

Three-loop effective potential for softly broken supersymmetry

Stephen P. Martin[✉]

Department of Physics, Northern Illinois University, DeKalb, Illinois 60115, USA



(Received 1 December 2023; accepted 29 December 2023; published 22 January 2024)

The effective potential has been previously calculated through three-loop order, in Landau gauge, for a general renormalizable theory using dimensional regularization. However, dimensional regularization is not appropriate for softly broken supersymmetric gauge theories, because it explicitly violates supersymmetry. In this paper, I obtain the three-loop effective potential using a supersymmetric regulator based on dimensional reduction. Checks follow from the vanishing of the effective potential in examples with supersymmetric vacua, and from renormalization-scale invariance in examples for which supersymmetry is broken, either spontaneously or explicitly by soft terms. As by-products, I obtain the three-loop Landau gauge anomalous dimension for the scalar component of a chiral supermultiplet, and the beta function for the field-independent vacuum energy.

DOI: 10.1103/PhysRevD.109.015019

I. INTRODUCTION

The quantitative analysis of vacuum expectation values and spontaneous symmetry breaking in quantum field theories can be formulated in terms of the Coleman-Weinberg effective potential [1–3]. The perturbative loop expansion of the effective potential is evaluated as the sum of all one-particle-irreducible vacuum diagrams, where the vertices and propagators depend on the scalar background fields. For a general field theory, the effective potential is known at two-loop order [4,5] and three-loop order [6,7]. These results are based on Landau gauge fixing, which greatly simplifies the expressions; other gauge-fixing choices have kinetic mixing between scalar and vector degrees of freedom. Complete effective potential results for a general field theory at two-loop order in a variety of other gauge-fixing prescriptions can be found in Ref. [8], which illustrates the unfortunate complications encountered. In the special case of the Standard Model, the four-loop contributions at leading order in QCD are also known [9].

The three-loop effective potential results of Ref. [7] were obtained using dimensional regularization (DREG) [10–15] followed by renormalization with modified minimal subtraction known as $\overline{\text{MS}}$ [16,17]. Although $\overline{\text{MS}}$ is the modern standard for loop calculations of all types in nonsupersymmetric theories, it is not appropriate for supersymmetric theories with or without explicit soft breaking terms. This is

because the DREG regularization procedure introduces explicit supersymmetry violation, due to the fact that in

$$d = 4 - 2\epsilon \quad (1.1)$$

dimensions there is a nonsupersymmetric mismatch between the numbers of gauge boson and gaugino degrees of freedom. Although this mismatch only has multiplicity 2ϵ , it is multiplied by poles in ϵ from loop diagrams. After renormalization, this leads to violations of the relationships among parameters that should be enforced by supersymmetry.

The purpose of this paper is to remedy this problem by providing a counterpart to the results of Ref. [7] but using Siegel's supersymmetric regularization by dimensional reduction (DRED) [18–20] followed by modified minimal subtraction.¹ In DRED, loop momenta are still in d dimensions, but each vector degree of freedom has four components, so as to avoid the nonsupersymmetric mismatch between gauginos and gauge bosons. The extra 2ϵ vector components are called ϵ scalars. When explicit soft supersymmetry breaking is present in supersymmetric gauge theories, there is an additional complication, because in general in the resulting $\overline{\text{DR}}$ renormalization scheme the ϵ scalars obtain nonzero squared mass contributions in excess of the corresponding vector squared masses, due to renormalization. These ϵ -scalar squared mass contributions are unphysical, in the sense that they have no observable

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¹Although there are technical problems [21–23] associated with simultaneously avoiding either inconsistencies or ambiguities of DRED at higher-loop orders while maintaining supersymmetry, these are not an issue for the three-loop vacuum diagrams considered in this paper, as demonstrated by the explicit calculations reported below.

counterparts. Accordingly, in Ref. [24] it was shown that these unphysical quantities can be simultaneously eliminated from the renormalization group equations and from the relations between on-shell physical quantities and the Lagrangian parameters, by a parameter redefinition of the type given in Ref. [25]. The resulting supersymmetric renormalization scheme based on regularization by dimensional reduction is known² as the $\overline{\text{DR}}'$ scheme, with the property that the ϵ -scalar squared masses appearing in propagators are exactly the same as those of the corresponding vector bosons. The two-loop results for the effective potential in a general softly broken supersymmetric gauge theory in $\overline{\text{DR}}'$ were obtained in Ref. [5], and in the present paper this will be extended to three-loop order.

The notations, conventions, and general strategies of this paper will follow closely those of Refs. [7, 26]. Therefore, to avoid needless (and lengthy) repetition, the reader is advised to consult those papers for the relevant definitions. In particular, the three-loop effective potential is given in terms of renormalized ϵ -finite basis integrals: $A(x)$ at one loop, $I(x, y, z)$ at two loops, and $F(w, x, y, z)$, $\bar{F}(0, x, y, z)$, $G(v, w, x, y, z)$, and $H(u, v, w, x, y, z)$ at three loops, along with convenient combinations $\bar{A}(x, y)$, $\bar{I}(w, x, y, z)$, and $K(u, v, w, x, y, z)$. Here, u, v, w, x, y, z denote propagator squared mass arguments, and the dependence on the common renormalization scale Q is suppressed in the lists of arguments, as it is typically the same everywhere within a given calculation. These basis functions were defined explicitly in Sec. II of [7] and Sec. II of [26], and the computer software library 3VIL provided with the latter reference provides for their fast and accurate numerical evaluation. Note that their definitions do not depend on whether one is using the $\overline{\text{MS}}$ or $\overline{\text{DR}}'$ scheme. They satisfy symmetry relations that reflect all of the invariances of the corresponding underlying Feynman diagrams under interchanges of squared mass arguments. They also satisfy special case relations, which are identities that occur when

the squared mass arguments are nongeneric, meaning that some of them are equal to each other, and/or vanish. Examples of these special case relations appeared in Eqs. (5.82)–(5.86) of Ref. [26] and (2.40)–(2.43) of Ref. [7]. There are many other identities reflecting the analytic special cases that occur when there is only one distinct nonzero squared mass found in Refs. [27–33], and listed in the notation of the present paper in Sec. V of Ref. [26]. For convenience, both the symmetry relations and the known special case relations are provided in the identities.anc file of the Supplemental Material [34].

Since the structure of the three-loop effective potential has been elucidated already in Ref. [7] at considerable length, the present paper will assume this as given and concentrate on the distinctions that are special to supersymmetric theories and DRED. Furthermore, the explicit results at three-loop order are extremely complicated, and therefore mostly useless to the human eye. Hence, they will be almost entirely relegated to ancillary electronic files, which are suitable for use with symbolic manipulation software and numerical evaluation with 3VIL.

II. EFFECTIVE POTENTIAL IN DIMENSIONAL REDUCTION

Consider a general renormalizable theory, which we will later assume to be a softly broken supersymmetric gauge theory. Suppose that the fields with diagonal tree-level squared masses consist of some real scalars R_j with squared masses m_j^2 , two-component fermions ψ_I with squared masses M_I^2 , and real vector fields A_a^μ with squared masses m_a^2 . In the case of the fermions, the masses need not be diagonal, but may include charged Dirac fermion fields consisting of pairs ψ_I and $\psi_{I'}$ with off-diagonal masses $M^{II'}$, where $M_I^2 = M_{I'}^2 = |M^{II'}|^2$. For Majorana fermions, one identifies I and I' . There are also field-dependent interactions

$$\begin{aligned} \mathcal{L}_{\text{int}} = & -\frac{1}{6}\lambda^{jkl}R_jR_kR_l - \frac{1}{24}\lambda^{jklm}R_jR_kR_lR_m - \frac{1}{2}(Y^{jIJ}R_j\psi_I\psi_J + \text{c.c.}) \\ & + g_I^{aJ}A^{\mu a}\psi^{\dagger I}\bar{\sigma}_\mu\psi_J - g^{ajk}A^{\mu a}R_j\partial_\mu R_k - \frac{1}{4}g^{abjk}A_\mu^aA^{\mu b}R_jR_k - \frac{1}{2}g^{abj}A_\mu^aA^{\mu b}R_j \\ & - g^{abc}A^{\mu a}A^{\nu b}\partial_\mu A_\nu^c - \frac{1}{4}g^{abe}g^{cde}A^{\mu a}A^{\nu b}A_\mu^cA_\nu^d - g^{abc}A^{\mu a}\omega^b\partial_\mu\bar{\omega}^c, \end{aligned} \quad (2.1)$$

where ω^a and $\bar{\omega}^c$ are ghost and antighost fields. The independent couplings are scalar cubic λ^{jkl} , scalar quartic

²Many sources elide the distinction between the $\overline{\text{DR}}'$ and $\overline{\text{DR}}$ schemes. It is hard to fault this practice, as the $\overline{\text{DR}}$ scheme as defined in Ref. [24], and in the present paper (including arbitrary independent unphysical ϵ -scalar squared masses) it is not of much practical use.

λ^{jklm} , Yukawa Y^{jIJ} , vector-fermion-fermion g_I^{aJ} , vector-scalar-scalar g^{ajk} , vector-vector-scalar g^{abj} , and vector-vector-vector g^{abc} . By convention, $Y_{jIJ} \equiv (Y^{jIJ})^*$ and $M_{II'} = (M^{II'})^*$. Note that the vector-vector-scalar-scalar and vector-vector-vector-vector interaction couplings are not independent of the cubic couplings, as they are given by

$$g^{abjk} = g^{ajl} g^{bkl} + g^{akl} g^{bjl}, \quad (2.2)$$

$$g^{abcd} = g^{abe} g^{cde}, \quad (2.3)$$

respectively. Each of these masses and couplings may depend on one or more background scalar fields φ , which correspond to the possible vacuum expectation values.

