a morphism of schemes and denote the divisorial part of the locus of non-reduced fibers on B by R_f, i.e., let

$$R_f^+ := \{b \in B \mid X_b \text{ is not reduced}\},$$

and let R_f be the reduced divisor corresponding to the union of those irreducible components of R_f^+ that are codimension one in B.

Note that if f is an snc morphism of relative dimension n, then by (2.5.1),

$$\Omega^n_{X/B}(\log \Delta_f) \simeq \omega_{X/B}(f^{-1}R_f - f^*R_f) \tag{2.12.1}$$

Further note that if f is strongly snc or stable, then $R_f^+ = \emptyset$ and hence $R_f = 0$.

Lemma 2.13. *Let $f : X \to B$ be a dominant morphism of regular schemes, $\tau : \widetilde{X} \to X$ a projective birational morphism, and let $\widetilde{f} := f \circ \tau$. Assume that Δ_f and $\Delta_{\widetilde{f}}$, defined in Notation 2.11, are snc divisors and that τ is an isomorphism outside Δ_f. Then, over the locus where f and $\widetilde{f}$ are strongly snc, there exists an injective morphism $\tau_* \Omega^n_{\widetilde{X}/B}(\log \Delta_{\widetilde{f}}) \hookrightarrow \Omega^n_{X/B}(\log \Delta_f)$.*

Proof. First, observe that $D_f \subseteq D_{\widetilde{f}}$ and hence $\tau^{-1}\Delta_f \subseteq \Delta_{\widetilde{f}}$. Further we note that by construction we have

$$\Delta_{\widetilde{f}} \leq \tau^* \Delta_f.$$

On the other hand, as X is non-singular, there exists another τ-exceptional effective divisor E_1 such that $\omega_{\widetilde{X}} \simeq \tau^* \omega_X(E_1)$. Putting everything together we obtain that

$$\omega_{\widetilde{X}/B}(\Delta_{\widetilde{f}} - \widetilde{f}^* D_{\widetilde{f}}) \subseteq \tau^* \left(\omega_{X/B}(\Delta_f - f^* D_f)\right) \otimes \mathscr{O}_{\widetilde{X}}(E_1),$$

and hence

$$\tau_* \omega_{\widetilde{X}/B}(\Delta_{\widetilde{f}} - \widetilde{f}^* D_{\widetilde{f}}) \subseteq \omega_{X/B}(\Delta_f - f^* D_f) \otimes \tau_* \mathscr{O}_{\widetilde{X}}(E_1).$$

By [KMM87, 1-3-2] $\tau_* \mathscr{O}_{\widetilde{X}}(E_1) \simeq \mathscr{O}_X$ and hence the above containment combined with (2.5.1) implies the desired statement. □

Notation 2.14. For a morphism of normal schemes $f : X \to B$, for any $r \in \mathbb{N}$, we denote the r-fold fiber product by

$$X^r := \underbrace{X \times_B \ldots \times_B X}_{r \text{ times}},$$

with the induced morphism $f^r : X^r \to B$. Furthermore, let $\pi : X^{(r)} \to X^r$ denote a strong resolution of X^r, with the naturally induced map $f^{(r)} : X^{(r)} \to B$. (Recall that a strong resolution $\pi : Y \to X$ is a resolution of X for which $\pi|_{\pi^{-1}(X_{\mathrm{reg}})}$ is an isomorphism, where X_{reg} denotes the regular locus of X.)

Proposition 2.15. *Given a stable family $f : X \to B$, f^r is also stable. Furthermore, for every m for which $\omega^{[m]}_{X/B}$ is invertible, $\omega^{[m]}_{X^r/B}$ is also invertible and, over the complement of a subscheme of B of* $\operatorname{codim}_B \geq 2$*, we have*

$$f^r_* \omega^{[m]}_{X^r/B} \simeq \bigotimes^r f_* \omega^{[m]}_{X/B}. \tag{2.15.1}$$

Proof. An iterated application of [BHPS13, Prop. 2.12] shows that f^r is stable. For the rest of the claim we use induction on r. Observe that $X^r = X \times_B X^{r-1}$ and let $\mathrm{pr} : X^r \to X^{r-1}$ and $\mathrm{pr}_r : X^r \to X$ denote the natural projections.

$$\begin{array}{ccc} X^r & \xrightarrow{\mathrm{pr}} & X^{r-1} \\ {\scriptstyle \mathrm{pr}_r}\downarrow & & \downarrow{\scriptstyle f^{r-1}} \\ X & \xrightarrow[f]{} & B \end{array}$$

Then $\omega^{[m]}_{X^r/X} \simeq \mathrm{pr}^* \omega^{[m]}_{X^{r-1}/B}$ by Remark 2.10 and hence $(\mathrm{pr}_r)_* \omega^{[m]}_{X^r/X} \simeq f^* f^{r-1}_* \omega^{[m]}_{X^{r-1}/B}$ by flat base change. Applying $f^r_* = f_*(\mathrm{pr}_r)_*$ to

$$\omega^{[m]}_{X^r/B} \simeq \omega^{[m]}_{X^r/X} \otimes \mathrm{pr}_r^* \, \omega^{[m]}_{X/B},$$

and using the induction hypothesis then yields

$$f_*(\mathrm{pr}_r)_*\omega^{[m]}_{X^r/B} \simeq f_*\left(f^*f_*^{r-1}\omega^{[m]}_{X^{r-1}/B}\otimes\omega^{[m]}_{X/B}\right) \simeq \left(\bigotimes^{r-1} f_*\omega^{[m]}_{X/B}\right)\otimes f_*\omega^{[m]}_{X/B}. \qquad \square$$

Corollary 2.16. *Let $f : X \to B$ be a family which is stable in codimension one. Assume that B is quasi-projective and that $\omega^{[m]}_{X/B}$ is invertible in codimension one. After removing a subscheme of B of $\mathrm{codim}_B \geq 2$ if necessary, there exists an injection*

$$\left(\det f_*\omega^{[m]}_{X/B}\right)^{\otimes k} \hookrightarrow f_*^{kr_m}\omega^{[m]}_{X^{kr_m}/B}, \tag{2.16.1}$$

for any $k \in \mathbb{N}$ where $r_m := \mathrm{rank}(f_*\omega^{[m]}_{X/B})$.

Proof. By removing a subscheme of B of $\mathrm{codim}_B \geq 2$ if necessary we may assume that f is stable, $\omega^{[m]}_{X/B}$ is invertible and that $f_*\omega^{[m]}_{X/B}$ is locally free on B. Raising the natural embedding

$$\det f_*\omega^{[m]}_{X/B} \hookrightarrow \bigotimes^{r_m} f_*\omega^{[m]}_{X/B}$$

to the k^{th} power yields

$$\left(\det f_*\omega^{[m]}_{X/B}\right)^{\otimes k} \hookrightarrow \bigotimes^{kr_m} f_*\omega^{[m]}_{X/B} \simeq f_*^{kr_m}\omega^{[m]}_{X^{kr_m}/B},$$

where the last isomorphism is simply (2.15.1). $\square$

2.A. **Determinants of direct image sheaves and base change.**

Definition-Notation 2.17. Let $f : X \to B$ be a morphism of finite type of normal schemes. Assume that B is regular, and fix an $m \in \mathbb{N}$. We define the sheaf $\mathscr{W}_m(f)$ as follows:

$$\mathscr{W}_m(f) := \det\left(f_*\omega^{[m]}_{X/B}(-mR_f)\right),$$

where R_f is as in Notation 2.12.

The following is a trivial observation, but we record it so we can easily cite it when needed.

Lemma 2.18. *Let $f : X \to B$ be a morphism of finite type of normal schemes. Assume that B is regular, and fix an $m \in \mathbb{N}$. Then $\mathscr{W}_m(f)$ is a line bundle and*

$$\mathscr{W}_m(f) \supseteq \det\left(f_*\omega^{[m]}_{X/B}(-mD_f)\right). \tag{2.18.1}$$

Furthermore, if f has reduced fibers (e.g., it is strongly snc or stable), then

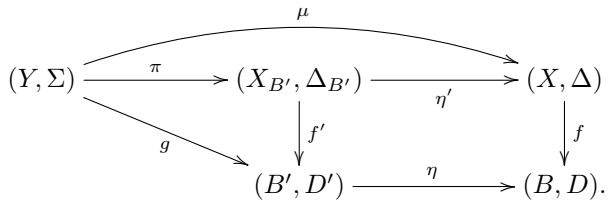
$$\mathscr{W}_m(f) = \det f_*\omega^{[m]}_{X/B}. \tag{2.18.2}$$

Lemma 2.19. *Let $f : (X, \Delta) \to (B, D)$ be an snc morphism of relative dimension n. Fix an $m \in \mathbb{N}$. Then (using Notation 2.11),*

$$\mathscr{W}_m(f) \subseteq \det f_*\left(\Omega^n_{X/B}(\log \Delta_f)^{\otimes m}\right).$$

Proof. This follows directly from (2.12.1). $\square$

Lemma 2.20. *Let (X, Δ) and (B, D) be two reduced pairs. Assume that (B, D) is snc. Let $f : (X, \Delta) \to (B, D)$ be a morphism of reduced pairs, with $\dim X/B = n \neq 0$. Further let B' be a regular variety and $\eta : B' \to B$ a flat surjective morphism. Let $\pi : Y \to X \times_B B'$ be a resolution of singularities, $D' = (\eta^* D)_{\mathrm{red}}$, and Σ a reduced divisor on Y such that $g = f' \circ \pi : (Y, \Sigma) \to (B', D')$ is an snc morphism. These objects and morphisms fit in the following commutative diagram of morphisms of pairs:*



(2.20.1) *If $f : (X, \Delta) \to (B, D)$ is an snc morphism, then there is a natural injective morphism*

$$\eta^* f_* \big(\Omega^n_{X/B}(\log \Delta)^{\otimes m}\big) \hookrightarrow g_* \big(\Omega^n_{Y/B'}(\log \Sigma)^{\otimes m}\big)$$

which is generically an isomorphism over B'.

(2.20.2) *If g is strongly snc, then for every projective birational morphism $\eta : \widetilde{X} \to X$ which is an isomorphism outside Δ_f and such that (using Notation 2.11) $\widetilde{f} := f \circ \eta : (\widetilde{X}, \Delta_{\widetilde{f}}) \to (B, D_{\widetilde{f}})$ is an snc morphism, there exists a natural injection*

$$\eta^* \mathscr{W}_m(\widetilde{f}) \hookrightarrow \det g_* \omega^m_{Y/B'},$$

which is generically an isomorphism over B'.

Proof. First, note that there is a natural injective morphism

$$\mu^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} \hookrightarrow \Omega^n_{Y/B'}(\log \Sigma)^{\otimes m},$$

and hence another one

$$f'_* \pi_* \pi^* (\eta')^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} = g_* \mu^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} \hookrightarrow g_* \Omega^n_{Y/B'}(\log \Sigma)^{\otimes m}, \tag{2.20.3}$$

which is an isomorphism over a dense open subset of B'.

