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MEMS Vol. 8 / 2024

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ISSN 2747-9080, eISSN 2747-9099
ISBN 978-3-98547-066-2, eISBN 978-3-98547-566-7, DOI 10.4171/MEMS/8

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Published by EMS Press, an imprint of the

European Mathematical Society – EMS – Publishing House GmbH
Institut für Mathematik
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin, Germany

https://ems.press

Typesetting: AfricanType, Cairo, Egypt
Printed in Germany
Printed on acid free paper

Abstract

A C^*-algebra satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet if it is equivalent in Kasparov's KK-theory to a commutative C^*-algebra. This paper is motivated by the problem of establishing the range of validity of the UCT, and in particular, whether the UCT holds for all nuclear C^*-algebras.

We introduce the idea of a C^*-algebra that "decomposes" over a class $\mathcal{C}$ of C^*-algebras. Roughly, this means that locally there are approximately central elements that approximately cut the C^*-algebra into two C^*-subalgebras from $\mathcal{C}$ that have well-behaved intersection. We show that if a C^*-algebra decomposes over the class of nuclear, UCT C^*-algebras, then it satisfies the UCT. The argument is based on a Mayer–Vietoris principle in the framework of controlled KK-theory; the latter was introduced by the authors in an earlier work. Nuclearity is used via Kasparov's Hilbert module version of Voiculescu's theorem, and Haagerup's theorem that nuclear C^*-algebras are amenable.

We say that a C^*-algebra has finite complexity if it is in the smallest class of C^*-algebras containing the finite-dimensional C^*-algebras, and closed under decomposability; our main result implies that all C^*-algebras in this class satisfy the UCT. The class of C^*-algebras with finite complexity is large, and comes with an ordinal-number invariant measuring the complexity level. We conjecture that a C^*-algebra of finite nuclear dimension and real rank zero has finite complexity; this (and several other related conjectures) would imply the UCT for all separable nuclear C^*-algebras. We also give new local formulations of the UCT, and some other necessary and sufficient conditions for the UCT to hold for all nuclear C^*-algebras.

Keywords. Amenable C^*-algebra, decomposition of a C^*-algebra, controlled KK-theory, Universal Coefficient Theorem

Mathematics Subject Classification (2020). Primary 19K35; Secondary 19K33, 46L05, 46L80, 46L85

Acknowledgements. We thank Wilhelm Winter for some helpful comments on a preliminary draft. We thank Claude Schochet for pointing out the work of Gray [28] on inverse limits; this reference allowed us to significantly improve the results from an earlier circulated draft of this memoir. Finally, we thank the anonymous referee for useful comments.

Funding. The authors gratefully acknowledge support from the US NSF (DMS 1564281, DMS 1700021, DMS 1901522, DMS 2000082, DMS 2247313, DMS 2247968), and the Simons Fellow Program.

Contents

Chapter 1

Introduction

Our aim in this memoir is to present some new techniques to establish the Universal Coefficient Theorem in C^*-algebra K-theory, and some new necessary and sufficient conditions for the Universal Coefficient Theorem to hold for all nuclear C^*-algebras.

Unless otherwise stated, anything in this introduction called A or B is a *separable* C^*-algebra.

1.1 The universal coefficient theorem

A C^*-algebra A satisfies the *Universal Coefficient Theorem* (UCT) of Rosenberg and Schochet [55] if for any C^*-algebra B, there is a canonical short exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_*(B)) \to KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0.$$

Equivalently (see [55, p. 456] or [60, Proposition 5.2]), A satisfies the UCT if it is KK-equivalent to a commutative C^*-algebra.

The UCT is known to hold for a large class of C^*-algebras. The fundamental examples are the C^*-algebras in the *bootstrap class* $\mathcal{N}$. This is the smallest collection of separable, nuclear C^*-algebras that contains all type I C^*-algebras, and that is closed under the following operations: extensions; stable isomorphisms; inductive limits; and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. Rosenberg and Schochet [55] showed that any C^*-algebra in $\mathcal{N}$ satisfies the UCT. Another important class of examples was established by Tu in [64, Proposition 10.7]; building on the work of Higson and Kasparov [35] on the Baum–Connes conjecture for a-T-menable groups, Tu showed that the groupoid[1] C^*-algebra of any a-T-menable groupoid satisfies the UCT. In particular, Tu's work applies to the groupoid C^*-algebras of amenable groupoids.

There has been other significant work giving sufficient conditions for the UCT to hold, and in some cases also necessary conditions as well as the work mentioned already, one has for example [60, Proposition 5.2], [53, Corollary 8.4.6], [21], [43, Remark 2.17], [6, Theorem 4.17], [4], and [5]. Nonetheless, the bootstrap class and the class of C^*-algebras of a-T-menable groupoids, which are defined in terms of *global* properties of the C^*-algebras involved, remain the most important classes of C^*-algebras known to satisfy the UCT.

[1] To be more precise, we need standard assumptions so that the groupoid C^*-algebra is defined and separable. Here, appropriate assumptions are that the groupoid is locally compact, Hausdorff, and second countable, and that it admits a Haar system.

On the other hand, Skandalis [60, p. 571] has shown[2] that there are C^*-algebras that do not satisfy the UCT. Skandalis's examples are quite concrete; they are reduced group C^*-algebras of countably infinite hyperbolic groups with property (T), and in particular are exact [44, Section 6.E]. Looking to more exotic examples, failures of exactness can also be used to produce non-UCT C^*-algebras; see for example [14, Remark 4.3].

Despite these counterexamples, there are no known *nuclear* C^*-algebras that do not satisfy the UCT. Whether or not the UCT holds for all nuclear C^*-algebras is a particularly important open problem. One reason for this is the spectacular recent progress (see for example [11, 23, 24, 26, 27, 42, 50, 63]) in the Elliott program [22] to classify simple, separable, nuclear C^*-algebras by K-theoretic invariants. Establishing the range of validity of the UCT is now the only barrier to getting the "best possible" classification result in this setting.

On the other hand, work inspired by the Elliott program has led to recent, and again spectacular, success in the general structure theory of nuclear C^*-algebras, including the recent solution of a large part of the Toms–Winter conjecture [12, 13]. Our motivation in the current paper is to try to bridge the gap between properties that are relevant in this structure theory – in particular the theory of nuclear dimension [70] introduced by Winter and Zacharias – and properties that imply the UCT. In particular, our aim is to give *local* conditions that imply the UCT, in contrast to the global conditions from the work of Rosenberg and Schochet [55] and Tu [64] mentioned above.

1.2 Decompositions and the main theorem

We now introduce our sufficient condition for the UCT. For the statement below, if X is a metric space, S is a subset of X, $x \in X$, and $\varepsilon > 0$ we write "$x \in_\varepsilon S$" if there exists $s \in S$ with $d(x, s) < \varepsilon$.

Definition 1.1. Let $\mathcal{C}$ be a class of unital C^*-algebras. A unital C^*-algebra[3] A *decomposes over* $\mathcal{C}$ if for every finite subset X of the unit ball of A and every $\varepsilon > 0$ there exist C^*-subalgebras C, D, and E of A that are in the class $\mathcal{C}$ and contain 1_A, and a positive contraction $h \in E$ such that

(i) $\|[h, x]\| < \varepsilon$ for all $x \in X$;

(ii) $hx \in_\varepsilon C$, $(1-h)x \in_\varepsilon D$, and $h(1-h)x \in_\varepsilon E$ for all $x \in X$;

(iii) for all e in the unit ball of E, $e \in_\varepsilon C$ and $e \in_\varepsilon D$.

[2] See also the exposition in [34, Sections 6.1 and 6.2].

[3] Not necessarily separable. For applications to the UCT, only the separable case is relevant, but the definition admits interesting examples in the non-separable case, and it seems plausible there will be other applications.

One should think of C and D as being approximately (unitizations of) ideals in A such that $C + D = A$, and E being approximately equal to (the unitization of) $C \cap D$. We will discuss examples later.

Here, is our main theorem, which was inspired by our earlier work on the Künneth formula (in collaboration with Oyono-Oyono) [48, 67], and by our earlier work on finite dynamical complexity and finite decomposition complexity (in collaboration with Guentner and Tessera) [29, 31]. See Corollary 7.5 below for the proof.

Theorem 1.2. *If A is a separable, unital C^*-algebra that decomposes over the class of separable, nuclear C^*-algebras that satisfy the UCT, then A is nuclear and satisfies the UCT.*

One can thus think of decomposability as an addition to the closure operations that are used in the definition of the bootstrap class $\mathcal{N}$.

1.3 C^*-algebras with finite complexity

Following the precedent established by [30] in coarse geometry, the notion of decomposability suggests a complexity hierarchy on C^*-algebras.

Definition 1.3. Let $\mathcal{D}$ denote a class of unital C^*-algebras. For an ordinal number α,

(i) if $\alpha = 0$, let $\mathcal{D}_0$ be the class of C^*-algebras D that are locally[4] in $\mathcal{D}$;

(ii) if $\alpha > 0$, let $\mathcal{D}_\alpha$ be the class of C^*-algebras that decompose over C^*-algebras in $\bigcup_{\beta<\alpha} \mathcal{D}_\beta$.

A unital C^*-algebra D has *finite complexity relative to* $\mathcal{D}$ if it is in $\mathcal{D}_\alpha$ for some α. If $\mathcal{D}$ is the class of finite-dimensional C^*-algebras, we just say that D has *finite complexity*.

If a unital C^*-algebra D has finite complexity relative to $\mathcal{D}$, the *complexity rank of D relative to* $\mathcal{D}$ is the smallest α such that D is in $\mathcal{D}_\alpha$. If $\mathcal{D}$ is the class of finite-dimensional C^*-algebras, we just say the *complexity rank of D* with no additional qualifiers.

The following result is equivalent to Theorem 1.2 above. However, we think the reframing in terms of complexity is quite suggestive.

Theorem 1.4. *Let $\mathcal{C}$ be a class of separable, unital, nuclear C^*-algebras that satisfy the UCT. Then, the class of separable, unital C^*-algebras that have finite complexity relative to $\mathcal{C}$ consists of nuclear C^*-algebras that satisfy the UCT.*

[4] A C^*-algebra is *locally* in a class $\mathcal{D}$ if for any finite subset X of D and any $\varepsilon > 0$ there is a C^*-subalgebra C of D that is in $\mathcal{D}$, and such that $x \in_\varepsilon C$ for all $x \in X$.

In particular, every separable C^*-algebra of finite complexity is nuclear and satisfies the UCT.

Examples 1.5. We can now give some non-trivial examples of C^*-algebras that decompose over natural, simpler, classes.

(i) In Proposition A.1, we show that for

$$2 \leq n < \infty,$$

the Cuntz algebra $\mathcal{O}_n$ has complexity rank one.

(ii) In [31], Guentner and the authors introduced "finite dynamical complexity" for groupoids, which also comes with a notion of complexity rank. In Proposition A.8 we show that if G is a locally compact, Hausdorff, étale, principal, ample groupoid with compact base space, then the complexity rank of $C_r^*(G)$ is bounded above by that of G. The class of groupoids with finite dynamical complexity is quite large; see Examples A.9 and A.11 below.

Combining part (ii) above with Theorem 1.4 gives a new proof of the UCT for the groupoid C^*-algebras of a large class of groupoids. However, we cannot claim any genuinely new examples; this is because the groupoids involved are all amenable, so the UCT for their C^*-algebras also follows from Tu's theorem [64] (see Remark A.13 below for more details).

1.4 Kirchberg algebras

Generalizing the Cuntz algebras from (i) above, recall that a *Kirchberg algebra* is a separable, nuclear C^*-algebra A such that for any non-zero $a \in A$, there are $b, c \in A$ such that $bac = 1_A$. Kirchberg algebras are closely connected to the UCT problem for nuclear C^*-algebras thanks to the following theorem of Kirchberg; see [53, Corollary 8.4.6] or [43, Remark 2.17].

Theorem 1.6 (Kirchberg). *To establish the UCT for all separable, nuclear C^*-algebras, it suffices to establish the UCT for any Kirchberg algebra with zero K-theory.* ∎

Theorems 1.4 and 1.6 imply that if any Kirchberg algebra with zero K-theory has finite complexity, then the UCT holds for all separable, nuclear C^*-algebras. Conversely, if the UCT holds for all separable, nuclear C^*-algebras, then from the Kirchberg–Phillips classification theorem [42, 50] (see also [53, Corollary 8.4.2] for the precise statement we want here), any unital Kirchberg algebra with zero K-theory

will be isomorphic to the Cuntz algebra $\mathcal{O}_2$, and so will have complexity rank one by Examples 1.5 (i). We summarize this discussion in the theorem below.

Theorem 1.7. *The following are equivalent:*

(i) *Any Kirchberg algebra with zero K-theory has complexity rank one.*

(ii) *All separable nuclear C^*-algebras satisfy the UCT.* ∎

Generalizing Examples 1.5 (i) above Jaime and the first author show in [37] that a Kirchberg algebra *that satisfies the UCT* has complexity rank one if and only if its K_1 group is torsion free, and that moreover any UCT Kirchberg algebra has complexity rank at most two. From Theorem 1.7, if one could prove this without the UCT assumption, then the UCT for all separable nuclear C^*-algebras would follow.

The paper [37] also discusses several other connections between complexity rank, real rank zero, and nuclear dimension. We will not go into this any more deeply here; suffice to say that these other connections inspired us to make the following conjectures.

Conjecture 1.8. *Any separable unital C^*-algebra with real rank zero and finite nuclear dimension has finite complexity.*

Conjecture 1.9. *Any separable unital C^*-algebra with finite nuclear dimension has finite complexity relative to the class of subhomogeneous*[5] *C^*-algebras.*

Thanks to Theorem 1.7 and the fact that all Kirchberg algebras have nuclear dimension one (see [9, Theorem G]) and real rank zero (see [72]), either of these conjectures implies the UCT for all separable, nuclear C^*-algebras. There are many other related conjectures one could reasonably make that imply the UCT for all nuclear C^*-algebras. About the strongest such conjecture would be that any separable, nuclear C^*-algebra with real rank zero has finite complexity[6]. One of the weakest is that any Kirchberg algebra with zero K-theory has finite complexity.

1.5 A local reformulation of the UCT

We now discuss the methods that go into the proof of Theorem 1.2.

[5] Recall that a C^*-algebra C is *subhomogeneous* if there is $N \in \mathbb{N}$ and a compact Hausdorff space X such that C is a C^*-subalgebra of $M_N(C(X))$; see for example [8, Section IV.1.4] for background.

[6] It would also be natural to drop the real rank zero assumption, and then only ask for finite complexity relative to the subhomogeneous C^*-algebras, or even just relative to the type I C^*-algebras.

In our earlier work [68], we introduced *controlled KK-theory* groups $KK_\varepsilon(X, B)$ associated to a C^*-algebra B, a finite subset X of a C^*-algebra A and a constant $\varepsilon > 0$. Very roughly (we give more details below), one defines these by representing A in "general position" inside the stable multiplier algebra $M(B \otimes \mathcal{K})$ of B. The group $KK_\varepsilon(X, B)$ then consists of the "part of the K-theory of B that commutes with X, up to ε".

To be more precise about this, assume that A and B are C^*-algebras, and assume for simplicity[7] that A is unital. Let $\pi : A \to M(B \otimes \mathcal{K})$ be a faithful, unital, and strongly unitally absorbing[8] representation. Fixing such a representation, identify A with a diagonal subalgebra of $M_2(M(B \otimes \mathcal{K}))$ via the representation $\pi \oplus \pi$. For a finite subset X of the unit ball of A and $\varepsilon > 0$, define $\mathcal{P}_\varepsilon(X, B)$ to be the set of projections in $M_2(M(B \otimes \mathcal{K}))$ such that $p\text{-}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $M_2(B \otimes \mathcal{K})$, and such that $\|[p, x]\| < \varepsilon$ for all $x \in X$. The associated *controlled KK-theory group*[9] is then defined to be the set

$$KK^0_\varepsilon(X, B) := \pi_0(\mathcal{P}_\varepsilon(X, B))$$

of path components in $\mathcal{P}_\varepsilon(X, B)$. One can show that this group is determined up to canonical isomorphism by the subset inclusion $X \subseteq A$, by B, and by ε; it does not depend on the choice of representation.

Note that if $X = \varnothing$, then $KK^0_\varepsilon(\varnothing, B)$ is canonically isomorphic to the usual K-theory group $K_0(B)$ (for any ε); this is what we mean when we say $KK_\varepsilon(X, B)$ consists of the "part of the K-theory of B that commutes with X, up to ε".

Now, if $0 < \delta \leq \varepsilon$ and if $Y \supseteq X$ are finite subsets of A_1, then there is an inclusion $\mathcal{P}_\delta(Y, B) \subseteq \mathcal{P}_\varepsilon(X, B)$ that induces a "forget control map"

$$KK_\delta(Y, B) \to KK_\varepsilon(X, B).$$

In [68, Theorem 1.1], we showed that there is a short exact "Milnor sequence" relating the inverse system built from these forget control maps to the usual KK-group $KK(A, B)$; see Theorem 2.13 below for details. This sequence is an analogue of the Milnor sequence appearing in Schochet's work [56, 57]; however, unlike Schochet's version, it is local in nature, and does not require the UCT.

Our first goal in this memoir is to use the Milnor sequence to establish the following "local reformulation" of the UCT.

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[7] The theory also works for C^*-algebras that are not unital, but the definitions are a little more complicated.

[8] Roughly, a strongly unitally absorbing representation is one that satisfies the conclusion of Voiculescu's theorem for all representations of A on Hilbert B-modules; for the current discussion, it is just important that such a representation always exists. See Definition 2.5 below for details.

[9] It is canonically a group, with the operation given by Cuntz sum in an appropriate sense.

Theorem 1.10. *Let A be a unital C^*-algebra. Then, the following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let B be a separable C^*-algebra such that $K_*(B) = 0$, and let $\pi : A \to M(SB \otimes \mathcal{K})$ be a strongly unitally absorbing representation into the stable multiplier algebra of the suspension of B. Then, for any finite subset X of A and any $\varepsilon > 0$ there exists a finite subset Y of A containing X and $\delta \leq \varepsilon$ such that the canonical forget control map*

$$KK_\delta(Y, SB) \to KK_\varepsilon(X, SB)$$

for the suspension of B is zero.

This is a key ingredient in our main results, but we hope it will prove to be useful in its own right. Note in particular that there are no assumptions on A other than that it is separable and unital[10].

There is a technical variation of Theorem 1.10 that applies to nuclear C^*-algebras, and that plays an important role in our arguments. The key point is one of order of quantifiers; condition (ii) from Theorem 1.10 starts with quantifiers of the form

$$\text{“}\forall B\ \forall \pi\ \forall X\ \forall \varepsilon\ \exists Y\ \exists \delta \ldots\text{”}.$$

If A is nuclear, the same statement is true with the order of quantifiers replaced with

$$\text{“}\forall \varepsilon\ \exists \delta\ \forall B\ \forall \pi\ \forall X\ \exists Y \ldots\text{”},$$

i.e., δ depends *only* on ε and not on any of the other choices involved. To establish this, we adapt an averaging argument due to Christensen, Sinclair, Smith, White, and Winter [17, Section 3], which is in turn based on Haagerup's theorem that nuclear C^*-algebras are amenable [33].

1.6 Strategy for the proof of the main theorem

Assume that A is a nuclear, unital C^*-algebra that decomposes with respect to the class of nuclear UCT C^*-algebras as in the statement of Theorem 1.2. Assume moreover that $K_*(B) = 0$. Thanks to Theorem 1.10 above, to establish the UCT for A it suffices to show that for any finite subset X of the unit ball A_1 of A, and any $\varepsilon > 0$ there exist $Y \supseteq X$ and $\delta \leq \varepsilon$ such that the canonical forget control map

$$KK^0_\delta(Y, SB) \to KK^0_\varepsilon(X, SB)$$

is zero.

[10] Unitality is not really necessary – we do not do it in this memoir, but similar techniques establish the result above for non-unital separable C^*-algebras, with appropriately reformulated controlled KK-groups.

Our approach to this is inspired directly by our earlier work with several collaborators; this includes the work on the Künneth formula of Oyono-Oyono and the second author [48], and separately by the first author [67]; the work of Guentner and the authors on the Baum–Connes conjecture for transformation groupoids with finite dynamical complexity [31]; and the work of Guentner, Tessera, and the second author on the stable Borel conjecture for groups of finite decomposition complexity [29]. These other papers all use controlled K-theory as opposed to KK-theory; the seminal result along these lines is the second author's work on the Novikov conjecture for groups with finite asymptotic dimension [71].

In the current context, we use decomposability and a Mayer–Vietoris argument. Let $\gamma > 0$ be a very small constant, which is in particular smaller than ε. Then, any suitably small[11] $\delta > 0$ will have the following property. Let h and C, D, and E be nuclear UCT algebras as in the definition of decomposability for the given set X and parameter δ. Let Y_C, Y_D and Y_E be finite subsets of the unit balls C_1, D_1, and E_1 respectively that contain $hX \cup \{h\}$, $(1-h)X \cup \{h\}$ and $h(1-h)X \cup \{h\}$ respectively up to δ-error, and so that Y_C and Y_D both contain Y_E up to δ-error. Let

$$Y = Y_C \cup Y_D \cup Y_E \cup X.$$

Then, one can construct a diagram[12] of the form

$$\begin{array}{ccc} KK^0_\delta(Y, SB) & \xrightarrow{\kappa_C \oplus \kappa_D} & KK^0_{2\delta}(Y_C, SB) \oplus KK^0_{2\delta}(Y_D, SB) \\ \downarrow & & \\ KK_\gamma(Y_E, S^2B) & \xrightarrow{\partial} & KK^0_\varepsilon(X, SB) \end{array} \tag{1.1}$$

where the vertical arrow is the canonical forget control map. This diagram has the "exactness" property that if $[p]$ goes to zero under the map

$$\kappa_C \oplus \kappa_D : KK^0_\delta(Y, B) \to KK^0_{2\delta}(Y_C, SB) \oplus KK^0_{2\delta}(Y_D, SB) \tag{1.2}$$

then the image of $[p]$ under the forget control map $KK^0_\delta(Y, SB) \to KK^0_\varepsilon(X, SB)$ is in the image of the map

$$\partial : KK_\gamma(Y_E, S^2B) \to KK^0_\varepsilon(X, SB). \tag{1.3}$$

[11] The size of γ depends linearly on ε and the size of δ depends linearly on γ; the constants involved are very large.

[12] The form of this diagram is not new; the basic idea is modeled on [29, Diagram (5.8)] from the work of the Guentner, Tessera, and the second author on the stable Borel conjecture for groups with finite decomposition complexity. See also [31, Proposition 7.6] from work of the Guentner and the authors in a more closely related context.

However, as $K_*(B) = 0$, if γ and δ are small enough, one can use Theorem 2.15 (in the stronger form for nuclear C^*-algebras) to choose Y_C, Y_D, and Y_E large enough so that the maps in lines (1.2) and (1.3) are zero. This completes the proof.

In the detailed exposition below we structure the proof to give it as "local" a flavor as possible, partly as we suspect that the ideas might be useful in other contexts. The two main "local"(ish) technical results are recorded as Propositions 7.1 and 7.2 below.

The argument above is directly inspired by the classical Mayer–Vietoris principle. Indeed, assume that C and D are nuclear *ideals* in A with intersection E, and such that

$$A = C + D.$$

Then, there is[13] an exact Mayer–Vietoris sequence

$$\cdots \to KK^0(E, SB) \to KK^0(A, B) \to KK^0(C, B) \oplus KK^0(D, B) \to \cdots .$$

In particular, if the groups at the left and right are zero, then the group in the middle is also zero. Our analysis of the diagram in line (1.1) is based on a concrete construction of this classical Mayer–Vietoris sequence that can be adapted to our controlled setting. The idea has its roots in algebraic K-theory, going back at least as far as [46, Chapter 2]. Having said this, there is significant work to be done adapting these classical ideas to the analytic superstructure that we built in [68], and the resulting formulas and arguments end up being quite different.

Remark 1.11. It would be very interesting to remove the nuclearity hypothesis from Theorem 1.2, or at least to replace it with something weaker such as exactness. Let us explain how nuclearity is used in the proof of Theorem 1.2, in the hope that some reader will see a way around it.

The first use of nuclearity is to show that any nuclear, unital C^*-algebra admits strongly unitally absorbing representations whose restriction to any nuclear, unital C^*-subalgebra is also strongly unitally absorbing; see Corollary 2.7 below. The proof of this is based on Kasparov's version of Voiculescu's theorem for Hilbert modules [40, Section 7]. It seems plausible from the discussion in Remark 2.8 below that some form of nuclearity is necessary for this to hold, but we do not know this.

The second place nuclearity is used is via an averaging argument due to Christensen, Sinclair, Smith, White, and Winter [17, Section 3]; this is applicable to nuclear C^*-algebras thanks to Haagerup's theorem that nuclear C^*-algebras are always amenable [33]. This lets us prove a stronger version of Theorem 1.10; see Corollary 2.22 below. We do not know if this result holds without nuclearity; see Remark 2.19 for a more detailed discussion.

[13] It is not in the literature as far as we can tell. For nuclear C^*-algebras, it can be derived from the usual long exact sequence in KK-theory using, for example, the argument of [69, Proposition 2.7.15].

1.7 Notation and conventions

For a subset S of a metric space X, $x \in X$ and $\varepsilon > 0$, we write "$x \in_\varepsilon S$" if there is $s \in S$ with $d(x, s) < \varepsilon$. For elements x, y of a metric space X, we write "$x \approx_\varepsilon y$" if $d(x, y) < \varepsilon$.

We write ℓ^2 for $\ell^2(\mathbb{N})$. Throughout, the letters A and B are reserved for *separable* C^*-algebras. The letter C will refer to a possibly non-separable C^*-algebra. The unit ball of C (or a more general normed space) is denoted by C_1, its unitization is C^+, its multiplier algebra is $M(C)$, its suspension is SC, and its n-fold suspension is S^nC. We write M_n or $M_n(\mathbb{C})$ for the $n \times n$ matrices, and $M_n(C)$ for the $n \times n$ matrices over a C^*-algebra C.

Our conventions on Hilbert modules follow those of Lance [45]. We will write $H_B := \ell^2 \otimes B$ for the standard Hilbert B-module, and $\mathcal{L}_B$, respectively $\mathcal{K}_B$, as shorthand for the C^*-algebra $\mathcal{L}(H_B)$ of adjointable operators on H_B, respectively the C^*-algebra $\mathcal{K}(H_B)$ of compact operators on H_B. We will typically identify $\mathcal{L}_B$ with the "diagonal subalgebra" $1_{M_n} \otimes \mathcal{L}_B$ of $M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$. Thus, we might write "$[x, y]$" for the commutator of $x \in \mathcal{L}_B$ and $y \in M_n(\mathcal{L}_B)$, when it would be more strictly correct to write something like "$[1_{M_n} \otimes x, y]$".

The symbol "$\otimes$" always denotes a completed tensor product: either the external tensor product of Hilbert modules (see [45, Chapter 4] for background on this), or the minimal tensor product of C^*-algebras (see for example [10, Chapter 3]).

We will sometimes write 0_n and 1_n for the zero matrix and identity matrix of size n when this seems helpful to avoid confusion, although we will generally omit the subscripts to avoid clutter. If $n \leq m$, we will also use $1_n \in M_m(\mathbb{C})$ for the rank n projection with n ones in the top-left part of the diagonal and zeros elsewhere. Given an $n \times n$ matrix a and an $m \times m$ matrix b, $a \oplus b$ denotes the "block sum" $(n + m) \times (n + m)$ matrix defined by

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Finally, $K_*(A) := K_0(A) \oplus K_1(A)$ denotes the graded K-theory group of a C^*-algebra, and $KK^*(A, B) := KK^0(A, B) \oplus KK^1(A, B)$ the graded KK-theory group. We will typically just write $KK(A, B)$ instead of $KK^0(A, B)$.

1.8 Outline of the paper

Chapter 2 gives our reformulation of the UCT in terms of a concrete vanishing condition for controlled KK-theory. The key ingredients for this are the Milnor sequence from [68, Theorem 1.1], and some ideas around the Mittag–Leffler condition from the theory of inverse limits (see for example [66, Section 3.5]). We also show that

a stronger vanishing result holds for nuclear, UCT C^*-algebras using an averaging argument of Christensen, Sinclair, Smith, White, and Winter [17, Section 3]; the averaging argument is in turn based on Haagerup's theorem [33] that nuclearity implies amenability.

Chapter 3 discusses our controlled KK^0-groups. We introduced these in [68], but we need a technical variation here. This is essentially because in [68] we were setting up general theory, and for this it is easier to work with projections in a fixed C^*-algebra. In this memoir we are doing computations with concrete algebraic formulas, where it is more convenient to work with general idempotents, and to allow taking matrix algebras. We will, however, use both versions in this memoir, as we need to relate our work back here to the general theory of [68]. We also introduce controlled KK^1-groups in a concrete formulation using invertible operators; in our earlier work [68] we (implicitly) defined controlled KK^1-groups using suspensions, but here we also need the more concrete version.

Chapter 4 collects together some technical facts. These are all analogues for controlled KK-theory of well-known results from K-theory; for example, we prove "controlled versions" of the statements that homotopic idempotents are similar, and that similar idempotents are homotopic (up to increasing matrix sizes). Some arguments in this chapter are adapted from the work of Oyono-Oyono and the second author [47] on controlled K-theory.

Chapter 5 revisits the vanishing conditions of Chapter 2. Using the techniques of Chapter 4, we reformulate these results in the more flexible setting allowed by Chapter 3. This gives us the vanishing conditions that are the first main technical ingredient needed for Theorem 1.2.

Chapter 6 establishes the second main technical ingredient needed for Theorem 1.2. Here, we construct a "Mayer–Vietoris boundary map" for controlled KK-theory, and prove that it has an exactness property. The construction is an analogue of the usual index map of operator K-theory (see for example [54, Chapter 9]), although concrete formulas for the Mayer–Vietoris boundary map unfortunately seem to be missing from the C^*-algebra literature. The formulas we use are instead inspired by classical formulas from algebraic K-theory [46, Chapter 2], adapted to reflect our analytic setting.

Finally, in the main body of the paper, Chapter 7 puts everything together and gives the proofs of Theorem 1.2 and Theorem 1.4. We also include technical "local" vanishing results that we hope to elucidate the structure of the proof, and might be useful in other contexts.

The paper concludes with Appendix A, which gives examples of C^*-algebras with finite complexity. We first use a technique of Winter and Zacharias [70, Section 7] to show that the Cuntz algebras $\mathcal{O}_n$ with $2 \leq n < \infty$ have complexity rank one. We then use our joint work with Guentner on dynamic complexity [31] to show that ample, principal, étale groupoids with finite dynamical complexity and compact

base space have C^*-algebras of finite complexity; we also get a similar result without the ampleness assumption if we allow C^*-algebras with finite complexity relative to subhomogeneous C^*-algebras.

Chapter 2

Reformulating the UCT

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ are respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$.

Our goal in this chapter is to recall the definition of the controlled KK-theory groups, and then to reformulate the universal coefficient theorem in these terms.