The loop expansion of the $\overline{\text{DR}}'$ effective potential can be written as

$$\begin{aligned} V_{\text{eff}} = V^{(0)} &+ \frac{1}{16\pi^2} V^{(1)} + \frac{1}{(16\pi^2)^2} V^{(2)} \\ &+ \frac{1}{(16\pi^2)^3} V^{(3)} + \dots \end{aligned} \quad (2.4)$$

The contribution $V^{(0)}$ is the tree-level background-field-dependent potential, and each $V^{(\ell)}$ is obtained by summing the contributions of ℓ -loop one-particle-irreducible diagrams. At one-loop order, the effective potential in the $\overline{\text{DR}}'$ scheme is given by the supertrace form

$$\begin{aligned} V^{(2)} = & \frac{1}{12} (\lambda^{jkl})^2 f_{SSS}(j, k, l) + \frac{1}{8} \lambda^{jjkk} f_{SS}(j, k) + \frac{1}{2} Y^{jIJ} Y_{jIJ} f_{FFS}(I, J, j) + \frac{1}{4} (Y^{jIJ} Y^{jlJ'} M_{II'} M_{JJ'} + \text{c.c.}) f_{\bar{F}\bar{F}S}(I, J, j) \\ & + \frac{1}{4} (g^{ajk})^2 f_{VSS}(a, j, k) + \frac{1}{4} (g^{abj})^2 f_{VVS}(a, b, j) + \frac{1}{2} g_I^{aj} g_J^{al} f_{FFV}(I, J, a) + \frac{1}{2} g_I^{aj} g_{I'}^{aj'} M^{II'} M_{JJ'} f_{\bar{F}\bar{F}V}(I, J, a) \\ & + \frac{1}{12} (g^{abc})^2 f_{\text{gauge}}(a, b, c), \end{aligned} \quad (2.8)$$

in terms of two-loop integral functions f_{SSS} , f_{SS} , f_{FFS} , $f_{\bar{F}\bar{F}S}$, f_{VSS} , f_{VVS} , f_{FFV} , $f_{\bar{F}\bar{F}V}$, and f_{gauge} . The functions f_{SSS} , f_{SS} , f_{FFS} , and $f_{\bar{F}\bar{F}S}$ do not involve vectors or ϵ scalars, and so are trivially the same in the $\overline{\text{MS}}$ and $\overline{\text{DR}}'$ schemes. In contrast, the functions f_{VSS} , f_{VVS} , f_{FFV} , $f_{\bar{F}\bar{F}V}$, and f_{gauge} are different in the two schemes. The $\overline{\text{DR}}'$ functions are constructed so as to include the contributions of the ϵ scalars corresponding to each vector field, with each ϵ -scalar mass equal to the corresponding field-dependent vector boson mass. They were obtained in³ Ref. [5]. The $\overline{\text{DR}}'$ results for the one-loop function f and the nine two-loop

$$V^{(1)} = \sum_j f(j) - 2 \sum_I f(I) + 3 \sum_a f(a), \quad (2.5)$$

where j , I , and a appearing as arguments of loop integral functions are shorthand notations for the corresponding $\overline{\text{DR}}'$ squared masses, and the one-loop integral function is

$$f(x) = \frac{1}{4} x^2 (\overline{\ln}(x) - 3/2), \quad (2.6)$$

which depends on the renormalization scale Q through the definition

$$\overline{\ln}(x) = \ln(x/Q^2). \quad (2.7)$$

As explained in Ref. [5], Eq. (2.5) differs from the $\overline{\text{MS}}$ result, which instead has a one-loop function $f_V(x) = f(x) + x^2/6$ for the vectors. The difference arises from the ϵ -scalar contribution to $f(x)$.

The two-loop contribution in either $\overline{\text{MS}}$ or $\overline{\text{DR}}'$ can be written in the form

functions are provided in the `functionsDRED.anc` file of the Supplemental Material [34].

As explained in Ref. [7], the three-loop contribution to the effective potential for a general renormalizable theory can be expressed in terms of 89 loop integral functions; see Eqs. (3.2)–(3.32) of that paper for the rather lengthy expression for $V^{(3)}$ in terms of the functions and the renormalized couplings. The 89 functions can be divided into three categories. First, there are 24 functions that do not involve vector fields or ϵ scalars at all, and so they are trivially the same in the $\overline{\text{DR}}'$ and $\overline{\text{MS}}$ schemes:

$$\begin{aligned} H_{SSSSSS}, \quad K_{SSSSSS}, \quad J_{SSSSS}, \quad G_{SSSSS}, \quad L_{SSSS}, \quad E_{SSSS}, \\ H_{FF\bar{F}SSS}, \quad H_{\bar{F}\bar{F}\bar{F}SSS}, \quad H_{FFSSFF}, \quad H_{FFSS\bar{F}\bar{F}}, \quad H_{F\bar{F}SSFF}, \quad H_{\bar{F}\bar{F}SS\bar{F}\bar{F}}, \\ K_{SSSSFF}, \quad K_{SSSS\bar{F}\bar{F}}, \quad K_{FFFSSF}, \quad K_{FFF\bar{F}\bar{F}}, \quad K_{\bar{F}\bar{F}FSSF}, \quad K_{\bar{F}\bar{F}\bar{F}SSF}, \\ K_{\bar{F}\bar{F}FSS\bar{F}}, \quad K_{SSFFF}, \quad K_{SSFF\bar{F}\bar{F}}, \quad K_{SS\bar{F}\bar{F}\bar{F}\bar{F}}, \quad J_{SSFFS}, \quad J_{SS\bar{F}\bar{F}S}. \end{aligned} \quad (2.9)$$

³Here we have adopted a slightly more efficient notation than in that paper, since $f_{VSS}(x, y, z) \equiv f_{SSV}(y, z, x) + F_{VS}(x, y) + F_{VS}(x, z)$, where the functions on the right side were the ones defined in Ref. [5], and the function f_{VSS} is the one used here. This takes advantage of Eq. (2.2).

In a second category are five functions which involve vector fields, but for which there are no corresponding ϵ -scalar contributions. This occurs when all Feynman diagram contributions to the function have only vector lines that terminate (at one end, at least) in a vector-scalar-scalar vertex, since in that case the vector index will be contracted with a momentum, which lives in only d dimensions, not four, thus projecting out the ϵ -scalar components. Therefore, these five functions are again the same in the $\overline{\text{DR}}'$ and $\overline{\text{MS}}$ schemes:

$$H_{FFFVSS}, \quad H_{F\bar{F}FVSS}, \quad H_{\bar{F}\bar{F}FVSS}, \quad H_{SSSSV}, \quad H_{SSSVV}. \quad (2.10)$$

The remaining 60 functions do involve ϵ -scalar contributions in at least one contributing diagram, and are therefore different in the $\overline{\text{DR}}'$ and $\overline{\text{MS}}$ schemes:

$$\begin{aligned} & H_{VVSSSS}, \quad H_{SSVVSS}, \quad H_{VVVSSS}, \quad H_{VVSSVS}, \quad H_{SSVVVV}, \quad H_{SVVVSV}, \\ & K_{SSSSSV}, \quad K_{SSSSVV}, \quad K_{SSSVVS}, \quad K_{VVSSSS}, \quad K_{SSSVVV}, \quad K_{VVSSVS}, \\ & K_{SSVVVV}, \quad K_{VVSVVS}, \quad J_{SSVSS}, \quad J_{SSVVS}, \quad G_{SVVVS}, \quad H_{\text{gauge},S}, \\ & K_{\text{gauge},S}, \quad K_{\text{gauge},SS}, \quad H_{FFVVFF}, \quad H_{FFVV\bar{F}\bar{F}}, \quad H_{F\bar{F}VV\bar{F}\bar{F}}, \quad H_{\bar{F}\bar{F}VV\bar{F}\bar{F}}, \\ & H_{FFFVVV}, \quad H_{F\bar{F}\bar{F}VVV}, \quad K_{FFFVVF}, \quad K_{FF\bar{F}VV\bar{F}}, \quad K_{\bar{F}\bar{F}FVVF}, \quad K_{\bar{F}\bar{F}\bar{F}VVF}, \\ & K_{\bar{F}\bar{F}\bar{F}VV\bar{F}}, \quad K_{VVFFFF}, \quad K_{VVF\bar{F}\bar{F}\bar{F}}, \quad K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}, \quad K_{\text{gauge},FF}, \quad K_{\text{gauge},\bar{F}\bar{F}}, \\ & H_{FFSVFF}, \quad H_{FFSV\bar{F}\bar{F}}, \quad H_{F\bar{F}SV\bar{F}\bar{F}}, \quad H_{F\bar{F}SVFF}, \quad H_{\bar{F}\bar{F}SV\bar{F}\bar{F}}, \quad H_{\bar{F}\bar{F}FSV\bar{V}}, \\ & H_{FF\bar{F}SVV}, \quad H_{\bar{F}\bar{F}\bar{F}SVV}, \quad K_{FFFSVF}, \quad K_{FF\bar{F}SV\bar{F}}, \quad K_{\bar{F}\bar{F}FSVF}, \quad K_{\bar{F}\bar{F}\bar{F}SVF}, \\ & K_{\bar{F}\bar{F}FSV\bar{F}}, \quad K_{\bar{F}\bar{F}\bar{F}SV\bar{F}}, \quad K_{SSSVFF}, \quad K_{SSSV\bar{F}\bar{F}}, \quad K_{SSVVFF}, \quad K_{SSVV\bar{F}\bar{F}}, \\ & K_{VVSSFF}, \quad K_{VVSS\bar{F}\bar{F}}, \quad K_{VVS\bar{F}\bar{F}}, \quad K_{VVS\bar{F}\bar{F}}, \quad H_{\text{gauge}}, \quad K_{\text{gauge}}. \end{aligned} \quad (2.11)$$

The main letters E, G, H, J, K, L correspond to the parent Feynman diagram topology, and the subscripts encode the information about the types of propagators in a canonical ordering, as shown in Fig. 1, and explained in detail in Ref. [7]. The distinction between F and \bar{F} is that the latter contains a chirality-flipping fermion mass insertion. Note that in many cases involving gauge boson interactions, more than one Feynman diagram contributes to a given function with a fixed structure of gauge invariants. For some of these, the word “gauge” in a subscript indicates combinations of diagram topologies involving multiple gauge vector boson or ghost propagators with a common group theoretic structure. The difference in the present paper is that for each vector propagator, one also includes the corresponding ϵ -scalar contribution in the $\overline{\text{DR}}'$ function. The results of evaluating all 89 functions appearing in $V^{(3)}$ are given in the `functionsDRED.anc` file of the Supplemental Material [34]. These constitute the main

new results of this paper. For any given softly broken supersymmetric gauge theory, one can plug in the results for the renormalized field-dependent masses and couplings, as specified above, into Eqs. (3.2)–(3.32) of Ref. [7] to evaluate the three-loop effective potential.