On the other hand, there exists a natural morphism,

$$(\eta')^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} \longrightarrow \pi_* \pi^* (\eta')^* \Omega^n_{X/B}(\log \Delta)^{\otimes m}$$

which is also an isomorphism over the preimage of a dense open subset of B'. This morphism, combined with the one in (2.20.3) gives a morphism

$$f'_* (\eta')^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} \longrightarrow f'_* \pi_* \pi^* (\eta')^* \Omega^n_{X/B}(\log \Delta)^{\otimes m} \longrightarrow g_* \Omega^n_{Y/B'}(\log \Sigma)^{\otimes m},$$

which is again an isomorphism over a dense open subset of B'. By flat base change, the left hand side is isomorphic to $\eta^* f_* \big(\Omega^n_{X/B}(\log \Delta)^{\otimes m}\big)$ and hence (2.20.1) follows.

For (2.20.2), eliminate the points of indeterminacy of the birational map $Y \dashrightarrow \widetilde{X}_{B'}$, let $\widetilde{Y}$ denote a resolution of singularities of the result and $\tau : \widetilde{Y} \to Y$ the induced projective birational morphism. We may assume that the induced morphism $\widetilde{g} : (\widetilde{Y}, \Delta_{\widetilde{g}}) \to (B', D_{\widetilde{g}})$

is snc (here we are using Notation 2.11), after removing a subset of B' of $\operatorname{codim}_{B'} \geq 2$, if necessary. We thus have the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{Y} & \longrightarrow & \widetilde{X}_{B'} = \widetilde{X} \times_B B' & \longrightarrow & (\widetilde{X}, \Delta_{\widetilde{f}}) \\ {\scriptstyle\tau}\downarrow & \overset{\widetilde{g}}{\searrow} & \downarrow & & \downarrow{\scriptstyle\widetilde{f}} \\ Y & \xrightarrow[\text{strong snc}]{g} & B' & \xrightarrow{\eta} & (B, D_{\widetilde{f}}). \end{array}$$

According to (2.20.1) there is an injection

$$\eta^* \widetilde{f}_* \big(\Omega^n_{\widetilde{X}/B}(\log \Delta_{\widetilde{f}})^{\otimes m}\big) \hookrightarrow \widetilde{g}_* \Omega^n_{\widetilde{Y}/B'}(\log \Delta_{\widetilde{g}})^{\otimes m}. \tag{2.20.4}$$

Moreover we have the following isomorphisms and containment:

$$\begin{aligned} \widetilde{g}_* \big(\Omega^n_{\widetilde{Y}/B'}(\log \Delta_{\widetilde{g}})^{\otimes m}\big) \simeq g_* \tau_* \big(\Omega^n_{\widetilde{Y}/B'}(\log \Delta_{\widetilde{g}})^{\otimes m}\big) &\overset{\text{Lemma 2.13}}{\hookrightarrow} \\ \hookrightarrow g_* \big(\Omega^n_{Y/B'}(\log \Delta_g)^{\otimes m}\big) &\overset{(2.11.1)}{\simeq} g_* \omega^m_{Y/B'}. \end{aligned} \tag{2.20.5}$$

Combining (2.20.4) and (2.20.5) and taking determinants implies (2.20.2). □

Corollary 2.21. *Under the assumptions and notation of Lemma 2.20 and (2.20.2), there exists a natural injective morphism,*

$$\eta^* \det \Big((\widetilde{f}_* \omega^m_{\widetilde{X}/B})(-m D_{\widetilde{f}}) \Big) \hookrightarrow \det g_* \omega^m_{Y/B'}.$$

Furthermore, if in addition $\widetilde{f}$ is strongly snc, then there exists a natural injective morphism,

$$\eta^* (\det \widetilde{f}_* \omega^m_{\widetilde{X}/B}) \hookrightarrow \det g_* \omega^m_{Y/B'}.$$

Proof. This follows directly from Lemma 2.18 and (2.20.2). □

Proposition 2.22. *In the situation of the Definition-Notation 2.17, assume that X and B are regular and quasi-projective. After removing a subscheme of B of $\operatorname{codim}_B \geq 2$ if necessary, there exists an injection*

$$\mathscr{W}_m(f)^{\otimes k} \hookrightarrow f_*^{(kr_m)} \omega^m_{X^{(kr_m)}/B}, \tag{2.22.1}$$

for any $k \in \mathbb{N}$ where $r_m := \operatorname{rank}(f_* \omega^m_{X/B})$.

Remark 2.23. Notice that by Corollary 2.16, we have that if in addition f is stable in codimension one, then after removing a subscheme of B of $\operatorname{codim}_B \geq 2$ if necessary, there exists an injection

$$\mathscr{W}_m(f)^{\otimes k} \hookrightarrow f_*^{kr_m} \omega^{[m]}_{X^{(kr_m)}/B}, \tag{2.23.1}$$

for any $k \in \mathbb{N}$ where $r_m := \operatorname{rank}(f_* \omega^{[m]}_{X/B})$.

Proof. Without loss of generality we may assume that $(X, \Delta_f) \to (B, D_f)$ is snc. Let $g : Y \to B'$ be a semistable reduction of f in codimension one, via the finite, flat, surjective and Galois morphism $\eta : B' \to B$, (cf. [KKMSD73] and [BG71], and also [AK00, §5], [ALT20], [Laz04, 4.1.6]). Let $G := \operatorname{Gal}(B'/B)$. By (2.20.2) we have

$$\eta^* \mathscr{W}_m(f) \hookrightarrow \det g_* \omega^m_{Y/B'}, \tag{2.23.2}$$

which is generically an isomorphism over B'.

By raising (2.23.2) to the power k we obtain the injections

$$\eta^*\mathscr{W}_m(f)^{\otimes k} \hookrightarrow \left(\det g_*\omega^m_{Y/B'}\right)^{\otimes k} \hookrightarrow \bigotimes^{kr_m} g_*\omega^m_{Y/B'}.$$

On the other hand, over the semistable locus of g we have

$$\bigotimes^{kr_m} g_*\omega^m_{Y/B'} \simeq g_*^{(kr_m)}\omega^m_{Y^{(kr_m)}/B'}.$$

Furthermore, [Vie83, §3, p. 336] implies that

$$g_*^{(kr_m)}\omega^m_{Y^{(kr_m)}/B'} \subseteq \eta^* f_*^{(kr_m)}\omega^m_{X^{(kr_m)}/B}$$

so

$$\eta^*\mathscr{W}_m(f)^{\otimes k} \hookrightarrow \eta^* f_*^{(kr_m)}\omega^m_{X^{(kr_m)}/B}. \tag{2.23.3}$$

The required injection (2.22.1) follows from applying the functor $\gamma_*(\_)^G$ to (2.23.3). □

2.B. **Positivity notions for families of varieties.**

Definition 2.24. An $\mathscr{O}_Y$-module $\mathscr{F}$ on a reduced scheme Y is called *globally generated over an open subset* $V \subseteq Y$, if the natural map

$$H^0(Y, \mathscr{F}) \otimes \mathscr{O}_V \longrightarrow \mathscr{F} \otimes \mathscr{O}_V$$

is surjective over V. When the open set V is not specified, we say $\mathscr{F}$ is *generically generated by global sections over* Y.

Recall that given a regular quasi-projective variety X and an open subset $U \subseteq X$, a torsion free sheaf $\mathscr{F}$ on X is called *weakly positive over* U, if $\mathscr{F}|_U$ is locally free and that for any ample line bundle $\mathcal{H}$ on X, and every $\alpha \in \mathbb{N}$, there exists a $\beta \in \mathbb{N}$ such that

$$\operatorname{Sym}^{[\alpha k]} \mathscr{F} \otimes \mathcal{H}^k$$

is globally generated over U, for any multiple k of β.

By [Kaw81] and [Vie83] (see also [Fuj78], [Zuc82] and [Kol86]) it is known that for a projective morphism $f : X \to B$ of regular quasi-projective varieties X and B, with connected fibers, $f_*\omega^m_{X/B}$ is weakly positive over $B \setminus \operatorname{disc}(f)$, for any $m \in \mathbb{N}$. One can then slightly generalize this to the case of mildly singular families as follows. For any projective morphism $f : X \to B$ of quasi-projective varieties X and B, if B is nonsingular and X has only canonical singularities, then the torsion free sheaf $f_*\omega^{[m]}_{X/B}$ is weakly positive for every $m \in \mathbb{N}$ for which $\omega^{[m]}_{X/B}$ is invertible (see also [Fuj18]).

2.C. **Singularities in linear systems of ample line bundles.**

Definition-Notation 2.25. Following the definition of Esnault-Viehweg [EV92, Def. 7.4] for a line bundle $\mathscr{L}$ on projective manifold X, with $H^0(X, \mathscr{L}) \neq 0$, we define

$$e(\mathscr{L}) := \sup\left\{ \left\lfloor \frac{1}{\operatorname{lct}(\Gamma)} \right\rfloor + 1 \;\middle|\; \Gamma \in |\mathscr{L}| \right\},$$

where $\operatorname{lct}(\_)$ denotes the log-canonical threshold.

If $\mathscr{L}$ is very ample, then

$$e(\mathscr{L}^m) \leq mc_1(\mathscr{L})^{\dim X} + 1, \tag{2.25.1}$$

for every integer $m > 0$ by [EV92, Lem. 7.7]. Therefore, for the set of very ample line bundles on projective manifolds with fixed Hilbert polynomial h, the number $e(\_)$ has an upperbound depending on h. As the moduli functor of canonically polarized manifolds with fixed canonical Hilbert polynomial h is bounded [Mat72] (see [Vie95, Def. 1.15 (1)] for the definition), one can find an integer

$$a_0 = a_0(h) \in \mathbb{N} \tag{2.25.2}$$

such that, for every integer multiple m of a_0, the line bundle ω_X^m is strongly ample (cf. Definition 2.2), and then by (2.25.1) there exists an $e_m := e_m(h) \in \mathbb{N}$, depending only on m and h, that satisfies the inequality

$$e(\omega_X^m) \leq e_m \tag{2.25.3}$$

for every manifold as above.

As we mentioned in Remark 1.1, we will be following the terminology of [Kol22] with regard to moduli functors and moduli spaces. In particular, instead of a Hilbert polynomial we will be using the dimension, n, and the canonical volume ν. So, we may rephrase the above statement using (n, ν) in place of h, and say that $a_0 = a_0(n, \nu)$ and $e_m = e_m(n, \nu)$ depend on these quantities.

Notation 2.26. For every projective morphism of normal schemes $f : X \to B$ whose general fiber is a canonically polarized manifold, every sufficiently divisible $m \in \mathbb{N}$ and any $a \in \mathbb{N}$, we set $t_{m,a} := m^a e_m r_m$, where e_m is as in (2.25.3) and $r_m := \operatorname{rank}(f_*\omega_{X/B}^{[m]})$.

