

HILBERT EXPANSION FOR COULOMB COLLISIONAL KINETIC MODELS

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Abstract. The relativistic Vlasov-Maxwell-Landau (r-VML) system and the relativistic Landau (r-LAN) equation are fundamental models that describe the dynamics of an electron gas. In this paper, we introduce a novel weighted energy method and establish the validity of the Hilbert expansion for the one-species r-VML system and r-LAN equation. As the Knudsen number shrinks to zero, we rigorously demonstrate the relativistic Euler-Maxwell limit and relativistic Euler limit, respectively. This successfully resolves the long-standing open problem regarding the hydrodynamic limits of Landau-type equations.

1. Introduction.

1.1. *Relativistic Vlasov-Maxwell-Landau system.* The relativistic Vlasov-Maxwell-Landau (r-VML) system is a fundamental and complete model describing the dynamics of a dilute collisional ionized plasma appearing in nuclear fusion and the interior of stars, etc. Correspondingly, the relativistic Euler-Maxwell system, the foundation of the two-fluid theory in plasma physics, describes the dynamics of two compressible ion and electron fluids interacting with their own self-consistent electromagnetic field. It is also the origin of many celebrated dispersive PDE such as NLS, KP, KdV, Zakharov, etc., as various scaling limits and approximations.

Received August 31, 2023, and, in revised form, February 11, 2024.

2020 *Mathematics Subject Classification.* Primary 82C40; Secondary 76P05, 35Q20.

Key words and phrases. Hilbert expansion, relativistic Landau equation, relativistic Vlasov-Maxwell-Landau system, local Maxwellian.

The first author was supported by an NSF Grant DMS-2202824. The second author was supported by an NSF Grant DMS-2104775. The third author was supported by NSFC Grants 11871469 and 12271506.

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Since the ion mass is far larger than the electron mass in a plasma, the dynamics of ions is negligible for simplification sometimes. In this special case, the plasma can be approximately described by the one-species r-VML system in the mesoscopic level and treated as a single fluid in the macroscopic level. It has been a long-term open question if the general relativistic Euler-Maxwell system can be derived rigorously from its kinetic counter-part, the r-VML system, as the Knudsen number approaches zero.

In this paper, we are able to answer this question in the affirmative. Consider the r-VML system for $F(t, x, p) \in \mathbb{R}$ (see [90]):

$$\partial_t F + c\hat{p} \cdot \nabla_x F - \frac{e}{m}(E + \hat{p} \times B) \cdot \nabla_p F = \mathcal{C}[F, F], \quad (1.1)$$

coupled with the Maxwell system for $(E(t, x), B(t, x)) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{cases} \partial_t E - c\nabla_x \times B = 4\pi e \int_{\mathbb{R}^3} \hat{p} F dp, \\ \partial_t B + c\nabla_x \times E = 0, \\ \nabla_x \cdot E = -4\pi e \left(\bar{n} - \int_{\mathbb{R}^3} F dp \right), \\ \nabla_x \cdot B = 0. \end{cases} \quad (1.2)$$

Here $F(t, x, p)$ is the number density function for electrons at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. $p^0 = \sqrt{m^2 c^2 + |p|^2}$ is the energy of an electron and $\hat{p} = \frac{p}{p^0}$. The constants $-e$ and m are the electrons' charge and rest mass, respectively. c is the speed of light, $\bar{n}$ is the uniform number density of ions, and $(E(t, x), B(t, x))$ are the electromagnetic fields.

Denote the four-momentums $p^\mu = (p^0, p)$ and $q^\mu = (q^0, q)$. We use the Einstein convention that repeated up-down indices be summed and we raise and lower indices using the Minkowski metric $g_{\mu\nu} := \text{diag}(-1, 1, 1, 1)$. The Lorentz inner product is then given by

$$p^\mu q_\mu := -p^0 q^0 + \sum_{i=1}^3 p_i q_i. \quad (1.3)$$

The collision operator $\mathcal{C}$ on the R.H.S. of (1.1), which registers binary collisions between particles, takes the following form:

$$\mathcal{C}[g, h] := \frac{2\pi}{c} e^2 \ln(\lambda) \nabla_p \cdot \left\{ \int_{\mathbb{R}^3} \Phi(p, q) \left[\nabla_p g(p) h(q) - g(p) \nabla_q h(q) \right] dq \right\}, \quad (1.4)$$

where the collision kernel $\Phi(p, q)$ is a 3×3 non-negative matrix

$$\Phi(p, q) := \left(\frac{mc}{p^0} \right) \left(\frac{mc}{q^0} \right) \Lambda(p, q) \mathcal{S}(p, q) \quad (1.5)$$

with

$$\begin{aligned}\Lambda(p, q) &:= \frac{1}{m^4 c^4} (p^\mu q_\mu)^2 \left(\frac{1}{m^4 c^4} (p^\mu q_\mu)^2 - 1 \right)^{-\frac{3}{2}}, \\ \mathcal{S}(p, q) &:= \left(\frac{1}{m^4 c^4} (p^\mu q_\mu)^2 - 1 \right) I_3 - \frac{1}{m^2 c^2} (p - q) \otimes (p - q) \\ &\quad - \frac{1}{m^2 c^2} \left(\frac{1}{m^2 c^2} (p^\mu q_\mu) + 1 \right) (p \otimes q + q \otimes p).\end{aligned}$$

Here $\ln(\lambda)$ denotes the Coulomb logarithm. From [90] and [32], for our purposes, we may simply regard $\ln(\lambda)$ as a fixed constant independent of other parameters.

It is well-known that $\Phi(p, q)$ satisfies

$$\sum_{i=1}^3 \Phi^{ij}(p, q) \left(\frac{q_i}{q^0} - \frac{p_i}{p^0} \right) = \sum_{j=1}^3 \Phi^{ij}(p, q) \left(\frac{q_j}{q^0} - \frac{p_j}{p^0} \right) = 0. \quad (1.6)$$

REMARK 1.1. Notice that as $c \rightarrow \infty$, using Taylor expansion based on the order of c^{-1}

$$p^0 \approx mc \left(1 + \frac{1}{2} \frac{|p|^2}{m^2 c^2} \right), \quad p^\mu q_\mu \approx -m^2 c^2 - \frac{1}{2} |p|^2 - \frac{1}{2} |q|^2 + p \cdot q = m^2 c^2 \left(-1 - \frac{1}{2} \frac{|p - q|^2}{m^2 c^2} \right).$$

Here the notation $\approx$ denotes the leading-order terms in the expansion. Hence, we have

$$\Lambda(p, q) \approx \frac{m^3 c^3}{|p - q|^3},$$

and

$$\mathcal{S}(p, q) \approx \left(\frac{|p - q|^2}{m^2 c^2} I_3 - \frac{(p - q) \otimes (p - q)}{m^2 c^2} \right).$$

Therefore, we know

$$\Phi(p, q) \approx mc \frac{1}{|p - q|} \left(I_3 - \frac{(p - q) \otimes (p - q)}{|p - q|^2} \right),$$

which reduces to the non-relativistic Landau collision operator (see [40]).

In order to introduce the quantity for the convenience of the hydrodynamic limit, we may follow the path in [8, 86, 87] to rescale the system (1.1). Define the reference time $\underline{t}$, length $\underline{x}$ and momentum $\underline{p}$. Redefining the variables

$$t \rightarrow \frac{t}{\underline{t}}, \quad x \rightarrow \frac{x}{\underline{x}}, \quad p \rightarrow \frac{p}{\underline{p}} \quad (1.7)$$

yields

$$\frac{\underline{x}}{\underline{t}} \partial_t F + c \hat{p} \cdot \nabla_x F - \frac{\underline{x}}{\underline{p}} \frac{e}{m} (E + \hat{p} \times B) \cdot \nabla_p F = \underline{x} \mathcal{C}[F, F], \quad (1.8)$$

and

$$\begin{cases} \frac{\underline{x}}{\underline{t}} \partial_t E - c \nabla_x \times B = \underline{x} \underline{p}^3 4\pi e \int_{\mathbb{R}^3} \hat{p} F dp, \\ \frac{\underline{x}}{\underline{t}} \partial_t B + c \nabla_x \times E = 0, \\ \nabla_x \cdot E = -\underline{x} 4\pi e \left(\bar{n} - \underline{p}^3 \int_{\mathbb{R}^3} F dp \right), \\ \nabla_x \cdot B = 0. \end{cases} \quad (1.9)$$

Assume that the reference spatial density is $\underline{\rho}$. Then we redefine the unknowns

$$F \rightarrow \frac{F}{\underline{\rho} \underline{p}^{-3}}, \quad E \rightarrow \frac{E}{\underline{\rho} \underline{x}}, \quad B \rightarrow \frac{B}{\underline{\rho} \underline{x}} \quad (1.10)$$

to arrive at

$$\frac{\underline{x}}{\underline{t}} \partial_t F + c \hat{p} \cdot \nabla_x F - \frac{\underline{\rho} \underline{x}^2}{\underline{p}} \frac{e}{m} (E + \hat{p} \times B) \cdot \nabla_p F = \frac{\underline{\rho} \underline{x}}{\underline{p}^3} \mathcal{C}[F, F], \quad (1.11)$$

and redefine $\bar{n} \rightarrow \bar{n} \underline{\rho}$

$$\begin{cases} \frac{\underline{x}}{\underline{t}} \partial_t E - c \nabla_x \times B = 4\pi e \int_{\mathbb{R}^3} \hat{p} F dp, \\ \frac{\underline{x}}{\underline{t}} \partial_t B + c \nabla_x \times E = 0, \\ \nabla_x \cdot E = -4\pi e \left(\bar{n} - \int_{\mathbb{R}^3} F dp \right), \\ \nabla_x \cdot B = 0. \end{cases} \quad (1.12)$$

The hydrodynamic limit corresponds to the scaling

$$\underline{x} \rightarrow 0, \quad \underline{t} \rightarrow 0, \quad \underline{p} \simeq 1, \quad \underline{\rho} \rightarrow \infty, \quad (1.13)$$

satisfying

$$\frac{\underline{x}}{\underline{t}} \simeq 1, \quad \underline{\rho} \underline{x}^2 \simeq 1. \quad (1.14)$$

Here the notation $\simeq$ denotes the same level of magnitude up to some physical constants. This indicates that the coefficient in front of the collision term

$$\frac{\underline{\rho} \underline{x}}{\underline{p}^3} \simeq \frac{1}{\underline{x}} \rightarrow \infty. \quad (1.15)$$

Then we define the Knudsen number ε ,

$$\varepsilon := \frac{\underline{p}^3}{\underline{\rho} \underline{x}} \rightarrow 0. \quad (1.16)$$

Physically, ε is proportional to the relative mean free path (see [87]). Then it suffices for us to consider the system for $F^\varepsilon(t, x, p) \in \mathbb{R}$:

$$\partial_t F^\varepsilon + c \hat{p} \cdot \nabla_x F^\varepsilon - \frac{e}{m} (E^\varepsilon + \hat{p} \times B^\varepsilon) \cdot \nabla_p F^\varepsilon = \frac{1}{\varepsilon} \mathcal{C}[F^\varepsilon, F^\varepsilon], \quad (1.17)$$

coupled with the Maxwell system for $(E^\varepsilon(t, x), B^\varepsilon(t, x)) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{cases} \partial_t E^\varepsilon - c \nabla_x \times B^\varepsilon = 4\pi e \int_{\mathbb{R}^3} \hat{p} F^\varepsilon dp, \\ \partial_t B^\varepsilon + c \nabla_x \times E^\varepsilon = 0, \\ \nabla_x \cdot E^\varepsilon = -4\pi e \left(\bar{n} - \int_{\mathbb{R}^3} F^\varepsilon dp \right), \\ \nabla_x \cdot B^\varepsilon = 0. \end{cases} \quad (1.18)$$

The collision operator $\mathcal{C}$ satisfies the orthogonality property:

$$\int_{\mathbb{R}^3} \left\{ \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} \mathcal{C}[g, h](p) \right\} dp = \mathbf{0}, \quad (1.19)$$

which, combined with (1.17) and (1.18), yields the conservation laws

$$\begin{aligned} \frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^\varepsilon(t, x, p) dp dx &= 0, \\ \frac{d}{dt} \left\{ \iint_{\mathbb{R}^3 \times \mathbb{R}^3} p F^\varepsilon(t, x, p) dp dx + \frac{1}{4\pi} \int_{\mathbb{R}^3} (E^\varepsilon(t, x) \times B^\varepsilon(t, x)) dx \right\} &= \mathbf{0}, \\ \frac{d}{dt} \left\{ \iint_{\mathbb{R}^3 \times \mathbb{R}^3} p^0 F^\varepsilon(t, x, p) dp dx + \frac{1}{4\pi} \int_{\mathbb{R}^3} (|E^\varepsilon(t, x)|^2 + |B^\varepsilon(t, x)|^2) dx \right\} &= 0. \end{aligned}$$

Corresponding to (1.17)–(1.18), at the hydrodynamic level, the electron gas obeys the relativistic Euler-Maxwell system for $(n(t, x), u(t, x), T(t, x)) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$:

$$\begin{cases} \frac{1}{c} \partial_t (nu^0) + \nabla_x \cdot (nu) = 0, \\ \frac{1}{c} \partial_t ((\mathfrak{e} + P)u^0 u) + \nabla_x \cdot ((\mathfrak{e} + P)u \otimes u) + c^2 \nabla_x P + cen(u^0 E + u \times B) = \mathbf{0}, \\ \frac{1}{c} \partial_t (\mathfrak{e}(u^0)^2 + P|u|^2) + \nabla_x \cdot ((\mathfrak{e} + P)u^0 u) + cen(u \cdot E) = 0, \end{cases} \quad (1.20)$$

coupled with the Maxwell system for $(E(t, x), B(t, x)) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{cases} \partial_t E - c \nabla_x \times B = 4\pi e \frac{nu}{c}, \\ \partial_t B + c \nabla_x \times E = 0, \\ \nabla_x \cdot E = -4\pi e \left(\bar{n} - \frac{nu^0}{c} \right), \\ \nabla_x \cdot B = 0, \end{cases} \quad (1.21)$$

where n is the electrons' number density, $u = (u_1, u_2, u_3)$, $u^0 = \sqrt{|u|^2 + c^2}$, and T is the temperature. In particular, $\mathfrak{e}(t, x)$ is the total energy (including the rest energy and internal energy) and $P(t, x)$ is the pressure given by

$$P := \frac{nmc^2}{\gamma} = \frac{k_B}{m} \rho T, \quad (1.22)$$

$$\mathfrak{e} := \frac{nmc^2}{K_2(\gamma)} \left\{ K_3(\gamma) - \frac{1}{\gamma} K_2(\gamma) \right\}, \quad (1.23)$$

where $\rho := nm$ is the mass density, $\gamma := mc^2(k_B T)^{-1}$ is a dimensionless variable, k_B is Boltzmann's constant, $K_j(\gamma)$ for $j = 0, 1, 2, \dots$ are the modified second-order Bessel functions:

$$K_j(\gamma) := \frac{(2^j j!) }{(2j)!} \frac{1}{\gamma^j} \int_{\gamma}^{\infty} e^{-\lambda} (\lambda^2 - \gamma^2)^{j-1/2} d\lambda, \quad (j \geq 0). \quad (1.24)$$

The system (1.21) has been well-studied in the irrotational context. Denote Faraday's tensor

$$\mathcal{F}^{ij} := \begin{pmatrix} 0 & -c^{-1}E_1 & -c^{-1}E_2 & -c^{-1}E_3 \\ c^{-1}E_1 & 0 & -B_3 & B_2 \\ c^{-1}E_2 & B_3 & 0 & -B_1 \\ c^{-1}E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \quad (1.25)$$

Let h be the specific enthalpy defined by $h'(x) = \frac{P(x)}{x}$ with $h > 0$. Then we say the solution to (1.20) and (1.21) is irrotational if

$$e\mathcal{F}_{jk} = -\partial_j(hnu_k) + \partial_k(hnu_j). \quad (1.26)$$

THEOREM 1.1 (Theorem 2.2 of [48]). Assume that the initial datum

$$(n(0, x), u(0, x), T(0, x), E(0, x), B(0, x))$$

satisfies (1.26) and is sufficiently close to the equilibrium $(\bar{n}, \mathbf{0}, \bar{T}, \mathbf{0}, \mathbf{0})$ for some constants $\bar{n} > 0$ and $\bar{T}$. Then there exists a unique global solution

$$(n(t, x), u(t, x), T(t, x), E(t, x), B(t, x))$$

to the one-fluid relativistic Euler-Maxwell system (1.20) and (1.21) that satisfies (1.26) for any $t > 0$ and

$$\begin{aligned} & \sup_{t \in [0, \infty)} \|(n(t) - \bar{n}, u(t), T(t) - \bar{T}, E(t), B(t))\|_{H^{N_c}} \\ & + \sup_{t \in [0, \infty)} \sup_{|\rho| \leq \bar{N}} ((1+t)^{\beta_0} \|\nabla_x^\rho (n(t) - \bar{n}, u(t), T(t) - \bar{T}, E(t), B(t))\|_{L^\infty}) \lesssim \bar{\varepsilon}_0, \end{aligned} \quad (1.27)$$

where $N_c \in \mathbb{N}$ is a sufficiently large constant, $\bar{N} \geq 3$ is a constant, $\beta_0 = \frac{101}{100}$ and $\bar{\varepsilon}_0$ is a sufficiently small positive constant.

In this article, we rigorously prove that solutions of the r-VML system (1.17)–(1.18) converge to solutions of the relativistic Euler-Maxwell system (1.20)–(1.21) globally in time, as the Knudsen number ε tends to zero.

THEOREM 1.2. Assume that $(n(t, x), u(t, x), T(t, x), E(t, x), B(t, x))$ is the global solution constructed in Theorem 1.1 and $\mathbf{M}(t, x, p) = \frac{n}{4\pi m^2 c k_B T K_2(\gamma)} \exp\left\{\frac{u^\mu p_\mu}{k_B T}\right\}$ is the corresponding local Maxwellian. Then there exists an $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, $k \geq 3$, and $0 < t \leq \bar{t}$ with $\bar{t} = \varepsilon^{-1/3}$, the asymptotic expansion (2.3) holds. Moreover, if $F^\varepsilon(0, x, p) \geq 0$, and

$$\|\mathbf{M}^{-\frac{1}{2}}(F^\varepsilon - \mathbf{M})(0)\|_{H_x^2 L_v^2} + \|(E^\varepsilon - E)(0)\|_{H^2} + \|(B^\varepsilon - B)(0)\|_{H^2} = O(\varepsilon), \quad (1.28)$$

then $F^\varepsilon(t, x, p) \geq 0$ and

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq \tilde{t}} \left\{ \left\| \mathbf{M}^{-\frac{1}{2}} (F^\varepsilon - \mathbf{M})(t) \right\|_{H_x^2 L_v^2} + \left\| (E^\varepsilon - E)(t) \right\|_{H^2} + \left\| (B^\varepsilon - B)(t) \right\|_{H^2} \right\} = 0. \quad (1.29)$$

1.2. Relativistic Landau equation. When the effects of electromagnetic fields are negligible, the relativistic Landau (r-LAN) equation provides a much easier yet still accurate description of the dynamics of a fast moving dilute plasma when the grazing collisions between particles are predominant in the collisions.

Let $F^\varepsilon = F^\varepsilon(t, x, p)$ be the number density function for particles at the phase-space position $(x, p) = (x_1, x_2, x_3, p_1, p_2, p_3) \in \mathbb{R}^3 \times \mathbb{R}^3$, at time $t \in \mathbb{R}_+$. Then F^ε satisfies the r-LAN equation

$$\partial_t F^\varepsilon + c \hat{p} \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{C}[F^\varepsilon, F^\varepsilon], \quad (1.30)$$

where $\hat{p} = \frac{p}{p^0}$, $p^0 = \sqrt{m^2 c^2 + |p|^2}$ is the energy of the particle, constants c, m are the speed of light and the rest mass of a particle, respectively. $0 < \varepsilon \ll 1$ is the Knudsen number.

Similar to (1.4), the collision operator $\mathcal{C}$ yields the conservation laws

$$\begin{aligned} \frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^\varepsilon(t, x, p) dp dx &= \frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} p^0 F^\varepsilon(t, x, p) dp dx = 0, \\ \frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} p F^\varepsilon(t, x, p) dp dx &= \mathbf{0}. \end{aligned}$$

Corresponding to (1.30), at the hydrodynamic level, the plasma obeys the relativistic Euler equations for $(n(t, x), u(t, x), T(t, x)) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$:

$$\begin{cases} \frac{1}{c} \partial_t (n u^0) + \nabla_x \cdot (n u) = 0, \\ \frac{1}{c} \partial_t \left\{ (\mathfrak{e} + P) u^0 u \right\} + \nabla_x \cdot \left\{ (\mathfrak{e} + P) (u \otimes u) \right\} + c^2 \nabla_x P = \mathbf{0}, \\ \frac{1}{c} \partial_t \left\{ (\mathfrak{e} + P) (u^0)^2 - c^2 |u|^2 P \right\} + \nabla_x \cdot \left\{ (\mathfrak{e} + P) u^0 u \right\} = 0, \end{cases} \quad (1.31)$$

where n is the particle number density, $u = (u_1, u_2, u_3)$, $u^0 = \sqrt{|u|^2 + c^2}$. Here, P and $\mathfrak{e}$ are defined as in (1.22) and (1.23) (see [88]).

THEOREM 1.3 (Theorem 1 of [88]). Under proper regularity conditions, if the initial data $(n(0), u(0), T(0))$ is sufficiently close to an equilibrium state $(\bar{n}, 0, \bar{T})$: for $N \geq 3$

$$\| (n - \bar{n}, u, T - \bar{T})(0) \|_{H^N} \leq \delta \ll 1, \quad (1.32)$$

then there exists a unique solution (n, u, T) to (1.31) for $t \in [0, \tilde{t}]$ with $\tilde{t} \geq \bar{c} \delta^{-1}$ for some constant $\bar{c} > 0$ satisfying

$$\sup_{0 \leq t \leq \tilde{t}, x \in \mathbb{R}^3, 0 \leq \ell \leq N-2} \{ |\nabla_{t,x}^\ell (n, u, T)| \} \ll 1. \quad (1.33)$$

In this article, we rigorously prove that solutions of the relativistic Landau equation (1.30) converge to solutions of the relativistic Euler equations (1.31) locally in time, as the Knudsen number ε tends to zero.