At three-loop order, there is a qualitatively new practical problem not encountered at one-loop and two-loop orders; the presence of doubled bosonic propagator lines carrying the same momentum (the propagators labeled 1 and 2 in topologies J, K , and L of Fig. 1) with small or vanishing squared masses can give rise to possible logarithmic infrared singularities. In the case of doubled Goldstone boson propagators, this problem was noted in the context of the three-loop effective potential in Ref. [6]. Besides causing infrared divergence problems when the renormalization-scale choice leads to small Goldstone boson squared masses, it manifests as imaginary parts of the effective potential at the minima of its real part when the

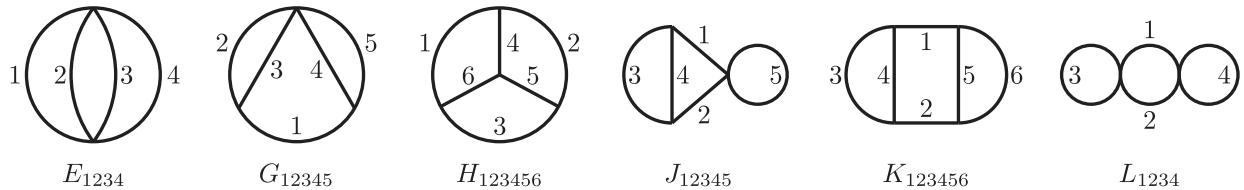


FIG. 1. Feynman diagram topologies that contribute to the effective potential at three-loop order. The numbers indicate the canonical ordering of subscripts denoting propagator types (S, F, \bar{F}, V), and the corresponding squared mass arguments.

tree-level Goldstone boson squared masses are negative. These imaginary parts are spurious, in the sense that they are not associated with any actual instability of the true vacuum state. This can occur for perfectly reasonable choices of the renormalization scale, including (see Ref. [6]) in the case of the actual parameters of the Standard Model. In general, this problem can be systematically defeated by resummation, as shown in Refs. [35,36], with further elucidations in Refs. [7,37–41]. Doubled propagators of massless gauge bosons can also [7] cause infrared divergences in the effective potential; these are benign, in the sense that they are also eliminated in the process of resumming the Goldstone boson contributions. It was also demonstrated in Ref. [7] that infrared divergences from doubled massless fermion lines do not occur at three-loop order.

Even in cases without infrared divergences due to doubled bosonic propagators, the presence of vanishing squared mass arguments can cause practical problems, because the three-loop integral functions for generic squared mass arguments will often contain individual terms with denominators with powers of the arguments when written in terms of the basis integrals. Although these do not give true infrared singularities, demonstrating this and obtaining expressions suitable for numerical evaluation often requires taking limits of basis integral functions that may not be immediately obvious.

To deal efficiently and systematically with these issues in particular cases, it is useful to have expansions of the basis integral functions for small squared mass arguments δ . A complete list of such expansions for every combination of squared mass arguments satisfying

$$0 < \delta \ll u, v, w, x, y, z \quad (2.12)$$

is provided in the `expdelta.anc` file of the Supplemental Material [34]. (Reference [7] provided an ancillary file `expzero.anc` with a subset of these expansions, which was less complete but sufficient for the special cases needed there, namely, those encountered in the Standard Model.) These expansions are derived using the differential equations that the basis integrals satisfy (obtained in Ref. [26]), and are given to order δ^5 for the I , F , and \bar{F} functions, order δ^4 for the \bar{I} and G functions, and order δ^3 for K and H . At three-loop order, the expansions can contain up to three powers of $\ln(\delta)$. Whenever a squared mass argument vanishes, or should be treated as small, one can replace it with δ , and then use these expansions to evaluate the leading order contribution to the effective potential as $\delta \rightarrow 0$. Poles in δ always cancel, and possible infrared divergences in individual diagrams then manifest themselves as residual powers of $\ln(\delta)$, which must also cancel from the minimization conditions for the effective potential, and from associated physically meaningful quantities. This provides a useful check in examples, including the ones mentioned below.

III. CHECKS FROM UNBROKEN SUPERSYMMETRY

Consider a supersymmetric theory, with no explicit supersymmetry-breaking terms. (For reviews of supersymmetry using notations and conventions consistent with the following, see Refs. [42,43].) As shown by Zumino in Ref. [44], at a supersymmetric minimum of the tree-level potential, the full effective potential must vanish at each order in perturbation theory. In the case of nongauge theories like the Wess-Zumino model, this was used in Ref. [7] as a check on the 24 contributions at three-loop order in Eq. (2.9) above, which do not involve vector bosons. I have now extended these checks to various supersymmetric gauge theory special cases.

For an example that is simple enough to analyze explicitly in text, consider a supersymmetric $U(1)$ gauge theory with gauge coupling g and two chiral superfields Φ_+ and Φ_- with charges +1 and -1, respectively, and a superpotential mass term

$$W = \mu \Phi_+ \Phi_- \quad (3.1)$$

The corresponding complex scalar fields can be written in terms of canonically normalized real components as

$$\phi_+ = \frac{1}{\sqrt{2}}(R_1 + iR_2), \quad \phi_- = \frac{1}{\sqrt{2}}(R_3 + iR_4). \quad (3.2)$$

There are three Weyl fermions, $\psi_1 = \psi_+$, $\psi_2 = \psi_-$, and $\psi_3 = \lambda$, the latter being the gaugino field. Finally, there is a single vector boson A^μ . In order to maintain unbroken supersymmetry, the background scalar field components for Φ_+ and Φ_- have been chosen to both vanish, leading to a tree-level potential $V^{(0)} = 0$. Then, the gaugino and gauge boson masses vanish, while the chiral fermion and scalar squared masses are all equal:

$$M_3^2 = m_A^2 = 0, \quad (3.3)$$

$$M_1^2 = M_2^2 = m_1^2 = m_2^2 = m_3^2 = m_4^2 = z, \quad (3.4)$$

with

$$M_{12} = M_{21} = \sqrt{z} \equiv \mu. \quad (3.5)$$

There are no scalar cubic interactions, and the nonvanishing quartic scalar interactions are

$$\lambda^{1111} = \lambda^{2222} = \lambda^{3333} = \lambda^{4444} = 3g^2, \quad (3.6)$$

$$\lambda^{1122} = \lambda^{3344} = g^2, \quad (3.7)$$

$$\lambda^{1133} = \lambda^{1144} = \lambda^{2233} = \lambda^{2244} = -g^2, \quad (3.8)$$

and permutations thereof, while the nonvanishing Yukawa couplings stemming from gaugino interactions with scalar and chiral fermion pairs are

$$Y^{113} = iY^{213} = -Y^{323} = -iY^{423} = g, \quad (3.9)$$

and equal values when the last two (fermion) indices are interchanged. The nonvanishing vector-scalar-scalar couplings are

$$g^{A21} = -g^{A12} = g^{A34} = -g^{A43} = g, \quad (3.10)$$

and the vector-fermion-fermion couplings are

$$g_1^{A1} = -g_2^{A2} = g. \quad (3.11)$$

There are no vector-vector-scalar interactions (because the gauge symmetry is not spontaneously broken) and no vector-vector-vector interactions (because the gauge symmetry is

Abelian). The one-loop part of the effective potential evaluated from Eq. (2.5) is

$$V^{(1)} = 4f(z) - 4\bar{f}(z) = 0, \quad (3.12)$$

where the two terms come from the scalar and fermion contributions to the supertrace, respectively, and there is no contribution from the massless vectors because $f(0) = 0$. The two-loop contribution from Eq. (2.8) is

$$\begin{aligned} V^{(2)} = & g^2[f_{SS}(z, z) + 4f_{FFS}(0, z, z) + f_{VSS}(0, z, z) \\ & + f_{FFV}(z, z, 0) - z\bar{f}_{\bar{F}\bar{F}V}(z, z, 0)]. \end{aligned} \quad (3.13)$$