We first recall the definition of the controlled KK-theory groups from [68]; to be precise, we need the version from [68, Sections A.1 and A.2] that is specific to unital C^*-algebras. We need a definition.

Definition 2.1. Let B be a separable C^*-algebra. Choose a unitary isomorphism $\ell^2 \cong \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2$, which induces a unitary isomorphism

$$\ell^2 \otimes B \cong (\mathbb{C}^2 \otimes \ell^2 \otimes \ell^2) \otimes B$$

of Hilbert B-modules. With respect to this isomorphism, let $e \in \mathcal{L}_B$ be the projection corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1_{\ell^2 \otimes \ell^2 \otimes B}$. We call e the *neutral projection*. A subset X of $\mathcal{L}_B$ is called *large* if every $x \in X$ is of the form $1_{\mathbb{C}^2 \otimes \ell^2} \otimes y$ for some $y \in \mathcal{L}(\ell^2 \otimes B)$ with respect to this decomposition.

Definition 2.2. Let B be a separable C^*-algebra. Let $\varepsilon > 0$, let X be a finite, large, subset of the unit ball of $\mathcal{L}_B$ and let $e \in \mathcal{L}_B$ be the neutral projection as in Definition 2.1. Let $\mathcal{P}_\varepsilon(X, B)$ consist of those projections p in $\mathcal{L}_B$ such that

(i) $p - e \in \mathcal{K}_B$; and

(ii) $\|[p, x]\| < \varepsilon$ for all $x \in X$.

Define $KK_\varepsilon(X, B)$ to be the set $\pi_0(\mathcal{P}_\varepsilon(X, B))$ of path components of $\mathcal{P}_\varepsilon(X, B)$. We write $[p] \in KK_\varepsilon(X, B)$ for the class of $p \in \mathcal{P}_\varepsilon(X, B)$.

Choose now isometries $t_1, t_2 \in \mathcal{B}(\ell^2)$ satisfying the *Cuntz relation*

$$t_1 t_1^* + t_2 t_2^* = 1,$$

and define $s_i := 1_{\mathbb{C}^2} \otimes t_i \otimes 1_{\ell^2 \otimes B} \in \mathcal{L}_B$. Define an operation on $KK_\varepsilon(X, B)$ by the *Cuntz sum*

$$[p] + [q] := [s_1 p s_1^* + s_2 q s_2^*].$$

The same proof as [68, Lemma A.4] shows that $KK_\varepsilon(X, B)$ is an abelian group, with identity element given by the class $[e]$ of the neutral projection.

We finish this section with two ancillary lemmas. The first is extremely well-known; we include an argument for completeness as we do not know a convenient reference.

Lemma 2.3. *Let a and b be elements of a unital C^*-algebra with b normal. Then, any z in the spectrum of a is contained within distance $\|a-b\|$ of the spectrum of b.*

Proof. We need to show that if z is further than $\|a-b\|$ from the spectrum of b, then $a-z$ is invertible. Indeed, in this case the continuous functional calculus implies that $\|(b-z)^{-1}\| < \|a-b\|^{-1}$. Hence,

$$\|(a-z)(b-z)^{-1}-1\| \leq \|(a-z)-(b-z)\|\|(b-z)^{-1}\| < 1,$$

whence $(a-z)(b-z)^{-1}$ is invertible, and so $a-z$ is invertible too. ∎

Lemma 2.4. *Let B be a separable C^*-algebra, let $\varepsilon > 0$, and let X be a finite, large, subset of the unit ball of $\mathcal{L}_B$. With notation as in Definition 2.2, the group $KK_\varepsilon(X,B)$ is countable.*

Proof. As B is separable $\mathcal{K}_B$ is separable, and so the set $\mathcal{P}_\varepsilon(X,B)$ is also separable. Let S be a countable dense subset of $\mathcal{P}_\varepsilon(X,B)$. It suffices to show that the map $S \to KK_\varepsilon(X,B)$ defined by $p \mapsto [p]$ is surjective.

Let $p \in \mathcal{P}_\varepsilon(X,B)$ be arbitrary, and define

$$\delta := \min\left\{\frac{1}{4}(\varepsilon - \max_{x\in X}\|[p,x]\|), \frac{1}{2}\right\}.$$

Let $q \in S$ be such that $\|p-q\| < \delta$, and let $p_t := (1-t)p + tq$ for $t \in [0,1]$. Then, for each $t \in [0,1]$, $\|p_t - p\| < \delta$, so Lemma 2.3 and that p_t is a positive contraction implies that the spectrum p_t is contained in $[0,\delta) \cup (1-\delta, 1]$. Let χ be the characteristic function of $(\frac{1}{2}, \infty)$. Then, $\|\chi(p_t) - p_t\| < \delta$ for all t, whence $\|\chi(p_t) - p\| < 2\delta$ for all t, from which it follows that $\|[\chi(p_t), x]\| < \varepsilon$ for all t and all $x \in X$. As $p_t - e \in \mathcal{K}_B$ for all t, it follows from the fact that $\mathcal{K}_B$ is an ideal in $\mathcal{L}_B$ that $\chi(p_t) - e \in \mathcal{K}_B$ too. Hence, $(\chi(p_t))_{t\in[0,1]}$ is a path connecting p and q within $\mathcal{P}_\varepsilon(X,B)$ so $[p] = [q]$, and we are done. ∎

2.1 The general case

We need a special class of representations on Hilbert B-modules, essentially taken from work of Thomsen [62, Definition 2.2] (see also [68, Definition A.11]). We do not need the details of the definition below, and only include it for completeness; all we really need are the facts about existence of such representations in Lemma 2.6 below.

Definition 2.5. Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. A representation $\sigma : A \to \mathcal{L}_B$ is *unitally absorbing* if for any unital completely positive map $\phi : A \to \mathcal{L}_B$ there exists a sequence of isometries (v_n) in $\mathcal{L}_B$

such that $\|v_n^*\sigma(a)v_n - \phi(a)\| \to 0$ as $n \to \infty$, and such that $v_n^*\sigma(a)v_n - \phi(a) \in \mathcal{K}_B$ for all $n \in \mathbb{N}$.

For a representation $\sigma : A \to \mathcal{L}_B = \mathcal{L}(H_B)$, let $\sigma^\infty : A \to \mathcal{L}(H_B^{\oplus\infty})$ be its infinite amplification, which we identify with a representation $\sigma^\infty : A \to \mathcal{L}_B$ via a choice of unitary isomorphism $(\ell^2)^{\oplus\infty} \cong \ell^2$ as in the string of identifications below

$$\mathcal{L}(H_B^{\oplus\infty}) = \mathcal{L}((\ell^2 \otimes B)^{\oplus\infty}) = \mathcal{L}((\ell^2)^{\oplus\infty} \otimes B) \cong \mathcal{L}(\ell^2 \otimes B) = \mathcal{L}_B$$

(all of the identifications labeled "=" are canonical). A unital representation $\pi : A \to \mathcal{L}_B$ is *strongly unitally absorbing* if there is a unitally absorbing representation

$$\sigma : A \to \mathcal{L}_B$$

such that $\pi = \sigma^{\oplus\infty}$.

Note that a (strongly) unitally absorbing representation is faithful. The following result is essentially due to Thomsen and Kasparov. Our main use of part (ii) occurs much later in the paper.

Lemma 2.6. *Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. Then,*

(i) *There exists a strongly unitally absorbing representation $\pi : A \to \mathcal{L}_B$.*

(ii) *Assume in addition that A or B is nuclear. Let $\sigma : A \to \mathcal{B}(\ell^2)$ be any faithful unital representation, let $\iota : \mathcal{B}(\ell^2) \to \mathcal{L}_B$ be the canonical inclusion arising from the decomposition $H_B = \ell^2 \otimes B$, and let $\pi : A \to \mathcal{L}_B$ be the infinite amplification of $\iota \circ \sigma$. Then, π is strongly unitally absorbing.*

Proof. For part (i), Thomsen shows in [62, Theorem 2.4] that a unitally absorbing representation $\sigma : A \to \mathcal{L}_B$ exists under the given hypotheses. Its infinite amplification π is then strongly unitally absorbing.

For part (ii), note first that identifying $(\iota \circ \sigma)^\infty$ with $(\iota \circ (\sigma^{\oplus\infty}))^\infty$ we may assume σ is the infinite amplification of some faithful unital representation $A \to \mathcal{B}(\ell^2)$. Having made this assumption, note that $\sigma(A) \cap \mathcal{K}(\ell^2) = \{0\}$. In [40, Theorem 5], Kasparov shows that if A is a separable, unital C^*-algebra and $\sigma : A \to \mathcal{B}(\ell^2)$ is a faithful representation such that $\sigma(A) \cap \mathcal{K}(\ell^2) = \{0\}$, and moreover if either A or B is nuclear, then the composition $\iota \circ \sigma$ satisfies the condition Thomsen gives in [62, Theorem 2.1, condition (4)]. Comparing [62, Theorem 2.1] and Definition 2.5, we see that $\iota \circ \sigma$ is unitally absorbing. Hence, $\pi = (\iota \circ \sigma)^{\oplus\infty}$ is strongly unitally absorbing. ∎

The following corollary is immediate from part (ii) of Lemma 2.6.

Corollary 2.7. *Let A be a separable, unital, nuclear C^*-algebra, and let B be a separable C^*-algebra. Then, there exists a strongly unitally absorbing representation*

$\pi : A \to \mathcal{L}_B$ such that the restriction of π to any unital, nuclear C^*-subalgebra of A is also strongly unitally absorbing. ∎

Remark 2.8. Corollary 2.7 is one of the two places nuclearity is used in the proof of Theorem 1.2, so it would be interesting to establish the corollary under some weaker assumption than nuclearity. The following observation shows that the method we used to establish Corollary 2.7 cannot extend beyond the nuclear case, however.

Let A be a separable, unital C^*-algebra, and let $A = B$. Let $\sigma : A \to \mathcal{B}(\ell^2)$ be a unital representation, and let $\pi := \iota \circ \sigma : A \to \mathcal{L}_A$ be as in Lemma 2.6 (ii). We claim that if π is unitally absorbing, then A is nuclear[1]. Let $\phi : A \to \mathcal{L}_A$ be the $*$-homomorphism $a \mapsto 1_{\ell^2} \otimes a$. If π is unitally absorbing then for any ε and finite subset X of A there is an isometry $v \in \mathcal{L}_A$ such that $\|v^*\pi(a)v - \phi(a)\| < \varepsilon$ for all $a \in X$. For each n, let $p_n \in \mathcal{B}(\ell^2)$ be the orthogonal projection onto $\ell^2(\{1, \ldots, n\})$, and let $q_n := p_n \otimes 1_A \in \mathcal{L}_A$. Note that $q_1\mathcal{L}_A q_1$ identifies canonically with A, and up to this identification $q_1\phi(a)q_1 = a$ for all $a \in A$, so in particular $\|q_1 v^*\pi(a)vq_1 - a\| < \varepsilon$ for all $a \in X$. As (q_n) converges strictly to the identity in $\mathcal{L}_A$, and as $q_1 v \in \mathcal{K}_A$, we have moreover that $q_1 v^* q_n \pi(a) q_n v q_1$ converges in norm to $q_1 v^* \pi(a) v^* q_1$, so there is n such that $\|q_1 v^* q_n \pi_n(a) q_n v q_1 - a\| < \varepsilon$ for all $a \in X$. We thus have ucp maps

$$A \xrightarrow{a \mapsto q_n \pi(a) q_n} q_n(\mathcal{B}(\ell^2) \otimes 1_A)q_n \cong M_n(\mathbb{C}) \xrightarrow{b \mapsto q_1 v^* b v q_1} A$$

whose composition agrees with the identity on X to within ε error. As X and ε were arbitrary, this implies nuclearity of A (see for example [10, Chapter 2]).

To state the main result of [68], we need some more definitions.

Definition 2.9. Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. A representation $\pi : A \to \mathcal{L}_B$ is *large* if there is a unitally absorbing representation $\sigma : A \to \mathcal{L}_B$ such that with respect to the choice of isomorphism

$$\ell^2 \otimes B \cong \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2 \otimes B$$

of Definition 2.1, we have $\pi(a) = 1_{\mathbb{C}^2 \otimes \ell^2} \otimes \sigma(a)$ for all $a \in A$.

Lemma 2.6 (i) implies that large representations exist for any (separable) A and B. Note that if π is large in the sense of Definition 2.9 then for any $X \subseteq A$, the subset $\pi(X) \subseteq \mathcal{L}_B$ is large in the sense of Definition 2.1. In particular, if we identify X with $\pi(X)$, the group $KK_\varepsilon(X, B)$ of Definition 2.2 makes sense.

Definition 2.10. Let C be a C^*-algebra, and let $\mathcal{X}_C$ consist of all pairs of the form (X, ε) where X is a finite subset of C_1, and $\varepsilon > 0$. Put a partial order on $\mathcal{X}_C$ by

[1] The following argument is inspired by [60, Théorème 1.5, Definition 1.6, and Remarque 1.7].

stipulating that $(X, \varepsilon) \leq (Y, \delta)$ if $\delta \leq \varepsilon$, and if for all $x \in X$ there exists $y \in Y$ with $\|x - y\| \leq \frac{1}{2}(\varepsilon - \delta)$.

A *good approximation* of C is a cofinal sequence[2] $((X_n, \varepsilon_n))_{n=1}^{\infty}$ of elements of $\mathcal{X}_C$.

Note that if $X \subseteq Y$ and $\delta \leq \varepsilon$, then $(X, \varepsilon) \leq (Y, \delta)$; in particular, this implies that $\mathcal{X}_C$ is a directed set. Note also that good approximations exist if and only if C is separable: if (ε_n) is a decreasing sequence that tends to zero, and (X_n) is an increasing sequence with dense union in C_1, then $((X_n, \varepsilon_n))_{n=1}^{\infty}$ is a good approximation; and if $((X_n, \varepsilon_n))_{n=1}^{\infty}$ is a good approximation, then $\bigcup_{n=1}^{\infty} X_n$ is a countable dense subset of C_1.

Definition 2.11. Let B be a separable C^*-algebra, and let $\mathcal{X}_{\mathcal{L}_B}$ be the directed set from Definition 2.10 above for the C^*-algebra $\mathcal{L}_B$. If $(X, \varepsilon) \leq (Y, \delta)$ and X and Y are both large in the sense of Definition 2.1, then with notation as in Definition 2.2 there is an inclusion

$$\mathcal{P}_\delta(Y, B) \subseteq \mathcal{P}_\varepsilon(X, B). \tag{2.1}$$

We call the canonical map

$$KK_\delta(Y, B) \to KK_\varepsilon(X, B)$$

induced by the inclusion in line (2.1) above a *forget control map*.

We now briefly recall some terminology from homological algebra; see for example [66, Section 3.5] or [58, Section 3] for more background on this material[3]. An *inverse system* of abelian groups consists of a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\phi_n} A_n \xrightarrow{\phi_{n-1}} A_{n-1} \xrightarrow{\phi_{n-2}} \cdots \xrightarrow{\phi_2} A_2 \xrightarrow{\phi_1} A_1.$$

Associated to such a system is a homomorphism

$$\phi : \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} A_n, \quad (a_n) \mapsto (\phi_n(a_{n+1})).$$

The *inverse limit*, denoted $\varprojlim A_n$, is defined to be the kernel of $\mathrm{id} - \phi$, and the $\varprojlim^1$-*group*, denoted $\varprojlim^1 A_n$, is defined to be the cokernel of $\mathrm{id} - \phi$. Note that if $m \geq n$, there is a canonical homomorphism $A_m \to A_n$ defined as $\phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{m-1}$. The inverse system satisfies the *Mittag–Leffler condition* if for any n there is $N \geq n$

[2] A sequence $(s_n)_{n=1}^{\infty}$ in a partially ordered set S is *cofinal* if $s_1 \leq s_2 \leq s_3 \leq \cdots$ and if for all $s \in S$ there is n such that $s \leq s_n$.

[3] Readers interested in a more sophisticated and general treatment can also see [38].

such that for all $m \geq N$, the image of the canonical map $A_m \to A_n$ equals the image of the canonical map $A_N \to A_n$.

Proposition 2.12. *Let (A_n) be an inverse system of abelian groups. If (A_n) satisfies the Mittag–Leffler condition, then $\varprojlim{}^1 A_n = 0$. Conversely, if $\varprojlim{}^1 A_n = 0$ and each A_n is countable, then the inverse system satisfies the Mittag–Leffler condition.*

Proof. It is well-known that the Mittag–Leffler condition implies vanishing of

$$\varprojlim{}^1 A_n = 0;$$

see for example [66, Proposition 3.5.7]. The converse in the case of countable groups follows from [28, Proposition on page 242]. ∎

Now, let A be a separable, unital C^*-algebra, let B be a separable C^*-algebra, and use a large representation $\pi : A \to \mathcal{L}_B$ (see Definition 2.9) to identify A with a C^*-subalgebra of $\mathcal{L}_B$. Let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a good approximation of A as in Definition 2.10, so the forget control maps of Definition 2.11 form an inverse system

$$\cdots \to KK_{\varepsilon_n}(X_n, B) \to KK_{\varepsilon_{n-1}}(X_{n-1}, B) \to \cdots \to KK_{\varepsilon_1}(X_1, B)$$

from which we define $\varprojlim KK_{\varepsilon_n}(X_n, B)$ and $\varprojlim{}^1 KK_{\varepsilon_n}(X_n, B)$ as above.

The following is [68, Proposition A.10].

Theorem 2.13. *Let A and B be separable C^*-algebras with A unital. Let*

$$\pi : A \to \mathcal{L}_B$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_B$. Let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a good approximation for A. Then, there is a short exact sequence

$$0 \to \varprojlim{}^1 KK_{\varepsilon_n}(X_n, SB) \to KK(A, B) \to \varprojlim KK_{\varepsilon_n}(X_n, B) \to 0.$$ ∎

We are now almost ready to state and prove our reformulation of the UCT. It will be convenient to use the following well-known reformulation of the UCT; see [55, p. 457] or [60, Proposition 5.3] for a proof.

Theorem 2.14. *A separable C^*-algebra A satisfies the UCT if and only if for any separable C^*-algebra B such that $K_*(B) = 0$ we have that*

$$KK(A, B) = 0.$$ ∎

Theorem 2.15. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let B be a separable C^*-algebra with $K_*(B) = 0$. Let*

$$\pi : A \to \mathfrak{L}_{SB}$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathfrak{L}_{SB}$. Then, for any (X, γ) in the set $\mathfrak{X}_A$ of Definition 2.10, there is $(Z, \varepsilon) \in \mathfrak{X}_A$ with $(X, \gamma) \preceq (Z, \varepsilon)$ and so that the forget control map

$$KK_\varepsilon(Z, SB) \to KK_\gamma(X, SB)$$

of Definition 2.11 is zero.

Proof. Assume first that A satisfies condition (i), and let X, ε, B and π be as in condition (ii). Let $((X_n, \varepsilon_n))_{n=1}^\infty$ be a good approximation of A with $X_1 = X$ and $\varepsilon_1 = \gamma$. As A satisfies the UCT and as $K_*(B) = 0$, we have $KK(A, B) = 0$. Hence, using Theorem 2.13, $\varprojlim{}^1 KK_{\varepsilon_n}(X_n, SB) = 0$. Lemma 2.4 implies that the groups $KK_{\varepsilon_n}(X_n, SB)$ are all countable, whence by Proposition 2.12, the inverse system $(KK_{\varepsilon_n}(X_n, SB))_{n=1}^\infty$ satisfies the Mittag–Leffler condition. On the other hand, as A satisfies the UCT and $K_*(SB) = 0$, we have $KK(A, SB) = 0$ by Theorem 2.14. Hence, by Theorem 2.13 again, $\varprojlim KK_{\varepsilon_n}(X_n, SB) = 0$, whence the definition of the inverse limit implies that for any n,

$$\bigcap_{m \geq n} \mathrm{Image}\big(KK_{\varepsilon_m}(X_m, SB) \to KK_{\varepsilon_n}(X_n, SB)\big) = 0.$$

The Mittag–Leffler condition implies that there is $N \geq n$ such that

$$\begin{aligned}\bigcap_{m \geq n} &\mathrm{Image}\big(KK_{\varepsilon_m}(X_m, SB) \to KK_{\varepsilon_n}(X_n, SB)\big) \\ &= \mathrm{Image}\big(KK_{\varepsilon_N}(X_N, SB) \to KK_{\varepsilon_n}(X_n, SB)\big)\end{aligned}$$

so we may conclude that the forget control map

$$KK_{\varepsilon_N}(X_N, SB) \to KK_{\varepsilon_n}(X_n, SB)$$

is zero. In particular, such an N exists for $n = 1$, and we may set $Z = X_N$ and $\varepsilon = \varepsilon_N$.

Conversely, say A satisfies condition (ii). Using Theorem 2.14, it suffices to show that if B is a separable C^*-algebra with $K_*(B) = 0$, then $KK(A, B) = 0$. Let $\pi_2 : A \to \mathfrak{L}_{S^2B}$ (respectively, $\pi_3 : A \to \mathfrak{L}_{S^3B}$) be a large representation, and use this to identify A with a C^*-subalgebra of $\mathfrak{L}_{S^2B}$ (respectively, $\mathfrak{L}_{S^3B}$). Using condition (ii) we may construct a good approximation $((X_n, \varepsilon_n))_{n=1}^\infty$ for A in the sense of Definition 2.10 such that for any n the maps

$$KK_{\varepsilon_{n+1}}(X_{n+1}, S^3B) \to KK_{\varepsilon_n}(X_n, S^3B) \tag{2.2}$$

and

$$KK_{\varepsilon_{n+1}}(X_{n+1}, S^2B) \to KK_{\varepsilon_n}(X_n, S^2B) \tag{2.3}$$

are zero. As the maps in line (2.2) are all zero, the inverse system

$$(KK_{\varepsilon_n}(X_n, S^3B))_{n=1}^{\infty}$$

satisfies the Mittag–Leffler condition, whence by Proposition 2.12 we have that

$$\varprojlim{}^1 KK_{\varepsilon_n}(X_n, S^3B) = 0.$$

On the other hand, the fact that the maps in line (2.3) are all zero and the definition of the inverse limit immediately imply that $\varprojlim KK_{\varepsilon_n}(X_n, S^2B) = 0$. Hence, in the short exact sequence

$$0 \to \varprojlim{}^1 KK_{\varepsilon_n}(X_n, S^3B) \to KK(A, S^2B) \to \varprojlim KK_{\varepsilon_n}(X_n, S^2B) \to 0$$

from Theorem 2.13 the left and right groups are zero, whence $KK(A, S^2B) = 0$. Hence, by Bott periodicity, $KK(A, B) = 0$ as desired. ∎

We include the following remark as the comparison to the existing literature might help orient some readers; it also gives a sense of why Corollary 2.7 is useful (our main use of that corollary will come later in the paper).

Remark 2.16. Theorem 2.15 can be used to deduce a weak version of a theorem of Dadarlat [21, Theorem 1.1]. Dadarlat shows that if A is a separable nuclear C^*-algebra such for any finite subset X of A and any $\varepsilon > 0$, one has a UCT subalgebra C of A such that $x \in_\varepsilon C$ for all $x \in X$, then A satisfies the UCT. Theorem 1.2 implies the special case of Dadarlat's theorem where the subalgebras C can also be taken nuclear.

To see this, note first that as a C^*-algebra satisfies the UCT (respectively, is nuclear) if and only if its unitization satisfies the UCT (respectively, is nuclear) by [55, Proposition 2.3 (a)] (respectively, by [10, Exercise 2.3.5]), we may assume that A is unital. We aim to establish the condition in Theorem 2.15 (ii). Let then B be a separable C^*-algebra with $K_*(B) = 0$. Using Corollary 2.7, there exists a large representation $\pi : A \to \mathcal{L}_{SB}$ such that the restriction of π to any unital nuclear C^*-subalgebra of A is also large. Let X be a finite subset of A_1, and let $\varepsilon > 0$. Let C be a nuclear, unital, UCT C^*-subalgebra of A such that $x \in_{\varepsilon/5} C$ for all $x \in X$. Let X' be a finite subset of C_1 such that for each $x \in X$ there is $x' \in X'$ such that

$$\|x - x'\| < 2\varepsilon/5.$$

Then, the forget control map

$$KK_{\varepsilon/5}(X', SB) \to KK_{\varepsilon}(X, B) \tag{2.4}$$

of Definition 2.11 is defined. As C satisfies the UCT, and as the restriction of π to C is also large, condition (ii) from Theorem 2.15 gives a finite subset Y of C_1 and $\delta > 0$ such that the forget control map

$$KK_\delta(Y, SB) \to KK_{\varepsilon/5}(X', SB) \tag{2.5}$$

is defined and zero. Composing the forget control maps in lines (2.4) and (2.5), we have established the condition from Theorem 2.15 (ii) for A, and are done.

It would be interesting if one could use these techniques to recover Dadarlat's theorem without the extra nuclearity assumption on the UCT subalgebras. This would seem to require better control over the representations involved; however, compare Remark 2.8 above.

2.2 The nuclear case

In this section, we prove a stronger version of Theorem 2.15 in the special case that the C^*-algebra A is nuclear. The key ingredient for this is an averaging argument due to Christensen, Sinclair, Smith, White, and Winter [17, Section 3], which in turn relies on Haagerup's theorem [33] that nuclear C^*-algebras are amenable.

Let us recall some terminology about bimodules.

Definition 2.17. Let A be a unital C^*-algebra. An *A-bimodule* is a Banach space E equipped with left and right module actions of A such that $1_A e = e 1_A = e$ for all $e \in E$, and such that $\|ae\|_E \leq \|a\|_A \|e\|_E$ and $\|ea\|_E \leq \|e\|_E \|a\|_A$ for all $a \in A$ and $e \in E$.

The following reformulation of nuclearity is implicit in [17, Section 3]; the reader is encouraged to see that reference for further background.

Lemma 2.18. *Let A be a unital C^*-algebra. Then, the following are equivalent:*

(i) *A is nuclear;*

(ii) *for any $\varepsilon > 0$ and any finite subset X of A, there exist contractions*

$$a_1, \ldots, a_n \in A$$

and scalars $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^n t_i a_i a_i^* \right\|_A < \varepsilon,$$

and such that for any A-bimodule E, any $e \in E_1$, and any $x \in X$,

$$\left\| x \left(\sum_{i=1}^n t_i a_i e a_i^* \right) - \left(\sum_{i=1}^n t_i a_i e a_i^* \right) x \right\|_E < \varepsilon. \tag{2.6}$$

Proof. We will need to recall the projective tensor product of Banach spaces. Let E and F be (complex) Banach spaces, and let $E \odot F$ denote their algebraic tensor product (over $\mathbb{C}$). The *projective norm* of $g \in E \odot F$ is defined by

$$\|g\| := \inf \sum_{i=1}^{n} \|e_i\|_E \|f_i\|_F, \tag{2.7}$$

where the infimum is taken over all ways of writing g as a sum $\sum_{i=1}^{n} e_i \otimes f_i$ of elementary tensors. The *projective tensor product* of E and F, denoted $E \mathbin{\hat{\otimes}} F$, is the completion of $E \odot F$ for the projective norm. If A is a C^*-algebra, we make $A \mathbin{\hat{\otimes}} A$ into an A-A-bimodule via the actions defined on elementary tensors by

$$a(b \otimes c) := ab \otimes c \quad \text{and} \quad (b \otimes c)a := b \otimes ca. \tag{2.8}$$

Now, it is shown in [17, Lemma 3.1][4] that a unital C^*-algebra is nuclear if and only if the following holds: "for any $\varepsilon > 0$ and any finite subset X of A, there exist contractions $a_1, \ldots, a_n \in A$ and scalars $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^{n} t_i a_i a_i^* \right\|_A < \varepsilon,$$

and such that

$$\left\| x\left(\sum_{i=1}^{n} t_i a_i \otimes a_i^* \right) - \left(\sum_{i=1}^{n} t_i a_i \otimes a_i^* \right) x \right\|_{A \hat{\otimes} A} < \varepsilon \tag{2.9}$$

for all $x \in X$." For the sake of this proof, let us call this the "CSSWW" condition. It suffices for us to show that condition (ii) is equivalent to the CSSWW condition.

First assume A satisfies condition (ii) above. Then, taking $E = A \mathbin{\hat{\otimes}} A$ and $e = 1_A \otimes 1_A$ shows that A satisfies the CSSWW condition. Conversely, say A satisfies the CSSWW condition. Let X be a finite subset of A and let $\varepsilon > 0$, and let $a_1, \ldots, a_n$ and $t_1, \ldots, t_n$ satisfy the properties in the CSSWW condition with respect to this X and ε. Let E be an A-bimodule, and $e \in E_1$. Consider the map

$$\pi : A \odot A \to E, \quad a \otimes b \mapsto aeb$$

from the algebraic tensor product (over $\mathbb{C}$) of A with itself to E. Using the definition of the projective tensor norm (line (2.7) above), it is straightforward to check that π

[4] This is based on several deep ingredients: the key points are the result of Connes [20, Corollary 2] that amenability for a C^*-algebra implies nuclearity; the converse to this due to Haagerup [33, Theorem 3.1]; and Johnson's foundational work on amenability and virtual diagonals [39, Section 1].

is contractive for that norm, whence it extends to a contractive linear map

$$\pi : A \mathbin{\hat{\otimes}} A \to E.$$

Moreover, the extended map π is clearly an A-bimodule map for the bimodule structure on $A \mathbin{\hat{\otimes}} A$ defined in line (2.8). Applying π to the expression inside the norm in line (2.9) therefore implies the inequality in line (2.6), so we are done. ∎

Remark 2.19. We will only need to apply Lemma 2.18 in the special case that the bimodule E in part (ii) is a C^*-algebra containing A as a unital C^*-subalgebra, with the bimodule actions defined by left and right multiplication. The corresponding, formally weaker, variant of condition (ii) still implies nuclearity, as we now sketch[5]. Let A be a unital C^*-algebra satisfying the variant of condition (ii) from Lemma 2.18, where E is a C^*-algebra containing A as a unital C^*-subalgebra. Let $\pi : A \to \mathcal{B}(H)$ be an arbitrary unital representation, which we use to make $\mathcal{B}(H)$ an A-bimodule. Let I be the directed set consisting of all pairs $i = (X, \varepsilon)$ where X is a finite subset of A, and $\varepsilon > 0$, and where $(X, \varepsilon) \leq (Y, \delta)$ if $X \subseteq Y$ and $\delta \leq \varepsilon$. For each $i = (X, \varepsilon) \in I$, let $a_1^{(i)}, \ldots, a_{n_i}^{(i)}$ and $t_1^{(i)}, \ldots, t_{n_i}^{(i)}$ have the properties in Lemma 2.18 (ii). For each i, define a ccp map

$$\phi_i : \mathcal{B}(H) \to \mathcal{B}(H), \quad b \mapsto \sum_{j=1}^{n_i} t_j^{(i)} \pi(a_j^{(i)}) b \pi(a_j^{(i)})^*,$$

and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be any point-ultraweak limit point of the net (ϕ_i) (such exists by [10, Theorem 1.3.7], for example). Then, one checks that ϕ is a conditional expectation from $\mathcal{B}(H)$ onto $\pi(A)'$, whence the latter is injective. As π was arbitrary, this implies that A is nuclear; indeed, applying this to the universal representation π implies that $\pi(A)'$ is injective, whence

$$A^{**} = \pi(A)''$$

is injective by [8, Theorem IV.2.2.7], whence A is nuclear by the main result of [16].