THEOREM 1.4. Assume that $(n(t, x), u(t, x), T(t, x))$ is the solution constructed in Theorem 1.3 and $\mathbf{M}(t, x, p) = \frac{n}{4\pi m^2 c k_B T K_2(\gamma)} \exp\left\{\frac{u^\mu p_\mu}{k_B T}\right\}$ is the corresponding local Maxwellian. Then there exists an $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, $k \geq 3$, and $0 < t \leq t_0$ with some t_0 depending on $(n(t, x), u(t, x), T(t, x))$ but independent of ε , the asymptotic expansion (2.39) holds. Moreover, if $F^\varepsilon(0, x, p) \geq 0$, and

$$\left\| \mathbf{M}^{-\frac{1}{2}} (F^\varepsilon - \mathbf{M}) (0) \right\|_{H_x^2 L_v^2} = O(\varepsilon), \quad (1.34)$$

then $F^\varepsilon(t, x, p) \geq 0$ and

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq t_0} \left\{ \left\| \mathbf{M}^{-\frac{1}{2}} (F^\varepsilon - \mathbf{M}) (t) \right\|_{H_x^2 L_v^2} \right\} = 0. \quad (1.35)$$

1.3. *Background and literature.* As a key ingredient to attack the well-known Hilbert's sixth problem, the rigorous derivation of fluid equations (Euler equations or Navier-Stokes equations, etc.) from the kinetic equations (Boltzmann equation, Landau equation, etc.) has attracted a lot of attention since the early twentieth century. The fundamental problem is to justify the asymptotic limits of kinetic solutions as the Knudsen number (which measures the relative mean free path) or the Strouhal number (which measures the relative time-varying speed) shrinks to zero.

There are mainly two genres to study hydrodynamic limits: kinetic-based approach or fluid-based approach. We refer to [63, 67] for more details.

The kinetic-based approach purely relies on the solution theory (well-posedness, regularity, etc.) of the kinetic equations and does *not* assume any a priori properties of the fluid limits. On one hand, in the context of the renormalized solution and entropy method, there are successful applications of this approach to the incompressible Euler/Navier-Stokes limit. We refer to Bardos-Golse [6], Golse-Saint-Raymond [36, 37], Saint-Raymond [83], Masmoudi-Saint-Raymond [78], Arsénio-Saint-Raymond [5], Bardos-Golse-Levermore [7–9], Lions-Masmoudi [74] and Masmoudi [77]. Interested readers may refer to the books by Saint-Raymond [84] and by Golse [35], and the references therein provide a nice summary of the progress. On the other hand, in the context of classical solutions, we refer to Nishida [79], Bardos-Ukai [10], Briant [11], Briant-Merino-Aceituno-Mouhot [12].

The fluid-based approach *does* assume a priori that we have a well-prepared fluid system with a unique smooth solution. And then we will justify that the kinetic solution converges to this fluid solution. In some sense, this is essentially “fluid-to-kinetic” limit and we avoid the complications of possible fluid ill-posedness, like blow up or shock wave. This approach typically provides hydrodynamic limits in the stronger sense and utilizes the so-called Hilbert expansion techniques. In this paper, we will focus on the fluid-based approach and discuss the progress in detail.

The Hilbert expansion dates back to 1912 by Hilbert [55], who proposed an asymptotic expansion of the distribution function solving the Boltzmann equation with respect to the Knudsen number and formally derived the limiting compressible Euler equations. The similar formal expansion can be naturally extended to treat the Landau equation, and collisional kinetic equations coupled with Poisson equation or Maxwell system.

The first rigorous justification of the compressible Euler limit of the Boltzmann equation was due to Cagliostro [13]. Later, with the $L^2 - L^\infty$ framework introduced in Guo [44], Guo-Jang-Jiang [51] improved Cagliostro's result and removed the assumption on the initial data $F_R^\varepsilon(0, x, v) = 0$. This framework was extended to treat the Vlasov-Poisson-Boltzmann (VPB) system in Guo-Jang [50] and the relativistic Boltzmann (r-BOL) equation in Speck-Strain [88]. Recently, this framework was further developed to the investigation of the relativistic Vlasov-Maxwellian-Boltzmann (r-VMB) system in Guo-Xiao [54] and the Boltzmann equation with boundary conditions in half-space in Guo-Huang-Wang [46], Jiang-Luo-Tang [64, 65]. We also refer to Grad [38], Ukai-Asano [93], De Masi-Esposito-Lebowitz [14], and the recent work Jang-Kim [62] and Kim-La [68] for the incompressible Euler limit.

For the convergence of the Boltzmann equation to the basic waves of the Euler equations: the shock waves, rarefaction waves and contact discontinuity, the interested readers may refer to Huang-Wang-Yang [57–59], Xin-Zeng [103] and Yu [107].

As for the incompressible Navier-Stokes limit of the Boltzmann equation, there are too many references and we only list some closely related works. The early development tracks back to De Masi-Esposito-Lebowitz [14] in 2D. Then Guo [43] justified the diffusive limit in the periodic domain via the non-linear energy method. This result was extended to the whole space in Liu-Zhao [75], to more general initial data with initial layer in Jiang-Xiong [66], and to the Vlasov-Maxwell-Boltzmann (VMB) system in Jang [61]. See also the recent work Gallagher-Tristani [33]. We also mention the very recent work Duan-Yang-Yu [24] for the compressible Euler-Maxwell limit of the one-species VMB system.

For stationary Boltzmann equation and other settings, we refer to Di Meo-Esposito [16], Arkeryd-Esposito-Marra-Nouri [4], Esposito-Lebowitz-Marra [30], Esposito-Guo-Marra [29], Esposito-Guo-Kim-Marra [28], Wu [96], Wu-Ouyang [97–99].

Despite the fruitful progress in the hydrodynamic limits of the Boltzmann-type equations, there are very limited works in this direction for Landau-type equations. For Landau equation, we refer to Guo [43] for the incompressible Navier-Stokes limit, Duan-Yang-Yu [22, 25] for the rarefaction wave limit and compressible Euler limit, and the recent work Rachid [82]. As far as we are aware of, our paper is the first result to justify the Hilbert expansion for r-LAN equation and r-VML system.

As for the well-posedness issue for fixed Knudsen number and Strouhal number, there are a huge number of literature. We list some closely related to this article. For the r-VML system, we refer to Strain-Guo [90], Yu [106], Yang-Yu [105], Liu-Zhao [76] and Xiao [100]. For the r-LAN equation, we refer to Hsiao-Yu [56] and Yang-Yu [104]. We also mention Guo-Strain [53] and some works in the non-relativistic framework: Villani [94], Guo [40, 42, 45], Strain [89], Duan-Strain [21], Duan [18], Duan-Lei-Yang-Zhao [19], Guo-Hwang-Jang-Ouyang [47], Duan-Liu-Sakamoto-Strain [20] for Landau equation and Dong-Guo-Ouyang [17] for Vlasov-Poisson-Landau (VPL) system.

Finally, we record some significant progress on the compressible fluid system. Sideris [85] justified the classical result on the compressible Euler equation that the solution might blow up even if the initial datum is small and irrotational. However, as a key observation, the electric field or the electromagnetic fields might help stabilize the system.

Based on the Klein-Gordon effect, Guo [39] and Germain-Masmoudi [34] constructed global classical solutions to the one-fluid Euler-Poisson system and Euler-Maxwell system, respectively. Using the combination of normal-form method and vector-field method to capture the so-called “null structure”, Guo-Ionescu-Pausader [49] justified the global well-posedness of 3D two-fluid Euler-Maxwell system, and the similar results were extended to treat 3D Euler-Poisson system, and 3D one-fluid/two-fluid relativistic Euler-Maxwell system in Guo-Ionescu-Pausader [48], which plays a key role in our proof of the hydrodynamic limits (as in Theorem 1.1). The 2D case was justified in Deng-Ionescu-Pausader [15]. More recently, the one-fluid Euler-Maxwell system in 3D with non-vanishing vorticity was studied in Ionescu-Lie [60].

2. Formulation and discussion. Without loss of generality, from now on, we will take the constants $c = e = m = k_B = 2\pi \ln(\lambda) = 1$.

2.1. Hilbert expansion for the relativistic Vlasov-Maxwell-Landau system. In this subsection, we will provide the Hilbert expansion of the r-VML system (1.17) and (1.18), and introduce necessary notations.

We consider the Hilbert expansion with respect to small Knudsen number ε and $k \geq 2$:

$$F^\varepsilon = F + \sum_{n=1}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon, \quad E^\varepsilon = E + \sum_{n=1}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_R^\varepsilon, \quad B^\varepsilon = B + \sum_{n=1}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_R^\varepsilon. \quad (2.1)$$

To determine the coefficients $F_n(t, x, p)$, $E_n(t, x)$, $B_n(t, x)$ for $0 \leq n \leq 2k-1$, we plug the formal expansions (2.1) into equations (1.17)–(1.18) and equate the coefficients on both sides in front of different powers of the parameter ε to obtain:

ε^{-1} -**Order**:

$$\mathcal{C}[F, F] = 0. \quad (2.2)$$

ε^0 -**Order**:

$$\partial_t F + \hat{p} \cdot \nabla_x F - (E + \hat{p} \times B) \cdot \nabla_p F = \mathcal{C}[F_1, F] + \mathcal{C}[F, F_1], \quad (2.3)$$

and

$$\begin{cases} \partial_t E - \nabla_x \times B = 4\pi \int_{\mathbb{R}^3} \hat{p} F dp, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = 4\pi \left(\bar{n} - \int_{\mathbb{R}^3} F dp \right), \\ \nabla_x \cdot B = 0. \end{cases} \quad (2.4)$$

ε^n -**Order** ($1 \leq n \leq 2k-1$):

$$\begin{aligned} & \partial_t F_n + \hat{p} \cdot \nabla_x F_n - (E_n + \hat{p} \times B_n) \cdot \nabla_p F - (E + \hat{p} \times B) \cdot \nabla_p F_n \\ &= \sum_{\substack{i+j=n+1 \\ i,j \geq 0}} \mathcal{C}[F_i, F_j] + \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + \hat{p} \times B_i) \cdot \nabla_p F_j, \end{aligned} \quad (2.5)$$

and

$$\begin{cases} \partial_t E_n - \nabla_x \times B_n = 4\pi \int_{\mathbb{R}^3} \hat{p} F_n dp, \\ \partial_t B_n + \nabla_x \times E_n = 0, \\ \nabla_x \cdot E_n = -4\pi \int_{\mathbb{R}^3} F_n dp, \\ \nabla_x \cdot B_n = 0. \end{cases} \quad (2.6)$$

Remainder equation: The remainder $(F_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$ satisfies

$$\begin{aligned} & \partial_t F_R^\varepsilon + \hat{p} \cdot \nabla_x F_R^\varepsilon - (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p F - (E + \hat{p} \times B) \cdot \nabla_p F_R^\varepsilon \\ & - \frac{1}{\varepsilon} \left\{ \mathcal{C}[F_R^\varepsilon, F] + \mathcal{C}[F, F_R^\varepsilon] \right\} \\ & = \varepsilon^{k-1} \mathcal{C}[F_R^\varepsilon, F_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \mathcal{C}[F_R^\varepsilon, F_i] + \mathcal{C}[F_i, F_R^\varepsilon] \right\} + \varepsilon^k (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p F_R^\varepsilon \\ & + \sum_{i=1}^{2k-1} \varepsilon^i \left\{ (E_i + \hat{p} \times B_i) \cdot \nabla_p F_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p F_i \right\} + \varepsilon^k S, \end{aligned} \quad (2.7)$$

and

$$\begin{cases} \partial_t E_R^\varepsilon - \nabla_x \times B_R^\varepsilon = 4\pi \int_{\mathbb{R}^3} \hat{p} F_R^\varepsilon dp, \\ \partial_t B_R^\varepsilon + \nabla_x \times E_R^\varepsilon = 0, \\ \nabla_x \cdot E_R^\varepsilon = -4\pi \int_{\mathbb{R}^3} F_R^\varepsilon dp, \\ \nabla_x \cdot B_R^\varepsilon = 0, \end{cases} \quad (2.8)$$

where

$$S = \sum_{\substack{i+j \geq 2k+1 \\ 2 \leq i, j \leq 2k-1}} \varepsilon^{i+j-2k-1} \left\{ \mathcal{C}[F_i, F_j] + \mathcal{C}[F_i, F_j] \right\} + \sum_{\substack{i+j \geq 2k \\ 1 \leq i, j \leq 2k-1}} \varepsilon^{i+j-2k} (E_i + \hat{p} \times B_i) \cdot \nabla_p F_j. \quad (2.9)$$

From (2.3), we conclude that F should be local Maxwellians:

$$F(t, x, p) = \mathbf{M} = \frac{n}{4\pi T K_2(\gamma)} \exp \left\{ \frac{u^\mu p_\mu}{T} \right\}, \quad (2.10)$$

where (n, u, T) is part of the solution to the relativistic Euler-Maxwell system (1.20). The other coefficients $F_n(t, x, p)$, $E_n(t, x)$, $B_n(t, x)$ for $0 \leq n \leq 2k-1$ can be derived in an inductive way (see Appendix A).

To prove Theorem 1.2, our main task is to solve (2.7)–(2.8). Define f_R^ε as

$$F_R^\varepsilon := \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon. \quad (2.11)$$

(2.7) and (2.8) can be rewritten as

$$\begin{aligned}
& \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} f_R^\varepsilon + \frac{u^0}{T} \hat{p} \mathbf{M}^{\frac{1}{2}} \cdot E_R^\varepsilon \\
& - \frac{u}{T} \mathbf{M}^{\frac{1}{2}} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) + \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon] \\
& = -f_R^\varepsilon \mathbf{M}^{-\frac{1}{2}} \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} + \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] \\
& + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right] \right\} + \varepsilon^k (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon \\
& - \varepsilon^k \frac{1}{2T} (u^0 \hat{p} - u) \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) f_R^\varepsilon \\
& + \sum_{i=1}^{2k-1} \varepsilon^i \left\{ (E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right\} \\
& - \sum_{i=1}^{2k-1} \varepsilon^i \left\{ (E_i + \hat{p} \times B_i) \cdot \frac{1}{2T} (u^0 \hat{p} - u) f_R^\varepsilon \right\} + \varepsilon^k \overline{S},
\end{aligned} \tag{2.12}$$

and

$$\begin{cases} \partial_t E_R^\varepsilon - \nabla_x \times B_R^\varepsilon = 4\pi \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon dp, \\ \partial_t B_R^\varepsilon + \nabla_x \times E_R^\varepsilon = 0, \\ \nabla_x \cdot E_R^\varepsilon = -4\pi \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon dp, \\ \nabla_x \cdot B_R^\varepsilon = 0. \end{cases} \tag{2.13}$$

Here $\overline{S} = \mathbf{M}^{-\frac{1}{2}} S$. The linearized collision operator $\mathcal{L}[f]$ and non-linear collision operator $\Gamma[f, g]$ are defined as follows:

$$\mathcal{L}[f] := \mathbf{M}^{-\frac{1}{2}} \left\{ \mathcal{C} \left[\mathbf{M}^{\frac{1}{2}} f, \mathbf{M} \right] + \mathcal{C} \left[\mathbf{M}, \mathbf{M}^{\frac{1}{2}} f \right] \right\} =: -\mathcal{A}[f] - \mathcal{K}[f], \tag{2.14}$$

and

$$\Gamma[f, g] := \mathbf{M}^{-\frac{1}{2}} \mathcal{C} \left[\mathbf{M}^{\frac{1}{2}} f, \mathbf{M}^{\frac{1}{2}} g \right]. \tag{2.15}$$

Note that the null space of the linearized operator $\mathcal{L}$ is given by

$$\mathcal{N} = \text{span} \left\{ \mathbf{M}^{\frac{1}{2}}, p_i \mathbf{M}^{\frac{1}{2}} (1 \leq i \leq 3), p^0 \mathbf{M}^{\frac{1}{2}} \right\}. \tag{2.16}$$

Denote $\mathbf{P}$ as the orthogonal projection from L_p^2 onto $\mathcal{N}$:

$$\mathbf{P}[f] = \left(a_f - \frac{\rho_2}{\rho_1} c_f \right) \mathbf{M}^{\frac{1}{2}} + b_f \cdot p \mathbf{M}^{\frac{1}{2}} + c_f p^0 \mathbf{M}^{\frac{1}{2}}, \tag{2.17}$$

where a_f , b_f and c_f are coefficients which will be written as a, b, c when there is no confusion, and

$$\rho_1 := \int_{\mathbb{R}^3} \mathbf{M} dp = nu^0, \quad \rho_2 := \int_{\mathbb{R}^3} p^0 \mathbf{M} dp = \epsilon(u^0)^2 + P|u|^2. \tag{2.18}$$

2.1.1. *Notation and convention.* Throughout the paper, C denotes a generic positive constant which may change line by line. The notation $A \lesssim B$ implies that there exists a positive constant C such that $A \leq CB$ holds uniformly over the range of parameters. The notation $A \approx B$ means $\frac{1}{C}A \leq B \leq CA$ for some constant $C > 1$.

Let $(\cdot, \cdot)$ denote the L^2 inner product in $p \in \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ the L^2 inner product in $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$(f, g) = \int_{\mathbb{R}_p^3} f g \, dp, \quad (2.19)$$

$$\langle f, g \rangle = \iint_{\mathbb{R}_x^3 \times \mathbb{R}_p^3} f g \, dp \, dx. \quad (2.20)$$

Let $|\cdot|_{L^2}$ denote the L^2 norm in $p \in \mathbb{R}^3$ and $\|\cdot\|$ the L^2 norm in $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$|f|_{L^2}^2 = (f, f), \quad \|f\|^2 = \langle f, f \rangle. \quad (2.21)$$

Note that for quantities related to E or B which do not depend on p , we also use $\|\cdot\|$ to denote the L^2 norm in $x \in \mathbb{R}^3$. Similarly, for $s = 0, 1, 2$, we define the Sobolev norms

$$\|f\|_{H^s}^2 = \sum_{|\alpha|=0}^s \iint_{\mathbb{R}_x^3 \times \mathbb{R}_p^3} |\partial_x^\alpha f|^2 \, dp \, dx, \quad (2.22)$$

where $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For $\mathbf{M}$ given in (2.10), we denote $(n, u, T)(t, x)$ as part of a solution to the relativistic Euler-Maxwell system (1.20)–(1.21), which was constructed in [48], and define the following 3×3 matrix-type collision frequency:

$$\sigma^{ij}(p) = \int_{\mathbb{R}^3} \Phi^{ij} \mathbf{M}(q) \, dq.$$

To measure the dissipation of the linearized relativistic Landau collision operator, we define the inner product:

$$(f, g)_\sigma = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \sigma^{ij} \partial_{p_i} f \partial_{p_j} g \, dp + \sum_{i,j=1}^3 \frac{1}{4T^2} \int_{\mathbb{R}^3} \sigma^{ij} \frac{p_i}{p^0} \frac{p_j}{p^0} f g \, dp. \quad (2.23)$$

Denote the corresponding σ norms:

$$|f|_\sigma^2 = (f, f)_\sigma, \quad \|f\|_\sigma^2 = \int_{\mathbb{R}^3} |f(x)|_\sigma^2 \, dx. \quad (2.24)$$

Similarly, for $s = 0, 1, 2$, we define the Sobolev σ norms

$$\|f\|_{H_\sigma^s}^2 = \sum_{|\alpha|=0}^s \int_{\mathbb{R}_x^3} |\partial_x^\alpha f|_\sigma^2 \, dx. \quad (2.25)$$

REMARK 2.1. Notice that all eigenvalues of $\sigma^{ij}(p)$ are positive and depend on p . Moreover, according to [73], the eigenvalues converge to positive constants as $|p| \rightarrow \infty$. Then, for the $|\cdot|_\sigma$ norm defined above, we have

$$\frac{1}{T} |f|_{L^2} + |\nabla_p f|_{L^2} \lesssim |f|_\sigma \lesssim \frac{1}{T} |f|_{L^2} + |\nabla_p f|_{L^2}. \quad (2.26)$$

Define the weight functions

$$w^\ell = (p^0)^{2(N_c - \ell)} \exp \left\{ \frac{p^0}{5 \ln(e + t) T_c} \right\}, \quad 0 \leq \ell \leq 2, \quad (2.27)$$

where N_c and T_c are constants satisfying $N_c \geq 3$ and

$$T_c \geq \sup_{t \in [0, \varepsilon^{-1/3}], x \in \mathbb{R}^3} T(t, x). \quad (2.28)$$

It should be pointed out that the weight functions in (2.27) are designed to make sure that

$$(w^\ell)^2 \mathbf{M}^{\frac{1}{2}} \lesssim e^{-c_0 p^0}, \quad (w^\ell)^2 (p^0)^{2\ell} \leq \frac{1}{2} \left((w^\ell)^2 + (w^0)^2 \right) \quad (2.29)$$

for some small constant $c_0 > 0$.

Correspondingly, define the weighted norms

$$\|f\|_{w^\ell} := \|w^\ell f\|, \quad \|f\|_{H_w^s} := \sum_{|\alpha|=0}^s \|w^{|\alpha|} \partial_x^\alpha f\|, \quad (2.30)$$

$$\|f\|_{w^\ell, \sigma} := \|w^\ell f\|_\sigma, \quad \|f\|_{H_{w, \sigma}^s} := \sum_{|\alpha|=0}^s \|w^{|\alpha|} \partial_x^\alpha f\|_\sigma. \quad (2.31)$$

Denote

$$\begin{aligned} W(t) &:= \exp \left(\frac{1}{5 \ln(e + t) T_c} \right), \quad Y(t) := -\frac{W'}{W} = \frac{1}{5 [\ln(e + t)]^2 (e + t) T_c}, \\ \mathcal{Z}(t) &:= \sup_{x \in \mathbb{R}^3, 0 \leq \ell \leq 2} \left\{ |\nabla_{t,x}^{1+\ell}(n, u, T)| + |\nabla_{t,x}^\ell(E, B)| + |\nabla_{t,x}^\ell u| \right\}. \end{aligned} \quad (2.32)$$

2.1.2. *Key proposition.* Theorem 1.2 follows naturally from Proposition 2.1.