This also vanishes due to nontrivial cancellations between these functions obtained by plugging in their expressions in terms of the basis integrals from the file `functionsDRED.anc` in the Supplemental Material [34]. Finally, the three-loop contribution obtained from the general form specified in Eqs. (3.2)–(3.32) of Ref. [7] is

$$\begin{aligned} V^{(3)} = & g^4 \left[L_{SSSS}(z, z, z, z) + \frac{3}{2}E_{SSSS}(z, z, z, z) + 4zH_{F\bar{F}SS\bar{F}}(0, z, z, z, 0, z) + 4J_{SSFFS}(z, z, 0, z, z) \right. \\ & + 4K_{FFFSS}(z, z, 0, z, z, 0) + 4zK_{\bar{F}\bar{F}FSS}(z, z, 0, z, z, 0) + 8K_{FFFSS}(0, 0, z, z, z, z) + 4K_{SSFFF}(z, z, 0, z, 0, z) \\ & - \frac{1}{2}H_{SSVVSS}(z, z, 0, 0, z, z) - 2J_{SSVSS}(z, z, 0, z, z) + K_{SSSVVS}(z, z, z, 0, 0, z) + K_{VVSSSS}(0, 0, z, z, z, z) \\ & + \frac{1}{2}z^2H_{\bar{F}\bar{F}VV\bar{F}\bar{F}}(z, z, 0, 0, z, z) + zH_{F\bar{F}VV\bar{F}\bar{F}}(z, z, 0, 0, z, z) - 2zH_{FFVV\bar{F}\bar{F}}(z, z, 0, 0, z, z) \\ & + \frac{1}{2}H_{FFVVFF}(z, z, 0, 0, z, z) + z^2K_{\bar{F}\bar{F}FVV\bar{F}}(z, z, z, 0, 0, z) + zK_{\bar{F}\bar{F}FVV\bar{F}}(z, z, z, 0, 0, z) \\ & - 4zK_{\bar{F}\bar{F}FVV\bar{F}}(z, z, z, 0, 0, z) + zK_{FF\bar{F}VV\bar{F}}(z, z, z, 0, 0, z) + K_{FFFVVF}(z, z, z, 0, 0, z) \\ & + z^2K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}(0, 0, z, z, z, z) - 2zK_{VVFFF\bar{F}}(0, 0, z, z, z, z) + K_{VVFFFF}(0, 0, z, z, z, z) \\ & + 4zH_{\bar{F}\bar{F}FVSS}(z, z, 0, 0, z, z) + 4H_{FFFVSS}(z, z, 0, 0, z, z) + 4zK_{\bar{F}\bar{F}FSVF}(z, z, 0, z, 0, z) \\ & - 8zK_{\bar{F}\bar{F}FSV\bar{F}}(z, z, 0, z, 0, z) + 4K_{FFFVF}(z, z, 0, z, 0, z) - 4K_{SSSVFF}(z, z, z, 0, 0, z) \\ & \left. + 2zK_{VVSS\bar{F}\bar{F}}(0, 0, z, z, z, z) - 2K_{VVSSFF}(0, 0, z, z, z, z) \right]. \end{aligned} \quad (3.14)$$

As noted at the end of the previous section, to evaluate this properly one may first change the 0 arguments to δ , then after using the results in `functionsDRED.anc`, apply the expansions in `expdelta.anc` to keep only nonvanishing terms as $\delta \rightarrow 0$. Most of the functions in Eq. (3.14) are individually completely smooth in the limit $\delta \rightarrow 0$. The exceptions, which have only simple logarithmic singularities, are

$$\begin{aligned} K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}(\delta, \delta, z, z, z, z) = & 12\bar{\ln}(\delta)[z + A(z)]^2/z^2 \\ & + \dots, \end{aligned} \quad (3.15)$$

$$K_{VVFFF}(\delta, \delta, z, z, z, z) = 12\bar{\ln}(\delta)[z + A(z)]^2 + \dots, \quad (3.16)$$

$$K_{VVFFFF}(\delta, \delta, z, z, z, z) = 12\bar{\ln}(\delta)[z + A(z)]^2 + \dots, \quad (3.17)$$

where the ellipses represent terms that are finite as $\delta \rightarrow 0$. Since these functions appear in Eq. (3.14) with coefficients proportional to z^2 , $-2z$, and 1, respectively, the $\bar{\ln}(\delta)$ terms are seen to successfully cancel in the complete expression. In fact, the whole expression for $V^{(3)}$ vanishes in the limit $\delta \rightarrow 0$, as required, due to nontrivial cancellations between

the various functions. This becomes apparent after expressing the results in terms of the renormalized three-loop basis integrals.

The simple example above does not come close to completely testing the results obtained in `functionsDRED.anc`, because of the absence of scalar-scalar-scalar, vector-vector-scalar, and vector-vector-vector interactions, and the absence of superpotential Yukawa couplings. I have carried out more detailed tests, each including many more terms, as follows:

- (i) Supersymmetric $U(1)$ gauge theory with three chiral superfields Φ , $\bar{\Phi}$, and Φ_0 , with charges $+1$, -1 , and 0 , respectively, and a superpotential $W = y\Phi_0\Phi\bar{\Phi} + \mu\Phi\bar{\Phi} + \frac{1}{2}\mu_0\Phi_0^2$. Supersymmetry is unbroken when the scalar background fields are taken to vanish.
- (ii) Supersymmetric $U(1)$ gauge theory with two chiral superfields Φ and $\bar{\Phi}$, with charges $+1$, -1 . There is no superpotential. The gauge symmetry is spontaneously broken by equal magnitude background fields for the scalars, $\bar{\varphi} = \varphi$. This is a D -flat direction, leaving supersymmetry unbroken.
- (iii) Supersymmetric $SU(n)$ gauge theory with $n = 2, 3$, with chiral superfields Φ_j and $\bar{\Phi}^j$ in the fundamental and antifundamental representations with $j = 1, \dots, n$, and a singlet chiral superfield Φ_0 . The superpotential is $W = y\Phi_0\Phi\bar{\Phi} + \mu\Phi\bar{\Phi} + \frac{1}{2}\mu_0\Phi_0^2$. The background scalar fields are taken to vanish, leaving the gauge symmetry unbroken and maintaining unbroken supersymmetry.
- (iv) Supersymmetric $SU(n)$ gauge theory with $n = 2, 3$, with chiral superfields Φ_j and $\bar{\Phi}^j$ in the fundamental and antifundamental representations, with no superpotential. The scalar fields obtain background values with equal magnitudes along a D -flat direction $\Phi_j = \bar{\Phi}^j = \varphi\delta_{j1}$, breaking the gauge symmetry but again maintaining unbroken supersymmetry.

In each of these cases, I have checked that $V^{(1)} = V^{(2)} = V^{(3)} = 0$ as required by unbroken supersymmetry at tree level. These are highly nontrivial consistency checks on the results obtained in `functionsDRED.anc`, relying on intricate cancellations between the individual contributions after writing them in terms of the renormalized basis integrals. (As one might expect, the cancellations of the individual contributions would not occur if one used the $\overline{\text{MS}}$ functions instead of the correct $\overline{\text{DR'}}$ ones.) These cancellations include terms proportional to the infrared regulator $\overline{\text{In}}(\delta)$ in the three-loop part, corresponding to massless vectors and massless scalars along flat directions.

IV. CHECKS FROM RENORMALIZATION GROUP INVARIANCE

Another class of checks, applicable for cases of non-supersymmetric vacua and softly broken supersymmetric

gauge theories, comes from renormalization group invariance. The invariance of the effective potential with respect to changes in the arbitrary renormalization scale Q can be expressed as

$$0 = Q \frac{dV_{\text{eff}}}{dQ} = \left(Q \frac{\partial}{\partial Q} + \sum_X \beta_X \frac{\partial}{\partial X} \right) V_{\text{eff}}, \quad (4.1)$$

where X runs over all of the independent $\overline{\text{DR'}}$ parameters of the theory, including the background scalar field(s) φ , the masses and couplings that may depend on the φ , and a field-independent contribution to the tree-level potential, which I will denote below by Λ . The beta functions for the parameters X are given in a loop expansion by

$$\beta_X = \frac{1}{16\pi^2} \beta_X^{(1)} + \frac{1}{(16\pi^2)^2} \beta_X^{(2)} + \frac{1}{(16\pi^2)^3} \beta_X^{(3)} + \dots, \quad (4.2)$$

and in the particular case of the background scalar fields, one writes $\beta_\varphi = -\gamma^S \varphi$, where γ^S is the scalar anomalous dimension, not to be confused with the chiral superfield anomalous dimension. Therefore, at each loop order ℓ , consistency requires

$$Q \frac{\partial}{\partial Q} V^{(\ell)} + \sum_{n=0}^{\ell-1} \left(\sum_X \beta_X^{(\ell-n)} \frac{\partial}{\partial X} V^{(n)} \right) = 0. \quad (4.3)$$

To evaluate the first term in Eq. (4.3), the derivatives with respect to Q of the basis integrals, and of the nine two-loop functions and the 89 three-loop functions, are given for convenience in the file `QdQDRED.anc` of the Supplemental Material [34]. Since most of the β_X functions are known from previous work, evaluating Eq. (4.3) for each ℓ in particular cases in principle gives nontrivial checks on the results of the present paper in the file `functionsDRED.anc`. However, there are two missing pieces of information. First, although the two-loop and three-loop contributions to the anomalous dimensions of the chiral superfields were calculated in Refs. [45,46], the anomalous dimensions of the scalar components are different and were only previously known to two-loop order. Second, the beta function of the field-independent vacuum energy Λ was only previously known at two-loop order. Therefore, by demanding that Eq. (4.3) holds for $\ell = 1, 2, 3$ in a variety of cases, I have been able to derive and then check these missing results. I will first provide these results and then briefly review the list of special case models used to infer and check them.