Variants of the next lemma we need are well-known; see for example the lemma on page 332 of [3], which we could have used for a purely qualitative version. For the sake of concreteness, we give a quantitative[6] version.

[5] This also gives an approach to the theorem of Connes that amenable C^*-algebras are nuclear that is maybe slightly more direct than the original argument from [20, Corollary 2]. However, it still factors through the theorem that injective von Neumann algebras are semi-discrete (see [19, Theorem 6] for the case of factors, and [65] for the general case), so cannot really be said to be genuinely simpler.

[6] The estimate it gives is optimal in some sense; to see this consider $C = M_2(\mathbb{C})$, $x = \left(\begin{smallmatrix} \delta & 0 \\ 0 & 1-\delta \end{smallmatrix}\right)$, and $c = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 2.20. *Let $\delta \in [0, \frac{1}{2})$, and let x be a self-adjoint element in a C^*-algebra C with spectrum that does not intersect the interval $(\delta, 1-\delta)$. Let χ be the characteristic function of $(\frac{1}{2}, \infty)$. Then, for any $c \in C$,*

$$\|[\chi(x), c]\| \leq \frac{1}{1-2\delta}\|[x, c]\|.$$

Proof. Let $N > \|x\|$. Let γ be the positively oriented rectangular contour in the complex plane with vertices at $\frac{1}{2} \pm iN$, and $2N \pm iN$. Then, by the holomorphic functional calculus, $\chi(x) = \frac{1}{2\pi i}\int_\gamma (z-x)^{-1}dz$. Hence, for any $c \in C$, $[\chi(x), c] = \frac{1}{2\pi i}\int_\gamma [(z-x)^{-1}, c]dz$. Applying the formula

$$[(z-x)^{-1}, c] = (z-x)^{-1}[c, x](z-x)^{-1}$$

and estimating gives

$$\|[\chi(x), c]\| \leq \frac{\|[c, x]\|}{2\pi}\int_\gamma \|(z-x)^{-1}\|^2 d|z|. \tag{2.10}$$

Let γ_1 be the side of γ described by $\{\frac{1}{2} + it \mid -N \leq t \leq N\}$, and let γ_2 be the union of the other three sides. Then, for z in the image of γ_2, the continuous functional calculus implies that

$$\|(z-x)^{-1}\| \leq (N - \|x\|)^{-1}.$$

As the length of γ_2 is $4N$, we thus see that

$$\int_{\gamma_2} \|(z-x)^{-1}\|^2 \|d|z| \leq \frac{4N}{(N-\|x\|)^2}. \tag{2.11}$$

On the other hand, for $z = \frac{1}{2} + it$ in the image of γ_1, the continuous functional calculus gives $\|(z-x)^{-1}\| \leq ((\frac{1}{2}-\delta)^2 + t^2)^{-1/2}$, whence

$$\begin{aligned}\int_{\gamma_2} \|(z-x)^{-1}\|^2 d|z| &\leq \int_{-N}^{N} \frac{1}{(\frac{1}{2}-\delta)^2 + t^2}dt \\ &\leq \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{2}-\delta)^2 + t^2}dt = \frac{\pi}{\frac{1}{2}-\delta}.\end{aligned} \tag{2.12}$$

Combining lines (2.10), (2.11), and (2.12) we get

$$\|[\chi(x), c]\| \leq \frac{\|[c, x]\|}{2\pi}\left(\frac{4N}{(N-\|x\|)^2} + \frac{\pi}{\frac{1}{2}-\delta}\right).$$

Letting $N \to \infty$ gives $\|[\chi(x), c]\| \leq \frac{\|[c,x]\|}{1-2\delta}$, which is the claimed estimate. ∎

The following lemma is our key application of Lemma 2.18.

Lemma 2.21. *Let $\varepsilon \in (0,1)$. Let B be a separable C^*-algebra, and let A be a separable, unital, nuclear C^*-algebra. Let $\pi : A \to \mathcal{L}_{SB}$ be a large representation (see Definition 2.9), and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$.*

Let X be a finite subset of A_1, and let (Y, δ) be an element of the set $\mathcal{X}_A$ of Definition 2.10 such that $(X, \varepsilon) \leq (Y, \delta)$. Then, there exists a finite subset Z of A_1 containing X and a homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z, B) \to KK_\delta(Y, B)$$

such that the following diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z,B) & & \\ \downarrow{\scriptstyle \phi_*} & \searrow & \\ KK_\delta(Y,B) & \longrightarrow & KK_\varepsilon(X,B) \end{array}$$

(where the unlabeled maps are forget control maps as in Definition 2.11) commutes.

Proof. Let X, Y, and δ be as in the statement. If $\delta \geq \varepsilon/8$, we may just take $Z = Y$ and ϕ_* the forget control map. Assume then that $\delta < \varepsilon/8$. According to Lemma 2.18 there exists contractions $a_1, \ldots, a_n \in A$ and $t_1, \ldots, t_n \in [0,1]$ such that $\sum_{i=1}^n t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^n t_i a_i a_i^* \right\|_A < \delta/4,$$

and such that for all $y \in Y$ and b in the unit ball of $\mathcal{L}_B$,

$$\left\| y\left(\sum_{i=1}^n t_i a_i b a_i^*\right) - \left(\sum_{i=1}^n t_i a_i b a_i^*\right) y \right\|_{\mathcal{L}_{SB}} < \delta/4. \tag{2.13}$$

We set $Z := X \cup \{a_1^*, \ldots, a_n^*\}$, and claim this works.

Let $p \in \mathcal{P}_{\varepsilon/8}(Z, B)$, let $e \in \mathcal{L}_B$ be the neutral projection (see Definition 2.1), and define

$$\alpha(p) := \sum_{i=1}^n t_i a_i p a_i^* + \left(e - \sum_{i=1}^n t_i a_i e a_i^*\right) \in \mathcal{L}_B.$$

As the representation is large, we may use the fixed isomorphism $\ell^2 \otimes B \cong \mathbb{C}^2 \otimes \ell^2 \otimes B$ to identify $\mathcal{L}_B$ with $M_2(\mathcal{L}_B)$ and have that with respect to this identification, operators in A are diagonal matrices, and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In particular, e commutes with all the a_i, and so we have

$$\begin{aligned} \|p - \alpha(p)\| &\leq \left\| \left(1 - \sum_{i=1}^n t_i a_i a_i^*\right) p \right\| + \sum_{i=1}^n t_i \|a_i [p, a_i^*]\| + \left\| \left(1 - \sum_{i=1}^n t_i a_i a_i^*\right) e \right\| \\ &< \frac{\delta}{4} + \frac{\varepsilon}{8} + \frac{\delta}{4}. \end{aligned} \tag{2.14}$$

As $\delta < \varepsilon/8$ and as $\varepsilon < 1$, we see that $\|p - \alpha(p)\| < \frac{1}{4}$. As p is a projection, Lemma 2.3 implies that

$$\text{spectrum}(\alpha(p)) \cap (1/4, 3/4) = \varnothing. \tag{2.15}$$

Let χ be the characteristic function of $(\frac{1}{2}, \infty)$, so χ is continuous on the spectrum of $\alpha(p)$ and we may define $\phi(p) := \chi(\alpha(p))$. The rest of the proof will be spent showing that the formula $[p] \mapsto [\phi(p)]$ defines a homomorphism

$$\phi_* : KK_{\varepsilon/6}(Z, B) \to KK_\delta(Y, B)$$

with the claimed properties.

We first claim that if $p \in \mathcal{P}_{\varepsilon/8}(Z, B)$, then $\phi(p)$ is in $\mathcal{P}_\delta(Y, B)$. Note first that

$$\alpha(p) - e = \sum_{i=1}^{n} t_i a_i (p - e) a_i^*,$$

which is in $\mathcal{K}_B$. As $\mathcal{K}_B$ is an ideal in $\mathcal{L}_B$, it follows $f(\alpha(p)) - f(e)$ is in $\mathcal{K}_B$ for any polynomial f. Letting (f_n) be a sequence of polynomials that converges uniform to χ on the spectrum of $\alpha(p)$ and letting $n \to \infty$, we see that $\chi(\alpha(p)) - e$ is in $\mathcal{K}_B$. Let now $y \in Y$ and apply the inequality in line (2.13) once with $b = p$ and once with $b = e$ (and use that $[e, y] = 0$) to deduce that

$$\|[\alpha(p), y]\| < \delta/2. \tag{2.16}$$

Lines (2.16), (2.15), and Lemma 2.20 imply that $\|[\chi(\alpha(p)), y]\| < \delta$, completing the proof that $\phi(p)$ is an element of $\mathcal{P}_\delta(Y, B)$. Moreover, it is straightforward to see that the assignment

$$\mathcal{P}_{\varepsilon/8}(Z, B) \to \mathcal{P}_\delta(Y, B), \quad p \mapsto \phi(p)$$

takes homotopies to homotopies and Cuntz sums to Cuntz sums. Hence, we do indeed get a well-defined homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z, B) \to KK_\delta(Y, B), \quad [p] \mapsto [\phi(p)]$$

as claimed.

It remains to show that the diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z, B) & & \\ \downarrow{\scriptstyle \phi_*} & \searrow & \\ KK_\delta(Y, B) & \longrightarrow & KK_\varepsilon(X, B) \end{array}$$

commutes. For this, let $p \in \mathcal{P}_{\varepsilon/8}(Z, B)$ represent a class in $KK_{\varepsilon/8}(Z, B)$, and for $t \in [0, 1]$, define

$$p_t := (1 - t)p + t\alpha(p).$$

Then, by line (2.14), we have that $\|p_t - p\| < \frac{\varepsilon}{8} + \frac{\delta}{2} < \frac{1}{4}$ for all $t \in [0,1]$, so in particular

$$\text{spectrum}(p_t) \cap (1/3, 3/4) = \varnothing \quad \text{for all } t \in [0,1]. \tag{2.17}$$

Hence, $\chi(p_t)$ is a well-defined projection for all $t \in [0,1]$. We claim that $\chi(p_t)$ is an element of $\mathcal{P}_\varepsilon(X, B)$ for all $t \in [0,1]$; as $\chi(p_1) = \chi(\alpha(p))$ and $\chi(p_0) = p$, this will complete the proof.

For this last claim, note first that $p_t - e \in \mathcal{K}_B$ for all $t \in [0,1]$, whence (analogously to the case of $\chi(\alpha(p))$ argued above) $\chi(p_t) - e \in \mathcal{K}_B$ for all $t \in [0,1]$. Moreover, for all $z \in Z$,

$$\|[p_t, z]\| \leq \|[p_t - p, z]\| + \|[p, z]\| < 2\Big(\frac{\varepsilon}{8} + \frac{\delta}{2}\Big) + \frac{\varepsilon}{8} < \frac{\varepsilon}{2},$$

where the last inequality used that $\delta < \varepsilon/8$. Hence, by line (2.17) and Lemma 2.20, $\|[\chi(p_t), z]\| < \varepsilon$ for all $z \in Z$, and so in particular for all $z \in X$. This completes the proof that $\chi(p_t) \in \mathcal{P}_\varepsilon(X, B)$ for all $t \in [0,1]$, so we are done. ∎

Corollary 2.22. *Let A be a separable, unital, nuclear C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let $\varepsilon \in (0,1)$, and let B be a separable C^*-algebra B with $K_*(B) = 0$. Let*

$$\pi : A \to \mathcal{L}_{SB}$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, for any finite subset X of A_1 there is a finite subset Z of A_1 such that $(X, \varepsilon) \leq (Z, \varepsilon/8)$ in the sense of Definition 2.10, and so that the forget control map

$$KK_{\varepsilon/8}(Z, SB) \to KK_\varepsilon(X, SB)$$

of Definition 2.11 is zero.

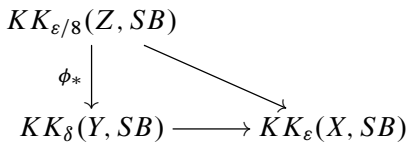
Proof. Using Theorem 2.15, it suffices to show that condition (ii) from that theorem implies condition (ii) from the current corollary (the converse is immediate). Let then ε, B, π, and X be as in the statement. Then, condition (ii) from Theorem 2.15 gives $(Y, \delta) \geq (X, \varepsilon)$ in the sense of Definition 2.10 such that the associated forget control map

$$KK_\delta(Y, SB) \to KK_\varepsilon(X, SB)$$

of Definition 2.11 is zero. Lemma 2.21 then gives a finite subset Z of A_1 containing X and a homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z, SB) \to KK_\delta(Y, SB), \quad [p] \mapsto [\phi(p)]$$

such that the following diagram



commutes (the unlabeled arrows are forget control maps). Hence, the diagonal forget control map in the above diagram is zero, which is what we wanted to show. ∎

Chapter 3

Flexible models for controlled KK-theory

In this section (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to give flexible models for controlled KK-theory that will be useful for computations. Contrary to the usual conventions of C^*-algebra K-theory, we base our new even and odd groups on idempotents and invertibles rather than projections and unitaries. The extra flexibility this allows is very useful for computations. The main reason for not writing the whole paper using the more flexible model is that we previously established Theorem 2.13 in [68] using the version of controlled KK-theory from Definition 2.2 above, so need to use that model where we are directly applying Theorem 2.13. Moreover, we need the results from Chapter 4 in the current paper (which are also independently needed in Chapter 6) to relate the two models.

3.1 The even case

Our goal in this section is to define a variant of the controlled KK-theory groups of Chapter 2, but based on idempotents rather than projections. For the next definition, we recall that C^+ denotes the unitization of a C^*-algebra C, and that if $a \in M_n(C)$ and $b \in M_m(C)$ are matrices over a C^*-algebra, then $a \oplus b$ denotes the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in $M_{n+m}(C)$.

Definition 3.1. Let B be a separable C^*-algebra, let X be a subset[1] of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ to be the collection of pairs (p, q) of idempotents in $M_n(\mathcal{K}_B^+)$ satisfying the following conditions:

(i) $\|p\| \leq \kappa$ and $\|q\| \leq \kappa$;

(ii) $\|[p, x]\| < \varepsilon$ and $\|[q, x]\| < \varepsilon$ for all $x \in X$;

(iii) the classes $[\sigma(p)], [\sigma(q)] \in K_0(\mathbb{C})$ defined by the images of p and q under the canonical quotient map $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ are the same.

[1] Unlike Definition 2.2, we do not require X to be "large" in the sense of Definition 2.1. Essentially, largeness is needed to ensure that the sets $KK_\varepsilon(X, B)$ of Definition 2.2 are groups; we show the sets we define in Definition 3.1 are groups by using matrix arguments and a weaker equivalence relation in this definition.

Define

$$\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B) := \bigsqcup_{n=1}^{\infty} \mathcal{P}_{n,\kappa,\varepsilon}(X,B),$$

i.e., $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ is the *disjoint* union of all the sets $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$.

Equip each $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$ with the norm topology it inherits from $M_n(\mathcal{L}_B) \oplus M_n(\mathcal{L}_B)$, and equip $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ with the disjoint union topology. Let $\sim$ be the equivalence relation on $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ generated by the following relations:

(i) $(p,q) \sim (p \oplus r, q \oplus r)$ for any element $(r,r) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ with both components the same;

(ii) $(p_1,q_1) \sim (p_2,q_2)$ whenever these elements are in the same path component of $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$.[2]

Define $KK^0_{\kappa,\varepsilon}(X,B)$ to be equal as a set to $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)/\sim$, and provisionally define a binary operation $+$ on $KK^0_{\kappa,\varepsilon}(X,B)$ by

$$[p_1,q_1] + [p_2,q_2] := [p_1 \oplus q_1, p_2 \oplus q_2].$$

The next lemma is essentially the same as [68, Lemma A.21].

Lemma 3.2. *With notation as in Definition 3.1, $KK^0_{\kappa,\varepsilon}(X,B)$ is a well-defined abelian group with identity element the class $[0,0]$ of the zero idempotent.*

Proof. Checking directly from the definitions shows that $KK^0_{\kappa,\varepsilon}(X,B)$ is a well-defined (associative) monoid with identity element the class $[0,0]$. A standard rotation homotopy shows that $KK^0_{\kappa,\varepsilon}(X,B)$ is commutative. To complete the proof we need to show that any element $[p,q]$ has an inverse. We claim that this is given by $[q,p]$. Indeed, applying the rotation homotopy

$$\left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\right), \quad t \in [0,\pi/2]$$

shows that $(p \oplus q, q \oplus p) \sim (p \oplus q, p \oplus q)$, and the element $(p \oplus q, p \oplus q)$ is equivalent to $(0,0)$ by definition of the equivalence relation. ∎

The following lemma gives a useful description of cycles $(p,q) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ that define the zero class in $KK^0_{\kappa,\varepsilon}(X,B)$.

Lemma 3.3. *With notation as in Definition 3.1, let $(p,q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X,B)$, and assume that $[p,q] = 0$ in $KK^0_{\kappa,\varepsilon}(X,B)$. Then, there is $m \in \mathbb{N}$ and an element (s,s) of $\mathcal{P}_{n+2m,\kappa,\varepsilon}(X,B)$ such that $(p \oplus 1_m \oplus 0_m, q \oplus 1_m \oplus 0_m)$ is in the same path component of $\mathcal{P}_{n+2m,2\kappa,\varepsilon}(X,B)$ as (s,s).*

[2] Equivalently, both are in the same $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$, and are in the same path component of this set.

Proof. For elements (p_1, q_1) and (p_2, q_2) in $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ let us write $(p_1, q_1) \to (p_2, q_2)$ if

$$(p_2, q_2) = (p_1 \oplus r, q_1 \oplus r)$$

for some $(r, r) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$; $(p_1, q_1) \overset{h}{\sim} (p_2, q_2)$ if there is a path connecting these elements; and $(p_1, q_1) \leftarrow (p_2, q_2)$ if $(p_2, q_2) \to (p_1, q_1)$. Then, $[p, q] = 0$ means that there is some sequence of moves from $\{\to, \leftarrow, \overset{h}{\sim}\}$ starting at (p, q) and finishing at $(0, 0)$. It is not difficult to see the following: any time a move from $\{\to, \leftarrow, \overset{h}{\sim}\}$ is consecutively repeated we may replace it by a single move of the same type; any occurrence of "$\overset{h}{\sim}\to$" may be replaced by an occurrence of "$\to\overset{h}{\sim}$"; any occurrence of "$\leftarrow\overset{h}{\sim}$" may be replaced by an occurrence of "$\overset{h}{\sim}\leftarrow$"; any occurrence of "$\leftarrow\to$" or "$\leftarrow\overset{h}{\sim}\to$" may be replaced by "$\to\overset{h}{\sim}\leftarrow$" (we leave the details to the reader in each case). Using these replacements, we see that our moves relating (p, q) to $(0, 0)$ may be assumed to be of the form

$$(p, q) \to\overset{h}{\sim}\leftarrow (0, 0),$$

or in other words that there are elements (r, r) and (t, t) in $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ such that $(p \oplus r, q \oplus r)$ is homotopic to (t, t).

To complete the proof, note then that $(p \oplus r \oplus 1 - r, q \oplus r \oplus 1 - r)$ is homotopic to $(t \oplus 1 - r, t \oplus 1 - r)$. For $t \in [0, \pi/2]$, define

$$r_t := \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 - r \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

so $(r_t)_{t\in[0,\pi/2]}$ is a path connecting $r \oplus 1 - r$ and $1 \oplus 0$. One computes that $\|r_t\| \leq 1 + \kappa \leq 2\kappa$ for all t, and that $\|[r_t, x]\| < \varepsilon$ for all $x \in X$. Hence, with $s = t \oplus 1 - r$ we get the claimed result. ∎

We will need a more general variation of Definitions 2.10 and 2.11.

Definition 3.4. Let C be a C^*-algebra. Let $\mathcal{X}'_C$ consist of all triples of the form (X, κ, ε) where X is a finite subset of the unit ball of C, $\kappa \geq 1$, and $\varepsilon > 0$. Put a partial order on $\mathcal{X}'_C$ by $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ if $\delta \leq \varepsilon$, $\lambda \leq \kappa$ and if for all $x \in X$ there exists $y \in Y$ with $\|x - y\| \leq \frac{1}{2\lambda}(\varepsilon - \delta)$.

Let now B be a separable C^*-algebra. Then, if $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ in $\mathcal{X}'_{\mathfrak{L}_B}$, one checks that for each n we have

$$\mathcal{P}_{n\lambda,\delta}(Y, B) \subseteq \mathcal{P}_{n,\kappa,\varepsilon}(X, B). \tag{3.1}$$

We call the canonical map

$$KK^0_{\lambda,\delta}(Y, B) \to KK^0_{\kappa,\varepsilon}(X, B)$$

induced by the inclusions in line (3.1) above a *forget control map*.

3.2 The odd case

Our goal in this section is to introduce an odd parity version of the controlled KK-theory groups of the previous section. For the statement, recall that C^+ denotes the unitization of a C^*-algebra C.

Definition 3.5. Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ to be the subset of those invertible elements u in $M_n(\mathcal{K}_B^+)$ satisfying the following conditions:

(i) $\|u\| \leq \kappa$ and $\|u^{-1}\| \leq \kappa$;

(ii) $\|[u, x]\| < \varepsilon$ and $\|[u^{-1}, x]\| < \varepsilon$ for all $x \in X$.

Define

$$\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B) := \bigsqcup_{n=1}^{\infty} \mathcal{U}_{n,\kappa,\varepsilon}(X, B),$$

i.e., $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ is the *disjoint* union of all the sets $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$.

Equip each $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ with the norm topology it inherits from $M_n(\mathcal{L}_B)$, and equip $\bigsqcup_{n=1}^{\infty} \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ with the disjoint union topology. Define an equivalence relation on $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ to be generated by the following relations:

(i) for any $k \in \mathbb{N}$, if $1_k \in \mathcal{U}_{k,\kappa,\varepsilon}(X, B)$ is the identity element, then

$$u \sim u \oplus 1_k;$$

(ii) $u_1 \sim u_2$ if both are elements of the same path component of $\mathcal{U}_{\infty,2\kappa,\varepsilon}(X,B)$.[3]

Define $KK^1_{\kappa,\varepsilon}(X, B)$ to be $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)/\sim$, and provisionally define a binary operation $+$ on $KK^1_{\kappa,\varepsilon}(X, B)$ by

$$[u_1] + [u_2] := [u_1 \oplus u_2].$$

Lemma 3.6. *With notation as in Definition 3.5, $KK^1_{\kappa,\varepsilon}(X,B)$ is a well-defined abelian group with identity element the class $[1_B]$ of the unit of B.*

Proof. It is straightforward to check that $KK^1_{\kappa,\varepsilon}(X, B)$ is a monoid, and the class $[1]$ is neutral by definition. A standard rotation homotopy shows that $KK^1_{\kappa,\varepsilon}(X, B)$ is commutative. It remains to show that inverses exist. We claim that for $u \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$, the inverse of the class $[u]$ is given by $[u^{-1}]$. Indeed, consider the homotopy

$$u_t := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}, \quad t \in [0, \pi/2].$$

[3] Equivalently, both are in the same $\mathcal{U}_{n,2\kappa,\varepsilon}(X, B)$, and are in the same path component of this set. Notice also the switch from κ to 2κ here, which is needed for our proof that $KK^1_{\kappa,\varepsilon}(X, B)$ is a group.

This connects $u \oplus u^{-1}$ and 1_{2k}, so it suffices to show that this passes through $\mathcal{U}_{2n,2\kappa,\varepsilon}(X, B)$. For the commutator condition, we compute that for $a \in X$ and $t \in [0, 2\pi]$

$$[a, u_t] = \begin{pmatrix} [a, u] & 0 \\ 0 & [u^{-1}, a] \end{pmatrix} \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) \\ \cos(t)\sin(t) & -\cos^2(t) \end{pmatrix}.$$

The scalar matrix on the right has norm $|\cos(t)|$, and the matrix on the left has norm at most $\max\{\|[a, u]\|, \|[a, u^{-1}]\|\} < \varepsilon$, so $\|[a, u_t]\| < \varepsilon$. For the norm condition, we compute that

$$u_t = \begin{pmatrix} u & 0 \\ 0 & -u^{-1} \end{pmatrix} \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) \\ \cos(t)\sin(t) & -\cos^2(t) \end{pmatrix} + \begin{pmatrix} \sin^2(t) & -\cos(t)\sin(t) \\ \cos(t)\sin(t) & \sin^2(t) \end{pmatrix}.$$

The first scalar matrix appearing above has norm $|\cos(t)|$, and the second has norm $|\sin(t)|$. We thus have that $\|u_t\| \leq \kappa|\cos(t)| + |\sin(t)|$, which is at most[4] 2κ as required. ∎

Definition 3.7. Let C be a C^*-algebra, and let $\mathcal{X}'_C$ be the directed set of Definition 3.4 above. Let B be a separable C^*-algebra. Then, if $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ in $\mathcal{X}'_{\mathcal{L}_B}$, one checks that for each n we have

$$\mathcal{U}_{n,\lambda,\delta}(Y, B) \subseteq \mathcal{U}_{n,\kappa,\varepsilon}(X, B) \tag{3.2}$$

for all n. We call the canonical map

$$KK^1_{\lambda,\delta}(Y, B) \subseteq KK^1_{\kappa,\varepsilon}(X, B)$$

induced by the inclusions in line (3.2) above a *forget control map*.

[4] We suspect the optimal estimate is κ – this is the case if u is normal, for example – but were unable to do better than $\sqrt{1 + \kappa^2}$ in general.

Chapter 4

Homotopies, similarities, and normalization

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal is to establish some technical lemmas about the controlled KK-groups $KK^0_{\kappa,\varepsilon}(X, B)$ and $KK^1_{\kappa,\varepsilon}(X, B)$ and the underlying sets of cycles $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ and $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ from Definitions 3.1 and 3.5 respectively. These are all variants of standard facts from C^*-algebra K-theory, but the arguments are more involved as we need to do extra work to control commutator estimates. Some of the material is adapted from the foundational work of Oyono-Oyono and the second author on controlled K-theory [47]; those authors work in the "dual" setting to us in some sense, and similar techniques are often useful.

Most of the results in this chapter come with explicit estimates. We have generally not tried to get optimal estimates, but as it might be useful for future work we have tried to point out where one might expect the estimates to be optimal where this is simple to do.

4.1 Background on idempotents

In this section we look at idempotents in C^*-algebras and their relationship to projections. Most of this is well-known; nonetheless, we give proofs for the sake of completeness where we could not find a good reference.

To establish notation, let us first note that if $p \in \mathcal{B}(H)$ is an idempotent, then with respect to the decomposition $H = \mathrm{Image}(p) \oplus \mathrm{Image}(p)^\perp$, p has a matrix representation

$$p = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

for some $a \in \mathcal{B}(\mathrm{Image}(p)^\perp, \mathrm{Image}(p))$; conversely, any operator admitting a matrix of this form with respect to some orthogonal direct sum decomposition of the underlying Hilbert space defines an idempotent.

Lemma 4.1. *If p is an idempotent bounded operator on a Hilbert space that is neither zero nor the identity, then*

$$\|1 - p\| = \|p\| \quad \textit{and} \quad \|p - p^*\| \leq \|p\|.$$

Proof. Writing p as in line (4.1) (and using that neither Image(p) nor Image$(p)^{\perp}$ are the zero subspace), we compute that

$$\|p\|^2 = \|pp^*\| = \|1 + aa^*\| = 1 + \|a\|^2 \tag{4.2}$$

and moreover that

$$\|1 - p\|^2 = \|(1-p)^*(1-p)\| = \|1 + a^*a\| = 1 + \|a\|^2 = \|p\|^2.$$

Looking now at $p - p^*$, we see that

$$(p-p^*)(p-p^*)^* = \begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix}\begin{pmatrix} 0 & -a \\ a^* & 0 \end{pmatrix} = \begin{pmatrix} aa^* & 0 \\ 0 & a^*a \end{pmatrix},$$

whence $\|p - p^*\|^2 = \|a\|^2 \leq \|p\|^2$. ∎

Corollary 4.2. *If $\kappa \geq 1$, and p is any idempotent in a C^*-algebra with $\|p\| \leq \kappa$, then $\|1 - p\| \leq \kappa$, $\|p - p^*\| \leq \kappa$, and $\|2p - 1\| \leq 2\kappa$.*

Proof. The estimates for $\|1 - p\|$ and $\|p - p^*\|$ are immediate from Lemma 4.1 (and direct checks for the degenerate cases $p = 0$ and $p = 1$). The estimate for $2p - 1$ follows as

$$2p - 1 = p - (1 - p).$$ ∎

It will be convenient to formalize a standard construction in C^*-algebra K-theory for turning idempotents into projections (compare for example [7, Proposition 4.6.2]).

Definition 4.3. Let p be an idempotent in a C^*-algebra C. Define

$$z := 1 + (p - p^*)(p^* - p) \in C^+,$$

and note that $z \geq 1_{C^+}$ so z is invertible. Define

$$r := pp^*z^{-1},$$

which is an element of C. We call r the *projection*[1] *associated to* p.

Remark 4.4. If C is a concrete C^*-algebra and p is an idempotent with matrix representation as in line (4.1), then one computes that the associated projection has matrix representation

$$r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

with respect to the same decomposition of the underlying Hilbert space. In particular, r is the projection with the same image as the idempotent p.

[1] It will be shown to be a projection in the next lemma.

Lemma 4.5. *Let p be an idempotent in a C^*-algebra C, and assume that $\|p\| \leq \kappa$ for some $\kappa \geq 1$. Let r be the projection associated to p as in Definition 4.3, and for $t \in [0,1]$ define $r_t := (1-t)p + tr$. Then, the following hold:*

(i) *The element r is a projection in C, and there is an invertible $u \in C^+$ such that $upu^{-1} = r$. Moreover, u and its inverse have norm at most $1 + \|p\|$, and u is connected to the identity through a path of invertibles such that all the invertibles in the path and all of their inverses have norm at most $1 + \|p\|$.*

(ii) *Each r_t is an idempotent such that $\|r_t\| \leq \kappa$ for all t, and the map $t \mapsto r_t$ is κ-Lipschitz.*

(iii) *For any $c \in C$ and $t \in [0,1]$ we have*

$$\|[r_t, c]\| \leq (1+2t)\|[p,c]\| + t\|[p,c^*]\|.$$

(iv) *The map*

$$\{p \in C \mid p = p^2\} \to \{p \in C \mid p = p^2 = p^*\}$$

that takes an idempotent to its associated projection is 1-Lipschitz.

