



INCOMPRESSIBLE NAVIER-STOKES-FOURIER LIMIT OF 3D STATIONARY BOLTZMANN EQUATION

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ABSTRACT. We consider the 3D stationary Boltzmann equation in convex domains with diffuse-reflection boundary condition. We rigorously derive the steady incompressible Navier-Stokes-Fourier system and justify the asymptotic convergence as the Knudsen number ε shrinks to zero. The proof is based on an intricate analysis of boundary layers with geometric correction and focuses on technical difficulties caused by the singularity in collision kernel $k(v, v')$ and the perturbed remainder estimates.

1. Introduction.

1.1. Problem Presentation. We consider the stationary Boltzmann equation in a three-dimensional smooth convex domain $\Omega \ni x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The density function $\mathfrak{F}^\varepsilon(x, v)$ satisfies

$$\begin{cases} \varepsilon v \cdot \nabla_x \mathfrak{F}^\varepsilon = Q[\mathfrak{F}^\varepsilon, \mathfrak{F}^\varepsilon] & \text{in } \Omega \times \mathbb{R}^3, \\ \mathfrak{F}^\varepsilon(x_0, v) = P^\varepsilon[\mathfrak{F}^\varepsilon](x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n(x_0) < 0, \end{cases} \quad (1)$$

where $n(x_0)$ is the unit outward normal vector at x_0 and the Knudsen number $0 < \varepsilon \ll 1$ characterizes the average distance a particle might travel between two collisions. We intend to study the behavior of $\mathfrak{F}^\varepsilon$ as $\varepsilon \rightarrow 0$.

Throughout this paper, we assume that Q is the symmetrized hard-sphere collision operator (see [13, Chapter 1]), and $\mathfrak{F}^\varepsilon$ satisfies the diffuse-reflection boundary condition

$$P^\varepsilon[\mathfrak{F}^\varepsilon](x_0, v) := \mu_b^\varepsilon(x_0, v) \int_{u \cdot n(x_0) > 0} \mathfrak{F}^\varepsilon(x_0, u) |u \cdot n(x_0)| \, du. \quad (2)$$

It describes that the particles are absorbed by the boundary and then re-emitted based on a boundary Maxwellian

$$\mu_b^\varepsilon(x_0, v) := \frac{\rho_b^\varepsilon(x_0)}{2\pi(\theta_b^\varepsilon(x_0))^2} \exp\left(-\frac{|v - u_b^\varepsilon(x_0)|^2}{2\theta_b^\varepsilon(x_0)}\right), \quad (3)$$

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where the wall density, velocity and temperature $(\rho_b^\varepsilon, u_b^\varepsilon, \theta_b^\varepsilon)$ is an ε -perturbation of $(1, 0, 1)$, which corresponds to the standard Maxwellian

$$\mu(v) := \frac{1}{2\pi} \exp\left(-\frac{|v|^2}{2}\right). \quad (4)$$

In detail, we write

$$\rho_b^\varepsilon(x_0) := 1 + \varepsilon \rho_{b,1}(x_0), \quad u_b^\varepsilon(x_0) := \varepsilon u_{b,1}(x_0), \quad \theta_b^\varepsilon(x_0) := 1 + \varepsilon \theta_{b,1}(x_0), \quad (5)$$

and thus we may further expand μ_b^ε into a power series with respect to ε ,

$$\mu_b^\varepsilon(x_0, v) := \mu(v) + \mu^{\frac{1}{2}}(v) \left(\sum_{k=1}^{\infty} \varepsilon^k \mu_k(x_0, v) \right). \quad (6)$$

In particular, the first-order perturbation has the form

$$\mu_1(x_0, v) := \mu^{\frac{1}{2}}(v) \left(\rho_{b,1}(x_0) + u_{b,1}(x_0) \cdot v + \theta_{b,1}(x_0) \frac{|v|^2 - 3}{2} \right). \quad (7)$$

We assume that both μ_b^ε and μ satisfies the normalization condition

$$\int_{v \cdot n(x_0) > 0} \mu_b^\varepsilon(x_0, v) |v \cdot n(x_0)| dv = \int_{v \cdot n(x_0) > 0} \mu(v) |v \cdot n(x_0)| dv = 1. \quad (8)$$

In addition, we require that the particles are only reflected on $\partial\Omega$ without in-flow or out-flow, i.e.

$$\int_{\mathbb{R}^3} \mu_b^\varepsilon(x_0, v) (v \cdot n(x_0)) dv = \int_{\mathbb{R}^3} \mu(v) (v \cdot n(x_0)) dv = 0. \quad (9)$$

Based on (8), (9) and (6), comparing the order of ε , we know

$$\int_{\mathbb{R}^3} \mu_k(x_0, v) \mu^{\frac{1}{2}}(v) |v \cdot n(x_0)| dv = 0 \quad \text{for } k \geq 1, \quad (10)$$

$$\int_{v \cdot n(x_0) \leq 0} \mu_k(x_0, v) \mu^{\frac{1}{2}}(v) |v \cdot n(x_0)| dv = 0 \quad \text{for } k \geq 1. \quad (11)$$

In particular for $k = 1$, we know $u_{b,1} \cdot n = 0$. In fluid mechanics, this corresponds to the non-penetration boundary condition.

We further assume that the perturbation is small, i.e.

$$\left| \langle v \rangle^\vartheta e^{\varrho|v|^2} \frac{\mu_b^\varepsilon - \mu}{\mu^{\frac{1}{2}}} \right| \leq C_0 \varepsilon, \quad (12)$$

for any $0 \leq \varrho < \frac{1}{4}$ and $3 < \vartheta \leq \vartheta_0$ with some given large ϑ_0 , and constant $C_0 > 0$ is sufficiently small.

Note that if $\mathfrak{F}^\varepsilon$ is a solution to (1), then for any constant $M \in \mathbb{R}$, $\mathfrak{F}^\varepsilon + M\mu_b^\varepsilon$ is also a solution. To guarantee uniqueness, we require the normalization condition

$$\iint_{\Omega \times \mathbb{R}^3} \mathfrak{F}^\varepsilon(x, v) dv dx = \iint_{\Omega \times \mathbb{R}^3} \mu(v) dv dx = \sqrt{2\pi} |\Omega|. \quad (13)$$

1.2. Perturbation equation. Considering (13), the solution $\mathfrak{F}^\varepsilon$ to (1) can be expressed as a perturbation of the standard Maxwellian

$$\mathfrak{F}^\varepsilon(x, v) = \mu(v) + \mu^{\frac{1}{2}}(v)f^\varepsilon(x, v), \quad (14)$$

with the normalization condition

$$\iint_{\Omega \times \mathbb{R}^3} f^\varepsilon(x, v) \mu^{\frac{1}{2}}(v) dv dx = 0. \quad (15)$$

Here $f^\varepsilon(x, v)$ satisfies the perturbation equation

$$\begin{cases} \varepsilon v \cdot \nabla_x f^\varepsilon + \mathcal{L}[f^\varepsilon] = \Gamma[f^\varepsilon, f^\varepsilon] & \text{in } \Omega \times \mathbb{R}^3, \\ f^\varepsilon(x_0, v) = \mathcal{P}^\varepsilon[f^\varepsilon](x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n(x_0) < 0, \end{cases} \quad (16)$$

where

$$\mathcal{L}[f^\varepsilon] := -2\mu^{-\frac{1}{2}}Q[\mu, \mu^{\frac{1}{2}}f^\varepsilon], \quad \Gamma[f^\varepsilon, f^\varepsilon] := \mu^{-\frac{1}{2}}Q[\mu^{\frac{1}{2}}f^\varepsilon, \mu^{\frac{1}{2}}f^\varepsilon], \quad (17)$$

and

$$\begin{aligned} \mathcal{P}^\varepsilon[f^\varepsilon](x_0, v) &:= \mu_b^\varepsilon(x_0, v) \mu^{-\frac{1}{2}}(v) \int_{\mathbf{u} \cdot \mathbf{n}(x_0) > 0} \mu^{\frac{1}{2}}(\mathbf{u}) f^\varepsilon(x_0, \mathbf{u}) |\mathbf{u} \cdot \mathbf{n}(x_0)| d\mathbf{u} \\ &\quad + \mu^{-\frac{1}{2}}(v) \left(\mu_b^\varepsilon(x_0, v) - \mu(v) \right). \end{aligned} \quad (18)$$

Hence, in order to study $\mathfrak{F}^\varepsilon$, it suffices to consider f^ε .

1.3. Linearized Boltzmann operator. To clarify, we specify the hard-sphere collision operator Q in (1) and (17)

$$\begin{aligned} Q[F, G] &:= \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\omega, |\mathbf{u} - v|) \left(F(\mathbf{u}_*) G(v_*) + G(\mathbf{u}_*) F(v_*) \right. \\ &\quad \left. - F(\mathbf{u}) G(v) - G(\mathbf{u}) F(v) \right) d\omega d\mathbf{u}, \end{aligned} \quad (19)$$

with

$$\mathbf{u}_* := \mathbf{u} + \omega((v - \mathbf{u}) \cdot \omega), \quad v_* := v - \omega((v - \mathbf{u}) \cdot \omega), \quad (20)$$

and the hard-sphere collision kernel

$$q(\omega, |\mathbf{u} - v|) := q_0 |\omega \cdot (v - \mathbf{u})|, \quad (21)$$

for a positive constant q_0 . Based on [13, Section 3.2–3.5], the linearized Boltzmann operator $\mathcal{L}$ can be rewritten

$$\mathcal{L}[f] = -2\mu^{-\frac{1}{2}}Q[\mu, \mu^{\frac{1}{2}}f] := \nu(v)f - K[f], \quad (22)$$

where

$$\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\omega, |\mathbf{u} - v|) \mu(\mathbf{u}) d\omega d\mathbf{u} \quad (23)$$

$$= \pi^2 q_0 \left(\left(2|v| + \frac{1}{|v|} \right) \int_0^{|v|} e^{-z^2} dz + e^{-|v|^2} \right),$$

$$K[f](v) = K_2[f](v) - K_1[f](v) = \int_{\mathbb{R}^3} k(\mathbf{u}, v) f(\mathbf{u}) d\mathbf{u}, \quad (24)$$

$$K_1[f](v) = \mu^{\frac{1}{2}}(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\omega, |\mathbf{u} - v|) \mu^{\frac{1}{2}}(\mathbf{u}) f(\mathbf{u}) d\omega d\mathbf{u} = \int_{\mathbb{R}^3} k_1(\mathbf{u}, v) f(\mathbf{u}) d\mathbf{u}, \quad (25)$$

$$K_2[f](v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\omega, |\mathbf{u} - v|) \mu^{\frac{1}{2}}(\mathbf{u}) \left(\mu^{\frac{1}{2}}(v_*) f(\mathbf{u}_*) + \mu^{\frac{1}{2}}(\mathbf{u}_*) f(v_*) \right) d\omega d\mathbf{u} \quad (26)$$

$$= \int_{\mathbb{R}^3} k_2(\mathbf{u}, v) f(\mathbf{u}) d\mathbf{u},$$

for kernels

$$k(\mathbf{u}, v) = k_2(\mathbf{u}, v) - k_1(\mathbf{u}, v), \quad (27)$$

$$k_1(\mathbf{u}, v) = \pi q_0 |\mathbf{u} - v| \exp\left(-\frac{1}{2} |\mathbf{u}|^2 - \frac{1}{2} |v|^2\right), \quad (28)$$

$$k_2(\mathbf{u}, v) = \frac{2\pi q_0}{|\mathbf{u} - v|} \exp\left(-\frac{1}{4} |\mathbf{u} - v|^2 - \frac{1}{4} \frac{(|\mathbf{u}|^2 - |v|^2)^2}{|\mathbf{u} - v|^2}\right). \quad (29)$$

$\mathcal{L}$ is self-adjoint in $L^2_\nu(\mathbb{R}^3)$ and the null space $\mathcal{N}$ is a five-dimensional space spanned by the orthonormal basis

$$\mu^{\frac{1}{2}} \left\{ 1, v, \frac{|v|^2 - 3}{2} \right\}. \quad (30)$$

We denote $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$. In addition, denote $\mathcal{L}^{-1} : \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp$ the quasi-inverse of $\mathcal{L}$.

1.4. Main result. Let $\langle \cdot, \cdot \rangle$ be the standard L^2 inner product for $v \in \mathbb{R}^3$. Define the L^p and L^∞ norms in $\mathbb{R}^3$:

$$|f(x)|_p := \left(\int_{\mathbb{R}^3} |f(x, v)|^p dv \right)^{\frac{1}{p}}, \quad |f(x)|_\infty := \operatorname{ess\,sup}_{v \in \mathbb{R}^3} |f(x, v)|. \quad (31)$$

Furthermore, we define the L^p and L^∞ norms in $\Omega \times \mathbb{R}^3$:

$$\|f\|_p := \left(\iint_{\Omega \times \mathbb{R}^3} |f(x, v)|^p dv dx \right)^{\frac{1}{p}}, \quad \|f\|_\infty := \operatorname{ess\,sup}_{(x, v) \in \Omega \times \mathbb{R}^3} |f(x, v)|. \quad (32)$$

Define the weighted L^2 norms:

$$|f(x)|_\nu := |\nu^{\frac{1}{2}} f(x)|_2, \quad \|f\|_\nu := \|\nu^{\frac{1}{2}} f\|_2. \quad (33)$$

Denote the Japanese bracket:

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}} \quad (34)$$

Define the weighted L^∞ norm for $\varrho, \vartheta \geq 0$:

$$|f(x)|_{\infty, \vartheta, \varrho} = \operatorname{ess\,sup}_{v \in \mathbb{R}^3} \left(\langle v \rangle^\vartheta e^{\varrho|v|^2} |f(x, v)| \right), \quad (35)$$

$$\|f\|_{\infty, \vartheta, \varrho} = \operatorname{ess\,sup}_{(x, v) \in \Omega \times \mathbb{R}^3} \left(\langle v \rangle^\vartheta e^{\varrho|v|^2} |f(x, v)| \right). \quad (36)$$

In (1) and (16), based on the flow direction, we can divide the boundary $\gamma := \{(x_0, v) : x_0 \in \partial\Omega, v \in \mathbb{R}^3\}$ into the in-flow boundary γ_- , the out-flow boundary γ_+ , and the grazing set γ_0 :

$$\gamma_- := \{(x_0, v) : x_0 \in \partial\Omega, v \cdot n(x_0) < 0\}, \quad (37)$$

$$\gamma_+ := \{(x_0, v) : x_0 \in \partial\Omega, v \cdot n(x_0) > 0\}, \quad (38)$$

$$\gamma_0 := \{(x_0, v) : x_0 \in \partial\Omega, v \cdot n(x_0) = 0\}. \quad (39)$$

In particular, the boundary condition is only given on γ_- .

Define $d\gamma = |v \cdot n| d\varpi dv$ on γ for the surface measure ϖ . Define the L^p and L^∞ norms on the boundary:

$$\|f\|_{\gamma,p} = \left(\iint_{\gamma} |f(x,v)|^p d\gamma \right)^{\frac{1}{p}}, \quad \|f\|_{\gamma,\infty} = \operatorname{ess\,sup}_{(x,v) \in \gamma} |f(x,v)|. \quad (40)$$

Also, define the weighted L^∞ norm for $\varrho, \vartheta \geq 0$:

$$\|f\|_{\gamma,\infty,\varrho,\vartheta} = \operatorname{ess\,sup}_{(x,v) \in \gamma} \left(\langle v \rangle^{\vartheta} e^{\varrho|v|^2} |f(x,v)| \right). \quad (41)$$

The similar notation also applies to $\gamma_{\pm}$.

We plan to construct a Boltzmann solution from the incompressible Navier-Stokes-Fourier (INSF) system. The well-posedness and regularity of INSF is classical and we may refer to [5, 6] for the following result:

Theorem 1.1. *For any integer $k \geq 1$ and real number $s \in [2, \infty)$, if the boundary data*

$$|u|_{W^{k-\frac{1}{s},s}} + |\theta|_{W^{k-\frac{1}{s},s}} \ll 1, \quad (42)$$

then the steady Navier-Stokes-Fourier system in the smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} u \cdot \nabla_x u - \gamma_1 \Delta_x u + \nabla_x p = 0, \\ \nabla_x \cdot u = 0, \\ u \cdot \nabla_x \theta - \gamma_2 \Delta_x \theta = 0, \end{cases} \quad (43)$$

with constants $\gamma_1 > 0$ and $\gamma_2 > 0$, admits a unique solution (u, p, θ) satisfying

$$\|u\|_{W^{k,s}} + \|p\|_{W^{k-1,s}} + \|\theta\|_{W^{k,s}} \ll 1. \quad (44)$$

Now we are ready to state our main theorem:

Theorem 1.2. *For given μ_b^ε satisfying (8), (9), (6) and (12), there exists a unique solution $\mathfrak{F}^\varepsilon(x, v) = \mu_b^\varepsilon(v) + \mu^{\frac{1}{2}}(v) f^\varepsilon(x, v)$ to the stationary Boltzmann equation (1) with (13) in the form of*

$$f^\varepsilon = (\varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3) + (\varepsilon \mathcal{F}_1 + \varepsilon^2 \mathcal{F}_2) + \varepsilon \mathfrak{R}. \quad (45)$$

Here the leading-order interior solution F_1 is defined as

$$F_1 = \mu^{\frac{1}{2}} \left(\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2} \right), \quad (46)$$

in which (ρ, u, θ) satisfies (43), with the boundary data

$$\rho(x_0) = \rho_{b,1}(x_0) + M(x_0), \quad u(x_0) = u_{b,1}(x_0), \quad \theta(x_0) = \theta_{b,1}(x_0). \quad (47)$$

$M(x_0)$ is a function chosen such that the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0, \quad (48)$$

and the normalization condition (15) hold. The higher-order interior solutions F_2, F_3 are defined in (62)(64), and the boundary layers $\mathcal{F}_1, \mathcal{F}_2$ are defined in (73)(74). The remainder $\mathfrak{R}$ satisfies for any $0 \leq \varrho < \frac{1}{4}$ and $3 < \vartheta \leq \vartheta_0$

$$\|\mathfrak{R}\|_{\infty,\vartheta,\varrho} + \|\mathfrak{R}\|_{\gamma_+,\infty,\varrho,\vartheta} \lesssim_\delta \varepsilon^{\frac{1}{3}-\delta} \quad (49)$$

for any $0 < \delta \ll 1$.

Remark 1.3. $\mathfrak{F}^\varepsilon$ is the solution to (1) both in the weak (L^2) and mild (weighted L^∞) sense. We refer to [19] for discussion of the Boltzmann solution in the bounded domain.

Remark 1.4. From Theorem 1.2, we know $f^\varepsilon \sim \varepsilon F_1$ is of order $O(\varepsilon)$. The difference $f^\varepsilon - \varepsilon F_1 = o(\varepsilon)$ as $\varepsilon \rightarrow 0$.

Remark 1.5. The case $\rho_{b,1}(x_0) = 0$, $u_{b,1}(x_0) = 0$ and $\theta_{b,1}(x_0) \neq 0$ is typically called the non-isothermal model, which represents a system that only has heat transfer through the boundary but has no particle exchange and no work done between the environment and the system. Based on Theorem 1.2, its hydrodynamic limit is a steady Navier-Stokes-Fourier system with non-slip boundary condition. This provides a rigorous derivation of this important fluid model.

Remark 1.6. The convexity of the domain plays a significant role in the boundary layer analysis. In smooth non-convex domains, we can show the well-posedness of the ε -Milne problem with geometric correction following the idea in [36]. However, the regularity proof cannot go through since the arguments to bound I_1 in Section 3.4 and II_5 in Section 3.5 will break down due to different shape of characteristics [43].

Remark 1.7. Our analysis of boundary layer using mild formulation relies on the hard-sphere collision kernel Q . As [19] pointed out, such method can be extended to treat hard potential with Grad's angular cutoff. However, it may not be directly applied to soft potential or non-cutoff case.

1.5. History and motivation. Hydrodynamic limits are central to connecting the kinetic theory and fluid mechanics. It provides rigorous derivation of fluid equations (like Euler equations or Navier-Stokes equations, etc.) from the kinetic equations (like Boltzmann equations, Landau equations, etc.). As an integrated step to tackle the well-known Hilbert's Sixth Problem, since early 20th century, this type of problems have been extensively studied in many different settings: stationary or evolutionary, linear or nonlinear, strong solution or weak solution, etc.

The early result by Hilbert [25] dates back to 1916, using the so-called Hilbert's expansion, i.e. an asymptotic series of the density function $\mathfrak{F}^\varepsilon$ as a power series of the Knudsen number ε .

The general theory of initial-boundary-value problems for hydrodynamic limits was first developed by Grad [16], and then extended by Sone [29, 30, 31] and Sone-Aoki [34], for both the evolutionary and stationary equations. The classical books by Sone [32, 33] provide a comprehensive summary of previous results and give a complete analysis of such approaches. However, the results in [32, 33] are only formal and lack rigorous justifications.

So far, the mainstream study of hydrodynamic limits can be put into two categories: renormalized solution and mild/strong solution.

The renormalized solution, introduced by DiPerna-Lions [9] to justify the global well-posedness of the Boltzmann equation, has shown to be a powerful tool to study the kinetic equation with general data. It has been proved that the hydrodynamic limits of renormalized solution is the Leray solution for fluid equation. Due to the huge number, it is almost impossible to give a complete list of all the related publications. Reader may refer to Golse-Saint-Raymond [14, 15], Saint-Raymond [27], Masmoudi-Saint-Raymond [26], Bardos-Golse-Levermore [1, 2, 3, 4]. It is also worth noting that the book by Saint-Raymond [28] and the references therein provide a nice summary of the progress in this direction.

Unfortunately, as [11] pointed out, this approach of renormalized solution does not work for stationary hydrodynamic limit problems due to the lack of L^1 and entropy estimates. Hence, it is necessary to develop a different theory based on mild/strong solutions and energy estimates. For this direction, reader may refer to Masi-Esposito-Lebowitz [8], Esposito-Lebowitz-Marra [12], Guo [18], Guo-Jang [20], Guo-Jang-Jiang [21, 22], Esposito-Guo-Kim-Marra [11] and the references therein.

For stationary Boltzmann equation where the state of gas is close to a uniform state at rest, the expansion of the perturbation $f^\varepsilon = O(\varepsilon)$ consists of two parts: the interior solution $f_{\text{in}}^\varepsilon = \sum_{k=1}^\infty \varepsilon^k F_k$, which is based on a hierarchy of linearized Boltzmann equations and satisfies a steady Navier-Stokes-Fourier system, and the boundary layer $f_{\text{bl}}^\varepsilon = \sum_{k=1}^\infty \varepsilon^k \mathcal{F}_k$, which is based on a half-space kinetic equation and decays rapidly when it is away from the boundary.