Consider a supersymmetric gauge theory with chiral superfields Φ_i and a superpotential

$$W = \frac{1}{6} y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j \quad (4.4)$$

involving Yukawa couplings y^{ijk} and supersymmetric masses μ^{ij} , and soft supersymmetry-breaking terms

$$-\mathcal{L}_{\text{soft}} = \left(\frac{1}{6} a^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} M_a \lambda^a \lambda^a \right) + \text{c.c.} + (m^2)_j^i \phi_i \phi^{*j} + \Lambda, \quad (4.5)$$

where ϕ_i are the scalar components of Φ_i , and λ^a are the gaugino fields. Here, a^{ijk} and b^{ij} are holomorphic scalar cubic and scalar squared mass terms, respectively, M_a are the gaugino masses, and $(m^2)_j^i$ are the nonholomorphic scalar squared masses. The last term, the field-independent vacuum energy Λ , is irrelevant to the (nongravitational) dynamics of the theory and therefore generally omitted, but its presence is necessary to maintain renormalization-scale invariance of V_{eff} . Note that in all checks below, I have assumed that there are no tadpole couplings in W or $-\mathcal{L}_{\text{soft}}$ associated with gauge-singlet chiral superfields. I also assume that there is at most one $U(1)$ component in the gauge group to avoid the complication of kinetic mixing between different Abelian gauge fields. Both of these assumptions hold in the case of the minimal supersymmetric Standard Model (MSSM).

The gauge group is assumed to have couplings g_a with generators $(\mathbf{t}^a)_i^j$. The notation for group theory invariants will closely follow that of the review in Chap. 11 of Ref. [43]. For each distinct group component, the dimension (number of Lie algebra generators) and the quadratic Casimir invariant are denoted d_a and G_a , respectively. The quadratic Casimir invariant of an irreducible representation carrying a flavor index i is denoted $C_a(i)$, where

$$(\mathbf{t}^a \mathbf{t}^a)_i^j = C_a(i) \delta_i^j. \quad (4.6)$$

For an irreducible representation r , the Dynkin index is $T_a(r)$ defined by

$$\text{Tr}_r[\mathbf{t}^a \mathbf{t}^b] = \delta_{ab} T_a(r), \quad (4.7)$$

and the sum of the $T_a(r)$ over all of the chiral supermultiplet representations is

$$S_a = \sum_r T_a(r). \quad (4.8)$$

Similarly, define

$$S_{ab} = \sum_r T_a(r) C_b(r), \quad (4.9)$$

$$S_{abc} = \sum_r T_a(r) C_b(r) C_c(r). \quad (4.10)$$

For example, for a supersymmetric $U(1)$ gauge theory with chiral superfields Φ_i with charges q_i , one has $d_a = 1$, $G_a = 0$, $C_a(i) = q_i^2$, $S_a = \sum_i q_i^2$, $S_{aa} = \sum_i q_i^4$, and $S_{aaa} = \sum_i q_i^6$. For a supersymmetric $SU(n_c)$ gauge theory with n_f flavors of fundamental and antifundamental chiral superfields, one has $d_a = n_c^2 - 1$, $G_a = n_c$, $C_a(i) = (n_c^2 - 1)/(2n_c) \equiv C_f$ for each i , and $S_a = n_f$, $S_{aa} = n_f C_f$, and $S_{aaa} = n_f C_f^2$.

The DRED beta functions for the gauge couplings g_a were found at two-loop order in [47,48], and at three-loop order in [49] by making use of results in [50,51]. Using the notations above, they are

$$\beta_{g_a}^{(1)} = g_a^3 (S_a - 3G_a), \quad (4.11)$$

$$\beta_{g_a}^{(2)} = 2g_a^5 G_a (S_a - 3G_a) + 4g_a^3 g_b^2 S_{ab} - g_a^3 y^{ijk} y_{ijk} C_a(i)/d_a, \quad (4.12)$$

$$\begin{aligned} \beta_{g_a}^{(3)} = & g_a^7 G_a (S_a - 3G_a) (7G_a - S_a) + 8g_a^5 g_b^2 G_a S_{ab} + 6g_a^3 g_b^4 S_{ab} (3G_b - S_b) - 8g_a^3 g_b^2 g_c^2 S_{abc} - 2g_a^5 y^{ijk} y_{ijk} C_a(i) G_a/d_a \\ & + g_a^3 g_b^2 y^{ijk} y_{ijk} [C_b(i) - 6C_b(j)] C_a(i)/d_a + g_a^3 y^{ijk} y_{ijl} y_{kmn} y^{lmn} [3C_a(i)/2 + C_a(k)/4]/d_a. \end{aligned} \quad (4.13)$$

The anomalous dimension of the chiral superfield Φ_i and the anomalous dimension of its scalar component ϕ_i have the same general form

$$\gamma_i^{(1)j} = \frac{1}{2} y_{ikl} y^{jkl} + n_1 \delta_i^j g_a^2 C_a(i), \quad (4.14)$$

$$\gamma_i^{(2)j} = -\frac{1}{2} y_{ikl} y^{jkm} y^{lnp} y_{mnp} + g_a^2 y_{ikl} y^{jkl} [n_2 C_a(k) + n_3 C_a(i)] + \delta_i^j g_a^2 C_a(i) [n_4 g_a^2 S_a + n_5 g_b^2 C_b(i) + n_6 g_a^2 G_a], \quad (4.15)$$

$$\begin{aligned}
\gamma_i^{(3)j} = & -\frac{1}{8}y_{ikl}y^{jpq}y^{kmn}y_{pmn}y^{lrs}y_{qrs} - \frac{1}{4}y_{ikl}y^{jkm}y^{lnp}y_{snp}y^{sqr}y_{mqr} + y_{ikl}y^{jkm}y^{lnp}y_{mnq}y^{qrs}y_{prs} + \frac{3}{2}\zeta_3 y_{ikl}y^{jpq}y^{kmn}y^{lrs}y_{pmr}y_{qns} \\
& + g_a^2 y_{ikl}y^{jkm}y^{lnp}y_{mnp}[n_7 C_a(i) + n_8 C_a(k) + n_9 C_a(l) + n_{10} C_a(p)] \\
& + g_a^2 g_b^2 y_{ikl}y^{jkl}[n_{11} C_a(i)C_b(i) + n_{12} C_a(i)C_b(k) + n_{13} C_a(k)C_b(k) + n_{14} C_a(k)C_b(l)] \\
& + g_a^4 y_{ikl}y^{jkl}[n_{15} S_a C_a(i) + n_{16} S_a C_a(k) + n_{17} G_a C_a(i) + n_{18} G_a C_a(k)] \\
& + \delta_i^j g_a^2 C_a(i) \{g_a^4 [n_{19} S_a^2 + n_{20} S_a G_a + n_{21} G_a^2] + g_a^2 g_b^2 [n_{22} S_a C_b(i) + n_{23} G_a C_b(i) + n_{24} S_{ab}] \\
& + n_{25} g_b^2 g_c^2 C_b(i)C_c(i) + n_{26} g_a^2 y^{klm}y_{klm}C_a(k)/d_a\}. \tag{4.16}
\end{aligned}$$

However, as is well known, some of the coefficients n_1, \dots, n_{26} of the gauge-coupling-dependent terms differ for the chiral superfield and its scalar component. Indeed, the coefficients for the scalar component are dependent on the choice of gauge fixing, while the coefficients for the chiral superfield are not. For the chiral superfield anomalous dimension, the results are [45,46]

$$\begin{aligned}
n_1 &= -2, & n_2 &= 2, & n_3 &= -1, & n_4 &= 2, & n_5 &= 4, & n_6 &= -6, & n_7 &= 2 + 3\zeta_3, & n_8 &= 1 - 3\zeta_3, \\
n_9 &= 4 - 6\zeta_3, & n_{10} &= -4 + 6\zeta_3, & n_{11} &= 4 - 15\zeta_3, & n_{12} &= -8 + 12\zeta_3, & n_{13} &= -12 - 6\zeta_3, \\
n_{14} &= -2 + 18\zeta_3, & n_{15} &= -1, & n_{16} &= -4, & n_{17} &= 3 - 3\zeta_3, & n_{18} &= 12 - 6\zeta_3, & n_{19} &= 2, \\
n_{20} &= -2 + 24\zeta_3, & n_{21} &= -12, & n_{22} &= -4, & n_{23} &= 12, & n_{24} &= 20 - 24\zeta_3, & n_{25} &= -16, & n_{26} &= -5. \tag{4.17}
\end{aligned}$$

For the scalar component, I find that the Landau gauge coefficients are instead,

$$\begin{aligned}
n_1 &= -1, & n_2 &= 2, & n_3 &= -1, & n_4 &= 1, & n_5 &= 2, & n_6 &= -9/4, & n_7 &= 2 + 3\zeta_3, & n_8 &= 1 - 3\zeta_3, \\
n_9 &= 4 - 6\zeta_3, & n_{10} &= -4 + 6\zeta_3, & n_{11} &= 4 - 12\zeta_3, & n_{12} &= -8 + 12\zeta_3, & n_{13} &= -12, & n_{14} &= -2 + 12\zeta_3, \\
n_{15} &= -1, & n_{16} &= -4, & n_{17} &= 7/2 - 9\zeta_3/2, & n_{18} &= 12 - 6\zeta_3, & n_{19} &= 1, & n_{20} &= -9/4 + 12\zeta_3, \\
n_{21} &= -13/16 - 63\zeta_3/8, & n_{22} &= -2, & n_{23} &= -1 + 15\zeta_3, & n_{24} &= 10 - 12\zeta_3, & n_{25} &= -12, & n_{26} &= -5/2. \tag{4.18}
\end{aligned}$$

The first six of these are not new, having been obtained in Ref. [5] from the \overline{DR}' two-loop effective potential.

The different roles played by the chiral superfield anomalous dimension and the scalar component field anomalous dimension are as follows. The former enters into the beta functions for superpotential parameters according to

$$\beta_{y^{ijk}} = \gamma_n^i y^{njk} + (i \leftrightarrow j) + (i \leftrightarrow k), \tag{4.19}$$

$$\beta_{\mu^{ij}} = \gamma_n^i \mu^{nji} + (i \leftrightarrow j), \tag{4.20}$$

valid at all orders in perturbation theory. The scalar component field anomalous dimension γ^S is instead related to the beta function of the background scalar fields φ_i according to

$$\beta_{\varphi_i} = -(\gamma^S)_i^j \varphi_j, \tag{4.21}$$

for use with $X = \varphi_i$ in Eq. (4.3).