Proof. Part (i) as in line (4.1), we may write $p = \left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$, and note as in line (4.2) that $\|p\| = \sqrt{1 + \|a\|^2}$, so in particular $\|a\| \leq \|p\|$. Using the discussion in Remark 4.4 we see that $u = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ satisfies $upu^{-1} = r$, and that the path $u_t = \left(\begin{smallmatrix} 1 & ta \\ 0 & 1 \end{smallmatrix}\right)$ connects u to the identity through invertibles of norm at most $1 + \|ta\| \leq 1 + \|p\|$. The claims on the norms of the inverses follow as $\left(\begin{smallmatrix} 1 & ta \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & -ta \\ 0 & 1 \end{smallmatrix}\right)$.

(Or see for example the proof of [7, Proposition 4.6.2]).

For part (ii), we write p as in line (4.1), note that $\|a\| \leq \kappa$, and also that r has the matrix representation as in line (4.3). This implies the claimed properties.

For part (iii), we again write p as a matrix as in line (4.1). Let $c \in C$, and with respect to the same decomposition of the underlying Hilbert space, let us write

$$c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Then, one computes that

$$[p,c] = \begin{pmatrix} ac_{21} & c_{12} + ac_{22} - c_{11}a \\ -c_{21} & -c_{21}a \end{pmatrix}. \tag{4.4}$$

As the conditional expectation that sends a matrix to its diagonal is contractive, we have

$$\left\| \begin{pmatrix} ac_{21} & 0 \\ 0 & -c_{21}a \end{pmatrix} \right\| \leq \|[p,c]\|$$

and combining this with line (4.4) gives

$$\left\| \begin{pmatrix} 0 & c_{12} + ac_{22} - c_{11}a \\ -c_{21} & 0 \end{pmatrix} \right\| \leq 2\|[p, c]\|. \tag{4.5}$$

One computes that the top right entry of $[p - p^*, c]$ is $ac_{22} - c_{11}a$, whence

$$\|ac_{22} - c_{11}a\| \leq \|[p - p^*, c]\| \leq \|[p, c]\| + \|[p, c^*]\|.$$

This and line (4.5) together imply that

$$\left\| \begin{pmatrix} 0 & c_{12} \\ -c_{21} & 0 \end{pmatrix} \right\| \leq 3\|[p, c]\| + \|[p, c^*]\|. \tag{4.6}$$

As r has the matrix representation from line (4.3), the left-hand side of the inequality in line (4.6) equals $\|[r, c]\|$, and so line (4.6) can be rewritten as the inequality $\|[r, c]\| \leq 3\|[p, c]\| + \|[p, c^*]\|$. As $r_t = (1 - t)p + tr$, this implies the claimed estimate.

For part (iv) we may assume that C is a concrete C^*-algebra. As noted in Remark 4.4, the projection r associated to an idempotent p is then simply the orthogonal projection with the same image as p. In this language, part (iv) is [41, Chapter One, Theorem 6.35]. ∎

4.2 From similarities to homotopies

Our goal in this short section is to establish an analogue of the standard K-theoretic fact that similar idempotents are homotopic, at least up to increasing matrix sizes. Compare for example [7, Proposition 4.4.1].

Proposition 4.6. *Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, and let $\kappa \geq 1$ and $\varepsilon > 0$. Let (p_0, q) and (p_1, q) be elements of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$, and let $u \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ be such that $up_0u^{-1} = p_1$. Then, the elements $(p_0 \oplus 0_n, q \oplus 0_n)$ and $(p_1 \oplus 0_n, q \oplus 0_n)$ are in the same path component of $\mathcal{P}_{2n,\kappa^3,3\kappa^2\varepsilon}(X, B)$, and in particular, (p_0, q) and (p_1, q) define the same class in $KK^0_{\kappa^3,3\kappa^2\varepsilon}(X, B)$.*

The analogous statement holds with the roles of the first ("p") and second ("q") components reversed.

Proof. Define

$$v_t := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in M_{2n}(\mathcal{K}_B^+).$$

Then, the path

$$t \mapsto (v_t(p_0 \oplus 0_n)v_t^{-1}, q \oplus 0_n), \quad t \in [0, \pi/2]$$

connects $(p_0 \oplus 0_n, q \oplus 0_n)$ to $(p_1 \oplus 0_n, q \oplus 0_n)$ through $\mathcal{P}_{2n,\kappa^3,3\kappa^2\varepsilon}(X, B)$. We leave the direct checks involved to the reader. ∎

4.3 Normalization

Our goal in this section is to show that cycles for $KK^0_{\kappa,\varepsilon}(X, B)$ and $KK^1_{\kappa,\varepsilon}(X, B)$ can be assumed to have prescribed "scalar part", at least up to some deterioration of κ and ε.

The following lemma is well-known without the Lipschitz condition[2]; see for example [7, Theorem 4.6.7] or [36, Corollary 4.1.8].

Lemma 4.7. *Let $L > 0$. Then, if $(p_t)_{t\in[0,1]}$ is an L-Lipschitz path of projections in a unital C^*-algebra C, there is a $(3L)$-Lipschitz path $(u_t)_{t\in[0,1]}$ of unitaries in C such that $u_0 = 1$, and such that $p_t = u_t p_0 u_t^*$ for all $t \in [0, 1]$.*

We need a preliminary lemma.

Lemma 4.8. *Let $\eta \geq 1$, and let C be a unital C^*-algebra. Then, the map*

$$\{c \in C \mid c \geq \eta^{-1}\} \to C, \quad c \mapsto c^{-1/2}$$

is $\frac{1}{2}\eta^{3/2}$-Lipschitz[3].

Proof. For any positive real number t, one has

$$t^{-1/2} = \frac{2}{\pi}\int_0^\infty (\lambda^2 + t)^{-1} d\lambda,$$

whence for any positive invertible elements $c, d \in C$

$$c^{-1/2} - d^{-1/2} = \frac{2}{\pi}\int_0^\infty \big((\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1}\big) d\lambda. \tag{4.7}$$

Using the formula

$$(\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1} = (\lambda^2 + c)^{-1}(d - c)(\lambda^2 + d)^{-1}$$

and assuming that $c \geq \eta^{-1}$ and $d \geq \eta^{-1}$, the continuous functional calculus implies that

$$\|(\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1}\| \leq \|c - d\|(\lambda^2 + \eta^{-1})^{-2}.$$

[2]The constant 3 appearing in the statement is not optimal; one can see from the proof that 3 can be replaced with $2 + \varepsilon$, for any $\varepsilon > 0$. We do not know what the optimal constant is.

[3]The constant is optimal in some sense; this follows as the absolute value if the derivative of the function $t \mapsto t^{-1/2}$ on $[\eta^{-1}, \infty)$ has maximum value $\frac{1}{2}\eta^{3/2}$.

This inequality and line (4.7) imply that

$$\|c^{-1/2} - d^{-1/2}\| \leq \frac{2\|c-d\|}{\pi} \int_0^\infty (\lambda^2 + \eta^{-1})^{-2} d\lambda.$$

The integral on the right-hand side equals $(\pi\eta^{3/2})/4$, whence the result. ∎

Proof of Lemma 4.7. We first claim that it suffices to show we can choose a $\delta > 0$ such that if $[t_1, t_2]$ is a sub-interval of $[0,1]$ of length at most δ, and $t \mapsto p_t$ is a projection-valued L-Lipschitz function on $[t_1, t_2]$, then there is a unitary-valued $(3L)$-Lipschitz function $t \mapsto u_t$ on $[t_1, t_2]$ such that $u_0 = 1$ and $p_t = u_t p_0 u_t^*$ for all $t \in [t_1, t_2]$. Indeed, if we can do this, then let $0 = t_0 < t_1 < \cdots < t_N = 1$ be a partition of the interval $[0,1]$ such that each subinterval has length at most δ, and for each $i \in \{0, \ldots, N-1\}$ choose a unitary-valued $(3L)$-Lipschitz function $t \mapsto u_t^{(i)}$ on $[t_i, t_{i+1}]$ such that $u_{t_i}^{(i)} = 1$ and $p_t = u_t^{(i)} p_{t_i} (u_t^{(i)})^*$ for all $t \in [t_i, t_{i+1}]$. The function on $[0,1]$ defined on each subinterval $[t_i, t_{i+1}]$ by

$$t \mapsto u_t^{(i)} u_{t_i}^{(i-1)} u_{t_{i-1}}^{(i-2)} \cdots u_{t_1}^{(0)}$$

then has the right properties to establish the lemma.

Let us then establish the statement in the claim. Let $\varepsilon > 0$ be small enough that

$$(1 - (2+\varepsilon)\varepsilon)^{-1/2} + (1+\varepsilon)^2 (1 - (2+\varepsilon)\varepsilon)^{-3/2} \leq 3,$$

and let $\delta > 0$ be such that if $t, s \in [0,1]$ satisfy $|t - s| \leq \delta$, then $\|p_s - p_t\| < \varepsilon$. Let $[t_1, t_2]$ be an interval of length at most δ. For $t \in [t_1, t_2]$, define

$$x_t := p_t p_{t_1} + (1 - p_t)(1 - p_{t_1})$$

and note that

$$\|x_t - 1\| = \|(2p_t - 1)(p_{t_1} - p_t)\| \leq \|2p_t - 1\| \|p_{t_1} - p_t\| < \varepsilon,$$

and so each x_t is invertible, $\|x_t\| < 1 + \varepsilon$, and also $\|x_t^{-1}\| < (1-\varepsilon)^{-1}$ by the Neumann series formula for the inverse. One computes that $x_t p_{t_1} = p_t p_{t_1} = p_t x_t$, and so $x_t p_{t_1} x_t^{-1} = p_t$. Moreover, $p_{t_1} x_t^* = x_t^* p_t$, and so $p_{t_1} x_t^* x_t = x_t^* p_t x_t = x_t^* x_t p_{t_1}$, i.e., $x_t^* x_t$ commutes with p_{t_1}. If we define $w_t := x_t (x_t^* x_t)^{-1/2}$, we have that w_t is unitary and moreover

$$w_t p_{t_1} w_t^{-1} = x_t (x_t^* x_t)^{-1/2} p_{t_1} (x_t^* x_t)^{1/2} x_t^{-1} = x_t p_{t_1} x_t^{-1} = p_t.$$

It remains to show that the path defined on $[t_1, t_2]$ by $t \mapsto w_t$ is $(3L)$-Lipschitz.

We first note that for $s, t \in [t_1, t_2]$, we have that

$$\|x_s - x_t\| = \|(p_t - p_s)(2p_{t_1} - 1)\| \leq \|p_t - p_s\| \leq L|s - t| \tag{4.8}$$

by assumption that (p_t) is L-Lipschitz. Using that $\|x_t\| < 1 + \varepsilon$, this implies that for any $s, t \in [t_1, t_2]$

$$\|x_t^* x_t - x_s^* x_s\| \leq \|x_t^* - x_s^*\|\|x_t\| + \|x_s^*\|\|x_t - x_s\| < 2(1+\varepsilon)L|s-t|.$$

Moreover, $\|1 - x_t^* x_t\| < (2+\varepsilon)\varepsilon$, whence $1 - (2+\varepsilon)\varepsilon \leq x_t^* x_t$ and so in particular

$$\|(x_t^* x_t)^{-1/2}\| \leq (1-(2+\varepsilon)\varepsilon)^{-1/2} \quad \text{for all } t \in [t_1, t_2]. \tag{4.9}$$

Hence, moreover Lemma 4.8 (with $\eta = (1-(2+\varepsilon))^{-1}$) implies that for any $s, t \in [t_1, t_2]$

$$\|(x_t^* x_t)^{-1/2} - (x_s^* x_s)^{-1/2}\| \leq (1-(2+\varepsilon)\varepsilon)^{-3/2}(1+\varepsilon)L|s-t|. \tag{4.10}$$

Lines (4.8), (4.10), and (4.9) combined with the fact that

$$\|x_t\| < 1 + \varepsilon$$

for all $t \in [t_1, t_2]$ implies that for any $s, t \in [t_1, t_2]$

$$\begin{aligned}\|w_t - w_s\| &\leq \|x_t - x_s\|\|(x_t^* x_t)^{-1/2}\| + \|x_s\|\|(x_t^* x_t)^{-1/2} - (x_s^* x_s)^{-1/2}\| \\ &\leq (1-(2+\varepsilon)\varepsilon)^{-1/2} L|s-t| + (1+\varepsilon)^2(1-(2+\varepsilon)\varepsilon)^{-3/2}L|s-t|\end{aligned}$$

which implies the desired estimate by choice of ε. ∎

For the statement of the next definition, recall that for $l \in \{1, \ldots, n\}$, we let $1_l \in M_n(\mathbb{C})$ be the rank l projection with l ones in the top-left part of the diagonal and zeros elsewhere.

Definition 4.9. With notation as in Definition 3.1, define

$$\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B) := \big\{(p,q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X, B) \mid \exists l \in \mathbb{N} \text{ such that } (p,q) - (1_l, 1_l) \text{ is in } M_n(\mathcal{K}_B) \oplus M_n(\mathcal{K}_B)\big\}.$$

Define $\mathcal{P}^1_{\infty,\kappa,\varepsilon}(X, B)$ to be the disjoint union of these sets as n ranges over $\mathbb{N}$.

Here, is the first of our main goals for this section; it allows control of the "scalar part" of cycles for $KK^0_{\kappa,\varepsilon}(X, B)$.

Proposition 4.10. *Let B be a separable C^*-algebra. Let X be a self-adjoint[4] subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, let $\kappa \geq 1$, and let $n \in \mathbb{N}$.*

(i) *Any element $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ is in the same path component of $\mathcal{P}_{n,4\kappa^3,\varepsilon}(X, B)$ as an element of $\mathcal{P}^1_{n,4\kappa^3,\varepsilon}(X, B)$[5].*

[4] We mean here that $X = X^*$, not the stronger assumption that every $x \in X$ is self-adjoint.

[5] If $\kappa = 1$, one can replace $4\kappa^3$ with 1 in the statement; we leave the details to the reader.

(ii) *If two elements* (p_0, q_0) *and* (p_1, q_1) *of* $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ *are connected by a path in* $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$*, then they are connected by a path in* $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$*. Moreover, if* $L \geq 1$ *is such that there is an* L*-Lipschitz path in* $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ *connecting* (p_0, q_0) *and* (p_1, q_1)*, then there is a* $(20\kappa L)$*-Lipschitz path in* $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$ *connecting* (p_0, q_0) *and* (p_1, q_1)*.*

Proof of Proposition 4.10. For part (i), we assume that (p, q) is an element of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. Hence, by definition of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$, if $\mathcal{K}_B^+$ is the unitization of $\mathcal{K}_B$ and $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ is the canonical quotient map then the classes $[\sigma(p)]$ and $[\sigma(q)]$ in $K_0(\mathbb{C})$ are the same, so in particular the idempotents $\sigma(p)$ and $\sigma(q)$ have the same rank. Using Lemma 4.5 (i), there are paths of invertibles $(u_t)_{t\in[0,1]}$ and $(v_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ and projections r, s such that $u_1 = v_1$ is the identity, such that $u_0 r u_0^{-1} = \sigma(p)$, such that $v_0 s v_0^{-1} = \sigma(q)$, and such that the norms of all the u_t, all the v_t and their inverses are all at most $1 + \kappa \leq 2\kappa$. On the other hand, r and s have the same rank, whence there are paths of unitaries $(u_t)_{t\in[1,2]}$ and $(v_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that $u_1 = v_1$ is the identity, and such that $u_2 r u_2^* = 1_l$, and $v_2 s v_2^* = 1_l$. As scalar matrices commute with X, the path $((u_t p u_t^{-1}, v_t q v_t^{-1}))_{t\in[0,2]}$ passes through $\mathcal{P}_{n,4\kappa^3,\varepsilon}(X, B)$, and connects (p, q) to an element of $\mathcal{P}^1_{n,4\kappa^3,\varepsilon}(X, B)$ as required.

For part (ii), we just look at the statement involving Lipschitz paths; the case of general continuous paths follows (in a simpler way) from the same arguments, and is left to the reader. Assume that (p_0, q_0) and (p_1, q_1) are elements of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ that are connected by an L-Lipschitz path that passes through $\mathcal{P}_{n,\kappa\cdot\varepsilon}(X, B)$. In particular, there exists $l \in \mathbb{N}$ such that $\sigma(p_0) = \sigma(q_0) = 1_l = \sigma(p_1) = \sigma(q_1)$. Let r_0 be the projection associated to p_0 as in Definition 4.3. As in Lemma 4.5 (ii), the path defined for $t \in [0, 1]$ by $t \mapsto (1 - t)p_0 + t r_0$ is κ-Lipschitz and connects p_0 and r_0 through idempotents of norm at most κ. Moreover, Lemma 4.5 (iii) implies that for all $x \in X$ and all $t \in [0, 1]$

$$\|[(1 - t)p_0 + t r_0, x]\| \leq (1 + 2t)\|[p_0, x]\| + t\|[p_0, x^*]\|.$$

As $X = X^*$, this implies that $\|[(1 - t)p_0 + t r_0, x]\| < 4\varepsilon$ for all $x \in X$, and all $t \in [0, 1]$. Note also that $\sigma((1 - t)p_0 + t r_0) = 1_l$ for all t. Similarly, we get s_0 which has the same properties with respect to q_0. We have thus shown that (p_0, q_0) is connected to the element (r_0, s_0) via a κ-Lipschitz path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$. Completely analogously, (p_1, q_1) is connected to its associated projection (r_1, s_1) via a κ-Lipschitz path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$. Moreover, using Lemma 4.5 (iv), we have that (r_0, s_0) and (r_1, s_1) are connected by an L-Lipschitz path of projections in $\mathcal{P}_{n,1,4\varepsilon}(X, B)$, say $((r_t, s_t))_{t\in[0,1]}$.

Now, consider the path $(\sigma(r_t), \sigma(s_t))_{t\in[0,1]}$ in $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, which is also L-Lipschitz. Lemma 4.7 gives $(3L)$-Lipschitz paths $(u_t)_{t\in[0,1]}$ and $(v_t)_{t\in[0,1]}$ of unitaries in $M_n(\mathbb{C})$ such that $\sigma(r_t) = u_t \sigma(r_0) u_t^*$ and $\sigma(s_t) = v_t \sigma(s_0) v_t^*$ for all

$t \in [0,1]$. The path $((u_t^* r_t u_t, v_t^* s_t v_t))_{t\in[0,1]}$ then passes through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, is $(6L)$-Lipschitz, and connects (r_0, s_0) to $(u_1^* r_1 u_1, v_1^* s_1 v_1)$.

Summarizing where we are, we have the following paths:

(i) A κ-Lipschitz path through $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (p_0, q_0) and (r_0, s_0).

(ii) A $(6L)$-Lipschitz path through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (r_0, s_0) and $(u_1^* r_1 u_1, v_1^* s_1 v_1)$.

(iii) A κ-Lipschitz path through $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (p_1, q_1) and (r_1, s_1).

We claim that there is a 2π-Lipschitz path passing through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, parametrized by $[0,1]$ and connecting $(u_1^* r_1 u_1, v_1^* s_1 v_1)$ and (r_1, s_1). Concatenating this new path with the three paths above (and using that $\kappa \geq 1$ and that $L \geq 1$), and rescaling the two κ-Lipschitz paths by $1/12$, the $6L$-Lipschitz path by $4/12$, and the 6π-Lipschitz by $6/12$, this will give us a $(20\kappa L)$-Lipschitz path connecting (p_0, q_0) and (p_1, q_1) through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, which will complete the proof.

To establish the claim note that u_1 commutes with 1_l, and is therefore connected to the identity in $M_n(\mathbb{C})$ via a π-Lipschitz path of unitaries that all commute with 1_l, say $(u_t)_{t\in[1,2]}$. Similarly, we get a π-Lipschitz path $(v_t)_{t\in[1,2]}$ with the same properties with respect to v_1. The path $((u_t^* r_1 u_t, v_t^* s_1 v_t))_{t\in[1,2]}$ then passes through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, is 2π-Lipschitz, and connects $(u_1^* r_1 u_1, v_1^* s_1 v_1)$ to (r_1, s_1), so we are done. ∎

We now move on to results that let us prescribe the "scalar part" of cycles for KK^1, which is much simpler.

Definition 4.11. With notation as in Definition 3.5, define

$$\mathcal{U}^1_{n,\kappa,\varepsilon}(X,B) := \{u \in \mathcal{U}_{n,\kappa,\varepsilon}(X,B) \mid u - 1 \in M_n(\mathcal{K}_B)\}.$$

Define $\mathcal{U}^1_{\infty,\kappa,\varepsilon}(X,B)$ to be the disjoint union of these sets as n ranges over $\mathbb{N}$.

We need a slight variant of the well-known fact that the group of invertibles in a C^*-algebra deform retracts onto the group of unitaries.

Lemma 4.12. *Let $\kappa \geq 1$, let C be a unital C^*-algebra, and let C_κ^{-1} be the set of invertible elements $u \in C$ such that $\|u\| \leq \kappa$ and $\|u^{-1}\| \leq \kappa$. Then, the unitary group of C is a deformation retract of C_κ^{-1}. In particular, $M_n(\mathbb{C})_\kappa^{-1}$ is connected.*

Proof. Let $u \in C_\kappa^{-1}$, and for $t \in [0, \frac{1}{2}]$ define $u_t := u(u^*u)^{-t}$. This is a homotopy between the identity $u \mapsto u_0$ on C_κ^{-1} and the map $u \mapsto u_{1/2}$; the latter is a retraction of C_κ^{-1} onto the unitary group of C, giving the first part. In particular, it follows that C_κ^{-1} is connected if and only if C_1^{-1} is connected; as the unitary group of $M_n(\mathbb{C})$ is connected, this gives the last statement. ∎

Proposition 4.13. *Let B be a separable C^*-space, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, let $\kappa \geq 1$, and let $n \in \mathbb{N}$.*

(i) *Any element $v \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ is connected to an element of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$ by a path in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(X, B)$.*

(ii) *If two elements $v_0, v_1 \in \mathcal{U}^1_{n,\kappa,\varepsilon}(X, B)$ are in the same path component of $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$, then they are in the same path component of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$.*

Proof. For part (i), let $\mathcal{K}_B^+$ be the unitization of $\mathcal{K}_B$, let $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ be the canonical quotient map, and set $w = \sigma(u^{-1})$. Using Lemma 4.12, there is a path $(w_t)_{t\in[0,1]}$ of invertibles connecting $w = w_1$ to the identity and all with norm at most κ. Then, the path $(w_t v)_{t\in[0,1]}$ is in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(X, B)$ and connects v to the element $u := w_1 v$, which satisfies $\sigma(u) = 1$, and so $1 - u \in M_n(\mathcal{K}_B)$.

For part (ii), let $(v_t)_{t\in[0,1]}$ be a path in $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ connecting v_0 and v_1. Let $w_t = \sigma(v_t^{-1})$, and note that $w_0 = w_1 = 1$. Moreover, $\|w_t\| \leq \kappa$ for all t. Then, $u_t := w_t v_t$ is a path connecting v_0 and v_1 in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$ as required. ∎

4.4 From homotopies to similarities

Our goal in this section is to establish a controlled variant of the fact that homotopic idempotents are similar; compare for example [7, Proposition 4.3.2]. This requires some work, as we need to control the "speed" of the homotopy in order to control the commutator estimates for the invertible element appearing in the similarity. The final target is Proposition 4.17 below; the other results build up to it.

Lemma 4.14. *Let $\kappa \geq 1$, and let p_0 and p_1 be idempotents in a C^*-algebra C with norm at most κ, and such that $\|p_0 - p_1\| \leq 1/(12\kappa^2)$. Then, there is a path $(p_t)_{t\in[0,1]}$ of idempotents connecting p_0 and p_1, and with the following properties:*

(i) *each p_t is an idempotent in C of norm at most 2κ;*

(ii) *for all $c \in C$ and $t \in [0, 1]$,*

$$\|[c, p_t]\| \leq 21\kappa^2 \max_{i=0,1} \|[c, p_i]\|;$$

(iii) *the function $t \mapsto p_t$ is 1-Lipschitz.*

Proof. For each $t \in [0, 1]$, define $r_t := (1 - t)p_0 + tp_1 \in C$, and define

$$u_t := (1 - r_t)(1 - p_0) + r_t p_0 \in C^+.$$

Corollary 4.2 implies that $\|2p_0 - 1\| \leq 2\kappa$, whence

$$\|1 - u_t\| = \|(2p_0 - 1)(p_0 - r_t)\| \leq 2\kappa\|p_0 - p_1\| \leq 1/6.$$

In particular, u_t is invertible, $\|u_t\| \le 7/6$, and $\|u_t^{-1}\| \le 6/5$ by the Neumann series formula of the inverse. Define $p_t := u_t p_0 u_t^{-1}$, which is an idempotent in C. We claim that the path $(p_t)_{t\in[0,1]}$ has the desired properties. Note first that $r_0 = p_0$, whence $u_0 = 1$, and so the path $(p_t)_{t\in[0,1]}$ does start at the original p_0. On the other hand, $u_1 p_0 = r_1 p_0 = p_1 p_0 = p_1 u_1$, whence $u_1 p_0 u_1^{-1} = p_1$. Thus, the path (p_t) does connect p_0 and p_1.

For part (i), note that as $u_t p_0 = r_t p_0$, we get

$$\|p_t\| = \|r_t p_0 u_t^{-1}\| \le \|(r_t - p_0)p_0 u_t^{-1}\| + \|p_0 u_t^{-1}\| \le \frac{1}{12\kappa^2}\kappa\frac{6}{5} + \kappa\frac{6}{5} \le 2\kappa.$$

For part (ii), let $\delta = \max_{i=0,1} \|[c, p_i]\|$. We compute using the identity $1 - u_t = (2p_0 - 1)(p_0 - r_t)$ that

$$\begin{aligned}\|[u_t, c]\| = \|[1 - u_t, c]\| &\le \|[2p_0 - 1, c]\|\|p_0 - r_t\| + \|2p_0 - 1\|\|[p_0 - r_t, c]\| \\ &\le 2\|[p_0, c]\|\|p_0 - r_t\| + \|2p_0 - 1\|(\|[p_0, c]\| + \|[r_t, c]\|).\end{aligned}$$

Using that $\|2p_0 - 1\| \le 2\kappa$ again, this implies that

$$\|[u_t, c]\| \le 2\delta\frac{1}{12\kappa^2} + 2\kappa \cdot 2\delta = \Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta.$$

Hence, also

$$\|[u_t^{-1}, c]\| = \|u_t^{-1}[c, u_t]u_t^{-1}\| \le \frac{36}{25}\Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta\|c\|$$

and so

$$\begin{aligned}\|[p_t, c]\| &= \|[u_t p_0 u_t^{-1}, c]\| \\ &\le \|[u_t, c]\|\|p_0\|\|u_t^{-1}\| + \|u_t\|\|[p_0, c]\|\|u_t^{-1}\| + \|u_t\|\|p_0\|\|[u_t^{-1}, c]\| \\ &\le \Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta\kappa\frac{6}{5} + \frac{7}{5}\delta + \frac{7}{6}\kappa\frac{36}{25}\Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta \\ &\le 21\kappa^2\delta\end{aligned}$$

as claimed. Finally, for part (iii), we again use that $\|2p_0 - 1\| \le 2\kappa$ to compute that for any $s, t \in [0, 1]$,

$$\begin{aligned}\|u_s - u_t\| = \|(2p_0 - 1)(r_s - r_t)\| &\le \|2p_0 - 1\||s - t|\|p_0 - p_1\| \le 2\kappa|s - t|\frac{1}{12\kappa^2} \\ &= \frac{1}{6\kappa}|s - t|\end{aligned}$$

and so

$$\|u_s^{-1} - u_t^{-1}\| = \|u_t^{-1}(u_t - u_s)u_s^{-1}\| \le \frac{36}{25}\frac{1}{6\kappa}|s - t| = \frac{6}{25\kappa}|s - t|.$$

Hence,

$$\begin{aligned}\|p_t - p_s\| &\leq \|(u_t - u_s)p_0u_t^{-1}\| + \|u_s p_0(u_t^{-1} - u_s^{-1})\| \\ &\leq \frac{1}{6\kappa}|s-t|\kappa\frac{6}{5} + \frac{7}{6}\kappa\frac{6}{25\kappa}|s-t| \\ &\leq |s-t|\end{aligned}$$

as claimed. ∎

The next lemma gives universal control over the "speed" of a homotopy between idempotents (at the price of moving to larger matrices). The basic idea is not new; see for example [47, Proposition 1.31]. We give a complete proof, however, as we need to incorporate commutator estimates and work with idempotents rather than projections.

Lemma 4.15. *Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Let (p_0, q_0) and (p_1, q_1) be elements of the same path component of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. Then, there is $k \in \mathbb{N}$ and a homotopy $((r_t, s_t))_{t\in[0,1]}$ in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$ such that $(r_i, s_i) = (p_i \oplus 1_{nk} \oplus 0_{nk}, q_i \oplus 1_{nk} \oplus 0_{nk})$ for $i \in \{0, 1\}$, and such that the map $t \mapsto (r_t, s_t)$ is (16κ)-Lipschitz.*

Proof. Let $((p_t, q_t))_{t\in[0,1]}$ be an arbitrary homotopy in $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ connecting (p_0, q_0) and (p_1, q_1). Let $\delta > 0$ be such that if $s, t \in [0, 1]$ satisfy $|s - t| \leq \delta$, then $\|p_s - p_t\| \leq 1/(12\kappa^2)$ and $\|q_s - q_t\| \leq 1/(12\kappa^2)$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a sequence of points in $[0, 1]$ such that $t_{i+1} - t_i \leq \delta$ for all i. We claim that this k works, and to show this we build an appropriate homotopy by concatenating the various steps below.

(i) Connect $(p_0 \oplus 1_{nk} \oplus 0_{nk}, q_0 \oplus 1_{nk} \oplus 0_{nk})$ to

$$\Big(p_0 \oplus \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}},\ q_0 \oplus \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}}\Big)$$

via a 2-Lipschitz rotation homotopy parametrized by $[0, \pi/2]$ and passing through $\mathcal{P}_{(2k+1)n,\kappa,\varepsilon}(X, B)$.

(ii) In the ith "block" $1_n \oplus 0_n$, use the homotopy

$$\begin{pmatrix} 1 - p_{t_i} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p_{t_i} \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

(parametrized by $t \in [0, \pi/2]$) to connect $1_n \oplus 0_n$ to $1 - p_{t_i} \oplus p_{t_i}$, and similarly for q. In order to compute commutator estimates, note that rearranging gives that the homotopy above is the same as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} p_{t_i} & 0 \\ 0 & p_{t_i} \end{pmatrix} \begin{pmatrix} -\cos^2(t) & -\sin(t)\cos(t) \\ -\sin(t)\cos(t) & \cos^2(t) \end{pmatrix}, \quad t \in [0, \pi/2].$$

The scalar matrix appearing on the right above has norm $|\cos(t)|$, whence every element in this homotopy has norm at most 2κ. Hence, our homotopy connects the result of the previous stage to

$$(p_0 \oplus 1 - p_{t_1} \oplus p_{t_1} \oplus \cdots \oplus 1 - p_{t_k} \oplus p_{t_k}, q_0 \oplus 1 - q_{t_1} \oplus q_{t_1} \oplus \cdots \oplus 1 - q_{t_k} \oplus q_{t_k})$$

through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$, and is 2κ-Lipschitz.