PROPOSITION 2.1. Let $F^\varepsilon(0, x, p) \geq 0$. Assume that $(n(t, x), u(t, x), T(t, x), E(t, x), B(t, x))$ is the global solution constructed in Theorem 1.1. Then for $k \geq 3$ in the Hilbert expansion (2.1) and $f_R^\varepsilon = \mathbf{M}^{-\frac{1}{2}} F_R^\varepsilon$ defined in (2.11), there exists an $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ and $0 < t \leq \bar{t}$ with $\bar{t} = \varepsilon^{-1/3}$, if

$$\mathcal{E}(0) \lesssim 1, \quad (2.33)$$

(2.12) and (2.13) admit a unique solution $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$ satisfying $F^\varepsilon(t, x, p) \geq 0$ and

$$\sup_{0 \leq t \leq \bar{t}} \mathcal{E}(t) + \int_0^{\bar{t}} \mathcal{D}(s) ds \lesssim \mathcal{E}(0) + 1, \quad (2.34)$$

where

$$\begin{aligned} \mathcal{E} \simeq & \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 \right) \\ & + \varepsilon \left(\|\nabla_x f_R^\varepsilon\|^2 + \|\nabla_x E_R^\varepsilon\|^2 + \|\nabla_x B_R^\varepsilon\|^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 \right) \\ & + \varepsilon^2 \left(\|\nabla_x^2 f_R^\varepsilon\|^2 + \|\nabla_x^2 E_R^\varepsilon\|^2 + \|\nabla_x^2 B_R^\varepsilon\|^2 + \varepsilon \|\nabla_x^2 f_R^\varepsilon\|_{w^2}^2 \right), \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} \mathcal{D} \simeq & \left(\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + Y \left\| \sqrt{p^0} (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 \right) \\ & + \left(\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 \right. \\ & \quad \left. + \varepsilon Y \left\| \sqrt{p^0} \nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 \right) \\ & + \left(\varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 + \varepsilon \|(\mathbf{I} - \mathbf{P})[\nabla_x^2 f_R^\varepsilon]\|_\sigma^2 + \varepsilon^2 \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 \right. \\ & \quad \left. + \varepsilon^3 Y \left\| \sqrt{p^0} \nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^2}^2 \right). \end{aligned} \quad (2.36)$$

REMARK 2.2. By (2.1), the estimates (A.7) of the coefficients F_n, E_n, B_n with $(1 \leq n \leq 2k-1)$ in Proposition A.1, and (2.34), we can obtain (1.29).

REMARK 2.3. In this paper, we will focus on deriving the a priori estimate (2.34) in Proposition 2.1. Then Theorem 1.2 naturally follows from a standard iteration/fixed-point argument. Based on the continuity argument (see [91]), from now on, we will assume that

$$\sup_{0 \leq t \leq \bar{t}} \mathcal{E}(t) \lesssim \varepsilon^{-\frac{1}{2}}, \quad (2.37)$$

and try to derive (2.34). Here we point out that the exponent $\frac{1}{2}$ in (2.37) can be replaced by any small positive constant and we choose it explicitly as $\frac{1}{2}$ simply for convenience of computation.

REMARK 2.4. The irrotational assumption (1.26) is necessary in the global well-posedness of Euler-Maxwell equation in [49]. Our proof does not rely on the irrotational assumption. Actually, as long as the fluid equation is well-posed and the solution enjoys proper time decay, our method should be able to justify the convergence.

2.2. *Hilbert expansion for the relativistic Landau equation.* In this subsection, we will derive the Hilbert expansion of the r-LAN equation (1.30), and introduce necessary notations.

We consider the Hilbert expansion for small Knudsen number ε ,

$$F^\varepsilon(t, x, p) := F + \sum_{n=1}^{2k-1} \varepsilon^n F_n(t, x, p) + \varepsilon^k F_R^\varepsilon(t, x, p), \quad (2.38)$$

for some $k \geq 2$. To determine the coefficients $F_n(t, x, p)$, we plug (2.38) into (1.30) and equate the coefficients on both sides of equation in front of different powers of the

parameter ε to obtain:

$$\begin{aligned}
 \varepsilon^{-1} : \quad & \mathcal{C}[F, F] = 0, \\
 \varepsilon^0 : \quad & \partial_t F + \hat{p} \cdot \nabla_x F = \mathcal{C}[F_1, F] + \mathcal{C}[F, F_1], \\
 & \dots\dots\dots \\
 \varepsilon^n : \quad & \partial_t F_n + \hat{p} \cdot \nabla_x F_n = \sum_{\substack{i+j=n+1 \\ i,j \geq 0}} \mathcal{C}[F_i, F_j], \\
 & \dots\dots\dots \\
 \varepsilon^{2k-1} : \quad & \partial_t F_{2k-1} + \hat{p} \cdot \nabla_x F_{2k-1} = \sum_{\substack{i+j=2k \\ i,j \geq 2}} \mathcal{C}[F_i, F_j].
 \end{aligned} \tag{2.39}$$

The remainder term F_R^ε satisfies the following equation:

$$\begin{aligned}
 & \partial_t F_R^\varepsilon + \hat{p} \cdot \nabla_x F_R^\varepsilon - \frac{1}{\varepsilon} \left\{ \mathcal{C}[F_R^\varepsilon, F] + \mathcal{C}[F, F_R^\varepsilon] \right\} \\
 &= \varepsilon^{k-1} \mathcal{C}[F_R^\varepsilon, F_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \mathcal{C}[F_i, F_R^\varepsilon] + \mathcal{C}[F_R^\varepsilon, F_i] \right\} + S,
 \end{aligned} \tag{2.40}$$

where

$$S := \sum_{\substack{i+j \geq 2k+1 \\ 2 \leq i, j \leq 2k-1}} \varepsilon^{i+j-k} \mathcal{C}[F_i, F_j]. \tag{2.41}$$

From the first equation in (2.39), we can obtain that F should be a local Maxwellian:

$$F(t, x, p) = \mathbf{M}(t, x, p) := \frac{n}{4\pi T K_2(\gamma)} \exp \left\{ \frac{u^\mu p_\mu}{T} \right\}, \tag{2.42}$$

where $(n, u, T)(t, x)$ is a solution to the relativistic Euler equations (1.31).

We define f_R^ε as

$$F_R^\varepsilon(t, x, p) := \mathbf{M}^{\frac{1}{2}}(t, x, p) f_R^\varepsilon(t, x, p). \tag{2.43}$$

Then the remainder equation (2.40) can be rewritten as

$$\begin{aligned}
 & \partial_t f_R^\varepsilon + \hat{p} \cdot \nabla_x f_R^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon] \\
 &= \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon] + \Gamma[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i] \right\} \\
 & \quad - \mathbf{M}^{-\frac{1}{2}} (\partial_t \mathbf{M}^{\frac{1}{2}} + \hat{p} \cdot \nabla_x \mathbf{M}^{\frac{1}{2}}) f_R^\varepsilon + \overline{S},
 \end{aligned} \tag{2.44}$$

where $\overline{S} := \mathbf{M}^{-\frac{1}{2}} S$.

2.2.1. Notation and convention. The notation here is mostly similar to that in Section 2.1.1. Define the weights

$$w^\ell := (p^0)^{2(N_c - \ell)} \exp \left(\frac{p^0}{5 \ln(e + t) T_c} \right), \quad 0 \leq \ell \leq 2, \tag{2.45}$$

where $N_c \geq 3$ is a constant and T_c is a constant satisfying

$$T_c \geq \sup_{t \in [0, t_0], x \in \mathbb{R}^3} T(t, x), \quad (2.46)$$

where t_0 satisfies (2.49).

For the classical solution $(n(t, x), u(t, x), T(t, x))$ to the relativistic Euler equations (1.31), denote W and Y as in (2.32) and

$$\begin{aligned} Z &:= \sup_{0 \leq t \leq t_0, x \in \mathbb{R}^3} \left\{ |\nabla_{t,x}(n, u, T)| \frac{(1+T)u^0}{T^2} \right\}, \\ \mathcal{Z} &:= \sup_{0 \leq t \leq t_0, x \in \mathbb{R}^3, 1 \leq \ell \leq 3} \{ |\nabla_{t,x}^\ell(n, u, T)| \}. \end{aligned}$$

2.2.2. Key proposition. Theorem 1.4 follows naturally from the Proposition 2.2.

PROPOSITION 2.2. Let $F^\varepsilon(0, x, p) \geq 0$, and let $F = \mathbf{M}$ as in (2.42). Assume $(n(t, x), u(t, x), T(t, x))$ is a sufficiently small solution to the relativistic Euler equations (1.31) satisfying

$$\mathcal{Z} \ll 1, \quad (2.47)$$

and

$$Z < \infty, \quad (2.48)$$

where $t_0 > 0$ fulfills

$$\frac{1}{10T_c(e + t_0)(\ln(e + t_0))^2} \geq Z \quad (2.49)$$

for T_c defined in (2.46). Then the Hilbert expansion (2.38) with $F_n, 1 \leq n \leq 2k-1$, defined in (B.1) holds for $k \geq 3$, and for the remainder $f_R^\varepsilon = \mathbf{M}^{-\frac{1}{2}} F_R^\varepsilon$ satisfying (2.44), there exists a constant $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq t_0$, if

$$\mathcal{E}(0) \lesssim 1, \quad (2.50)$$

then there exists a solution $F^\varepsilon(t, x, p) \geq 0$ to (1.30) satisfying

$$\sup_{0 \leq t \leq t_0} \mathcal{E}(t) + \int_0^t \mathcal{D}(s) ds \leq \mathcal{E}(0) + \varepsilon^{2k+3}, \quad (2.51)$$

where

$$\begin{aligned} \mathcal{E} &\sim \left(\|f_R^\varepsilon\|^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 \right) \\ &\quad + \varepsilon \left(\|\nabla_x f_R^\varepsilon\|^2 + \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 \right) \\ &\quad + \varepsilon^2 \left(\|\nabla_x^2 f_R^\varepsilon\|^2 + \varepsilon \|\nabla_x^2 f_R^\varepsilon\|_{w^2}^2 \right), \end{aligned} \quad (2.52)$$

and

$$\begin{aligned} \mathcal{D} \sim & \left(\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + Y \left\| \sqrt{p^0}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 \right) \\ & + \left(\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 \right. \\ & \quad \left. + \varepsilon Y \left\| \sqrt{p^0} \nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 \right) \\ & + \left(\varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 + \varepsilon \|\nabla_x^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^2 \|\nabla_x^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 \right. \\ & \quad \left. + \varepsilon^3 Y \left\| \sqrt{p^0} \nabla_x^2 f_R^\varepsilon \right\|_{w^2}^2 \right). \end{aligned} \quad (2.53)$$

REMARK 2.5. Theorem 1.3 and the additional assumption (2.49) actually dictate that for $0 \leq t \leq t_0$

$$Z \leq \frac{1}{2}Y \quad \text{and} \quad \mathcal{Z} \ll 1. \quad (2.54)$$

This will play a key role in the energy estimates, since the solution to Euler equations does not have time decay.

REMARK 2.6. In this paper, we will focus on deriving the a priori estimate (2.51). Then Proposition 2.2 naturally follows from a standard iteration/fixed-point argument. Based on the continuity argument (see [91]), for the energy estimates in Section 8, we will assume that

$$\sup_{0 \leq t \leq t_0} \mathcal{E}(t) \lesssim \varepsilon^{-\frac{1}{2}}, \quad (2.55)$$

and try to derive (2.51). Then in Section 8.4, we will in turn verify the validity of (2.55) with the help of (2.51).

2.3. Technical overview. In this paper, we will develop a new time-dependent energy method to study the Hilbert expansion of the Landau-type equation in the relativistic framework, which combines our preprints [80, 81]. This is inspired by Caglioli's pioneering work [13]. It is well known that in the study of the Hilbert expansion of the Boltzmann/Landau-type equation, the main task is to solve the remainder term $F_R^\varepsilon = \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon$, and one of the most challenging difficulties is from the linear term with one power moment growth

$$\mathbf{M}^{-\frac{1}{2}}(t, x, v) f_R^\varepsilon(t, x, p) \left\{ \partial_t + \hat{p} \cdot \nabla_x \right\} \mathbf{M}^{\frac{1}{2}}(t, x, p) \quad (2.56)$$

in the relativistic frame, or the linear term with cubic velocity growth

$$\mathbf{M}^{-\frac{1}{2}}(t, x, v) f_R^\varepsilon(t, x, v) \left\{ \partial_t + v \cdot \nabla_x \right\} \mathbf{M}^{\frac{1}{2}}(t, x, v) \quad (2.57)$$

in the non-relativistic one.

To tame the velocity growth, Caglioli decomposed the remainder $F_R^\varepsilon(t, x, v)$ into low- and high-velocity parts, which satisfy a coupled system and can be separately estimated via a weighted energy method.

This approach motivates us to design a time-dependent weight function

$$w(t, x) = \exp \left\{ \frac{p^0}{5 \ln(e+t)T_c} \right\}. \quad (2.58)$$

Then this exponential momentum function generates an additional dissipation term to control the moment growth terms in (2.56)

$$w^2 f_R^\varepsilon \partial_t f_R^\varepsilon = \frac{1}{2} \frac{d}{dt} (w f_R^\varepsilon)^2 + \frac{1}{5(e+t)[\ln(e+t)]^2 T_c} p^0 (w f_R^\varepsilon)^2. \quad (2.59)$$

Correspondingly, the troublesome term in (2.56) is roughly

$$w^2 f_R^\varepsilon \mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x \right\} \mathbf{M}^{\frac{1}{2}} \leq \left\{ |\nabla_{t,x}(n, u, T)| \frac{(1+T)u^0}{T^2} \right\} p^0 (w f_R^\varepsilon)^2. \quad (2.60)$$

As long as

$$\left\{ |\nabla_{t,x}(n, u, T)| \frac{(1+T)u^0}{T^2} \right\} < \frac{1}{5(e+t)[\ln(e+t)]^2 T_c} \quad (2.61)$$

holds for t, x under consideration, we can suppress the momentum growth in (2.56). Therefore, if $|\nabla_{t,x}(n, u, T)|$ is sufficiently small for all $x \in \mathbb{R}^3$, (2.61) holds locally in time; if $|\nabla_{t,x}(n, u, T)|$ further enjoys suitably fast time decay, (2.61) holds globally in time.

Since 1980s, time-dependent exponential weight functions have been widely used in the study of the collisional kinetic equations. In 1986, Ukai [92] introduced a weight function $w(t, v) \approx \exp \{(\alpha - \kappa t)(1 + |v|^2)\}$ with $\alpha, \kappa > 0, t \in [0, \frac{1}{2}\alpha\kappa^{-1}]$ to study the local well-posedness of the cutoff Boltzmann equation. Later, this technique was extended by AMUXY [1–3] for constructing local solutions to the non-cutoff Boltzmann equation in Sobolev spaces. In these works, the weight function provides an extra gain of velocity weight at the expense of the loss of the decay in the time-dependent Maxwellian.

In the exploration of global classical solutions to the one-species VPB system for cutoff hard potentials and moderately soft potentials, to control the large velocity growth in the non-linear term due to the Coulomb force, Duan-Yang-Zhao [26, 27] introduced another type of weight function $w(t, v) \approx \exp \left\{ \frac{\lambda(1+|v|^2)}{(1+t)^\vartheta} \right\}$, where λ, ϑ are small positive constants. By introducing a new time weighted energy framework, Xiao-Xiong-Zhao [101, 102] removed the so-called neutral condition assumption on the initial datum in previous work [27], and extended this well-posedness result to the very soft potentials case. We point out that the non-linear energy method and macro-micro decomposition technique employed in [101, 102] play an essential role in the proof of the main results of this paper. Recently, such techniques were further applied in constructing global classical solutions to the cutoff VMB system, non-cutoff VMB system, and VML system [18, 19, 31, 72, 95].

More recently, a new weight function $w(t, v) \approx \exp \left\{ (q_1 - q_2 \int_0^t q_3(s) ds)(1 + |v|)^2 \right\}$, with constants $q_1, q_2 > 0$ and q_3 being a dissipation energy functional, was used in Duan-Yang-Yu [23] to justify the asymptotic convergence in Landau equation.

Technically, the introduction of weight function w in (2.58) brings multi-level complications. In order to handle the non-linear term Γ , we have to control L^∞ norm of f_R^ε , which in turn requires spatial regularity up to H^2 . The more derivatives hit $\mathbf{M}$, the more p^0 will be generated. Hence, we have to carefully design a hierarchy of weighted functions w^ℓ to control all kinds of interactions and non-linear terms in the energy-dissipation

structure. In particular, T_c satisfies (2.28) and (2.46) so that

$$\frac{2p^0}{5\ln(e+t)T_c} + \frac{p^\mu u_\mu}{2T} < 0,$$

due to the smallness of u . This yields that for some constant $c_0 > 0$

$$w^{2\ell} \mathbf{M}^{\frac{1}{2}} \lesssim e^{-c_0 p^0},$$

which helps to control the cross terms with both $w^\ell \mathbf{M}^{\frac{1}{2}}$ and polynomial growth in p^0 .

Nevertheless, this hierarchy of weighted energy method produces new difficulties, especially from the linear collision operator term $\varepsilon^{-1} \mathcal{L}[f_R^\varepsilon]$ and the macroscopic part $\mathbf{P}[f_R^\varepsilon]$.

On the one hand, the linear collision operator term $\varepsilon^{-1} \mathcal{L}[f_R^\varepsilon]$ and $\mathbf{P}[f_R^\varepsilon]$ do not commute with the spatial derivative operator ∇_x , and thus we have to bound the commutator $[\mathcal{L}, \nabla_x]$. This difficulty was also present in the third author's previous work with Guo [54]. In the derivative estimate, we have

$$\begin{aligned} \left\langle \frac{1}{\varepsilon} \nabla_x \mathcal{L}[f_R^\varepsilon], \nabla_x f_R^\varepsilon \right\rangle &= \left\langle \frac{1}{\varepsilon} \mathcal{L}[\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], (\mathbf{I} - \mathbf{P}) \nabla_x [f_R^\varepsilon] \right\rangle \\ &\quad - \left\langle \frac{1}{\varepsilon} [\mathcal{L}, \nabla_x] (\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \nabla_x f_R^\varepsilon \right\rangle \\ &\geq \frac{\delta}{\varepsilon} \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \frac{CZ}{\varepsilon^2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - CZ \|f_R^\varepsilon\|_{H^1}^2. \end{aligned} \quad (2.62)$$

Noting that only $\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2$ is included in the no-weight energy estimate, this implies that we have to pay the cost of $\frac{1}{\varepsilon}$ for the derivative estimate $\|\nabla_x f_R^\varepsilon\|^2$. This is the very reason to include $\varepsilon \|\nabla_x f_R^\varepsilon\|^2$ and $\varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2$ in $\mathcal{E}(t)$.

On the other hand, in the weighted energy estimate

$$\begin{aligned} \left\langle \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], (w^0)^2 f_R^\varepsilon \right\rangle &= \left\langle \frac{1}{\varepsilon} \mathcal{L}[(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], (w^0)^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] + (w^0)^2 \mathbf{P}[f_R^\varepsilon] \right\rangle \\ &\geq \frac{\delta}{\varepsilon} \|w^0 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \frac{C}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \frac{C}{\varepsilon} \|w^0 \mathbf{P}[f_R^\varepsilon]\|_\sigma^2 \\ &\geq \frac{\delta}{\varepsilon} \|w^0 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \frac{C}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \frac{C}{\varepsilon} \|\mathbf{P}[f_R^\varepsilon]\|^2, \end{aligned} \quad (2.63)$$

while $\|\mathbf{P}[f_R^\varepsilon]\|^2$ cannot be controlled by the dissipation terms in $\mathcal{D}$. Noting that due to the Maxwellian in the macroscopic part $\mathbf{P}[f_R^\varepsilon]$, we naturally have $\|w^0 \mathbf{P}[f_R^\varepsilon]\| \lesssim \|f_R^\varepsilon\|$, and thus the weight function only takes effect for the microscopic part $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$. Therefore, we may first apply the microscopic projection $(\mathbf{I} - \mathbf{P})$ onto the f_R^ε equation (2.44), and directly estimate $w^0 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$ and $w^1 \nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$. Then the trouble term $C\varepsilon^{-1} \|\mathbf{P}[f_R^\varepsilon]\|^2$ in (2.63) would not appear.

However, this $(\mathbf{I} - \mathbf{P})$ projection in turn generates another commutator $[\mathbf{P}, \hat{p} \cdot \nabla_x]$, which may be controlled as

$$\left\langle [\mathbf{P}, \hat{p} \cdot \nabla_x] [f_R^\varepsilon], w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \lesssim \frac{1}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2. \quad (2.64)$$

The term $\varepsilon \|\nabla_x f_R^\varepsilon\|^2$ cannot be controlled by the dissipation term $\varepsilon \|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma^2$ in $\mathcal{D}$. Similar to (2.64), in the weighted first-order derivative estimate, linear term $\varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2$ arises. This reveals that the $(\mathbf{I} - \mathbf{P})$ projection argument requires estimate of one more derivative (e.g. in order to bound $w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$, we need the control of $\|\nabla_x f_R^\varepsilon\|$), and thus the microscopic projection cannot be applied to the highest-order derivatives. Hence, we have to directly perform the weighted energy estimate for $\nabla_x^2 f_R^\varepsilon$, which in turn calls for $\varepsilon^3 \|(w^2)^2 \nabla_x^2 f_R^\varepsilon\|^2$ in $\mathcal{E}(t)$ and leads to the trouble term $\varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|$ again.