3. Twisted direct image sheaves and Viehweg numbers

3.A. **Generic global generation of twisted direct image sheaves in stable families.**

We define a notion of twisted direct images sheaves, which is closely connected to our notion of Viehweg numbers, to be introduced later in this section.

Definition 3.1 (Twisted direct image sheaves)**.** In the settings of Definition-Notation 2.17, Notation 2.26, set $V := B \backslash D$ and assume that for all $v \in V$ each fiber X_v is regular of dimension n and of canonical volume ν. For each multiple $m \in \mathbb{N}$ of $a_0(n, \nu)$ (cf. (2.25.2)), we define the *twisted direct image sheaf* by the following (recall, that $t_{m,2} = m^2 e_m r_m$):

$$\mathscr{K}_m^\nu(f) := \begin{cases} f_*^{t_{m,2}} \omega_{X^{t_{m,2}}/B}^{[m]} \otimes \mathscr{W}_m(f)^{-m} & \text{if } f \text{ is stable, and} \\ f_*^{(t_{m,2})} \omega_{X^{(t_{m,2})}/B}^{m} \otimes \mathscr{W}_m(f)^{-m} & \text{if } f \text{ is not stable, but } X \text{ is regular.} \end{cases}$$

To avoid cumbersome notation, when there is no ambiguity, we omit f from $\mathscr{W}_m(f)$ in the notation.

3.B. **The stable case.**

Proposition 3.2 (Base change for $\mathscr{K}_m^\nu$)**.** *Let $f : X \to B$ be a stable family with a normal general fiber. Assume that B is a regular quasi-projective variety and $\psi : B' \to B$ is a morphism of finite type, with B' also regular. Let Z be the main component of $X_{B'}$, equipped with the natural projection $g : Z \to B'$. Finally, let $m \in \mathbb{N}$ be such that $\omega_{X/B}^{[m]}$ is a strongly f-ample line bundle (cf. Definition 2.2). Then $\psi^* \mathscr{K}_m^\nu(f) \cong \mathscr{K}_m^\nu(g)$.*

Proof. This directly follows from Remark 2.10. That is, we consider the two morphisms

$$f^{t_{m,2}} : X^{t_{m,2}} \to B \quad , \quad g^{t_{m,2}} : Z^{t_{m,2}} \to B',$$

and observe that by Proposition 2.15 and Remark 2.10 we have

$$\psi^* f^{t_{m,2}}_* \omega^{[m]}_{X^{t_{m,2}}/B} \cong g^{t_{m,2}}_* \omega^{[m]}_{Z^{t_{m,2}}/B'}.$$

On the other hand, if $\omega^{[m]}_{X/B}$ is strongly f-ample, then $\psi^* \mathscr{W}_m(f) \cong \mathscr{W}_m(g)$, which gives the desired isomorphism. □

Proposition 3.3. *Let $f : X \to B$ be a stable family, where B is a regular quasi-projective variety. Assume that there is an open subset $V \subseteq B$ such that for every $v \in V$, the fiber X_v is regular of dimension n and canonical volume ν. Then $\mathscr{K}^{\nu}_m(f)$ is weakly positive for every multiple m of $a_0(n, \nu)$ (cf. (2.25.2)).*

Proof. Note that as B is regular, it follows that $X^{t_{m,2}}_V = \left(f^{t_{m,2}}\right)^{-1} V \subseteq X^{t_{m,2}}$ is regular. Let $\mu : X^{(t_{m,2})} \to X^{t_{m,2}}$ be a resolution which is an isomorphism over V and let $f^{(t_{m,2})} = f^{t_{m,2}} \circ \mu : X^{(t_{m,2})} \to B$ denote the induced family. Next, let E be an exceptional divisor on $X^{(t_{m,2})}$ such that $K_{X^{(t_{m,2})}} + E \sim_{\mathbb{Q}} \mu^* K_{X^{t_{m,2}}}$.

Recall that by Corollary 2.16 (cf. Lemma 2.18) there is an injection

$$\mathscr{W}_m^{m^2 e_m} \hookrightarrow f^{t_{m,2}}_* \omega^{[m]}_{X^{t_{m,2}}/B}. \tag{3.3.1}$$

Let $\Gamma \subset X^{t_{m,2}}$ be the effective Cartier divisor corresponding to the global section of $\omega^{[m]}_{X^{t_{m,2}}/B} \otimes (f^{t_{m,2}})^* \mathscr{W}_m^{-m^2 e_m}$ induced by the adjoint morphism of (3.3.1). In particular we have

$$\mathscr{O}_{X^{t_{m,2}}}(\Gamma) \simeq \omega^{[m]}_{X^{t_{m,2}}/B} \otimes (f^{t_{m,2}})^* \mathscr{W}_m^{-m^2 e_m}.$$

Setting $\widetilde{\Gamma} := \mu^* \Gamma$ leads to:

$$\mathscr{O}_{X^{(t_{m,2})}}(\widetilde{\Gamma}) \simeq \mu^* \omega^{[m]}_{X^{t_{m,2}}/B} \otimes \left(f^{(t_{m,2})}\right)^* \mathscr{W}_m^{-m^2 e_m} \simeq \omega^m_{X^{(t_{m,2})}/B}(mE) \otimes \left(f^{(t_{m,2})}\right)^* \mathscr{W}_m^{-m^2 e_m}.$$

Next, we define the invertible sheaf $\widetilde{\mathscr{M}}$ on $X^{(t_{m,2})}$ by

$$\widetilde{\mathscr{M}} := \left(\omega_{X^{(t_{m,2})}/B}(\lceil E \rceil)\right)^{m-1} \otimes \left(f^{(t_{m,2})}\right)^* \mathscr{W}_m^{-m}.$$

We thus have

$$\begin{aligned} \widetilde{\mathscr{M}}^{me_m}(-\widetilde{\Gamma}) &\simeq \omega^{m(e_m(m-1)-1)}_{X^{(t_{m,2})}/B} \left(me_m(m-1)\lceil E \rceil - mE\right) \simeq \\ &\simeq \mu^* \omega^{[m(e_m(m-1)-1)]}_{X^{t_{m,2}}/B} \left(me_m(m-1)(\lceil E \rceil - E)\right). \end{aligned} \tag{3.3.2}$$

Notice that $\lceil E \rceil - E$ is an effective (exceptional) divisor and hence

$$\mu^* \omega^{[m(e_m(m-1)-1)]}_{X^{t_{m,2}}/B} \subseteq \widetilde{\mathscr{M}}^{me_m}(-\widetilde{\Gamma}). \tag{3.3.3}$$

Recall that $\omega^m_{X^{t_{m,2}}_v}$ is strongly ample for every $v \in V$ by the choice of a_0. Then, it follows from Grauert's theorem [Har77, Cor. 12.9] that the natural morphism

$$\left(f^{t_{m,2}}\right)^* \underbrace{f^{t_{m,2}}_* \omega^{[m(e_m(m-1)-1)]}_{X^{t_{m,2}}/B}}_{:=\mathscr{F}} \longrightarrow \omega^{[m(e_m(m-1)-1)]}_{X^{t_{m,2}}/B}$$

is surjective over V. Note that by the choice of μ, the μ-exceptional divisor E, and hence $\lceil E\rceil - E$ is disjoint from X_V, which implies that after pulling back by μ and using (3.3.2), the obtained morphism

$$\left(f^{(t_{m,2})}\right)^*\mathscr{F} \longrightarrow \mu^*\omega_{X^{t_{m,2}}/B}^{[m(e_m(m-1)-1)]} \hookrightarrow \widetilde{\mathscr{M}}^{me_m}(-\widetilde{\Gamma}),$$

is surjective onto $\widetilde{\mathscr{M}}^{me_m}(-\widetilde{\Gamma})$ over V and hence generically surjective over B (cf. (3.3.3)).

On the other hand, for every $v \in V$, we have

$$\begin{aligned} me_m \geq e_m &\geq e(\omega_{X_v}^{[m]}) \\ &= e(\omega_{X_v^{t_{m,2}}}^{[m]}) \quad \text{by [Vie95, Cor. 5.21]} \\ &\geq \left\lfloor \frac{1}{\operatorname{lct}(\Gamma|_{X_v^{t_{m,2}}})} \right\rfloor + 1 = \left\lfloor \frac{1}{\operatorname{lct}(\widetilde{\Gamma}|_{\widetilde{X}_v^{t_{m,2}}})} \right\rfloor + 1. \end{aligned}$$

Now, as $\mathscr{F}$ is weakly positive, by vanishing results due to Kollár and Kawamata-Viehweg cf. [Vie95, §2.4] and more precisely by [VZ03, Prop. 3.3], we find that

$$\left(f^{(t_{m,2})}\right)_*\left(\omega_{X^{(t_{m,2})}/B} \otimes \widetilde{\mathscr{M}}\right) \tag{3.3.4}$$

is weakly positive.

Claim 3.4. Let $f_U : U \to V$ denote the restriction of $f : X \to B$ to V. There is a natural injection

$$\mu_*\left(\omega_{X^{(t_{m,2})}/B} \otimes \widetilde{\mathscr{M}}\right) \hookrightarrow \omega_{X^{t_{m,2}}/B}^{[m]} \otimes \left(f^{t_{m,2}}\right)^*\mathscr{W}_m^{-m},$$

that is an isomorphism over $U^{t_{m,2}}$.