(iii) From Corollary 4.2, each idempotent $1 - p_{t_i}$ has norm at most κ. For each $i \in \{1, \ldots, k\}$, using that $\|(1 - p_{t_i}) - (1 - p_{t_{i-1}})\| \leq 1/(12\kappa^2)$, Lemma 4.14 gives a path of idempotents connecting $1 - p_{t_i}$ and $1 - p_{t_{i-1}}$ and with the following properties: it is 1-Lipschitz; it consists of idempotents of norm at most 2κ; each idempotent r in the path satisfies $\|[r, x]\| \leq 21\kappa^2\varepsilon$ for all $x \in X$. We get similar paths with respect to the elements $1 - q_{t_i}$, and use these paths to connect the result of the previous stage to

$$(p_0 \oplus 1 - p_{t_0} \oplus p_{t_1} \oplus \cdots \oplus 1 - p_{t_{k-1}} \oplus p_{t_k}, q_0 \oplus 1 - q_{t_0} \oplus q_{t_1} \oplus \cdots \oplus 1 - q_{t_{k-1}} \oplus q_{t_k})$$

via a 1-Lipschitz path in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$.

(iv) Use an analog of the homotopy in step (ii) in each block of the form $p_{t_i} \oplus 1 - p_{t_i}$ (and similarly for q) to connect the result of the previous stage to

$$\Big(\underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}} \oplus p_{t_k}, \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}} \oplus q_{t_k}\Big).$$

This passes through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$, and is 2κ-Lipschitz.

(v) Finally, noting that $p_{t_k} = p_1$ and $q_{t_k} = q_1$, use a rotation homotopy parametrized by $[0, \pi/2]$ to connect the result of the previous stage to $(p_1 \oplus 1_{nk} \oplus 0_{nk}, q_1 \oplus 1_{nk} \oplus 0_{nk})$. This passes through $\mathcal{P}_{(2k+1)n,\kappa,\varepsilon}(X, B)$ and is 2κ-Lipschitz.

Concatenating the five homotopies above gives a 2κ-Lipschitz homotopy, parametrized by $[0, 2\pi + 1]$, that passes through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$ and connects $(p_0 \oplus 1_{nk} \oplus 0_{nk}, q_0 \oplus 1_{nk} \oplus 0_{nk})$ and $(p_1 \oplus 1_{nk} \oplus 0_{nk}, q_1 \oplus 1_{nk} \oplus 0_{nk})$. Reparametrizing by $[0, 1]$, we get a (16κ)-Lipschitz homotopy as required. ∎

Before we get to the main result of this section, we give one more elementary lemma; we record it as it will be used multiple times below.

Lemma 4.16. *Say x and $y_1, \ldots, y_n$ are elements of a C^*-algebra such that*

$$\|[x, y_i]\| \leq \delta \quad \text{and} \quad \|y_i\| \leq m$$

for all i. Then, if $y := y_1 y_2 \cdots y_n$, we have $\|[x, y]\| \leq n m^{n-1} \delta$.

Proof. This follows from the formula

$$[x, y] = \sum_{i=1}^{n} \Big(\prod_{1 \leq j < i} y_j \Big) [x, y_i] \Big(\prod_{i < j \leq n} y_j \Big),$$

which itself follows from induction on n and the usual Leibniz formula $[x, y_1y_2] = y_1[x, y_2] + [x, y_1]y_2$. ∎

Here, is the main result of this section. The basic idea of the proof is contained in [47, Corollary 1.32], but as usual we need to do more work in order to get our estimates.

Proposition 4.17. *Let B be a separable C^*-algebra, let X be a self-adjoint subset of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, and let $\varepsilon > 0$. Let $M = 2^{(100\kappa)^3}$. With notation as in Definition 4.9, let $n \in \mathbb{N}$, and let (p, q) be in the same path component of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ as an element (r, r) with both entries the same. Then, there is $m \in \mathbb{N}$ and (with notation as in Definition 4.11) an element $u \in \mathcal{U}^1_{n+2m,M,M\varepsilon}(X, B)$ such that*

$$u(p \oplus 1_m \oplus 0_m)u^{-1} = q \oplus 1_m \oplus 0_m.$$

Proof. Let $k \in \mathbb{N}$ be as in the conclusion of Lemma 4.15, so there exists a (16κ)-Lipschitz homotopy in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$ between $(p \oplus 1_{nk} \oplus 0_{nk}, q \oplus 1_{nk} \oplus 0_{nk})$ and $(r \oplus 1_{nk} \oplus 0_{nk}, r \oplus 1_{nk} \oplus 0_{nk})$. Set $m = kn$. Proposition 4.10 gives a $(20\kappa \cdot 16\kappa)$-Lipschitz path $((p_t, q_t))_{t\in[0,1]}$ passing through $\mathcal{P}^1_{n+2m,2\kappa,84\kappa^2\varepsilon}(X, B)$ that connects $(p \oplus 1_{nk} \oplus 0_{nk}, q \oplus 1_{nk} \oplus 0_{nk})$ and $(r \oplus 1_{nk} \oplus 0_{nk}, r \oplus 1_{nk} \oplus 0_{nk})$. To simplify notation, note this path is $(2^9\kappa^2)$-Lipschitz, and that it passes through $\mathcal{P}^1_{n+2m,2\kappa,2^7\kappa^2\varepsilon}(X, B)$.

Define $N := \lceil 2^{13}\kappa^3 \rceil$ (where $\lceil y \rceil$ is the least integer at least as large as y), and define $t_i = i/N$ for $i \in \{0, \ldots, N\}$. As the path $((p_t, q_t))_{t\in[0,1]}$ is $(2^9\kappa^2)$-Lipschitz, for any $i \in \{1, \ldots, N\}$, $\|p_{t_i} - p_{t_{i-1}}\| \leq (16\kappa)^{-1}$. For $i \in \{1, \ldots, N\}$, define

$$v_i := p_{t_{i-1}}p_{t_i} + (1 - p_{t_{i-1}})(1 - p_{t_i}).$$

As $\|p_{t_i}\| \leq 2\kappa$ for all i, Corollary 4.2 implies that

$$\|2p_{t_i} - 1\| \leq 4\kappa \tag{4.11}$$

for all i, and so

$$\|1 - v_i\| = \|(2p_{t_{i-1}} - 1)(p_{t_{i-1}} - p_{t_i})\| \leq 4\kappa \cdot (16\kappa)^{-1} \leq \frac{1}{2}.$$

It follows that each v_i is invertible, $\|v_i\| \leq 2$, and (by the Neumann series formula for the inverse) $\|v_i^{-1}\| \leq 2$. Note also that as the homotopy $((p_t, q_t))_{t\in[0,1]}$ passes through $\mathcal{P}^1_{(2k+1)n,2\kappa,2^7\kappa^2\varepsilon}(X, B)$ all the elements p_{t_i} must have the same "scalar part" (i.e., the same image under the canonical map $M_{n+2m}(\mathcal{K}_B^+) \to M_{n+2m}(\mathbb{C})$), and so the elements v_i must satisfy

$$1 - v_i \in M_{n+2m}(\mathcal{K}_B).$$

Moreover, for $x \in X$, using line (4.11) again we see that

$$\begin{aligned}\|[v_i, x]\| &= \|[v_i - 1, x]\| \\ &= \|[(2p_{t_{i-1}} - 1)(p_{t_{i-1}} - p_{t_i}), x]\| \\ &\leq 2\|[p_{t_{i-1}}, x]\|(\|p_{t_{i-1}}\| + \|p_{t_i}\|) + \|2p_{t_{i-1}} - 1\|(\|[p_{t_{i-1}}, x]\| + \|[p_{t_i}, x]\|) \\ &\leq 12\kappa \cdot 2^7\kappa^2\varepsilon.\end{aligned}$$

Hence, moreover

$$\|[v_i^{-1}, x]\| = \|v_i^{-1}[x, v_i]v_i^{-1}\| \leq 4 \cdot 12\kappa \cdot 2^7\kappa^2\varepsilon \leq 2^{13}\kappa^3\varepsilon.$$

At this point we have that each v_i is an element of $\mathcal{U}^1_{n+2m,2,2^{13}\kappa^3\varepsilon}$.

Note also that $v_i p_{t_i} = p_{t_{i-1}} p_{t_i} = p_{t_{i-1}} v_i$, and so $v_i p_{t_i} v_i^{-1} = p_{t_{i-1}}$ for each i. Define v to be the product $v_1 v_2 \cdots v_N$, so v satisfies $v^{-1} p_0 v = p_1$, or in other words

$$v^{-1}(p \oplus 1_m \oplus 0_m)v = r \oplus 1_m \oplus 0_m.$$

Note that $1 - v \in M_{n+2m}(\mathcal{K}_B)$. As $\|v_i\| \leq 2$ and $\|v_i^{-1}\| \leq 2$ for each i, we have that $\|v\| \leq 2^N$ and similarly $\|v^{-1}\| \leq 2^N$. Moreover, for any $x \in X$, Lemma 4.16 gives $\|[v, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ and similarly $\|[v^{-1}, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$. Applying the same construction with (q_t) in place of (p_t), we get an invertible element w such that $w^{-1}(q \oplus 1_m \oplus 0_m)w = r \oplus 1_m \oplus 0_m$, such that $1 - w \in M_{n+2m}(\mathcal{K}_B)$, such that $\|w\| \leq 2^N$, $\|w^{-1}\| \leq 2^N$, and such that $\|[w, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ and $\|[w^{-1}, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ for all $x \in X$. Define $u = wv^{-1}$. As $N = \lceil 2^{13}\kappa^3 \rceil$, this has the claimed properties. ∎

Chapter 5

Reformulating the UCT II

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to reformulate the vanishing results on the UCT of Chapter 2 in terms of the groups $KK^i_{\kappa,\varepsilon}(X, B)$ of Chapter 3. We look at the even ($i = 0$) and odd ($i = 1$) cases separately.

5.1 The even case

Lemma 5.1. *Let $\kappa \geq 1$ and $\varepsilon > 0$. Let B be a separable C^*-algebra, and let X be a self-adjoint subset of the unit ball of $\mathcal{L}_B$. Then, there is a homomorphism $\psi_* : KK^*_{\kappa,\varepsilon/4}(X, B) \to KK^*_{1,\varepsilon}(X, B)$ such that the diagrams*

$$\begin{array}{ccc} KK^0_{1,\varepsilon}(X, B) & & \\ \uparrow{\scriptstyle \psi_*} & \searrow & \\ KK^0_{\kappa,\varepsilon/4}(X, B) & \longrightarrow & KK^0_{\kappa,\varepsilon}(X, B) \end{array} \tag{5.1}$$

and

$$\begin{array}{ccc} KK^0_{1,\varepsilon/4}(X, B) & \longrightarrow & KK^0_{1,\varepsilon}(X, B) \\ & \searrow & \uparrow{\scriptstyle \psi_*} \\ & & KK^0_{\kappa,\varepsilon/4}(X, B) \end{array} \tag{5.2}$$

commute, where the unlabeled arrows are the forget control maps of Definition 3.4.

Proof. Let (p, q) be an element of $\mathcal{P}_{n,\kappa,\varepsilon/4}(X, B)$. Let r and s be the projections associated to p and q respectively as in Definition 4.3. Using Lemma 4.5 parts (i) and (iii) we may define a map

$$\psi : \mathcal{P}_{n,\kappa,\varepsilon/4}(X, B) \to \mathcal{P}_{n,1,\varepsilon}(X, B), \quad (p, q) \mapsto (r, s).$$

Allowing n to vary, and noting that the process of taking associated projections takes homotopies to homotopies (by part (iv) of Lemma 4.5) and block sums to block sums, we get a well-defined homomorphism

$$\psi_* : KK^0_{\kappa,\varepsilon/4}(X, B) \to KK^0_{1,\varepsilon}(X, B).$$

To check commutativity of the diagram in line (5.1), it suffices to show that if $(r, s) \in \mathcal{P}_{n,1,\varepsilon}(X, B)$ is the pair of projections associated to $(p, q) \in \mathcal{P}_{n,\kappa,\varepsilon/4}(X, B)$ as above, then (r, s) and (p, q) are in the same path component of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. This follows from parts (ii) and (iii) of Lemma 4.5. Commutativity of the diagram in line (5.2) is immediate; if (p, q) is in $\mathcal{P}_{n,1,\varepsilon}(X, B)$ for some n, then p and q are themselves projections, so equal their associated projections. ∎

The following lemma records some results from [68, Section A.3] that we will need. For the statement, recall the notion of a unitally strongly absorbing representation from Definition 2.5 above.

Lemma 5.2. *In the statement of this lemma, all unlabeled arrows are forget control maps as in Definitions 2.11 and 3.4. Let A be a separable unital C^*-algebra, and let B be a separable C^*-algebra. Let $\pi : A \to \mathcal{L}_B$ be a strongly unitally absorbing representation of A, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_B$.*

Let $\varepsilon > 0$, and let X be a finite subset of A_1. Then, there exist homomorphisms

$$\alpha : KK^0_{1,\varepsilon}(X, B) \to KK_{5\varepsilon}(X, B)$$

and

$$\beta : KK_{\varepsilon}(X, B) \to KK^0_{1,\varepsilon}(X, B)$$

that are natural with respect to forget control maps; more precisely if $(X, \varepsilon) \leq (Y, \delta)$ in $\mathcal{X}_A$ as in Definition 2.10 then the diagrams

$$\begin{array}{ccc} KK^0_{1,\delta}(Y, B) & \longrightarrow & KK^0_{1,\varepsilon}(X, B) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ KK^0_{\delta}(Y, B) & \longrightarrow & KK^0_{\varepsilon}(X, B) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} KK^0_{1,5\delta}(Y, B) & \longrightarrow & KK^0_{1,5\varepsilon}(X, B) \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha} \\ KK^0_{\delta}(Y, B) & \longrightarrow & KK^0_{\varepsilon}(X, B) \end{array}$$

are defined and commute.

Moreover, the diagrams

$$\begin{array}{ccc} KK^0_{1,\varepsilon}(X, B) & \longrightarrow & KK^0_{1,5\varepsilon}(X, B) \\ \downarrow{\scriptstyle \alpha} & \nearrow_{\beta} & \\ KK_{5\varepsilon}(X, B) & & \end{array}$$

and

$$\begin{array}{ccc} KK_{\varepsilon}(X, B) & \longrightarrow & KK_{5\varepsilon}(X, B) \\ \downarrow{\scriptstyle \beta} & \nearrow_{\alpha} & \\ KK^0_{1,\varepsilon}(X, B) & & \end{array}$$

commute.

Proof. Let $\pi : A \to M_2(\mathcal{L}_B)$ be (the amplification of) our fixed representation. In the language of [68, Appendix A.2], the groups $KK_\varepsilon(X, B)$ are the same as the groups that are called there $KK_\varepsilon^{\pi,p}(X, B)$, while in the language of [68, Appendix A.3], the groups $KK^0_{1,\varepsilon}(X, B)$ would there be called $KK_\varepsilon^{\pi_0,m}(X, B)$. The lemma thus follows from the arguments of [68, Lemmas A.22, A.23, and A.24]. ∎

We are now able to deduce a version of Corollary 2.22 for the groups of Definition 3.1.

Corollary 5.3. *Let A be a separable, unital, nuclear C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let $\kappa \geq 1$ and $\varepsilon \in (0, 1)$. Let B be a separable C^*-algebra with $K_*(B) = 0$. Let $\pi : A \to \mathcal{L}_{SB}$ be a strongly unitally absorbing representation, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, for any finite subset X of A_1, there is a finite subset Z of A_1 such that*

$$(X, \kappa, \varepsilon) \leq (Z, \kappa, \varepsilon/160)$$

in the sense of Definition 3.4, and such that the forget control map

$$KK^0_{\kappa,\varepsilon/160}(Z, SB) \to KK^0_{\kappa,\varepsilon}(X, SB)$$

of Definition 3.4 is zero.

(iii) *There exist $\kappa \geq 1$ and $\nu \geq \kappa$ with the following property. Let $\gamma > 0$, let B be a separable C^*-algebra with $K_*(B) = 0$, and let X be a finite subset of A_1. Let*

$$\pi : A \to \mathcal{L}_{SB}$$

be a strongly unitally absorbing representation, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, there is $\varepsilon > 0$ and a finite subset Z of A_1 such that $(X, \nu, \gamma) \leq (Z, \kappa, \varepsilon)$ in the sense of Definition 3.4, and such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

of Definition 3.4 is zero.

Proof. In the following proof, all unlabeled arrows are forget control maps as in Definition 2.11, or Definition 3.4. Assume first that condition (i) from the statement holds, and let $\kappa \geq 1$ and $\varepsilon > 0$; we may assume moreover that $\varepsilon < 1$. Let a finite subset X be given as in condition (ii). Then, by the equivalence from Corollary 2.22, there is a finite subset Z of A_1 such that the forget control map

$$KK_{\varepsilon/8}(Z, SB) \to KK_\varepsilon(X, SB)$$

is zero. Replacing Z by $Z \cup Z^*$ if necessary, we may assume that Z is self-adjoint. Lemma 5.2 gives a commutative diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z, SB) & \xrightarrow{0} & KK_{\varepsilon}(X, SB) \\ \uparrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ KK^0_{1,\varepsilon/40}(Z, SB) & \longrightarrow & KK^0_{1,\varepsilon}(X, SB), \end{array}$$

whence the bottom horizontal map is zero. On the other hand, Lemma 5.1 (see in particular line (5.1)) gives a map ψ_* such that the bottom triangle in the diagram below

$$\begin{array}{ccc} KK^0_{1,\varepsilon/40}(Z, SB) & \xrightarrow{0} & KK^0_{1,\varepsilon}(X, SB) \\ \uparrow{\scriptstyle \psi_*} & \searrow & \downarrow \\ KK^0_{\kappa,\varepsilon/160}(Z, SB) & \longrightarrow & KK_{\kappa,\varepsilon}(X, SB) \end{array}$$

commutes. The top triangle also commutes as all the maps involved are forget control maps, whence the bottom horizontal map is zero. This gives us condition (ii) from the statement.

Condition (ii) clearly implies condition (iii), so it remains to show that condition (iii) implies condition (i). For this, it suffices to establish condition (ii) from Theorem 2.15, so let $\gamma > 0$ and a finite subset X of A_1 be given. Then, according to condition (iii) there are $\nu \geq \kappa \geq 1$, $\varepsilon > 0$ and a finite subset Z of A_1 such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma/20}(X, SB)$$

is defined and zero. Replacing Z with $Z \cup Z^*$ if necessary, we may assume Z is self-adjoint. Using Lemma 5.1 (see in particular line (5.2)) there is a map ψ_* such that the top right triangle in the diagram below commutes

$$\begin{array}{ccccc} KK^0_{1,\varepsilon}(Z, SB) & \longrightarrow & KK_{1,\gamma/20}(X, SB) & \longrightarrow & KK_{1,\gamma/5}(X, SB) \\ \downarrow & & \downarrow & \nearrow & \uparrow{\scriptstyle \psi_*} \\ KK^0_{\kappa,\varepsilon}(Z, SB) & \xrightarrow[0]{} & KK_{\nu,\gamma/20}(X, SB) & \longrightarrow & KK^0_{\nu,\gamma/20}(X, SB). \end{array}$$

The rest of the diagram also commutes, as all the arrows are forget control maps, whence the composition

$$KK^0_{1,\varepsilon}(Z, SB) \to KK_{1,\gamma/20}(X, SB) \to KK_{1,\gamma/5}(X, SB)$$

of the two top horizontal maps is zero. Using Lemma 5.2, there is a commutative diagram

$$\begin{array}{ccc} KK_\varepsilon(Z, SB) & \longrightarrow & KK_\gamma(X, SB) \\ {\scriptstyle\beta}\downarrow & & \uparrow{\scriptstyle\alpha} \\ KK^0_{1,\varepsilon}(Z, SB) & \xrightarrow[0]{} & KK_{1,\gamma/5}(X, SB). \end{array}$$

The top horizontal map is therefore zero; this is the conclusion we need for Theorem 2.15, condition (ii), so we are done. ∎

5.2 The odd case

For the statement of the next lemma, consider the Hilbert module $\ell^2 \otimes SB$ associated to the suspension $SB = C_0((0,1), B)$ of a separable C^*-algebra B. Let $C_{sb}(X, M(C))$ denote the C^*-algebra of bounded and strictly continuous functions from a locally compact space X to the multiplier algebra $M(C)$ of a C^*-algebra C. For any C^*-algebra C there are canonical identifications

$$\mathcal{L}_C = M(C \otimes \mathcal{K})$$

(see for example [45, Theorem 2.4]) and $M(C_0(X, C)) = C_{sb}(X, M(C))$ (see for example [1, Corollary 3.4]). Hence, there is a canonical identification

$$\mathcal{L}_{SB} = C_{sb}((0,1), \mathcal{L}_B). \tag{5.3}$$

We identify $\mathcal{L}_B = \mathcal{L}(\ell^2 \otimes B)$ with a C^*-subalgebra of $\mathcal{L}_{SB} = \mathcal{L}(\ell^2 \otimes B \otimes C_0(0,1))$ via the $*$-homomorphism $a \mapsto a \otimes 1_{C_0(0,1)}$. We recall also that $\mathcal{K}_B^+$ denotes the unitization of $\mathcal{K}_B$.

Lemma 5.4. *Let B be a separable C^*-algebra. Let $\kappa \geq 1$, $\varepsilon > 0$, and let X be a subset of the unit ball of $\mathcal{L}_B$. Then,*

(i) *Elements of $\mathcal{P}_{n,\kappa,\varepsilon}(X, SB)$ (see Definition 3.1) identify canonically with continuous paths $(p_t, q_t)_{t\in[0,1]}$ of idempotents in $M_n(\mathcal{K}_B^+) \oplus M_n(\mathcal{K}_B^+)$ satisfying the following conditions:*

   (a) *for all $t \in [0,1]$, $\|p_t\| \leq \kappa$ and $\|q_t\| \leq \kappa$;*

   (b) *for all $t \in [0,1]$ and all $x \in X$, $\|[p_t, x]\| < \varepsilon$ and $\|[q_t, x]\| < \varepsilon$;*

   (c) *there are $p, q \in M_n(\mathbb{C})$ such that $p_0 = p_1 = p$, $q_0 = q_1 = q$ and such that if $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ is the canonical quotient map then $\sigma(p_t) = p$ and $\sigma(q_t) = q$ for all $t \in [0,1]$.*

   Moreover, the element (p, q) is in the subset $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, SB)$ of Definition 4.9 if and only if p and q are equal to 1_l for some $l \in \mathbb{N}$.

(ii) *Elements of* $\mathcal{U}_{n,\kappa,\varepsilon}(X, SB)$ *(see Definition 3.5) identify with continuous paths* $(u_t)_{t\in[0,1]}$ *of invertibles in* $M_n(\mathcal{K}_B^+)$ *satisfying the following conditions:*

(a) *for all* $t \in [0,1]$, $\|u_t\| \leq \kappa$ *and* $\|u_t^{-1}\| \leq \kappa$*;*

(b) *for all* $t \in [0,1]$ *and all* $x \in X$, $\|[u_t, x]\| < \varepsilon$ *and* $\|[u_t^{-1}, x]\| < \varepsilon$*;*

(c) *there is* $u \in M_n(\mathbb{C})$ *such that* $u_0 = u_1 = u$ *and such that if*

$$\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$$

is the canonical quotient map then $\sigma(u_t) = u$ *for all* $t \in [0,1]$.

Moreover, the element is in the subset $\mathcal{U}^1_{n,\kappa,\varepsilon}(X, SB)$ *of Definition 4.11 if and only if* u *is the identity.*

Proof. We have a canonical identification

$$\mathcal{K}_{SB}^+ = \{f \in C([0,1], \mathcal{K}_B^+) \mid \sigma(f(t)) = f(0) = f(1) \text{ for all } t \in [0,1]\}.$$

Part (i) follows directly by comparing this with Definitions 3.1 and 4.9; similarly, part (ii) follows from comparing this with Definitions 3.5 and 4.11. We leave the details to the reader. ∎

Lemma 5.5. *For any* $\kappa \geq 1$ *there exists a positive constant* M_1 *with the following property. Let* $\varepsilon > 0$*, let* A *be a separable, unital, nuclear* C^**-algebra that satisfies the UCT, and let* B *be a separable* C^**-algebra with* $K_*(B) = 0$*. Let* $\pi : A \to \mathcal{L}_{SB}$ *be a strongly unitally absorbing representation that factors through the subalgebra* $\mathcal{B}(\ell^2)$ *(such exists by Lemma 2.6), and use this to identify* A *with a* C^**-subalgebra of* $\mathcal{L}_{SB}$*.*

Then, for any finite subset X *of* A_1 *there exists a finite subset* Z *of* A_1 *such that the forget control map*

$$KK^1_{\kappa,\varepsilon}(Z, SB) \to KK^1_{M_1,M_1\varepsilon}(X, SB)$$

of Definition 3.7 is defined and zero.

Proof. We claim $M_1 = 2^{(200\kappa^8)^3} \cdot 320\kappa^7$ works. Using Corollary 5.3 there is a finite subset Z of A_1 such that the forget control map

$$KK^0_{\kappa^8,2\kappa^6\varepsilon}(Z, SB) \to KK^0_{\kappa^8,320\kappa^6\varepsilon}(X, SB) \tag{5.4}$$

of Definition 3.4 is zero. We claim this set Z works.

Let u be an arbitrary element of $\mathcal{U}_{n,\kappa,\varepsilon}(Z, B)$. Using Proposition 4.13 (i), and with notation as in Definition 4.11, there is an element v of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(Z, B)$ in the same path component of $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(Z, B)$ as u. Define now a path $(v_t)_{t\in[0,1]}$ by

$$v_t := \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

Note that each v_t is an element of $\mathcal{U}^1_{2n,\kappa^4,2\kappa^3\varepsilon}(Z,B)$. Define

$$p_t := v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1}.$$

Write $\underline{p}$ for the path (p_t), and note that according to Lemma 5.4 (i), we may identify the pair $(\underline{p}, 1_n \oplus 0_n)$ with (using the notation of Definition 4.9) an element of $\mathcal{P}^1_{2n,\kappa^8,2\kappa^7\varepsilon}(Z, SB)$, and therefore also a class $[\underline{p}, 1_n \oplus 0_n] \in KK^0_{\kappa^8,2\kappa^7\varepsilon}(Z, SB)$. By assumption, the forget control map in line (5.4) is zero, and therefore the image of $[\underline{p}, 1_n \oplus 0_n]$ in $KK^0_{\kappa^8,320\kappa^7\varepsilon}(X, SB)$ is zero. For notational simplicity, at this point let us define $\varepsilon_1 := 320\kappa^7\varepsilon$.

Now, Lemma 3.3 gives $m \in \mathbb{N}$ and $(s,s) \in \mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X, SB)$ such that $(\underline{p} \oplus 1_m \oplus 0_m, 1_n \oplus 0_n \oplus 1_m \oplus 0_m)$ and (s,s) are in the same path component of the set $\mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X, SB)$. Let x be a unitary matrix in $M_{2(n+m)}(\mathbb{C})$ such that $x(1_n \oplus 0_n \oplus 1_m \oplus 0_m)x^* = 1_{n+m} \oplus 0_{n+m}$. As x is connected to the identity through unitaries, the element $(x(\underline{p} \oplus 1_m \oplus 0_m)x^*, 1_{n+m} \oplus 0_{n+m})$ is also homotopic to (s,s) in $\mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X, SB)$; moreover (with notation as in Definition 4.9), it is in $\mathcal{P}^1_{2(n+m),2\kappa^8,\varepsilon_1}(X, SB)$. We may now apply Proposition 4.17 to see that if

$$M = 2^{(200\kappa^8)^3}$$

then there is $k \in \mathbb{N}$ and an element $\underline{w}$ of $\mathcal{U}^1_{2(n+m+k),M,M\varepsilon_1}(X, SB)$ such that

$$\underline{w}(x(\underline{p} \oplus 1_m \oplus 0_m)x^* \oplus 1_k \oplus 0_k)\underline{w}^{-1} = 1_{n+m} \oplus 0_{n+m} \oplus 1_k \oplus 0_k.$$

Write $\underline{v}$ for the path defined in line (5.5) above, which naturally defines an element of $\mathcal{L}_{SB}$ using the identification in line (5.3). Then, if we define

$$\underline{y} := \underline{w}(x \oplus 1_{2k})(\underline{v} \oplus 1_{2(m+k)}) \in \mathcal{L}_{SB},$$

we have

$$\underline{y}(1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k)\underline{y}^{-1} = 1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k.$$

In other words, the element $\underline{y}$ commutes with $1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k$. Define

$$\underline{z} := (1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k)\underline{y}(1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k).$$

Using Lemma 5.4 (ii), we may think of $\underline{z}$ as a continuous path $(z_t)_{t\in[0,1]}$ in $\mathcal{U}_{n+m+k,M,M\varepsilon_1}(X, B)$. Now, write $\underline{w}$ as a path $(w_t)_{t\in[0,1]}$, and note that as $\underline{w}$ is in $\mathcal{U}^1_{2(n+m+k),M,M\varepsilon_1}(X, SB)$, then by Lemma 5.4 (ii), $w_0 = w_1 = 1_{2(n+m)}$. Moreover, $v_0 = 1_{2n}$ by definition. Hence, $z_0 = x \oplus 1_k$. On the other hand $v_1 = u \oplus u^{-1} \oplus 1_{2(m+k)}$ and so $z_1 = (x \oplus 1_k)(u \oplus 1_{m+k})$. Hence, $(x \oplus 1_k)^*\underline{z}$ defines a homotopy in $\mathcal{U}_{n+m+k,M,M\varepsilon_1}(X, B)$ between 1_{n+m+k} and $u \oplus 1_{m+k}$. This implies $[u]$ maps to zero in $KK^1_{M,M\varepsilon_1}(X, SB)$, which completes the proof. ∎

Chapter 6

A Mayer–Vietoris boundary map

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to construct and analyse a "Mayer–Vietoris boundary map" in controlled KK-theory. The main results of the chapter prove the existence of this boundary map (Proposition 6.1) and show it has an exactness property (Proposition 6.6). These results are the technical heart of the paper.

6.1 Existence

Here, is the construction of the boundary map.

Proposition 6.1. *Define an increasing function $N_0 : [1,\infty) \to [0,\infty)$ by the formula $N_0(\kappa) = 2^{27}\kappa^{24}$. This function has the following properties.*

Let $\kappa \geq 1$, let $N_0 = N_0(\kappa)$, let $\varepsilon > 0$, let B be a separable C^*-algebra, and let X be a subset of the unit ball of $\mathcal{L}_B$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$. Then, there is a homomorphism

$$\partial : KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0,N_0\varepsilon}(X \cup \{h\}, B)$$

defined by applying the following process to a class from $KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$:

(i) *Choose a representative $w \in \mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ for the class, and use Proposition 4.13 (i) to find an element*

$$u \in \mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

that is in the same path component as w in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$.