To control the worrisome linear terms $\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|$ and $\varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|$, we need to capture the macroscopic structure of the remainder equations (2.12) and (2.44). The macroscopic dissipation estimates for $\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|$ and $\varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|$ are given in Section 6 and Section 8.3. Motivated by [43], we give the proof combining the local conservation laws and the macroscopic equations. We write the macroscopic quantities (2.17), and obtain the conservation law equations for $a^\varepsilon, b^\varepsilon, c^\varepsilon$. Although these equations are very complicated, we only need to focus on main terms corresponding to the global Maxwellian case as in [76] and [90] without the electromagnetic field.

For the r-VML system, we discovered a new phenomenon related to the dissipation of the electromagnetic field. Through an intricate analysis of the macroscopic variables $\|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2$, we conclude that $a^\varepsilon, E_R^\varepsilon$ and $\nabla_x a^\varepsilon, \nabla_x E_R^\varepsilon, \nabla_x B_R^\varepsilon$ belong to the dissipation $\mathcal{D}$. In the near-global-Maxwellian case (see [18, 105]), these dissipation terms are stronger than the energy $\mathcal{E}$. However, in our near-local-Maxwellian case, due to the Hilbert expansion, these dissipation terms are much weaker. Actually, we show that

$$\varepsilon \left(\|a^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 \right) + \varepsilon^2 \left(\|\nabla_x a^\varepsilon\|^2 + \|\nabla_x E_R^\varepsilon\|^2 + \|\nabla_x B_R^\varepsilon\|^2 \right) \leq \varepsilon \mathcal{E}, \quad (2.65)$$

which can be absorbed by $\mathcal{E}$ after integration w.r.t. time t for $t \in [0, \varepsilon^{-1/3}]$.

In addition, motivated by [41] and [47], we justify the positivity of the solution F^ε . We first perform a careful analysis of the construction of the initial data and prove that $F^\varepsilon(0) \geq 0$. Then by analyzing the elliptic structure of the relativistic Landau operator, we show the validity of maximum principle and conclude that $F^\varepsilon(t) \geq 0$ for all $t \geq 0$.

Compared with the L^2 - L^∞ framework as in [51, 52, 54], our new method has several advantages. Firstly, we don't require an explicit lower bound of the temperature T :

$$T_M < \max_{t,x} T(t, x) < 2T_M \quad (2.66)$$

for some constant $T_M > 0$. This extra restriction on T is a technical requirement in the L^2 - L^∞ method, and seems artificial from the physical viewpoint. Secondly, our method works for more general settings, including both the Landau-type and cutoff/non-cutoff Boltzmann-type equations in the relativistic frame. The L^2 - L^∞ framework heavily relies on the analysis of the characteristic, which fails for the presence of the diffusion effect.

Finally, we briefly discuss the possible applications of our new method. First, we are hopeful to apply this method to the relativistic non-cutoff Boltzmann equation. Due to the absence of momentum derivative estimate and weak dissipation of $\mathcal{L}$, the estimation of non-linear terms related to the electromagnetic field would be critical. Then we can make use of our time-dependent exponential weight function and ideas in [101, 102]. Second, it is also very interesting to extend our method to deal with the bounded domain problem. Recently, Duan-Liu-Sakamoto-Strain [20] proved the global existence of mild solutions to the non-relativistic Landau equation and non-cutoff Boltzmann equation for $x \in \mathbb{T}^3$ or $x \in (-1, 1) \times \mathbb{T}^2$ with boundary conditions. Further investigation of the corresponding Hilbert expansion should be a good future direction.

As a follow-up, recently the third author and collaborators have extended the techniques in this article to treat the non-relativistic Euler and Euler-Maxwell limits in [70, 71].

This paper is organized as follows: in Section 3, we will present some preliminary lemmas regarding the linear and non-linear relativistic Landau operators; in Sections 4–6, we will prove the a priori estimate for the no-weight and weighted estimates as well as the macroscopic estimates of the r-VML system; in Section 7, we justify Proposition 2.1; finally, in Section 8, we consider the r-LAN equation and prove Proposition 2.2.

3. Preliminaries. In this part, we will write down explicit forms of the operators $\mathcal{A}$, $\mathcal{K}$ and Γ and further prove the coercive estimate of the linear collision operator $\mathcal{L}$ and trilinear estimates about Γ for both the r-LAN equation and the r-VML system. Weighted energy estimates for these collision operators will also be established.

LEMMA 3.1. For the local Maxwellian $\mathbf{M}$, it holds that

$$\left| \mathbf{M}^{-\frac{1}{2}} \partial_t \mathbf{M}^{\frac{1}{2}} \right| + \left| \mathbf{M}^{-\frac{1}{2}} \nabla_x \mathbf{M}^{\frac{1}{2}} \right| \leq p^0 Z, \quad (3.1)$$

$$\left| \nabla_x^\ell \left(\mathbf{M}^{-\frac{1}{2}} \partial_t \mathbf{M}^{\frac{1}{2}} \right) \right| + \left| \nabla_x^\ell \left(\mathbf{M}^{-\frac{1}{2}} \nabla_x \mathbf{M}^{\frac{1}{2}} \right) \right| \lesssim \frac{1}{T^2} (p^0)^{\ell+1} Z, \quad \ell \geq 1. \quad (3.2)$$

Proof. Direct computation and the assumption (2.61) can justify this. $\square$

LEMMA 3.2. For the operators $\mathcal{A}$, $\mathcal{K}$ and Γ , we have

$$\mathcal{A}[f] = \partial_{p_i} (\sigma^{ij} \partial_{p_j} f) - \frac{\sigma^{ij}}{4T^2} (u^0 \hat{p}_i - u_i) (u^0 \hat{p}_j - u_j) f + \frac{1}{2T} \partial_{p_i} \left(\sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f, \quad (3.3)$$

$$\mathcal{K}[f] = \left(\partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2T} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) \left(\frac{-u^0 \hat{q}_j + u_j}{2T} f(q) - \partial_{q_j} f(q) \right) dq, \quad (3.4)$$

and

$$\Gamma[f, g] = \left(\partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2T} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq. \quad (3.5)$$

Proof. This corresponds to [90, Lemma 6]. We first prove (3.3). From the definition of the operator $\mathcal{A}$ and (1.6), we have

$$\begin{aligned}
 \mathcal{A}[f] &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \left(\partial_{p_j} [\mathbf{M}^{\frac{1}{2}} f](p) \mathbf{M}(q) - [\mathbf{M}^{\frac{1}{2}} f](p) \partial_{q_j} \mathbf{M}(q) \right) dq \quad (3.6) \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}(q) \left(\left(-\frac{u^0 \hat{p}_j - u_j}{2T} + \frac{u^0 \hat{q}_j - u_j}{T} \right) f(p) + \partial_{p_j} f(p) \right) dq \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}(q) \left(\frac{u^0 \hat{p}_j - u_j}{2T} f(p) + \partial_{p_j} f(p) \right) dq \\
 &= \partial_{p_i} \left(\sigma^{ij}(p) \left(\frac{u^0 \hat{p}_j - u_j}{2T} f(p) + \partial_{p_j} f(p) \right) \right) - \sigma^{ij}(p) \frac{u^0 \hat{p}_i - u_i}{2T} \left(\partial_{p_j} f(p) + \frac{u^0 \hat{p}_j - u_j}{2T} \right) \\
 &= \partial_{p_i} \left(\sigma^{ij} \partial_{p_j} f \right) - \frac{\sigma^{ij}}{4T^2} (u^0 \hat{p}_i - u_i) (u^0 \hat{p}_j - u_j) f + \frac{1}{2T} \partial_{p_i} \left(\sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f.
 \end{aligned}$$

Similarly, for (3.4), we can obtain that

$$\begin{aligned}
 \mathcal{K}[f] &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \left(\partial_{p_j} \mathbf{M}(p) [\mathbf{M}^{\frac{1}{2}} f](q) - \mathbf{M}(p) \partial_{q_j} [\mathbf{M}^{\frac{1}{2}} f](q) \right) dq \quad (3.7) \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) \mathbf{M}(p) \left(\left(-\frac{u^0 \hat{p}_j - u_j}{T} + \frac{u^0 \hat{q}_j - u_j}{2T} \right) f(q) - \partial_{q_j} f(q) \right) dq \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) \mathbf{M}(p) \left(\frac{-u^0 \hat{p}_j + u_j}{2T} f(q) - \partial_{q_j} f(q) \right) dq \\
 &= \left(\partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2T} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) \left(\frac{-u^0 \hat{q}_j + u_j}{2T} f(q) - \partial_{q_j} f(q) \right) dq.
 \end{aligned}$$

For (3.5), we use (1.6) again to have

$$\begin{aligned}
 \Gamma[f, g] &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \left(\partial_{p_j} [\mathbf{M}^{\frac{1}{2}} f](p) [\mathbf{M}^{\frac{1}{2}} g](q) \right. \quad (3.8) \\
 &\quad \left. - [\mathbf{M}^{\frac{1}{2}} f](p) \partial_{q_j} [\mathbf{M}^{\frac{1}{2}} g](q) \right) dq \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq \\
 &\quad + \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) f(p) g(q) \left(-\frac{u^0 \hat{p}_j - u_j}{2T} + \frac{u^0 \hat{q}_j - u_j}{2T} \right) dq \\
 &= \mathbf{M}^{-\frac{1}{2}}(p) \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq \\
 &= \left(\partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2T} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq.
 \end{aligned}$$

□

REMARK 3.1. From Lemma 3.2, we know that when taking x_i derivatives on $\mathcal{L}$ and Γ , although there will be p or q popped out from $\mathbf{M}$ and $\mathbf{M}^{\frac{1}{2}}$, they can be absorbed by $\mathbf{M}$ or $\mathbf{M}^{\frac{1}{2}}$.

LEMMA 3.3. The linearized collision operator $\mathcal{L}$ is self-adjoint in L^2 . It satisfies

$$(\mathcal{L}[f], f) \gtrsim |(\mathbf{I} - \mathbf{P})[f]|_\sigma^2. \quad (3.9)$$

Proof. Using Lemma 3.2, compared with [90, Lemma 6], for any large constant R , it holds that

$$(\mathcal{L}[f], f) \gtrsim |(\mathbf{I} - \mathbf{P})[f]|_{\tilde{\sigma}}^2, \quad (3.10)$$

where the norm $|\cdot|_{\tilde{\sigma}}$ is defined as

$$|f|_{\tilde{\sigma}}^2 = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \sigma^{ij} \partial_{p_i} f \partial_{p_j} f \, dp + \sum_{i,j=1}^3 \frac{1}{4T^2} \int_{\mathbb{R}^3} \sigma^{ij} (u^0 \hat{p}_i - u_i) (u^0 \hat{p}_j - u_j) |f|^2 \, dp. \quad (3.11)$$

Now we show the equivalence of the norm $|\cdot|_{\sigma}$ and $|\cdot|_{\tilde{\sigma}}$ under the smallness assumption of u . By the simple inequality

$$(A - B)^2 \geq \frac{1}{2} A^2 - B^2, \quad (3.12)$$

we have

$$\sum_{i,j=1}^3 \sigma^{ij} (u^0 \hat{p}_i - u_i) (u^0 \hat{p}_j - u_j) \geq \sum_{i,j=1}^3 \frac{1}{2} (u^0)^2 \sigma^{ij} \hat{p}_i \hat{p}_j - \sum_{i,j=1}^3 \sigma^{ij} u_i u_j. \quad (3.13)$$

Combining (3.11) and (3.13), we use the smallness assumption of u to obtain

$$\begin{aligned} |f|_{\tilde{\sigma}}^2 &\geq \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \sigma^{ij} \partial_{p_i} f \partial_{p_j} f \, dp + \sum_{i,j=1}^3 \frac{(u^0)^2}{8T^2} \int_{\mathbb{R}^3} \sigma^{ij} \hat{p}_i \hat{p}_j |f|^2 \, dp - \sum_{i,j=1}^3 \frac{1}{4T^2} \int_{\mathbb{R}^3} \sigma^{ij} u_i u_j |f|^2 \, dp \\ &\geq \frac{1}{8} |f|_{\sigma}^2 - \frac{C}{T^2} \|u\|_{L_{t,x}^{\infty}}^2 |f|_{L^2}^2 \gtrsim |\nabla_p f|_{L^2}^2 + |f|_{L^2}^2 \gtrsim |f|_{\sigma}^2. \end{aligned}$$

On the other hand, from (2.26), we have $|f|_{\sigma} \lesssim |f|_{\tilde{\sigma}}$. (3.9) follows from (3.10) and the above inequality. $\square$

LEMMA 3.4. The non-linear collision operator Γ satisfies

$$|(\Gamma[f, g] + \Gamma[g, f], h)| \lesssim (|f|_{L^2} |g|_{\sigma} + |g|_{L^2} |f|_{\sigma}) |(\mathbf{I} - \mathbf{P})[h]|_{\sigma}. \quad (3.14)$$

Moreover, for $|\alpha| \leq 2$, we have

$$\begin{aligned} &|\langle \partial_x^{\alpha} \Gamma[f, g] + \partial_x^{\alpha} \Gamma[g, f], \partial_x^{\alpha} h \rangle| \\ &\lesssim (\|f\|_{H^2} \|g\|_{H_{\sigma}^{|\alpha|}} + \|g\|_{H^2} \|f\|_{H_{\sigma}^{|\alpha|}}) [\|(\mathbf{I} - \mathbf{P})[\partial_x^{\alpha} h]\|_{\sigma} + \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|h\|_{H^{|\alpha|-1}}] \\ &\quad + \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 (\|f\|_{H^2} \|g\|_{H_{\sigma}^{|\alpha|-1}} + \|g\|_{H^2} \|f\|_{H_{\sigma}^{|\alpha|-1}}) \|\partial_x^{\alpha} h\|_{\sigma}. \end{aligned} \quad (3.15)$$

Here and below χ_{Ω} denotes the characteristic function on the set Ω .

Proof. The proof of (3.14) is similar to [90, Theorem 4], so we omit it here. Compared with [90, Theorem 4], we need the smallness of $\|u\|_{L_{t,x}^{\infty}}$ to handle the term $-\frac{u \hat{p}_i - u_i}{2T}$ in (3.5).

Denote

$$\Gamma_{\alpha_2}[f, g] = \left(\partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2T} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \partial_x^{\alpha_2} [\mathbf{M}^{\frac{1}{2}}(q)] \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq.$$

For (3.15), we use (3.14) and (1.27) to have

$$\begin{aligned}
& |\langle \partial_x^\alpha \Gamma[f, g] + \partial_x^\alpha \Gamma[g, f], \partial_x^\alpha h \rangle| \\
& \leq \sum_{\alpha_1 \leq \alpha} |\langle \Gamma[\partial_x^{\alpha_1} f, \partial_x^{\alpha-\alpha_1} g] + \Gamma[\partial_x^{\alpha_1} g, \partial_x^{\alpha-\alpha_1} f], \partial_x^\alpha h \rangle| \\
& \quad + \sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} |\langle \Gamma_{\alpha_2}[\partial_x^{\alpha_1} f, \partial_x^{\alpha-\alpha_1-\alpha_2} g] + \Gamma_{\alpha_2}[\partial_x^{\alpha_1} g, \partial_x^{\alpha-\alpha_1-\alpha_2} f], \partial_x^\alpha h \rangle|, \\
& \lesssim \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} (|\partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |\partial_x^{\alpha-\alpha_1} f|_\sigma) |(\mathbf{I} - \mathbf{P})[\partial_x^\alpha h]|_\sigma \\
& \quad + \sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} \int_{\mathbb{R}^3} (|\partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1-\alpha_2} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |\partial_x^{\alpha-\alpha_1-\alpha_2} f|_\sigma) |\partial_x^\alpha h|_\sigma.
\end{aligned} \tag{3.16}$$

Now we estimate the first term in (3.16). Note that by the algebra estimate for $H^2(\mathbb{R}_x^3)$,

$$\begin{aligned}
& \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} (|\partial_x^{\alpha_1} f|_{L^2}^2 |\partial_x^{\alpha-\alpha_1} g|_\sigma^2 + |\partial_x^{\alpha_1} g|_{L^2}^2 |\partial_x^{\alpha-\alpha_1} f|_\sigma^2) \\
& \lesssim \|f\|_{H^2}^2 \|g\|_{H_\sigma^{|\alpha|}}^2 + \|g\|_{H^2}^2 \|f\|_{H_\sigma^{|\alpha|}}^2,
\end{aligned} \tag{3.17}$$

and by (1.27),

$$\begin{aligned}
& |(\mathbf{I} - \mathbf{P})[\partial_x^\alpha h]|_\sigma \leq |\partial_x^\alpha (\mathbf{I} - \mathbf{P})[h]|_\sigma + \chi_{|\alpha| \geq 1} |[[\mathbf{P}, \partial_x^\alpha]] [h]|_\sigma \\
& \lesssim |\partial_x^\alpha (\mathbf{I} - \mathbf{P})[h]|_\sigma + \chi_{|\alpha| \geq 1} \mathcal{Z} \|\mathbf{P}[h]\|_{H_\sigma^{|\alpha|-1}} \\
& \lesssim |\partial_x^\alpha (\mathbf{I} - \mathbf{P})[h]|_\sigma + \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|h\|_{H^{|\alpha|-1}}.
\end{aligned} \tag{3.18}$$

Then we combine (3.17) and (3.18) to have the upper bound of the first term in (3.16)

$$\begin{aligned}
& \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} (|\partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |\partial_x^{\alpha-\alpha_1} f|_\sigma) |(\mathbf{I} - \mathbf{P})[\partial_x^\alpha h]|_\sigma \\
& \lesssim (\|f\|_{H^2} \|g\|_{H_\sigma^{|\alpha|}} + \|g\|_{H^2} \|f\|_{H_\sigma^{|\alpha|}}) [\|(\mathbf{I} - \mathbf{P})[\partial_x^\alpha h]\|_\sigma + \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|h\|_{H^{|\alpha|-1}}].
\end{aligned}$$

Similarly, for the second term in (3.16), we can obtain

$$\begin{aligned}
& \sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} \int_{\mathbb{R}^3} (|\partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1-\alpha_2} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |\partial_x^{\alpha-\alpha_1-\alpha_2} f|_\sigma) |\partial_x^\alpha h|_\sigma \\
& \lesssim \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 (\|f\|_{H^2} \|g\|_{H_\sigma^{|\alpha|-1}} + \|g\|_{H^2} \|f\|_{H_\sigma^{|\alpha|-1}}) \|\partial_x^\alpha h\|_\sigma.
\end{aligned}$$

We collect the above two estimates in (3.16) to derive (3.15). $\square$

LEMMA 3.5. For the weight functions defined in (2.27), it holds that

$$((w^\ell)^2 \mathcal{L}[f], f) \gtrsim |(w^\ell) f|_\sigma^2 - C |f|_\sigma^2.$$

Proof. We split $\mathcal{L}$ as $-\mathcal{A}$ and $-\mathcal{K}$ and use the expression of $\mathcal{A}$ in (3.3) to integrate by parts w.r.t. p to have

$$\begin{aligned}
 (\mathcal{L}[f], (w^\ell)^2 f) &= -(\partial_i(\sigma^{ij}\partial_j f), (w^\ell)^2 f) + \left(\frac{\sigma^{ij}}{4T^2} (u^0 \hat{p}_i - u_i)(u^0 \hat{p}_j - u_j) f, (w^\ell)^2 f \right) \\
 &\quad - \frac{1}{2T} \left(\partial_i \left(\sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f, (w^\ell)^2 f \right) - (\mathcal{K}[f], (w^\ell)^2 f) \\
 &= (\sigma^{ij}\partial_j f, (w^\ell)^2 \partial_i f) + \left(\frac{\sigma^{ij}}{4T^2} (u^0 \hat{p}_i - u_i)(u^0 \hat{p}_j - u_j) f, (w^\ell)^2 f \right) \\
 &\quad - \frac{1}{2T} \left(\partial_i \left(\sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f, (w^\ell)^2 f \right) \\
 &\quad - (\sigma^{ij}\partial_j f, \partial_i((w^\ell)^2 f)) - (\mathcal{K}[f], (w^\ell)^2 f).
 \end{aligned} \tag{3.19}$$

Now we estimate the terms in the R.H.S. of the second equal sign in (3.19). From [90, Lemma 5], we know

$$|\nabla_p^k \sigma^{ij}(p)| \lesssim (p^0)^{-k}, \tag{3.20}$$

for any integer $k \geq 0$. Then for any large constant R , we have

$$\begin{aligned}
 &\frac{1}{2T} \left| \left(\partial_i \left(\sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f, (w^\ell)^2 f \right) \right| \lesssim \frac{1}{2T} \int_{\mathbb{R}^3} \frac{(w^\ell)^2}{p^0} |f|^2 dp \\
 &= \frac{1}{2T} \int_{p^0 \leq R} \frac{(w^\ell)^2}{p^0} |f|^2 dp + \frac{1}{2T} \int_{p^0 > R} \frac{(w^\ell)^2}{p^0} |f|^2 dp \lesssim \frac{C_R}{T} |f|_{L^2}^2 + \frac{1}{RT} |(w^\ell)f|_{L^2}^2.
 \end{aligned} \tag{3.21}$$

Noting that

$$\partial_i((w^\ell)^2) = \frac{4(N_c - \ell)}{p^0} \hat{p}_i (w^\ell)^2 + \frac{2\hat{p}_i}{5T_c \ln(e+t)} (w^\ell)^2, \tag{3.22}$$

we use Cauchy's inequality to have

$$\begin{aligned}
 &|(\sigma^{ij}\partial_j f, \partial_i[(w^\ell)^2]f)| \\
 &\leq C \int_{\mathbb{R}^3} \frac{(w^\ell)^2}{p^0} (|\nabla_p f|^2 + |f|^2) dp + \left| \int_{\mathbb{R}^3} \frac{2(w^\ell)^2}{5T_c \ln(e+t)} (\sigma^{ij} \hat{p}_i \hat{p}_j |f|^2)^{\frac{1}{2}} (\sigma^{ij} \partial_i f \partial_j f)^{\frac{1}{2}} dp \right| \\
 &\leq C_R |f|_\sigma^2 + \frac{C}{R} |(w^\ell)f|_\sigma^2 + \frac{1}{2} \int_{\mathbb{R}^3} (w^\ell)^2 \sigma^{ij} \partial_i f \partial_j f dp \\
 &\quad + \frac{2}{25T_c^2 \ln^2(e+t)} \int_{\mathbb{R}^3} (w^\ell)^2 \sigma^{ij} \hat{p}_i \hat{p}_j |f|^2 dp.
 \end{aligned} \tag{3.23}$$

For the term $(\mathcal{K}[f], (w^\ell)^2 f)$, we integrate with respect to p and use (3.4) and (3.22) to have

$$\begin{aligned}
 |(\mathcal{K}[f], (w^\ell)^2 f)| &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\Phi(p, q)| \mathbf{M}^{\frac{1}{2}}(p) \mathbf{M}^{\frac{1}{2}}(q) (w^\ell)^2 \left(|\nabla_p f(p)| + \frac{1}{T_c} |f(p)| \right) \\
 &\quad \times \left(|\nabla_q f(q)| + \frac{1}{T} |f(q)| \right) dp dq \\
 &\lesssim \left(\int_{\mathbb{R}^3} |\Phi(p, q)|^2 \mathbf{M}(q) dq \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \left(|\nabla_q f(q)|^2 + \frac{1}{T^2} |f(q)|^2 \right) dq \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{\mathbb{R}^3} (w^\ell)^4 \mathbf{M}(p) dp \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \left(|\nabla_p f(p)|^2 + \frac{1}{T_c^2} |f(p)|^2 \right) dq \right)^{\frac{1}{2}} \\
 &\lesssim \left(\int_{\mathbb{R}^3} |\Phi(p, q)|^2 \mathbf{M}(q) dp \right)^{\frac{1}{2}} |f|_\sigma^2.
 \end{aligned} \tag{3.24}$$

Here we used (2.29) to deduce that

$$\int_{\mathbb{R}^3} (w^\ell)^4 \mathbf{M}(p) dp \lesssim 1. \tag{3.25}$$

As in [90, Lemma 2], we can obtain

$$\int_{\mathbb{R}^3} |\Phi(p, q)|^2 \mathbf{M}(q) dq \lesssim \int_{\mathbb{R}^3} (1 + |p - q|^{-2}) \mathbf{M}^{\frac{1}{2}}(q) dq \lesssim 1. \tag{3.26}$$

Then we can further bound $|(\mathcal{K}[f], (w^\ell)^2 f)|$ by $|f|_\sigma^2$.