For the sake of completeness, I also review the beta functions for the soft supersymmetry-breaking parameters as needed below, again following closely the notation of the

review in Chap. 11 of Ref. [43]. This can be done most efficiently in terms of differential operators in coupling-constant space that act on the chiral superfield anomalous dimensions:

$$\Omega = \frac{1}{2} M_a g_a \frac{\partial}{\partial g_a} - a^{ijk} \frac{\partial}{\partial y^{ijk}}, \tag{4.22}$$

$$\Omega^* = \frac{1}{2} M_a^* g_a \frac{\partial}{\partial g_a} - a_{ijk} \frac{\partial}{\partial y_{ijk}}. \tag{4.23}$$

The beta functions for M_a , a^{ijk} , and b^{ij} were found at two-loop order results in Refs. [25,52–54] and extended by Refs. [55,56] to all orders in perturbation theory,

$$\beta_{M_a} = 2\Omega(\beta_{g_a}/g_a), \tag{4.24}$$

$$\beta_{a^{ijk}} = [\gamma_n^i a^{njk} - 2y^{njk} \Omega(\gamma_n^i)] + (i \leftrightarrow j) + (i \leftrightarrow k), \tag{4.25}$$

$$\beta_{b^{ij}} = [\gamma_n^i b^{nji} - 2\mu^{nji} \Omega(\gamma_n^i)] + (i \leftrightarrow j) \tag{4.26}$$

using spurion methods as proposed in Ref. [54]. However, in cases with gauge-singlet chiral superfields, $\beta_{b^{ij}}$ contains extra terms not captured by the above. The results were given in Refs. [25,53,54] at two-loop order:

$$\beta_{b^{ij}}^{(1)} = \frac{1}{2} b^{il} y_{lmn} y^{mnj} + \frac{1}{2} y^{ijl} y_{lmn} b^{mn} + \mu^{il} y_{lmn} a^{mnj} - 2(b^{ij} - 2M_a \mu^{ij}) g_a^2 C_a(i) + (i \leftrightarrow j), \quad (4.27)$$

$$\begin{aligned} \beta_{b^{ij}}^{(2)} = & -\frac{1}{2} b^{il} y_{lmn} y^{pqn} y_{pqr} y^{mrj} - \frac{1}{2} y^{ijl} y_{lmn} b^{mr} y_{pqr} y^{pqn} - \frac{1}{2} y^{ijl} y_{lmn} \mu^{mr} y_{pqr} a^{pqn} \\ & - \mu^{il} y_{lmn} a^{npq} y_{pqr} y^{mrj} - \mu^{il} y_{lmn} y^{npq} y_{pqr} a^{mrj} + 2y^{ijl} y_{lpq} (b^{pq} - \mu^{pq} M_a) g_a^2 C_a(p) \\ & + (b^{il} y_{lpq} y^{pjq} + 2\mu^{il} y_{lpq} a^{pjq} - 2\mu^{il} y_{lpq} y^{pjq} M_a) g_a^2 [2C_a(p) - C_a(i)] \\ & + (2b^{ij} - 8\mu^{ij} M_a) g_a^2 C_a(i) [g_a^2 S_a + 2g_b^2 C_b(i) - 3g_a^2 G_a] + (i \leftrightarrow j). \end{aligned} \quad (4.28)$$

This will be sufficient for the examples considered below. A way of finding $\beta_{b^{ij}}$ at arbitrary loop order in terms of the chiral superfield anomalous dimension is given in Ref. [57]. For the nonholomorphic soft squared masses $(m^2)_i^j$, the result is [58]

$$\begin{aligned} \beta_{(m^2)_i^j} = & g_a^2 (\mathbf{t}^a)_i^j A_a + \left\{ 2\Omega\Omega^* + (|M_a|^2 + X_a) g_a \frac{\partial}{\partial g_a} + [(m^2)_k^n y^{kpq} + (m^2)_k^p y^{nkp} + (m^2)_k^q y^{npk}] \frac{\partial}{\partial y^{npq}} \right. \\ & \left. + [(m^2)_n^k y_{kpq} + (m^2)_p^k y_{nkq} + (m^2)_q^k y_{npk}] \frac{\partial}{\partial y_{npq}} \right\} \gamma_i^j, \end{aligned} \quad (4.29)$$

with

$$X_a = \frac{1}{16\pi^2} X_a^{(1)} + \frac{1}{(16\pi^2)^2} X_a^{(2)} + \dots, \quad (4.30)$$

$$A_a = \frac{1}{16\pi^2} A_a^{(1)} + \frac{1}{(16\pi^2)^2} A_a^{(2)} + \frac{1}{(16\pi^2)^3} A_a^{(3)} + \dots, \quad (4.31)$$

where the results needed for three-loop order $\beta_{(m^2)_i^j}$ are [58,59]

$$X_a^{(1)} = 2g_a^2 [G_a |M_a|^2 - (m^2)_k^k C_a(k) / d_a], \quad (4.32)$$

$$\begin{aligned} X_a^{(2)} = & g_a^4 (10G_a - 2S_a) G_a |M_a|^2 - 4g_a^4 G_a (m^2)_k^k C_a(k) / d_a - 4g_a^2 g_b^2 S_{ab} |M_b|^2 + g_a^2 y^{kpq} y_{npq} (m^2)_k^n \left[C_a(p) + \frac{1}{2} C_a(k) \right] / d_a \\ & + g_a^2 a^{kpq} a_{kpq} C_a(k) / 2d_a, \end{aligned} \quad (4.33)$$

and the special contributions from Abelian group factors are [60]

$$A_a^{(1)} = 2(\mathbf{t}^a)_k^l (m^2)_l^k, \quad (4.34)$$

$$A_a^{(2)} = (\mathbf{t}^a)_k^l [8g_b^2 C_b(k) (m^2)_l^k - 2(m^2)_n^k y^{npq} y_{lpq}], \quad (4.35)$$

$$\begin{aligned} A_a^{(3)} = & (\mathbf{t}^a)_k^l \left\{ 3(m^2)_l^n y^{kpq} y_{npr} y^{rst} y_{qst} - \frac{3}{2} (m^2)_l^n y^{kpq} y_{pqr} y^{rst} y_{nst} - 4y^{kn} y_{pqr} y^{rst} y_{lst} (m^2)_n^q - 2a^{kn} a_{npq} y_{lrs} y^{qrs} \right. \\ & - \frac{5}{2} a^{kn} a_{lrs} y^{qrs} y_{npq} + 16g_b^2 C_b(k) y^{kn} y_{lnp} |M_b|^2 + (8 - 24\zeta_3) g_b^2 C_b(k) a^{kn} a_{lnp} \\ & + (12\zeta_3 - 10) g_b^2 C_b(k) [a^{kn} y_{lnp} M_b^* + y^{kn} a_{lnp} M_b] + g_b^2 (m^2)_l^n y^{kpq} y_{npq} [(10 - 24\zeta_3) C_b(k) - 12C_b(p)] \\ & + (16 - 48\zeta_3) g_b^2 C_b(k) y^{kn} y_{lrs} (m^2)_n^r - 16g_b^2 g_c^2 C_b(k) C_c(k) (m^2)_l^k + g_b^2 g_c^2 C_b(k) C_c(k) [(96\zeta_3 - 64) |M_b|^2 \\ & \left. + (48\zeta_3 - 40) M_b M_c^* \delta_l^k + 12g_b^4 C_b(k) (3G_b - S_b) (m^2)_l^k \right\}. \end{aligned} \quad (4.36)$$

Finally, renormalization group invariance of the effective potential requires nontrivial running of the field-independent vacuum energy $X = \Lambda$ in Eq. (4.3). The one-loop and two-loop contributions were found in Ref. [5] from the two-loop $\overline{\text{DR}}'$ effective potential:

$$\beta_{\Lambda}^{(1)} = (m^2)_i^j (m^2)_j^i + 2(m^2)_i^j \mu^{ik} \mu_{jk} + b^{ij} b_{ij} - d_a |M_a|^4, \quad (4.37)$$

$$\begin{aligned} \beta_{\Lambda}^{(2)} = & -y^{ijk} y_{ijl} [(m^2)_k^m (m^2)_m^l + (m^2)_k^m \mu_{mn} \mu^{ln} + \mu_{kn} \mu^{mn} (m^2)_m^l + \mu_{km} (m^2)_n^m \mu^{nl} + b^{ml} b_{km}] \\ & - a^{ijk} a_{ijl} [(m^2)_k^l + \mu_{km} \mu^{lm}] - 2y^{ijk} y_{ilm} (m^2)_j^l \mu_{kn} \mu^{mn} - y^{ijk} a_{ijl} \mu_{km} b^{ml} - y_{ijk} a^{ijl} \mu^{km} b_{ml} \\ & + 4g_a^2 C_a(i) [(m^2)_i^j (m^2)_j^i + b^{ij} b_{ij} - M_a \mu^{ij} b_{ij} - M_a^* \mu_{ij} b^{ij} + 2(m^2)_i^j \mu^{ik} \mu_{jk} + 2\mu^{ij} \mu_{ij} |M_a|^2] \\ & + g_a^2 d_a (4S_a - 8G_a) |M_a|^4. \end{aligned} \quad (4.38)$$