Note that boundary layer plays a significant role in proving the asymptotic convergence in the L^∞ sense. If instead we consider L^p convergence for $1 \leq p < \infty$ which is technically easier, then the boundary layer $\mathcal{F}_1$ is of order $\varepsilon^{\frac{1}{p}}$ due to rescaling, which is negligible compared with F_1 as $\varepsilon \rightarrow 0$. As far as we are aware of, at this stage the best result regarding hydrodynamic limits of mild/strong solutions for the 3D stationary problem in bounded domains is [11], which justifies the L^p convergence without boundary layer analysis. As for the 2D problem, the best result is [38], which justifies the L^∞ convergence with a detailed discussion of boundary layers.

In this paper, we will fill the last piece and focus on the most difficult case, the 3D problem with L^∞ convergence. This paper is the first half of our monograph [41] on the incompressible Navier-Stokes-Fourier limit of kinetic equations (the second half [42] focuses on the evolutionary problems). As far as we are aware of, our theorem is the first result to rigorously justify the hydrodynamic limits of 3D stationary Boltzmann equation with boundary layer effects in L^∞ .

1.6. Ideas and methodology. The geometric effects in boundary layer analysis has been observed for a long time (see [12]). Inspired by [7], a new formulation of boundary layer based on the Milne problem with geometric correction was proposed in [40] to study a simple kinetic model – neutron transport equations, in a 2D plate domain. The key component of the proof is the L^∞ well-posedness and decay of the boundary layer equation. Furthermore, through a careful discussion of the $W^{1,\infty}$ regularity and quasi- $W^{2,\infty}$ regularity, such results were extended in [23, 24, 37, 39] to treat more general 2D/3D domains and boundary conditions.

Neutron transport equation is a linear equation with homogeneous collision kernels. In contrast, Boltzmann equation poses more technical complications due to the higher dimension of null space and more singular collision kernels. For 2D boundary layers, the L^∞ well-posedness and decay were discussed in [35] and the $W^{1,\infty}$ regularity estimates were proved in [38]. However, such results are completely absent for 3D domains.

As [11, Section 1] and [38, Section 2.2] reveal, 3D problems and L^∞ convergence have several key difficulties: boundary geometry is more complicated; remainder estimates is not strong enough to close the proof; the matching between interior solution and boundary layer is unclear.

Among all these, the most serious issue is that 3D collision kernel $k(v, v')$ contains the singularity $\frac{1}{|v-v'|}$. Such singularity is absent in 2D, in which we can freely manipulate the integrals involving k . For example, in 2D we may use Hölder's

inequality to bound

$$\int_{\mathbb{R}^2} k(v, v') g(v') dv' \lesssim \|k(v)\|_p \|g\|_q, \quad (50)$$

with $p \gg 1$ and $q \sim 1$ (such techniques are also utilized in analyzing the neutron transport equation). However, such approach naturally fails in 3D since k^p might not be integrable. The consequence is devastating. The preliminary lemmas in [38, Section 2.1] showing the $W^{1,\infty}$ regularity will not work any more. Hence, the key arguments [38, (5.81), (5.82), (5.95)] cannot be adapted to the singular kernel k . This kind of issues are so common in the regularity proof, so we have to reconsider all the details.

In this paper, our major upshots focus on tackling the challenging technical difficulties in justifying regularity of 3D boundary layers. The basic idea is to introduce an intricate characteristic analysis to capture the interaction of spacial and velocity derivatives, and analyze the singularity generated by the weighted non-local operators. We need several important preliminary results: 3.7, 3.8 and Lemma 3.9. Roughly speaking, we carefully analyze all kinds of the integrals involving k and $\nabla_v k$ to tame the singularity. Then we implement these lemmas in different regions of the characteristics. Certainly, our argument also works in 2D, but is more subtle than that in [38, Section 5].

In addition, as a minor contribution, we modify the remainder estimate in [11] and [38] to include the boundary layer terms and close the proof. In a non-rigorous fashion, for the remainder equation

$$\varepsilon v \cdot \nabla_x R + \mathcal{L}[R] = S + \Gamma[R, H] + \varepsilon^3 \Gamma[R, R], \quad (51)$$

we intend to justify $R \sim o(\varepsilon^{-2+\delta})$ in terms of $S \sim o(1)$ and $H \sim o(\varepsilon)$. In 2D, based on the $L^{2m} - L^\infty$ framework and L^∞ nonlinear estimate, we arrive at

$$\|R\|_{\infty, \vartheta, \varrho} \lesssim \frac{1}{\varepsilon^2} \|S\|_{L^1} + \frac{1}{\varepsilon^2} \|H\|_{L^\infty}. \quad (52)$$

For S part, due to rescaling in boundary layers in L^1 , which offers another ε , this is sufficient to close the proof. However, as one of the key steps in the remainder estimate, the embedding theorem is much worse in 3D than in 2D. For example, the result as [38, (4.14)] is only true when $1 \leq m < 3$. This restricts our choice of $m \sim 3$ and thus in 3D we only have the bound

$$\|R\|_{\infty, \vartheta, \varrho} \lesssim \frac{1}{\varepsilon^{\frac{5}{2}}} \|S\|_{L^{\frac{6}{5}}} + \frac{1}{\varepsilon^3} \|H\|_{L^\infty}. \quad (53)$$

Now S part is still controllable. However, there is no clear mechanism to improve the bounds of H , so we get stuck if following the above argument.

To overcome this difficulty, we have to dig into more details of the proof of remainder estimates. In our modified $L^{2m} - L^\infty$ framework, the main strategy is to combine

$$\|\mathbb{P}[R]\|_{L^6} \lesssim \frac{1}{\varepsilon^2} \|S\|_{L^{\frac{6}{5}}} + \frac{1}{\varepsilon} \|\Gamma[R, H]\|_{L^2} \quad \text{and} \quad \|R\|_{\infty, \vartheta, \varrho} \lesssim \frac{1}{\varepsilon^{\frac{1}{2}}} \|\mathbb{P}[R]\|_{L^6}. \quad (54)$$

Our key idea is to regard $\Gamma[R, H]$ as a perturbation of the linear term and absorb it into the left-hand side estimate at the L^6 level rather than at the L^∞ level. This requires a detailed proof of the nonlinear bound $\|\Gamma[R, H]\|_{L^2} \lesssim \|R\|_{L^2} \|H\|_{L^\infty}$ and the interaction of $\mathbb{P}[R]$ and $(\mathbb{I} - \mathbb{P})[R]$. A similar argument is also provided for the boundary terms.

Finally, it is worthwhile to mention that we give detailed derivation of boundary layer equations in 3D quasi-spherical coordinates and provide a full description of the matching procedure, which are absent in previous works.

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω , but does not depend on the data or ε . It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z . We write $a \lesssim b$ to denote $a \leq Cb$.

This paper is organized as follows: in Section 2, we perform the asymptotic expansion and matching procedure; Section 3 focuses on the well-posedness and regularity of the boundary layer equation, i.e. the ε -Milne problem with geometric correction; in Section 4, we study the remainder estimates in both non-perturbed and perturbed cases, and finally prove the main theorem.

2. Asymptotic expansion.

2.1. Interior expansion. We define the interior expansion

$$f_{\text{in}}^\varepsilon(x, v) := \sum_{k=1}^3 \varepsilon^k F_k(x, v). \quad (55)$$

Plugging it into the equation (16) and comparing the order of ε , we obtain

$$\mathcal{L}[F_1] = 0, \quad (56)$$

$$\mathcal{L}[F_2] = -v \cdot \nabla_x F_1 + \Gamma[F_1, F_1], \quad (57)$$

$$\mathcal{L}[F_3] = -v \cdot \nabla_x F_2 + 2\Gamma[F_1, F_2]. \quad (58)$$

The analysis of F_k solvability is standard and well-known. As [32, Chapter 4] and [33, Chapter 3] reveal

$$F_1 = \mu^{\frac{1}{2}} \left\{ \rho + u \cdot v + \theta \left(\frac{|v|^2 - 3}{2} \right) \right\}, \quad (59)$$

where (ρ, u, θ) satisfies the Navier-Stokes-Fourier system

$$\begin{cases} u \cdot \nabla_x u - \gamma_1 \Delta_x u + \nabla_x p = 0, \\ \nabla_x \cdot u = 0, \\ u \cdot \nabla_x \theta - \gamma_2 \Delta_x \theta = 0, \end{cases} \quad (60)$$

for constants $\gamma_1 > 0$ and $\gamma_2 > 0$ and the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0. \quad (61)$$

Similarly,

$$\begin{aligned} F_2 = & \mu^{\frac{1}{2}} \left\{ \rho_2 + u_2 \cdot v + \theta_2 \left(\frac{|v|^2 - 3}{2} \right) \right\} + \mu^{\frac{1}{2}} \left\{ \rho u \cdot v + (\rho \theta + |u|^2) \left(\frac{|v|^2 - 3}{2} \right) \right\} \\ & + \mathcal{L}^{-1}[-v \cdot \nabla_x F_1 + \Gamma[F_1, F_1]] \end{aligned} \quad (62)$$

where (ρ_2, u_2, θ_2) satisfies the fluid system

$$\begin{cases} \nabla_x(p - (\rho_2 + \theta_2 + \rho \theta)) = 0, \\ u_1 \cdot \nabla_x u_2 + (\rho u + u_2) \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_2 + \nabla_x p = -\gamma_2 \nabla_x \cdot \Delta_x \theta - \gamma_4 \nabla_x \cdot (\theta_1 (\nabla_x u + (\nabla_x u)^T)), \\ \nabla_x \cdot u_2 = -u \cdot \nabla_x \rho, \\ u \cdot \nabla_x \theta_2 + (\rho u + u_2) \cdot \nabla_x \theta - u \cdot \nabla_x p = \gamma_1 (\nabla_x u + (\nabla_x u)^T)^2 + \Delta_x (\gamma_2 \theta_2 + \gamma_5 \theta^2), \end{cases} \quad (63)$$

for constants $\gamma_3, \gamma_4, \gamma_5$. Since we do not expand the interior solution beyond F_3 , we cannot fully determine F_3 and it suffices to take

$$F_3 = \mathcal{L}^{-1}[-v \cdot \nabla_x F_2 + 2\Gamma[F_1, F_2]]. \quad (64)$$

2.2. Boundary layer expansion. In order to define boundary layer, we need to design a coordinate system based on the normal and tangential directions on the boundary surface.

For smooth manifold $\partial\Omega$, there exists an orthogonal curvilinear coordinates system (ι_1, ι_2) such that the coordinate lines locally coincide with the principal directions at x_0 .

Assume $\partial\Omega$ is parameterized by $r = r(\iota_1, \iota_2)$. Let $|\cdot|$ denote the length and ∂_i denote the derivative with respect to ι_i for $i = 1, 2$. Hence, $\partial_1 r$ and $\partial_2 r$ represent two orthogonal tangential vectors. Denote $P_i = |\partial_i r|$ for $i = 1, 2$. Then define the two orthogonal unit tangential vectors

$$\varsigma_1 := \frac{\partial_1 r}{P_1}, \quad \varsigma_2 := \frac{\partial_2 r}{P_2}. \quad (65)$$

Also, the outward unit normal vector is

$$n := \frac{\partial_1 r \times \partial_2 r}{|\partial_1 r \times \partial_2 r|} = \varsigma_1 \times \varsigma_2. \quad (66)$$

Obviously, $(\varsigma_1, \varsigma_2, n)$ forms a new orthogonal frame. Hence, consider the corresponding new coordinate system $(\iota_1, \iota_2, \mathfrak{N})$, where $\mathfrak{N}$ denotes the normal distance to boundary surface $\partial\Omega$, i.e.

$$x = r - \mathfrak{N}n. \quad (67)$$

Through a length computation, we arrive at

$$v \cdot \nabla_x = -(v \cdot n) \frac{\partial}{\partial \mathfrak{N}} - \frac{v \cdot \varsigma_1}{P_1(\kappa_1 \mathfrak{N} - 1)} \frac{\partial}{\partial \iota_1} - \frac{v \cdot \varsigma_2}{P_2(\kappa_2 \mathfrak{N} - 1)} \frac{\partial}{\partial \iota_2}, \quad (68)$$

where κ_1 and κ_2 are two principal curvatures.

Next, define the orthogonal velocity substitution for $\mathbf{v} := (v_\eta, v_\phi, v_\psi)$ as

$$\begin{cases} -v \cdot n := v_\eta, \\ -v \cdot \varsigma_1 := v_\phi, \\ -v \cdot \varsigma_2 := v_\psi. \end{cases} \quad (69)$$

Then the transport operator in (68) becomes

$$\begin{aligned} v \cdot \nabla_x = & v_\eta \frac{\partial}{\partial \mathfrak{N}} - \frac{1}{R_1 - \mu} \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{1}{R_2 - \mu} \left(v_\psi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\psi \frac{\partial}{\partial v_\psi} \right) \\ & - \frac{1}{P_1 P_2} \left(\frac{\partial_{11} r \cdot \partial_2 r}{P_1(\kappa_1 \mathfrak{N} - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_2 r}{P_2(\kappa_2 \mathfrak{N} - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\ & - \frac{1}{P_1 P_2} \left(\frac{\partial_{22} r \cdot \partial_1 r}{P_2(\kappa_2 \mathfrak{N} - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_1 r}{P_1(\kappa_1 \mathfrak{N} - 1)} v_\phi^2 \right) \frac{\partial}{\partial v_\psi} \\ & - \left(\frac{v_\phi}{P_1(\kappa_1 - 1)\mathfrak{N}} \frac{\partial}{\partial \tau_1} + \frac{v_\psi}{P_2(\kappa_2 \mathfrak{N} - 1)} \frac{\partial}{\partial \tau_2} \right), \end{aligned} \quad (70)$$

where $R_1 = \frac{1}{\kappa_1}$ and $R_2 = \frac{1}{\kappa_2}$ represent the radius of principal curvature.

Finally, we define the scaled variable $\eta = \frac{\eta_1}{\varepsilon}$. Then the equation (16) is transformed into

$$\left\{ \begin{aligned} & v_\eta \frac{\partial f^\varepsilon}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left(v_\phi^2 \frac{\partial f^\varepsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f^\varepsilon}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left(v_\psi^2 \frac{\partial f^\varepsilon}{\partial v_\eta} - v_\eta v_\psi \frac{\partial f^\varepsilon}{\partial v_\psi} \right) \\ & - \frac{\varepsilon}{P_1 P_2} \left(\frac{\partial_{11} r \cdot \partial_2 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_2 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\psi^2 \right) \frac{\partial f^\varepsilon}{\partial v_\phi} \\ & - \frac{\varepsilon}{P_1 P_2} \left(\frac{\partial_{22} r \cdot \partial_1 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_1 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial f^\varepsilon}{\partial v_\psi} \\ & - \varepsilon \left(\frac{v_\phi}{P_1(\varepsilon \kappa_1 \eta - 1)} \frac{\partial f^\varepsilon}{\partial \iota_1} + \frac{v_\psi}{P_2(\varepsilon \kappa_2 \eta - 1)} \frac{\partial f^\varepsilon}{\partial \iota_2} \right) + \mathcal{L}[f^\varepsilon] = \Gamma[f^\varepsilon, f^\varepsilon] \quad \text{in } \Omega \times \mathbb{R}^3, \\ & f^\varepsilon(0, \iota_1, \iota_2, \mathbf{v}) = \mathcal{P}^\varepsilon[f^\varepsilon](0, \iota_1, \iota_2, \mathbf{v}) \quad \text{for } v_\eta > 0. \end{aligned} \right. \quad (71)$$

We define the boundary layer expansion:

$$f_{\text{bl}}^\varepsilon(\eta, \iota_1, \iota_2, \mathbf{v}) = \sum_{k=1}^2 \varepsilon^k \mathcal{F}_k(\eta, \iota_1, \iota_2, \mathbf{v}), \quad (72)$$

where $\mathcal{F}_k$ can be defined by comparing the order of ε via plugging (72) into the equation (71). Thus, in a neighborhood of the boundary, we have

$$v_\eta \frac{\partial \mathcal{F}_1}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left(v_\psi^2 \frac{\partial \mathcal{F}_1}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{F}_1}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{F}_1] = 0, \quad (73)$$

$$v_\eta \frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left(v_\psi^2 \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{F}_2] = Z, \quad (74)$$

where

$$\begin{aligned} Z := & 2\Gamma[F_1, \mathcal{F}_1] + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + \frac{1}{P_1 P_2} \left(\frac{\partial_{11} r \cdot \partial_2 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_2 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\psi^2 \right) \frac{\partial \mathcal{F}_1}{\partial v_\phi} \\ & + \frac{1}{P_1 P_2} \left(\frac{\partial_{22} r \cdot \partial_1 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_1 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_1}{\partial v_\psi} + \frac{v_\phi}{P_1(\varepsilon \kappa_1 \eta - 1)} \frac{\partial \mathcal{F}_1}{\partial \iota_1} \\ & + \frac{v_\psi}{P_2(\varepsilon \kappa_2 \eta - 1)} \frac{\partial \mathcal{F}_1}{\partial \iota_2}. \end{aligned} \quad (75)$$

2.3. Boundary condition expansion. The bridge between the interior solution and boundary layer is the boundary condition. Define

$$\mathcal{P}[f](x_0, v) := \mu^{\frac{1}{2}}(v) \int_{\mathbf{u} \cdot \mathbf{n}(x_0) > 0} \mu^{\frac{1}{2}}(\mathbf{u}) f(x_0, \mathbf{u}) |\mathbf{u} \cdot \mathbf{n}(x_0)| d\mathbf{u}. \quad (77)$$

Plugging the combined expansion from (55) and (72) into the boundary condition (16) and (18), and comparing the order of ε , we obtain

$$F_1 + \mathcal{F}_1 = \mathcal{P}[F_1 + \mathcal{F}_1] + \mu_1(x_0, v), \quad (78)$$

$$(79)$$

$$F_2 + \mathcal{F}_2 = \mathcal{P}[F_2 + \mathcal{F}_2] + \mu_1(x_0, v) \int_{\mathbf{u} \cdot \mathbf{n}(x_0) > 0} \mu^{\frac{1}{2}}(\mathbf{u})(F_1 + \mathcal{F}_1) |\mathbf{u} \cdot \mathbf{n}(x_0)| d\mathbf{u} + \mu_2(x_0, v).$$

2.4. Matching procedure. Based on the analysis in Section 2.1, if

$$\rho_1(x_0) = \rho_{b,1}(x_0) + M_1(x_0), \quad u_1(x_0) = u_{b,1}(x_0), \quad \theta_1(x_0) = \theta_{b,1}(x_0), \quad (80)$$

where $M_1(x_0)$ is chosen such that the Boussinesq relation (61) and the normalization condition (based on (15))

$$\iint_{\Omega \times \mathbb{R}^3} F_1(x, v) \mu^{\frac{1}{2}}(v) dv dx = 0. \quad (81)$$

hold. By standard fluid estimates, for sufficiently smooth μ_1 , we have $F_1 \in W^{k,s}$ for any $k \in \mathbb{N}$ and $2 \leq s \leq \infty$. Also, since we have $F_1 = \mathcal{P}[F_1] + \mu_1$, we may take $\mathcal{F}_1 = 0$ which means the leading-order boundary layer vanishes.

Then we go to the next order F_2 and $\mathcal{F}_2$. Note that key observation that F_2 defined through (63) cannot satisfy the boundary condition (79) alone, and thus we have to introduce the non-vanishing boundary layer $\mathcal{F}_2$.

Let $\mathcal{F}_2$ satisfy the ε -Milne problem with geometric correction

$$(82)$$

$$\begin{cases} v_\eta \frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left(v_\psi^2 \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{F}_2] = 0, \\ \mathcal{F}_2(0, \iota_1, \iota_2, \mathbf{v}) = h(\iota_1, \iota_2, \mathbf{v}) - \tilde{h}(\iota_1, \iota_2, \mathbf{v}) \quad \text{for } v_\eta > 0, \\ \mathcal{F}_2(L, \iota_1, \iota_2, \mathbf{v}) = \mathcal{F}_2(L, \iota_1, \iota_2, \mathcal{R}[\mathbf{v}]), \end{cases}$$

where the length of boundary layer $L := \varepsilon^{-\frac{1}{2}}$ and $\mathcal{R}[v_\eta, v_\phi, v_\psi] = (-v_\eta, v_\phi, v_\psi)$, with the in-flow boundary data

$$h(\iota_1, \iota_2, \mathbf{v}) = M_1 \mu_1(x_0, v) + \mu_2(x_0, v) - (B - \mathcal{P}[B]), \quad (83)$$

for

$$B := \mu^{\frac{1}{2}} \left\{ \rho \mathbf{u} \cdot \mathbf{v} + (\rho \theta + |\mathbf{u}|^2) \left(\frac{|\mathbf{v}|^2 - 3}{2} \right) \right\} + \mathcal{L}^{-1}[-v \cdot \nabla_x F_1 + \Gamma[F_1, F_1]]. \quad (84)$$

Using (10), considering B given in Section 2.1 and using symmetry, we may directly check that

$$\begin{aligned} & \int_{v_\eta > 0} \mu^{\frac{1}{2}}(\mathbf{v}) h(\iota_1, \iota_2, \mathbf{v}) |v_\eta| d\mathbf{v} \\ &= - \int_{v \cdot \mathbf{n}(x_0) < 0} B(x_0) (v \cdot \mathbf{n}(x_0)) dv + \int_{v \cdot \mathbf{n}(x_0) < 0} \mathcal{P}[B](x_0) (v \cdot \mathbf{n}(x_0)) dv \\ &= - \int_{\mathbb{R}^3} B(x_0) (v \cdot \mathbf{n}(x_0)) dv = 0. \end{aligned} \quad (85)$$

Here the last equality holds due to orthogonality of $\mathcal{N}$ and $\mathcal{N}^\perp$, and $u_{b,1} \cdot \mathbf{n} = 0$. Based on Theorem 3.1, there exists a unique $\tilde{h}(\iota_1, \iota_2, \mathbf{v}) \in \mathcal{N}$ such that (82) is well-posed and the solution decays exponentially fast. Then we further prescribe the boundary conditions for (ρ_2, u_2, θ_2)

$$\mu^{\frac{1}{2}} \left\{ \rho_2(x_0) + u_2(x_0) \cdot \mathbf{v} + \theta_2(x_0) \left(\frac{|\mathbf{v}|^2 - 3}{2} \right) \right\} = \tilde{h}(\iota_1, \iota_2, \mathbf{v}) + M_2(x_0) \mu^{\frac{1}{2}}(v). \quad (86)$$

Here x_0 corresponds to (ι_1, ι_2) and v corresponds to $\mathbf{v}$, based on substitution in Section 2.2. Here, the constant $M_2(x_0)$ is chosen to enforce the Boussinesq relation

$$p - (\rho_2 + \theta_2 + \rho_1 \theta_1) = \text{constant}, \quad (87)$$

and the normalization condition

$$\iint_{\Omega \times \mathbb{R}^3} (F_2 + \mathcal{F}_2)(x, v) \mu^{\frac{1}{2}}(v) dv dx = 0, \quad (88)$$

where p is the pressure solved from (60). Then using the zero mass-flux condition

$$\int_{\mathbb{R}^3} \mu^{\frac{1}{2}}(\mathbf{u}) \mathcal{F}_2(x, \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) d\mathbf{u} = 0, \quad (89)$$

and (10) for μ_1, μ_2 , we obtain

$$\mathcal{P} \left[\mu^{\frac{1}{2}} \left\{ \rho_2(x_0) + u_2(x_0) \cdot v + \theta_2(x_0) \left(\frac{|v|^2 - 3}{2} \right) \right\} + \mathcal{F}_2 \right] = M_2 \mu^{\frac{1}{2}}, \quad (90)$$

which further yields (79) holds.