(ii) *Define*

$$c = c(u,h) := hu + (1-h), \quad d = d(u,h) := hu^{-1} + (1-h) \tag{6.1}$$

in $M_n(\mathcal{L}_B)$, and

$$v = v(u,h) := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_{2n}(\mathcal{L}_B). \tag{6.2}$$

(iii) *Define*

$$\partial[w] := \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

Moreover, the boundary map is "natural with respect to forget control maps"; precisely, if for some $\kappa \leq \lambda$ and $\varepsilon \leq \delta$, the boundary maps

$$\partial : KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0(\kappa),N_0(\kappa)\varepsilon}(X \cup \{h\}, B)$$

and

$$\partial : KK^1_{\lambda,\delta}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0(\lambda),N_0(\lambda)\delta}(X \cup \{h\}, B)$$

both exist, then the diagram

$$\begin{array}{ccc} KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0(\kappa),N_0(\kappa)\varepsilon}(X \cup \{h\}, B) \\ \downarrow & & \downarrow \\ KK^1_{\lambda,\delta}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0(\lambda),N_0(\lambda)\delta}(X \cup \{h\}, B) \end{array}$$

(with vertical maps the forget control maps of Definitions 3.4 and 3.7) commutes.

In order to make the proof more palatable, we split off some computations as lemmas. The proofs of these lemmas are elementary, but the second one is quite lengthy. We record them for the sake of completeness, but recommend the reader skips the proofs.

Lemma 6.2. *Let B be a separable C^*-algebra. Let $u \in M_n(\mathcal{L}_B)$ be an invertible element such that $1-u \in M_n(\mathcal{K}_B)$, and let $h \in \mathcal{L}_B$ be a positive contraction. Then, the elements $c = c(u,h)$ and $d = d(u,h)$ from line (6.1) above have the following properties.*

(i) *The elements $cd-1$ and $dc-1$ are in $M_n(\mathcal{K}_B)$.*

(ii) *If $\kappa \geq 1$ and $\varepsilon > 0$ are such that $\|u\| \leq \kappa$, $\|u^{-1}\| \leq \kappa$, $\|[h,u]\| < \varepsilon$, and $\|[h,u^{-1}]\| < \varepsilon$, then $cd-1$ and $dc-1$ are both closer than $(\kappa+1)\varepsilon$ to $h(1-h)(u+u^{-1}-2)$.*

Proof. We just look at the case of $cd-1$ for both parts (i) and (ii); the case of $dc-1$ is similar. Note first that because $1-u$ is in $M_n(\mathcal{K}_B)$ and $M_n(\mathcal{K}_B)$ is an ideal in $M_n(\mathcal{L}_B)$, we must have that $1-u^{-1}$ is in $M_n(\mathcal{K}_B)$ also. We compute that

$$\begin{aligned} cd-1 &= huhu^{-1} + (1-h)hu^{-1} + hu(1-h) - 2h + h^2 \\ &= h^2 + hu[h,u^{-1}] + h(1-h)u^{-1} \\ &\quad + h(1-h)u + [h,u](1-h) - 2h + h^2. \end{aligned} \tag{6.3}$$

Using that u and u^{-1} equal 1 modulo the ideal $M_n(\mathcal{K}_B)$, we compute that this equals 0 modulo $M_n(\mathcal{K}_B)$. Hence, $cd-1$ is in $M_n(\mathcal{K}_B)$

Looking at part (ii), note that the terms $hu[h,u^{-1}]$ and $[h,u](1-h)$ in line (6.3) above have norms at most $\kappa\varepsilon$ and ε respectively. Hence, $cd-1$ is within $(\kappa+1)\varepsilon$ of $h^2+h(1-h)u^{-1}+h(1-h)u-2h+h^2$, which equals $h(1-h)(u+u^{-1}-2)$. ∎

Lemma 6.3. *Let B be a separable C^*-algebra. Let $\kappa\geq 1$, $\varepsilon>0$, and let X be a subset of the unit ball of $\mathcal{L}_B$. Let $h\in\mathcal{L}_B$ be a positive contraction such that $\|[h,x]\|<\varepsilon$ for all $x\in X$, and let u be an element of the set $\mathcal{U}^1_{n,\kappa,\varepsilon}(h(1-h)X\cup\{h\},B)$ from Definition 4.11. Let $c=c(u,h)$ and $d=d(u,h)$ be as in line (6.1) above, and let $v=v(u,h)$ be as in line (6.2).*

Then, $\|v\|\leq(\kappa+2)^3$, $\|v^{-1}\|\leq(\kappa+2)^3$, and the pair

$$\left(v\begin{pmatrix}1&0\\0&0\end{pmatrix}v^{-1},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)$$

is an element of $\mathcal{P}^1_{2n,3^6\kappa^6,2^{16}\kappa^5\varepsilon}(X\cup\{h\},B)$ from Definition 4.9.

Proof. From the definition of v in line (6.2) above,

$$v=\begin{pmatrix}c(dc-2)&1-cd\\dc-1&-d\end{pmatrix}\tag{6.4}$$

and

$$v^{-1}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&-c\\0&1\end{pmatrix}\begin{pmatrix}1&0\\d&1\end{pmatrix}\begin{pmatrix}1&-c\\0&1\end{pmatrix}=\begin{pmatrix}-d&dc-1\\1-cd&c(dc-2)\end{pmatrix}.$$

Hence,

$$v\begin{pmatrix}1&0\\0&0\end{pmatrix}v^{-1}=\begin{pmatrix}cd(2-cd)&c(dc-2)(dc-1)\\(1-dc)d&(dc-1)^2\end{pmatrix}$$

and so

$$v\begin{pmatrix}1&0\\0&0\end{pmatrix}v^{-1}-\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}-(cd-1)^2&(cd-1)c(dc-2)\\(1-dc)d&(dc-1)^2\end{pmatrix}.\tag{6.5}$$

This formula, part (i) of Lemma 6.2, and the fact that $M_n(\mathcal{K}_B)$ is an ideal in $M_n(\mathcal{L}_B)$ imply that

$$v\begin{pmatrix}1&0\\0&0\end{pmatrix}v^{-1}-\begin{pmatrix}1&0\\0&0\end{pmatrix}\in M_{2n}(\mathcal{K}_B),$$

whence $v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)v^{-1}$ is in $M_{2n}(\mathcal{K}_B^+)$, and $v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)v^{-1}$ and $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ have the same image under the image of the canonical quotient map

$$\sigma:M_{2n}(\mathcal{K}_B^+)\to M_{2n}(\mathbb{C}).$$

Note moreover that $\|v\| \leq (\kappa+2)^3$ and $\|v^{-1}\| \leq (\kappa+2)^3$ from the formula for v (whence also v^{-1}) as a product of four matrices in line (6.2). As $\kappa \geq 1$, this implies that

$$\left\| v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\| \leq (\kappa+2)^6 \leq 3^6\kappa^6.$$

To complete the proof that the pair

$$\left(v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

defines an element of $\mathcal{P}^1_{2n,3^6\kappa^6,2^{16}\kappa^5\varepsilon}(X,B)$ it remains to check the relevant commutator estimates, i.e., condition (ii) from Definition 3.1 with x in $X \cup \{h\}$ and ε replaced by $2^{16}\kappa^5\varepsilon$. As $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (and indeed, any scalar matrix) commutes with elements of $X \cup \{h\}$ exactly, it suffices to show that

$$\left\| \left[x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| \leq 2^{16}\kappa^5\varepsilon \tag{6.6}$$

for all $x \in X \cup \{h\}$. We focus on the case when x is in X; the case when $x = h$ follows from similar (and much simpler) estimates that we leave to the reader.

Working towards the estimate in line (6.6), we compute that the element in line (6.5) equals

$$\begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix}. \tag{6.7}$$

The second matrix above satisfies

$$\begin{aligned} \left\| \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right\| &\leq \|1-cd\| + \|c\|\|dc-2\| + \|d\| + \|dc-1\| \\ &\leq ((\kappa+1)^2+1) + (\kappa+1)((\kappa+1)^2+2) \\ &\quad + (\kappa+1) + ((\kappa+1)^2+1). \end{aligned}$$

As $\kappa + 1 \geq 1$, we therefore see that

$$\left\| \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right\| \leq 8(\kappa+1)^4. \tag{6.8}$$

On the other hand, using part (ii) of Lemma 6.2, the first matrix in line (6.7) above is closer than $\varepsilon(\kappa+1)$ to $h(1-h)(u+u^{-1}-2)$ (we identify this as usual with the diagonal matrix with both entries equal to $h(1-h)(u+u^{-1}-2)$). Hence, the difference in line (6.5) is closer than $8(\kappa+1)^5\varepsilon$ to

$$h(1-h)(u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix}.$$

Hence, for $x \in X$,

$$\left\| \left[x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\|$$
$$< 16(\kappa+1)^5\varepsilon + \left\| \left[x, h(1-h)(u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] \right\|. \quad (6.9)$$

As $\|[x,h]\| < \varepsilon$, we have $\|[x,h(1-h)]\| < 2\varepsilon$; combining this with line (6.8) gives

$$\left\| \left[x, h(1-h)(u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] \right\|$$
$$< 2\varepsilon \cdot 8(\kappa+1)^5 + \left\| h(1-h) \left[x, (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] \right\|.$$

Combining this with line (6.9) gives

$$\left\| \left[x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\|$$
$$< 32(\kappa+1)^5\varepsilon + \left\| h(1-h) \left[x, (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] \right\|. \quad (6.10)$$

Every entry of the matrix $(u+u^{-1}-2)\left(\begin{smallmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{smallmatrix}\right)$ can be written as a sum of at most 30 terms, each of which is a product of at most 5 elements from the set $\{u, u^{-1}, h, 1\}$, each of which has norm at most κ. As $\|[h(1-h)x, y]\| < \varepsilon$ for all $y \in \{u, u^{-1}, h, 1\}$, Lemma 4.16 gives

$$\left\| \left[h(1-h)x, (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] \right\| < 4 \cdot 30 \cdot 5 \cdot \kappa^4 \varepsilon. \quad (6.11)$$

On the other hand, $\|[h(1-h), y]\| < 2\varepsilon$ for all $y \in \{u, u^{-1}, h, 1\}$, whence

$$\left\| \left[h(1-h), (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] x \right\| < 4 \cdot 30 \cdot 5 \cdot \kappa^4 \varepsilon. \quad (6.12)$$

Finally, note that

$$h(1-h) \left[x, (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right]$$
$$= \left[h(1-h)x, (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right]$$
$$+ \left[h(1-h), (u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right] x,$$

so combining lines (6.10), (6.11), and (6.12) implies

$$\left\| \left[x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| < 1232(\kappa+1)^5 \varepsilon.$$

Recalling that $\kappa \geq 1$, this is enough for the estimate in line (6.6). ∎

We are now ready for the proof of Proposition 6.1.

Proof of Proposition 6.1. Assume that $w \in \mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$, and let

$$u \in \mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

be in the same path component as w in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$; u is guaranteed to exist by Proposition 4.13 (i). Define $v := v(u,h)$ as in line (6.2), so Lemma 6.3 gives an element

$$\left(v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \in \mathcal{P}_{2n,3^6\kappa^{12},2^{16}\kappa^{11}\varepsilon}(X \cup \{h\}, B).$$

Moreover, if $u_0 := u$, and u_1 is another choice of element in

$$\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

that is connected to w in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ then Proposition 4.13 (ii) implies that there is a homotopy $(u_t)_{t\in[0,1]}$ that connects u_0 and u_1 through

$$\mathcal{U}^1_{n,\kappa^4,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B).$$

Let $v_t := v(u_t, h)$ be as in line (6.2). Then, Lemma 6.3 implies that the path

$$t \mapsto \left(v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad t \in [0,1]$$

has image in $\mathcal{P}^1_{2n,3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$. In particular, the class

$$\partial[w] \in KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$$

does not depend on the choice of u, so at this point we have a well-defined set map

$$\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B).$$

We next claim that this map sends block sums on the left to sums on the right.

For this, assume that w_1 and w_2 are elements of $\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. Let u_1 and u_2 be elements of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are connected to w_1 and

w_2 respectively in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$. For $i \in \{1,2\}$ let $v_i = v(u_i, h)$ be as in line (6.2), and let $v := v(u_1 \oplus u_2, h) \in M_{4n}(\mathcal{L}_B)$. Then, the pairs

$$\left(v_1 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_1^{-1} \oplus v_2 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_2^{-1},\ \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right)$$

and

$$\left(v \begin{pmatrix} 1_{2n} & 0 \\ 0 & 0 \end{pmatrix} v^{-1},\ \begin{pmatrix} 1_{2n} & 0 \\ 0 & 0 \end{pmatrix}\right)$$

in $M_{4n}(\mathcal{K}_B^+) \oplus M_{4n}(\mathcal{K}_B^+)$ differ by conjugation by the same (scalar) permutation matrix in each component, and so define the same class in $KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$.

At this point, we have a semigroup homomorphism

$$\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B).$$

We claim that it respects the equivalence relation defining $KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. First, we check that $w \oplus 1_k$ goes to the same class as w. As we already know we have a semigroup homomorphism, it suffices to show that 1_k goes to zero in $KK^0_{3^6\kappa^{24},2^{16}\kappa^{20}\varepsilon}(X \cup \{h\}, B)$. For this, note that if $v := v(1_k, h)$ is as in line (6.2), then $v = 1_{2k}$, whence the image of 1_k in $KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$ is the class $[1_k \oplus 0_k, 1_k \oplus 0_k]$, which is zero by definition.

Let us now show that elements of $\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are homotopic through $\mathcal{U}_{n,2\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ go to the same class. For this, say that w_0 and w_1 are homotopic through $\mathcal{U}_{n,2\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. Choose u_0 and u_1 in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are connected to w_0 and w_1 respectively in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ as in Proposition 4.13 (i). Using Proposition 4.13 (ii), u_0 and u_1 are connected by a homotopy $(u_t)_{t\in[0,1]}$ in $\mathcal{U}^1_{n,4\kappa^4,2\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$. Let $v_t := v(u_t, h)$ be as in line (6.2). Then, Lemma 6.3 implies that the path

$$\left(v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1},\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

defines a homotopy between the images of w_0 and w_1 in $\mathcal{P}^1_{2n,3^{14}\kappa^{24},2^{27}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$. We thus see that $N_0(\kappa) := 2^{27}\kappa^{24}$ has the desired property, and we are done with the existence of ∂.

As the formulas for the boundary map ∂ do not depend on the constants κ and ε the naturality statement is clear. ∎

6.2 Exactness

We now turn to the exactness property of the boundary map. In order to state this, we need two lemmas.

Lemma 6.4. *Let B be a separable C^*-algebra. Let X and Y be subsets of the unit ball of $\mathcal{L}_B$, $\varepsilon > 0$ and $\kappa \geq 1$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h, x]\| < \varepsilon$ for all $x \in X$. With notation as in Definition 3.1, let*

$$(p, q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X \cup Y \cup \{h\}, B)$$

(respectively, with notation as in Definition 4.9, let $(p, q) \in \mathcal{P}^{(1)}_{n,\kappa,\varepsilon}(X \cup Y \cup \{h\}, B)$). Then,

$$(p, q) \in \mathcal{P}_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$$

(respectively, $(p, q) \in \mathcal{P}^1_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$).

In particular, there are homomorphisms

$$\eta_h : KK^0_{\kappa,\varepsilon}(X \cup Y \cup \{h\}, B) \to KK^0_{\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$$

and

$$\eta_{1-h} : KK^0_{\kappa,\varepsilon}(X \cup Y \cup \{h\}, B) \to KK^0_{\kappa,2\varepsilon}((1-h)X \cup Y \cup \{h\}, B)$$

induced by the identity map on cycles (p, q).

Proof. We compute that for $x \in X$,

$$\|[p, hx]\| \leq \|h\|\|[p, x]\| + \|[p, h]\|\|x\| < \varepsilon + \varepsilon.$$

These estimates hold similarly for q so $(p, q) \in \mathcal{P}^1_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$. As the identity map on cycles takes homotopies to homotopies, and block sums to block sums, existence of the homomorphism η_h is clear. Existence of η_{1-h} follows on noting that the assumptions on h also holds for $1 - h$. ∎

We leave the direct checks needed for the proof of the next lemma for the reader.

Lemma 6.5. *Let B be a separable C^*-algebra. Let X and Y be subsets of the unit ball of $\mathcal{L}_B$, $\varepsilon > 0$ and $\kappa \geq 1$. Assume moreover that there is $\delta > 0$ such that for all $y \in Y$, $x \in_\delta X$. Then, for any $\gamma \geq \kappa\delta + \varepsilon$ and $\lambda \geq \kappa$, the forget control map of Definition 3.4*

$$KK^0_{\kappa,\varepsilon}(X, B) \to KK_{\lambda,\gamma}(Y, B)$$

is well-defined. ∎

The next proposition is the exactness property of the Mayer–Vietoris boundary map that we are aiming for. We refer the reader to Section 1.6 for motivation behind the statement. For the statement, recall that for an element x and subset Y of a metric space, and for $\varepsilon > 0$, we write "$x \in_\varepsilon S$" to mean that there is $y \in Y$ with $d(x, y) < \varepsilon$. Moreover, in the statement below, all unlabeled arrows between controlled KK-groups are the forget control maps of Definition 3.4 or Definition 3.7.

Proposition 6.6. *The increasing functions $N_1, N_2 : [1,\infty) \to [1,\infty)$ defined by*

$$N_1(\lambda) = 2^{9000000\lambda^3} \quad \text{and} \quad N_2(\mu) = 2^3 7\mu^{25}.$$

satisfy the following properties.

Let $\kappa \geq 1$, and let $\varepsilon > 0$. Let $\lambda \geq \kappa$, and let $\delta \geq 3\kappa\varepsilon$. Let $N_1 := N_1(\lambda)$, and let $\mu \geq N_1$ and $\gamma \geq N_1\delta$. With notation as in Proposition 6.1, define

$$N_0 := N_0(\mu),$$

and let $N_2 := N_2(\mu)$.

Let B be a separable C^*-algebra, and let X be a self-adjoint subset of the unit ball of $\mathcal{L}_B$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$. Let Y_h, Y_{1-h}, and Y be self-adjoint subsets of the unit ball of $\mathcal{L}_B$ such that $y \in_\varepsilon Y_h$ and $y \in_\varepsilon Y_{1-h}$ for all $y \in Y$. With notation as in Definition 4.9, let (p,q) be an element of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B)$. With η_h and η_{1-h} as in Lemma 6.4, and suing Lemma 6.5 to define the right hand vertical maps in each case, assume that the images of $[p,q]$ under the maps

$$\begin{array}{ccc} KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup \{h\}, B) & \xrightarrow{\eta_h} & KK^0_{\kappa,2\varepsilon}(hX \cup Y_h \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B) \end{array} \tag{6.13}$$

and

$$\begin{array}{ccc} KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{\kappa,\varepsilon}(X \cup Y_{1-h} \cup \{h\}, B) & \xrightarrow{\eta_{1-h}} & KK^0_{\kappa,2\varepsilon}((1-h)X \cup Y_{1-h} \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B) \end{array} \tag{6.14}$$

are zero.

Then, with notation as in Definition 4.11, there exists an element

$$u \in \mathcal{U}^1_{\infty,N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Y, B)$$

such that in the diagram below

$$\begin{array}{ccc} KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Y, B) & & KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B) \\ \downarrow & & \downarrow \\ KK^1_{\mu,\gamma}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0,N_0\gamma}(X \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B) \end{array}$$

the images of the classes $[u] \in KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Y)$ *and*

$$[p,q] \in KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B)$$

in the bottom right group $KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B)$ *are the same.*

Just as for Proposition 6.1, to make the argument more palatable, we split off some computations as two technical lemmas. As in that earlier case, the arguments we give for these lemmas are elementary, but quite lengthy (in fact, much longer than the earlier ones). We record them for the sake of completeness, but again recommend that the reader skips the proofs.

Lemma 6.7. *Let* B *be a separable* C^**-algebra. Let* $\nu \geq 1$ *and let* $\gamma > 0$*. Let* X *and* Y *be self-adjoint subsets of the unit ball of* $\mathcal{L}_B$*. Let* $h \in \mathcal{L}_B$ *be a positive contraction such that* $\|[h,x]\| < \gamma$ *for all* $x \in X$*. Let* $(p,q) \in \mathcal{P}^1_{n,\nu,\gamma}(X \cup Y \cup \{h\}, B)$ *(see Definition 4.9 for notation), and let* $u_h \in \mathcal{U}^1_{n,\nu,\gamma}(hX \cup \{h\} \cup Y, B)$ *and* $u_{1-h} \in \mathcal{U}^1_{n,\nu,\gamma}((1-h)X \cup \{h\} \cup Y, B)$ *(see Definition 4.11 for notation).*

Then, the element

$$u := u_{1-h}(1-p) + u_h p \tag{6.15}$$

is in $\mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X \cup \{h\} \cup Y, B)$.

Proof. We split the computations into the points labeled (i), (ii), (iii), (iv), and (v) below.

(i) As $u_h - 1 \in M_n(\mathcal{K}_B)$ and $u_{1-h} - 1 \in M_n(\mathcal{K}_B)$, we compute from line (6.15) that $u - 1 \in M_n(\mathcal{K}_B)$.

(ii) Note that

$$\|1-p\| \leq \nu \tag{6.16}$$

by Corollary 4.2. Hence, $\max\{\|u_h\|, \|u_{1-h}\|, \|p\|, \|1-p\|\} \leq \nu$, and so by line (6.15), $\|u\| \leq 2\nu^2$.

(iii) Let $y \in Y$. Then, by definition, $\|[a,y]\| < \gamma$ for all $a \in \{u_h, u_{1-h}, p, 1-p\}$. Hence, the definition of u from line (6.15) implies that $\|[y,u]\|$ is bounded above by

$$\|[y,u_{1-h}]\|\|1-p\| + \|u_{1-h}\|\|[y,1-p]\| + \|[y,u_h]\|\|p\| + \|u_h\|\|[y,p]\| < 4\nu\gamma.$$

(iv) Using the definition of u from line (6.15) and the assumptions on u_h, u_{1-h} and p directly together with line (6.16) implies that

$$\begin{aligned}\|[u,h]\| &\leq \|[h,u_{1-h}]\|\|1-p\| + \|u_{1-h}\|\|[h,1-p]\| \\ &\quad + \|[h,u_h]\|\|p\| + \|u_h\|\|[h,p]\| \\ &< 4\nu\gamma.\end{aligned}$$

(v) Let $x \in X$ and note that

$$[h(1-h)x,u_h] = (1-h)[hx,u_h] + [h,u_h](1-h)x.$$

As $\|[hx,u_h]\| < \gamma$, as $\|[h,u_h]\| < \gamma$, as h is a positive contraction, and as x is a contraction, we get

$$\|[h(1-h)x,u_h]\| \leq \|[hx,u_h]\|\|1-h\| + \|hx\|\|[1-h,u_h]\| < 2\gamma. \tag{6.17}$$

Completely analogously, we see that

$$\|[h(1-h)x,u_{1-h}]\| < 2\gamma. \tag{6.18}$$

We see also that

$$\begin{aligned}\|[h(1-h)x,p]\| &\leq \|[x,p]\|\|h(1-h)\| + \|[1-h,p]\|\|hx\| + \|[h,p]\|\|(1-h)x\| \\ &< 3\gamma.\end{aligned}$$

Combining this with lines (6.16), (6.17), (6.18), we get

$$\begin{aligned}\|[h(1-h)x,u]\| &\leq \|[h(1-h)x,u_{1-h}]\|\|1-p\| + \|u_{1-h}\|\|[h(1-h)x,1-p]\| \\ &\quad + \|[h(1-h)x,u_h]\|\|p\| + \|u_h\|\|[h(1-h)x,p]\| \\ &< 2\nu\gamma + 3\nu\gamma + 2\nu\gamma + 3\nu\gamma \\ &= 10\nu\gamma.\end{aligned}$$

Putting the points (i), (ii), (iii), (iv), and (v) above together (and using that $\nu \geq 1$) we conclude that, u is an element of $\mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X \cup \{h\} \cup Y, B)$ as claimed. ∎

Lemma 6.8. *With assumptions as in Lemma 6.7, let*

$$u := u_{1-h}(1-p) + u_h p \in \mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X \cup \{h\} \cup Y, B)$$

be the element considered there. Let $v := v(u,h)$ be as in line (6.2) above, and define

$$w := \begin{pmatrix} u_{1-h}(1-p) & -q \\ p & (1-p)u_{1-h}^{-1} \end{pmatrix} \in M_{2n}(\mathcal{L}_B).$$

Then, w is invertible, and vw^{-1} is in $\mathcal{U}_{2n,(2\nu)^8,2^{37}\nu^{25}\gamma}(X \cup \{h\}, B)$.

Proof. Using the assumptions on $\|p\|$, $\|u_{1-h}\|$, $\|u_{1-h}^{-1}\|$ and line (6.16) to estimate $\|1-p\|$, we have

$$\|w\| \leq \|u_{1-h}(1-p)\| + \|q\| + \|p\| + \|(1-p)u_{1-h}^{-1}\| \leq 4\nu^2.$$

A direct computation shows that w is invertible with inverse

$$w^{-1} = \begin{pmatrix} (1-p)u_{1-h}^{-1} & p \\ -q & u_{1-h}(1-p) \end{pmatrix}. \tag{6.19}$$

This satisfies the same norm estimate as w, and so we get the norm estimates

$$\|w\| \leq (2\nu)^2 \quad \text{and} \quad \|w^{-1}\| \leq (2\nu)^2. \tag{6.20}$$

Lemma 6.3 and the fact that $\|u\| \leq 2\nu^2$ implies that $\|v\| \leq (2\nu^2+2)^3$ and $\|v^{-1}\| \leq (2\nu^2+2)^3$. As $\nu \geq 1$, we thus see that

$$\|v\| \leq (2\nu)^6 \quad \text{and} \quad \|v^{-1}\| \leq (2\nu)^6. \tag{6.21}$$

Lines (6.20) and (6.21) then imply

$$\|vw^{-1}\| \leq (2\nu)^8 \quad \text{and} \quad \|wv^{-1}\| \leq (2\nu)^8. \tag{6.22}$$

To complete the proof, we need to show that for all $x \in X \cup \{h\}$, we have $\|[vw^{-1}, x]\| < 2^{37}\nu^{25}\gamma$ and $\|[wv^{-1}, x]\| < 2^{37}\nu^{25}\gamma$. We focus first on the case of vw^{-1}, and look first at $[h, vw^{-1}]$.

Let $c := hu + (1-h)$ and $d := hu^{-1} + (1-h)$ be as in line (6.1). It will be technically convenient to define

$$S := \{h, 1-h, p, q, 1-p, 1-q, u_h, u_h^{-1}, u_{1-h}, u_{1-h}^{-1}, u, u^{-1}, c, d\}, \tag{6.23}$$

and to define S^n to be the set of all products of at most n elements from S. Note that for every $s \in S$ we have $\|s\| \leq (2\nu)^2$, and $\|[s,h]\| < 10\nu\gamma$. Hence, by Lemma 4.16, for all $n \in \mathbb{N}$ we have

$$s \in S^n \Rightarrow \|[h,s]\| < n(2\nu)^{2(n-1)}10\nu\gamma. \tag{6.24}$$

Using the formula in line (6.4) above,

$$[h,v] = \begin{pmatrix} [cdc,h] - 2[c,h] & [cd,h] \\ [h,dc] & [d,h] \end{pmatrix}$$

and so

$$\|[h,v]\| \leq \|[cdc,h]\| + 2\|[c,h]\| + \|[cd,h]\| + \|[h,dc]\| + \|[d,h]\|.$$

Each summand on the right-hand side above is of the form $\|[h,s]\|$ where $s \in S^3$ for S as in line (6.23). Hence, line (6.24) implies that

$$\|[h,v]\| < 6 \cdot 3 \cdot (2\nu)^4 \cdot 10\nu\gamma \leq 2^{11}\nu^5\gamma. \tag{6.25}$$

We also compute that

$$[h,w^{-1}] = \begin{pmatrix} [h,(1-p)u_{1-h}^{-1}] & [h,p] \\ [q,h] & [h,u_{1-h}(1-p)] \end{pmatrix},$$

whence

$$\|[h,w^{-1}]\| \leq \|[h,(1-p)u_{1-h}^{-1}]\| + \|[h,p]\| + \|[q,h]\| + \|[h,u_{1-h}(1-p)]\|.$$

Each commutator appearing above is of the form $[h,s]$ for some $s \in S^2$ as in line (6.23), whence line (6.24) gives

$$\|[h,w^{-1}]\| < 4 \cdot (2\nu)^2 \cdot 10\nu\gamma \leq 2^7\nu^3\gamma. \tag{6.26}$$

On the other hand,

$$\|[h,vw^{-1}]\| \leq \|[h,v]\|\|w^{-1}\| + \|v\|\|[h,w^{-1}]\|.$$

Combining this with lines (6.20), (6.21), (6.25), and (6.26), as well as that $\nu \geq 1$, we see that

$$\|[h,vw^{-1}]\| < 2^{11}\nu^5\gamma \cdot (2\nu)^2 + (2\nu)^6 \cdot 2^7\nu^3\gamma \leq 2^{14}\nu^9\gamma. \tag{6.27}$$

Now, let us look at $[x,vw^{-1}]$ for $x \in X$. The definition of v from line (6.2) gives

$$\begin{aligned} vw^{-1} &= \begin{pmatrix} c(dc-1) & 1-cd \\ dc-1 & 0 \end{pmatrix} w^{-1} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} w^{-1} \\ &= \begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} w^{-1} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} w^{-1}. \end{aligned}$$

Hence, the formula for w^{-1} from line (6.19) gives

$$\begin{aligned} vw^{-1} = &\underbrace{\begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} c(1-p)u_{1-h}^{-1} & cp - u_{1-h}(1-p) \\ (1-p)u_{1-h}^{-1} & p \end{pmatrix}}_{y_1} \\ &\underbrace{- h \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}}_{y_2} \\ &\underbrace{- (1-h) \begin{pmatrix} (1-p)u_{1-h}^{-1} & p \\ -q & u_{1-h}(1-p) \end{pmatrix}}_{y_3}. \end{aligned} \tag{6.28}$$

We now estimate $\|[vw^{-1}, x]\|$ for each $x \in X$ by looking at each of the terms y_1, y_2, and y_3 separately.