Collecting the above estimates in (3.19), we use (2.28), (2.46) and (3.13) to get

$$\begin{aligned}
 &(\mathcal{L}[f], (w^\ell)^2 f) \\
 &\geq \frac{1}{2} ((w^\ell)^2 \sigma^{ij} \partial_j f, \partial_i f) - C_R |f|_\sigma^2 - C \left(\frac{1}{R} + \|u\|_{L_{t,x}^\infty}^2 \right) |(w^\ell) f|_\sigma^2 \\
 &\quad + \left(\frac{(u^0)^2}{8T^2} - \frac{2}{25T_c^2 \ln^2(e+t)} \right) \int_{\mathbb{R}^3} (w^\ell)^2 \sigma^{ij} \hat{p}_i \hat{p}_j |f|^2 dp \\
 &\geq \frac{1}{2} ((w^\ell)^2 \sigma^{ij} \partial_j f, \partial_i f) - C_R |f|_\sigma^2 - C \left(\frac{1}{R} + \|u\|_{L_{t,x}^\infty}^2 \right) |(w^\ell) f|_\sigma^2 \\
 &\quad + \frac{9(u^0)^2}{200T^2} \int_{\mathbb{R}^3} (w^\ell)^2 \sigma^{ij} \hat{p}_i \hat{p}_j |f|^2 dp \\
 &\gtrsim |(w^\ell) f|_\sigma^2 - C_R |f|_\sigma^2
 \end{aligned} \tag{3.27}$$

by choosing R large enough. □

LEMMA 3.6. For the weight functions defined in (2.27), it holds that

$$(\Gamma[f, g], (w^\ell)^2 h) \lesssim \left(|w^\ell f|_{L^2} |g|_\sigma + |g|_{L^2} |w^\ell f|_\sigma \right) |w^\ell h|_\sigma. \tag{3.28}$$

Moreover, for $|\alpha| \leq 2$, we have

$$\begin{aligned} & \left| \left\langle \partial_x^\alpha \Gamma[f, g] + \partial_x^\alpha \Gamma[g, f], (w^{|\alpha|})^2 \partial_x^\alpha h \right\rangle \right| \\ & \lesssim (\|f\|_{H_w^2} \|g\|_{H_\sigma^{|\alpha|}} + \|g\|_{H^2} \|f\|_{H_{w,\sigma}^{|\alpha|}}) \|w^{|\alpha|} \partial_x^\alpha h\|_\sigma \\ & \quad + \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 (\|f\|_{H_w^2} \|g\|_{H_{w,\sigma}^{|\alpha|-1}} + \|g\|_{H^2} \|f\|_{H_{w,\sigma}^{|\alpha|-1}}) \|w^{|\alpha|} \partial_x^\alpha h\|_\sigma. \end{aligned} \quad (3.29)$$

Proof. From (3.5), we integrate by parts with respect to p and use (3.22) to get

$$\begin{aligned} & |\Gamma[f, g], (w^\ell)^2 h| \\ & = \left| \left(\int_{\mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) \left(\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q) \right) dq, \left(\partial_{p_i} - \frac{u \hat{p}_i - u_i}{2T} \right) ((w^\ell)^2 h) \right) \right| \\ & \lesssim \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \Phi^{ij}(p, q) \mathbf{M}^{\frac{1}{2}}(q) |\partial_{p_j} f(p) g(q) - f(p) \partial_{q_j} g(q)| (w^\ell)^2 \left(\frac{|h|}{T_c} + |\partial_{p_i} h| \right) dp dq. \end{aligned} \quad (3.30)$$

By Hölder's inequality, we can use (3.28) to further estimate (3.30) as

$$\begin{aligned} |\Gamma[f, g], (w^\ell)^2 h| & \lesssim |\Phi(p, q) \mathbf{M}^{\frac{1}{2}}(q)|_{L_p^\infty L_q^2} (|w^\ell \partial_{p_j} f|_{L^2} |g|_{L^2} + |w^\ell f|_{L^2} |\partial_{q_j} g|_{L^2}) |w^\ell h|_\sigma \\ & \lesssim (|w^\ell f|_\sigma |g|_{L^2} + |w^\ell f|_{L^2} |g|_\sigma) |w^\ell h|_\sigma. \end{aligned} \quad (3.31)$$

Now we turn to derive (3.29). As in (3.16), we use (3.30) to have

$$\begin{aligned} & \left| \left\langle \partial_x^\alpha \Gamma[f, g], (w^{|\alpha|})^2 \partial_x^\alpha h \right\rangle \right| \\ & \leq \sum_{\alpha_1 \leq \alpha} \left| \left\langle \Gamma[\partial_x^{\alpha_1} f, \partial_x^{\alpha-\alpha_1} g], \partial_x^{\alpha-\alpha_1} f, (w^{|\alpha|})^2 \partial_x^\alpha h \right\rangle \right| \\ & \quad + \sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} \left| \left\langle \Gamma_{\alpha_2}[\partial_x^{\alpha_1} f, \partial_x^{\alpha-\alpha_1-\alpha_2} g], \partial_x^{\alpha-\alpha_1-\alpha_2} f, (w^{|\alpha|})^2 \partial_x^\alpha h \right\rangle \right| \\ & \lesssim \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} (|w^{|\alpha|} \partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |w^{|\alpha|} \partial_x^{\alpha-\alpha_1} f|_\sigma) |w^{|\alpha|} \partial_x^\alpha h|_\sigma \\ & \quad + \sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} \int_{\mathbb{R}^3} (|w^{|\alpha|} \partial_x^{\alpha_1} f|_{L^2} |\partial_x^{\alpha-\alpha_1-\alpha_2} g|_\sigma + |\partial_x^{\alpha_1} g|_{L^2} |w^{|\alpha|} \partial_x^{\alpha-\alpha_1-\alpha_2} f|_\sigma) |w^{|\alpha|} \partial_x^\alpha h|_\sigma. \end{aligned} \quad (3.32)$$

Now we estimate the first term in (3.32). Note that there is no weight for norms of g in (3.30) and the weighted norms may cost more ε as in (2.35). We will always try best to not raise the derivative of f in our estimates. By the algebra estimate for $H^2(\mathbb{R}_x^3)$, we have

$$\begin{aligned} & \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} (|w^{|\alpha|} \partial_x^{\alpha_1} f|_{L^2}^2 |\partial_x^{\alpha-\alpha_1} g|_\sigma^2 + |\partial_x^{\alpha_1} g|_{L^2}^2 |w^{|\alpha|} \partial_x^{\alpha-\alpha_1} f|_\sigma^2) \\ & \lesssim \|f\|_{H_w^2}^2 \|g\|_{H_\sigma^{|\alpha|}}^2 + \|g\|_{H^2}^2 \|f\|_{H_{w,\sigma}^{|\alpha|}}^2. \end{aligned} \quad (3.33)$$

Then the upper bound of the first term in (3.32) is

$$\sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} \left(\left| w^{|\alpha|} \partial_x^{\alpha_1} f \right|_{L^2} \left| \partial_x^{\alpha - \alpha_1} g \right|_{\sigma} + \left| \partial_x^{\alpha_1} g \right|_{L^2} \left| w^{|\alpha|} \partial_x^{\alpha - \alpha_1} f \right|_{\sigma} \right) \left| w^{|\alpha|} \partial_x^{\alpha} h \right|_{\sigma} \\ \lesssim \left(\|f\|_{H_w^2} \|g\|_{H_{\sigma}^{|\alpha|}} + \|g\|_{H^2} \|f\|_{H_{w,\sigma}^{|\alpha|}} \right) \left\| w^{|\alpha|} \partial_x^{\alpha} h \right\|_{\sigma}.$$

Similarly, for the second term in (3.32), we can obtain

$$\sum_{\alpha_1 + \alpha_2 \leq \alpha} \chi_{|\alpha_2| > 0} \int_{\mathbb{R}^3} \left(\left| w^{|\alpha|} \partial_x^{\alpha_1} f \right|_{L^2} \left| \partial_x^{\alpha - \alpha_1 - \alpha_2} g \right|_{\sigma} + \left| \partial_x^{\alpha_1} g \right|_{L^2} \left| w^{|\alpha|} \partial_x^{\alpha - \alpha_1 - \alpha_2} f \right|_{\sigma} \right) \left| w^{|\alpha|} \partial_x^{\alpha} h \right|_{\sigma} \\ \lesssim \chi_{|\alpha| \geq 1} (1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|f\|_{H_w^2} \|g\|_{H_{\sigma}^{|\alpha|-1}} + \|g\|_{H^2} \|f\|_{H_{w,\sigma}^{|\alpha|-1}} \right) \left\| w^{|\alpha|} \partial_x^{\alpha} h \right\|_{\sigma}.$$

We collect the above two estimates in (3.32) to derive (3.29). $\square$

4. No-weight energy estimates. In the following two sections, we will focus on the a priori estimates on the remainders $(f_R^{\varepsilon}, E_R^{\varepsilon}, B_R^{\varepsilon})$. Compared with the corresponding estimates in the existing results Strain-Guo [90] and Yang-Yu [105], ours faces new difficulties due to the key structural differences: the replacement of a global Maxwellian by a local Maxwellian and the appearance of the singularity coefficient ε^{-1} of the relativistic Landau collision operator. Now we derive the L^2 energy estimates for the remainders.

4.1. Basic L^2 estimates. We first perform the simplest L^2 energy estimate for the remainders. In this part, the main difficulty is to estimate the term $\mathbf{M}^{-\frac{1}{2}} f_R^{\varepsilon} [\partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p] \mathbf{M}^{\frac{1}{2}}$ with momentum growth and the term $\{\Gamma[\mathbf{M}^{-\frac{1}{2}} F_1, f_R^{\varepsilon}] + \Gamma[f_R^{\varepsilon}, \mathbf{M}^{-\frac{1}{2}} F_1]\}$ with time growth according to (A.7). In order to control the momentum growth term, we will divide $\mathbb{R}_p^3$ into two regions depending on ε and estimate them separately. The key point is to transform the corresponding estimate in the unbounded region to the weighted norm $\|(\mathbf{I} - \mathbf{P})[f_R^{\varepsilon}]\|_{w^0, \sigma}^2$ with a small coefficient ε . While for the estimate of the time growth term $\Gamma[\mathbf{M}^{-\frac{1}{2}} F_1, f_R^{\varepsilon}] + \Gamma[f_R^{\varepsilon}, \mathbf{M}^{-\frac{1}{2}} F_1]$, we use $1 = \varepsilon \times \frac{1}{\varepsilon}$ and absorb the time growth by ε , as shown in the **Seventh Term on the R.H.S. of (4.3)** for more details.

PROPOSITION 4.1. For the remainders $(f_R^{\varepsilon}, E_R^{\varepsilon}, B_R^{\varepsilon})$, it holds that

$$\frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} f_R^{\varepsilon} \right\|^2 + \|E_R^{\varepsilon}\|^2 + \|B_R^{\varepsilon}\|^2 \right) + \frac{\delta}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^{\varepsilon}]\|_{\sigma}^2 \\ \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right] \mathcal{E} + \varepsilon^2 \mathcal{D} + \varepsilon^{2k+1} (1+t)^{4k+2} + \varepsilon^k (1+t)^{2k} \sqrt{\mathcal{E}}. \quad (4.1)$$

Proof. From (2.13), we have

$$\frac{1}{2} \frac{d}{dt} \left(\|E_R^{\varepsilon}\|^2 + \|B_R^{\varepsilon}\|^2 \right) = 4\pi \left\langle \hat{p} \mathbf{M}^{\frac{1}{2}} f_R^{\varepsilon}, E_R^{\varepsilon} \right\rangle. \quad (4.2)$$

We take the L^2 inner product of (2.12) with $\frac{4\pi T}{u^0} f_R^\varepsilon$ and use (3.9), (4.2) to have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} f_R^\varepsilon \right\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) + \frac{\delta}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
& \leq \left| \left\langle \mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon [\partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p] \mathbf{M}^{\frac{1}{2}}, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \frac{1}{2} \left\langle \left\{ (\partial_t + \hat{p} \cdot \nabla_x) \left[\frac{T}{u^0} \right] \right\} f_R^\varepsilon, 4\pi f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \left\langle -u \mathbf{M}^{\frac{1}{2}} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon), \frac{4\pi}{u^0} f_R^\varepsilon \right\rangle \right| + \left| \varepsilon^{k-1} \left\langle \Gamma[f_R^\varepsilon, f_R^\varepsilon], \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \Gamma[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon] + \Gamma[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i], \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \varepsilon^k \left\langle (u^0 \hat{p} - u) \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) f_R^\varepsilon, \frac{2\pi f_R^\varepsilon}{u^0} \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (E_i + \hat{p} \times B_i) \cdot (u^0 \hat{p} - u) f_R^\varepsilon, \frac{2\pi}{u^0} f_R^\varepsilon \right\rangle \right| + \left| \varepsilon^k \left\langle \overline{S}, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right|.
\end{aligned} \tag{4.3}$$

Now we estimate each term on the R.H.S. of (4.3).

First term on the R.H.S. of (4.3): Note that

$$\mathbf{M}^{-\frac{1}{2}} \{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \} \mathbf{M}^{\frac{1}{2}}$$

are the first-order polynomials of p . Then, for a given sufficiently small positive constant κ , we have

$$\begin{aligned}
& \left| \left\langle \mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon [\partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p] \mathbf{M}^{\frac{1}{2}}, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \lesssim \left(\|E\|_{L^\infty} + \|B\|_{L^\infty} \right) \|f_R^\varepsilon\|^2 + \|\nabla_x(n, u, T)\|_{L^\infty} \left\| \sqrt{p^0} f_R^\varepsilon \right\|^2 \\
& \lesssim \mathcal{Z} \left(\left\| \sqrt{p^0} \mathbf{P}[f_R^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{p^0 \leq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 + \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right) \\
& \lesssim (1+t)^{-\beta_0} \|f_R^\varepsilon\|^2 + o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2.
\end{aligned} \tag{4.4}$$

Here we have used (1.27) and $\varepsilon(w^0)^2 \gtrsim \varepsilon(p^0)^2 \gtrsim p^0$ for $p^0 \gtrsim \varepsilon^{-1}$.

REMARK 4.1. The decay estimate $\mathcal{Z} \lesssim (1+t)^{-\beta_0}$ is crucial here.

Second and third terms on the R.H.S. of (4.3): From (1.27) again, we have

$$\begin{aligned}
& \left| \frac{1}{2} \left\langle \left\{ (\partial_t + \hat{p} \cdot \nabla_x) \left[\frac{T}{u^0} \right] \right\} f_R^\varepsilon, 4\pi f_R^\varepsilon \right\rangle \right| + \left| \left\langle -u \mathbf{M}^{\frac{1}{2}} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon), \frac{4\pi}{u^0} f_R^\varepsilon \right\rangle \right| \\
& \lesssim \|\nabla_x(n, u, T)\|_{L^\infty} \|f_R^\varepsilon\|^2 + \|u\|_{L^\infty} \|f_R^\varepsilon\| \left(\|E_R^\varepsilon\| + \|B_R^\varepsilon\| \right) \\
& \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + (1+t)^{-\beta_0} \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right).
\end{aligned}$$

Fourth term on the R.H.S. of (4.3): We use (3.15) in Lemma 3.4 to bound it by

$$\begin{aligned} & C\varepsilon^{k-1} \|f_R^\varepsilon\|_{H^2} \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma + \|\mathbf{P}[f_R^\varepsilon]\|_\sigma \right) \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \\ & \lesssim \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\mathbf{P}[f_R^\varepsilon]\|_\sigma^2 \lesssim \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|f_R^\varepsilon\|^2. \end{aligned}$$

Fifth term on the R.H.S. of (4.3): Similarly, considering that F_i decay fast in p by (A.7) in Proposition A.1 and $t \leq \bar{t} = \varepsilon^{-1/3}$, we have

$$\begin{aligned} & \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left(\left\| \mathbf{M}^{-\frac{1}{2}} F_i \right\|_{L_x^\infty L_p^2} \|f_R^\varepsilon\|_\sigma + \left\| \mathbf{M}^{-\frac{1}{2}} F_i \right\|_\sigma \|f_R^\varepsilon\|_{L_x^\infty} \right) \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^{i-1} (1+t)^i \|f_R^\varepsilon\|_\sigma \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon(1+t)^2 \|f_R^\varepsilon\|_\sigma^2 \\ & \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon(1+t)^2 \|\mathbf{P}[f_R^\varepsilon]\|_\sigma^2 \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|^2. \end{aligned}$$

Since $\|f_R^\varepsilon\|^2$ enjoys no time-integrability according to the a priori assumption (2.37), we need the assumption $t \leq \bar{t} = \varepsilon^{-1/3}$ to assure that $\int_0^{\bar{t}} \varepsilon^{\frac{1}{3}} dt = 1$. This is the very reason why the life-span of Hilbert expansion for the r-VML system is $\varepsilon^{-1/3}$. We can also see that the index $1/3$ in the life-span $\bar{t} = \varepsilon^{-1/3}$ should be optimal.

Sixth term on the R.H.S. of (4.3): According to the assumption (2.37), its upper bound is

$$\varepsilon^k \left(\|E_R^\varepsilon\|_{H^2} + \|B_R^\varepsilon\|_{H^2} \right) \|f_R^\varepsilon\|^2 \lesssim \varepsilon \|f_R^\varepsilon\|^2.$$

Seventh term on the R.H.S. of (4.3): We use (A.7) in Proposition A.1 to obtain

$$\begin{aligned} & \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\ & = \sum_{i=1}^{2k-1} \varepsilon^i \left| \left\langle (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^i \left\| \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right\|_{L_x^\infty L_p^2} \left(\|E_R^\varepsilon\| + \|B_R^\varepsilon\| \right) \|f_R^\varepsilon\| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1+t)^i \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) \\ & \lesssim \varepsilon(1+t) \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) \lesssim \varepsilon^{\frac{2}{3}} \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right). \end{aligned}$$

Eighth and ninth terms on the R.H.S. of (4.3): Similarly, we estimate the last two terms as

$$\begin{aligned} & \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (E_i + \hat{p} \times B_i) \cdot (u^0 \hat{p} - u) f_R^\varepsilon, \frac{2\pi}{u^0} f_R^\varepsilon \right\rangle \right| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^i \left(\|E_i\|_{L^\infty} + \|B_i\|_{L^\infty} \right) \|f_R^\varepsilon\|^2 \lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1+t)^i \|f_R^\varepsilon\|^2 \lesssim \varepsilon^{\frac{2}{3}} \|f_R^\varepsilon\|^2 \end{aligned}$$

for $t \leq \bar{t} = \varepsilon^{-1/3}$, and

$$\begin{aligned} \varepsilon^k \left| \left\langle \bar{S}, \frac{4\pi T}{u^0} f_R^\varepsilon \right\rangle \right| & \leq o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + C \varepsilon^{2k+1} \sum_{\substack{i+j \geq 2k+1 \\ 2 \leq i, j \leq 2k-1}} \varepsilon^{2(i+j-2k-1)} (1+t)^{2(i+j)} \\ & \quad + C \sum_{\substack{i+j \geq 2k \\ 1 \leq i, j \leq 2k-1}} \varepsilon^{i+j-k} (1+t)^{i+j} \|f_R^\varepsilon\| \\ & \leq o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + C \varepsilon^{2k+1} (1+t)^{4k+2} + C \varepsilon^k (1+t)^{2k} \|f_R^\varepsilon\|. \end{aligned} \quad (4.5)$$

Summary: We collect these estimates in (4.3) to obtain (4.1). $\square$

4.2. First-order derivatives estimates. In this part, we continue to perform the L^2 energy estimates for the first-order derivatives of the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$. To this end, we first apply $\partial_x^\alpha (1 \leq |\alpha| \leq 2)$ to (2.12) to have

$$\begin{aligned} & \partial_x^\alpha \left(\left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} [f_R^\varepsilon] \right) + \partial_x^\alpha \left(\frac{u^0}{T} \hat{p} \mathbf{M}^{\frac{1}{2}} \cdot E_R^\varepsilon \right) \\ & - \partial_x^\alpha \left(\frac{u}{T} \mathbf{M}^{\frac{1}{2}} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \right) + \frac{\partial_x^\alpha \mathcal{L}[f_R^\varepsilon]}{\varepsilon} \\ & = -\partial_x^\alpha \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} \right) + \varepsilon^{k-1} \partial_x^\alpha \Gamma[f_R^\varepsilon, f_R^\varepsilon] \\ & + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \partial_x^\alpha \left(\Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right] \right) + \varepsilon^k \partial_x^\alpha \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon \right) \\ & - \varepsilon^k \partial_x^\alpha \left(\frac{1}{2T} (u^0 \hat{p} - u) \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) f_R^\varepsilon \right) \\ & + \sum_{i=1}^{2k-1} \varepsilon^i \partial_x^\alpha \left((E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right) \\ & - \sum_{i=1}^{2k-1} \varepsilon^i \partial_x^\alpha \left((E_i + \hat{p} \times B_i) \cdot \frac{1}{2T} (u^0 \hat{p} - u) f_R^\varepsilon \right) + \varepsilon^k \partial_x^\alpha \bar{S}. \end{aligned} \quad (4.6)$$

Due to the appearance of the local Maxwellian $\mathbf{M}$, the linear collision operator term $\varepsilon^{-1} \mathcal{L}[f_R^\varepsilon]$ and $\mathbf{P}[f_R^\varepsilon]$ do not commute with the spatial derivative operator ∇_x . Moreover, in the term $\partial_x^\alpha \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} \right)$, more p^0 are generated.