Due to Theorem 3.16, we know that $(\rho_2, u_2, \theta_2) \in W^{1,\infty}(\partial\Omega)$, and thus by standard fluid estimates, we have $(\rho_2, u_2, \theta_2) \in W^{3,s}$ for any $2 \leq s < \infty$ and thus $F_2 \in W_x^{3,s} L_v^\infty$. Further, we have $F_3 \in W_x^{2,s} L_v^\infty$.

3. Analysis of boundary layers.

3.1. Well-posedness and decay. In this section, we will study the well-posedness and decay of the ε -Milne problem with geometric correction. We will only record the main results without the proofs since it is rather similar to those in [38].

Note the null space $\mathcal{N}$ of the operator $\mathcal{L}$ is spanned by $\mu^{\frac{1}{2}} \left\{ 1, v_\eta, v_\phi, v_\psi, \frac{|v|^2 - 3}{2} \right\}$. Given the boundary data $h(\mathbf{v})$ and source term $S(\eta, \mathbf{v})$ satisfying for some constant $K > 0$

$$|h|_{\infty, \vartheta, \varrho} \lesssim 1, \quad \|e^{K\eta} S\|_{\infty, \vartheta, \varrho} \lesssim 1, \quad (91)$$

we intend to find $\tilde{h}(\mathbf{v}) \in \mathcal{N}$ such that the ε -Milne problem with geometric correction for $\mathcal{G}(\eta, \mathbf{v})$ in the domain $(\eta, \mathbf{v}) \in [0, L] \times \mathbb{R}^3$ as

$$\begin{cases} v_\eta \frac{\partial \mathcal{G}}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left(v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left(v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{G}}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{G}] = S, \\ \mathcal{G}(0, \mathbf{v}) = h(\mathbf{v}) - \tilde{h}(\mathbf{v}) \quad \text{for } v_\eta > 0, \\ \mathcal{G}(L, \mathbf{v}) = \mathcal{G}(L, \mathcal{R}[\mathbf{v}]), \end{cases} \quad (92)$$

with the zero mass-flux condition

$$\int_{\mathbb{R}^3} v_\eta \mathcal{G}(0, \mathbf{v}) d\mathbf{v} = 0, \quad (93)$$

is well-posed, and $\mathcal{G}$ decays to zero as $\eta \rightarrow \infty$. Here $\mathcal{R}[\mathbf{v}] = (-v_\eta, v_\phi, v_\psi)$ and $L = \varepsilon^{-\frac{1}{2}}$. For simplicity, we temporarily ignore the dependence of ι_1, ι_2 , but our estimates are uniform in these variables. Also, the estimates and decaying rate should be uniform in ε .

Theorem 3.1 (Well-Posedness and decay). *Assume (91) holds. Then there exists a unique $\tilde{h}(\mathbf{v}) \in \mathcal{N}$ and $0 < K_0 < K$ such that there exists a unique solution $\mathcal{G}(\eta, \mathbf{v})$ to the equation (92) satisfying for $\varrho \geq 0$ and $\vartheta > 3$,*

$$\|e^{K_0\eta}\mathcal{G}\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (94)$$

3.2. Preliminaries for regularity estimates. Now we begin to study the regularity of the solution $\mathcal{G}$ to (92). From now on, denote the boundary data $p := h - \tilde{h}$. Besides (91), we further require

$$\|\nabla_{\mathbf{v}} p\|_{\infty, \vartheta, \varrho} \lesssim 1, \quad \|e^{K\eta}\partial_{\eta} S\|_{\infty, \vartheta, \varrho} + \|e^{K\eta}\nabla_{\mathbf{v}} S\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (95)$$

Let $G_i(\eta) := \frac{\varepsilon}{R_i - \varepsilon\eta}$ for $i = 1, 2$. Denote the potential function $W_i(\eta) := \ln\left(\frac{R_i}{R_i - \varepsilon\eta}\right)$ and $W(\eta) := W_1(\eta) + W_2(\eta)$. It is easy to check that

$$\frac{dW_i}{d\eta} = \frac{\varepsilon}{R_i - \varepsilon\eta} = -G_i, \quad W_i(0) = 0. \quad (96)$$

Define a weight function

$$\zeta(\eta; \mathbf{v}) = \left((v_{\eta}^2 + v_{\phi}^2 + v_{\psi}^2) - \left(\frac{R_1 - \varepsilon\eta}{R_1} \right)^2 v_{\phi}^2 - \left(\frac{R_2 - \varepsilon\eta}{R_2} \right)^2 v_{\psi}^2 \right)^{\frac{1}{2}}. \quad (97)$$

It is easy to see that the closer a point $(\eta; v_{\eta}, v_{\phi}, v_{\psi})$ is to the grazing set $(\eta; v_{\eta}, v_{\phi}, v_{\psi}) = (0; 0, v_{\phi}, v_{\psi})$, the smaller ζ is. In particular, at the grazing set, $\zeta(0; 0, v_{\phi}, v_{\psi}) = 0$. In particular, direct computation justifies the following commutativity property.

Lemma 3.2. *Let ζ be defined as in (97). We have*

$$v_{\eta} \frac{\partial \zeta}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta} \left(v_{\phi}^2 \frac{\partial \zeta}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial \zeta}{\partial v_{\phi}} \right) - \frac{\varepsilon}{R_2 - \varepsilon\eta} \left(v_{\psi}^2 \frac{\partial \zeta}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial \zeta}{\partial v_{\psi}} \right) = 0. \quad (98)$$

Remark 3.3. This lemma indicates the commutativity of ζ and the ε -Milne operator, i.e.

$$\begin{aligned} & v_{\eta} \frac{\partial(\zeta f)}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta} \left(v_{\phi}^2 \frac{\partial(\zeta f)}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial(\zeta f)}{\partial v_{\phi}} \right) - \frac{\varepsilon}{R_2 - \varepsilon\eta} \left(v_{\psi}^2 \frac{\partial(\zeta f)}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial(\zeta f)}{\partial v_{\psi}} \right) \\ &= \zeta \left(v_{\eta} \frac{\partial f}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta} \left(v_{\phi}^2 \frac{\partial f}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial f}{\partial v_{\phi}} \right) - \frac{\varepsilon}{R_2 - \varepsilon\eta} \left(v_{\psi}^2 \frac{\partial f}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial f}{\partial v_{\psi}} \right) \right). \end{aligned} \quad (99)$$

Lemma 3.4. *For Boltzmann collision frequency $\nu = \nu(|\mathbf{v}|)$, we have*

$$\left| \frac{d\nu}{d|\mathbf{v}|} \right| \lesssim 1. \quad (100)$$

Proof. Based on [13, Chapter 3], we know

$$\nu(|\mathbf{v}|) \sim \left(2|\mathbf{v}| + \frac{1}{|\mathbf{v}|} \right) \int_0^{|\mathbf{v}|} e^{-z^2} dz + e^{-|\mathbf{v}|^2}. \quad (101)$$

Then for $|\mathbf{v}| \geq 1$, we have

$$\left| \frac{d\nu}{d|\mathbf{v}|} \right| \lesssim \left(1 + \frac{1}{|\mathbf{v}|^2} \right) \int_0^{|\mathbf{v}|} e^{-z^2} dz + \left(|\mathbf{v}| + \frac{1}{|\mathbf{v}|} \right) e^{-|\mathbf{v}|^2} \lesssim 1. \quad (102)$$

For $|\mathbf{v}| \leq 1$, the key difficulty is the fractional term. Taylor expansion implies

$$\frac{1}{|\mathbf{v}|} \int_0^{|\mathbf{v}|} e^{-z^2} dz \sim \frac{1}{|\mathbf{v}|} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} |\mathbf{v}|^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} |\mathbf{v}|^{2k} \lesssim 1. \quad (103)$$

Hence, the desired result naturally follows. $\square$

Lemma 3.5 (Lemma 3 of [19]). *Let $0 \leq \varrho < \frac{1}{4}$ and $\vartheta \geq 0$. Then for $\delta > 0$ sufficiently small and any $\mathbf{v} \in \mathbb{R}^3$, we have*

$$|k(\mathbf{u}, \mathbf{v})| \lesssim \left(|\mathbf{u} - \mathbf{v}| + \frac{1}{|\mathbf{u} - \mathbf{v}|} \right) e^{-\frac{1}{8}|\mathbf{u} - \mathbf{v}|^2 - \frac{1}{8} \frac{||\mathbf{u}|^2 - |\mathbf{v}|^2|^2}{|\mathbf{u} - \mathbf{v}|^2}}, \quad (104)$$

and thus

$$\int_{\mathbb{R}^3} e^{\delta|\mathbf{u} - \mathbf{v}|^2} |k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim \frac{1}{\langle \mathbf{v} \rangle}. \quad (105)$$

Lemma 3.6. *Let $0 \leq \varrho < \frac{1}{4}$ and $\vartheta \geq 0$. We have*

$$\int_{\mathbb{R}^3} |\nabla_{\mathbf{v}} k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim 1. \quad (106)$$

Proof. Based on [13, Chapter 3], for hard-sphere gas, $k = k_1 + k_2$, where

$$k_1(\mathbf{u}, \mathbf{v}) \sim |\mathbf{u} - \mathbf{v}| e^{-\frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2}|\mathbf{v}|^2}, \quad (107)$$

$$k_2(\mathbf{u}, \mathbf{v}) \sim \frac{1}{|\mathbf{u} - \mathbf{v}|} e^{-\frac{1}{4}|\mathbf{u} - \mathbf{v}|^2 - \frac{1}{4} \frac{||\mathbf{u}|^2 - |\mathbf{v}|^2|^2}{|\mathbf{u} - \mathbf{v}|^2}}. \quad (108)$$

Following the similar argument as in Lemma 3.5, we have

$$|\nabla_{\mathbf{v}} k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} \lesssim (1 + |\mathbf{u} - \mathbf{v}|^2)^{\frac{\vartheta}{2}} |\nabla_{\mathbf{v}} k(\mathbf{u}, \mathbf{v})| e^{-\varrho(|\mathbf{u}|^2 - |\mathbf{v}|^2)} \quad (109)$$

Here, the key is to bound $|\nabla_{\mathbf{v}} k(\mathbf{u}, \mathbf{v})|$. Substituting $\mathbf{u} \rightarrow \sigma = \mathbf{u} - \mathbf{v}$, we get

$$k_1(\sigma, \mathbf{v}) = |\sigma| e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2}|\sigma|^2}, \quad (110)$$

$$k_2(\sigma, \mathbf{v}) = \frac{1}{|\sigma|} e^{-\frac{1}{4}|\sigma|^2 - \frac{1}{4} \frac{||\sigma|^2 - 2\sigma \cdot \mathbf{v}|^2}{|\sigma|^2}}. \quad (111)$$

Then we compute

$$\nabla_{\mathbf{v}} k_1(\sigma, \mathbf{v}) = |\sigma| (-2\mathbf{v} - \sigma) e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2}|\sigma|^2}, \quad (112)$$

which implies

$$|\nabla_{\mathbf{v}} k_1(\sigma, \mathbf{v})| \lesssim |\sigma|^2 e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2}|\sigma|^2} + |\sigma| |\mathbf{v}| e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2}|\sigma|^2} := I_1 + I_2. \quad (113)$$

Here, I_1 is covered by similar techniques as in the proof of Lemma 3.7, I_2 is covered in Lemma 3.5. We obtain

$$I_1 \lesssim 1, \quad I_2 \lesssim \frac{|\mathbf{v}|}{1 + |\mathbf{v}|} \lesssim 1, \quad (114)$$

which implies

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{v}} k_1(\mathbf{u}, \mathbf{v}) \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim 1. \quad (115)$$

On the other hand, we compute

$$|\nabla_{\mathbf{v}} k_2(\sigma, \mathbf{v})| = \frac{1}{|\sigma|} \left(\sigma - \frac{2\sigma \cdot \mathbf{v}}{|\sigma|^2} \sigma \right) e^{-\frac{1}{4}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2}}, \quad (116)$$

which implies

$$|\nabla_{\mathbf{v}} k_2(\sigma, \mathbf{v})| \lesssim e^{-\frac{1}{4}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2}} + \frac{|\mathbf{v}|}{|\sigma|} e^{-\frac{1}{4}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2}} =: II_1 + II_2. \quad (117)$$

Still, II_1 is covered by similar techniques as in the proof of Lemma 3.7, II_2 is covered in Lemma 3.5. We obtain

$$II_1 \lesssim 1, \quad II_2 \lesssim \frac{|\mathbf{v}|}{1 + |\mathbf{v}|} \lesssim 1, \quad (118)$$

which implies

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{v}} k_2(\mathbf{u}, \mathbf{v}) \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim 1. \quad (119)$$

Then the desired results follow from (115) and (119). $\square$

3.3. Mild formulation. Taking η derivative in (92) and multiplying ζ defined in (97) on both sides, we obtain the ε -transport problem for $\mathcal{A} := \zeta \frac{\partial \mathcal{G}}{\partial \eta}$

$$\begin{cases} v_\eta \frac{\partial \mathcal{A}}{\partial \eta} + G_1(\eta) \left(v_\phi^2 \frac{\partial \mathcal{A}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{A}}{\partial v_\phi} \right) + G_2(\eta) \left(v_\psi^2 \frac{\partial \mathcal{A}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{A}}{\partial v_\psi} \right) + \nu \mathcal{A} = \widetilde{\mathcal{A}} + S_{\mathcal{A}}, \\ \mathcal{A}(0, \mathbf{v}) = p_{\mathcal{A}}(\mathbf{v}) \text{ for } v_\eta > 0, \\ \mathcal{A}(L, \mathbf{v}) = \mathcal{A}(L, \mathcal{R}[\mathbf{v}]), \end{cases} \quad (120)$$

where the crucial non-local term

$$\widetilde{\mathcal{A}}(\eta, \mathbf{v}) = \int_{\mathbb{R}^3} \frac{\zeta(\eta, \mathbf{v})}{\zeta(\eta, \mathbf{u})} k(\mathbf{u}, \mathbf{v}) \mathcal{A}(\eta, \mathbf{u}) d\mathbf{u}. \quad (121)$$

Here we utilize Lemma 3.2 to move ζ inside the derivative. $p_{\mathcal{A}}$ and $S_{\mathcal{A}}$ will be specified later. We need to derive the a priori estimate of $\mathcal{A}$. Note that $\mathcal{A}$ is different from $K[\mathcal{A}]$ since the denominator $\zeta(\eta, \mathbf{u})$ is possibly zero. Thus, this creates a strong singularity and becomes the major difficulty in this section.

Define the characteristics $(\eta(s), v_\eta(s), v_\phi(s), v_\psi(s))$ for some parameter $s \in \mathbb{R}$ satisfying

$$\begin{aligned} \frac{d\eta}{ds} &= v_\eta, \quad \frac{dv_\eta}{ds} = G_1(\eta) v_\phi^2 + G_2(\eta) v_\psi^2, \quad \frac{dv_\phi}{ds} = -G_1(\eta) v_\eta v_\phi, \quad \frac{dv_\psi}{ds} = -G_2(\eta) v_\eta v_\psi, \\ \text{which leads to} \end{aligned} \quad (122)$$

$$v_\eta^2(s) + v_\phi^2(s) + v_\psi^2(s) := E_1, \quad v_\phi(s) e^{-W_1(\eta(s))} := E_2, \quad v_\psi(s) e^{-W_2(\eta(s))} := E_3,$$

where the conserved quantities E_i are constants depending on the starting point. We can easily check that the weight function satisfies $\zeta = \sqrt{E_1 - E_2^2 - E_3^2}$. Along the characteristics, the equation (120) can be rewritten as:

$$v_\eta \frac{d\mathcal{A}}{d\eta} + \mathcal{A} = \widetilde{\mathcal{A}} + S_{\mathcal{A}}. \quad (124)$$

Let

$$v'_\phi(\eta, \mathbf{v}; \eta') := v_\phi e^{W_1(\eta') - W_1(\eta)}, \quad v'_\psi(\eta, \mathbf{v}; \eta') := v_\psi e^{W_2(\eta') - W_2(\eta)}. \quad (125)$$

On the characteristics, we should always have $E_1 \geq v_\phi'^2 + v_\psi'^2$. Define

$$v'_\eta(\eta, \mathbf{v}; \eta') := \sqrt{E_1 - v_\phi'^2(\eta, \mathbf{v}; \eta') - v_\psi'^2(\eta, \mathbf{v}; \eta')}, \quad (126)$$

$$\mathbf{v}'(\eta, \mathbf{v}; \eta') := \left(v'_\eta(\eta, \mathbf{v}; \eta'), v'_\phi(\eta, \mathbf{v}; \eta'), v'_\psi(\eta, \mathbf{v}; \eta') \right), \quad (127)$$

$$\mathcal{R}[\mathbf{v}'(\eta, \mathbf{v}; \eta')] := \left(-v'_\eta(\eta, \mathbf{v}; \eta'), v'_\phi(\eta, \mathbf{v}; \eta'), v'_\psi(\eta, \mathbf{v}; \eta') \right). \quad (128)$$

Basically, this means $(\eta, v_\eta, v_\phi, v_\psi)$ and $(\eta', v'_\eta, v'_\phi, v'_\psi)$, $(\eta', -v'_\eta, v'_\phi, v'_\psi)$ are on the same characteristics.

We write the mild solution to (120) as

$$\mathcal{A}(\eta, \mathbf{v}) = \mathcal{K}[p_\mathcal{A}] + \mathcal{T}[\widetilde{\mathcal{A}} + S_\mathcal{A}], \quad (129)$$

where the operators $\mathcal{K}$ and $\mathcal{T}$ are defined as follows:

Region I: $v_\eta > 0$: The characteristics directly tracks back to the in-flow boundary $\eta = 0$ and $v_\eta > 0$, i.e.

$$\mathcal{K}[h](\eta, \mathbf{v}) := h\left(\mathbf{v}'(\eta, \mathbf{v}; 0)\right) \exp(-H_{\eta,0}), \quad (130)$$

$$\mathcal{T}[Q](\eta, \mathbf{v}) := \int_0^\eta \frac{Q\left(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta')\right)}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(-H_{\eta, \eta'}) d\eta'. \quad (131)$$

Here

$$H_{\eta, \eta'} := \int_{\eta'}^\eta \frac{\nu\left(\mathbf{v}'(\eta, \mathbf{v}; y)\right)}{v'_\eta(\eta, \mathbf{v}; y)} dy. \quad (132)$$

Region II: $v_\eta < 0$ and $v_\eta^2 + v_\phi^2 + v_\psi^2 \geq v_\phi'^2(\eta, \mathbf{v}; L) + v_\psi'^2(\eta, \mathbf{v}; L)$: The characteristics first goes a bit farther to the boundary $\eta = L$, then gets reflected and tracks back to the in-flow boundary, i.e.

$$\mathcal{K}[h](\eta, \mathbf{v}) := h\left(\mathbf{v}'(\eta, \mathbf{v}; 0)\right) \exp(-H_{L,0} - H_{L,\eta}), \quad (133)$$

$$\begin{aligned} \mathcal{T}[Q](\eta, \mathbf{v}) := & \left(\int_0^L \frac{Q\left(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta')\right)}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(-H_{L,\eta'} - H_{L,\eta}) d\eta' \right. \\ & \left. + \int_\eta^L \frac{Q\left(\eta', \mathcal{R}[\mathbf{v}'(\eta, \mathbf{v}; \eta')]\right)}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(H_{\eta, \eta'}) d\eta' \right). \end{aligned} \quad (134)$$

Region III: $v_\eta < 0$ and $v_\eta^2 + v_\phi^2 + v_\psi^2 \leq v_\phi'^2(\eta, \mathbf{v}; L) + v_\psi'^2(\eta, \mathbf{v}; L)$: The characteristics reaches the line $v_\eta = 0$ before reaching the boundary $\eta = L$, and then directly tracks back to the in-flow boundary, i.e.

$$\mathcal{K}[h](\eta, \mathbf{v}) := h\left(\mathbf{v}'(\eta, \mathbf{v}; 0)\right) \exp(-H_{\eta^+,0} - H_{\eta^+, \eta}), \quad (135)$$

$$\begin{aligned} \mathcal{T}[Q](\eta, \mathbf{v}) := & \left(\int_0^{\eta^+} \frac{Q\left(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta')\right)}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(-H_{\eta^+, \eta'} - H_{\eta^+, \eta}) d\eta' \right. \\ & \left. + \int_\eta^{\eta^+} \frac{Q\left(\eta', \mathcal{R}[\mathbf{v}'(\eta, \mathbf{v}; \eta')]\right)}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(H_{\eta, \eta'}) d\eta' \right). \end{aligned} \quad (136)$$

Here $\eta^+(\eta, \mathbf{v})$ defined by

$$E_1(\eta, \mathbf{v}) = v_\phi'^2(\eta, \mathbf{v}; \eta^+) + v_\psi'^2(\eta, \mathbf{v}; \eta^+) \quad (137)$$

locates the position that the characteristics touch $v_\eta = 0$ line, i.e. $(\eta^+, 0, v_\phi', v_\psi')$ is on the same characteristics as $(\eta, v_\eta, v_\phi, v_\psi)$. Based on [38], we can directly obtain

$$\|\mathcal{K}[p_{\mathcal{A}}]\|_{\infty, \vartheta, \varrho} \lesssim |p_{\mathcal{A}}|_{\infty, \vartheta, \varrho}, \quad (138)$$

$$\|\mathcal{T}[S_{\mathcal{A}}]\|_{\infty, \vartheta, \varrho} \lesssim \|\nu^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \varrho}. \quad (139)$$

Since we always assume that $(\eta, \mathbf{v})$ and $(\eta', \mathbf{v}')$ are on the same characteristics, in the following, we will simply write $\mathbf{v}'(\eta')$ or even $\mathbf{v}'$ instead of $\mathbf{v}'(\eta, \mathbf{v}; \eta')$ when there is no confusion. In addition, we will use δ or δ_0 to represent small quantities. They may depend on ε and need to be chosen later.

The next three subsections will be devoted to the estimate of $\mathcal{T}[\widetilde{\mathcal{A}}]$. In the analysis below, we will repeatedly use the following packages of simple facts (PSF):

1. Based on Theorem 3.1, we know $\|e^{K_0 \eta} \mathcal{G}\|_{\infty, \vartheta, \varrho} \lesssim 1$.
2. Based on Lemma 3.5, for $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3$, we have $\|e^{K_0 \eta} K[\mathcal{G}]\|_{\infty, \vartheta, \varrho} \lesssim \|e^{K_0 \eta} \nu^{-1} \mathcal{G}\|_{\infty, \vartheta, \varrho} \lesssim 1$.
3. Based on Lemma 3.6, we know $\|e^{K_0 \eta} \nabla_v K[\mathcal{G}]\|_{\infty, \vartheta, \varrho} \lesssim \|e^{K_0 \eta} \mathcal{G}\|_{\infty, \vartheta, \varrho} \lesssim 1$.
4. Since E_1 is conserved along the characteristics, we have $|\mathbf{v}| = |\mathbf{v}'|$ and further $\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} = \langle \mathbf{v}' \rangle^\vartheta e^{\varrho |\mathbf{v}'|^2}$.