Proof of Claim 3.4. By the definition of $\widetilde{\mathscr{M}}$ we have

$$\begin{aligned} \omega_{X^{(t_{m,2})}/B} \otimes \widetilde{\mathscr{M}} &= \omega_{X^{(t_{m,2})}/B} \otimes \left(\omega_{X^{(t_{m,2})}/B}(\lceil E\rceil)\right)^{m-1} \otimes \left(f^{(t_{m,2})}\right)^*\mathscr{W}_m^{-m} \simeq \\ &\simeq \omega_{X^{(t_{m,2})}/B}^m((m-1)\lceil E\rceil) \otimes \left(f^{(t_{m,2})}\right)^*\mathscr{W}_m^{-m} \simeq \\ &\simeq \mu^*\left(\omega_{X^{t_{m,2}}/B}^{[m]} \otimes (f^{t_{m,2}})^*\mathscr{W}_m^{-m}\right) \otimes \mathscr{O}_{X^{(t_{m,2})}/B}((m-1)\lceil E\rceil - mE) \subseteq \\ &\subseteq \mu^*\left(\omega_{X^{t_{m,2}}/B}^{[m]} \otimes (f^{t_{m,2}})^*\mathscr{W}_m^{-m}\right) \otimes \mathscr{O}_{X^{(t_{m,2})}/B}(m(\lceil E\rceil - E)), \end{aligned} \tag{3.4.1}$$

from which the required injection follows, because $\lceil E\rceil - E$ is an effective μ-exceptional divisor and hence $\mu_*\mathscr{O}_{X^{(t_{m,2})}/B}(m(\lceil E\rceil - E)) \simeq \mathscr{O}_{X^{t_{m,2}}/B}$, cf. [KMM87, Lem. 1-3-2]. □

Now, using Claim 3.4 and the fact that (3.3.4) is weakly positive, it follows that so is

$$f_*^{t_{m,2}}\left(\omega_{X^{t_{m,2}}/B}^{[m]} \otimes (f^{t_{m,2}})^*\mathscr{W}_m^{-m}\right) \simeq \mathscr{K}_m^{\nu}(f). \qquad \square$$

In the situation of Proposition 3.3, let $\mathcal{H}$ be any ample line bundle on B. Then, for every m as in Proposition 3.3, there is a $\beta_m \in \mathbb{N}$ such that, for every multiple k of β_m,

$$\operatorname{Sym}^k(\mathscr{K}_m^{\nu}(f)) \otimes \mathcal{H}^{mk} \text{ is generically generated by global sections.} \tag{3.4.2}$$

Remark 3.5. The conclusion of Proposition 3.3 also holds for any snc morphism $f : (X, \Delta_f) \to (B, D_f)$, with quasi-projective B and canonically polarized regular fibers. More precisely, using Proposition 2.22, taking $\mathscr{O}_{X^{(t_{m,2})}} = \omega^m_{X^{(t_{m,2})}} \otimes (f^{(t_{m,2})})^* \mathscr{W}_m^{-m^2 e_m}$ and replacing $\widetilde{\mathscr{M}}$ by

$$\mathscr{M} = \omega^{m-1}_{X^{(t_{m,2})}/B} \otimes \left(f^{(t_{m,2})}\right)^* \mathscr{W}_m^{-m},$$

from the proof of Proposition 3.3 it follows that $\mathscr{K}^\nu_m(f)$ is weakly positive.

3.C. **The strongly snc case.**

Lemma 3.6 (Twisted direct image sheaves under semistable reductions)**.** *In the setting of Lemma 2.20, assume that $X \setminus \Delta$ is regular. Let $\widetilde{X} \to X$ be a strong resolution such that $\widetilde{f} : (\widetilde{X}, \Delta_{\widetilde{f}}) \to (B, D_{\widetilde{f}})$ is an snc morphism and $g : Y \to B'$ a strongly snc morphism as in (2.20.2). Then, there is an injection*

$$\mathscr{K}^\nu_m(g) = g^{(t_{m,2})}_* \omega^m_{Y^{(t_{m,2})}/B'} \otimes \mathscr{W}_m^{-m}(g) \hookrightarrow \eta^* \mathscr{K}^\nu_m(\widetilde{f}),$$

naturally defined by a generic isomorphism over B'.

Proof. By (2.20.2) there is an injection

$$\mathscr{W}_m(g)^{-1} \hookrightarrow \eta^* \mathscr{W}_m(\widetilde{f})^{-1}. \tag{3.6.1}$$

For a suitable choice of strong resolutions $\widetilde{X}^{(t_{m,2})}$ and $Y^{(t_{m,2})}$ there is a commutative diagram

$$\begin{array}{ccc} Y^{(t_{m,2})} & \longrightarrow & \widetilde{X}^{(t_{m,2})} \\ {\scriptstyle g^{(t_{m,2})}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}^{(t_{m,2})}} \\ B' & \xrightarrow{\eta} & B. \end{array}$$

By [Vie83, §3, p. 336] we have

$$g^{(t_{m,2})}_* \omega^m_{Y^{(t_{m,2})}/B'} \subseteq \eta^* \widetilde{f}^{(t_{m,2})}_* \omega^m_{\widetilde{X}^{(t_{m,2})}/B}. \tag{3.6.2}$$

This inclusion and (3.6.1), raised to the power m, gives the required injection. □

3.D. **Viehweg numbers.**

Definition 3.7. In the setting of Definition 3.1, assuming that B is quasi-projective, let $\mathcal{H}$ be any ample line bundle on B. We define the line bundle $\mathscr{M}$ as follows.

(3.7.1) If f is stable and $\omega_{X^{t_{m,2}}/B}$ is a line bundle, e.g., if f is Gorenstein, then

$$\mathscr{M} := \omega_{X^{t_{m,2}}/B} \otimes (f^{t_{m,2}})^* (\mathscr{W}_m^{-1} \otimes \mathcal{H}),$$

and

(3.7.2) if X is regular, but f is not stable, then

$$\mathscr{M} := \omega_{X^{(t_{m,2})}/B} \otimes (f^{(t_{m,2})})^* (\mathscr{W}_m^{-1} \otimes \mathcal{H}),$$

Now, for some $\beta \in \mathbb{N}$, assume that $H^0(\mathscr{M}^{m\beta}) \neq 0$. Fix a non-zero section $0 \neq s_m \in H^0(\mathscr{M}^{m\beta})$. Let $\sigma_{s_m} : Z'_{s_m} \to X^{t_{m,2}}$, respectively $\sigma_{s_m} : Z'_{s_m} \to X^{(t_{m,2})}$, be the cyclic covering associated to s_m cf. [Laz04, Prop. 4.1.6], for $\mathscr{M}$ as in (3.7.1) and (3.7.2).

Let $\mu : Z_{s_m} \to Z'_{s_m}$ be a resolution with the induced family $g_{s_m} : Z_{s_m} \to B$ and the commutative diagram

$$\begin{array}{ccccc} Z_{s_m} & \xrightarrow{\mu} & Z'_{s_m} & \xrightarrow{\sigma_{s_m}} & X^{t_{m,2}} \text{ or } X^{(t_{m,2})} \\ & \searrow^{g_{s_m}} & & & \downarrow f \\ & & & & B. \end{array} \tag{3.7.3}$$

Definition 3.8. With a fixed ample line bundle $\mathcal{H}$ on B, and using the notation of Definition 3.7, we define the *Viehweg number* of f associated to the triple (m, β, s_m) to be the rank of the following local system

$$\alpha_{s_m}(f) = \alpha_{m,\beta,s_m}(f) := \operatorname{rank}\left(\mathcal{R}^{\dim(X^{t_{m,2}}/B)}(g_{s_m})_*\mathbb{C}_{(Z_{s_m}\setminus\Delta_{g_{s_m}})}\right).$$

In the setting of Definition 3.7 assume in addition that the smooth and Gorenstein fibers are canonically polarized. Then Proposition 3.3, (3.4.2) and Remark 3.5 imply that, after removing a subset of B of $\operatorname{codim}_B \geq 2$, if necessary, there exist m, and $\beta_m \in \mathbb{N}$ such that $H^0(\mathscr{M}^{m\beta_m}) \neq 0$ (on the respective varieties in the stable and the regular case). Therefore Definition 3.8 is relevant for all such families.

Notation 3.9. In order to keep the notation manageable, we will suppress some of the parameters in Definition 3.8 and use the notation $\alpha_{s_m}(f)$ instead of $\alpha_{m,\beta,\mathcal{H},s_m}(f)$, but will keep in mind the choices that we have made to define s_m.

Remark 3.10. Let $f_{U'} : U' \to V'$ be a subfamily of $f : X \to B$, that is, $f_{U'} = f|_{U'}$. Then, we clearly have (with the same m and β),

$$\alpha_{s_m}(f) = \alpha_{s'_m}(f_{U'})$$

where $s_m|_{U'} = s'_m$.

Proposition 3.11. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two line bundles on a variety U and assume that there is an isomorphism $\eta : \mathscr{L}_1 \to \mathscr{L}_2$. Let $0 \neq s_{m,2} \in H^0(U, \mathscr{L}_2^m)$, for some $m \in \mathbb{N}$, and set $s_{m,1} := \eta^* s_{m,2} \in H^0(U, \mathscr{L}_1^m)$. Let $\sigma_{s_{m,1}} : Z'_1 \to U$ and $\sigma_{s_{m,2}} : Z'_2 \to U$ be the cyclic coverings associated to $s_{m,1}$ and $s_{m,2}$. Then, there is a natural isomorphism $Z'_1 \simeq Z'_2$, induced by η, that commutes with $\sigma_{s_{m,1}}$ and $\sigma_{s_{m,2}}$.*

Proof. This directly follows from the construction of Z'_i cf. [Laz04, Prop. 4.1.6]. More precisely, let

$$\mathbf{L}_i := \operatorname{Spec}_{\mathscr{O}_U}(\operatorname{Sym}\mathscr{L}_i^*),$$

with the natural projection $p_i : \mathbf{L}_i \to U$, and set $T_i \in H^0(\mathbf{L}_i, p_i^*\mathscr{L}_i)$ to be the global section associated to the tautological map $\mathscr{O}_{\mathbf{L}_i} \to p_i^*\mathscr{L}_i$. Let $\overline{\eta} : \mathbf{L}_2 \to \mathbf{L}_1$ denote the natural isomorphism induced by η. By construction we have $Z'_i = (T_i^m - p_i^* s_i)_0$, and that $\overline{\eta}^*(T_1^m - p_1^* s_{m,1}) = T_2^m - p_2^* s_{m,2}$, inducing the desired isomorphism. □

The next lemma now follows by combining Proposition 3.11 with Remark 3.10.

Lemma 3.12. *Let $f : X \to B$ and $\mathscr{M}$ be as in Definition 3.7 and let $V \subseteq B$ an open set and $f_U : U = X_V \to V$ the corresponding subfamily. Consider a line bundle $\mathscr{N}$ on U that is equipped with an isomorphism $\eta : \mathscr{N} \xrightarrow{\simeq} \mathscr{M}|_U$. Let $s_{\mathscr{N}} := \eta^* s_m|_U$, $Z_\eta \to U$ a resolution of singularities of the cyclic covering associated to $s_{\mathscr{N}}$, and $g_{\mathscr{N}} : Z_{\mathscr{N}} \to V$ the induced morphism. Then,*

$$\operatorname{rank}\left(\mathcal{R}^{\dim(X^{t_{m,2}}/B)}(g_{\mathscr{N}})_*\mathbb{C}_{Z_{\mathscr{N}}\setminus\Delta_{g_{\mathscr{N}}}}\right) = \alpha_{s_m}(f).$$

4. Finite-type substacks of the stack of canonically polarized manifolds and boundedness of Viehweg numbers

In this section we first recall the main results of [KL10] and [BV00] regarding parametrizing spaces for canonically polarized families. As we have already mentioned in Remark 1.1 we will follow the terminology of [Kol22] and use the dimension and the canonical volume instead of the Hilbert polynomial. The above papers used Hilbert polynomials, but by [Kol22, 5.1,6.19] the two approaches are equivalent.