Therefore, in the derivative estimates of the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, we need to pay special attentions to the collision terms and the momentum growth term.

PROPOSITION 4.2. For the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that

$$\begin{aligned} & \varepsilon \left[\frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \nabla_x f_R^\varepsilon \right\|^2 + \|\nabla_x E_R^\varepsilon\|^2 + \|\nabla_x B_R^\varepsilon\|^2 \right) + \frac{\delta}{\varepsilon} \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right] \\ & \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right] \mathcal{E} + \bar{\varepsilon}_0 \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \mathcal{D} + \varepsilon^{2k+2} (1+t)^{4k+2} \\ & \quad + \varepsilon^{k+1} (1+t)^{2k} \sqrt{\mathcal{E}}. \end{aligned} \quad (4.7)$$

Proof. From (2.13), we have

$$\frac{1}{2} \frac{d}{dt} \left(\|\partial_x E_R^\varepsilon\|^2 + \|\partial_x B_R^\varepsilon\|^2 \right) = 4\pi \left\langle \partial_x \left(\hat{p} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon \right), \partial_x E_R^\varepsilon \right\rangle. \quad (4.8)$$

Here and below we use ∂_x to denote ∂_{x_i} , $1 \leq i \leq 3$, for simplicity. Note that

$$\begin{aligned} \llbracket \mathcal{L}, \partial_x \rrbracket [f_R^\varepsilon] &= \mathcal{L}[\partial_x f_R^\varepsilon] - \partial_x \left(\mathcal{L}[f_R^\varepsilon] \right) = \mathcal{L}[(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]] - \partial_x \left(\mathcal{L}[(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] \right) \\ &= -\mathcal{L}[\llbracket \mathbf{P}, \partial_x \rrbracket [f_R^\varepsilon]] + \llbracket \mathcal{L}, \partial_x \rrbracket [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]]. \end{aligned}$$

Hence, naturally we have

$$\begin{aligned} \varepsilon^{-1} \left| \left\langle \llbracket \mathcal{L}, \partial_x \rrbracket [f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| &\lesssim \varepsilon^{-1} \left| \left\langle \mathcal{L}[\llbracket \mathbf{P}, \partial_x \rrbracket [f_R^\varepsilon]], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\ &\quad + \varepsilon^{-1} \left| \left\langle \llbracket \mathcal{L}, \partial_x \rrbracket [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right|. \end{aligned}$$

For the first term, we have

$$\varepsilon^{-1} \left| \left\langle \mathcal{L}[\llbracket \mathbf{P}, \partial_x \rrbracket [f_R^\varepsilon]], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| = \varepsilon^{-1} \left| \left\langle \mathcal{L}[\llbracket \mathbf{P}, \partial_x \rrbracket [f_R^\varepsilon]], \frac{4\pi T}{u^0} (\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon] \right\rangle \right|.$$

Note that $\llbracket \mathbf{P}, \partial_x \rrbracket$ only contains terms that ∂_x hits the Maxwellian but not f_R^ε . Hence, we have

$$\begin{aligned} \varepsilon^{-1} \left| \left\langle \mathcal{L}[\llbracket \mathbf{P}, \partial_x \rrbracket [f_R^\varepsilon]], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| &\lesssim o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-1} \mathcal{Z} \|f_R^\varepsilon\|^2 \\ &\lesssim o(1) \varepsilon^{-1} \|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \bar{\varepsilon}_0 \varepsilon^{-1} (1+t)^{-\beta_0} \|f_R^\varepsilon\|^2. \end{aligned}$$

For the second term, since $\llbracket \mathcal{L}, \partial_x \rrbracket$ indicates that ∂_x only hits the Maxwellian in $\mathcal{L}$ but not on $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$, we directly bound

$$\begin{aligned} \varepsilon^{-1} \left| \left\langle \llbracket \mathcal{L}, \partial_x \rrbracket [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| &\lesssim \varepsilon^{-1} \mathcal{Z} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \|\partial_x f_R^\varepsilon\|_\sigma \\ &\lesssim (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|\nabla_x f_R^\varepsilon\|^2 + o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]\|_\sigma^2 + \bar{\varepsilon}_0 \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2. \end{aligned}$$

As in (3.18), we have

$$\|(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]\|_\sigma \geq \|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma - (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|f_R^\varepsilon\|. \quad (4.9)$$

In total, we have

$$\begin{aligned}
& \varepsilon^{-1} \left\langle \partial_x \mathcal{L}[f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \\
& \geq \varepsilon^{-1} \left\langle \mathcal{L}[\partial_x f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle - \varepsilon^{-1} \left| \left\langle [\mathcal{L}, \partial_x][f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \geq \delta \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]\|_\sigma^2 - C\bar{\varepsilon}_0 \varepsilon^{-2} \mathcal{Z} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - C\mathcal{Z} \|\nabla_x f_R^\varepsilon\|^2 - \varepsilon^{-1} \mathcal{Z} \|f_R^\varepsilon\|^2 \\
& \geq \delta \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - C\bar{\varepsilon}_0 \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
& \quad - C(1+t)^{-\beta_0} \bar{\varepsilon}_0 (\|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^{-1} \|f_R^\varepsilon\|^2).
\end{aligned} \tag{4.10}$$

Taking $|\alpha| = 1$ in (4.6) and denoting ∂_x as ∂_{x_i} , $i = 1, 2, 3$ for convenience, we multiply the equation by $\frac{4\pi T}{u^0} \partial_x f_R^\varepsilon$, and use (4.8), (4.10) to have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \partial_x f_R^\varepsilon \right\|^2 + \|\partial_x E_R^\varepsilon\|^2 + \|\partial_x B_R^\varepsilon\|^2 \right) + \frac{\delta}{\varepsilon} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
& \leq \left| \left\langle \partial_x \left(\frac{u^0}{T} \hat{p} \mathbf{M}^{\frac{1}{2}} \right) \cdot E_R^\varepsilon, \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| + 4\pi \left| \left\langle \hat{p}(\partial_x \mathbf{M}^{\frac{1}{2}}) f_R^\varepsilon, \partial_x E_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \left\langle \partial_x \left(\{ (E + \hat{p} \times B) \cdot \nabla_p \} [f_R^\varepsilon] \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \left\langle \partial_x \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \} [\mathbf{M}^{\frac{1}{2}}] \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \frac{1}{2} \left\langle (\partial_t + \hat{p} \cdot \nabla_x) \left[\frac{T}{u^0} \right], 4\pi |\partial_x f_R^\varepsilon|^2 \right\rangle \right| \\
& \quad + \left| \left\langle \partial_x \left(u \mathbf{M}^{\frac{1}{2}} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| + \varepsilon^{k-1} \left| \left\langle \partial_x \Gamma[f_R^\varepsilon, f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x \Gamma[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon] + \partial_x \Gamma[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \varepsilon^k \left| \left\langle \partial_x \left(\frac{(u^0 \hat{p} - u)}{2T} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \varepsilon^k \left| \left\langle \partial_x \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x \left((E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x \left((E_i + \hat{p} \times B_i) \cdot \frac{(u^0 \hat{p} - u)}{2T} f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + \varepsilon^k \left| \left\langle \partial_x \bar{S}, \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\
& \quad + C\bar{\varepsilon}_0 \left[\varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + (1+t)^{-\beta_0} (\|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^{-1} \|f_R^\varepsilon\|^2) \right].
\end{aligned} \tag{4.11}$$

Now we estimate each term on the R.H.S. of (4.11).

First three terms on the R.H.S. of (4.11): The three terms can be bounded by

$$(1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|f_R^\varepsilon\|_{H^1}^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right).$$

Fourth term on the R.H.S. of (4.11): For this term, we should note that more p^0 will be generated when ∂_x hits $\mathbf{M}^{\frac{1}{2}}$ or $\mathbf{M}^{-\frac{1}{2}}$. Using $w^0 \gtrsim (p^0)^3$, and noticing that for $p^0 \gtrsim \varepsilon^{-1}$, we have $\varepsilon(w^1)^2 \gtrsim \varepsilon(p^0)^2 \gtrsim p^0$. Then, for κ sufficiently small, we have

$$\begin{aligned} & \left| \left\langle \partial_x \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \left[\mathbf{M}^{\frac{1}{2}} \right] \right), \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\ & \lesssim \mathcal{Z} \langle p^0 \partial_x f_R^\varepsilon, \partial_x f_R^\varepsilon \rangle + \mathcal{Z} | \langle (p^0)^2 f_R^\varepsilon, \partial_x f_R^\varepsilon \rangle | \lesssim \mathcal{Z} \left\| \sqrt{p^0} \partial_x f_R^\varepsilon \right\|^2 + \mathcal{Z} \left\| \sqrt{p^0} p^0 f_R^\varepsilon \right\|^2 \\ & \lesssim \mathcal{Z} \left(\left\| \sqrt{p^0} \partial_x \mathbf{P}[f_R^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{p^0 \leq \varepsilon^{-1} \kappa} p^0 |\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right. \\ & \quad \left. + \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1} \kappa} p^0 |\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right) \\ & \quad + \mathcal{Z} \left(\left\| \sqrt{p^0} p^0 \mathbf{P}[f_R^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (p^0)^3 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right) \\ & \lesssim \mathcal{Z} \|\partial_x f_R^\varepsilon\|^2 + o(1) \varepsilon^{-1} \|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \mathcal{Z} \|f_R^\varepsilon\|^2 \\ & \quad + \mathcal{Z} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 \\ & \lesssim o(1) \varepsilon^{-1} \|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 \\ & \quad + \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + (1+t)^{-\beta_0} \|f_R^\varepsilon\|_{H^1}^2. \end{aligned}$$

Fifth and sixth terms on the R.H.S. of (4.11): They can be bounded by

$$\begin{aligned} & C \mathcal{Z} \left(\|\partial_x f_R^\varepsilon\|^2 + \left(\|E_R^\varepsilon\|_{H^1} + \|B_R^\varepsilon\|_{H^1} \right) \|\partial_x f_R^\varepsilon\| \right) \\ & \leq C(1+t)^{-\beta_0} \left(\|\partial_x f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 \right). \end{aligned}$$

Seventh term on the R.H.S. of (4.11): Using (3.15) in Lemma 3.4, we have

$$\begin{aligned} & \left| \varepsilon^{k-1} \left\langle \partial_x \Gamma[f_R^\varepsilon, f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\ & \lesssim \varepsilon^{k-1} \|f_R^\varepsilon\|_{H^2} \|f_R^\varepsilon\|_{H_\sigma^1} \|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma + \mathcal{Z} \varepsilon^{k-1} \|f_R^\varepsilon\|_{H^2} \|f_R^\varepsilon\|_\sigma \|\nabla_x f_R^\varepsilon\|_\sigma \\ & \lesssim \varepsilon^{\frac{1}{2}} \|f_R^\varepsilon\|_{H_\sigma^1} \|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma + \varepsilon^{\frac{1}{2}} \mathcal{Z} \|f_R^\varepsilon\|_\sigma \|\nabla_x f_R^\varepsilon\|_\sigma \\ & \lesssim \varepsilon^{\frac{1}{2}} \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1} + \|f_R^\varepsilon\|_{H^1} \right) \left(\|\partial_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma + \mathcal{Z} \|f_R^\varepsilon\| \right) \\ & \quad + \varepsilon^{\frac{1}{2}} \mathcal{Z} \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma + \|f_R^\varepsilon\| \right) \left(\|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma + \|\nabla_x f_R^\varepsilon\| \right) \\ & \lesssim \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 + (1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|f_R^\varepsilon\|^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2 \right). \end{aligned}$$

Eighth term on the R.H.S. of (4.11): Similarly, considering that F_i decay fast in p , we know

$$\begin{aligned} & \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \partial_x \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], \frac{4\pi T}{u^0} \partial_x f_R^\varepsilon \right\rangle \right| \\ & \lesssim o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[\partial_x f_R^\varepsilon]\|_\sigma^2 + \varepsilon(1+t)^2 \|f_R^\varepsilon\|_{H_\sigma^1}^2 \\ & \quad + \mathcal{Z}(1+t) \left(\|f_R^\varepsilon\| + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \right) \left(\|f_R^\varepsilon\|_{H^1} + \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma \right) \\ & \lesssim o(1) \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \bar{\varepsilon}_0 \left[\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|^2 \right]. \end{aligned}$$

Other terms on the R.H.S. of (4.11): Similar to the corresponding estimates in Proposition 4.1, these terms can be controlled by

$$\begin{aligned} & o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 + C \varepsilon^{-1} (1+t)^{-\beta_0} \|f_R^\varepsilon\|^2 \\ & + \varepsilon^{\frac{2}{3}} \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 + \|\nabla_x f_R^\varepsilon\|^2 \right) \\ & + \varepsilon^{2k+1} (1+t)^{4k+2} + \varepsilon^k (1+t)^{2k} \|\nabla_x f_R^\varepsilon\|. \end{aligned}$$

Summary: By collecting all the above estimates in (4.11), we multiply the resulting inequality by ε to derive (4.7). $\square$

4.3. Second-order derivatives estimates. In this subsection, we proceed to the L^2 estimate of $\nabla_x^2(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$. Although the form of second-order derivatives estimates of the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$ is more complicated, they are quite similar to the corresponding first-order derivatives estimates in Section 4.2.

PROPOSITION 4.3. For the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that

$$\begin{aligned} & \varepsilon^2 \frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \nabla_x^2 f_R^\varepsilon \right\|^2 + \|\nabla_x^2 E_R^\varepsilon\|^2 + \|\nabla_x^2 B_R^\varepsilon\|^2 \right) + \frac{\delta}{\varepsilon} \|\nabla_x^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \quad (4.12) \\ & \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right] \mathcal{E} + \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 + \varepsilon \mathcal{D} + \varepsilon^{2k+3} (1+t)^{4k+2} + \varepsilon^{k+2} (1+t)^{2k} \sqrt{\mathcal{E}}. \end{aligned}$$

Proof. From (2.13), we have

$$\frac{1}{2} \frac{d}{dt} \left(\|\partial_x^2 E_R^\varepsilon\|^2 + \|\partial_x^2 B_R^\varepsilon\|^2 \right) = 4\pi \left\langle \partial_x^2 \left(\hat{p} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon \right), \partial_x^2 E_R^\varepsilon \right\rangle. \quad (4.13)$$

Here and below we denote ∂_x^2 as $\partial_{x_i} \partial_{x_j}$ with $i, j = 1, 2, 3$ for convenience. Using a similar derivation of (4.10), we can obtain

$$\begin{aligned} & \varepsilon^{-1} \left\langle \partial_x^2 \mathcal{L} [f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \quad (4.14) \\ & \geq \varepsilon^{-1} \left\langle \mathcal{L} [\partial_x^2 f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle - \varepsilon^{-1} \left| \left\langle [\mathcal{L}, \partial_x^2] [f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \geq \delta \varepsilon^{-1} \|\partial_x^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - \varepsilon^{-2} \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 \\ & \quad - (1+t)^{-\beta_0} \left(\|\nabla_x^2 f_R^\varepsilon\|^2 + \varepsilon^{-1} \|f_R^\varepsilon\|_{H^1}^2 \right). \end{aligned}$$

Taking $|\alpha| = 2$ in (4.6), we multiply the equation by $\frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon$, and further use (4.13), (4.14) to obtain

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \partial_x^2 f_R^\varepsilon \right\|^2 + \|\partial_x^2 E_R^\varepsilon\|^2 + \|\partial_x^2 B_R^\varepsilon\|^2 \right) + \frac{\delta}{\varepsilon} \|\partial_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
 & \leq \left| \sum_{|\alpha| < 2} \left\langle \partial_x^{2-|\alpha|} \left(\frac{u^0}{T} \hat{\mathbf{p}} \mathbf{M}^{\frac{1}{2}} \right) \cdot \partial_x^\alpha E_R^\varepsilon, \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + 4\pi \left| \sum_{|\alpha| < 2} \left\langle \hat{\mathbf{p}} (\partial_x^{2-|\alpha|} \mathbf{M}^{\frac{1}{2}}) \partial_x^\alpha f_R^\varepsilon, \partial_x^2 E_R^\varepsilon \right\rangle \right| \\
 & \quad + \left| \left\langle \partial_x^2 \left((E + \hat{\mathbf{p}} \times B) \cdot \nabla_p f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \left| \left\langle \partial_x^2 \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{\mathbf{p}} \cdot \nabla_x - (E + \hat{\mathbf{p}} \times B) \cdot \nabla_p \right\} [\mathbf{M}^{\frac{1}{2}}] \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \frac{1}{2} \left| \left\langle (\partial_t + \hat{\mathbf{p}} \cdot \nabla_x) \left[\frac{T}{u^0} \right], 4\pi |\partial_x^2 f_R^\varepsilon|^2 \right\rangle \right| \\
 & \quad + \left| \left\langle \partial_x^2 \left(u \mathbf{M}^{\frac{1}{2}} \cdot (\hat{\mathbf{p}} \times B_R^\varepsilon) \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| + \varepsilon^{k-1} \left| \left\langle \partial_x^2 \Gamma[f_R^\varepsilon, f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x^2 \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \partial_x^2 \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \varepsilon^k \left| \left\langle \partial_x^2 \left(\frac{u^0 \hat{\mathbf{p}} - u}{2T} \cdot (E_R^\varepsilon + \hat{\mathbf{p}} \times B_R^\varepsilon) f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \varepsilon^k \left| \left\langle \partial_x^2 \left((E_R^\varepsilon + \hat{\mathbf{p}} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x^2 \left((E_i + \hat{\mathbf{p}} \times B_i) \cdot \nabla_p f_R^\varepsilon + (E_R^\varepsilon + \hat{\mathbf{p}} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x^2 \left((E_i + \hat{\mathbf{p}} \times B_i) \cdot \frac{u^0 \hat{\mathbf{p}} - u}{2T} f_R^\varepsilon \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\
 & \quad + \varepsilon^k \left| \left\langle \partial_x^2 \bar{S}, \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| + C \varepsilon^{-2} \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 \\
 & \quad + C(1+t)^{-\beta_0} (\|\nabla_x^2 f_R^\varepsilon\|^2 + \varepsilon^{-1} \|f_R^\varepsilon\|_{H^1}^2).
 \end{aligned} \tag{4.15}$$

Now we estimate each term on the R.H.S. of (4.15).