3.4. **Region I:** $v_\eta > 0$. Based on (130), we need to bound

$$I = \mathcal{T}[\widetilde{\mathcal{A}}] = \int_0^\eta \frac{\widetilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta'))}{v_\eta'(\eta, \mathbf{v}; \eta')} \exp(-H_{\eta, \eta'}) d\eta'. \quad (140)$$

Based on (123) and (96), we have

$$E_2(\eta', v_\phi') = \frac{R_1 - \varepsilon \eta'}{R_1} v_\phi', \quad E_3(\eta', v_\psi') = \frac{R_2 - \varepsilon \eta'}{R_2} v_\psi'. \quad (141)$$

Then we can directly obtain for $0 < \eta' < L = \varepsilon^{-\frac{1}{2}}$,

$$\begin{aligned} \zeta(\eta', \mathbf{v}') &= \sqrt{(v_\eta'^2 + v_\phi'^2 + v_\psi'^2) - \left(\frac{R_1 - \varepsilon \eta'}{R_1}\right)^2 v_\phi'^2 - \left(\frac{R_2 - \varepsilon \eta'}{R_2}\right)^2 v_\psi'^2} \\ &\lesssim |v_\eta'| + \sqrt{\varepsilon \eta'} |v_\phi'| + \sqrt{\varepsilon \eta'} |v_\psi'| \lesssim |\mathbf{v}'|, \end{aligned} \quad (142)$$

and

$$\begin{aligned} \zeta(\eta', \mathbf{v}') &\geq \frac{1}{2} \left(\sqrt{v_\eta'^2} + \frac{1}{R_1} \sqrt{(R_1^2 - (R_1 - \varepsilon \eta')^2) v_\phi'^2} + \frac{1}{R_2} \sqrt{(R_2^2 - (R_2 - \varepsilon \eta')^2) v_\psi'^2} \right) \\ &\gtrsim |v_\eta'| + \sqrt{\varepsilon \eta'} |v_\phi'| + \sqrt{\varepsilon \eta'} |v_\psi'| \gtrsim \sqrt{\varepsilon \eta'} |\mathbf{v}'|. \end{aligned} \quad (143)$$

Also, considering (123) and (96), we know for $0 \leq \eta' \leq \eta$,

$$\begin{aligned} v_\eta &\leq v_\eta' = \sqrt{v_\eta'^2 + v_\phi'^2 + v_\psi'^2 - v_\phi'^2 \left(\frac{R_1 - \varepsilon \eta}{R_1 - \varepsilon \eta'}\right)^2 - v_\psi'^2 \left(\frac{R_2 - \varepsilon \eta}{R_2 - \varepsilon \eta'}\right)^2} \\ &\lesssim \sqrt{v_\eta'^2 + \varepsilon(\eta - \eta') v_\phi'^2 + \varepsilon(\eta - \eta') v_\psi'^2}, \end{aligned} \quad (144)$$

which means

$$\begin{aligned} -\int_{\eta'}^{\eta} \frac{1}{v_{\eta}'(y)} dy &\lesssim -\int_{\eta'}^{\eta} \frac{1}{2\sqrt{v_{\eta}^2 + \varepsilon(\eta-y)v_{\phi}^2 + \varepsilon(\eta-y)v_{\psi}^2}} dy \\ &= -\frac{\eta - \eta'}{v_{\eta} + \sqrt{v_{\eta}^2 + \varepsilon(\eta - \eta')v_{\phi}^2 + \varepsilon(\eta - \eta')v_{\psi}^2}} \\ &\lesssim -\frac{\eta - \eta'}{\sqrt{v_{\eta}^2 + \varepsilon(\eta - \eta')v_{\phi}^2 + \varepsilon(\eta - \eta')v_{\psi}^2}}. \end{aligned} \quad (145)$$

Define a C^{∞} cut-off function $\chi \in C^{\infty}[0, \infty)$ satisfying

$$\chi(v_{\eta}) = \begin{cases} 1 & \text{for } |v_{\eta}| \leq \delta, \\ 0 & \text{for } |v_{\eta}| \geq 2\delta. \end{cases} \quad (146)$$

We use χ instead of a sharp cut-off for the convenience of integration by parts. In the following, we will divide the estimate of I in (140) into several cases based on the value of v_{η} , v_{η}' , $\varepsilon\eta'$ and $\varepsilon(\eta - \eta')$. Assume the dummy variable $\mathbf{u} = (u_{\eta}, u_{\phi}, u_{\psi}) = (u_{\eta}, \tilde{u})$. The similar notation also applies to $\mathbf{v} = (v_{\eta}, v_{\phi}, v_{\psi}) = (v_{\eta}, \tilde{v})$.

Estimate of $I_1 : v_{\eta} \geq \delta_0$: In this step, we will not resort to $\mathcal{A}$ equation (120), but rather directly bound

$$\left| \langle \mathbf{v} \rangle^{\vartheta} e^{e|\mathbf{v}|^2} I_1 \right| \lesssim |\zeta| \left| \langle \mathbf{v} \rangle^{\vartheta} e^{e|\mathbf{v}|^2} \frac{\partial \mathcal{G}}{\partial \eta} \right| \lesssim \left| \langle \mathbf{v} \rangle^{\vartheta+1} e^{e|\mathbf{v}|^2} \frac{\partial \mathcal{G}}{\partial \eta} \right|. \quad (147)$$

Hence, the key is to estimate $\frac{\partial \mathcal{G}}{\partial \eta}$. As in (130), we rewrite the equation (92) along the characteristics as

$$\mathcal{G}(\eta, \mathbf{v}) = \exp(-H_{\eta,0}) \left(p(\mathbf{v}'(0)) + \int_0^{\eta} \frac{(K[\mathcal{G}] + S)(\eta', \mathbf{v}'(\eta'))}{v_{\eta}'(\eta')} \exp(H_{\eta',0}) d\eta' \right). \quad (148)$$

Taking η derivative on both sides of (148), we have

$$\frac{\partial \mathcal{G}}{\partial \eta} := X_1 + X_2 + X_3 + X_4 + X_5 + X_6, \quad (149)$$

where

$$X_1 = -\exp(-H_{\eta,0}) \frac{\partial H_{\eta,0}}{\partial \eta} \left(p(\mathbf{v}'(0)) + \int_0^{\eta} \frac{K[\mathcal{G}](\eta', \mathbf{v}'(\eta'))}{v_{\eta}'(\eta')} \exp(H_{\eta',0}) d\eta' \right), \quad (150)$$

$$X_2 = \exp(-H_{\eta,0}) \frac{\partial p(\mathbf{v}'(0))}{\partial \eta}, \quad (151)$$

$$X_3 = \frac{(K[\mathcal{G}] + S)(\eta, \mathbf{v})}{v_{\eta}}, \quad (152)$$

$$X_4 = -\exp(-H_{\eta,0}) \int_0^{\eta} \left((K[\mathcal{G}] + S)(\eta', \mathbf{v}'(\eta')) \exp(H_{\eta',0}) \frac{1}{v_{\eta}'^2(\eta')} \frac{\partial v_{\eta}'(\eta')}{\partial \eta} d\eta' \right) \quad (153)$$

$$X_5 = \exp(-H_{\eta,0}) \int_0^{\eta} \frac{(K[\mathcal{G}] + S)(\eta', \mathbf{v}'(\eta'))}{v_{\eta}'(\eta')} \exp(H_{\eta',0}) \frac{\partial H_{\eta',0}}{\partial \eta} d\eta', \quad (154)$$

$$(155)$$

$$X_6 = \exp(-H_{\eta,0}) \int_0^\eta \frac{1}{v'_\eta(\eta')} \left(\nabla_{\mathbf{v}'}(K[\mathcal{G}] + S) \left(\eta', \mathbf{v}'(\eta') \right) \frac{\partial \mathbf{v}'(\eta')}{\partial \eta} \right) \exp(H_{\eta',0}) d\eta'.$$

We need to estimate each term. Below is the package of preliminary estimates (PPE):

1. For $\eta' \leq \eta$, we must have $v'_\eta \geq v_\eta \geq \delta_0$, which means $\frac{1}{v'_\eta} \leq \frac{1}{v_\eta} \leq \frac{1}{\delta_0}$.
2. Using substitution $y = H_{\eta,\eta'}$, we know

$$\left| \int_0^\eta \frac{\nu(\mathbf{v}'(\eta'))}{v'_\eta(\eta')} \exp(-H_{\eta,\eta'}) d\eta' \right| \leq \left| \int_0^\infty e^{-y} dy \right| = 1. \quad (156)$$

3. For $t, s \in [0, \eta]$, based on (PSF), we have

$$|H_{t,s}| \lesssim \left| \int_s^t \frac{\nu(\mathbf{v}'(y))}{v'_\eta(y)} dy \right| \lesssim \frac{|\mathbf{v}|}{\delta_0} |t - s|. \quad (157)$$

4. Direct computation reveals that

$$\frac{\partial v'_\phi(\eta')}{\partial \eta} = -\frac{\varepsilon v_\phi}{R_1 - \varepsilon \eta'}, \quad \frac{\partial v'_\psi(\eta')}{\partial \eta} = -\frac{\varepsilon v_\psi}{R_2 - \varepsilon \eta'}, \quad (158)$$

$$\frac{\partial v'_\eta(\eta')}{\partial \eta} = \frac{2\varepsilon}{v'_\eta(\eta)} \left(v_\phi^2 \frac{R_1 - \varepsilon \eta}{R_1 - \varepsilon \eta'} + v_\psi^2 \frac{R_2 - \varepsilon \eta}{R_2 - \varepsilon \eta'} \right), \quad (159)$$

which implies

$$\left| \frac{\partial v'_\phi(\eta')}{\partial \eta} \right| \lesssim \varepsilon |\mathbf{v}|, \quad \left| \frac{\partial v'_\psi(\eta')}{\partial \eta} \right| \lesssim \varepsilon |\mathbf{v}|, \quad \left| \frac{\partial v'_\eta(\eta')}{\partial \eta} \right| \lesssim \frac{\varepsilon |\mathbf{v}|^2}{v'_\eta(\eta')} \lesssim \frac{\varepsilon |\mathbf{v}|^2}{\delta_0}. \quad (160)$$

5. For $t, s \in [0, \eta]$, note that

$$\begin{aligned} \frac{\partial H_{t,s}}{\partial \eta} &= \int_s^t \frac{\partial}{\partial \eta} \left(\frac{\nu(\mathbf{v}'(y))}{v'_\eta(y)} \right) dy \\ &= \int_s^t \frac{1}{v'_\eta(y)} \frac{\partial \nu(|\mathbf{v}'|)}{\partial |\mathbf{v}'|}(y) \frac{1}{|\mathbf{v}'(y)|} \left(v'_\eta(y) \frac{\partial v'_\eta(y)}{\partial \eta} + v'_\phi(y) \frac{\partial v'_\phi(y)}{\partial \eta} + v'_\psi(y) \frac{\partial v'_\psi(y)}{\partial \eta} \right) dy \\ &\quad - \int_s^t \frac{\nu(|\mathbf{v}'|(y))}{v_\eta'^2(y)} \frac{\partial v'_\eta(y)}{\partial \eta} dy. \end{aligned} \quad (161)$$

Based on (157), Lemma 3.4 and (PSF), we obtain

$$\begin{aligned} \left| \frac{\partial H_{t,s}}{\partial \eta} \right| &\lesssim \left| \int_s^t \frac{\nu(\mathbf{v}'(y))}{v'_\eta(y)} \left(\varepsilon + \frac{\varepsilon |\mathbf{v}|}{\delta_0} \right) dy \right| + \left| \int_s^t \frac{\nu(|\mathbf{v}'|(y)) \varepsilon |\mathbf{v}|^2}{v'_\eta(y) \delta_0^2} dy \right| \\ &\lesssim \frac{\varepsilon \langle \mathbf{v} \rangle^2}{\delta_0^2} |H_{t,s}| \lesssim \frac{\varepsilon \langle \mathbf{v} \rangle^3}{\delta_0^3} |t - s| \lesssim \frac{\varepsilon \eta \langle \mathbf{v} \rangle^3}{\delta_0^3} \lesssim \frac{\langle \mathbf{v} \rangle^3}{\delta_0^3}. \end{aligned} \quad (162)$$

We estimate each X_i based on (PSF) and (PPE). Using (148) and (162), we have

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} X_1 \right| &\lesssim \left| \frac{\partial H_{\eta,0}}{\partial \eta} \right| \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} \mathcal{G} \right| \lesssim \left(\frac{|\mathbf{v}|}{\delta_0} + \frac{\langle \mathbf{v} \rangle^3}{\delta_0^3} \right) \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} \mathcal{G} \right| \\ &\lesssim \frac{1}{\delta_0^3} \|\mathcal{G}\|_{\infty, \vartheta+3, \varrho} \lesssim \frac{1}{\delta_0^3}. \end{aligned} \quad (163)$$

Based on (160) and (95), we know

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} X_2 \right| \lesssim \left(\varepsilon |\mathbf{v}| + \frac{\varepsilon |\mathbf{v}|^2}{\delta_0} \right) \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} \nabla_{\mathbf{v}} p \right| \lesssim \frac{\varepsilon}{\delta_0^2} |\nabla_{\mathbf{v}} p|_{\infty, \vartheta+2, \varrho} \lesssim \frac{\varepsilon}{\delta_0^2}. \quad (164)$$

Also, using (91) and Lemma 3.5, we have

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} X_3 \right| &\lesssim \left| \frac{1}{v_\eta} \right| \left(\left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} K[\mathcal{G}] \right| + \left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} S \right| \right) \\ &\lesssim \frac{1}{\delta_0} \left(1 + \left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} \nu^{-1} \mathcal{G} \right| \right) \lesssim \frac{1}{\delta_0}. \end{aligned} \quad (165)$$

On the other hand, using (160), (156) and (91), we obtain

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} X_4 \right| \lesssim \frac{\varepsilon}{\delta_0^3} \left(\left\| \nu^{-1} \mathcal{G} \right\|_{\infty, \vartheta+2, \varrho} + \|S\|_{\infty, \vartheta+2, \varrho} \right) \left(\int_0^\eta \exp(-H_{\eta, \eta'}) d\eta' \right) \lesssim \frac{\varepsilon}{\delta_0^3}. \quad (166)$$

Using (162), (156) and (91), we know

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} X_5 \right| \lesssim \frac{1}{\delta_0^4} \left(\left\| \nu^{-1} \mathcal{G} \right\|_{\infty, \vartheta+3, \varrho} + \|S\|_{\infty, \vartheta+3, \varrho} \right) \left(\int_0^\eta \exp(-H_{\eta, \eta'}) d\eta' \right) \lesssim \frac{1}{\delta_0^4}. \quad (167)$$

Finally, using (160), (156) and (95), we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} X_6 \right| \lesssim \frac{\varepsilon}{\delta_0^3} \left(\left\| \mathcal{G} \right\|_{\infty, \vartheta+2, \varrho} + \|S\|_{\infty, \vartheta+2, \varrho} \right) \left(\int_0^\eta \exp(-H_{\eta, \eta'}) d\eta' \right) \lesssim \frac{\varepsilon}{\delta_0^3}. \quad (168)$$

Collecting all X_i estimates, we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{|\mathbf{v}|^2} I_1 \right| \lesssim \frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4}. \quad (169)$$

Estimate of I_2 : $0 \leq v_\eta \leq \delta_0$ **with $1 - \chi(\mathbf{u}_\eta)$:** We naturally decompose $1 = (1 - \chi(\mathbf{u}_\eta)) + \chi(\mathbf{u}_\eta)$. In this step, we focus on $1 - \chi(\mathbf{u}_\eta)$ part, while $\chi(\mathbf{u}_\eta)$ part will be handled in following steps involving I_3, I_4, I_5 . Based on (146), the cut-off $1 - \chi(\mathbf{u}_\eta)$ is nonzero only when $|\mathbf{u}_\eta| \geq \delta$. We have

$$\begin{aligned} I_2 &:= \int_0^\eta \left(\int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \left(1 - \chi(\mathbf{u}_\eta) \right) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right) \frac{1}{v_\eta'} \exp(-H_{\eta, \eta'}) d\eta' \\ &= \int_0^\eta \left(\int_{\mathbb{R}^3} \left(1 - \chi(\mathbf{u}_\eta) \right) k(\mathbf{u}, \mathbf{v}') \frac{\mathcal{G}(\eta', \mathbf{u})}{\partial \eta'} d\mathbf{u} \right) \frac{\zeta(\eta', \mathbf{v}')}{v_\eta'} \exp(-H_{\eta, \eta'}) d\eta'. \end{aligned} \quad (170)$$

We first handle the inner integral. Based on (92), we have

$$\begin{aligned} \frac{\partial \mathcal{G}(\eta', \mathbf{u})}{\partial \eta'} &= -\frac{1}{\mathbf{u}_\eta} \left(G_1(\eta') \left(\mathbf{u}_\phi^2 \frac{\partial \mathcal{G}(\eta', \mathbf{u})}{\partial \mathbf{u}_\eta} - \mathbf{u}_\eta \mathbf{u}_\phi \frac{\partial \mathcal{G}(\eta', \mathbf{u})}{\partial \mathbf{u}_\phi} \right) \right. \\ &\quad + G_2(\eta') \left(\mathbf{u}_\psi^2 \frac{\partial \mathcal{G}(\eta', \mathbf{u})}{\partial \mathbf{u}_\eta} - \mathbf{u}_\eta \mathbf{u}_\psi \frac{\partial \mathcal{G}(\eta', \mathbf{u})}{\partial \mathbf{u}_\psi} \right) \\ &\quad \left. + \nu \mathcal{G}(\eta', \mathbf{u}) - K[\mathcal{G}](\eta', \mathbf{u}) - S(\eta', \mathbf{u}) \right). \end{aligned} \quad (171)$$

Hence, inserting (171) into the inner integral in (170), we have the decomposition

$$\begin{aligned} J &:= \int_{\mathbb{R}^3} \left(1 - \chi(\mathbf{u}_\eta) \right) k(\mathbf{u}, \mathbf{v}') \frac{\mathcal{G}(\eta', \mathbf{u})}{\partial \eta'} d\mathbf{u} = J_1 + J_2 + J_3 \\ &:= - \int_{\mathbb{R}^3} \left(1 - \chi(\mathbf{u}_\eta) \right) k(\mathbf{u}, \mathbf{v}') \frac{1}{\mathbf{u}_\eta} \left(\nu \mathcal{G}(\eta', \mathbf{u}) - K[\mathcal{G}](\eta', \mathbf{u}) - S(\eta', \mathbf{u}) \right) d\mathbf{u} \end{aligned} \quad (172)$$

$$\begin{aligned}
& - \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) k(u, v') \frac{1}{u_\eta} G_1(\eta') \left(u_\phi^2 \frac{\partial \mathcal{G}(\eta', u)}{\partial u_\eta} - u_\eta u_\phi \frac{\partial \mathcal{G}(\eta', u)}{\partial u_\phi} \right) du \\
& - \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) k(u, v') \frac{1}{u_\eta} G_2(\eta') \left(u_\psi^2 \frac{\partial \mathcal{G}(\eta', u)}{\partial u_\eta} - u_\eta u_\psi \frac{\partial \mathcal{G}(\eta', u)}{\partial u_\psi} \right) du.
\end{aligned}$$

Since $|u_\eta| \geq \delta$, using Lemma 3.5, (91) and (PSF), we obtain

$$\begin{aligned}
& \left| \langle v' \rangle^\vartheta e^{\varrho |v'|^2} J_1 \right| \lesssim \left| \langle v' \rangle^\vartheta e^{\varrho |v'|^2} \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) k(u, v') \frac{1}{u_\eta} \left(\nu \mathcal{G}(\eta', u) - K[\mathcal{G}](\eta', u) - S(\eta', u) \right) du \right| \\
& \lesssim \frac{1}{\delta} \left(\|\mathcal{G}\|_{\infty, \vartheta+1, \varrho} + \|S\|_{\infty, \vartheta, \varrho} \right) \left| \int_{\mathbb{R}^3} k(u, v') \frac{\langle v' \rangle^\vartheta e^{\varrho |v'|^2}}{\langle u \rangle^\vartheta e^{\varrho |u|^2}} du \right| \lesssim \frac{1}{\delta}.
\end{aligned} \tag{173}$$

On the other hand, an integration by parts yields

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^3} \left(-\frac{u_\phi^2}{u_\eta^2} \left(1 - \chi(u_\eta)\right) - \frac{u_\phi^2}{u_\eta} \chi'(u_\eta) - \left(1 - \chi(u_\eta)\right) \right) G_1(\eta') k(u, v') \mathcal{G}(\eta', u) du \\
&+ \int_{\mathbb{R}^3} G_1(\eta') \left(1 - \chi(u_\eta)\right) \left(-\frac{u_\phi^2}{u_\eta} \frac{\partial k(u, v')}{\partial u_\eta} + u_\phi \frac{\partial k(u, v')}{\partial u_\phi} \right) \mathcal{G}(\eta', u) du := J_{2,1} + J_{2,2}.
\end{aligned} \tag{174}$$

Here the key difficulty of $J_{2,1}$ is the integral singularity due to $\frac{1}{u_\eta}$. We need the following lemma.