From the special case examples described below, I was able to deduce the general three-loop result, which is divided into parts with 0, 2, and 4 powers of gauge couplings:

$$\beta_{\Lambda}^{(3)} = \beta_{\Lambda}^{(3a)} + \beta_{\Lambda}^{(3b)} + \beta_{\Lambda}^{(3c)}, \quad (4.39)$$

where

$$\begin{aligned} \beta_{\Lambda}^{(3a)} = & 2y^{ilm} y_{kln} y^{npq} y_{mpq} [(m^2)_i^j (m^2)_j^k + (m^2)_i^j \mu_{jr} \mu^{kr} + \mu_{ir} \mu^{jr} (m^2)_j^k + \mu_{ij} (m^2)_r^j \mu^{kr} + b_{ij} b^{jk}] \\ & + y^{ikm} y_{jln} y^{npq} y_{mpq} (m^2)_k^l \left[\frac{1}{2} (m^2)_i^j + 2\mu_{ir} \mu^{jr} \right] + y^{imn} y_{jmp} y^{kqp} y_{lqn} (m^2)_k^l [3(m^2)_i^j + 4\mu_{ir} \mu^{jr}] \\ & - \frac{1}{2} y^{ilm} y_{knp} y_{lmq} y^{npq} [(m^2)_i^j (m^2)_j^k + (m^2)_i^j \mu_{jr} \mu^{kr} + \mu_{ir} \mu^{jr} (m^2)_j^k + \mu_{ij} (m^2)_r^j \mu^{kr} + b_{ij} b^{jk}] \\ & - \frac{1}{4} y^{imn} y_{lmn} y_{jqp} y^{kqp} (m^2)_k^l [(m^2)_i^j + 2\mu_{ir} \mu^{jr}] - y^{ikl} y_{jkm} y^{nqr} y_{pqr} \mu_{ln} \mu^{mp} (m^2)_i^j \\ & - y^{ikm} y_{jkl} (m^2)_i^j [\mu_{mr} \mu^{nr} y_{npq} y^{lpq} + \mu_{nr} \mu^{lr} y_{mpq} y^{npq}] \\ & + 2y^{ikm} y_{jkl} \mu_{ir} \mu^{jr} [(m^2)_n^m y_{npq} y^{lpq} + (m^2)_n^l y_{mpq} y^{npq}] \\ & - \frac{1}{2} y^{ijk} y_{ijl} y^{mnp} y_{mnq} [\mu^{lq} \mu_{kr} (m^2)_p^r + \mu_{kp} \mu^{lr} (m^2)_r^q] \\ & + 12\zeta_3 y^{ijk} y^{lpn} y_{lmr} y_{knq} (m^2)_r^i \mu_{jp} \mu^{mq} + 6\zeta_3 y^{ijk} y^{lmn} y_{imp} y_{knq} [\mu_{jl} \mu^{qr} (m^2)_r^p + \mu^{qp} \mu_{jr} (m^2)_l^r] \\ & + 2[a^{ikl} a_{jkm} y^{mpq} y_{lpq} + y^{ikl} y_{jkm} a^{mpq} a_{lpq} + a^{ikl} y_{jkm} y^{mpq} a_{lpq} + y^{ikl} a_{jkm} a^{mpq} y_{lpq}] \mu_{ir} \mu^{jr} \\ & - \frac{1}{2} [a^{ijk} a_{ijl} y^{lmn} y_{pmn} + a^{ijk} y_{ijl} y^{lmn} a_{pmn} + y^{ijk} a_{ijl} a^{lmn} y_{pmn} + y^{ijk} y_{ijl} a^{lmn} a_{pmn}] \mu_{kr} \mu^{pr} \\ & - \frac{1}{2} [a^{ijk} a_{ijl} y^{mpq} y_{npq} + a^{ijk} y_{ijl} y^{mpq} a_{npq}] \mu^{ln} \mu_{km} - \frac{1}{4} y^{imn} y_{kmn} y^{jlpq} y_{lpq} b_{ij} b^{kl} \\ & + 2[a^{ikl} a_{jkm} y^{mpq} y_{lpq} + a^{ikl} y_{jkm} y^{mpq} a_{lpq} + y^{ikl} a_{jkm} a^{mpq} y_{lpq}] (m^2)_i^j \\ & - \frac{1}{2} [a^{ijk} a_{ijl} y^{lmn} y_{pmn} + a^{ijk} y_{ijl} y^{lmn} a_{pmn} + y^{ijk} y_{ijl} a^{lmn} a_{pmn}] (m^2)_k^p \\ & + [2y^{klm} a_{iln} y^{npq} y_{mpq} + 2y^{klm} y_{iln} y^{npq} a_{mpq} - \frac{1}{2} y_{imn} y^{kqp} y^{mn} a_{pql} - \frac{1}{2} a_{imn} y^{kqp} y^{mn} a_{pql}] b^{ij} \mu_{jk} \\ & + [2y_{klm} a^{iln} y_{npq} y^{mpq} + 2y_{klm} y^{iln} y_{npq} a^{mpq} - \frac{1}{2} y^{imn} y_{kpq} y_{mn} a^{pql} - \frac{1}{2} a^{imn} y_{kpq} y_{mn} a^{pql}] b_{ij} \mu^{jk} \\ & - \frac{1}{2} [y^{ijk} a_{ijl} b^{lm} \mu_{kn} y^{npq} y_{mpq} + y_{ijk} a^{ijl} b_{lm} \mu^{kn} y_{npq} y^{mpq}] - \frac{1}{4} a^{ijk} a_{ijl} a^{lmn} a_{kmn} \\ & + 3\zeta_3 [(y^{ijk} b^{lm} + 2a^{ijk} \mu^{lm}) (y_{iln} b_{jp} + 2a_{iln} \mu_{jp}) y_{kmq} y^{npq}], \end{aligned} \quad (4.40)$$

$$\begin{aligned}
\beta_{\Lambda}^{(3b)} = & 6g_a^2 C_a(i) a^{ijk} a_{ijk} |M_a|^2 + g_a^2 a^{ijk} a_{ijl} (m^2)_k^l [(12\zeta_3 - 8) C_a(i) + (4 - 18\zeta_3) C_a(k)] \\
& + g_a^2 (y^{ijk} a_{ijl} M_a + a^{ijk} y_{ijl} M_a^*) (m^2)_k^l [(8 - 12\zeta_3) C_a(i) + (6\zeta_3 - 6) C_a(k)] \\
& + 5g_a^2 C_a(i) (y^{ijk} a_{ijk} M_a + a^{ijk} y_{ijk} M_a^*) |M_a|^2 - 10g_a^2 y^{ijk} y_{ijk} C_a(i) |M_a|^4 \\
& + g_a^2 y^{ijk} y_{ijl} (m^2)_k^l |M_a|^2 [14C_a(k) - 4C_a(i)] \\
& + g_a^2 y^{ijk} y_{ijl} (m^2)_k^n (m^2)_n^l [(12\zeta_3 - 8) C_a(i) + (8 - 18\zeta_3) C_a(k)] \\
& + g_a^2 y^{ijk} y_{iln} (m^2)_j^n (m^2)_k^l [(12\zeta_3 - 2) C_a(i) + (12 - 24\zeta_3) C_a(j)] \\
& + g_a^2 [(12\zeta_3 - 8) C_a(i) + (10 - 18\zeta_3) C_a(k)] \{ (a^{ijk} - y^{ijk} M_a) (a_{ijl} - y_{ijl} M_a^*) \mu_{kn} \mu^{ln} \\
& + y^{ijk} y_{ijl} [(b^{nl} - \mu^{nl} M_a) (b_{kn} - \mu_{kn} M_a^*) + (m^2)_k^n \mu_{np} \mu^{lp} + \mu_{kp} \mu^{np} (m^2)_n^l + \mu_{kn} (m^2)_p^n \mu^{pl}] \\
& + [y^{ijk} y_{iln} (m^2)_j^n + y^{ijk} y_{jln} (m^2)_i^l] \mu_{kp} \mu^{np} + y^{ijk} a_{ijl} \mu_{kn} b^{ln} + a^{ijk} y_{ijl} b_{kn} \mu^{ln} \}, \tag{4.41}
\end{aligned}$$

$$\begin{aligned}
\beta_{\Lambda}^{(3c)} = & g_a^4 d_a [(4 - 24\zeta_3) G_a^2 + (96\zeta_3 - 26) G_a S_a + 10 S_a^2] |M_a|^4 \\
& + g_a^2 g_b^2 d_a S_{ab} |M_a|^2 [(20 - 24\zeta_3) (2|M_a|^2 + M_a M_b^* + M_a^* M_b) - 24|M_b|^2] \\
& + g_a^2 C_a(i) \{ g_a^2 [(84 - 120\zeta_3) G_a - 36 S_a] + g_b^2 C_b(i) (96\zeta_3 - 64) \} (m^2)_i^l |M_a|^2 \\
& + g_a^2 g_b^2 C_a(i) C_b(i) [(48\zeta_3 - 40) (m^2)_i^l + (48\zeta_3 - 56) \mu^{ij} \mu_{ij}] M_a M_b^* \\
& + g_a^2 C_a(i) [g_a^2 (18G_a - 6S_a) - 8g_b^2 C_b(i)] (m^2)_i^j (m^2)_j^l + g_a^4 C_a(i) [(56 - 24\zeta_3) G_a - 16 S_a] \mu^{ij} \mu_{ij} |M_a|^2 \\
& + g_a^2 C_a(i) [(24 - 12\zeta_3) g_a^2 G_a - 8g_a^2 S_a + (24\zeta_3 - 28) g_b^2 C_b(i)] [2(m^2)_i^j \mu_{jk} \mu^{ik} \\
& + (b^{ij} - 2M_a \mu^{ij}) (b_{ij} - 2M_a^* \mu_{ij})] + 8g_a^4 (m^2)_i^l [(m^2)_j^l - \mu^{jk} \mu_{jk}] C_a(i) C_a(j) / d_a. \tag{4.42}
\end{aligned}$$

Note that β_{Λ} vanishes in the case of no supersymmetry-breaking terms.