Subsequently, we will use these to establish certain uniform global generation results for the twisted direct image sheaves $\mathscr{K}_m^{\nu}(f)$ of canonically polarized families introduced in Definition 3.1. We start by reviewing numerical bounds arising from Arakelov inequalities over curves.

Let M° be a connected and finite type scheme (not necessarily irreducible) with a *compactification* $M^\circ \subseteq M$ as a subscheme of a projective scheme M, equipped with a fixed line bundle $\mathscr{L}$, which is ample on M°. Depending on $\mathscr{L}$, let $b_{\mathscr{L}} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function in two variables.

Definition 4.1 (Weak bound, [KL10, Def. 2.4])**.** Let V be a regular quasi-projective variety (of arbitrary dimension). Then a morphism $\mu_V : V \to M^\circ$ is called *weakly bounded with respect to* $b_{\mathscr{L}}$, if any regular curve C° equipped with a morphism $i : C^\circ \to V$ satisfies the following property: Let C be the regular compactification of C° of genus g and $d := \deg(C \setminus C^\circ)$. Then the extension $\mu_C : C \to M$ of the naturally induced map $\mu_{C^\circ} : C^\circ \to M^\circ$ satisfies the inequality

$$\deg(\mu_C^* \mathscr{L}) \leq b_{\mathscr{L}}(g, d).$$

Notation 4.2. Let $n, \nu \in \mathbb{N}$. Recall that $\mathcal{Sm}_{n,\nu}(V)$ denotes the class of smooth, projective and canonically polarized families $f_U : U \to V$ of varieties of dimension n and of canonical volume $\nu = K^n_{U_t}$. This is naturally a subclass of $\mathcal{St}_{n,\nu}(V)$, the class of stable families of varieties of dimension n and of canonical volume ν. This class defines a *good moduli theory* as defined in [Kol22, 6.10], cf. [Kol22, 6.16, Thm. 6.18] and admits a coarse moduli space which is projective [Kol22, Thm. 8.1]. We will denote this coarse moduli space of stable n-dimensional varieties X with canonical volume $K^n_X = \nu$ by $M_{n,\nu}$.

Definition 4.3 (Weakly bounded families)**.** A family $(f_U : U \to V) \in \mathcal{Sm}_{n,\nu}(V)$ is called *weakly bounded*, if there exists an ample line bundle $\mathscr{L}$ on $M_{n,\nu}$ and a function $b_{\mathscr{L}}$ with respect to which the induced moduli map $\mu_V : V \to M_{n,\nu}$ is weakly bounded.

By combining [Kol90, Thm. 2.5], [Vie95, Thm. 7.17], [Vie10, Thm. 5] and [BV00, Thm. 1.4] we have the following important fact.

Fact 4.4. There exist an ample line bundle $\mathscr{L}$ on $M_{n,\nu}$ and a function $b_{\mathscr{L}}$ for which every $f_U \in \mathcal{Sm}_{n,\nu}(V)$ is weakly bounded for any smooth quasi-projective variety V.

4.A. Parameterizing spaces of weakly bounded families.

Set-up 4.5. Let V be a smooth quasi-projective variety and (B, D) an snc compactificaiton, that is $B \setminus D \simeq V$, where $D \subset B$ is a reduced divisor with simple normal crossing support. Let $\mathcal{H}_B$ be an ample line bundle on B.

According to [KL10, Cor. 2.23, Thm. 1.6, Thm. 1.7] with fixed n (dimension) and ν (canonical volume) and a suitable choice of a weak bound $b := b_{\mathscr{L}}$ as in Fact 4.4, there is a reduced, connected finite type scheme $W := W^b$ and a projective family $\mathsf{f} : \mathscr{X} \to$

$W \times V =: W_V$ of canonically polarized manifolds such that every canonically polarized family $f_U : U \to V$ of relative dimension n and relative canonical volume ν appears in a fiber over W, i.e., for every such f_U there exists a closed point $w \in W$ such that $\mathsf{f}_w : \mathscr{X}|_{\mathsf{f}^{-1}(\{w\}\times V)} \to \{w\} \times V$ is isomorphic to f_U. The irreducible components of $\mathscr{X}$ and the corresponding families will be denoted by

$$\{f_{W_{V,i}} : X_{W_{V,i}} \to W_V\}_{1\leq i\leq k}.$$

Note that $f_{W_V,i}$ is projective and there exists a closed subscheme $W_i \subseteq W$ such that

$$f_{W_{V,i}} : X_{W_{V,i}} \to W_i \times V \tag{4.5.1}$$

is surjective for each $i \in \{1, \dots, k\}$.

Notation 4.6. We will write $(f_U : U \to V) \subseteq X_{W_{V,i}}$, if there is a closed point $w \in W$ such that $(X_{W_{V,i}})_{\{w\}\times V} \simeq_V U$.

By the above we have that

$$f_U \in \mathcal{Sm}_{n,\nu}(V) \implies \exists\, i, 1 \leq i \leq k, f_U \subseteq X_{W_{V,i}}. \tag{4.6.1}$$

Remark 4.7 (Regularity assumption for W_i). Note that one may assume that W_i is smooth. Indeed, given a resolution $\widetilde{W}_i$ of W_i, after replacing $X_{W_V,i}$ by

$$X_{W_{V,i}} \times_{(W_i\times V)} \left(\widetilde{W}_i \times V\right)$$

and W_i by $\widetilde{W}_i$, and using the fact that W_i is reduced, we can see that the property (4.6.1) is preserved.

Notation 4.8. We will denote $W_i \times V$ by $W_{V,i}$.

4.B. **Uniform exponents for global generation of twisted direct image sheaves.**

We will follow the conventions and notation of the previous subsection.

Lemma 4.9. *In the situation of and 4.A, for every $1 \leq i \leq k$, there exist*

(4.9.1) *a quasi-projective subscheme $W_i^\circ \subseteq W_i$ with $W_{V,i}^\circ := W_i^0 \times V$, $X_{W_{V,i}^0} := (X_{W_{V,i}})_{W_{V,i}^0}$, $W_{B,i} := W_i^\circ \times B$, surjective morphism $f_{W_{V,i}^0} : X_{W_{V,i}^0} \to W_{V,i}^0$, and divisor $D_{W_{B,i}} := W_i^\circ \times D$,*

(4.9.2) *an ample line bundle $\mathcal{H}$ on $W_{B,i}$ such that $\mathcal{H}|_{\{w_i^0\}\times B} \simeq \mathcal{H}_B$, for any $w_i^0 \in W_i^0$, and*

(4.9.3) *a positive integer $m_0 = m_0(n, \nu)$ such that for each multiple m of m_0, there are integers β_m^i, $\widetilde{\beta}_m^i$ and a_i, where $\beta_m^i = \widetilde{\beta}_m^i a_i$, with the following properties:*

(4.9.4) *ω_F^m is globally generated, for every fiber F of $f_{W_{V,i}^0}$.*

(4.9.5) *For every $(f_U : U \to V) \subseteq X_{W_{V,i}^0}$ and any smooth compactification $f : X \to B$, the torsion free sheaf*

$$\bigotimes^{a_i} \left(\operatorname{Sym}^{\widetilde{\beta}_m^i} \mathscr{K}_m^\nu(f)\right) \otimes \mathcal{H}_B^{m\beta_m^i}$$

is generically generated by global sections.

(4.9.6) *$\bigotimes^{a_i} \left(\operatorname{Sym}^{\widetilde{\beta}_m^i} \mathscr{K}_m^\nu(f_{W_{V,i}^0})\right) \otimes \mathcal{H}^{m\beta_m^i} \otimes \mathscr{O}_{W_{V,i}^0}$ is generically generated by global sections.*

Proof. Let $n = \dim X/B$. We may assume that $\dim W_i \neq 0$. Let $W_i^0 \subseteq W_i$ be an open quasi-projective subset and let $\mathcal{H} := \pi_B^* \mathcal{H}_B$, where $\pi_B : W_{B,i} \to B$ is the projection. This $\mathcal{H}$ is an ample line bundle on $W_{B,i}$ satisfying (4.9.2).

Let $f_{W_B,i} : X_{W_B,i} \to W_{B,i}$ denote a proper closure of $f_{W^0_{V,i}}$. After removing a subset of $W_{B,i}$ of $\operatorname{codim}_{W_{B,i}} \geq 2$, let

$$\eta_{W_{B,i}} : W'_{B,i} \longrightarrow W_{B,i}$$

be the cyclic, flat morphism arising from a semistable reduction, cf. [KKMSD73] and [KM98, §7.17]. Set $Y_{W_B,i} := X_{W_B,i} \times_{W_{B,i}} W'_{B,i}$, which is irreducible as $\eta_{W_{B,i}}$ is flat, with $g_{W_{B,i}} : Y_{W_B,i} \to W'_{B,i}$ denoting the natural projection. Let $D_{W'_{B,i}} := \eta^{-1}_{W_{B,i}} D_{W_{B,i}}$.

For any $f_U \subseteq X_{W^0_V,i}$, let $w \in W_i^0$ be a parametrizing closed point as in Notation 4.6. Let $B' \subset W_{B',i}$ be the subscheme defined by $\eta^{-1}_{W_{B,i}}(\{w\} \times B)$ and $\eta : B' \to B$ the induced cyclic morphism. For a generic choice of $w \in W_i^0$, η is flat, ramified only along a very ample divisor and $\operatorname{supp} D'$, for some $D' \geq D$, and with the same ramification indices as those of $\eta_{W_{B,i}}$. Denote the pullback of f_U via η by $f_{U_{B'}} : U_{B'} \to B' \setminus \eta^{-1}(D)$.

Now, let $f'_{W_{B,i}} : X'_{W_{B,i}} \to W'_{B,i}$ be the stable reduction of $f_{W_{B,i}}$ through $\eta_{W_B,i}$, resulting in the commutative diagram:

$$\begin{array}{ccccc} X'_{W_{B,i}} & \overset{\text{birational}}{\dashleftarrow\!\!\dashrightarrow} & Y_{W_{B,i}} & \longrightarrow & X_{W_{B,i}} \\ & \searrow^{f'_{W_{B,i}}} & \downarrow{\scriptstyle g_{W_{B,i}}} & & \downarrow{\scriptstyle f_{W_{B,i}}} \\ & & W'_{B,i} & \xrightarrow[\text{flat and Galois}]{\eta_{W_{B,i}}} & W_{B,i}, \end{array}$$

that is there is a resolution $\pi : Z_{W_{B,i}} \to \widehat{Y}_{W_{B,i}}$ of the normalization $\widehat{Y}_{W_{B,i}}$ of $Y_{W_{B,i}}$ such that $Z_{W_{B,i}} \to W'_{B,i}$ is semistable in codimension one, and that $X'_{W_{B,i}}$ is the canonical model of $Z_{W_{B,i}}$ over $W'_{B,i}$, cf. Kollár [Kol10, §3] and Hacon-Xu [HX13, Cor. 1.4]. In particular $f'_{W_{B,i}}$ is stable in codimension one. With no loss of generality, after possibly removing another subset of codimension two, we may assume that $Z_{W_{B,i}} \to W'_{B,i}$ is semistable and $f'_{W_{B,i}}$ is stable over $W'_{B,i}$. Let n_0 be the smallest integer for which $\omega^{[n_0]}_{X_{W'_{B,i}}/W'_{B,i}}$ is invertible and set $m_0 := \operatorname{lcm}(\mathrm{m}_0, \mathrm{n}_0)$, with a_0 as in (2.25.2). Then (4.9.4) holds for this choice of m, and Proposition 3.3 is applicable.