Lemma 3.7. *Let $0 \leq \varrho < \frac{1}{4}$ and $\vartheta \geq 0$. Then for $\delta > 0$ sufficiently small and any $v \in \mathbb{R}^3$,*

$$\int_{\mathbb{R}^3} e^{\delta |u-v|^2} \frac{1}{|u|} |k(u, v)| \frac{\langle v \rangle^\vartheta e^{\varrho |v|^2}}{\langle u \rangle^\vartheta e^{\varrho |u|^2}} du \lesssim 1. \tag{175}$$

Proof. From Lemma 3.5, we observe that

$$|k(u, v)| \frac{\langle v \rangle^\vartheta e^{\varrho |v|^2}}{\langle u \rangle^\vartheta e^{\varrho |u|^2}} \lesssim \left(1 + |u-v|^2\right)^{\frac{\vartheta}{2}} \left(|u-v| + \frac{1}{|u-v|} \right) e^{-\frac{1}{8}|u-v|^2 - \frac{1}{8} \frac{|u|^2 - |v|^2|^2}{|u-v|^2} - \varrho(|u|^2 - |v|^2)}. \tag{176}$$

We first handle the exponential term in (176). Let $\sigma = u - v$, so $u = \sigma + v$. Then we have

$$\begin{aligned}
& -\frac{1}{8} |u-v|^2 - \frac{1}{8} \frac{|u|^2 - |v|^2|^2}{|u-v|^2} - \varrho \left(|u|^2 - |v|^2 \right) \\
&= -\frac{1}{8} |\sigma|^2 - \frac{1}{8} \frac{|\sigma + v|^2 - |v|^2|^2}{|\sigma|^2} - \varrho \left(|\sigma + v|^2 - |v|^2 \right) \\
&= -\frac{1}{8} |\sigma|^2 - \frac{1}{8} \frac{|\sigma|^2 - 2\sigma \cdot v}{|\sigma|^2} - \varrho \left(|\sigma|^2 - 2\sigma \cdot v \right) \\
&= -\frac{1}{4} |\sigma|^2 + \frac{1}{2} \sigma \cdot v - \frac{1}{2} \frac{|\sigma \cdot v|^2}{|\sigma|^2} - \varrho \left(|\sigma|^2 - 2\sigma \cdot v \right) \\
&= \left(-\frac{1}{4} - \varrho \right) |\sigma|^2 + \left(\frac{1}{2} + 2\varrho \right) \sigma \cdot v - \frac{1}{2} \frac{|\sigma \cdot v|^2}{|\sigma|^2}.
\end{aligned} \tag{177}$$

For $0 \leq \varrho \leq \frac{1}{4}$, the discriminant

$$\Delta = \left(\frac{1}{2} + 2\varrho\right)^2 + 2\left(-\frac{1}{4} - \varrho\right) = 4\varrho^2 - \frac{1}{4} < 0, \quad (178)$$

so the above quadratic form for $|\sigma|$ and $\frac{\sigma \cdot \mathbf{v}}{|\sigma|}$ is negative definite. In particular, for δ small, the perturbed form is still negative definite, i.e.

$$\begin{aligned} & -\left(\frac{1}{8} - \delta\right)|\mathbf{u} - \mathbf{v}|^2 - \frac{1}{8} \frac{||\mathbf{u}|^2 - |\mathbf{v}|^2|^2}{|\mathbf{u} - \mathbf{v}|^2} - \varrho(|\mathbf{u}|^2 - |\mathbf{v}|^2) \\ & \lesssim -\left(|\sigma|^2 + \frac{|\sigma \cdot \mathbf{v}|^2}{|\sigma|^2}\right) \lesssim -|\mathbf{u} - \mathbf{v}|^2. \end{aligned} \quad (179)$$

Hence, using Hölder's inequality, (176) and (179), we may bound

$$\begin{aligned} & \int_{\mathbb{R}^3} e^{\delta|\mathbf{u}-\mathbf{v}|^2} \frac{1}{|\mathbf{u}|} |k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \\ & \lesssim \int_{\mathbb{R}^3} \left(1 + |\mathbf{u} - \mathbf{v}|^2\right)^{\frac{\vartheta}{2}} \frac{1}{|\mathbf{u}|} \left(|\mathbf{u} - \mathbf{v}| + \frac{1}{|\mathbf{u} - \mathbf{v}|}\right) e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ & \lesssim \int_{\mathbb{R}^3} \frac{1}{|\mathbf{u}|} \left(|\mathbf{u} - \mathbf{v}| + \frac{1}{|\mathbf{u} - \mathbf{v}|}\right) e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ & \lesssim \left(\int_{\mathbb{R}^3} \frac{1}{|\mathbf{u}|^2} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \left(|\mathbf{u} - \mathbf{v}|^2 + \frac{1}{|\mathbf{u} - \mathbf{v}|^2}\right) e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u}\right)^{\frac{1}{2}} := I \times II. \end{aligned} \quad (180)$$

Using spherical coordinates and substitution $\mathbf{u} \rightarrow \sigma = \mathbf{u} - \mathbf{v}$, we have

$$\begin{aligned} I & \lesssim \left(\int_{|\mathbf{u}| \leq 1} \frac{1}{|\mathbf{u}|^2} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u}\right)^{\frac{1}{2}} + \left(\int_{|\mathbf{u}| \geq 1} \frac{1}{|\mathbf{u}|^2} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u}\right)^{\frac{1}{2}} \\ & \lesssim \left(\int_{|\mathbf{u}| \leq 1} \frac{1}{|\mathbf{u}|^2} d\mathbf{u}\right)^{\frac{1}{2}} + \left(\int_{|\mathbf{u}| \geq 1} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u}\right)^{\frac{1}{2}} \lesssim 1 + \left(\int_{\mathbb{R}^3} e^{-|\sigma|^2} d\sigma\right)^{\frac{1}{2}} \lesssim 1, \end{aligned} \quad (181)$$

and

$$II \lesssim \left(\int_{\mathbb{R}^3} \left(|\sigma|^2 + \frac{1}{|\sigma|^2}\right) e^{-|\sigma|^2} d\sigma\right)^{\frac{1}{2}} \lesssim 1. \quad (182)$$

In summary, inserting (181) and (182) into (180), we obtain the desired result. $\square$

Since $|\mathbf{u}_\eta| \geq \delta$, using Lemma 3.7 and Lemma 3.5, we have

$$\left|\langle \mathbf{v}' \rangle^\vartheta e^{\varrho|\mathbf{v}'|^2} J_{2,1}\right| \lesssim \frac{\varepsilon}{\delta^2} \|\mathcal{G}\|_{\infty, \vartheta+2, \varrho} \lesssim \frac{\varepsilon}{\delta^2}. \quad (183)$$

On the other hand, the estimate of $J_{2,2}$ has further complication due to derivatives of k . We need the following lemma.

Lemma 3.8. *Let $0 \leq \varrho < \frac{1}{4}$ and $\vartheta \geq 0$. We have*

$$\int_{\mathbb{R}^3} |\nabla_{\mathbf{u}} k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim \langle \mathbf{v} \rangle^2. \quad (184)$$

Proof. This is an improved version of Lemma 3.6. Following a similar argument, we have

$$|\nabla_{\mathbf{u}} k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho |\mathbf{u}|^2}} \lesssim (1 + |\mathbf{u} - \mathbf{v}|^2)^{\frac{\vartheta}{2}} |\nabla_{\mathbf{u}} k(\mathbf{u}, \mathbf{v})| e^{-\varrho(|\mathbf{u}|^2 - |\mathbf{v}|^2)} \quad (185)$$

Here, the key is to bound $|\nabla_{\mathbf{u}} k(\mathbf{u}, \mathbf{v})|$. Substituting $\mathbf{u} \rightarrow \sigma = \mathbf{u} - \mathbf{v} = (\sigma_\eta, \sigma_\phi, \sigma_\psi)$, we get (110) and (111). Also, note that $\nabla_{\mathbf{u}} = \nabla_\sigma$. Then we compute

$$\nabla_\sigma k_1(\sigma, \mathbf{v}) = |\sigma| \left(-\mathbf{v} - \sigma \right) e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2} |\sigma|^2} + \frac{\sigma}{|\sigma|} e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2} |\sigma|^2}, \quad (186)$$

which implies

$$|\nabla_\sigma k_1(\sigma, \mathbf{v})| \lesssim (|\sigma|^2 + 1) e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2} |\sigma|^2} + |\sigma| |\mathbf{v}| e^{-|\mathbf{v}|^2 - \sigma \cdot \mathbf{v} - \frac{1}{2} |\sigma|^2} := I_1 + I_2. \quad (187)$$

Here, using similar techniques as in the proof of Lemma 3.7, we obtain $I_1 \lesssim 1$ and $I_2 \lesssim \langle \mathbf{v} \rangle$, which imply

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{u}} k_1(\mathbf{u}, \mathbf{v}) \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho |\mathbf{u}|^2}} d\mathbf{u} \lesssim |\mathbf{v}|. \quad (188)$$

On the other hand, we compute

$$\begin{aligned} |\nabla_\sigma k_2(\sigma, \mathbf{v})| &= \frac{1}{|\sigma|} \left(-\sigma + \mathbf{v} - \frac{2\sigma \cdot \mathbf{v}}{|\sigma|^2} (\mathbf{v} \cdot \mathcal{T}) \right) e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2} |\mathbf{v}|^2} \\ &\quad - \frac{\sigma}{|\sigma|^3} e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2} |\mathbf{v}|^2}, \end{aligned} \quad (189)$$

for tensor

$$\mathcal{T} := \frac{1}{|\sigma|^3} \begin{pmatrix} \sigma_\phi^2 + \sigma_\psi^2 & -\sigma_\eta \sigma_\phi & -\sigma_\eta \sigma_\psi \\ -\sigma_\eta \sigma_\phi & \sigma_\eta^2 + \sigma_\psi^2 & -\sigma_\phi \sigma_\psi \\ -\sigma_\eta \sigma_\psi & -\sigma_\phi \sigma_\psi & \sigma_\eta^2 + \sigma_\phi^2 \end{pmatrix}, \quad (190)$$

which implies

$$\begin{aligned} |\nabla_\sigma k_2(\sigma, \mathbf{v})| &\lesssim \left(1 + \frac{1}{|\sigma|^2} \right) e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2} |\mathbf{v}|^2} \\ &\quad + \left(\frac{|\mathbf{v}|}{|\sigma|} + \frac{|\mathbf{v}|^2}{|\sigma|^2} \right) e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot \mathbf{v}}{|\sigma|^2} |\mathbf{v}|^2} := II_1 + II_2. \end{aligned} \quad (191)$$

Still, using similar techniques as in the proof of Lemma 3.7, we obtain $II_1 \lesssim 1$ and $II_2 \lesssim \langle \mathbf{v} \rangle^2$, which imply

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{u}} k_2(\mathbf{u}, \mathbf{v}) \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho |\mathbf{u}|^2}} d\mathbf{u} \lesssim \langle \mathbf{v} \rangle^2. \quad (192)$$

Then the desired results follow from (188) and (192). $\square$

Also, using Lemma 3.7 and Lemma 3.8, we have

$$\left| \langle \mathbf{v}' \rangle^\vartheta e^{\varrho |\mathbf{v}'|^2} J_{2,2} \right| \lesssim \frac{\varepsilon}{\delta} \|\mathcal{G}\|_{\infty, \vartheta+4, \varrho} \lesssim \frac{\varepsilon}{\delta}. \quad (193)$$

(183) and (193) yield

$$\left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J_2 \right| \lesssim \frac{\varepsilon}{\delta^2}. \quad (194)$$

Combined with a similar argument for J_3 , we estimate the inner integral in (170) (195)

$$\left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J \right| \lesssim \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J_1 \right| + \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J_2 \right| + \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J_3 \right| \lesssim \frac{1}{\delta} + \frac{\varepsilon}{\delta^2}.$$

Then for the outer integral in (170), we can use (142) and (156) to show that

$$\left| \int_0^\eta \frac{\zeta(\eta', \mathbf{v}')}{v'_\eta(\eta')} \exp(-H_{\eta, \eta'}) d\eta' \right| \leq \left| \int_0^\eta \frac{\nu(\mathbf{v}')}{v'_\eta(\eta')} \exp(-H_{\eta, \eta'}) d\eta' \right| \lesssim 1. \quad (196)$$

Then we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{e|\mathbf{v}|^2} I_2 \right| \lesssim \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} J \right| \lesssim \frac{1}{\delta} + \frac{\varepsilon}{\delta^2}. \quad (197)$$

Estimate of I_3 : $0 \leq v_\eta \leq \delta_0$, with $\chi(\mathbf{u}_\eta)$, and $\sqrt{\varepsilon\eta'} |\tilde{v}'| \geq v'_\eta$: Based on (146) and (170), we are left with $\chi(\mathbf{u}_\eta)$ part, which is nonzero only when $|\mathbf{u}_\eta| \leq 2\delta$, i.e.

$$\int_0^\eta \left(\int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(\mathbf{u}_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right) \frac{1}{v'_\eta} \exp(-H_{\eta, \eta'}) d\eta'. \quad (198)$$

We will further decompose this integral into I_3, I_4, I_5 . In this step, based on (142), $\sqrt{\varepsilon\eta'} |\tilde{v}'| \geq v'_\eta$ implies

$$\zeta(\eta', \mathbf{v}') \lesssim |v'_\eta| + \sqrt{\varepsilon\eta'} |\tilde{v}'| \lesssim \sqrt{\varepsilon\eta'} |\tilde{v}'|. \quad (199)$$

On the other hand, (143) implies

$$\zeta(\eta', \mathbf{u}) \gtrsim \sqrt{\varepsilon\eta'} |\mathbf{u}|. \quad (200)$$

Then considering (199) and (200) and using Lemma 3.7 the inner integral in (198)

$$\begin{aligned} & \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(\mathbf{u}_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \\ & \lesssim |\tilde{v}'| \left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{u}|} \chi(\mathbf{u}_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \lesssim |\tilde{v}'| \|\mathcal{A}\|_{\infty, \vartheta, e}. \end{aligned} \quad (201)$$

This bound is weaker than desired since we have not used the smallness $|\mathbf{u}_\eta| \leq 2\delta$, which means the integral is actually over a very small domain. We naturally modify the proof of Lemma 3.7. The key step is (181). Here for either $|\mathbf{u}| \leq 1$ or $|\mathbf{u}| \geq 1$, the small domain of $\mathbf{u}_\eta$ produces an extra smallness in integral. In precise,

$$\left| \langle \mathbf{v}' \rangle^\vartheta e^{e|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(\mathbf{u}_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \lesssim \delta |\tilde{v}'| \|\mathcal{A}\|_{\infty, \vartheta, e}. \quad (202)$$

Here, this $|\tilde{v}'|$ will be handled by the outer integral of (198) as in (196)

$$\int_0^\eta \frac{|\tilde{v}'|}{v'_\eta} \exp(-H_{\eta, \eta'}) d\eta' \lesssim \int_0^\eta \frac{\nu(\mathbf{v}')}{v'_\eta} \exp(-H_{\eta, \eta'}) d\eta' \lesssim 1. \quad (203)$$

In total, we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{e|\mathbf{v}|^2} I_3 \right| \lesssim \delta \|\mathcal{A}\|_{\infty, \vartheta, e}. \quad (204)$$

Estimate of I_4 : $0 \leq v_\eta \leq \delta_0$, **with** $\chi(u_\eta)$, $\sqrt{\varepsilon\eta'}|\tilde{v}'| \leq v'_\eta$ **and** $v_\eta^2 \leq \varepsilon(\eta - \eta')|\tilde{v}|^2$: I_4 is defined similar as (198). Based on (142), $\sqrt{\varepsilon\eta'}|\tilde{v}'| \leq v'_\eta$ implies

$$\zeta(\eta', \mathbf{v}') \lesssim |v'_\eta| + \sqrt{\varepsilon\eta'}|\tilde{v}'| \lesssim v'_\eta. \quad (205)$$

Hence, similar to the derivation for I_3 in (201) and (202), using (200) and (205), we have

$$\begin{aligned} & \left| \langle \mathbf{v}' \rangle^\vartheta e^{\varrho|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(u_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \\ & \lesssim \frac{v'_\eta}{\sqrt{\varepsilon\eta'}} \left| \langle \mathbf{v}' \rangle^\vartheta e^{\varrho|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{u}|} \chi(u_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \lesssim \delta \frac{v'_\eta}{\sqrt{\varepsilon\eta'}} \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (206)$$

Hence, we must handle $\frac{v'_\eta}{\sqrt{\varepsilon\eta'}}$ with the outer integral in (198). Based on (145), $v_\eta^2 \leq \varepsilon(\eta - \eta')|\tilde{v}|^2$ leads to

$$-H_{\eta, \eta'} = - \int_{\eta'}^\eta \frac{\nu(\mathbf{v})}{v'_\eta(y)} dy \lesssim - \frac{\nu(\mathbf{v})(\eta - \eta')}{|\tilde{v}| \sqrt{\varepsilon(\eta - \eta')}} \lesssim - \frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta - \eta'}{\varepsilon}}. \quad (207)$$

Therefore, we know

$$\begin{aligned} & \int_0^\eta \frac{v'_\eta}{\sqrt{\varepsilon\eta'}} \frac{1}{v'_\eta} \exp(-H_{\eta, \eta'}) d\eta' \lesssim \int_0^\eta \frac{1}{\sqrt{\varepsilon\eta'}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta - \eta'}{\varepsilon}}\right) d\eta' \\ & = \int_0^{\frac{\eta}{\varepsilon}} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz \\ & = \int_0^1 \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz + \int_1^{\frac{\eta}{\varepsilon}} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz, \end{aligned} \quad (208)$$

where we use the substitution $\eta' \rightarrow z = \frac{\eta'}{\varepsilon}$. We can estimate these two terms separately.

$$\int_0^1 \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz \leq \int_0^1 \frac{1}{\sqrt{z}} dz = 2. \quad (209)$$

$$\begin{aligned} \int_1^{\frac{\eta}{\varepsilon}} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz & \leq \int_1^{\frac{\eta}{\varepsilon}} \exp\left(-\frac{\nu(\mathbf{v})}{|\tilde{v}|} \sqrt{\frac{\eta}{\varepsilon} - z}\right) dz \\ & \stackrel{t^2 = \frac{\eta}{\varepsilon} - z}{\lesssim} \int_0^\infty t e^{-\frac{\nu(\mathbf{v})}{|\tilde{v}|} t} dt \lesssim \left(\frac{|\tilde{v}|}{\nu(\mathbf{v})}\right)^2 \lesssim 1. \end{aligned} \quad (210)$$

Inserting (209) and (210) into (208), we know the outer integral in (198) is bounded and thus

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2} I_4 \right| \lesssim \delta \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (211)$$

Estimate of I_5 : $0 \leq v_\eta \leq \delta_0$, **with** $\chi(u_\eta)$, $\sqrt{\varepsilon\eta'}|\tilde{v}'| \leq v'_\eta$, $v_\eta^2 \geq \varepsilon(\eta - \eta')|\tilde{v}|^2$: I_5 is defined similar as (198). Using (205), we have

$$\left| \int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(u_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \lesssim \left| \int_{\mathbb{R}^3} \frac{v'_\eta}{\zeta(\eta', \mathbf{u})} \chi(u_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right|. \quad (212)$$

Here the key difficulty is the integral singularity due to $\zeta(\eta', \mathbf{u})$. We need the following lemma.

Lemma 3.9. *For any $\mathbf{v} \in \mathbb{R}^3$, we have*

$$\int_{\mathbb{R}^3} \frac{1}{\zeta(\eta; \mathbf{u})} |k(\mathbf{u}, \mathbf{v})| \frac{\langle \mathbf{v} \rangle^\vartheta e^{\varrho|\mathbf{v}|^2}}{\langle \mathbf{u} \rangle^\vartheta e^{\varrho|\mathbf{u}|^2}} d\mathbf{u} \lesssim 1 + |\ln(\varepsilon\eta)|. \quad (213)$$

Proof. Based on (97), letting $\mathbf{u} = (\mathbf{u}_\eta, \mathbf{u}_\phi, \mathbf{u}_\psi)$, we directly obtain

$$\zeta(\eta; \mathbf{u}) \gtrsim \sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}. \quad (214)$$

Combined with (176) and (179), we bound

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{\zeta(\eta; \mathbf{u})} |k(\mathbf{u}, \mathbf{v})| d\mathbf{u} &\lesssim \int_{\mathbb{R}^3} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} |\mathbf{u} - \mathbf{v}| e^{-\delta|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ &+ \int_{\mathbb{R}^3} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} \frac{1}{|\mathbf{u} - \mathbf{v}|} e^{-\delta|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} := I + II. \end{aligned} \quad (215)$$

Since exponential term decays much faster than polynomial term, we have

$$\begin{aligned} I &\lesssim \int_{\mathbb{R}^3} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ &\lesssim \int_{|\mathbf{u}| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} + \int_{|\mathbf{u}| \geq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ &\lesssim \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(\int_{|\mathbf{u}_\eta| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} d\mathbf{u}_\eta \right) d\mathbf{u}_\phi d\mathbf{u}_\psi + \int_{|\mathbf{u}| \geq 1} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ &\lesssim 1 + \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(\int_{|\mathbf{u}_\eta| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} d\mathbf{u}_\eta \right) d\mathbf{u}_\phi d\mathbf{u}_\psi. \end{aligned} \quad (216)$$

The key is to bound the inner integral for $|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1, 0 < \eta \leq L = \varepsilon^{-\frac{1}{2}}$,

$$\begin{aligned} J &:= \int_{|\mathbf{u}_\eta| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} d\mathbf{u}_\eta \\ &= 2 \ln \left(1 + \sqrt{1 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2} \right) - 2 \ln \left(\sqrt{(\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2} \right) \\ &\lesssim \sqrt{1 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2} + \left| \ln \left((\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2 \right) \right| \\ &\lesssim 1 + \left| \ln \left((\varepsilon\eta)\mathbf{u}_\phi^2 \right) \right| + \left| \ln \left((\varepsilon\eta)\mathbf{u}_\psi^2 \right) \right| \lesssim 1 + |\ln(\varepsilon\eta)| + |\ln |\mathbf{u}_\phi|| + |\ln |\mathbf{u}_\psi||. \end{aligned} \quad (217)$$

Inserting (217) into (216), we obtain

$$\begin{aligned} I &\lesssim 1 + \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(1 + |\ln(\varepsilon\eta)| + |\ln |\mathbf{u}_\phi|| + |\ln |\mathbf{u}_\psi|| \right) d\mathbf{u}_\phi d\mathbf{u}_\psi \\ &\lesssim 1 + |\ln(\varepsilon\eta)| + \int_{|\mathbf{u}_\phi| \leq 1} |\ln |\mathbf{u}_\phi|| d\mathbf{u}_\phi + \int_{|\mathbf{u}_\psi| \leq 1} |\ln |\mathbf{u}_\psi|| d\mathbf{u}_\psi \lesssim 1 + |\ln(\varepsilon\eta)|. \end{aligned} \quad (218)$$

On the other hand, similar to (216), we have

$$\begin{aligned} II &\lesssim \int_{|\mathbf{u}| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} \frac{1}{|\mathbf{u} - \mathbf{v}|^2} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \\ &+ \int_{|\mathbf{u}| \geq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} \frac{1}{|\mathbf{u} - \mathbf{v}|^2} e^{-|\mathbf{u}-\mathbf{v}|^2} d\mathbf{u} \end{aligned} \quad (219)$$

$$\begin{aligned}
&\lesssim \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(\int_{|\mathbf{u}_\eta| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} d\mathbf{u}_\eta \right) \frac{1}{\sqrt{(\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2}} d\mathbf{u}_\phi d\mathbf{u}_\psi \\
&\quad + \int_{|\mathbf{u}| \geq 1} \frac{1}{|\mathbf{u} - \mathbf{v}|^2} e^{-|\mathbf{u} - \mathbf{v}|^2} d\mathbf{u} \\
&\lesssim 1 + \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(\int_{|\mathbf{u}_\eta| \leq 1} \frac{1}{\sqrt{\mathbf{u}_\eta^2 + (\varepsilon\eta)\mathbf{u}_\phi^2 + (\varepsilon\eta)\mathbf{u}_\psi^2}} d\mathbf{u}_\eta \right) \frac{1}{\sqrt{(\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2}} d\mathbf{u}_\phi d\mathbf{u}_\psi.
\end{aligned}$$