First three terms on the R.H.S. of (4.15): The three terms can be bounded by

$$C(1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|f_R^\varepsilon\|_{H^2}^2 + \|E_R^\varepsilon\|_{H^2}^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 \right).$$

Fourth term on the R.H.S. of (4.15): Noting that $\varepsilon(w^2)^2 \gtrsim \varepsilon(p^0)^2 \gtrsim p^0$ for $p^0 \gtrsim \varepsilon^{-1}$, we use similar arguments in (4.4) to have

$$\begin{aligned} & \left| \left\langle \partial_x^2 \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \left[\mathbf{M}^{\frac{1}{2}} \right] \right), \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \lesssim \mathcal{Z} \langle (p^0)^3 |f_R^\varepsilon| + (p^0)^2 |\partial_x f_R^\varepsilon| + p^0 |\partial_x^2 f_R^\varepsilon|, |\partial_x^2 f_R^\varepsilon| \rangle \\ & \lesssim \mathcal{Z} \left(\left\| \sqrt{p^0} \partial_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|^2 + \left\| \sqrt{p^0} \partial_x^2 \mathbf{P} [f_R^\varepsilon] \right\|^2 \right) + \mathcal{Z} \|f_R^\varepsilon\|_{w^0, \sigma}^2 + \mathcal{Z} \|\partial_x f_R^\varepsilon\|_{w^1, \sigma}^2 \\ & \lesssim o(1) \varepsilon^{-1} \left\| \partial_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_\sigma^2 + \varepsilon \bar{\varepsilon}_0 \left\| \nabla_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_{w^2}^2 + (1+t)^{-\beta_0} \left\| \nabla_x^2 f_R^\varepsilon \right\|^2 \\ & \quad + \bar{\varepsilon}_0 \left(\|w_0 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon]\|_\sigma^2 + \|w_1 \nabla_x (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon]\|_\sigma^2 \right) + (1+t)^{-\beta_0} \|f_R^\varepsilon\|_{H^1}^2. \end{aligned}$$

Fifth and sixth terms on the R.H.S. of (4.15): The upper bound of the two terms is

$$C\mathcal{Z} \left(\left\| \partial_x^2 f_R^\varepsilon \right\|^2 + \|E_R^\varepsilon\|_{H^2}^2 + \|B_R^\varepsilon\|_{H^2}^2 \right) \lesssim (1+t)^{-\beta_0} \left(\left\| \partial_x^2 f_R^\varepsilon \right\|^2 + \|E_R^\varepsilon\|_{H^2}^2 + \|B_R^\varepsilon\|_{H^2}^2 \right).$$

Seventh term on the R.H.S. of (4.15): Using a similar estimation as the seventh term on the R.H.S. of (4.11), we have

$$\begin{aligned} & \left| \varepsilon^{k-1} \left\langle \partial_x^2 \Gamma [f_R^\varepsilon, f_R^\varepsilon], \frac{4\pi T}{u^0} \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \lesssim \varepsilon^{k-1} \int_{x \in \mathbb{R}^3} \left[\left(|\partial_x^2 f_R^\varepsilon|_\sigma |f_R^\varepsilon|_{L^2} + |\partial_x^2 f_R^\varepsilon|_{L^2} |f_R^\varepsilon|_\sigma + |\partial_x f_R^\varepsilon|_\sigma |\partial_x f_R^\varepsilon|_{L^2} \right) |(\mathbf{I} - \mathbf{P}) [\partial_x^2 f_R^\varepsilon]|_\sigma \right. \\ & \quad \left. \lesssim \varepsilon^{k-1} \|f_R^\varepsilon\|_{H^2} \|f_R^\varepsilon\|_{H_\sigma^2} \|(\mathbf{I} - \mathbf{P}) [\nabla_x^2 f_R^\varepsilon]\|_\sigma + \varepsilon^{k-1} \mathcal{Z} \|f_R^\varepsilon\|_{H^2} \|f_R^\varepsilon\|_{H_\sigma^1} \left\| \partial_x^2 f_R^\varepsilon \right\|_\sigma \right. \\ & \quad \left. \lesssim \varepsilon^{\frac{1}{2}} \|f_R^\varepsilon\|_{H_\sigma^2} \left(\left\| \partial_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_\sigma + \mathcal{Z} \|f_R^\varepsilon\|_{H^1} \right) + \varepsilon^{\frac{1}{2}} \mathcal{Z} \|f_R^\varepsilon\|_{H_\sigma^1} \left\| \partial_x^2 f_R^\varepsilon \right\|_\sigma \right. \\ & \quad \left. \lesssim \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P}) [f_R^\varepsilon]\|_{H_\sigma^2}^2 + \left[(1+t)^{-\beta_0} + \varepsilon \right] \|f_R^\varepsilon\|_{H^2}^2. \right. \end{aligned}$$

Eighth term on the R.H.S. of (4.15): Similar to the seventh term, it can be bounded by

$$\begin{aligned} & o(1) \varepsilon^{-1} \left\| (\mathbf{I} - \mathbf{P}) [\partial_x^2 f_R^\varepsilon] \right\|_\sigma^2 + \varepsilon (1+t)^2 \|f_R^\varepsilon\|_{H_\sigma^2}^2 \\ & \quad + \mathcal{Z} (1+t) \left(\|f_R^\varepsilon\|_{H^1} + \|(\mathbf{I} - \mathbf{P}) [f_R^\varepsilon]\|_{H_\sigma^1} \right) \left(\|f_R^\varepsilon\|_{H^2} + \left\| \partial_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_\sigma \right) \\ & \lesssim o(1) \varepsilon^{-1} \left\| \partial_x^2 (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_\sigma^2 + \bar{\varepsilon}_0 \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P}) [f_R^\varepsilon]\|_{H_\sigma^1}^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|_{H^2}^2 + \bar{\varepsilon}_0 \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2. \end{aligned}$$

Other terms on the R.H.S. of (4.15): As in Proposition 4.2, we have the following upper bound of these terms:

$$\begin{aligned} & o(1) \varepsilon^{-1} \left\| \partial_x (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] \right\|_{H_\sigma^1}^2 + C \varepsilon^{-1} (1+t)^{-\beta_0} \|f_R^\varepsilon\|_{H^1}^2 \\ & \quad + C \varepsilon^{\frac{2}{3}} \left(\left\| \nabla_x^2 f_R^\varepsilon \right\|^2 + \|E_R^\varepsilon\|_{H^2}^2 + \|B_R^\varepsilon\|_{H^2}^2 \right) \\ & \quad + C \varepsilon^{2k+1} (1+t)^{4k+2} + C \varepsilon^k (1+t)^{2k} \left\| \nabla_x^2 f_R^\varepsilon \right\|. \end{aligned}$$

Summary: We collect the above estimates in (4.15), and multiply the resulting inequality by ε^2 to derive (4.12). $\square$

5. Weighted energy estimates. In this section, we are devoted to the weighted energy estimates of the remainder term f_R^ε . This is the essential part of our energy method. The key point is to use the additional dissipation term generated by our special exponential weight function and the smallness of $\bar{\varepsilon}_0$ to control the momentum growth term (2.56).

5.1. *Weighted estimate.* Due to the exponential decay of momentum for $\mathbf{P}[f_R^\varepsilon]$, we only need to proceed to weighted estimates for $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$. Therefore, we first apply microscopic projection $(\mathbf{I} - \mathbf{P})$ onto (2.12) to have

$$\begin{aligned}
 & \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \\
 & - (\mathbf{I} - \mathbf{P}) \left[(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \frac{-u^0 \hat{p} + u}{T} \mathbf{M}^{\frac{1}{2}} \right] + \frac{\mathcal{L}[f_R^\varepsilon]}{\varepsilon} \\
 = & -\mathbf{M}^{-\frac{1}{2}} (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} + \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] \\
 & + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left(\Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right] \right) + \varepsilon^k (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \\
 & - \varepsilon^k \frac{1}{2T} (u^0 \hat{p} - u) \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^i (E_i + \hat{p} \times B_i) \cdot \nabla_p (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \\
 & + \sum_{i=1}^{2k-1} \varepsilon^i (\mathbf{I} - \mathbf{P}) \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right) \\
 & - \sum_{i=1}^{2k-1} \varepsilon^i \left[(E_i + \hat{p} \times B_i) \cdot \frac{1}{2T} (u^0 \hat{p} - u) (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right] + \varepsilon^k (\mathbf{I} - \mathbf{P})[\bar{S}] + \llbracket \mathbf{P}, \tau_B \rrbracket f_R^\varepsilon,
 \end{aligned} \tag{5.1}$$

where $\llbracket \mathbf{P}, \tau_B \rrbracket = \mathbf{P} \tau_B - \tau_B \mathbf{P}$ denotes the commutator of two operators $\mathbf{P}$ and τ_B :

$$\begin{aligned}
 \tau_B := & \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \\
 & + \mathbf{M}^{-\frac{1}{2}} \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} \\
 & + \varepsilon^k \frac{1}{2T} (u^0 \hat{p} - u) \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) - (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p \\
 & - \sum_{i=1}^{2k-1} \varepsilon^i \left\{ (E_i + \hat{p} \times B_i) \cdot \left[\nabla_p - \frac{1}{2T} (u^0 \hat{p} - u) \right] \right\}.
 \end{aligned}$$

In Proposition 5.1, we will derive the weighted estimate for the remainder f_R^ε . The most important estimate is to control the momentum growth as in (5.6). Another estimate of which we should take care is (5.7), where $\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2$ appears and will be further estimated in Section 6. Other estimates are similar to the basic estimate of the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$ in Subsection 4.1 or have been estimated in Lemmas 3.5 and 3.6.

PROPOSITION 5.1. For the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that

$$\begin{aligned} & \frac{d}{dt} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 + \frac{\delta}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + Y \left\| \sqrt{p^0}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 \\ & \lesssim \frac{1}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+1}(1+t)^{4k+2}. \end{aligned} \quad (5.2)$$

Proof. Noting

$$(w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \cdot \partial_t \{(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\} = \frac{1}{2} \partial_t |w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 + p^0 Y (w^0)^2 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2, \quad (5.3)$$

we take the L^2 inner product of (5.1) with $(w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$ and use Lemma 3.5 to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 + \frac{\delta}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + Y \left\| \sqrt{p^0}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 \\ & \leq \frac{C}{\varepsilon} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \left| \left\langle (E + \hat{p} \times B) \cdot \nabla_p (\mathbf{I} - \mathbf{P})[f_R^\varepsilon], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \left\langle (\mathbf{I} - \mathbf{P}) \left[(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \frac{-u^0 \hat{p} + u}{T} \mathbf{M}^{\frac{1}{2}} \right], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \left\langle \mathbf{M}^{-\frac{1}{2}} \{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \} \left[\mathbf{M}^{\frac{1}{2}} \right], |w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right\rangle \right| \\ & \quad + |\varepsilon^{k-1} \langle \Gamma[f_R^\varepsilon, f_R^\varepsilon], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \varepsilon^k \left\langle \frac{(u^0 \hat{p} - u)}{2T} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon), |w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right\rangle \right| \\ & \quad + \left| \left\langle (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon, (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (E_i + \hat{p} \times B_i) \cdot \nabla_p (\mathbf{I} - \mathbf{P})[f_R^\varepsilon], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle (\mathbf{I} - \mathbf{P}) \left[(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \left((E_i + \hat{p} \times B_i) \cdot \frac{u^0 \hat{p} - u}{2T} \right), |w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right\rangle \right| \\ & \quad + |\varepsilon^k \langle (\mathbf{I} - \mathbf{P})[\bar{S}], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| + |\langle [\mathbf{P}, \tau_B][f_R^\varepsilon], (w^0)^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle|. \end{aligned} \quad (5.4)$$

Second and third terms on the R.H.S. of (5.4): They are bounded by

$$o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + C\varepsilon (\|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2).$$

Fourth term on the R.H.S. of (5.4): By the smallness of $\bar{\varepsilon}_0$, it holds that

$$\mathcal{Z} \lesssim \bar{\varepsilon}_0 (1+t)^{-\beta_0} \ll Y. \quad (5.5)$$

Then, (2.61) holds true uniformly and we have

$$\begin{aligned} & \left| \left\langle \mathbf{M}^{-\frac{1}{2}} \{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \} \left[\mathbf{M}^{\frac{1}{2}} \right], |w^0(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right\rangle \right| \\ & \lesssim \mathcal{Z} \left(\left\| \sqrt{p^0}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 \right) \\ & \leq \frac{Y}{2} \left\| \sqrt{p^0}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 + C\varepsilon_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2. \end{aligned} \quad (5.6)$$

Fifth term on the R.H.S. of (5.4): From (3.29) in Lemma 3.6, it can be controlled by

$$\begin{aligned} & \varepsilon^{k-1} \left(\|f_R^\varepsilon\|_{H^2} \|f_R^\varepsilon\|_{w^0, \sigma} + \|f_R^\varepsilon\|_{H_\sigma^2} \|f_R^\varepsilon\|_{w^0} \right) \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma} \\ & \lesssim \left\{ \varepsilon^{\frac{1}{2}} \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma} + \|\mathbf{P}[f_R^\varepsilon]\|_{w^0, \sigma} \right) \right. \\ & \quad \left. + \varepsilon \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^2} + \|\mathbf{P}[f_R^\varepsilon]\|_{H_\sigma^2} \right) \right\} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma} \\ & \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + \varepsilon^3 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^2}^2 + \varepsilon^3 \|f_R^\varepsilon\|_{H^2}^2 + \varepsilon^2 \|f_R^\varepsilon\|^2. \end{aligned}$$

Sixth term on the R.H.S. of (5.4): Similarly, we use (2.29), (3.28) and (A.7) in Proposition A.1 to obtain that for $t \leq \bar{t} = \varepsilon^{-1/3}$,

$$\begin{aligned} & \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left| \left\langle \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], (w^0)^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\| |w^0 \mathbf{M}^{-\frac{1}{2}} F_i|_\sigma \right\|_{L_x^\infty} \|f_R^\varepsilon\|_{w^0, \sigma} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 \\ & \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + \varepsilon(1+t)^2 \|f_R^\varepsilon\|_{w^0, \sigma}^2 \\ & \lesssim o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|^2. \end{aligned}$$

Last term on the R.H.S. of (5.4): For this term, we have

$$\begin{aligned} & \left| \left\langle [\mathbf{P}, \tau_B][f_R^\varepsilon], (w^0)^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \lesssim \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \left(\|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 + \|f_R^\varepsilon\|_{H^1}^2 \right). \end{aligned} \quad (5.7)$$

Note that $\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2$ is included in $\mathcal{E}$. In order to make it time-integrable, we will derive its dissipativeness in Section 6.

Other terms on the R.H.S. of (5.4): Similarly, these terms can be bounded by

$$o(1)\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + C\varepsilon \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) + C\varepsilon^{2k+1} (1+t)^{4k+2}.$$

Summary: We collect the above estimates in (5.4) to derive (5.2). $\square$

5.2. *Weighted first-order derivatives estimates.* In this subsection, we proceed to the weighted L^2 estimate of $\nabla_x f_R^\varepsilon$. Note that in the term $\partial_x \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} \right)$, $(p^0)^2$ can be generated. To solve this kind of problem, the algebraic part of our weight function is designed to depend on the order of derivatives.

PROPOSITION 5.2. For the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that

$$\begin{aligned} & \varepsilon \left(\frac{d}{dt} \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \frac{\delta}{\varepsilon} \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 + Y \left\| \sqrt{p^0} \nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 \right) \\ & \lesssim \left((1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right) \mathcal{E} + \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 + \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \mathcal{D} + C\varepsilon^{2k+2}(1+t)^{4k+2}. \end{aligned} \quad (5.8)$$

Proof. From Lemma 3.5 and (4.9), we have

$$\begin{aligned} & \varepsilon^{-1} \langle \partial_x \mathcal{L}[(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle \\ & \geq \varepsilon^{-1} \langle \mathcal{L}[\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle \\ & \quad - \varepsilon^{-1} |\langle [\mathcal{L}, \partial_x](\mathbf{I} - \mathbf{P})[f_R^\varepsilon], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| \\ & \geq \delta \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 - C\varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 - C\varepsilon_0 \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2. \end{aligned} \quad (5.9)$$

Apply ∂_x to (5.1) and take the L^2 inner product of the resulting equation with $(w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$. Then, by similar arguments as in (5.3), we use (5.9) to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \frac{\delta}{\varepsilon} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 + Y \left\| \sqrt{p^0} \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 \\ & \leq C\varepsilon^{-1} \left(\|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon_0 \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 \right) \\ & \quad + \left| \left\langle \partial_x \left((E + \hat{p} \times B) \cdot \nabla_p(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \left\langle \partial_x(\mathbf{I} - \mathbf{P}) \left[\left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \frac{-u^0 \hat{p} + u}{T} \mathbf{M}^{\frac{1}{2}} \right) \right], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \left\langle \partial_x \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \mathbf{M}^{-\frac{1}{2}} \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \left[\mathbf{M}^{\frac{1}{2}} \right] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \varepsilon^{k-1} |\langle \partial_x \Gamma[f_R^\varepsilon, f_R^\varepsilon], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \partial_x \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \varepsilon^k \left| \left\langle \partial_x \left(\frac{(u^0 \hat{p} - u)}{2T} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon)(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \varepsilon^k \left| \left\langle \partial_x \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x \left((E_i + \hat{p} \times B_i) \cdot \nabla_p(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x \left((\mathbf{I} - \mathbf{P}) \left[(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x \left((E_i + \hat{p} \times B_i) \cdot \frac{u^0 \hat{p} - u}{2T} (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right), (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\rangle \right| \\ & \quad + \varepsilon^k |\langle \partial_x(\mathbf{I} - \mathbf{P})[\bar{S}], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| + |\langle \partial_x[\mathbf{P}, \tau_B] f_R^\varepsilon, (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle|. \end{aligned} \quad (5.10)$$

Now we estimate each term on the R.H.S. of (5.10).

Second term on the R.H.S. of (5.10): It can be bounded by

$$C \left(\|\nabla_x E\|_{W^{1,\infty}} + \|\nabla_x B\|_{W^{1,\infty}} \right) \left(\|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0,\sigma}^2 \right) \\ \lesssim (1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0,\sigma}^2 \right).$$

Third term on the R.H.S. of (5.10): It can be controlled by

$$C\varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + C\varepsilon \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 \right).$$

Fourth term on the R.H.S. of (5.10): Similar to (5.6), we bound it by

$$CZ \left\| \sqrt{p^0} \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 + CZ \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 \\ \leq \frac{Y}{2} \left\| \sqrt{p^0} \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 + (1+t)^{-\beta_0} \bar{\varepsilon}_0 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0,\sigma}^2.$$

Fifth term on the R.H.S. of (5.10): Similar to the estimation of the fifth term on the R.H.S. of (5.4), we have

$$\left| \langle \varepsilon^{k-1} \partial_x \Gamma[f_R^\varepsilon, f_R^\varepsilon], (w^1)^2 \partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle \right| \\ \lesssim \varepsilon^{k-1} \left(\|f_R^\varepsilon\|_{H^2} \|w^1 f_R^\varepsilon\|_{H_\sigma^1} + \|f_R^\varepsilon\|_{H_\sigma^2} \|w^1 f_R^\varepsilon\|_{H^1} \right) \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma} \\ \lesssim \left(\varepsilon^{\frac{1}{2}} \|w^1 f_R^\varepsilon\|_{H_\sigma^1} + \varepsilon \|f_R^\varepsilon\|_{H_\sigma^2} \right) \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma} \\ \lesssim o(1) \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma}^2 + \varepsilon^2 \|w^1 f_R^\varepsilon\|_{H_\sigma^1}^2 + \varepsilon^3 \|f_R^\varepsilon\|_{H_\sigma^2}^2 \\ \lesssim o(1) \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma}^2 + \varepsilon^2 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0,\sigma}^2 \\ + \varepsilon^3 \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^2}^2 + \varepsilon^3 \|f_R^\varepsilon\|_{H^2}^2 + \varepsilon^2 \|f_R^\varepsilon\|_{H^1}^2.$$

Sixth term on the R.H.S. of (5.10): Similarly, for $t \in [0, \varepsilon^{-1/3}]$, we use (3.28) and (A.7) in Proposition A.1 to bound it by

$$\sum_{i=1}^{2k-1} \varepsilon^{i-1} (1+t)^i \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma} \|f_R^\varepsilon\|_{H_{w,\sigma}^1} \quad (5.11) \\ \lesssim o(1) \varepsilon^{-1} \|\partial_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1,\sigma}^2 + C \sum_{i=1}^{2k-1} \varepsilon^{2i-1} (1+t)^{2i} \|f_R^\varepsilon\|_{H_{w,\sigma}^1}^2 \\ \lesssim o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^1}^2 + \varepsilon^{\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2.$$

Other terms on the R.H.S. of (5.10): Similarly, these terms can be controlled by

$$o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^1}^2 + C\varepsilon \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 + \|f_R^\varepsilon\|_{H^2}^2 \right) + C\varepsilon^{2k+1} (1+t)^{4k+2}.$$

Summary: We collect the above estimates in (5.10) and multiply them by ε to obtain (5.8). $\square$

5.3. *Weighted second-order derivatives estimates.* In this subsection, we proceed to the weighted L^2 estimate of $\nabla_x^2 f_R^\varepsilon$. Since $\nabla_x^2 f_R^\varepsilon$ is the highest-order derivative, we can't apply microscopic projection to the equation of f_R^ε as in the previous two subsections.

PROPOSITION 5.3. For the remainders $(f_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)$, it holds that

$$\begin{aligned} & \varepsilon^3 \left(\frac{d}{dt} \|\nabla_x^2 f_R^\varepsilon\|_{w^2}^2 + Y \left\| \sqrt{p^0} \nabla_x^2 f_R^\varepsilon \right\|_{w^2}^2 + \frac{\delta}{\varepsilon} \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 \right) \\ & \lesssim \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|_{w^2}^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+4}(1+t)^{4k+2}. \end{aligned} \quad (5.12)$$

Proof. We first use Lemma 3.5 to obtain

$$\begin{aligned} & \varepsilon^{-1} \langle \partial_x^2 \mathcal{L}[f_R^\varepsilon], (w^2)^2 \partial_x^2 f_R^\varepsilon \rangle \\ & \geq \varepsilon^{-1} \langle \mathcal{L}[\partial_x^2 f_R^\varepsilon], (w^2)^2 \partial_x^2 f_R^\varepsilon \rangle - \varepsilon^{-1} |\langle [\mathcal{L}, \partial_x^2] [f_R^\varepsilon], (w^2)^2 \partial_x^2 f_R^\varepsilon \rangle| \\ & \geq \delta \varepsilon^{-1} \|\partial_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 - C \bar{\varepsilon}_0 \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w, \sigma}^1}^2 - C \varepsilon^{-1} \|f_R^\varepsilon\|_{H^2}^2. \end{aligned} \quad (5.13)$$

Now we take $|\alpha| = 2$ in (4.6) and multiply the equation by $(w^2)^2 \partial_x^2 f_R^\varepsilon$ to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^2 f_R^\varepsilon\|_{w^2}^2 + Y \left\| \sqrt{p^0} \partial_x^2 f_R^\varepsilon \right\|_{w^2}^2 + \frac{\delta}{\varepsilon} \|\partial_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 \\ & \leq \left| \left\langle \partial_x^2 \left(\mathbf{M}^{\frac{1}{2}} (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \frac{u^0 \hat{p} - u}{T} \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \left| \left\langle \partial_x^2 \left((E + \hat{p} \times B) \cdot \nabla_p f_R^\varepsilon \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \left| \left\langle \partial_x^2 \left(\mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \left[\mathbf{M}^{\frac{1}{2}} \right] \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \varepsilon^{k-1} |\langle \partial_x^2 \Gamma[f_R^\varepsilon, f_R^\varepsilon], (w^2)^2 \partial_x^2 f_R^\varepsilon \rangle| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x^2 \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \partial_x^2 \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \varepsilon^k \left| \left\langle \partial_x^2 \left(\frac{(u^0 \hat{p} - u)}{2T} \cdot (E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) f_R^\varepsilon \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \varepsilon^k \left| \left\langle \partial_x^2 \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \nabla_p f_R^\varepsilon \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x^2 \left((E_i + \hat{p} \times B_i) \cdot \nabla_p f_R^\varepsilon \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x^2 \left((E_R^\varepsilon + \hat{p} \times B_R^\varepsilon) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left\langle \partial_x^2 \left((E_i + \hat{p} \times B_i) \cdot \frac{(u^0 \hat{p} - u)}{2T} f_R^\varepsilon \right), (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \quad + \varepsilon^k |\langle \partial_x^2 \bar{S}, (w^2)^2 \partial_x^2 f_R^\varepsilon \rangle| + C \bar{\varepsilon}_0 \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w, \sigma}^1}^2 + C \varepsilon^{-1} \|f_R^\varepsilon\|_{H^2}^2. \end{aligned} \quad (5.14)$$

Now we treat the terms in the R.H.S. of (5.14).