According to (3.4.2), for every multiple m of m_0, there is an integer $\widetilde{\beta}^i_m$ such that, for every multiple r of $\widetilde{\beta}^i_m$,

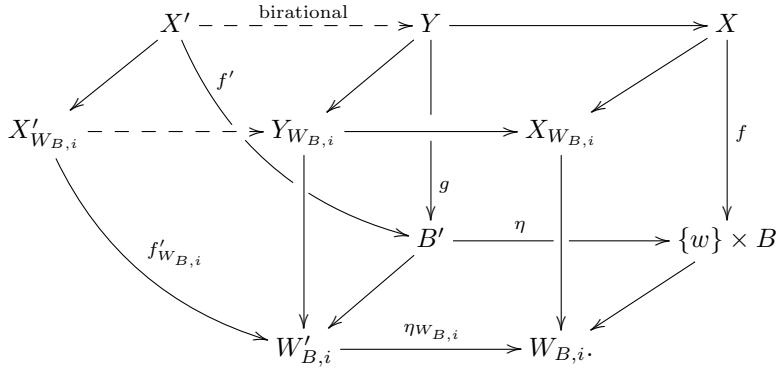
(4.9.7) the sheaf $\operatorname{Sym}^r \mathscr{K}^{\nu}_m(f'_{W_{B,i}}) \otimes (\mathcal{H}')^{mr}$ is globally generated over $W'_{B,i}$,

where $\mathcal{H}' := \eta^*_{W_{B,i}} \mathcal{H}$ (recall that we have removed a subset of $W'_{B,i}$ of $\operatorname{codim}_{W'_{B,i}} \geq 2$).

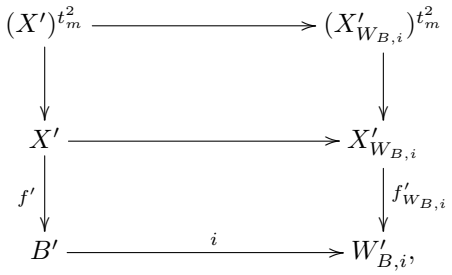
We next consider the pullback $(X_{W_{B,i}})_{\{w\} \times B}$, which gives a morphism of reduced pairs $f : (X, \Delta) \to (B, D)$. Define $Y := X \times_{X_{W_{B,i}}} Y_{W_{B,i}}$ with the natural projection $g : Y \to B'$ and set $X' := (X'_{W_{B,i}})_{B'}$, with the induced family $f' : X' \to B'$, which is stable. We

summarize these constructions in the following commutative diagram.

(4.9.8)



Now, we consider the t_m^2-fold fiber product of (4.9.8) with the induced stable maps



where $i : B' \to W'_{B,i}$ denotes the natural inclusion. According to Proposition 3.2, after removing a subset of B' of $\operatorname{codim}_{B'} \geq 2$, we have

$$i^* \mathscr{K}_m^{\nu}(f'_{W_{B,i}}) \simeq \mathscr{K}_m^{\nu}(f'). \tag{4.9.9}$$

Claim 4.10. After replacing W_i^0 by an open subset, if necessary, for every multiple r of $\widetilde{\beta}_m^i$, the torsion free sheaf $\operatorname{Sym}^r \mathscr{K}_m^{\nu}(f') \otimes \mathcal{H}_{B'}^{mr}$ is generically generated by global sections, where $\mathcal{H}_{B'} := \eta^* \mathcal{H}_B$.

Proof of Claim 4.10. We have already established that $\operatorname{Sym}^r \mathscr{K}_m^{\nu}(f'_{W_{B,i}}) \otimes (\mathcal{H}')^{mr}$ is globally generated over $W'_{B,i}$ cf. (4.9.7). By pulling back and using (4.9.9) we find that

$$i^* \left(\operatorname{Sym}^r \mathscr{K}_m^{\nu}(f'_{W_{B,i}}) \otimes (\mathcal{H}')^{mr} \right) \simeq \operatorname{Sym}^r \mathscr{K}_m^{\nu}(f') \otimes \mathcal{H}_{B'}^{mr}$$

is generically generated by global sections, for the family f parametrized by a general closed point $w \in W_i^0$. □

From now on we will replace W_i^0 by its open subset provided by Claim 4.10. We note that by construction, for the general family $f : X \to B$ parametrized by $w \in W_i^0$, and for every codimension one point $b' \in B'$, $Y_{b'}$ is reduced. With no loss of generality we will assume that this holds for every $w \in W_i^0$. Thus, there is a strong resolution $\widetilde{Y} \to Y$ of the normalization of Y such that the induced morphism $\widetilde{g} : \widetilde{Y} \to B'$ is semistable [KKMSD73].

Following the above construction, there is also a birational map $X' \dashrightarrow \widetilde{Y}$. Let $\mu : \widetilde{X} \to X'$ denote the birational morphism obtained by eliminating the indeterminacy of that

rational map. Let $\sigma : \widetilde{X} \to \widetilde{Y}$ be the induced birational morphism, and let $\widetilde{f}' : \widetilde{X} \to B'$ denote the resulting family, all fitting in the following commutative diagram:

$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & \swarrow^{\mu} & & \searrow^{\sigma} & \\ X' & \dashrightarrow & & & \widetilde{Y} \longrightarrow X \\ & \searrow_{f'} & & & \downarrow \widetilde{g} \qquad \downarrow f \\ & & & & B' \xrightarrow{\eta} B. \end{array} \tag{4.10.1}$$

Now, as X' has only canonical singularities [KS16], for every multiple m of m_0, we have

$$f'_*\omega^{[m]}_{X'/B'} \simeq \widetilde{f}'_*\omega^m_{\widetilde{X}/B'} \simeq \widetilde{g}_*\omega^m_{\widetilde{Y}/B'}. \tag{4.10.2}$$

It follows that

$$\mathscr{W}_m(f') \simeq \mathscr{W}_m(\widetilde{g}). \tag{4.10.3}$$

Moreover, considering the t_m^2-fold fiber product of (4.10.1) we find that

$$\begin{array}{ccccc} & & \widetilde{X}^{(t_m^2)} & & \\ & \swarrow^{\mu'} & & \searrow^{\sigma'} & \\ (X')^{t_m^2} & \dashrightarrow & & & \widetilde{Y}^{(t_m^2)} \longrightarrow X^{(t_m^2)} \\ & \searrow_{(f')^{t_m^2}} & & & \downarrow \widetilde{g}^{(t_m^2)} \qquad \downarrow f^{(t_m^2)} \\ & & & & B' \xrightarrow{\eta} B. \end{array}$$

Again, with $(f')^{t_m^2} : (X')^{t_m^2} \to B'$ being stable, $(X')^{t_m^2}$ has only canonical singularities and therefore we have

$$(f')^{t_m^2}_*\omega^{[m]}_{(X')^{t_m^2}/B'} \simeq (\widetilde{f}')^{t_m^2}\omega^m_{\widetilde{X}^{(t_m^2)}/B'} \simeq \widetilde{g}^{(t_m^2)}_*\omega^m_{\widetilde{Y}^{(t_m^2)}/B'}. \tag{4.10.4}$$

Combining (4.10.3) and (4.10.4) now leads to

$$\mathscr{K}^{\nu}_m(f') \simeq \mathscr{K}^{\nu}_m(\widetilde{g}).$$

On the other hand, according to Lemma 3.6, for a log resolution $(\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$, we have an injection

$$\mathscr{K}^{\nu}_m(\widetilde{g}) \hookrightarrow \eta^*\mathscr{K}^{\nu}_m(\widetilde{f}),$$

that is an isomorphism over $\eta^{-1}V$. With $\mathcal{H}_{B'} = \eta^*\mathcal{H}_B$, this implies that for every multiple r of $\widetilde{\beta}^i_m$ we have

$$\operatorname{Sym}^r \mathscr{K}^{\nu}_m(\widetilde{g}) \otimes \mathcal{H}^{mr}_{B'} \hookrightarrow \eta^*\big(\operatorname{Sym}^r \mathscr{K}^{\nu}_m(\widetilde{f}) \otimes \mathcal{H}^{mr}_B\big). \tag{4.10.5}$$

We saturate the image of the injection (4.10.5), call it $\mathscr{G}'$ and consider $\bigotimes_{g\in G} g^*\mathscr{G}'$, where $G = \operatorname{Gal}(B'/B)$, with $|G| = |\operatorname{Gal}(W'_{B,i}/W_{B,i})|$. By [HL10, Thm. 4.2.15] and by construction, after removing a subset of B of $\operatorname{codim}_B \geq 2$ if necessary, there is a locally free sheaf with an injection

$$\jmath : \mathscr{G} \hookrightarrow \bigotimes^{|G|}\big(\operatorname{Sym}^r \mathscr{K}^{\nu}_m(\widetilde{f}) \otimes \mathcal{H}^{mr}_B\big) \tag{4.10.6}$$

on B such that

(4.10.7) $\eta^*\mathscr{G} \simeq \bigotimes_g g^*\mathscr{G}'$, and
(4.10.8) $\jmath$ is an isomorphism over V.