Inserting (217) into (219), and applying Hölder's inequality, we obtain

$$\begin{aligned}
II &\lesssim 1 + \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \frac{1 + |\ln(\varepsilon\eta)| + |\ln|\mathbf{u}_\phi|| + |\ln|\mathbf{u}_\psi||}{\sqrt{(\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2}} d\mathbf{u}_\phi d\mathbf{u}_\psi \quad (220) \\
&\lesssim 1 + |\ln(\varepsilon\eta)| + \int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \frac{|\ln|\mathbf{u}_\phi|| + |\ln|\mathbf{u}_\psi||}{\sqrt{(\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2}} d\mathbf{u}_\phi d\mathbf{u}_\psi \\
&\lesssim 1 + |\ln(\varepsilon\eta)| + \left(\int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \frac{1}{((\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2)^{\frac{3}{4}}} d\mathbf{u}_\phi d\mathbf{u}_\psi \right)^{\frac{2}{3}} \\
&\quad \times \left(\int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(|\ln|\mathbf{u}_\phi|| + |\ln|\mathbf{u}_\psi|| \right)^3 d\mathbf{u}_\phi d\mathbf{u}_\psi \right)^{\frac{1}{3}}.
\end{aligned}$$

Note that using polar coordinates, we have

$$\int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \frac{1}{((\mathbf{u}_\phi - v_\phi)^2 + (\mathbf{u}_\psi - v_\psi)^2)^{\frac{3}{4}}} d\mathbf{u}_\phi d\mathbf{u}_\psi \lesssim 1, \quad (221)$$

$$\int_{|\mathbf{u}_\phi| \leq 1, |\mathbf{u}_\psi| \leq 1} \left(|\ln|\mathbf{u}_\phi|| + |\ln|\mathbf{u}_\psi|| \right)^3 d\mathbf{u}_\phi d\mathbf{u}_\psi \lesssim 1. \quad (222)$$

Hence, inserting (222) and (221) into (220), we get

$$II \lesssim 1 + |\ln(\varepsilon\eta)|. \quad (223)$$

Inserting (218) and (223) into (215), we obtain the desired result. $\square$

Using Lemma 3.9, we may bound

$$\left| \langle \mathbf{v}' \rangle^\vartheta e^{\varrho|\mathbf{v}'|^2} \int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} \chi(\mathbf{u}_\eta) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right| \lesssim v'_\eta \left(1 + |\ln(\varepsilon\eta')| \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (224)$$

Hence, we must handle $v'_\eta (1 + |\ln(\varepsilon\eta')|)$ with the outer integral in (198). Based on (145), $v_\eta^2 \geq \varepsilon(\eta - \eta') |\tilde{v}|^2$ implies

$$-H_{\eta, \eta'} = - \int_{\eta'}^\eta \frac{\nu(\mathbf{v}')}{v'_\eta(y)} dy \lesssim - \frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}. \quad (225)$$

Therefore, we know

$$\int_0^\eta v'_\eta \left(1 + |\ln(\varepsilon\eta')| \right) \frac{1}{v'_\eta} \exp(-H_{\eta, \eta'}) d\eta' \lesssim \int_0^\eta \left(1 + |\ln(\varepsilon\eta')| \right) \exp\left(- \frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta} \right) d\eta'. \quad (226)$$

Naturally,

$$1 + |\ln(\varepsilon\eta')| \lesssim \left(1 + |\ln(\varepsilon)| \right) + |\ln(\eta')|. \quad (227)$$

Since $0 \leq v_\eta \leq \delta_0$, direct computation reveals that

$$(228)$$

$$\int_0^\eta (1 + |\ln(\varepsilon)|) \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta' \lesssim (1 + |\ln(\varepsilon)|) \frac{v_\eta}{\nu(\mathbf{v})} \lesssim \delta_0 (1 + |\ln(\varepsilon)|).$$

Hence, it suffices to consider

$$Q = \int_0^\eta |\ln(\eta')| \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta'. \quad (229)$$

If $0 \leq \eta \leq 2$, applying Hölder's inequality, we have

$$Q \lesssim \left(\int_0^2 |\ln(\eta')|^2 d\eta'\right)^{\frac{1}{2}} \left(\int_0^2 \exp\left(-\frac{2\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta'\right)^{\frac{1}{2}} \lesssim \sqrt{\frac{v_\eta}{\nu(\mathbf{v})}} \lesssim \sqrt{\delta_0}. \quad (230)$$

If $2 \leq \eta \leq L = \varepsilon^{-\frac{1}{2}}$, we decompose and apply Hölder's inequality to obtain

$$\begin{aligned} Q &\lesssim \int_0^2 |\ln(\eta')| \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta' \\ &\quad + \int_2^\eta |\ln(\eta')| \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta' \\ &\lesssim \left(\int_0^2 |\ln(\eta')|^2 d\eta'\right)^{\frac{1}{2}} \left(\int_0^2 \exp\left(-\frac{2\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta'\right)^{\frac{1}{2}} \\ &\quad + \ln(L) \int_2^\eta \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta' \\ &\lesssim \sqrt{\delta_0} (1 + |\ln(\varepsilon)|). \end{aligned} \quad (231)$$

In summary, we have

$$\int_0^\eta |\ln(\eta')| \exp\left(-\frac{\nu(\mathbf{v}')(\eta - \eta')}{v_\eta}\right) d\eta' \lesssim \sqrt{\delta_0} (1 + |\ln(\varepsilon)|). \quad (232)$$

This completes the bound of outer integral of (198). Hence, we know

$$|\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} I_5| \lesssim \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (233)$$

Synthesis: Collecting all estimates related to I_i in (169), (197), (204), (211) and (233), we have proved

$$|\langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} I| \lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|)\right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\varepsilon}{\delta^2} + \frac{1}{\delta}\right). \quad (234)$$

3.5. Region II: $v_\eta < 0$ and $v_\eta^2 + v_\phi^2 + v_\psi^2 \geq v_\phi^2(L) + v_\psi^2(L)$. Based on (133), we decompose

$$\begin{aligned} \mathcal{T}[\widetilde{\mathcal{A}}] &= \int_0^\eta \frac{\widetilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v'_\eta(\eta')} \exp(-H_{L, \eta'} - H_{L, \eta}) d\eta' \\ &\quad + \int_\eta^L \frac{\widetilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v'_\eta(\eta')} \exp(-H_{L, \eta'} - H_{L, \eta}) d\eta' + \int_\eta^L \frac{\widetilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v'_\eta(\eta')} \exp(H_{\eta, \eta'}) d\eta'. \end{aligned} \quad (235)$$

The integral $\int_0^\eta$ part can be estimated as in Region I due to $\exp(-H_{L, \eta'} - H_{L, \eta}) \lesssim \exp(-H_{\eta', \eta})$, so we focus on the integral $\int_\eta^L$ part. Analogously, it suffices

to estimate

$$II = \int_{\eta}^L \frac{\widetilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v'_{\eta}(\eta')} \exp(-H_{\eta', \eta}) d\eta'. \quad (236)$$

Here the proof is similar to that in Region I, so we only point out the key differences.

(142) and (143) still holds, but the key result (145) needs to be updated. For $0 \leq \eta \leq \eta'$,

$$v'_{\eta} = \sqrt{E_1 - v_{\phi}^2 - v_{\psi}^2} = \sqrt{v_{\eta}^2 + v_{\phi}^2 + v_{\psi}^2 - \left(\frac{R_1 - \varepsilon\eta}{R_1 - \varepsilon\eta'}\right)^2 v_{\phi}^2 - \left(\frac{R_2 - \varepsilon\eta}{R_2 - \varepsilon\eta'}\right)^2 v_{\psi}^2} \leq |v_{\eta}|. \quad (237)$$

Then we have

$$-\int_{\eta}^{\eta'} \frac{1}{v'_{\eta}(y)} dy \leq -\int_{\eta}^{\eta'} \frac{1}{|v_{\eta}|} dy = -\frac{\eta' - \eta}{|v_{\eta}|}. \quad (238)$$

Here, note that $v_{\eta} < 0$ but $v'_{\eta} \geq 0$ defined in (126).

Estimate of II_1 : $v_{\eta} \leq -\delta_0$ and $v'_{\eta} \geq \frac{\delta_0}{2}$ for all $\eta' \in [0, L]$: Since $\eta' \geq \eta$, we must have $v'_{\eta} \leq |v_{\eta}|$, so it is unclear whether $|v'_{\eta}| \geq \frac{\delta_0}{2}$ directly from $v_{\eta} \leq \delta_0$. Hence, we must put this as an additional requirement. If there exists some $v'_{\eta} \leq \frac{\delta_0}{2}$, it will be handled in II_5 estimate later.

The estimate is in the same spirit as that of I_1 , so we only point out the package of preliminary estimates (PPE):

1. For $\eta' \geq \eta$, we have $\frac{1}{v'_{\eta}} \lesssim \frac{1}{\delta_0}$.
2. Using substitution $y = H_{\eta, \eta'}$, we know

$$\left| \int_{\eta}^L \frac{\nu(\mathbf{v}'(\eta'))}{v'_{\eta}(\eta')} \exp(H_{\eta, \eta'}) d\eta' \right| \leq \left| \int_{-\infty}^0 e^y dy \right| = 1. \quad (239)$$

3. For $t, s \in [\eta, L]$, based on (PSF), we have

$$|H_{t,s}| \lesssim \frac{|\mathbf{v}|}{\delta_0} |t - s|. \quad (240)$$

4. We have

$$\left| \frac{\partial v'_{\phi}(\eta')}{\partial \eta} \right| \lesssim \varepsilon |\mathbf{v}|, \quad \left| \frac{\partial v'_{\psi}(\eta')}{\partial \eta} \right| \lesssim \varepsilon |\mathbf{v}|, \quad \left| \frac{\partial v'_{\eta}(\eta')}{\partial \eta} \right| \lesssim \frac{\varepsilon |\mathbf{v}|^2}{v'_{\eta}(\eta')} \lesssim \frac{\varepsilon |\mathbf{v}|^2}{\delta_0}. \quad (241)$$

5. For $t, s \in [\eta, L]$, we obtain

$$\left| \frac{\partial H_{t,s}}{\partial \eta} \right| \lesssim \frac{\varepsilon \langle \mathbf{v} \rangle^3}{\delta_0^3} |t - s| \lesssim \frac{\varepsilon L \langle \mathbf{v} \rangle^3}{\delta_0^3} \lesssim \frac{\langle \mathbf{v} \rangle^3}{\delta_0^3}. \quad (242)$$

With the help of (PSF) and (PPE), we have

$$\left| \langle \mathbf{v} \rangle^{\vartheta} e^{e|\mathbf{v}|^2} II_1 \right| \lesssim \frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4}. \quad (243)$$

Estimate of II_2 : $-\delta_0 \leq v_\eta \leq 0$ **with** $1 - \chi(u_\eta)$: We decompose $1 = (1 - \chi(u_\eta)) + \chi(u_\eta)$.

$$II_2 := \int_\eta^L \left(\int_{\mathbb{R}^3} \frac{\zeta(\eta', \mathbf{v}')}{\zeta(\eta', \mathbf{u})} (1 - \chi(u_\eta)) k(\mathbf{u}, \mathbf{v}') \mathcal{A}(\eta', \mathbf{u}) d\mathbf{u} \right) \frac{1}{v'_\eta} \exp(H_{\eta, \eta'}) d\eta'. \quad (244)$$

Then by a similar argument as estimating I_2 , we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} II_2 \right| \lesssim \frac{1}{\delta} + \frac{\varepsilon}{\delta^2}. \quad (245)$$

Estimate of II_3 : $-\delta_0 \leq v_\eta \leq 0$, **with** $\chi(u_\eta)$, **and** $\sqrt{\varepsilon \eta'} v'_\phi \geq v'_\eta$: This is similar to the estimate of I_3 , we have

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} II_3 \right| \lesssim \delta \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (246)$$

Estimate of II_4 : $-\delta_0 \leq v_\eta \leq 0$, **with** $\chi(u_\eta)$, **and** $\sqrt{\varepsilon \eta'} v'_\phi \leq v'_\eta$: This step is different. We do not need to further decompose the cases like I_4 and I_5 . Based on (238), we have,

$$-H_{\eta, \eta'} \lesssim -\frac{\nu(\mathbf{v})(\eta' - \eta)}{v_\eta}. \quad (247)$$

Then following the same argument in estimating I_5 , we know

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} II_4 \right| \lesssim \sqrt{\delta_0} \left(1 + |\ln(\varepsilon)| \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (248)$$

Estimate of II_5 : $v_\eta \leq -\delta_0$ **and** $v'_\eta \leq \frac{\delta_0}{2}$ **for some** $\eta' \in [0, L]$: Now we come back to study the leftover in Step 1, i.e. though the characteristic starts from a point with $|v_\eta| \geq \delta_0$, as it goes, we finally arrive at the region that $v'_\eta \leq \frac{\delta_0}{2}$.

Let $(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^*)$ be on the same characteristic as $(\eta, \mathbf{v})$, i.e. this is the first time that the characteristic enters the region $v'_\eta \leq \frac{\delta_0}{2}$. In detail, we have

$$v_\phi^* = \frac{R_1 - \varepsilon \eta}{R_1 - \varepsilon \eta^*} v_\phi, \quad v_\psi^* = \frac{R_2 - \varepsilon \eta}{R_2 - \varepsilon \eta^*} v_\psi, \quad (249)$$

$$v_\eta^2 + v_\phi^2 + v_\psi^2 = \frac{\delta_0^2}{4} + \left(\frac{R_1 - \varepsilon \eta}{R_1 - \varepsilon \eta^*} \right)^2 v_\phi^2 + \left(\frac{R_2 - \varepsilon \eta}{R_2 - \varepsilon \eta^*} \right)^2 v_\psi^2. \quad (250)$$

Taking η derivative in (250), we obtain

$$\frac{\partial \eta^*}{\partial \eta} = \frac{\frac{R_1 - \varepsilon \eta}{(R_1 - \varepsilon \eta^*)^2} v_\phi^2 + \frac{R_2 - \varepsilon \eta}{(R_2 - \varepsilon \eta^*)^2} v_\psi^2}{\frac{(R_1 - \varepsilon \eta)^2}{(R_1 - \varepsilon \eta^*)^3} v_\phi^2 + \frac{(R_2 - \varepsilon \eta)^2}{(R_2 - \varepsilon \eta^*)^3} v_\psi^2}, \quad (251)$$

Here we do not need to compute η^* explicitly. Since $\eta < \eta^* \leq L$, we know $0 \leq \varepsilon \eta < \varepsilon \eta^* \leq \varepsilon L = \varepsilon^{\frac{1}{2}}$, which implies

$$\frac{R_1}{2} \leq R_1 - \varepsilon \eta^* < R_1 - \varepsilon \eta \leq R_1, \quad \frac{R_2}{2} \leq R_2 - \varepsilon \eta^* < R_2 - \varepsilon \eta \leq R_2. \quad (252)$$

Inserting (252) into (251), we have

$$\left| \frac{\partial \eta^*}{\partial \eta} \right| \lesssim 1. \quad (253)$$

Taking η derivative in (249) and using (253) and (252), we obtain

$$\left| \frac{\partial v_\phi^*}{\partial \eta} \right| = \varepsilon |v_\phi| \left| \frac{R_1 - \varepsilon \eta}{(R_1 - \varepsilon \eta^*)^2} \frac{\partial \eta^*}{\partial \eta} - \frac{1}{R_1 - \varepsilon \eta^*} \right| \lesssim \varepsilon \nu(\mathbf{v}), \quad (254)$$

$$\left| \frac{\partial v_\psi^*}{\partial \eta} \right| = \varepsilon |v_\psi| \left| \frac{R_2 - \varepsilon \eta}{(R_2 - \varepsilon \eta^*)^2} \frac{\partial \eta^*}{\partial \eta} - \frac{1}{R_2 - \varepsilon \eta^*} \right| \lesssim \varepsilon \nu(\mathbf{v}). \quad (255)$$

Then we have the mild formulation between η and η^* as

$$\begin{aligned} \mathcal{G}(\eta, \mathbf{v}) = & \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right) \exp(-H_{\eta^*, \eta}) \\ & + \int_\eta^{\eta^*} \frac{(K[\mathcal{G}] + S)(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta'))}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(H_{\eta', \eta}) d\eta'. \end{aligned} \quad (256)$$

Similar to the estimate of II_1 , taking η derivative in (256) and multiplying ζ on both sides, we obtain

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varepsilon |\mathbf{v}|^2} II_5 \right| = \left| \langle \mathbf{v} \rangle^\vartheta e^{\varepsilon |\mathbf{v}|^2} \zeta(\eta, \mathbf{v}) \frac{\partial \mathcal{G}}{\partial \eta} \right| \lesssim \left| \langle \mathbf{v} \rangle^\vartheta e^{\varepsilon |\mathbf{v}|^2} \zeta(P_1 + P_2) \right|, \quad (257)$$

where

$$P_1 = \frac{\partial \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial \eta} \exp(-H_{\eta^*, \eta}), \quad (258)$$

$$\begin{aligned} P_2 = & -\mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right) \exp(-H_{\eta^*, \eta}) \frac{\partial H_{\eta^*, \eta}}{\partial \eta} \\ & + \frac{\partial}{\partial \eta} \left(\int_\eta^{\eta^*} \frac{(K[\mathcal{G}] + S)(\eta', \mathbf{v}'(\eta, \mathbf{v}; \eta'))}{v'_\eta(\eta, \mathbf{v}; \eta')} \exp(H_{\eta', \eta}) d\eta' \right). \end{aligned} \quad (259)$$

Since for $\eta' \in [\eta, \eta^*]$, we always have $v'_\eta \geq \frac{\delta_0}{2}$, mimicking Step 1 to estimate II_1 and using (253), we may bound

$$\left| \langle \mathbf{v} \rangle^\vartheta e^{\varepsilon |\mathbf{v}|^2} \zeta P_2 \right| \lesssim \frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4}. \quad (260)$$

The key is the estimate of P_1 : considering $|\exp(-H_{\eta^*, \eta})| \lesssim 1$ and using (253), (254) and (255), we have

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varepsilon |\mathbf{v}|^2} \zeta P_1 \right| & \lesssim \left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \zeta \frac{\partial \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial \eta} \right| \\ & \leq \left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \zeta \frac{\partial \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial \eta^*} \frac{\partial \eta^*}{\partial \eta} \right| + \left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \zeta \frac{\partial \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial v_\phi^*} \frac{\partial v_\phi^*}{\partial \eta} \right| \\ & \quad + \left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \zeta \frac{\partial \mathcal{G} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial v_\psi^*} \frac{\partial v_\psi^*}{\partial \eta} \right| \\ & \lesssim \left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \mathcal{A} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right) \right| + \varepsilon \left\| \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varepsilon} + \varepsilon \left\| \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varepsilon}. \end{aligned} \quad (261)$$

The estimate of $\left| \langle v \rangle^\vartheta e^{\varepsilon |v|^2} \mathcal{A} \left(\eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right) \right|$ is achieved as in II_2, II_3, II_4 since now $|v_\eta^*| \leq \frac{\delta_0}{2}$. However, we have to preserve the latter two terms related

to $\frac{\partial \mathcal{G}}{\partial v_\phi}$ and $\frac{\partial \mathcal{G}}{\partial v_\psi}$. Hence, we have

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} \zeta P_1 \right| &\lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta^2} + \frac{1}{\delta} \right) \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (262)$$

Inserting (260) and (262) into (257), we obtain

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} II_5 \right| &\lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\varepsilon}{\delta^2} + \frac{1}{\delta} \right) \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (263)$$

Synthesis: Collecting all estimates related to II_i in (243), (245), (246), (248) and (263), we have proved

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} II \right| &\lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\varepsilon}{\delta^2} + \frac{1}{\delta} \right) \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (264)$$

3.6. Region III: $v_\eta < 0$ and $v_\eta^2 + v_\phi^2 + v_\psi^2 \leq v_\phi'^2(L) + v_\psi'^2(L)$. Based on (135), we decompose

$$\begin{aligned} \mathcal{T}[\tilde{\mathcal{A}}] &= \int_0^\eta \frac{\tilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v_\eta'(\eta')} \exp(-H_{\eta^+, \eta'} - H_{\eta^+, \eta}) d\eta' \\ &\quad + \int_\eta^{\eta^+} \frac{\tilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v_\eta'(\eta')} \exp(-H_{\eta^+, \eta'} - H_{\eta^+, \eta}) d\eta' + \int_\eta^{\eta^+} \frac{\tilde{\mathcal{A}}(\eta', \mathbf{v}'(\eta'))}{v_\eta'(\eta')} \exp(H_{\eta, \eta'}) d\eta'. \end{aligned} \quad (265)$$

The integral $\int_0^\eta$ part can be estimated as in Region I and the integral $\int_\eta^{\eta^+}$ part can be estimated as in Region II, so we omit the details here. At the end of the day, we have

$$\begin{aligned} \left| \langle \mathbf{v} \rangle^\vartheta e^{\varrho |\mathbf{v}|^2} III \right| &\lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\varepsilon}{\delta^2} + \frac{1}{\delta} \right) \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (266)$$

3.7. Estimates of normal and velocity derivatives. Collecting estimates (234), (264), (266) in these three regions, and inserting (138) and (139) into (129), we have

$$\begin{aligned} \|\mathcal{A}\|_{\infty, \vartheta, \varrho} &\lesssim \left(\delta + \sqrt{\delta_0} (1 + |\ln(\varepsilon)|) \right) \|\mathcal{A}\|_{\infty, \vartheta, \varrho} + \left(\frac{\varepsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\varepsilon}{\delta^2} + \frac{1}{\delta} \right) \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho} + |p_{\mathcal{A}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (267)$$

Then we choose these constants to perform absorbing argument. First we choose $0 < \delta \ll 1$ sufficiently small such that $C\delta \leq \frac{1}{4}$. Then we take $\delta_0 = \delta^2(1 + |\ln(\varepsilon)|)^{-2}$ such that $C(1 + |\ln(\varepsilon)|)\sqrt{\delta_0} \leq C\delta \leq \frac{1}{4}$, for ε sufficiently small. Hence, we can absorb

all the term related to $\|\mathcal{A}\|_{\infty, \vartheta, \varrho}$ on the right-hand side of (267) to the left-hand side to obtain the desired result.