(i) First, we look at y_1 from line (6.28). Let $x \in X$. Lemma 6.2 implies that

$$\left\| \begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} - h(1-h)(u+u^{-1}-2) \right\| < (v+1)\gamma \tag{6.29}$$

(where, as usual, we identify $h(1-h)(u+u^{-1}-2)$ with the corresponding diagonal matrix). Let

$$z_1 := \begin{pmatrix} c(1-p)u_{1-h}^{-1} & cp - u_{1-h}(1-p) \\ (1-p)u_{1-h}^{-1} & p \end{pmatrix}. \tag{6.30}$$

As in line (6.16), $\|1-p\| \leq v$, whence using that $v \geq 1$,

$$\begin{aligned} \|z_1\| &\leq \|c\|\|1-p\|\|u_{1-h}^{-1}\| + \|c\|\|p\| + \|u_{1-h}\|\|1-p\| \\ &\quad + \|1-p\|\|u_{1-h}^{-1}\| + \|p\| \\ &\leq (2v^2+1)v^2 + (2v^2+1)v + v^2 + v^2 + v \\ &\leq 9v^4. \end{aligned} \tag{6.31}$$

Combining this with line (6.29), we see that

$$\begin{aligned} &\|y_1 - h(1-h)(u+u^{-1}-2)z_1\| \\ &\quad \leq \left\| \begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} - h(1-h)(u+u^{-1}-2) \right\| \|z_1\| \\ &\quad < 9v^4(v+1)\gamma \leq (2v)^5\gamma. \end{aligned}$$

As $\|x\| \leq 1$, this implies that

$$\begin{aligned} \|[x, y_1]\| &\leq \|[x, y_1 - h(1-h)(u+u^{-1}-2)z_1]\| \\ &\quad + \|[x, h(1-h)(u+u^{-1}-2)z_1]\| \\ &< (2v)^5\gamma + \|[x, h(1-h)(u+u^{-1}-2)z_1]\|. \end{aligned}$$

Hence, we see that

$$\begin{aligned} \|[x, y_1]\| &< (2v)^5\gamma + \|[[x, h(1-h)], (u+u^{-1}-2)z_1]\| \\ &\quad + \|[h(1-h)x, (u+u^{-1}-2)z_1]\| \\ &\quad + \|[h(1-h), (u+u^{-1}-2)z_1]x\|. \end{aligned} \tag{6.32}$$

Looking at line (6.30), every entry of the matrix $(u+u^{-1}-2)z_1$ is a sum of at most 8 elements from the set S^4, where S is as in line (6.23). Hence, by line (6.24), we see that

$$\|[h(1-h), (u+u^{-1}-2)z_1]\| < 4 \cdot 2 \cdot 8 \cdot 4 \cdot (2v)^6 \cdot 12v^2\gamma \leq 2^{18}v^8\gamma. \tag{6.33}$$

We have $\|[x, h(1-h)]\| < 2\gamma$, and line (6.31) implies

$$\|(u + u^{-1} - 2)z_1\| \leq (4\nu^2 + 2) \cdot 9\nu^4 \leq 2^6\nu^6,$$

whence

$$\|[[x, h(1-h)], (u + u^{-1} - 2)z_1]\| \leq 2^8\nu^6\gamma. \tag{6.34}$$

Combining lines (6.32), (6.33), and (6.34) thus implies that

$$\|[x, y_1]\| \leq 2^{19}\nu^8\gamma + \|[h(1-h)x, (u + u^{-1} - 2)z_1]\|. \tag{6.35}$$

Note now that for every element $s \in S$ we have that at least one of the following holds: (a) $\|[s, x]\| < 16\nu^2\gamma$ for all $x \in X$; or (b) $\|[s, (1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$; or (c) $\|[s, (1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$; or (d) $\|[s, h(1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$. In any of these cases, using that $\|[s, h]\| \leq 12\nu^2\gamma$ for any $s \in S$, we get that for any $s \in S$ and $x \in X$, $\|[s, h(1-h)x]\| < 40\nu^2\gamma$. Applying Lemma 4.16, we therefore see that

$$s \in S^n \Rightarrow \|[h(1-h)x, s]\| < n(2\nu)^{2(n-1)}40\nu^2\gamma. \tag{6.36}$$

As we have observed above already, every entry in the matrix $(u + u^{-1} - 2)z_1$ is a sum of at most 8 elements from the set S^4, where S is as in line (6.23). From line (6.36) we therefore see that

$$\|[h(1-h)x, (u + u^{-1} - 2)z_1]\| < 4 \cdot 4 \cdot (2\nu)^4 \cdot 40\nu^2\gamma \leq 2^{14}\nu^6\gamma.$$

Combining this with line (6.35) above therefore implies

$$\|[x, y_1]\| < 2^{20}\nu^8\gamma.$$

(ii) Now, we look at the element y_2 from line (6.28) above. If $x \in X$, we see that

$$[x, y_2] = \left[xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right] + \left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right]x. \tag{6.37}$$

We have that

$$\left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right] = \begin{pmatrix} [q, h] & [h, u_h p] \\ [u_h^{-1}q, h] & [p, h] \end{pmatrix}.$$

Each entry in the matrix on the right is the commutator of h with an element of S^2, where S is as in line (6.23) above. Hence, by line (6.24), we see that

$$\left\|\left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right]\right\| < 4 \cdot 2 \cdot (2\nu)^2 \cdot 12\nu^2\gamma \leq 2^9\nu^4\gamma.$$

Combining this with line (6.37) gives

$$\|[x, y_2]\| < \left\| \left[xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| + 2^9 \nu^4 \gamma. \tag{6.38}$$

On the other hand

$$\begin{aligned} \left[xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] &= \left[[x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \\ &\quad + \left[hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right]. \end{aligned} \tag{6.39}$$

As $\|[h, x]\| < \gamma$, we have

$$\left\| \left[[x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| \leq 2\gamma \left\| \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right\|.$$

As $\|1-p\| \leq \nu$ and $\|1-q\| \leq \nu$ by Corollary 4.2, every entry of the matrix on the right has norm at most ν^2, and so

$$\left\| \left[[x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < 2^3 \nu^2 \gamma.$$

Hence, line (6.39) implies that

$$\left\| \left[xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < \left\| \left[hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| + 2^3 \nu^2 \gamma. \tag{6.40}$$

The commutator appearing on the right above equals

$$\begin{pmatrix} [q, hx] & [hx, u_h]p + u_h[hx, p] \\ [u_h^{-1}, hx]q - u_h^{-1}[hx, q] & [p, hx] \end{pmatrix}.$$

Using that $u_h \in \mathcal{U}^1_{n,\nu,\gamma}(hX, B)$, and applying Lemma 6.4, the norm of each entry above is at most $2\nu\gamma$, whence

$$\left\| \left[hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < 2^3 \nu \gamma.$$

Combining this with lines (6.38) and (6.40) therefore implies that

$$\|[x, y_2]\| < 2^{10} \nu^4 \gamma.$$

(iii) Finally, we look at y_3 from line (6.28). This can be handled very similarly to the case of y_2, giving the estimate $\|[x, y_3]\| < 2^{10} \nu^4 \gamma$ for all $x \in X$; we leave the details to the reader.

Putting together the concluding estimates of points (i), (ii), and (iii) above, we see that $\|[x, vw^{-1}]\| < 2^{21}\nu^8\gamma$ for all $x \in X$. Combining this with line (6.27), we see that

$$\|[x, vw^{-1}]\| < 2^{21}\nu^9\gamma \tag{6.41}$$

for all $x \in X \cup \{h\}$.

To complete the proof, let us estimate $\|[x, wv^{-1}]\|$ for $x \in X \cup \{h\}$. Using the formula $[x, wv^{-1}] = wv^{-1}[vw^{-1}, x]wv^{-1}$, we see that

$$\|[x, wv^{-1}]\| \leq \|wv^{-1}\|\|[vw^{-1}, x]\|\|wv^{-1}\|.$$

Lines (6.41) and (6.22) therefore imply that

$$\|[x, wv^{-1}]\| \leq 2^{37}\nu^{25}\gamma$$

and we are finally done. ∎

Finally, we are ready for the proof of Proposition 6.6.

Proof of Proposition 6.6. With notation as in the statement, let

$$(p, q) \in \mathcal{P}^1_{n,\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B),$$

and assume that the images of $[p, q]$ in $KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ and $KK^0_{\lambda,\delta}((1-h)X \cup Y \cup \{h\}, B)$ under the maps in lines (6.13) and (6.14) are zero.

Note first that the map in line (6.13) is induced by the identity map on cycles, so Lemma 3.3 applied to the cycle (p, q) in $\mathcal{P}_{n,\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ implies that there exists $k \in \mathbb{N}$ such that $(p \oplus 1_k \oplus 0_k, q \oplus 1_k \oplus 0_k)$ is in the same path component of $\mathcal{P}_{n+2k,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ as an element of the form (r, r). Replacing (r, r) with (yry^*, yry^*) for some appropriate unitary $y \in M_{n+2k}(\mathbb{C})$ and using that the unitary group of $M_{n+2k}(\mathbb{C})$ is connected, we may assume that (r, r) is in $\mathcal{P}^1_{n+2k,2\lambda,\delta}(hX \cup Y_h \cup \{h\}, B)$ (see Definition 4.9 for notation). Moreover, as

$$(p, q) \in \mathcal{P}^1_{n,\lambda,\delta}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B)$$

there is a unitary $z \in M_{n+2k}(\mathbb{C})$ such that $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ is in $\mathcal{P}^1_{n,\lambda,\delta}(hX \cup Y \cup \{h\}, B)$. As the elements (r, r) and $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ of $\mathcal{P}^1_{n,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ are connected by a path $\mathcal{P}_{n,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$, we may use Proposition 4.10 (ii) to connect them by a path in $\mathcal{P}^1_{n,2\lambda,4\delta}(hX \cup Y \cup \{h\}, B)$. Precisely analogously (increasing k if necessary), we may assume that $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ is in the same path component of

$$\mathcal{P}^1_{n,2\lambda,4\delta}((1-h)X \cup Y_{1-h} \cup \{h\}, B)$$

as an element of the form (s, s).

For notational simplicity, write $m = n + 2k$, and let $M := 4 \cdot 2^{(200\lambda)^3}$. Then, (with notation as in Definition 4.11), Proposition 4.17 gives $j \in \mathbb{N}$ and elements

$$u_h \in \mathcal{U}^1_{m+2j,M,M\delta}(hX \cup \{h\} \cup Y, B)$$

and

$$u_{1-h} \in \mathcal{U}^1_{m+2j,M,M\delta}((1-h)X \cup \{h\} \cup Y, B)$$

such that

$$u_h(z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j)u_h^{-1} = z(q \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j \qquad (6.42)$$

and

$$u_{1-h}(z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j)u_{1-h}^{-1} = z(q \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j. \qquad (6.43)$$

For notational simplicity, rename $z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j$ and $z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j$ as p and q respectively and rewrite $m + 2j$ as n; if the conclusion of the proposition holds for this new pair then it also holds for the original pair thanks to the definition of the controlled KK^0 groups (see Definition 3.1), so this makes no real difference. In this new language, lines (6.42) and (6.43) can be rewritten $u_h p u_h^{-1} = q$ and $u_{1-h} p u_{1-h}^{-1} = q$ respectively.

Define now

$$u := u_{1-h}(1-p) + u_h p,$$

which we claim has the properties in the statement. Using Lemma 6.7 with $\nu = M$ and $\gamma = M\delta$, we see that (with notation as in Definition 4.11), u is an element of $\mathcal{U}^1_{n,2M^2,10M^2\delta}(h(1-h)X \cup \{h\} \cup Y, B)$. Recalling that $M = 4 \cdot 2^{(200\lambda)^3}$, we see that

$$N_1(\lambda) = 2^{9000000\lambda^3}$$

has the desired property.

To complete the proof, it remains to show that if $N_2 = N_2(\mu) = 2^{252000000\mu^3}$, then $\partial[u] = [p, q]$ in $KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B)$.

Now, $v := v(u, h)$ is as in line (6.2), we have

$$\partial[u] = \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

Define now

$$w := \begin{pmatrix} u_{1-h}(1-p) & -q \\ p & (1-p)u_{1-h}^{-1} \end{pmatrix} \in M_{2n}(\mathfrak{L}_B).$$

Applying Lemma 6.8 with

$$\nu = M \quad \text{and} \quad \gamma = M\delta,$$

we see that w is in $\mathcal{U}_{2n,(2M)^8,2^{37}M^{25}\delta}(X \cup \{h\}, B)$. For notational simplicity, set $M_1 := 2^{37}M^{25}$. Proposition 4.6 implies that in $KK^0_{M_1^3,3M_1^3\delta}(X \cup \{h\}, B)$

$$\begin{aligned}\partial[u] &= \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= \left[(vw^{-1})^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}(vw^{-1}), \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].\end{aligned}$$

Computing, we see that

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} = \begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix},$$

whence

$$\partial[u] = \left[\begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \tag{6.44}$$

in the group $KK^0_{M_1^3,3M_1^3\delta}(X \cup \{h\}, B)$.

Note now that the matrix $\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \in M_{2n}(\mathcal{K}_B^+)$ has norm at most 2λ (as $\|q\| \leq \kappa \leq \lambda$, and so $\|1-q\| \leq \lambda$ by Corollary 4.2), and that it satisfies

$$\left\| \left[x, \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \right] \right\| < \varepsilon < \delta$$

for all $x \in X \cup \{h\}$. Hence, $\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \in \mathcal{U}_{2n,2\lambda,\delta}(X \cup \{h\}, B)$. Applying Proposition 4.6 again and using that $\lambda \leq M_1$, the identity

$$\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} = \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix}$$

shows that the class on the right-hand side of line (6.44) is the same as the class

$$\left[\begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix} \right]$$

in $KK^0_{M_1^6,9M_1^9\delta}(X \cup \{h\}, B)$. Using a rotation homotopy, this is the same as $[p, q]$ by definition of $KK^0_{M_1^6,9M_1^9\delta}(X \cup \{h\}, B)$; recalling that

$$M_1 := 2^{37}M^{25}, \quad M = 4 \cdot 2^{(200\lambda)^3},$$

and that $\mu \geq 2^{9000000\lambda^3}$ we see that $N_2(\mu) = 2^{37}\mu^{25}$ indeed has the right properties. ∎

Chapter 7

Main theorems

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ are respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. We identify $\mathcal{L}_B$ with the "diagonal subalgebra" $1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$ for each n.

In this chapter we prove our main result; that is, the class of separable and nuclear C^*-algebras with the UCT is closed under decomposability.

7.1 Two technical "local" controlled vanishing results

In order to make the structure of the proof of Theorem 1.2 as clear as we can, in this section we split off two "local" technical results. These are based on our work in Chapters 5 and 6; given the material in these earlier chapters, at this point the proofs are essentially bookkeeping.

The next result is the first key technical ingredient we need; it is based on the material from Chapter 5. For the statement, recall that if x and S are respectively an element and subset of a metric space, and $\varepsilon > 0$, then "$x \in_\varepsilon S$" means that there is $s \in S$ with $d(x, s) < \varepsilon$.

Proposition 7.1. *There exists a function $M : [1, \infty) \to [1, \infty)$ with the following property. Let $\kappa \geq 1$, and let $M := M(\kappa)$. Let B be a separable C^*-algebra such that*

$$K_*(B) = 0.$$

Let $\varepsilon > 0$, and let X be a finite subset of the unit ball of $\mathcal{L}_{SB}$. Let $F \subseteq \mathcal{L}_{SB}$ be a separable, nuclear, unital C^*-subalgebra of $\mathcal{L}_{SB}$ such that the identity representation $F \to \mathcal{L}_{SB}$ is strongly unitally absorbing (see Definition 2.5), such that for all $x \in X$, $x \in_\varepsilon F$, and such that F satisfies the UCT.

Then, for each $i \in \{0, 1\}$ there exists a finite subset Z of F_1 such that the forget control map

$$KK^i_{\kappa,\varepsilon}(Z, B) \to KK^i_{M,M\varepsilon}(X, B)$$

of Definition 3.4 (for $i = 0$) or Definition 3.7 (for $i = 1$) is zero.

Proof. Let us focus on the case of $i = 0$ first. Let Y be a finite subset of F_1 such that for all $x \in X$ there exists $y \in Y$ with $\|x - y\| < \varepsilon$. Then, for any n, any $\delta > 0$, we see that with notation in Definition 3.1

$$\mathcal{P}_{n,\kappa,\delta}(Y, SB) \subseteq \mathcal{P}_{n,\kappa,\delta+2\kappa\varepsilon}(X, SB).$$

Hence, the forget control map

$$KK^0_{\kappa,\delta}(Y, SB) \to KK^0_{\kappa,\delta+2\kappa\varepsilon}(X, SB) \tag{7.1}$$

is defined. On the other hand, Corollary 5.3 implies that there is a finite subset Z of F_1 such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\kappa,160\varepsilon}(Y, SB)$$

is defined and zero. Taking $\delta = 160\varepsilon$, and composing this with the forget control map in line (7.1) above, we see that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\kappa,(160+2\kappa)\varepsilon}(Y, SB)$$

is well-defined and zero. We are therefore done in the case $i = 0$; any function M satisfying $M(\kappa) \geq 160 + 2\kappa$ will work. The case of $i = 1$ is similar (although requiring a much larger $M(\kappa)$), using Lemma 5.5 in place of Corollary 5.3. ∎

The second key technical result we need is as follows; it is based on the material from Chapter 6.

Proposition 7.2. *Let X be a finite subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, and let $\kappa \geq 1$. Assume there exists a positive contraction $h \in \mathcal{L}_B$, finite self-adjoint subsets Z_h, Z_{1-h}, and $Z_{h(1-h)}$ of the unit ball of $\mathcal{L}_B$, and $\lambda, \mu \geq 1$ and $\delta, \gamma > 0$ with the following properties:*

(i) $\|[h, x]\| < \varepsilon$ *for all* $x \in X$;

(ii) *for each* $z \in Z_{h(1-h)}$, $z \in_\varepsilon Z_h$ *and* $z \in_\varepsilon Z_{1-h}$;

(iii) *with* $N_1 := N_1(\lambda)$ *as in Proposition 6.6, the forget control map*

$$KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}) \to KK^1_{\mu,\gamma}(h(1-h)X \cup \{h\}, B)$$

of Definition 3.7 is defined and zero;

(iv) *the forget control map*

$$KK^0_{4\kappa^2,2\varepsilon}(Z_h \cup hX \cup \{h\}, B) \to KK^0_{\lambda,\delta}(hX \cup \{h\} \cup Z_{h(1-h)}, B)$$

of Definition 3.4 is defined and zero;

(v) *the forget control map*

$$KK^0_{4\kappa^2,2\varepsilon}(Z_{1-h} \cup (1-h)X \cup \{h\}, B) \to KK^0_{\lambda,\delta}((1-h)X \cup \{h\} \cup Z_{h(1-h)}, B)$$

of Definition 3.4 is defined and zero.

Then, if $Z := Z_h \cup Z_{1-h} \cup X \cup \{h\}$ *and* $N_2 := N_2(\mu)$ *is as in Proposition 6.6, we have that the forget control map*

$$KK^0_{\kappa,\varepsilon}(Z, B) \to KK^0_{N_2,N_2\gamma}(X, B)$$

of Definition 3.4 is zero.

Proof. We need to show that an arbitrary class $\alpha \in KK^0_{\kappa,\varepsilon}(X, B)$ vanishes under the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, B) \to KK^0_{N_2,N_2\gamma}(X, B).$$

Using Proposition 4.10 (i), with notation as in Definition 4.9, we may assume that there is a cycle $(p, q) \in \mathcal{P}^1_{n,4\kappa^3,\varepsilon}(Z, B)$ such that $[p, q] \in KK^0_{4\kappa^3,\varepsilon}(Z, B)$ agrees with the image of α under the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, B) \to KK^0_{4\kappa^3,\varepsilon}(Z, B).$$

It thus suffices to show that $[p, q] \in KK^0_{4\kappa^3,\varepsilon}(Z, B)$ vanishes under the forget control map

$$KK^0_{4\kappa^2,\varepsilon}(Z, B) \to KK^0_{N_2,N_2\gamma}(X, B)$$

(we leave the check that this map is defined under our assumptions to the reader). Now, with notation as in Proposition 6.6, the composition

$$\begin{array}{ccc} KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup Z_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup \{h\}, B) & \xrightarrow{\eta_h} & KK^0_{4\kappa^2,2\varepsilon}(hX \cup Z_h \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}(hX \cup Z_{h(1-h)} \cup \{h\}, B) \end{array}$$

(compare line (6.13)) is the zero map; indeed, the right-hand vertical map is zero by assumption (iv). Similarly, using assumption (v), we see that the composition

$$\begin{array}{ccc} KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup Z_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{4\kappa^2,\varepsilon}(X \cup Z_{1-h} \cup \{h\}, B) & \xrightarrow{\eta_{1-h}} & KK^0_{4\kappa^2,2\varepsilon}((1-h)X \cup Z_{1-h} \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}((1-h)X \cup Z_{h(1-h)} \cup \{h\}, B) \end{array}$$

(compare line (6.14)) is zero. Hence, Proposition 6.6 gives us an element

$$u \in \mathcal{U}^1_{\infty,N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}, B)$$

such that in the diagram below (with $N_0 = N_0(\mu)$ as in Proposition 6.1)

$$\begin{array}{ccc} KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}, B) & & KK^0_{4\kappa^2,2\varepsilon}(Z, B) \\ \downarrow & & \downarrow \\ KK^1_{\mu,\gamma}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0,N_0\gamma}(X \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B) \end{array} \tag{7.2}$$

the images of the classes $[u] \in KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)})$ and $[p,q] \in KK^0_{\kappa,\varepsilon}(Z, B)$ in the bottom right group $KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B)$ are the same; a fortiori their images are also the same if we further compose with the forget control map

$$KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B) \to KK^0_{N_2,N_2\gamma}(X, B).$$

Assumption (iii) implies, however, that the left-hand vertical map in line (7.2) is zero, however, so we are done. ∎

7.2 Proof of the main theorems

We are now ready for our main results. For the statement of the first of these, we recall what it means for a C^*-algebra to decompose over a class of C^*-algebras from Definition 1.1 above. After giving a proof of this, we will use it to establish the theorems from the introduction.

Theorem 7.3. *Let $\kappa \geq 1$ and $\gamma > 0$. Let $M_1 := M(4)$ be as in Proposition 7.1. Let $N_1 := N_1(M_1)$ be as in Proposition 6.6. Let $M_2 := M(N_1)$ be as in Proposition 7.1. Let $N_2 := N_2(M_2)$ be as in Proposition 6.6. Then, any $\nu \geq N_2$ and $\varepsilon \in (0, \gamma(2N_2M_2N_1M_1)^{-1})$ have the following property.*

Let A be a separable, unital C^*-algebra that decomposes over the class of nuclear C^*-algebras that satisfy the UCT. Let B be any separable C^*-algebra such that $K_*(B) = 0$. Then, for any finite subset X of A_1, and $\varepsilon > 0$, there is a finite subset Z of A_1, such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

of Definition 3.4 is defined and zero.

In particular, A satisfies the UCT.

Proof. The claim that A satisfies the UCT follows as the vanishing property in the statement of Theorem 7.3 implies condition (iii) from Corollary 5.3. It thus suffices to prove the vanishing property. Let ν and ε satisfy the given assumptions.

As A is decomposable with respect to the family of nuclear C^*-subalgebras that satisfy the UCT, there are nuclear, UCT C^*-subalgebras C, D and E of A and a positive contraction $h \in E$ such that for all $x \in X$, $\|[h, x]\| < \varepsilon$, $hx \in_\varepsilon C$, $(1-h)x \in_\varepsilon D$, and $h(1-h)x \in_\varepsilon E$, and such that all $e \in E$ we have that $e \in_\varepsilon C$, and $e \in_\varepsilon D$. Replacing C, D, and E by the C^*-subalgebra of A spanned by the algebra and the unit of A, we may assume that C, D, and E are unital subalgebras of A (note that the unitization of a nuclear C^*-algebra that satisfies the UCT is nuclear and satisfies the UCT; see [10, Exercise 2.3.5] for nuclearity and [55, Proposition 2.3 (a)] for the UCT).

Represent A on $\mathcal{L}_{SB}$ using a representation with the properties in Corollary 2.7 (with B replaced by SB), and identify A (therefore also C, D, and E) with unital C^*-subalgebras of $\mathcal{L}_{SB}$ using this representation. Note that the restrictions of this representation to each of E, C, D, (and the representation of A itself) are strongly unitally absorbing.

Throughout the rest of the proof, all unlabeled arrows are forget control maps as in Definitions 3.4 or 3.7 as appropriate.

Using Proposition 7.1 there exists a finite self-adjoint subset $Z_{h(1-h)}$ of E_1 such that the forget control map

$$\begin{aligned} &KK^1_{N_1,2N_1M_1\varepsilon}(h(1-h)X \cup Z_{h(1-h)} \cup \{h\}, SB) \\ &\quad \to KK^1_{M_2,2M_2N_1M_1\varepsilon}(h(1-h)X \cup \{h\}, SB) \end{aligned} \tag{7.3}$$

is zero. Similarly, Proposition 7.1 and the facts that for all $z \in Z_{h(1-h)} \subseteq E_1$, $z \in_\varepsilon C$ and $z \in_\varepsilon D$ gives finite self-adjoint subsets Z_h and Z_{1-h} of C_1 and D_1 respectively such that the forget control maps

$$\begin{aligned} &KK^0_{4,2\varepsilon}(hX \cup Z_h \cup \{h\}, SB) \\ &\quad \to KK^0_{M_1,2M_1\varepsilon}(hX \cup Z_{h(1-h)} \cup \{h\}, SB) \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} &KK^0_{4,2\varepsilon}((1-h)X \cup Z_{1-h} \cup \{h\}, SB) \\ &\quad \to KK^0_{M_1,2M_1\varepsilon}((1-h)X \cup Z_{h(1-h)} \cup \{h\}, SB) \end{aligned} \tag{7.5}$$

are defined and zero. Expanding Z_h and Z_{1-h} if necessary (using that for all $e \in E$, $e \in_\varepsilon C$, and $e \in_\varepsilon D$), we may assume that,

$$\text{for all } z \in Z,\ z \in_\varepsilon Z_h \text{ and } z \in_\varepsilon Z_{1-h}. \tag{7.6}$$

We are now in a position to apply Proposition 7.2 with the given ε and κ, $\lambda = M_1$, $\delta = 2M_1\varepsilon$, $\mu = M_2$ and γ as given: assumption (i) follows by choice of h; assumption (ii) follows from line (7.6); assumption (iii) follows as the map in line (7.3) is

zero; assumption (iv) follows as the map in line (7.4) is zero; and assumption (v) follows as the map in line (7.5) is zero. Therefore, Proposition 7.2 implies that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

is zero and we are done. ∎

To establish the main results as stated in the introduction, we need a basic lemma.

Lemma 7.4. *The class of unital, nuclear C^*-algebras is closed under decomposability.*

Proof. Let A be a unital C^*-algebra that decomposes over the class of unital nuclear C^*-algebras. Let a finite subset X of A and $\varepsilon \in (0, 1)$ be given. To show that A is nuclear, it will suffice to construct a finite rank ccp map

$$\phi : A \to A$$

such that $\phi(x) \approx_\varepsilon x$ for all $x \in X$ (compare for example [8, Lemma IV.3.1.6, (iii)]). We may assume that X contains the unit of A.

Let then C, D, E[1], and h be as in the definition of decomposability (Definition 1.1) with respect to the finite set X and the parameter $\delta = \frac{1}{18}(\varepsilon/(1+\varepsilon))^2$, and with C and D nuclear. Note that for any $x \in X$, $\|[h^{1/2}, x]\| \leq \frac{5}{4}\|[h, x]\|^{1/2}$ by the main result of [49], whence

$$\|hx - h^{1/2}xh^{1/2}\| \leq \frac{5}{4}\|[h, x]\|^{1/2} < \frac{5}{4}\delta^{1/2} < 2\delta^{1/2}; \tag{7.7}$$

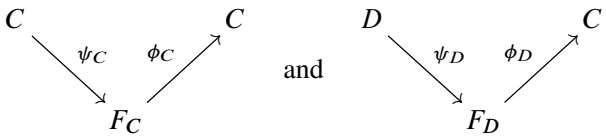
as $hx \in_\delta C$, and as $\delta < 1$, this implies that $h^{1/2}xh^{1/2} \in_{3\delta^{1/2}} C$. Choose a finite subset Y of C such that for all $x \in X$ there is $y_x \in Y$ with

$$\|y_x - h^{1/2}xh^{1/2}\| < 3\delta^{1/2}. \tag{7.8}$$

Similarly, there is a finite subset Z of D such that for all $x \in X$ there is $z_x \in Z$ with

$$\|z_x - (1-h)^{1/2}x(1-h)^{1/2}\| < 3\delta^{1/2}.$$

Now, as C and D are nuclear there are diagrams



[1] One does not actually need E at all in the proof.

where all the arrows are ccp maps; F_C and F_D are finite dimensional C^*-algebras; and for all $y \in Y$, and all $z \in Z$,

$$\phi_C(\psi_C(y)) \approx_{\delta^{1/2}} y \quad \text{and} \quad \psi_D(\phi_D(z)) \approx_{\delta^{1/2}} z. \tag{7.9}$$

Using Arveson's extension theorem (see for example [10, Theorem 1.6.1]), extend ψ_C and ψ_D to ccp maps defined on all of A, which we keep the same notation for. Define

$$\phi_0 : A \to A, \quad a \mapsto \phi_C(\psi_C(h^{1/2}xh^{1/2})) + \phi_D(\psi_D((1-h)^{1/2}x(1-h)^{1/2})),$$

and note that ϕ_0 is completely positive. For any $x \in X$, let y_x have the property in line (7.8). As ψ_C is contractive, this and lines (7.9) and (7.7) imply that

$$\phi_C(\psi_C(h^{1/2}xh^{1/2})) \approx_{3\delta^{1/2}} \phi(\psi_C(y_x)) \approx_{\delta^{1/2}} y_x \approx_{3\delta^{1/2}} h^{1/2}xh^{1/2} \approx_{2\delta^{1/2}} hx.$$

Precisely analogously, for any $x \in X$,

$$\phi_D(\psi_D((1-h)^{1/2}x(1-h)^{1/2})) \approx_{9\delta^{1/2}} (1-h)x$$

and so for any $x \in X$, $\phi_0(x) \approx_{18\delta^{1/2}} x$. Applying this to $x = 1$ implies in particular that $\|\phi_0\| = \|\phi_0(1)\| \geq 1 - 18\delta^{1/2}$. Hence, if we define

$$\phi : A \to A, \quad a \mapsto \frac{\phi_0(a)}{\|\phi_0(1)\|}$$

then ϕ is a ccp map such that

$$\|\phi(x) - x\| \leq \frac{18\delta^{1/2}}{1 - 18\delta^{1/2}}$$

for all $x \in X$. Using the choice of δ, this completes the proof. ∎

The next corollary is Theorem 1.2 from the introduction; it is an immediate consequence of Lemma 7.4 and Theorem 7.3.

Corollary 7.5. *If a separable, unital C^*-algebra decomposes over the class of nuclear, unital C^*-algebras that satisfies the UCT, then it is nuclear and satisfies the UCT.* ∎

The next result is Theorem 1.4 from the introduction. For the definition of finite complexity and the classes $\mathcal{D}_\alpha$ used below, see Definition 1.3.

Corollary 7.6. *Let $\mathcal{C}$ be a class of separable, unital, nuclear C^*-algebras that satisfy the UCT. Then, the class of separable unital C^*-algebras that have finite complexity relative to $\mathcal{C}$ consists of nuclear C^*-algebras that satisfy the UCT.*

In particular, every separable C^*-algebra of finite complexity is nuclear and satisfies the UCT.