Now, as the functor $\gamma_*(\_)^G$ is exact, and since $\mathscr{G}'$ is globally generated over $\eta^{-1}V$, we find that $\mathscr{G} \simeq \gamma_*\big(\eta^*\mathscr{G}\big)^G$ is generically generated by global sections and thus so is

$$\bigotimes^{|G|} \big(\operatorname{Sym}^r \mathscr{K}_m^\nu(\widetilde{f}) \otimes \mathcal{H}_B^{mk}\big) \simeq \bigotimes^{|G|} \big(\operatorname{Sym}^r \mathscr{K}_m^\nu(\widetilde{f})\big) \otimes \mathcal{H}_B^{mk|G|}.$$

We conclude the proof by setting $a_i := |\operatorname{Gal}(W'_{B,i}/W_{B,i})|$. □

4.C. **Generic deformation invariance and an upper-bound for $\alpha_{m,\beta_m,s_m}(\_)$.**

Theorem 4.11. *In the situation of Set-up 4.5, for each $m, \widetilde{\beta}_m^i$ and β_m^i as in Lemma 4.9, for each i, there exists an $\alpha_m^i \in \mathbb{N}$ with the following property. For every $f_U \subseteq X_{W_{V,i}^0}$ and smooth compactification $f : X \to B$ there is a section*

$$0 \neq s_m \in H^0\left(X^{(t_m^2)}, \big(\omega_{X^{(t_m^2)}/B} \otimes (f^{(t_m^2)})^*\big(\mathscr{W}_m^{-1} \otimes \mathcal{H}_B\big)\big)^{m\beta_m^i}\right),$$

such that

$$\alpha_{s_m}(f) \leq \alpha_m^i.$$

Proof. According to Lemma 4.9 there exists an $m_0(n,\nu) \in \mathbb{N}$ such that, for any multiple m of m_0, the two sheaves

$$\begin{aligned} &\bigotimes^{a_i} \big[\operatorname{Sym}^{\widetilde{\beta}_m^i}\big((f_{W_{B,i}}^{(t_m^2)})_*\omega^m_{X^{t_m^2}_{W_{B,i}}/W_{B,i}} \otimes \mathscr{W}_m^{-m}(f_{W_{B,i}})\big)\big] \otimes \mathcal{H}^{\beta_m^i m}, \text{ and} \\ &\bigotimes^{a_i} \big[\operatorname{Sym}^{\widetilde{\beta}_m^i}\big(f_*^{(t_m^2)}\omega^m_{X^{(t_m^2)}/B} \otimes \mathscr{W}_m^{-m}(f)\big)\big] \otimes \mathcal{H}_B^{\beta_m^i m} \end{aligned} \tag{4.11.1}$$

are generically generated by global sections, for some for β_m^i, $\widetilde{\beta}_m^i$ and a_i as in Lemma 4.9.

On the other hand, by the invariance of plurigenera [Siu98] and (4.9.4) the two natural maps

$$\begin{aligned} (f^{t_m^2}_{W_V^0,i})^*(f^{t_m^2}_{W_V^0,i})_*\omega^m_{X^{t_m^2}_{W_V^0,i}/W_{V,i}} &\longrightarrow \omega^m_{X^{t_m^2}_{W_{V,i}^0}/W_{V,i}^0}, \text{ and} \\ (f^{t_m^2})^* f_*^{t_m^2}\omega^m_{U^{t_m^2}/V} &\longrightarrow \omega^m_{U^{t_m^2}/V} \end{aligned} \tag{4.11.2}$$

are surjective. Therefore, by pulling back the sheaves in (4.11.1), and using the surjectivity of the maps in (4.11.2), it follows that

$$\begin{aligned} \bigotimes^{a_i} \operatorname{Sym}^{\widetilde{\beta}_m^i}\Big[\omega^m_{X^{(t_m^2)}_{W_{B,i}}/W_{B,i}} \otimes (f^{(t_m^2)}_{X_{W_{B,i}}})^*\mathscr{W}_m^{-m}(f_{W_{B,i}})\Big] \otimes (f^{(t_m^2)}_{W_{B,i}})^*\mathcal{H}^{\beta_m^i m} \simeq \\ \simeq \Big[\omega_{X^{(t_m^2)}_{W_{B,i}}/W_{B,i}} \otimes (f^{(t_m^2)}_{W_{B,i}})^*\big(\mathscr{W}_m^{-1}(f_{W_{B,i}}) \otimes \mathcal{H}\big)\Big]^{\beta_m^i m}, \end{aligned}$$

and similarly

$$\Big[\omega_{X^{(t_m^2)}/B} \otimes (f^{(t_m^2)})^*\big(\mathscr{W}_m^{-1}(f) \otimes \mathcal{H}_B\big)\Big]^{\beta_m^i m} \tag{4.11.3}$$

are globally generated over the (preimage of the) smooth locus of the family. Next, let

$$0 \neq s_{m,W_{V,i}^0} \in H^0\left(X^{t_m^2}_{W_{V,i}^0}, \Big[\omega_{X^{t_m^2}_{W_V,i}/W_{V,i}} \otimes (f^{t_m^2}_{W_{V,i}^0})^*\big(\mathscr{W}_m^{-1}(f_{W_{V,i}^0}) \otimes \mathcal{H}\big)\Big]^{\beta_m^i m}\right) \tag{4.11.4}$$

be such that the intersection of $D_{X_{W^0_{V,i}}} := (s_{m,W^0_{V,i}})_0$ with the general subfamily $f_U \subset X_{W^0_{V,i}}$ is smooth. Denote the (finite type) subscheme of W^0_i over which this intersection is not smooth by T^0_i. Next, for each f_U as above, set

$$s^0_m \in H^0\left(U^{t^2_m}, \left[\omega^{t^2_m}_{U^{t^2_m}/V} \otimes (f^{t^2_m}_U)^*\big(\mathscr{W}^{-1}_m(f_U) \otimes \mathcal{H}_B\big)\right]\right) \tag{4.11.5}$$

to be the section induced by the pullback $s_{m,W^0_{V,i}}\big|_{U^{t^2_m}}$. Noting that the pullback of the line bundle in (4.11.4) is isomorphic to the one in (4.11.5), by the construction of Viehweg numbers Definition 3.8 and Lemma 3.12, we find that

$$\alpha_{s_{m,W^0_{V,i}}}(f_{W^0_{V,i}}) = \alpha_{s^0_m}(f_U). \tag{4.11.6}$$

Without loss of generality, we may assume that the sheaf in (4.11.3) is globally generated over $U^{t^2_m}$. Consequently, there exists an

$$s_m \in H^0\left(X^{(t^2_m)}, \left[\omega_{X^{(t^2_m)}/B} \otimes (f^{(t^2_m)})^*\big(\mathscr{W}^{-1}_m(f) \otimes \mathcal{H}_B\big)\right]^{\beta^i_m m}\right) \tag{4.11.7}$$

such that $s_m|_{U^{t^2_m}} = s^0_m$. We have that $\alpha_{s_m}(f) = \alpha_{s^0_m}(f_U)$ (see Remark 3.10), so by (4.11.6), for every family $(f_U : U \to V) \subseteq X_{W^0_{V,i}} \setminus f^{-1}_{W^0_{V,i}}(T^0_i \times V)$, regardless of a choice of compactification $f : X \to B$ over B, there exists a section s_m as in (4.11.7) such that

$$\alpha_{s_m}(f) = \alpha_{s_{m,\beta^i_m,W^0_{V,i}}}(f_{W^0_{V,i}}).$$

After pulling back $f_{W^0_{V,i}}$ over each irreducible component $T^0_{ij} \times V$ of $T^0_i \times V$, and replacing W_i in Lemma 4.9 by T^0_{ij}, the conclusions of Lemma 4.9 are again valid and we can repeat the above argument. As T^0_i is of finite type, we can find an α^i_m, by induction on the dimension of W_i, such that for every $f_U \subseteq X_{W^0_{V,i}}$ we have

$$\alpha_{s_m}(f) \leq \alpha^i_m,$$

with s_m being as in (4.11.7). □

Corollary 4.12. *In the situation of Set-up 4.5 there are integers m_0 and $\overline{\alpha}_m$, depending only on n and ν, with the following property. For every multiple m of m_0 and every $f_U \in \mathcal{Sm}_{n,\nu}(V)$ and smooth compactification $f : X \to B$ we have*

$$\alpha_{s_m}(f) \leq \overline{\alpha}_m,$$

for some $\beta_m \in \mathbb{N}$ and $0 \neq s_m \in H^0(X^{(t^2_m)}, \mathscr{M}^{m\beta_m})$ (as defined in Definition 3.7).

Proof. By Theorem 4.11 we know that for each fixed i and every $f_U \in \mathcal{Sm}_{n,\nu}(V)$ with $f_U \subseteq X_{W^0_{V,i}}$ there are $\alpha^i_m, \beta^i_m \in \mathbb{N}$ and s_m such that

$$\alpha_{s_m}(f) \leq \alpha^i_m. \tag{4.12.1}$$

Now, for each irreducible component T_{ij} of the scheme $W_i \setminus W^0_i$, we replace W_i in (4.5.1) by T_{ij}, $1 \leq j \leq l$, for some $l \in \mathbb{N}$. Again, by Lemma 4.9 and Theorem 4.11 we find that, for suitable choices of β^i_m, s_m and α^i_m, the inequality (4.12.1) is valid. As W_i is of finite type, by induction on $\dim(W_i)$, the existence of $\overline{\alpha}^i_m$ such that $\alpha_m(f) \leq \overline{\alpha}^i_m$ holds for every $f_U \subseteq X_{W_V,i}$. We conclude the proof by setting $\overline{\alpha}_m := \max_{1\leq i\leq k}\{\overline{\alpha}^i_m\}$. □

5. Higher dimensional Arakelov inequalities

5.A. **Reflexive systems of Hodge sheaves containing $\mathscr{W}_m(-D)$.** Following [Taj20, Def. 2.2], given sheaves of $\mathscr{O}_B$-modules $\mathscr{W}$ and $\mathscr{F}$, a *$\mathscr{W}$-valued system* means a splitting $\mathscr{F} = \bigoplus \mathscr{F}_i$ and a sheaf homomorphism $\tau : \mathscr{F} \to \mathscr{W} \otimes \mathscr{F}$ that is Griffiths-transversal. If we assume further that $\mathscr{W} = \Omega^1_B(\log D)$, τ is integrable and $\mathscr{F}$ is reflexive, then $(\mathscr{F}, \tau)$ is referred to as a *reflexive logarithmic-system of Hodge sheaves*.

Now, let V be a smooth quasi-projective variety with a smooth compactification (B, D) and let $\mathcal{H}_B$ be an ample line bundle on B. Fix $h \in \mathbb{Z}[x]$. According to Theorem 4.11 and Corollary 4.12 there are integers $m_0 = m_0(n, \nu)$, β_m (suppressing the unnecessary superscript i), and $\overline{\alpha}_m$ such that, for every multiple m of m_0, and every smooth compactification $f : X \to B$ of any smooth projective family $f_U \in \mathcal{Sm}_{n,\nu}(V)$, over an open subset $B^0 \subseteq B$, with $\operatorname{codim}_B(B \backslash B^0) \geq 2$, there exists an $0 \neq s_m \in H^0(X^{(t_{m,2})}, \mathscr{M}^{m\beta_m})$, where (recall from (3.7.2))

$$\mathscr{M} := \omega_{X^{(t_{m,2})}/B} \otimes (f^{(t_{m,2})})^*(\mathscr{W}_m^{-1} \otimes \mathcal{H}_B), \tag{5.0.1}$$

for which we have $\alpha_{s_m}(f) \leq \overline{\alpha}_m$.