Lemma 3.10. *Assume (91) and (95) holds. We have*

$$\begin{aligned} \|\mathcal{A}\|_{\infty, \vartheta, \varrho} &\lesssim |\ln(\varepsilon)|^8 + |p_{\mathcal{A}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \varrho} \\ &\quad + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_{\phi}} \right) \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \left(\zeta \frac{\partial \mathcal{G}}{\partial v_{\psi}} \right) \right\|_{\infty, \vartheta, \varrho}. \end{aligned} \quad (268)$$

We may apply the similar techniques to estimate the velocity derivatives. Taking v_{η} derivative in (92) and multiplying ζ defined in (97) on both sides, we obtain the ε -transport problem for $\mathcal{B} := \zeta \frac{\partial \mathcal{G}}{\partial v_{\eta}}$

$$\begin{cases} v_{\eta} \frac{\partial \mathcal{B}}{\partial \eta} + G_1(\eta) \left(v_{\phi}^2 \frac{\partial \mathcal{B}}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial \mathcal{B}}{\partial v_{\phi}} \right) + G_2(\eta) \left(v_{\psi}^2 \frac{\partial \mathcal{B}}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial \mathcal{B}}{\partial v_{\psi}} \right) + \nu \mathcal{B} = \tilde{\mathcal{B}} + S_{\mathcal{B}}, \\ \mathcal{B}(0, \mathbf{v}) = p_{\mathcal{B}}(\mathbf{v}) \quad \text{for } v_{\eta} > 0, \\ \mathcal{B}(L, \mathbf{v}) = -\mathcal{B}(L, \mathcal{R}[\mathbf{v}]), \end{cases} \quad (269)$$

where the crucial non-local term

$$\tilde{\mathcal{B}}(\eta, \mathbf{v}) := \int_{\mathbb{R}^3} \zeta(\mathbf{u}) \partial_{v_{\eta}} k(\mathbf{u}, \mathbf{v}) \mathcal{G}(\eta, \mathbf{u}) d\mathbf{u}. \quad (270)$$

Here we utilize Lemma 3.2 to move ζ inside the derivative. $p_{\mathcal{B}}$ and $S_{\mathcal{B}}$ will be specified later. Compared with $\tilde{\mathcal{A}}$ defined in (121), the key difference is that $\tilde{\mathcal{B}}$ does not contain $\mathcal{B}$ directly but rather $\mathcal{G}$. Hence, we no longer need the analysis in previous sections to tackle the strong singularities. Then directly tracking along the characteristics, by a similar but much simpler argument using Theorem 3.1, Lemma 3.6 and (91), (95), we obtain the desired result.

Lemma 3.11. *Assume (91) and (95) holds. We have*

$$\|\mathcal{B}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{B}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{B}}\|_{\infty, \vartheta, \varrho}. \quad (271)$$

In a similar fashion, $\mathcal{C} := \zeta \frac{\partial \mathcal{G}}{\partial v_{\phi}}$ and $\mathcal{D} := \zeta \frac{\partial \mathcal{G}}{\partial v_{\psi}}$ can be estimated. Assume the boundary data and source terms $p_{\mathcal{C}}, S_{\mathcal{C}}, p_{\mathcal{D}}, S_{\mathcal{D}}$ will be specified later.

Lemma 3.12. *Assume (91) and (95) holds. We have*

$$\|\mathcal{C}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{C}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{C}}\|_{\infty, \vartheta, \varrho} \quad (272)$$

$$\|\mathcal{D}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{D}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{D}}\|_{\infty, \vartheta, \varrho}. \quad (273)$$

Then we combine above a priori estimates of normal and velocity derivatives.

Theorem 3.13. *Assume (91) and (95) holds. We have*

$$\left\| \zeta \frac{\partial \mathcal{G}}{\partial \eta} \right\|_{\infty, \vartheta, \varrho} + \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_{\eta}} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8, \quad (274)$$

$$\left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_{\phi}} \right\|_{\infty, \vartheta, \varrho} + \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_{\psi}} \right\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (275)$$

Proof. Collecting the estimates for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ in Lemma 3.10, Lemma 3.11, and Lemma 3.12, we have

$$(276)$$

$$\|\mathcal{A}\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8 + |p_{\mathcal{A}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \varrho} + \varepsilon \left(\|\nu \mathcal{C}\|_{\infty, \vartheta, \varrho} + \|\nu \mathcal{D}\|_{\infty, \vartheta, \varrho} \right),$$

$$\|\mathcal{B}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{B}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{B}}\|_{\infty, \vartheta, \varrho}, \quad (277)$$

$$\|\mathcal{C}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{C}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{C}}\|_{\infty, \vartheta, \varrho}, \quad (278)$$

$$\|\mathcal{D}\|_{\infty, \vartheta, \varrho} \lesssim 1 + |p_{\mathcal{D}}|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_{\mathcal{D}}\|_{\infty, \vartheta, \varrho}. \quad (279)$$

Now we clear up these boundary terms and source terms. At $\eta = 0$, we know $\zeta = v_{\eta}$. Hence, we may solve from (92) to get

$$\begin{aligned} p_{\mathcal{A}} &= v_{\eta} \frac{\partial \mathcal{G}}{\partial \eta}(0, \mathbf{v}) \\ &= -\frac{\varepsilon}{R_1} \left(v_{\phi}^2 \frac{\partial p}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial p}{\partial v_{\phi}} \right) - \frac{\varepsilon}{R_2} \left(v_{\psi}^2 \frac{\partial p}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial p}{\partial v_{\psi}} \right) + \nu p - K[\mathcal{G}](0, \mathbf{v}). \end{aligned} \quad (280)$$

Therefore, using Theorem 3.1, Lemma 3.5, (91) and (95), we have

$$|p_{\mathcal{A}}|_{\infty, \vartheta, \varrho} \lesssim \varepsilon |\nabla_{\mathbf{v}} p|_{\infty, \vartheta+2, \varrho} + |p|_{\infty, \vartheta+1, \varrho} + \|\nu^{-1} \mathcal{G}\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (281)$$

On the other hand, we can directly take derivative in the boundary data p to get

$$p_{\mathcal{B}} = v_{\eta} \frac{\partial p}{\partial v_{\eta}}, \quad p_{\mathcal{C}} = v_{\eta} \frac{\partial p}{\partial v_{\phi}}, \quad p_{\mathcal{D}} = v_{\eta} \frac{\partial p}{\partial v_{\psi}}, \quad (282)$$

which, using (95), yield

$$|p_{\mathcal{B}}|_{\infty, \vartheta, \varrho} + |p_{\mathcal{C}}|_{\infty, \vartheta, \varrho} + |p_{\mathcal{D}}|_{\infty, \vartheta, \varrho} \lesssim |\nabla_{\mathbf{v}} p|_{\infty, \vartheta+1, \varrho} \lesssim 1. \quad (283)$$

Directly taking η and $\mathbf{v}$ derivatives on both sides of (92) and multiplying ζ , we obtain

$$S_{\mathcal{A}} = \frac{dG_1}{d\eta} \left(v_{\phi}^2 \mathcal{B} - v_{\eta} v_{\phi} \mathcal{C} \right) + \frac{dG_2}{d\eta} \left(v_{\psi}^2 \mathcal{B} - v_{\eta} v_{\psi} \mathcal{D} \right), \quad (284)$$

$$S_{\mathcal{B}} = \mathcal{A} - G_1 v_{\phi} \mathcal{C} - G_2 v_{\psi} \mathcal{D}, \quad (285)$$

$$S_{\mathcal{C}} = G_1 \left(2v_{\phi} \mathcal{B} - v_{\eta} \mathcal{C} \right), \quad S_{\mathcal{D}} = G_2 \left(2v_{\psi} \mathcal{B} - v_{\eta} \mathcal{D} \right). \quad (286)$$

Note that fact that $|G_1| + |G_2| \lesssim \varepsilon$ and $\left| \frac{dG_1}{d\eta} \right| + \left| \frac{dG_2}{d\eta} \right| \lesssim \varepsilon^2$. We have

$$\|\nu^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon^2 \left(\|\nu \mathcal{B}\|_{\infty, \vartheta, \varrho} + \|\nu \mathcal{C}\|_{\infty, \vartheta, \varrho} + \|\nu \mathcal{D}\|_{\infty, \vartheta, \varrho} \right), \quad (287)$$

$$\|\nu^{-1} S_{\mathcal{B}}\|_{\infty, \vartheta, \varrho} \lesssim \|\nu^{-1} \mathcal{A}\|_{\infty, \vartheta, \varrho} + \varepsilon \left(\|\mathcal{C}\|_{\infty, \vartheta, \varrho} + \|\mathcal{D}\|_{\infty, \vartheta, \varrho} \right), \quad (288)$$

$$\|\nu^{-1} S_{\mathcal{C}}\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon \left(\|\mathcal{B}\|_{\infty, \vartheta, \varrho} + \|\mathcal{C}\|_{\infty, \vartheta, \varrho} \right), \quad (289)$$

$$\|\nu^{-1} S_{\mathcal{D}}\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon \left(\|\mathcal{B}\|_{\infty, \vartheta, \varrho} + \|\mathcal{D}\|_{\infty, \vartheta, \varrho} \right). \quad (290)$$

Inserting (283) and (289) into (278), and absorbing $\varepsilon \|\mathcal{C}\|_{\infty, \vartheta, \varrho}$ into the left-hand side, we get

$$\|\mathcal{C}\|_{\infty, \vartheta, \varrho} \lesssim 1 + \varepsilon \|\mathcal{B}\|_{\infty, \vartheta, \varrho}. \quad (291)$$

Similarly, inserting (283) and (290) into (279), and absorbing $\varepsilon \|\mathcal{D}\|_{\infty, \vartheta, \varrho}$ into the left-hand side, we get

$$\|\mathcal{D}\|_{\infty, \vartheta, \varrho} \lesssim 1 + \varepsilon \|\mathcal{B}\|_{\infty, \vartheta, \varrho}. \quad (292)$$

Inserting (291) and (292) into (288), and further with (283) into (277), after absorbing $\varepsilon^2 \|\mathcal{B}\|_{\infty, \vartheta, \varrho}$ into the left-hand side, we have

$$\|\mathcal{B}\|_{\infty, \vartheta, \varrho} \lesssim 1 + \|\nu^{-1} \mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (293)$$

Then inserting (293) into (291) and (292), we obtain

$$\|\mathcal{C}\|_{\infty, \vartheta, \varrho} \lesssim 1 + \varepsilon \|\nu^{-1} \mathcal{A}\|_{\infty, \vartheta, \varrho}, \quad \|\mathcal{D}\|_{\infty, \vartheta, \varrho} \lesssim 1 + \varepsilon \|\nu^{-1} \mathcal{A}\|_{\infty, \vartheta, \varrho}. \quad (294)$$

Finally, inserting (293) and (294) into (287), and further with (281) into (276), after absorbing $\varepsilon^2 \|\mathcal{A}\|_{\infty, \vartheta, \varrho}$ into the left-hand side, we obtain

$$\|\mathcal{A}\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (295)$$

Hence, inserting (295) into (293) and (294), we get the desired result. $\square$

Theorem 3.14. Assume (91) and (95) holds. For $K_0 > 0$ sufficiently small, we have

$$\left\| e^{K_0 \eta} \zeta \frac{\partial \mathcal{G}}{\partial \eta} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \eta} \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8, \quad (296)$$

$$\left\| e^{K_0 \eta} \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \eta} \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (297)$$

Proof. This proof is analogous to that of Theorem 3.13. In each step, we need to multiple $e^{K_0 \eta}$ on both sides. When K_0 is sufficiently small, we can close the proof. $\square$

Corollary 3.15. Assume (91) and (95) holds. We have

$$\varepsilon \left\| e^{K_0 \eta} v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| e^{K_0 \eta} v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (298)$$

Proof. We rearrange the terms in (92) to obtain

$$\left(G_1 v_\phi^2 + G_2 v_\psi^2 \right) \frac{\partial \mathcal{G}}{\partial v_\eta} = \left(S - \nu \mathcal{G} + K[\mathcal{G}] \right) - v_\eta \frac{\partial \mathcal{G}}{\partial \eta} + G_1 v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} + G_2 v_\eta v_\psi \frac{\partial \mathcal{G}}{\partial v_\psi}. \quad (299)$$

Recall ζ definition in (97), we know $|v_\eta| \leq \zeta$. Therefore, using (91) and Theorem 3.13, we know

$$\begin{aligned} & \left\| \left(G_1 v_\phi^2 + G_2 v_\psi^2 \right) \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \\ & \lesssim \|S - \nu \mathcal{G} + K[\mathcal{G}]\|_{\infty, \vartheta, \varrho} + \left\| \zeta \frac{\partial \mathcal{G}}{\partial \eta} \right\|_{\infty, \vartheta, \varrho} + \left\| G_1 \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} + \left\| G_2 \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \\ & \lesssim \|S\|_{\infty, \vartheta, \varrho} + \|\mathcal{G}\|_{\infty, \vartheta+2, \varrho} + \left\| \zeta \frac{\partial \mathcal{G}}{\partial \eta} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \end{aligned} \quad (300)$$

Since G_1 and G_2 have the same sign and $\varepsilon \lesssim |G_1| \lesssim \varepsilon$, $\varepsilon \lesssim |G_2| \lesssim \varepsilon$, we can separate the two terms in the left-hand side of (300) to obtain

$$\varepsilon \left\| v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (301)$$

We can perform the same analysis with an extra $e^{K_0\eta}$ term. Hence, our result naturally follows. $\square$

3.8. Estimates of tangential derivatives. Now we pull the tangential variables ι_1 and ι_2 dependence back and study the tangential derivatives.

Theorem 3.16. *Assume (91) and (95) holds. We have*

$$\left\| e^{K_0\eta} \frac{\partial \mathcal{G}}{\partial \iota_1} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8, \quad \left\| e^{K_0\eta} \frac{\partial \mathcal{G}}{\partial \iota_2} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (302)$$

Proof. Let $\mathcal{W} := \frac{\partial \mathcal{G}}{\partial \iota_i}$ for $i = 1, 2$. Taking ι_i derivative on both sides of (92), we know that $\mathcal{W}$ satisfies the equation

$$\begin{cases} v_\eta \frac{\partial \mathcal{W}}{\partial \eta} + G_1(\eta) \left(v_\phi^2 \frac{\partial \mathcal{W}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{W}}{\partial v_\phi} \right) + G_2(\eta) \left(v_\psi^2 \frac{\partial \mathcal{W}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{W}}{\partial v_\psi} \right) + \nu \mathcal{W} - K[\mathcal{W}] = S_{\mathcal{W}}, \\ \mathcal{W}(0, \iota_1, \iota_2, \mathbf{v}) = \frac{\partial p}{\partial \iota_i}(\iota_1, \iota_2, \mathbf{v}) \quad \text{for } \sin \phi > 0, \\ \mathcal{W}(L, \iota_1, \iota_2, \mathbf{v}) = \mathcal{W}(L, \iota_1, \iota_2, \mathcal{R}[\mathbf{v}]), \end{cases} \quad (303)$$

where

$$S_{\mathcal{W}} := \frac{\partial S}{\partial \iota_i} + \frac{\partial_{\iota_i} R_1}{R_1 - \varepsilon \eta} G_1(\eta) \left(v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right) + \frac{\partial_{\iota_i} R_2}{R_2 - \varepsilon \eta} G_2(\eta) \left(v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{G}}{\partial v_\psi} \right). \quad (304)$$

For $\eta \in [0, L]$, we have

$$\frac{\partial_{\iota_i} R_j}{R_j - \varepsilon \eta} \lesssim \max_{i,j=1,2} \partial_{\iota_i} R_j \lesssim 1. \quad (305)$$

Therefore, noting that $|v_\eta| \leq \zeta$, using (95), Theorem 3.14 and Corollary 3.15, we have

$$\begin{aligned} \|S_{\mathcal{W}}\|_{\infty, \vartheta, \varrho} &\lesssim \left\| \frac{\partial S}{\partial \iota_i} \right\|_{\infty, \vartheta, \varrho} + \left\| G_1(\eta) \left(v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right) \right\|_{\infty, \vartheta, \varrho} \\ &\quad + \left\| G_2(\eta) \left(v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \vartheta, \varrho} \\ &\lesssim 1 + \varepsilon \left\| v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} + \varepsilon \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \\ &\lesssim |\ln(\varepsilon)|^8. \end{aligned} \quad (306)$$

By a similar argument, we can add $e^{K_0\eta}$ contribution to obtain

$$\left\| e^{K_0\eta} S_{\mathcal{W}} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (307)$$

Therefore, applying Theorem 3.1 to (303), we have that

$$\left\| e^{K_0\eta} \mathcal{W}(\eta, \iota_1, \iota_2, \mathbf{v}) \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (308)$$

$\square$

Theorem 3.17. *Assume (91) and (95) holds. We have*

$$\left\| e^{K_0\eta} \nu \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8, \quad \left\| e^{K_0\eta} \nu \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (309)$$

Proof. Let $\mathcal{V} := v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi}$. Taking v_ϕ derivative on both sides of (92) and multiplying v_ϕ , we know that $\mathcal{V}$ satisfies the equation

$$\begin{cases} v_\eta \frac{\partial \mathcal{V}}{\partial \eta} + G_1(\eta) \left(v_\phi^2 \frac{\partial \mathcal{V}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{V}}{\partial v_\phi} \right) + G_2(\eta) \left(v_\psi^2 \frac{\partial \mathcal{V}}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{V}}{\partial v_\psi} \right) + \nu \mathcal{V} = S_\mathcal{V}, \\ \mathcal{V}(0, \iota_1, \iota_2, \mathbf{v}) = v_\phi \frac{\partial p}{\partial v_\phi}(\iota_1, \iota_2, \mathbf{v}) \quad \text{for } \sin \phi > 0, \\ \mathcal{V}(L, \iota_1, \iota_2, \mathbf{v}) = \mathcal{V}(L, \iota_1, \iota_2, \mathcal{R}[\mathbf{v}]), \end{cases} \quad (310)$$

where

$$S_\mathcal{V} := \int_{\mathbb{R}^3} v_\phi \partial_{v_\phi} k(\mathbf{u}, \mathbf{v}) d\mathbf{u} + v_\phi \frac{\partial S}{\partial v_\phi} + 2G_1 v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - 2G_1 v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi}. \quad (311)$$

Based on (95), Lemma 3.6 and Theorem 3.1, we have

$$\left\| \int_{\mathbb{R}^3} v_\phi \partial_{v_\phi} k(\mathbf{u}, \mathbf{v}) d\mathbf{u} \right\|_{\infty, \vartheta, \varrho} + \left| v_\phi \frac{\partial S}{\partial v_\phi} \right| \lesssim 1. \quad (312)$$

Using Corollary 3.15, we get

$$\left\| 2G_1 v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon \left\| v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (313)$$

Using Theorem 3.14, we obtain

$$\left\| 2G_1 v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon \left\| \nu \zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim 1. \quad (314)$$

Hence, collecting all above, we have proved that

$$\|S_\mathcal{V}\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (315)$$

Based on the analysis in the well-posedness of Milne problem, we have

$$\|\mathcal{V}\|_{\infty, \vartheta, \varrho} \lesssim \left| v_\phi \frac{\partial p}{\partial v_\phi} \right|_{\infty, \vartheta, \varrho} + \|\nu^{-1} S_\mathcal{V}\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (316)$$

By a similar argument, we can add $e^{K_0 \eta}$ contribution to obtain

$$\left\| e^{K_0 \eta} v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (317)$$

Similarly, we can show

$$\left\| e^{K_0 \eta} v_\psi \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (318)$$

Since $|v_\eta| \lesssim \zeta$, Theorem 3.14 implies

$$\left\| e^{K_0 \eta} v_\eta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (319)$$

Then our result naturally follows. The $\frac{\partial \mathcal{G}}{\partial v_\psi}$ bounds can be shown in a similar fashion. $\square$

4. Hydrodynamic limit.

4.1. Linearized remainder estimates. We consider the linearized stationary Boltzmann equation

$$\begin{cases} \varepsilon v \cdot \nabla_x f + \mathcal{L}[f] = S(x, v) & \text{in } \Omega \times \mathbb{R}^3, \\ f(x_0, v) = \mathcal{P}[f](x_0, v) + h(x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n < 0. \end{cases} \quad (320)$$

The data S and h satisfy the compatibility condition

$$\iint_{\Omega \times \mathbb{R}^3} S(x, v) \mu^{\frac{1}{2}}(v) dv dx + \int_{\gamma_-} h(x, v) \mu^{\frac{1}{2}}(v) d\gamma = 0. \quad (321)$$

It is easy to see if f is a solution to (320), then $f + C\mu^{\frac{1}{2}}$ is also a solution for arbitrary $C \in \mathbb{R}$. Hence, to guarantee uniqueness, the solution should satisfy the normalization condition

$$\iint_{\Omega \times \mathbb{R}^3} f(x, v) \mu^{\frac{1}{2}}(v) dv dx = 0. \quad (322)$$

Our analysis is based on the ideas in [10], [19], [35] and [38]. We will only record the main results and skip the detailed proof.

We first introduce the well-known micro-macro decomposition. Define $\mathbb{P}$ as the orthogonal projection onto $\mathcal{N}$:

$$\mathbb{P}[f] := \mu^{\frac{1}{2}}(v) \left(a_f(x) + v \cdot b_f(x) + \frac{|v|^2 - 3}{2} c_f(x) \right) \in \mathcal{N}. \quad (323)$$

When there is no confusion, we will simply write a, b, c . Definitely, $\mathcal{L}[\mathbb{P}[f]] = 0$. Then the operator $\mathbb{I} - \mathbb{P}$ is naturally

$$(\mathbb{I} - \mathbb{P})[f] := f - \mathbb{P}[f], \quad (324)$$

which satisfies $(\mathbb{I} - \mathbb{P})[f] \in \mathcal{N}^\perp$, i.e. $\mathcal{L}[f] = \mathcal{L}[(\mathbb{I} - \mathbb{P})[f]]$.

In the following, let $\frac{3}{2} < m < 3$. Denote $o(1)$ a sufficiently small constant.