To obtain the three-loop beta function for Λ , I found that it was more than sufficient to consider Eq. (4.3) for the following example models, chosen somewhat arbitrarily. Since the goal here was only to obtain the beta function for the field-independent vacuum energy, the background values of all scalar fields were simply set to 0.

- (i) Supersymmetric theory with no gauge symmetry and six chiral superfields $\Phi_{1,2,3,4,5,6}$, with superpotential $W = y\Phi_1\Phi_2\Phi_3 + y'\Phi_1\Phi_4\Phi_5 + y''\Phi_2\Phi_4\Phi_6 + \frac{1}{2}\mu_1\Phi_1^2 + \frac{1}{2}\mu_2\Phi_2^2 + \frac{1}{2}\mu_3\Phi_3^2 + \frac{1}{2}\mu_4\Phi_4^2 + \frac{1}{2}\mu_5\Phi_5^2 + \frac{1}{2}\mu_6\Phi_6^2$ and soft supersymmetry-breaking Lagrangian $-\mathcal{L}_{\text{soft}} = m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + m_3^2|\phi_3|^2 + m_4^2|\phi_4|^2 + m_5^2|\phi_5|^2 + m_6^2|\phi_6|^2 + \Lambda$.
- (ii) Supersymmetric theory with no gauge symmetry and six chiral superfields $\Phi_{1,2,3,4,5,6}$, with superpotential $W = y\Phi_1\Phi_2\Phi_3 + y'\Phi_1\Phi_4\Phi_5 + y''\Phi_2\Phi_4\Phi_6 + \mu_{12}\Phi_1\Phi_2 + \mu_{34}\Phi_3\Phi_4 + \mu_{56}\Phi_5\Phi_6$ and soft supersymmetry-breaking Lagrangian $-\mathcal{L}_{\text{soft}} = m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + m_3^2|\phi_3|^2 + m_4^2|\phi_4|^2 + m_5^2|\phi_5|^2 + m_6^2|\phi_6|^2 + \Lambda$.
- (iii) Supersymmetric theory with no gauge symmetry and two chiral superfields, with the most general superpotential (including four independent Yukawa couplings and three mass terms) and the most general soft supersymmetry-breaking Lagrangian.

- (iv) Supersymmetric $U(1)$ gauge theory with six chiral superfields with charges $q_1, -q_1, q_2, -q_2, 0$, and 0, with the most general allowed superpotential and the most general soft supersymmetry-breaking Lagrangian consistent with these charge assignments.
- (v) Supersymmetric $U(1)$ gauge theory with six chiral superfields with charges $q_1, -q_1, q_2, -q_2, (q_1 + q_2)$, and $-(q_1 + q_2)$, with the most general allowed superpotential and soft supersymmetry-breaking Lagrangian consistent with these charge assignments.
- (vi) Supersymmetric $SU(2)$ gauge theory, with three doublet chiral superfields Φ_1, Φ_2 , and Φ_3 and one singlet S , with superpotential $W = y\Phi_1\Phi_2S$, and soft supersymmetry-breaking Lagrangian $-\mathcal{L}_{\text{soft}} = (a\phi_1\phi_2s + \frac{1}{2}M\lambda\lambda) + \text{c.c.} + m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + m_3^2|\phi_3|^2 + m_s^2|s|^2 + \Lambda$.
- (vii) Supersymmetric $SU(2)$ gauge theory, with four doublet chiral superfields $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, and one singlet S , with superpotential $W = y\Phi_1\Phi_2S + y'\Phi_3\Phi_4S$, and soft supersymmetry-breaking Lagrangian $-\mathcal{L}_{\text{soft}} = (a\phi_1\phi_2s + a'\phi_3\phi_4s + \frac{1}{2}M\lambda\lambda) + \text{c.c.} + m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + m_3^2|\phi_3|^2 + m_4^2|\phi_4|^2 + m_s^2|s|^2 + \Lambda$.
- (viii) Supersymmetric $SU(2) \times U(1)$ gauge theory, with chiral superfields transforming as $(\mathbf{2}, +1)$, $(\mathbf{2}, -1)$, and $(\mathbf{1}, 0)$, and the most general allowed superpotential and soft supersymmetry-breaking Lagrangian.

(ix) Supersymmetric $SU(3)$ gauge theory, with triplet and antitriplet chiral superfields $\Phi, \bar{\Phi}$, and one singlet S , with superpotential $W = y\Phi\bar{\Phi}S$, and soft supersymmetry-breaking Lagrangian $-\mathcal{L}_{\text{soft}} = (a\phi\bar{\phi}s + \frac{1}{2}M\lambda\bar{\lambda}) + \text{c.c.} + m^2|\phi|^2 + \bar{m}^2|\bar{\phi}|^2 + m_s^2|s|^2 + \Lambda$.

The expression for $\beta_{\Lambda}^{(3)}$ in Eqs. (4.39)–(4.42) was obtained by writing the most general possible form for it with unknown coefficients, and then solving for the coefficients by demanding the vanishing of Eq. (4.3) for $\ell = 1, 2, 3$. These examples also produced numerous redundant checks.

To obtain the previously unknown values of the three-loop scalar field anomalous dimension coefficients n_7, \dots, n_{26} in Eq. (4.18), I found that it was again more than sufficient to consider Eq. (4.3) for each of the following example models:

- (i) Supersymmetric $U(1)$ gauge theory with six chiral superfields $\Phi_{1,2,3,4,5,6}$, with charges $q_1, q_2, q_3, q_4, -(q_1 + q_2)$, and $-(q_1 + q_3)$, and a superpotential $W = y\Phi_1\Phi_2\Phi_5 + y'\Phi_1\Phi_3\Phi_6$. To avoid a gauge anomaly, $q_4^3 = q_1^3 + 3q_1^2q_2 + 3q_1q_2^2 + 3q_1^2q_3 + 3q_1q_3^2$. The effective potential is a function of the Yukawa couplings y, y' , the gauge coupling g , and the background values of the scalar components of Φ_1 and Φ_2 , which are taken to be independent.
- (ii) Supersymmetric $SU(2) \times U(1)$ gauge theory with chiral superfields transforming as $\Phi_1 = (\mathbf{2}, q_1)$, $\Phi_2 = (\mathbf{2}, q_2)$, $\Phi_3 = (\mathbf{2}, q_3)$, $\Phi_4 = (\mathbf{1}, q_4)$, $\Phi_5 = (\mathbf{1}, -q_1 - q_2)$. There is a Yukawa interaction $W = y\Phi_1\Phi_2\Phi_5$. To avoid gauge anomalies, $q_3 = -q_1 - q_2$, and $q_4^3 = q_1^3 + 9q_1^2q_2 + 9q_1q_2^2 + q_2^3$. The effective potential is a function of the Yukawa coupling y , the gauge couplings g and g' , and the background values of the scalar components of Φ_1 and Φ_2 , which are taken to be independent.
- (iii) Supersymmetric $SU(2)$ gauge theory with chiral superfields consisting of one doublet Φ and one triplet Σ , with superpotential $W = y\Phi\Phi\Sigma$. The effective potential is a function of the Yukawa coupling y , the gauge couplings g , and the background values of the scalar components Φ and Σ , which are taken to be independent.
- (iv) Supersymmetric $U(1)$, $SU(2)$, and $SU(3)$ gauge theories with chiral superfields Φ and $\bar{\Phi}$ in fundamental and antifundamental representations, with superpotential $W = \mu\Phi\bar{\Phi}$ and $-\mathcal{L}_{\text{soft}} = \Lambda + m^2|\phi|^2 + \bar{m}^2|\bar{\phi}|^2 + (b\phi\bar{\phi} + \frac{1}{2}M\lambda^a\bar{\lambda}^a + \text{c.c.})$. The background scalar field components of ϕ and $\bar{\phi}$ were taken to

be nonzero and equal along a D -flat direction, but supersymmetry is explicitly broken by soft terms, so the effective potential does not vanish.

In addition to determining the scalar field anomalous dimension coefficients, these models again produced numerous redundant checks of Eq. (4.3).

V. OUTLOOK

In this paper, I have provided the three-loop effective potential in Landau gauge for a general softly broken supersymmetric theory, using a regularization and renormalization scheme that respects supersymmetry. As by-products, the beta function for the field-independent vacuum energy and the Landau gauge anomalous dimension of scalars have been obtained.

It should be noted that the results obtained in this paper apply only to models with softly broken supersymmetry. This is because if there is supersymmetry violation in the dimensionless couplings (or simply in the field content) of the theory, then it was shown in Refs. [61,62] that while dimensional reduction can be applied in a consistent way, renormalization requires that there are evanescent couplings that are different (at all but at most one renormalization scale) for ϵ scalars and vectors. This is inconsistent with the procedure followed in the present paper, where the contributions of ϵ scalars and vectors have been combined due to always having the same gauge interactions. This is a feature only of softly broken supersymmetry.

I have checked the consistency of the three-loop effective potential for numerous toy models, as described above. An obvious more practical application of the results obtained here is to the MSSM, which could well describe our world even though there are increasingly stringent bounds on superpartners coming from direct searches at the Large Hadron Collider. This would extend the two-loop results of Ref. [63] and allow a more precise determination of the relations between the Higgs vacuum expectation values and the other renormalized Lagrangian parameters. Implementing the general results found here in the special case of the MSSM is in principle straightforward, although the combinatorics appear to be somewhat intimidating. This is left as an exercise for the clever and courageous reader.

ACKNOWLEDGMENTS

This work is supported in part by the National Science Foundation Grants No. 2013340 and No. 2310533.

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