Proof. With notation as in Definition 1.3, let $\mathcal{D}_0 = \mathcal{C}$, and for each ordinal α, let $\mathcal{D}_{\alpha,sep}$ consist of the separable C^*-algebras in the class $\mathcal{D}_\alpha$ from Definition 1.3. We proceed by transfinite induction to show that each $\mathcal{D}_{\alpha,sep}$ consists of nuclear, UCT C^*-algebras. If $\alpha = 0$, this is just the well-known fact that AF C^*-algebras satisfy the UCT. If $\alpha > 0$ (and either a successor or limit ordinal) then any C^*-algebra in $\mathcal{D}_{\alpha,sep}$ decomposes over C^*-algebras in $\bigcup_{\beta<\alpha} \mathcal{D}_{\beta,sep}$, and so is nuclear and UCT by Corollary 7.5 and the inductive hypothesis. ∎

Appendix A

Examples

In this appendix we give some examples of C^*-algebras with finite complexity.

A.1 Cuntz algebras

The material in this section is based closely on work of Winter and Zacharias [70, Section 7][1]. Our aim is to establish the following result.

Proposition A.1. *For any n with $2 \leq n < \infty$, the Cuntz algebra $\mathcal{O}_n$ has complexity rank one.*

We should remark that the proof of Proposition A.1 uses classification results for Cuntz algebras, and so depends on prior knowledge of the UCT; it therefore cannot be said that Proposition A.1 gives a new proof of the UCT for Cuntz algebras (and even if it did, it would be quite a complicated one!). Indeed, the main point of establishing Proposition A.1 for us is to use it as an ingredient in Theorem 1.7 from the introduction, not to establish the UCT.

We should also remark that Proposition A.1 was subsequently generalized in [37, Theorem 1.5]; nonetheless, we hope that the different argument given here still has some interest.

We now embark on the proof of Proposition A.1. We will follow the notation from [70, Section 7]. Fix $n \in \mathbb{N}$ with $n \geq 2$. Let H be an n-dimensional Hilbert space, with fixed orthonormal basis $\{e_1, \ldots, e_n\}$. Define

$$\Gamma(n) := \bigoplus_{l=0}^{\infty} H^{\otimes l}, \tag{A.1}$$

where $H^{\otimes l}$ is the lth tensor power of H (and $H^{\otimes 0}$ is by definition a copy of $\mathbb{C}$). Let W_n be the set of all finite words based on the alphabet $\{1, \ldots, n\}$. In symbols

$$W_n := \bigsqcup_{k=0}^{\infty} \{1, \ldots, n\}^k$$

(with $\{1, \ldots, n\}^0$ by definition consisting only of the empty word). For each $\mu = (i_1, \ldots, i_k) \in W_n$, define $e_\mu := e_{i_1} \otimes \cdots \otimes e_{i_k}$, and define $e_\varnothing$ to be any unit-length

[1] More specifically, it is based on the slightly different approach to the material in [70, Section 7] suggested in [70, Remark 7.3].

element of $H^{\otimes 0} = \mathbb{C}$. Then, the set $\{e_\mu \mid \mu \in W_n\}$ is an orthonormal basis of $\Gamma(n)$. For $\mu \in W_n$, write $|\mu|$ for the length of μ, i.e., $|\mu| = k$ means that

$$\mu = (i_1, \ldots, i_k)$$

for some $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Then, the canonical copy of $H^{\otimes k}$ inside $\Gamma(n)$ from line (A.1) has orthonormal basis $\{e_\mu \mid |\mu| = k\}$.

For each $i \in \{1, \ldots, n\}$ let T_i be the bounded operator on $\Gamma(n)$ that acts on basis elements via the formula

$$T_i : e_\mu \mapsto e_i \otimes e_\mu.$$

The *Cuntz–Toeplitz algebra* $\mathcal{T}_n$ is defined to be the C^*-subalgebra of $\mathcal{B}(\Gamma(n))$ generated by $T_1, \ldots, T_n$. We note that each T_i is an isometry, and that $1 - \sum_{i=1}^n T_i T_i^*$ is the projection onto the span of $e_\varnothing$. It follows directly from this that $\mathcal{T}_n$ contains all matrix units with respect to the basis $\{e_\mu\}$ of $\Gamma(n)$, and therefore contains the compact operators $\mathcal{K}$ on $\Gamma(n)$. Moreover, in the quotient $\mathcal{T}_n/\mathcal{K}$, the images s_i of the generators T_i satisfy the Cuntz relations $s_i^* s_i = 1$ and $\sum_{i=1}^n s_i s_i^* = 1$, and therefore the quotient is a copy of the Cuntz algebra $\mathcal{O}_n$.

Now, for $x \in \mathbb{R}_+$, define $\lceil x \rceil := \min\{n \in \mathbb{N} \mid n \geq x\}$, and define[2]

$$\Gamma_{0,k} := \bigoplus_{l=k}^{2k-1} H^{\otimes l} \quad \text{and} \quad \Gamma_{1,k} := \bigoplus_{l=k+\lceil k/2 \rceil}^{2k+\lceil k/2 \rceil} H^{\otimes l}. \tag{A.2}$$

For $i \in \{0, 1\}$, define $B_{i,k}^{(0)} := \mathcal{B}(\Gamma_{i,k})$. For each $l, m \in \mathbb{N}$, we identify $H^{\otimes l} \otimes H^{\otimes m}$ with $H^{\otimes(l+m)}$ via the bijection of orthonormal bases

$$\left(e_{i_1} \otimes \cdots \otimes e_{i_l}\right) \otimes \left(e_{j_1} \otimes \cdots \otimes e_{j_m}\right) \leftrightarrow e_{i_1} \otimes \cdots \otimes e_{i_l} \otimes e_{j_1} \otimes \cdots \otimes e_{j_m}.$$

Fix for the moment $k \in \mathbb{N}$ (it will stay fixed until Lemma A.2 below). Then, for each $j \in \mathbb{N}$ we get a canonical identification

$$\Gamma_{0,k} \otimes H^{\otimes jk} = \bigoplus_{l=k}^{2k-1} H^{\otimes l} \otimes H^{\otimes jk} = \bigoplus_{l=jk}^{(j+1)k-1} H^{\otimes l}.$$

Combining this with line (A.1) we get a canonical identification

$$\Gamma(n) = \underbrace{\left(\bigoplus_{l=0}^{k-1} H^{\otimes l}\right)}_{=:H_0} \oplus \left(\bigoplus_{j=0}^{\infty} \Gamma_{0,k} \otimes H^{\otimes jk}\right).$$

[2] In [70, Section 7], $\Gamma_{0,k}$ is written $\Gamma_{k,2k}$ and $\Gamma_{1,k}$ is written $\Gamma_{k+\lceil k/2 \rceil, 2k+\lceil k/2 \rceil}$.

Let id be the identity representation of $B_{0,k}^{(0)}$ on $\Gamma_{0,k}$ and write $B_{0,k}$ for the image of $B_{0,k}^{(0)}$ in the representation on $\Gamma(n)$ that is given by

$$0_{H_0} \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{id} \otimes 1_{H^{\otimes jk}} \right)$$

with respect to the above decomposition above. Similarly, we get a decomposition

$$\Gamma(n) = \underbrace{\left(\bigoplus_{l=0}^{k+\lceil k/2 \rceil - 1} H^{\otimes l} \right)}_{=:H_1} \oplus \left(\bigoplus_{j=0}^{\infty} \Gamma_{1,k} \otimes H^{\otimes jk} \right)$$

and define $B_{1,k}$ to be the image of $B_{1,k}^{(0)}$ under the representation

$$0_{H_1} \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{id} \otimes 1_{H^{\otimes jk}} \right).$$

Now, let $f : [0,1] \to [0,1]$ be the piecewise linear function that takes the value 0 on $[0, 1/6]$ and $[5/6, 0]$, the value 1 on $[2/6, 4/6]$, and interpolates linearly between 0 and 1 on $[1/6, 2/6]$ and $[4/6, 5/6]$. Let $h_{0,k}^{(0)} \in B_{0,k}^{(0)}$ be the operator on $\Gamma_{0,k}$ that acts on the summand $H^{\otimes l}$ from line (A.2) by multiplication by the scalar $f((l-k)/(k-1))$. Similarly, let $h_{1,k}^{(0)} \in B_{1,k}^{(1)}$ be the operator on $\Gamma_{1,k}$ that acts on the summand $H^{\otimes l}$ from line (A.2) by multiplication by the scalar $1 - f((l-k-\lceil k/2 \rceil)/(k-1))$. Let $h_{0,k}$ and $h_{1,k}$ be the images of $h_{0,k}^{(0)}$ and $h_{1,k}^{(0)}$ in $B_{0,k}$ and $B_{1,k}$ respectively. Note that the operator on $h_{0,k} + h_{1,k}$ on $\Gamma(n)$ acts on the summand on $H^{\otimes l}$ from line (A.1) by multiplication by 1 as long as $l \geq k + \lceil k/2 \rceil$. In particular,

$$h_{0,k} + h_{1,k} \text{ equals the identity on } \Gamma(n) \text{ up to a finite rank perturbation.} \tag{A.3}$$

We will need two technical lemmas about these operators.

Lemma A.2. *For any T in the Cuntz–Toeplitz algebra $\mathcal{T}_n$ and $i \in \{0,1\}$, we have that $\|[h_{i,k}, T]\| \to 0$ as $k \to \infty$.*

Proof. We will focus on $h_{0,k}$; the case of $h_{1,k}$ is essentially the same. It suffices to consider the case where T is one of the canonical generators T_i of the Cuntz–Toeplitz algebra. Let e_μ be a basis element with $|\mu| = jk + l$ for some $j, l \in \mathbb{N}$ with $l \in \{0, \ldots, k-1\}$. Then, we compute that $[h_{0,k}, T_i]e_\mu = 0$ if $j = 0$, and that otherwise

$$[h_{0,k}, T_i]e_\mu = \big(f((l+1)/(k-1)) - f(l/(k-1))\big)e_i \otimes e_\mu.$$

As the elements $\{e_i \otimes e_\mu \mid \mu \in W_n\}$ are an orthonormal set, this implies that

$$\|[h_{0,k}, T_i]\| \leq \max_{l \in \{0,\ldots,k-1\}} |f((l+1)/(k-1)) - f(l/(k-1))|.$$

The choice of function f implies that the right-hand side above is approximately $6/k$, so we are done. ∎

Lemma A.3. *For any T in the Cuntz–Toeplitz algebra $\mathcal{T}_n$ we have that*

(i) *for $i \in \{0,1\}$, $d(h_{i,k}T, B_{i,k}) \to 0$ as $k \to \infty$;*

(ii) $d(h_{0,k}h_{1,k}T, B_{0,k} \cap B_{1,k}) \to 0$ *as* $k \to \infty$.

Proof. We will focus on the case of $h_{0,k}$; the other cases are similar. It suffices to consider T a finite product $S_1 \cdots S_m$, where each S_j is either one of the generators T_i or its adjoint. Using Lemma A.2, we see that $[h_{0,k}^{1/l}, S_j] \to 0$ as $k \to \infty$ for any j, and any $l \in \mathbb{N}$ with $l \geq 1$. Hence, the difference

$$h_{0,k}S_1 \cdots S_m - \left(h_{0,k}^{1/(2m)} S_1 h_{0,k}^{1/(2m)}\right)\left(h_{0,k}^{1/(2m)} S_2 h_{0,k}^{1/(2m)}\right) \cdots \left(h_{0,k}^{1/(2m)} S_m h_{0,k}^{1/(2m)}\right)$$

tends to zero as $k \to \infty$. It thus suffices to prove that the distance between each of the terms $h_{0,k}^{1/(2m)} S_j h_{0,k}^{1/(2m)}$ and $B_{0,k}$ tends to zero as $k \to \infty$. Define p_k to be the strong operator topology limit of $h_{0,k}^{1/l}$ as $l \to \infty$; in other words, p_k is the support projection of $h_{0,k}$. Then, we have that $h_{0,k}^{1/(2m)} S_j h_{0,k}^{1/(2m)} = h_{0,k}^{1/(2m)} p_k S_j p_k h_{0,k}^{1/(2m)}$. As $h_{0,k}^{1/(2m)}$ is in $B_{0,k}$, it suffices to prove that the distance between $p_k T_i p_k$ and $B_{0,k}$ tends to zero as $k \to \infty$. However, $p_k T_i p_k$ is actually in $B_{0,k}$, so we are done. ∎

Now, as in the discussion on [70, p. 488], define

$$\Gamma_k(n) := \bigoplus_{l=0}^{k-1} H^{\otimes l}.$$

For a word $\mu \in W_n$ in $\{1, \ldots, n\}$, we may uniquely write $\mu = \mu_0\mu_1$, where the lengths $|\mu_0|$ and $|\mu_1|$ satisfy $|\mu_0| \in \{0, \ldots, k-1\}$, and $|\mu_1| \in k\mathbb{N}$. Then, the bijective correspondence of orthonormal bases

$$e_\mu \leftrightarrow e_{\mu_0} \otimes e_{\mu_1}$$

gives rise to a decomposition

$$\Gamma(n) = \Gamma_k(n) \otimes \Gamma(n^k).$$

Identify the C^*-algebra $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ with its image in the representation on $\Gamma(n)$ arising from the above decomposition. The following is essentially part of [70, Lemma 7.1].

Lemma A.4. *With notation as above, $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ contains the finite-dimensional C^*-algebras we have called $B_{0,k}$ and $B_{1,k}$, and in particular also contains $h_{0,k}$ and $h_{1,k}$.*

Proof. In the notation of [70, Lemma 7.1], $B_{0,k} = \Lambda_k(\mathcal{B}(\Gamma_{k,2k}))$, and

$$B_{1,k} = \Lambda_k(\mathcal{B}(\Gamma_{k+\lceil k/2\rceil,2k+\lceil k/2\rceil})).$$

Part (i) of [70, Lemma 7.1] says exactly that the image of Λ_k is contained in

$$\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k},$$

however, so we are done. ∎

It is explained on [70, p. 488] that $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ contains $\mathcal{T}_n$, so we get a canonical inclusion.

$$\mathcal{T}_n \to \mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}. \tag{A.4}$$

The dimension of $\Gamma_k(n)$ is $d_k := 1 + n + n^2 + \cdots + n^{k-1}$, so we may make the identification $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k} = M_{d_k}(\mathcal{T}_{n^k})$. With respect to this identification, the inclusion in line (A.4) takes the compact operators on $\Gamma(n)$ to $M_{d_k}(\mathcal{K}(\Gamma(n^k)))$. Taking the quotient by the compacts on both sides of line (A.4) thus gives rise to an inclusion

$$\iota : \mathcal{O}_n \to M_{d_k}(\mathcal{O}_{n^k}). \tag{A.5}$$

In this language, we get the following immediate corollary of Lemmas A.2 and A.3. To state it, let $q : \mathcal{B}(\Gamma(n)) \to \mathcal{Q}(\Gamma(n))$ be the quotient map from the bounded operators on $\Gamma(n)$ to the Calkin algebra.

Corollary A.5. *For any $a \in \mathcal{O}_n$, we have that the following all tend to zero as $k \to \infty$: $\|[q(h_{0,k}), \iota(a)]\|$, $\|[q(h_{1,k}), \iota(a)]\|$, $d(q(h_{0,k})\iota(a), q(B_{0,k}))$, $d(q(h_{1,k})\iota(a), q(B_{1,k}))$, and $d(q(h_{0,k}h_{1,k})\iota(a), q(B_{0,k} \cap B_{1,k}))$.* ∎

We are finally ready for the proof of Proposition A.1.

Proof of Proposition A.1. Let $\varepsilon > 0$, and let X be a finite subset of the unit ball of $\mathcal{O}_n$. Corollary A.5 implies that for any large k we have that for all $a \in X$ and $i \in \{0, 1\}$, the quantities $\|[q(h_{i,k}), \iota(a)]\|$, $d(q(h_{i,k})\iota(a), q(B_{i,k}))$, and $d(q(h_{0,k}h_{1,k})\iota(a), q(B_{0,k} \cap B_{1,k}))$ are smaller than $\varepsilon/2$. We may assume moreover that $k \equiv 1$ modulo $n - 1$. Fix this k for the remainder of the proof.

As discussed on [70, p. 488], we have a canonical unital inclusion $\mathcal{O}_{n^k} \to \mathcal{O}_n$ by treating suitable products of the generators of $\mathcal{O}_n$ as generators of $\mathcal{O}_{n^k}$. Moreover, d_k is equal to k modulo $n - 1$. It follows that the K-theory of $M_{d_k}(\mathcal{O}_n)$ is given by $\mathbb{Z}/(n-1)\mathbb{Z}$ in dimension zero and zero in dimension one, with the class $[1]$ of the unit in K_0 represented by the residue of k in $\mathbb{Z}/(n-1)\mathbb{Z}$. Hence, the K-theory

invariants of $M_{d_k}(\mathcal{O}_n)$ and $\mathcal{O}_n$ agree, as we are assuming that $k \equiv 1$ modulo $n-1$. In particular, the Kirchberg–Phillips classification theorem (see for example [53, Corollary 8.4.8]) gives a unital isomorphism $M_{d_k}(\mathcal{O}_n) \cong \mathcal{O}_n$. Combining this with the inclusion $\mathcal{O}_{n^k} \to \mathcal{O}_n$ mentioned above gives a unital inclusion

$$\beta : M_{d_k}(\mathcal{O}_{n^k}) \to \mathcal{O}_n. \tag{A.6}$$

Now, the composition $\beta \circ \iota : \mathcal{O}_n \to \mathcal{O}_n$ of β as in line (A.6) and ι as in line (A.5) is a unital inclusion, whence necessarily induces an isomorphism on K-theory. As $\mathcal{O}_n$ satisfies the UCT, $\beta \circ \iota$ is therefore a KK-equivalence (see for example [55, Proposition 7.3]). Hence, the uniqueness part of the Kirchberg–Phillips classification theorem (see for example [53, Theorem 8.3.3, (iii)]) implies that $\beta \circ \iota : \mathcal{O}_n \to \mathcal{O}_n$ is approximately unitarily equivalent to the identity. Thus, there is a sequence (u_m) of unitaries in $\mathcal{O}_n$ such that

$$\|a - u_m \beta\iota(a) u_m^*\| \to 0$$

for all $a \in \mathcal{O}_n$. Choose m large enough so that $\|a - u_m \beta\iota(a) u_m^*\| < \varepsilon/2$ for all $a \in X$.

Set $h := u_m \beta(q(h_{0,k})) u_m^*$, $C_0 := u_m \beta(q(B_{0,k})) u_m^*$, $D_0 := u_m \beta(q(B_{1,k})) u_m^*$, and $E_0 := u_m \beta(q(B_{1,k} \cap B_{0,k})) u_m^*$. Set C to be the C^*-subalgebra of $\mathcal{O}_n$ spanned by C_0 and the unit, and similarly for D and E. Our choices, plus the fact that

$$q(h_{0,k} + h_{1,k}) = 1$$

(see line (A.3)), imply that this data satisfies the definition of decomposability (Definition 1.1), so we are done. ∎

A.2 Groupoids with finite dynamical complexity

In this section, we give another interesting class of C^*-algebras with finite complexity, that is, C^*-algebras of groupoids with finite dynamical complexity. To avoid repeating the same assumptions, let us stipulate that throughout this appendix the word "groupoid" means "locally compact, Hausdorff, étale groupoid"; we will often also assume that G has compact base space, but not always. For background on this class of groupoids and their C^*-algebras, we recommend [10, Section 5.6], [51, Section 2.3], or [59].

Note that if G is a groupoid in this sense, then any open subgroupoid H of G (i.e., H is an open subset of G that is algebraically a groupoid with the inherited operations) is also a groupoid in this sense. Again, to avoid too much repetition, let us say that the word "subgroupoid" means "open subgroupoid".

The following definitions are essentially contained in the authors' joint work with Guentner [31, Definition A.4].

Definition A.6. Let G be a groupoid, let H be a subgroupoid of G, and let $\mathcal{C}$ be a set of subgroupoids of G. We say that H is *decomposable* over $\mathcal{C}$ if for any compact subset K of H there exists an open cover $\{U_0, U_1\}$ of $r(K) \cup s(K)$ such that for each $i \in \{0, 1\}$ the subgroupoid of H generated by

$$\{h \in K \mid s(h) \in U_i\}$$

is contained in an element of $\mathcal{C}$.

Definition A.7. For an ordinal number α,

(i) if $\alpha = 0$, let $\mathcal{C}_0$ be the class of groupoids G such that for any compact subset K of G there is a subgroupoid H of G such that $K \subseteq H$, and such that the closure of H is compact;

(ii) if $\alpha > 0$, let $\mathcal{C}_\alpha$ be the class of groupoids that decompose over the collection of their subgroupoids in the class $\bigcup_{\beta<\alpha} \mathcal{C}_\beta$.

We say that a groupoid G has *finite dynamical complexity* if G is contained in $\mathcal{C}_\alpha$ for some ordinal α. If G has finite dynamical complexity, the *complexity rank* of G is the smallest α such that G is in $\mathcal{C}_\alpha$.

The main result of this section is as follows. For the statement, recall that a groupoid is *ample* if it has totally disconnected base space, and *principal* if the units are the elements $g \in G$ that satisfy $s(g) = r(g)$. Recall also that a C^*-algebra is subhomogeneous if it is isomorphic to a C^*-subalgebra of $M_N(C(X))$ for some $N \in \mathbb{N}$ and compact Hausdorff space X. Recall finally the notion of complexity rank relative to a class of C^*-algebras from Definition 1.3.

Proposition A.8. *Let G be a groupoid with compact base space.*

(i) *The complexity rank of $C_r^*(G)$ relative to the class of subhomogeneous C^*-algebras is bounded above by the complexity rank of G.*

(ii) *If G is ample and principal, then the complexity rank of $C_r^*(G)$ (relative to the class of finite-dimensional C^*-algebras) is bounded above by the complexity rank of G.*

In particular, if G is second countable and has finite dynamical complexity, then $C_r^*(G)$ satisfies the UCT.

Before getting into the proof of this, let us discuss some remarks and examples.

Example A.9. Let $G(X)$ be the coarse groupoid associated to a bounded geometry metric space X; see [61, Section 3] or [52, Chapter 10] for background. For such spaces X, Guentner, Tessera and Yu [29] introduced a notion called *finite decomposition complexity*; it comes with a natural complexity rank, defined to be the smallest ordinal α such that X is in the class $\mathfrak{D}_\alpha$ of [30, Definition 2.2.1]. Then, [31, Theorem A.7] shows that $G(X)$ has finite dynamical complexity if and only if X has

finite decomposition complexity[3]; moreover, inspection of the proof shows that the two complexity ranks agree. It follows from this and [30, Theorem 4.1] that for any $n \in \mathbb{N}$ there are spaces X such that $G(X)$ is not in $\mathcal{C}_n$, but is in $\mathcal{C}_N$ for some finite $N > n$. Moreover, it follows from [30, discussion below Definition 2.2.1] or the main result of [15] that there are spaces X such that $G(X)$ is in $\mathcal{C}_\alpha$ for some infinite α, but not for any finite α.

Example A.9 shows that the range of possible values of the complexity rank for groupoids is quite rich. As we do not know the corresponding fact for C^*-algebras, the following question is natural.

Question A.10. *Are there any circumstances when the complexity rank of $C_r^*(G)$ is bounded above by that of G?*

It seems very unlikely that there is a positive answer in general, but it is conceivable that there could be a positive answer for coarse groupoids.

Example A.11. Transformation groupoids provide natural examples with finite complexity rank. Using the main result of [2], the complexity rank of the transformation groupoid associated to any free action of a virtually cyclic group on a finite-dimensional space is one. We guess that the techniques used in the proof of [18, Theorem 1.3] should show that for many discrete groups Γ, any free action on the Cantor set X gives rise to a groupoid $X \rtimes \Gamma$ with finite dynamical complexity; however, we did try to look into the details, and would be interested in any progress here. These ideas lead to the following conjecture.

Conjecture A.12. *If Γ has finite decomposition complexity then $X \rtimes \Gamma$ has finite dynamical complexity for* any *free action of Γ on the Cantor set.*

Remark A.13. Proposition A.8 does not give new information on the UCT; this is because all groupoids with finite dynamical complexity are amenable by [31, Theorem A.9], whence their groupoid C^*-algebras satisfy the UCT by Tu's theorem [64, Proposition 10.7]. However, it seems interesting to have an approach to the UCT for a large class of groupoids that does not factor through the Dirac-dual-Dirac machinery employed by Tu.

We now turn to the proof of Proposition A.8. For a subgroupoid H of a groupoid G, write

$$H' := H \cup G^{(0)},$$

which is also a subgroupoid of G.

[3]This result was one of the key motivations for the definition of finite dynamical complexity, and also motivates the terminology.

Lemma A.14. *Let G be a groupoid with compact base space, and let H be a subgroupoid in $\mathcal{C}_\alpha$. Then, $H \cup G^{(0)}$ is a subgroupoid of G that is also in $\mathcal{C}_\alpha$.*

Proof. We proceed by transfinite induction on α. For the base case $\alpha = 0$, let H be a subgroupoid of G in $\mathcal{C}_0$, and let K' be a compact subset of H'. As the base space in an étale groupoid is open, $K := K' \setminus G^{(0)}$ is also a compact set, and is contained in H. As H is in $\mathcal{C}_0$, there exists a subgroupoid L of H that contains K, and that has compact closure. Hence, L' is a subgroupoid of H' that contains K' and has compact closure. Thus, H' is in $\mathcal{C}_0$ too. The inductive step follows the same idea. ∎

The lemma below is very similar to [67, Lemma B.3].

Lemma A.15. *Let G be a groupoid with compact base space. Let H be a subgroupoid of G that decomposes over some class $\mathcal{C}$ of subgroupoids of G. Then, H' decomposes over the collection of subgroupoids L', where L is a subgroupoid of H that is in $\mathcal{C}$.*

Proof. Let X be a finite subset of the unit ball of $C_r^*(H')$, and $\varepsilon > 0$. As $C_c(H) + C(G^{(0)})$ is dense in $C_r^*(H')$, perturbing X slightly, we may assume that X is contained in a subset of $C_r^*(H')$ of the form $C_c(K) + C(G^{(0)})$, where K is an open and relatively compact subset of H. The proof of [67, Lemma B.3] gives us open subgroupoids H_1 and H_2 of H and a positive contraction h in $C_c(H_1^{(0)}) \subseteq C_r^*(H_1)$ such that H_1, H_2 and $H_1 \cap H_2$ are in the class $\mathcal{C}$, and such that for all $x \in X$, $hx \in C_r^*(H_1)$, $(1-h)x \in C_r^*(H_2)$, and $(1-h)hx \in C_r^*(H_1 \cap H_2)$. Then, the data h, $C := C_r^*(H_1')$, $D = C_r^*(H_2')$, and $E = C_r^*(H_1' \cap H_2')$ give the desired decomposability statement. ∎

Proof of Proposition A.8. For part (i), fix a groupoid G. We show by transfinite induction on α that if H is an open subgroupoid of G in the class $\mathcal{C}_\alpha$, and if

$$H' = H \cup G^{(0)},$$

then $C_r^*(H')$ is in the class $\mathcal{D}_\alpha$ of Definition 1.3, where we define $\mathcal{D}_\alpha$ relative to the class of subhomogeneous C^*-algebras. Applying this to $H = G$ then gives the desired conclusion for $C_r^*(G)$.

For the base case, we need to show that if H is an open subgroupoid of G in the class $\mathcal{C}_0$ and if $H' = H \cup G^{(0)}$, then $C_r^*(H')$ is locally subhomogeneous. Let a finite subset X of $C_r^*(H')$ and $\varepsilon > 0$ be given. As $C_c(H')$ is dense in $C_r^*(H')$, up to a perturbation, we may assume X is contained in $C_c(K)$ for some open and relatively compact subset K of H'. Lemma A.14 implies that H' is in $\mathcal{C}_0$, whence there is an open subgroupoid L of H' with compact closure that contains K, and therefore so that X is contained in $C_r^*(L)$. On the other hand, $C_r^*(L)$ is subhomogeneous by the proof [32, Lemma 8.14], so we are done with the base case.

Assume now that $\alpha > 0$ (and is either a successor ordinal or limit ordinal), and let H be a subgroupoid of G in the class $\mathcal{C}_\alpha$. According to Lemma A.15, we have that H' decomposes over

$$\left\{C_r^*(L') \mid L \text{ an open subgroupoid of } H' \text{ in } \bigcup_{\beta<\alpha} \mathcal{C}_\beta\right\}$$

which completes the proof of part (i) by inductive hypothesis.

We now look at part (ii), so let G be principal and ample. We will show that if G is in $\mathcal{C}_0$, then $C_r^*(G)$ is locally finite dimensional; thanks to our work in part (i), this will suffice for the proof.

Let then G be an element of $\mathcal{C}_0$. We claim that for any compact subset K of G there is a compact open subgroupoid of H of G that contains K. The claim shows that $C_r^*(G)$ is locally finite-dimensional. Indeed, up to a perturbation we can assume any finite subset of $C_r^*(G)$ is contained in $C_c(K)$ for some open and relatively compact subset K of G, and so in $C_r^*(H)$ for some compact, open subgroupoid of G. It is well-known that a compact, Hausdorff, étale, principal groupoid with totally disconnected base space has a locally finite-dimensional C^*-algebra; for example, this follows directly from the structure theorem for "CEERs" in [25, Lemma 3.4].

To establish the claim, let a compact subset K of G be given. According to the definition of $\mathcal{C}_0$ there exists an open subgroupoid L of G with compact closure such that K is contained in L. Note first that as L has compact closure, there is some $m \in \mathbb{N}$ such that L is covered by m open bisections from G. Hence, in particular, for any $x \in L^{(0)}$, we have that the range fibre L^x has at most m elements. Working entirely inside L, it suffices to prove that if K is a compact subset of a principal, ample groupoid L such that $\sup_{x\in L^{(0)}} |L^x| = m < \infty$, then there is a compact, open subgroupoid H of L that contains K.

Now, as L is ample (and étale), each point $l \in K$ is contained in a compact, open subset of L. As finitely many of these compact, open subsets cover K, there is a compact, open subset K' of L such that $K \subseteq K'$. Let H be the subgroupoid of L generated by K'. A subgroupoid generated by an open subset is always open (see for example [32, Lemma 5.2]), so it suffices to prove that H is compact. Let $(h_i)_{i\in I}$ be an arbitrary net consisting of elements from H. Each h_i can be written as a finite product $h_i = k_i^{(1)} \cdots k_i^{(n_i)}$, with $k_i^{(j)}$ in $K'' := K' \cup (K')^{-1} \cup s(K') \cup r(K')$. As each range fibre from L has at most m elements, we may assume that $n_i \leq m$ for all m; in fact we may assume it is exactly m, as otherwise we can just "pad" it with identity elements. Write then $h_i = k_i^{(1)} \cdots k_i^{(m)}$. As K'' is compact, we may pass to a subnet of I, and thus assume that each net $(k_i^{(j)})_{i\in I}$ has a convergent subnet, converging to some $k^{(j)}$ in K''. It follows on passing to this subnet that (h_i) converges to $k^{(1)} \cdots k^{(m)}$. As we have shown that every net in H has a convergent subnet, H is compact, completing the proof. ∎

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ISBN 978-3-98547-067-9

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