Theorem 4.1. Assume (321) and (322) hold. The solution $f(x, v)$ to the equation (320) satisfies

$$\begin{aligned} & \frac{1}{\varepsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[f]\|_{\gamma_+, 2} + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_\nu + \|\mathbb{P}[f]\|_{2m} \\ & \lesssim o(1) \varepsilon^{\frac{3}{2m}} \left(\|f\|_{\gamma_+, \infty} + \|f\|_\infty \right) \\ & \quad + \frac{1}{\varepsilon^2} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|h\|_{\gamma_-, 2}. \end{aligned} \quad (325)$$

Theorem 4.2. Assume (321) and (322) hold. The solution $f(x, v)$ to the equation (320) satisfies for $\vartheta \geq 0$ and $0 \leq \varrho < \frac{1}{4}$,

$$\begin{aligned} & \|f\|_{\infty, \vartheta, \varrho} + \|f\|_{\gamma_+, \infty, \varrho, \vartheta} \\ & \lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|\nu^{-1} S\|_{\infty, \vartheta, \varrho} \\ & \quad + \frac{1}{\varepsilon^{\frac{3}{2m}}} \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \varrho, \vartheta}. \end{aligned} \quad (326)$$

Remark 4.3. Inserting Theorem 4.2 into Theorem 4.1, we actually have

$$\frac{1}{\varepsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[f]\|_{\gamma_+, 2} + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_\nu + \|\mathbb{P}[f]\|_{2m} \quad (327)$$

$$\begin{aligned}
&\lesssim \frac{1}{\varepsilon^2} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 \\
&\quad + \left\| \nu^{-1} S \right\|_{\infty, \vartheta, \varrho} + \|h\|_{\gamma-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|h\|_{\gamma-, 2} + \|h\|_{\gamma-, \infty, \varrho, \vartheta}.
\end{aligned} \tag{328}$$

4.2. Nonlinear estimates.

Lemma 4.4. *The nonlinear term Γ defined in (17) satisfies $\Gamma[f, g] \in \mathcal{N}^\perp$. Also, for $0 \leq \varrho < \frac{1}{4}$ and $\vartheta \geq 0$,*

$$\|\Gamma[f, g]\|_2 \lesssim \left(\sup_{x \in \Omega} |\nu g(x)|_2 \right) \|\nu f\|_2, \tag{329}$$

$$\left\| \nu^{-1} \Gamma[f, g] \right\|_{\infty, \vartheta, \varrho} \lesssim \|f\|_{\infty, \vartheta, \varrho} \|g\|_{\infty, \vartheta, \varrho}. \tag{330}$$

Proof. The orthogonality is shown in [13, Section 3.8]. (329) can be shown following the idea in [17, Lemma 2.3]. From (17),

$$\Gamma[f, g] := \mu^{-\frac{1}{2}} Q[\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g] = \Gamma_{\text{gain}}[f, g] - \Gamma_{\text{loss}}[f, g], \tag{331}$$

where we use the energy conservation $|\mathbf{u}|^2 + |v|^2 = |\mathbf{u}_*|^2 + |v_*|^2$, and

$$\Gamma_{\text{gain}}[f, g] := q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|\mathbf{u}|^2}{2}} (\omega \cdot (v - \mathbf{u})) f(\mathbf{u}_*) g(v_*) d\omega d\mathbf{u}, \tag{332}$$

$$\Gamma_{\text{loss}}[f, g] := q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|\mathbf{u}|^2}{2}} (\omega \cdot (v - \mathbf{u})) f(\mathbf{u}) g(v) d\omega d\mathbf{u}, \tag{333}$$

with

$$\mathbf{u}_* := \mathbf{u} + \omega((v - \mathbf{u}) \cdot \omega), \quad v_* := v - \omega((v - \mathbf{u}) \cdot \omega). \tag{334}$$

For the loss term, we substitute $u = v - \mathbf{u}$, so we know

$$\Gamma_{\text{loss}}[f, g] = q_0 g(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v - \mathbf{u}|^2}{2}} (\omega \cdot \mathbf{u}) f(v - \mathbf{u}) d\omega d\mathbf{u}. \tag{335}$$

Hence, using Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\Gamma_{\text{loss}}[f, g])^2 dv \\
&= q_0^2 \int_{\mathbb{R}^3} g^2(x, v) \left(\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v - \mathbf{u}|^2}{2}} (\omega \cdot \mathbf{u}) f(x, v - \mathbf{u}) d\omega d\mathbf{u} \right)^2 dv \\
&\lesssim \int_{\mathbb{R}^3} g^2(x, v) \left(\int_{\mathbb{R}^3} e^{-|v - \mathbf{u}|^2} |\mathbf{u}|^2 d\mathbf{u} \right) \left(\int_{\mathbb{R}^3} f^2(x, v - \mathbf{u}) d\mathbf{u} \right) dv \lesssim |f(x)|_2^2 |\nu g(x)|_2^2,
\end{aligned} \tag{336}$$

where we utilize the fact that

$$\int_{\mathbb{R}^3} e^{-|v - \mathbf{u}|^2} |\mathbf{u}|^2 d\mathbf{u} \lesssim \nu^2(v). \tag{337}$$

On the other hand, for the gain term, after substituting $u = v - \mathbf{u}$, we know

$$\Gamma_{\text{gain}}[f, g] = q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v - \mathbf{u}|^2}{2}} (\omega \cdot \mathbf{u}) f(v - \mathbf{u}_\perp) g(v - \mathbf{u}_\parallel) d\omega d\mathbf{u}, \tag{338}$$

where

$$u_\perp = u - \omega(u \cdot \omega), \quad u_\parallel = \omega(u \cdot \omega). \tag{339}$$

Hence, using Hölder's inequality, we have

$$\int_{\mathbb{R}^3} \left(\Gamma_{\text{gain}}[f, g](x) \right)^2 dv \tag{340}$$

$$\begin{aligned}
&= q_0^2 \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v-u|^2}{2}} (\omega \cdot u) f(x, v - u_\perp) g(x, v - u_\parallel) d\omega du \right)^2 dv \\
&\lesssim \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} e^{-|v-u|^2} |u|^2 du \right) \left(\int_{\mathbb{R}^3} f^2(x, v - u_\perp) g^2(x, v - u_\parallel) du \right) dv \\
&\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nu^2(v) f^2(x, v - u_\perp) g^2(x, v - u_\parallel) du dv
\end{aligned}$$

Denote $u' = v - u_\perp$ and $v' = v - u_\parallel$. Consider substitution $(u, v) \rightarrow (u', v')$. It is well-known (see the proof of [17, Lemma 2.3]) that $du dv = du' dv'$ and $|v| \lesssim |u'| + |v'|$. Hence, we have

$$\int_{\mathbb{R}^3} (\Gamma_{\text{gain}}[f, g])^2 dv \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\nu^2(u') + \nu^2(v') \right) f^2(x, u') g^2(x, v') du' dv' \quad (341)$$

$$\begin{aligned}
&\lesssim \left(\int_{\mathbb{R}^3} \nu^2(u') f^2(x, u') du' \right) \left(\int_{\mathbb{R}^3} \nu^2(v') g^2(x, v') dv' \right) \quad (342) \\
&\lesssim |\nu f(x)|_2^2 |\nu g(x)|_2^2.
\end{aligned}$$

Combining (336) and (341), we know

$$\int_{\mathbb{R}^3} (\Gamma[f, g])^2 dv \lesssim |\nu f(x)|_2^2 |\nu g(x)|_2^2. \quad (343)$$

Therefore, (329) naturally follows. Also, (330) is proved in [19, Lemma 5]. $\square$

4.3. Perturbed linearized remainder estimates. We consider the perturbed linearized stationary Boltzmann equation

$$\begin{cases} \varepsilon v \cdot \nabla_x f + \mathcal{L}[f] = \Gamma[f, g] + S(x, v) & \text{in } \Omega \times \mathbb{R}^3, \\ f(x_0, v) = \mathcal{P}[f](x_0, v) + (\mu_b^\varepsilon - \mu) \mu^{-1} \mathcal{P}[f] + h(x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n < 0. \end{cases} \quad (344)$$

Assume that a priori

$$\iint_{\Omega \times \mathbb{R}^3} f(x, v) \mu^{\frac{1}{2}}(v) dv dx = 0. \quad (345)$$

and

$$\|g\|_{\infty, \vartheta, \varrho} = o(1)\varepsilon. \quad (346)$$

The data S and h satisfy the compatibility condition

$$\iint_{\Omega \times \mathbb{R}^3} S(x, v) \mu^{\frac{1}{2}}(v) dv dx + \int_{\gamma_-} h(x, v) \mu^{\frac{1}{2}}(v) d\gamma = 0. \quad (347)$$

Theorem 4.5. *Assume (347) and (345) hold. The solution $f(x, v)$ to the equation (344) satisfies*

$$\frac{1}{\varepsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+,2}} + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_\nu + \|\mathbb{P}[f]\|_{2m} \quad (348)$$

$$\begin{aligned}
&\leq o(1)\varepsilon^{\frac{3}{2m}} \left(\|f\|_{\gamma_{+,\infty}} + \|f\|_\infty \right) + \frac{1}{\varepsilon^2} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} \\
&\quad + \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|h\|_{\gamma_-, 2}.
\end{aligned} \quad (349)$$

Proof. Since the perturbed term $\Gamma[f, g] \in \mathcal{N}^\perp$, we apply Theorem 4.1 to (344) to obtain

$$\frac{1}{\varepsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[f]\|_{\gamma_+, 2} + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_\nu + \|\mathbb{P}[f]\|_{2m} \quad (350)$$

$$\leq o(1)\varepsilon^{\frac{3}{2m}} \left(\|f\|_{\gamma_+, \infty} + \|f\|_\infty \right) + \frac{1}{\varepsilon^2} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} \quad (351)$$

$$\begin{aligned} &+ \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|h\|_{\gamma_-, 2} \\ &+ \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} \Gamma[f, g] \right\|_2 + \|(\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[f]\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|(\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[f]\|_{\gamma_-, 2}. \end{aligned}$$

Using Lemma 4.4 and (346), we have

$$\frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} \Gamma[f, g] \right\|_2 \lesssim o(1) \left\| \nu^{\frac{1}{2}} f \right\|_2 \lesssim o(1) \|\mathbb{P}[f]\|_\nu + o(1) \|(\mathbb{I} - \mathbb{P})[f]\|_\nu. \quad (352)$$

Note that direct computation reveals that

$$\|\mathbb{P}[f]\|_{2m} \gtrsim \|\mathbb{P}[f]\|_\nu, \quad (353)$$

so inserting (352) into (350), we can absorb $o(1) \|\mathbb{P}[f]\|_\nu$ and $o(1) \|(\mathbb{I} - \mathbb{P})[f]\|_\nu$ into the left-hand side. On the other hand, due to (12), we know

$$\begin{aligned} &\|(\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[f]\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|(\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[f]\|_{\gamma_-, 2} \\ &\lesssim o(1)\varepsilon \|\mathcal{P}[f]\|_{\gamma_-, \frac{4m}{3}} + o(1) \|\mathcal{P}[f]\|_{\gamma_-, 2} \lesssim o(1)\varepsilon \|f\|_{\gamma_+, \infty} + o(1) \|\mathcal{P}[f]\|_{\gamma_+, 2}. \end{aligned} \quad (354)$$

Here, $o(1) \|f\|_{\gamma_+, \infty}$ can be combined with the corresponding term on the right-hand side of (350). Also, the bound of $\|\mathcal{P}[f]\|_{\gamma_+, 2}$ has been achieved in the proof of Theorem 4.1 (see [38])

$$\|\mathcal{P}[f]\|_{\gamma_+, 2}^2 \lesssim \|\mathbb{P}[f]\|_2^2 + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_2^2 + \frac{1}{\varepsilon} \left| \iint_{\Omega \times \mathbb{R}^3} fS \right|. \quad (355)$$

Inserting (355) into (350), using (354), Hölder's inequality and Theorem 4.1, we know

$$\begin{aligned} \|\mathcal{P}[f]\|_{\gamma_+, 2} &\lesssim \|\mathbb{P}[f]\|_2 + \frac{1}{\varepsilon} \|(\mathbb{I} - \mathbb{P})[f]\|_2 + \frac{1}{\varepsilon^{\frac{1}{2}}} \left(\left| \iint_{\Omega \times \mathbb{R}^3} fS \right| \right)^{\frac{1}{2}} \\ &\lesssim o(1)\varepsilon^{\frac{3}{2m}} \left(\|f\|_{\gamma_+, \infty} + \|f\|_\infty \right) + \frac{1}{\varepsilon^2} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 \\ &\quad + \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon} \|h\|_{\gamma_-, 2} + o(1) \|\mathcal{P}[f]\|_{\gamma_+, 2}. \end{aligned} \quad (356)$$

Then absorbing $o(1) \|\mathcal{P}[f]\|_{\gamma_+, 2}$ into the left-hand side, we get control of $\|\mathcal{P}[f]\|_{\gamma_+, 2}$. Then inserting it into (354) and further (350), we get the desired result. $\square$

Theorem 4.6. Assume (347) and (345) hold. The solution $f(x, v)$ to the equation (344) satisfies for $\vartheta \geq 0$ and $0 \leq \varrho < \frac{1}{4}$,

$$\begin{aligned} &\|f\|_{\infty, \vartheta, \varrho} + \|f\|_{\gamma_+, \infty, \vartheta, \varrho} \\ &\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|\nu^{-1} S\|_{\infty, \vartheta, \varrho} \\ &\quad + \frac{1}{\varepsilon^{\frac{3}{2m}}} \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \vartheta, \varrho}. \end{aligned} \quad (357)$$

Proof. Since we already have bounds for f in L^{2m} as Theorem 4.5, following the proof of Theorem 4.2 (see [38]), we obtain

$$\begin{aligned} & \|f\|_{\infty, \vartheta, \varrho} + \|f\|_{\gamma_+, \infty, \varrho, \vartheta} \\ & \lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S] \right\|_2 + \|\nu^{-1} S\|_{\infty, \vartheta, \varrho} + \|\nu^{-1} \Gamma[f, g]\|_{\infty, \vartheta, \varrho} \\ & \quad + \frac{1}{\varepsilon^{\frac{3}{2m}}} \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \varrho, \vartheta} + \|(\mu_b^\varepsilon - \mu) \mu^{-1} \mathcal{P}[f]\|_{\gamma_-, \infty, \varrho, \vartheta}. \end{aligned} \quad (358)$$

Using Lemma 4.4 and (346), we have

$$\|\nu^{-1} \Gamma[f, g]\|_{\infty, \vartheta, \varrho} \lesssim \|f\|_{\infty, \vartheta, \varrho} \|g\|_{\infty, \vartheta, \varrho} \lesssim o(1) \|f\|_{\infty, \vartheta, \varrho}. \quad (359)$$

Inserting (359) into (358), we can absorb $o(1) \|f\|_{\infty, \vartheta, \varrho}$ into the left-hand side. Also, using (12), we have

$$\|(\mu_b^\varepsilon - \mu) \mu^{-1} \mathcal{P}[f]\|_{\gamma_-, \infty, \varrho, \vartheta} \lesssim o(1) \|f\|_{\gamma_+, \infty, \varrho, \vartheta}. \quad (360)$$

Inserting (360) into (358) and absorbing $o(1) \|f\|_{\gamma_+, \infty, \varrho, \vartheta}$ into the left-hand side, we obtain the desired result. $\square$

4.4. Proof of the main Theorem. Now we turn to the proof of the main result, Theorem 1.2.

Denote the remainder

$$\varepsilon \mathfrak{R} = \varepsilon^3 R := f^\varepsilon - (\varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3) - (\varepsilon \mathcal{F}_1 + \varepsilon^2 \mathcal{F}_2) = f^\varepsilon - Q - \mathcal{Q}, \quad (361)$$

where

$$Q := \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3, \quad \mathcal{Q} := \varepsilon \mathcal{F}_1 + \varepsilon^2 \mathcal{F}_2. \quad (362)$$

We write $\mathcal{L} := \varepsilon v \cdot \nabla_x f + \mathcal{L}[f]$ to denote the linearized Boltzmann operator.

The equation (16) is actually

$$\mathcal{L}[Q + \mathcal{Q} + \varepsilon^3 R] = \Gamma[Q + \mathcal{Q} + \varepsilon^3 R, Q + \mathcal{Q} + \varepsilon^3 R]. \quad (363)$$

Based on the construction in Section 2.4, we obtain

$$\mathcal{L}[R] = \varepsilon^3 \Gamma[R, R] + 2\Gamma[R, Q + \mathcal{Q}] + S_1 + S_2, \quad (364)$$

where

$$S_1 = -\varepsilon v \cdot \nabla_x F_3 + \frac{1}{P_1 P_2} \left(\frac{\partial_{11} r \cdot \partial_2 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_2 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\psi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} \quad (365)$$

$$\begin{aligned} & + \frac{1}{P_1 P_2} \left(\frac{\partial_{22} r \cdot \partial_1 r}{P_2(\varepsilon \kappa_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_1 r}{P_1(\varepsilon \kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\psi} \\ & + \left(\frac{v_\phi}{P_1(\varepsilon \kappa_1 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_1} + \varepsilon^3 \frac{v_\psi}{P_2(\varepsilon \kappa_2 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_2} \right), \end{aligned}$$

$$S_2 = 2\Gamma[F_1, \mathcal{F}_2] + 2\varepsilon \Gamma[F_1, F_3] + \text{higher-order } \Gamma \text{ terms up to } \varepsilon^3. \quad (366)$$

The boundary condition of (16) is essentially

$$(367)$$

$$Q + \mathcal{Q} + \varepsilon^3 R = \mathcal{P}[Q + \mathcal{Q} + \varepsilon^3 R] + (\mu_b^\varepsilon - \mu) \mu^{-1} \mathcal{P}[Q + \mathcal{Q} + \varepsilon^3 R] + \mu^{-\frac{1}{2}} (\mu_b^\varepsilon - \mu).$$

Based on the boundary condition expansion in Section 2.3, we have

$$R - \mathcal{P}[R] = H[R] + h, \quad (368)$$

where

$$H[R](x_0, v) = (\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[R], \quad (369)$$

and

$$\begin{aligned} h = & \varepsilon^{-2}(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac{1}{2}}\mu_1)\mu^{-1}\mathcal{P}[F_1 + \mathcal{F}_1] + \varepsilon^{-1}(\mu_b^\varepsilon - \mu)\mu^{-1}\mathcal{P}[F_2 + \mathcal{F}_2] \\ & + \mathcal{P}[F_3] - F_3 + \varepsilon^{-3}\mu^{-\frac{1}{2}}(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac{1}{2}}\mu_1 - \varepsilon^2\mu^{\frac{1}{2}}\mu_2). \end{aligned} \quad (370)$$

The equation (364) and boundary condition (368) forms a system that fits into (344):

$$\begin{cases} \varepsilon v \cdot \nabla_x R + \mathcal{L}[R] = \Gamma[R, 2(Q + \mathcal{Q}) + \varepsilon^3 R] + S_1(x, v) + S_2(x, v) & \text{in } \Omega \times \mathbb{R}^3, \\ R(x_0, v) = \mathcal{P}[R](x_0, v) + H[R](x_0, v) + h(x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n < 0. \end{cases} \quad (371)$$

Then we can verify (371) satisfies the assumptions (346) (since Q and $\mathcal{Q}$ are small), (347). Also, the construction in Section 2.4 implies that the solution satisfies (345). Applying Theorem 4.6 to (371), we obtain

$$\begin{aligned} & \|R\|_{\infty, \vartheta, \varrho} + \|R\|_{\gamma_+, \infty, \varrho, \vartheta} \\ & \lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S_1]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S_1] \right\|_2 + \|\nu^{-1} S_1\|_{\infty, \vartheta, \varrho} \\ & \quad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S_2]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S_2] \right\|_2 + \|\nu^{-1} S_2\|_{\infty, \vartheta, \varrho} \\ & \quad + \frac{1}{\varepsilon^{\frac{3}{2m}}} \|h\|_{\gamma_-, \frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \varrho, \vartheta} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} (\varepsilon^3 \Gamma[R, R]). \end{aligned} \quad (372)$$

Based on the analysis in Section 2.4, we know

$$\|\varepsilon v \cdot \nabla_x F_3\|_{\frac{2m}{2m-1}} + \left\| \nu^{-\frac{1}{2}} (\varepsilon v \cdot \nabla_x F_3) \right\|_2 + \|\nu^{-1} (\varepsilon v \cdot \nabla_x F_3)\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon. \quad (373)$$

On the other hand, using the rescaling $\eta = \frac{\mathfrak{N}}{\varepsilon}$, we have

$$\begin{aligned} & \left\| \nu \frac{\partial \mathcal{F}_2}{\partial \iota_1} \right\|_{\frac{2m}{2m-1}} + \left\| \nu \frac{\partial \mathcal{F}_2}{\partial \iota_2} \right\|_{\frac{2m}{2m-1}} + \left\| \nu^2 \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right\|_{\frac{2m}{2m-1}} + \left\| \nu^2 \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right\|_{\frac{2m}{2m-1}} \lesssim \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)|^8, \\ & \left\| \nu^{\frac{1}{2}} \frac{\partial \mathcal{F}_2}{\partial \iota_1} \right\|_2 + \left\| \nu^{\frac{1}{2}} \frac{\partial \mathcal{F}_2}{\partial \iota_2} \right\|_2 + \left\| \nu^{\frac{3}{2}} \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right\|_2 + \left\| \nu^{\frac{3}{2}} \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right\|_2 \lesssim \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)|^8, \end{aligned} \quad (375)$$

$$\left\| \nu^{\frac{1}{2}} \frac{\partial \mathcal{F}_2}{\partial \iota_1} \right\|_2 + \left\| \nu^{\frac{1}{2}} \frac{\partial \mathcal{F}_2}{\partial \iota_2} \right\|_2 + \left\| \nu \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right\|_{\infty, \vartheta, \varrho} + \left\| \nu \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (376)$$

Collecting all terms, we have

$$\|\mathbb{P}[S_1]\|_{\frac{2m}{2m-1}} \lesssim \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)|^8, \quad \left\| \nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S_1] \right\|_2 \lesssim \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)|^8, \quad (377)$$

$$\|\nu^{-1} S_1\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (378)$$

Since S_2 are all nonlinear terms, Lemma 4.4 implies that $\mathbb{P}[S_2] = 0$. Then the leading-order term is $\Gamma[F_1, \mathcal{F}_2]$. Hence, using Lemma 4.4, we have

$$\left\| \nu^{-\frac{1}{2}} \Gamma[F_1, \mathcal{F}_2] \right\|_2 \lesssim \sup_{x \in \Omega} (|\nu F_1(x)|_2) \|\nu \mathcal{F}_2\|_2 \lesssim \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)|^8, \quad (379)$$

$$\|\nu^{-1}\Gamma[F_1, \mathcal{F}_2]\|_{\infty, \vartheta, \varrho} \lesssim \|F_1\|_{\infty, \vartheta, \varrho} \|\mathcal{F}_2\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (380)$$

In total, we have

$$\|\mathbb{P}[S_2]\|_{\frac{2m}{2m-1}} = 0, \quad \left\| \nu^{-\frac{1}{2}}(\mathbb{I} - \mathbb{P})[S_2] \right\|_2 \lesssim \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)|^8, \quad \|\nu^{-1}S_2\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\varepsilon)|^8. \quad (381)$$

Note that all terms in h are at least of $O(1)$. Hence, we directly bound

$$\|h\|_{\gamma_-, \frac{4m}{3}} \lesssim 1, \quad \|h\|_{\gamma_-, 2} \lesssim 1, \quad \|h\|_{\gamma_-, \infty, \varrho, \vartheta} \lesssim 1. \quad (382)$$

Inserting (377), (381) and (382) into (372), we have

$$\|R\|_{\infty, \vartheta, \varrho} + \|R\|_{\gamma_+, \infty, \varrho, \vartheta} \lesssim \varepsilon^{-1-\frac{2}{m}} |\ln(\varepsilon)|^8 + \varepsilon^{1-\frac{3}{2m}} \|R\|_{\infty, \vartheta, \varrho}^2. \quad (383)$$

By standard iteration/fixed-point argument, for ε small, we have

$$\|R\|_{\infty, \vartheta, \varrho} + \|R\|_{\gamma_+, \infty, \varrho, \vartheta} \lesssim \varepsilon^{-1-\frac{2}{m}} |\ln(\varepsilon)|^8. \quad (384)$$

Therefore, we know for $\frac{3}{2} < m < 3$

$$\|f^\varepsilon - \varepsilon F_1\|_{\infty, \vartheta, \varrho} \lesssim \varepsilon^{2-\frac{2}{m}} |\ln(\varepsilon)| \lesssim C(\delta) \varepsilon^{\frac{4}{3}-\delta}, \quad (385)$$

for any $0 < \delta \ll 1$.

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