Proposition 5.1. *For every projective morphism $f : X \to B$ as above there is a reflexive system of Hodge sheaves $(\mathscr{G} = \bigoplus_{i=0}^{w} \mathscr{G}_i, \theta)$ of weight $w \in \mathbb{N}$ on B, with logarithmic poles along D, satisfying the following properties.*

(5.1.1) $w \leq \dim(X/B)t_{m,2}$.
(5.1.2) $\operatorname{rank}(\mathscr{G}) \leq \overline{\alpha}_m$.
(5.1.3) *There is an injection $\mathscr{W}_m(-D) \otimes \mathcal{H}_B^{-1} \hookrightarrow \mathscr{G}_0$.*
(5.1.4) *The torsion free sheaf $\mathscr{N}_j := \ker(\theta|_{\mathscr{G}_j})$ is weakly negative, for every $0 \leq j \leq w$.*

Proof. Following [VZ03], [Taj21, 2.2] and [Taj20, 2.2] (see also [KT21, §4] for a general construction for flat families) we consider the $\Omega^1_{B^0}(\log D)$-valued system $(\mathscr{F} = \bigoplus \mathscr{F}_i, \tau)$ of weight equal to $\dim(X^{(t_{m,2})}/B)$, with each $\mathscr{F}_i$ being defined by

$$\mathcal{R}^i f_*^{(t_{m,2})} \left(\Omega^{t_{m,2}-i}_{X^{(t_{m,2})}/B}(\log \Delta_{f^{(t_{m,2})}}) \otimes \mathscr{M}^{-1} \right).$$

With s_m as above, over B_0, there is a surjective morphism $Z_{s_m} \to X^{(t_{m,2})}$, as in (3.7.3). Let $\mathcal{V}^0$ denote Deligne's extension of the $\mathbb{C}$-VHS of weight $\dim(Z_{s_m}/B)$ underlying the smooth locus of g_{s_m}, and set $(\mathscr{E}^0 = \bigoplus \mathscr{E}^0_i, \theta^0)$ to be the associated Hodge bundle. By [Taj21, pp. 8–9] there is a morphism of systems $\Phi : (\mathscr{F}, \tau) \to (\mathscr{E}^0, \theta^0)$ such that Φ_0 is injective and its image $(\mathscr{G}^0 = \bigoplus_{i=0}^{w} \mathscr{G}^0_i, \theta^0)$ has the following property: the natural extension of $(\mathscr{G}^0, \theta^0)^{**}$ to the reflexive system of Hodge sheaves $(\mathscr{G}, \theta)$ on B satisfies (5.1.4). The construction and the injectivity of Φ_0 implies (5.1.3). Moreover, by the construction of $\mathscr{G}$ and Definition 3.8, we have that $w \leq \dim(X/B)t_{m,2}$, so (5.1.1) follows. Furthermore, $\operatorname{rank}(\mathscr{G}) \leq \alpha_{m,\beta_m,s_m}(f)$ by construction, so (5.1.2) follows from Corollary 4.12. □

5.B. **Proof of Theorem 1.3.** We are now ready to prove the main theorem.

Theorem 5.2 (the precise version of Theorem 1.3)**.** *Let (B, D) be a smooth compactification of a smooth quasi-projective variety V of dimension d and H an ample Cartier divisor on B. Further let $n, \nu \in \mathbb{N}$ and assume that $K_B + D$ is pseudo-effective. Then, there exists an $m_0 = m_0(n, \nu) \in \mathbb{N}$, such that for every integer multiple m of m_0, there exist $a_m, b_m, \gamma_m \in \mathbb{N}$ with the following property. For each smooth compactifcation $f : X \to B$*

of an arbitrary $f_U \in \mathcal{Sm}_{n,\nu}(V)$ we have

$$c_1(\det f_*\omega_{X/B}^m)\cdot H^{d-1} \leq (d^{\gamma_m}a_m(K_B+D)+b_mD)\cdot H^{d-1}+H^d. \tag{5.2.1}$$

Proof. Let $\mathscr{W}_m = \det\left((f_*\omega_{X/B}^m)(-mD)\right)$ (cf. Definition-Notation 2.17). Further let $\mathscr{A}_m := \mathscr{W}_m(-D-H)$. As before, set $r_m := \operatorname{rank}(f_*\omega_{X/B}^m)$. We will distinguish cases based on the sign of $c_1(\mathscr{A}_m)\cdot H^{d-1}$.

First, assume that $c_1(\mathscr{A}_m)\cdot H^{d-1}\leq 0$. Then

$$c_1\left(\det(f_*\omega_{X/B}^m)\,(-(mr_m)D-H)\right)\cdot H^{d-1}\leq D\cdot H^{d-1},$$

which implies that

$$c_1(\det f_*\omega_{X/B}^m)\cdot H^{d-1}\leq (1+mr_m)D\cdot H^{d-1}+H^d. \tag{5.2.2}$$

Next, assume that

$$c_1(\mathscr{A}_m)\cdot H^{d-1}>0. \tag{5.2.3}$$

According to Proposition 5.1, there exists a reflexive system of Hodge sheaves ($\mathscr{G}=\bigoplus_{i=1}^w\mathscr{G}_i,\theta$) such that $\mathscr{A}_m\hookrightarrow\mathscr{G}_0$. Now, define

$$\theta^l := \underbrace{(\mathrm{id}\otimes\theta)\circ\ldots\circ(\mathrm{id}\otimes\theta)}_{(l-1)\text{-times}}\circ\theta : \mathscr{G}_0\longrightarrow\left(\Omega_B^1(\log D)\right)^{\otimes l}\otimes\mathscr{G}_l.$$

Claim 5.3. $\theta(\mathscr{A}_m)\neq 0$.

Proof of Claim 5.3. The proof is the same as in [KT21, Claim 5.5]. □

Clearly, $\theta^l(\mathscr{A}_m)=0$ for $l\gg 0$ and Claim 5.3 implies that such an l has to be larger than 1. Let k be the largest integer for which $\theta^k(\mathscr{A}_m)\neq 0$. In particular, then $\theta^{k+1}(\mathscr{A}_m)=0$, and hence

$$\theta^k\in\Gamma\left((\Omega_B^1(\log D))^{\otimes k}\otimes\mathcal{H}om(\mathscr{G}_0,\mathscr{N}_k)\right),$$

where $\mathscr{N}_k := \ker(\theta|_{\mathscr{G}_k})$ as in (5.1.4). As $\mathscr{A}_m$ is of rank one, the nontrivial map

$$\mathscr{A}_m\longrightarrow\left(\Omega_B^1(\log D)\right)^{\otimes k}\otimes\mathscr{N}_k$$

must be an injection. We may assume with no loss of generality that the image of this injection is saturated. After raising it to the power $s := \operatorname{rank}(\mathscr{N}_k)$, we find that

$$\mathscr{A}_m^s\otimes(\det\mathscr{N}_k)^{-1}\hookrightarrow\left(\Omega_B^1(\log D)\right)^{\otimes ks}.$$

i.e., by using the definition of $\mathscr{A}_m$, we have

$$(\mathscr{W}_m(-D-H))^s\otimes(\det\mathscr{N}_k)^{-1}\hookrightarrow\left(\Omega_B^1(\log D)\right)^{\otimes ks}. \tag{5.3.1}$$

Now, as $(\det\mathscr{N}_k)^{-1}$ is pseudo-effective by (5.1.4), we have the inequality

$$sc_1\left(\mathscr{W}_m(-D-H)\right)\cdot H^{d-1}\leq c_1\left((\mathscr{W}_m(-D-H))^s\otimes(\det\mathscr{N}_k)^{-1}\right)\cdot H^{d-1}.$$

On the other hand, by [CP19, Thm. 1.3] it follows from (5.3.1) that

$$c_1\left((\mathscr{W}_m(-D-H))^s\otimes(\det\mathscr{N}_k)^{-1}\right)\cdot H^{d-1}\leq c_1\left(\left(\Omega_B^1(\log D)\right)^{\otimes ks}\right)\cdot H^{d-1}.$$

By combining these latter two inequalities we find that

$$c_1\left(\mathscr{W}_m(-D-H)\right)\cdot H^{d-1}\leq d^{ks-1}k(K_B+D)\cdot H^{d-1}.$$

Now, substituting the definition $\mathscr{W}_m = \det\left((f_*\omega^m_{X/B})(-mD)\right)$, this implies that

$$c_1\left(\det f_*\omega^m_{X/B}\left((-mr_m-1)D\right)\right)\cdot H^{d-1} \leq d^{ks-1}k(K_B+D)\cdot H^{d-1}+H^d. \tag{5.3.2}$$

Furthermore, by Proposition 5.1 we have that $k \leq w \leq a_m$, where $a_m := t_{m,2}\cdot \dim X/B$, and $s \leq \overline{\alpha}_m$. Therefore, from (5.3.2) we find

$$c_1(\det f_*\omega^m_{X/B})\cdot H^{d-1} \leq d^{\overline{\alpha}_m a_m - 1}a_m(K_B+D)\cdot H^{d-1}+(1+mr_m)D\cdot H^{d-1}+H^d. \tag{5.3.3}$$

The statement now follows from combining (5.2.2) and (5.3.3), with $\gamma_m = \overline{\alpha}_m a_m - 1$,

$$a_m = (r_m m^2 e_m)\dim X/B \quad \text{and} \quad b_m = 1+mr_m. \tag{5.3.4}$$

□

5.C. **Proof of Theorem 1.5.** After removing a subset of B of $\operatorname{codim}_B \geq 2$, let $f' : X' \to B'$ be the stable reduction associated to $\eta : B' \to B$, as in (4.10.1). By (4.10.2) and (3.6.2) there is an embedding $f'_*\omega^{[m]}_{X'/B'} \hookrightarrow \eta^* f_*\omega^m_{X/B}$, such that

$$\underbrace{\left[\det f'_*\omega^{[m]}_{X'/B'}\right]^{p_m}}_{\simeq\Phi^*(\lambda_m)\text{ cf. [Kol90, 2.5],[Vie95, Thm. 7.17]}} \hookrightarrow \eta^*(\det f_*\omega^m_{X/B})^{p_m}.$$

By Teissier's inequality [Laz04, Thm. 1.6.1] (and references therein) and Theorem 1.3 it follows that

$$(\operatorname{vol}(\Phi^*\lambda_m))^{\frac{1}{d}}\left[\left(\eta^*H^d\right)^{\frac{1}{d}}\right]^{d-1} \leq p_m \deg\eta\left((d^{\gamma_m}a_m(K_B+D)+b_mD)\cdot H^{d-1}+H^d\right),$$

which implies that

$$(\operatorname{vol}(\Phi^*\lambda_m))^{\frac{1}{d}} \leq \frac{p_m\deg\gamma}{(H^d)^{1-\frac{1}{d}}}\left((d^{\gamma_m}a_m(K_B+D)+b_mD)\cdot H^{d-1}+H^d\right).$$

Finally, setting

$$c_m(x_1,x_2,x_3) = \frac{p_m^d}{x_3^{d-1}}\left(d^{\gamma_m}a_mx_1+b_mx_2+x_3\right)^d$$

completes the proof. □

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