First term on the R.H.S. of (5.14): By (2.29), it can be bounded by

$$C \left(\|f_R^\varepsilon\|_{H^2}^2 + \|E_R^\varepsilon\|_{H^2}^2 + \|B_R^\varepsilon\|_{H^2}^2 \right).$$

Second terms on the R.H.S. of (5.14): Its upper bound is

$$\mathcal{Z} \|f_R^\varepsilon\|_{H_{w,\sigma}^2}^2 \lesssim (1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^2}^2 + \|f_R^\varepsilon\|_{H^2}^2 \right).$$

Third term on the R.H.S. of (5.14): Similar to (4.4), it can be bounded by

$$\begin{aligned} & \mathcal{Z} \left\| \sqrt{p^0} \partial_x^2 f_R^\varepsilon \right\|_{w^2}^2 + \mathcal{Z} \|f_R^\varepsilon\|_{H_{w,\sigma}^2}^2 \\ & \leq \frac{Y}{2} \left\| \sqrt{p^0} \partial_x^2 f_R^\varepsilon \right\|_{w^2}^2 + C(1+t)^{-\beta_0} \bar{\varepsilon}_0 \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^2}^2 + \|f_R^\varepsilon\|_{H^2}^2 \right). \end{aligned}$$

Fourth term on the R.H.S. of (5.14): We use (2.29) and (3.29) in Lemma 3.6 to have

$$\begin{aligned} & \left| \left\langle \varepsilon^{k-1} \partial_x^2 \Gamma[f_R^\varepsilon, f_R^\varepsilon], (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \lesssim \varepsilon^{k-1} \left(\|f_R^\varepsilon\|_{H^2} \|w^2 f_R^\varepsilon\|_{H_\sigma^2} + \|f_R^\varepsilon\|_{H_\sigma^2} \|w^2 f_R^\varepsilon\|_{H^2} \right) \|\partial_x^2 f_R^\varepsilon\|_{w^2, \sigma} \\ & \lesssim \left(\varepsilon^{\frac{1}{2}} \|w^2 f_R^\varepsilon\|_{H_\sigma^2} + \|f_R^\varepsilon\|_{H_\sigma^2} \right) \|\partial_x^2 f_R^\varepsilon\|_{w^2, \sigma} \\ & \lesssim \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^2}^2 + \|f_R^\varepsilon\|_{H^2}^2. \end{aligned}$$

Fifth term on the R.H.S. of (5.14): Similarly, we use (2.29), (3.28) and (A.7) in Proposition A.1 to obtain that for $t \leq \bar{t} = \varepsilon^{-1/3}$,

$$\begin{aligned} & \left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\langle \partial_x^2 \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \partial_x^2 \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right], (w^2)^2 \partial_x^2 f_R^\varepsilon \right\rangle \right| \\ & \lesssim \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left(\left\| w^2 \mathbf{M}^{-\frac{1}{2}} F_i \right\|_{W_x^{2,\infty} L_p^2} \|f_R^\varepsilon\|_{H_{w,\sigma}^2} + \left\| w^2 \mathbf{M}^{-\frac{1}{2}} F_i \right\|_{W_x^{2,\infty}} \|f_R^\varepsilon\|_{H_w^2} \right) \|w^2 f_R^\varepsilon\|_{\sigma}^2 \\ & \lesssim (1+t) \|f_R^\varepsilon\|_{H_{w,\sigma}^2}^2 \lesssim o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^2}^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|_{H^2}^2. \end{aligned}$$

Other terms on the R.H.S. of (5.14): Similarly, these terms can be controlled by

$$\begin{aligned} & o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_{w,\sigma}^2}^2 + C \varepsilon^{-1} \|f_R^\varepsilon\|_{H^2}^2 \\ & + C \varepsilon \left(\|E_R^\varepsilon\|_{H^2}^2 + \|B_R^\varepsilon\|_{H^2}^2 \right) + C \varepsilon^{2k+1} (1+t)^{4k+2}. \end{aligned}$$

Summary: We collect the above estimates in (5.14) and multiply the resulting inequality by ε^3 to derive (5.12). $\square$

6. Macroscopic estimates and electromagnetic dissipation. In this section, we study the macroscopic estimates of f_R^ε and electromagnetic dissipation. With these estimates and the estimates obtained in the previous two Sections, we can finally close the whole energy estimate.

6.1. *Macroscopic estimates.* To capture the dissipation of the macroscopic part of f_R^ε which can be seen as a perturbation around a local Maxwellian, as in (2.17), we write $\mathbf{P}[f_R^\varepsilon]$ as

$$\mathbf{P}[f_R^\varepsilon] = \left(a^\varepsilon - \frac{\rho_2}{\rho_1}c^\varepsilon\right)\mathbf{M}^{\frac{1}{2}} + b^\varepsilon \cdot p\mathbf{M}^{\frac{1}{2}} + c^\varepsilon p^0\mathbf{M}^{\frac{1}{2}}, \quad (6.1)$$

where ρ_1 and ρ_2 are defined in (2.18).

PROPOSITION 6.1. There are two functionals $\mathcal{E}_i^{mac}$ for $i = 1, 2$ satisfying

$$\mathcal{E}_i^{mac} \lesssim \|\nabla_x^{i-1} f_R^\varepsilon\| \|\nabla_x^i f_R^\varepsilon\|, \quad (6.2)$$

such that

$$\begin{aligned} & -\frac{d}{dt} \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) + \varepsilon \left(\|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \|a^\varepsilon\|^2 + \|(\nabla_x \cdot E_R^\varepsilon)\|^2 \right) \\ & + \varepsilon^2 \left(\|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 + \|\nabla_x a^\varepsilon\|^2 + \|\nabla_x (\nabla_x \cdot E_R^\varepsilon)\|^2 \right) \\ & \lesssim \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{\frac{2}{3}} (\mathcal{E} + \mathcal{D}) \\ & \quad + \varepsilon^{k+1} (1+t)^{2k+1}. \end{aligned} \quad (6.3)$$

Proof. Motivated by [43, Lemma 6.1], we will prove this proposition by two key ingredients: local conservation laws and the macroscopic equations of f_R^ε . For convenience, we write (2.12) as

$$\partial_t f_R^\varepsilon + \hat{p} \cdot \nabla_x f_R^\varepsilon + \frac{u^0}{T} \hat{p} \mathbf{M}^{\frac{1}{2}} \cdot E_R^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon] = \bar{h}^\varepsilon, \quad (6.4)$$

where

$$\begin{aligned} \bar{h}^\varepsilon = & \left(E + \hat{p} \times B \right) \cdot \nabla_p f_R^\varepsilon + \frac{u}{T} \mathbf{M}^{\frac{1}{2}} \cdot \left(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon \right) \\ & - \mathbf{M}^{-\frac{1}{2}} f_R^\varepsilon \left\{ \partial_t + \hat{p} \cdot \nabla_x - (E + \hat{p} \times B) \cdot \nabla_p \right\} \mathbf{M}^{\frac{1}{2}} + \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] \\ & + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma \left[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon \right] + \Gamma \left[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i \right] \right\} + e\varepsilon^k \left(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon \right) \cdot \nabla_p f_R^\varepsilon \\ & - \varepsilon^k \frac{1}{2T} \left(u^0 \hat{p} - u \right) \cdot \left(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon \right) f_R^\varepsilon \\ & + \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \left(E_i + \hat{p} \times B_i \right) \cdot \nabla_p f_R^\varepsilon + \left(E_R^\varepsilon + \hat{p} \times B_R^\varepsilon \right) \cdot \mathbf{M}^{-\frac{1}{2}} \nabla_p F_i \right\} \\ & - \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \left(E_i + \hat{p} \times B_i \right) \cdot \frac{1}{2T} \left(u^0 \hat{p} - u \right) f_R^\varepsilon \right\} + \varepsilon^k \bar{S}. \end{aligned} \quad (6.5)$$

Local conservation laws: Firstly, we derive the local conservation laws of $a^\varepsilon, b^\varepsilon, c^\varepsilon$. Note that

$$\frac{\rho_2}{\rho_1} = \frac{\mathfrak{e}(u^0)^2 + P|u|^2}{nu^0}. \quad (6.6)$$

Projecting (6.4) onto the null space $\mathcal{N}$, similar to the derivation of (A.3), (A.4) and (A.5), we can obtain

$$nu^0 \partial_t a^\varepsilon + P \nabla_x \cdot b^\varepsilon \quad (6.7)$$

$$\begin{aligned} &= \Xi_1[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] dp + \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \\ &\quad \frac{nu^0 K_3(\gamma)}{\gamma K_2(\gamma)} \partial_t b^\varepsilon + P \nabla_x \left(a^\varepsilon - \frac{\mathbf{e} u^0}{n} c^\varepsilon \right) + \frac{nu^0 K_3(\gamma)}{\gamma K_2(\gamma)} \nabla_x c^\varepsilon + nu^0 E_R^\varepsilon \end{aligned} \quad (6.8)$$

$$\begin{aligned} &= \Xi_2[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} p \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] dp + \int_{\mathbb{R}^3} p \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \\ &\quad \mathbf{e} (u^0)^2 \partial_t \left(a^\varepsilon - \frac{\mathbf{e} u^0}{n} c^\varepsilon \right) + \frac{n(u^0)^2 [3K_3(\gamma) + \gamma K_2(\gamma)]}{\gamma K_2(\gamma)} \partial_t c^\varepsilon + \frac{nu^0 K_3(\gamma)}{\gamma K_2(\gamma)} \nabla_x \cdot b^\varepsilon \quad (6.9) \\ &= \Xi_3[a^\varepsilon, b^\varepsilon, c^\varepsilon] + \int_{\mathbb{R}^3} p^0 \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \end{aligned}$$

where $\Xi_j[a^\varepsilon, b^\varepsilon, c^\varepsilon]$ for $j = 1, 2, 3$ denotes a combination of linear terms of $a^\varepsilon, b^\varepsilon, c^\varepsilon$ with coefficients $\nabla_{t,x}(n, u, T)$, and derivatives of $a^\varepsilon, b^\varepsilon, c^\varepsilon$ with coefficient u . Since they are small perturbations and thus will not affect the estimates, we will ignore the details for clarity.

Noting that

$$P = \frac{n}{\gamma}, \quad \mathbf{e} = n \left(\frac{K_3(\gamma)}{K_2(\gamma)} - \frac{1}{\gamma} \right) = n \left(\frac{K_1(\gamma)}{K_2(\gamma)} + \frac{3}{\gamma} \right), \quad (6.10)$$

we can further write the above system as

$$nu^0 \partial_t a^\varepsilon + \frac{n}{\gamma} \nabla_x \cdot b^\varepsilon \quad (6.11)$$

$$\begin{aligned} &= \Xi_1[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] dp + \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \\ &\quad n \left(\frac{K_1(\gamma)}{\gamma K_2(\gamma)} + \frac{4}{\gamma^2} \right) \partial_t b^\varepsilon + \frac{n}{\gamma} \nabla_x a^\varepsilon + \frac{n}{\gamma^2} \nabla_x c^\varepsilon + nu^0 E_R^\varepsilon \end{aligned} \quad (6.12)$$

$$\begin{aligned} &= \Xi_2[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} p \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] dp + \int_{\mathbb{R}^3} p \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \\ &\quad n \left(-\frac{K_1^2(\gamma)}{K_2^2(\gamma)} - \frac{3}{\gamma} \frac{K_1(\gamma)}{K_2(\gamma)} + 1 + \frac{3}{\gamma^2} \right) \partial_t c^\varepsilon + \frac{n}{\gamma^2} \nabla_x \cdot b^\varepsilon \quad (6.13) \\ &= \Xi_3[a^\varepsilon, b^\varepsilon, c^\varepsilon] + \int_{\mathbb{R}^3} p^0 \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp \\ &\quad - u^0 \left(\frac{K_1(\gamma)}{K_2(\gamma)} + \frac{3}{\gamma} \right) \left(-\nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] dp + \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp \right). \end{aligned}$$

This system fully describes the evolution of $a^\varepsilon, b^\varepsilon$ and c^ε .

Macroscopic equations: Secondly, we turn to the macroscopic equations of f_R^ε . Splitting f_R^ε as the macroscopic part $\mathbf{P}[f_R^\varepsilon]$ and the microscopic $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$ part in (6.4),

we have

$$\begin{aligned} & \left\{ \partial_t \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + p \cdot \partial_t b^\varepsilon + p^0 \partial_t c^\varepsilon \right\} \mathbf{M}^{\frac{1}{2}} \\ & + \hat{p} \cdot \left\{ \nabla_x \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + \nabla_x b^\varepsilon \cdot p + p^0 \nabla_x c^\varepsilon \right\} \mathbf{M}^{\frac{1}{2}} + \frac{u^0}{T} \hat{p} \mathbf{M}^{\frac{1}{2}} \cdot E_R^\varepsilon = \ell^\varepsilon + h^\varepsilon, \end{aligned} \quad (6.14)$$

where

$$\begin{aligned} \ell^\varepsilon &:= - \left(\partial_t + \hat{p} \cdot \nabla_x \right) \left[(\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \\ h^\varepsilon &:= - \left\{ \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + p \cdot b^\varepsilon + p^0 c^\varepsilon \right\} (\partial_t + \hat{p} \cdot \nabla_x) \mathbf{M}^{\frac{1}{2}} + \bar{h}^\varepsilon. \end{aligned}$$

For fixed t, x , we compare the coefficients in front of

$$\left\{ \mathbf{M}^{\frac{1}{2}}, p_i \mathbf{M}^{\frac{1}{2}}, p^0 \mathbf{M}^{\frac{1}{2}}, \frac{p_i}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i p_j}{p^0} \mathbf{M}^{\frac{1}{2}} \right\}, \quad 1 \leq i, j \leq 3, \quad (6.15)$$

on both sides of (6.14) and get the following macroscopic equations:

$$\partial_t a^\varepsilon - \frac{\rho_2}{\rho_1} \partial_t c^\varepsilon = \ell_a^\varepsilon + h_a^\varepsilon + \partial_t \left(\frac{\rho_2}{\rho_1} \right) c^\varepsilon, \quad (6.16)$$

$$\begin{aligned} \partial_t b_i^\varepsilon + \partial_i c^\varepsilon &= \ell_{bi}^\varepsilon + h_{bi}^\varepsilon, \\ \partial_t c^\varepsilon &= \ell_c^\varepsilon + h_c^\varepsilon, \end{aligned} \quad (6.17)$$

$$\partial_i a^\varepsilon - \frac{\rho_2}{\rho_1} \partial_i c^\varepsilon + \frac{u^0}{T} E_{R,i}^\varepsilon = \ell_{ai}^\varepsilon + h_{ai}^\varepsilon + \partial_i \left(\frac{\rho_2}{\rho_1} \right) c^\varepsilon, \quad (6.18)$$

$$\partial_i b_i^\varepsilon = \ell_{ii}^\varepsilon + h_{ii}^\varepsilon,$$

$$\partial_i b_j^\varepsilon + \partial_j b_i^\varepsilon = \ell_{ij}^\varepsilon + h_{ij}^\varepsilon, \quad i \neq j.$$

Here $\ell_a^\varepsilon, h_a^\varepsilon, \ell_{bi}^\varepsilon, h_{bi}^\varepsilon, \ell_c^\varepsilon, h_c^\varepsilon, \ell_{ai}^\varepsilon, h_{ai}^\varepsilon, \ell_{ii}^\varepsilon, h_{ii}^\varepsilon$ and $\ell_{ij}^\varepsilon, h_{ij}^\varepsilon$ take the form

$$(\ell^\varepsilon, \zeta) \quad \text{and} \quad (h^\varepsilon, \zeta),$$

where ζ is linear combinations of vectors in (6.15). Combing (6.16) and (6.17) yields

$$\partial_t a^\varepsilon = \ell_a^\varepsilon + h_a^\varepsilon + \frac{\rho_2}{\rho_1} \left(\ell_c^\varepsilon + h_c^\varepsilon \right) + \partial_t \left(\frac{\rho_2}{\rho_1} \right) c^\varepsilon. \quad (6.19)$$

For $m = 0, 1$, we have the following estimates:

$$\begin{aligned} & \|\nabla_x^m h_a^\varepsilon\| + \|\nabla_x^m h_{bi}^\varepsilon\| + \|\nabla_x^m h_c^\varepsilon\| + \|\nabla_x^m h_{ai}^\varepsilon\| + \|\nabla_x^m h_{ii}^\varepsilon\| + \|\nabla_x^m h_{ij}^\varepsilon\| \\ & \lesssim \mathcal{Z} \left(\|E_R^\varepsilon\|_{H^m} + \|B_R^\varepsilon\|_{H^m} \right) + \left(\mathcal{Z} + \varepsilon^{\frac{1}{2}} \right) \|f_R^\varepsilon\|_{H_\sigma^m} \\ & + \sum_{l=1}^{2k-1} \varepsilon^{l-1} \left(\left\| \mathbf{M}^{-\frac{1}{2}} F_l \right\|_{W_x^{m,\infty} L_p^2} \|f_R^\varepsilon\|_{H_\sigma^m} + \left\| \mathbf{M}^{-\frac{1}{2}} F_l \right\|_{W_x^{m,\infty}} \|f_R^\varepsilon\|_{H^m} \right) \\ & + \sum_{l=1}^{2k-1} \varepsilon^l (1+t)^l \left(\|f_R^\varepsilon\|_{H_\sigma^m} + \|E_R^\varepsilon\|_{H^m} + \|B_R^\varepsilon\|_{H^m} \right) + \varepsilon^k (1+t)^{2k+1} \\ & \lesssim \varepsilon^{-\frac{1}{3}} \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^m} + \|f_R^\varepsilon\|_{H^m} \right) \\ & + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|_{H^m} + \|B_R^\varepsilon\|_{H^m} \right) + \varepsilon^k (1+t)^{2k+1}, \end{aligned} \quad (6.20)$$

for $t \in [0, \varepsilon^{-1/3}]$. For brevity, we only give the estimate of $\|\nabla_x a^\varepsilon\|$ and $\|a^\varepsilon\|$. The estimates w.r.t. b^ε and c^ε in (6.3) can be derived similarly as in [43, Lemma 6.1]. From (6.18), we have

$$\begin{aligned} & -\Delta a^\varepsilon + \frac{\rho_2}{\rho_1} \Delta c^\varepsilon - \frac{u^0}{T} \nabla_x \cdot E_R^\varepsilon \\ &= -\sum_{i=1}^3 \partial_i (\ell_{ai}^\varepsilon + h_{ai}^\varepsilon) - \nabla_x \cdot \left[c^\varepsilon \nabla_x \left(\frac{\rho_2}{\rho_1} \right) \right] - \nabla_x \left(\frac{u^0}{T} \right) \cdot E_R^\varepsilon. \end{aligned} \quad (6.21)$$

On the other hand, by (2.13), we have

$$\begin{aligned} \nabla_x \cdot E_R^\varepsilon &= -4\pi \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon dp \\ &= -4\pi n a^\varepsilon - 4\pi n a^\varepsilon (u^0 - 1) - 4\pi(\mathfrak{e} + P) u^0 u \cdot b^\varepsilon \\ &= -4\pi \bar{n} a^\varepsilon - 4\pi(n - \bar{n}) a^\varepsilon - 4\pi n a^\varepsilon (u^0 - 1) - 4\pi(\mathfrak{e} + P) u^0 u \cdot b^\varepsilon. \end{aligned} \quad (6.22)$$

Collecting (6.22) in (6.21), we have

$$-\Delta a^\varepsilon + \frac{\mathfrak{e}}{n} \Delta c^\varepsilon + \frac{4\pi \bar{n}}{T} a^\varepsilon = -\sum_{i=1}^3 \partial_i (\ell_{ai}^\varepsilon + h_{ai}^\varepsilon) + \Xi_5[a^\varepsilon, b^\varepsilon, c^\varepsilon], \quad (6.23)$$

where

$$\begin{aligned} \Xi_5[a^\varepsilon, b^\varepsilon, c^\varepsilon] &= -\frac{\mathfrak{e}(u^0 - 1)}{n} \Delta c^\varepsilon - 4\pi(n - \bar{n}) a^\varepsilon - 4\pi n a^\varepsilon (u^0 - 1) \\ &\quad - 4\pi(\mathfrak{e} + P) u^0 u \cdot b^\varepsilon - \nabla_x \cdot \left[c^\varepsilon \nabla_x \left(\frac{\rho_2}{\rho_1} \right) \right] + \nabla_x \left(\frac{u^0}{T} \right) \cdot E_R^\varepsilon. \end{aligned}$$

Note that

$$\left| \left\langle \frac{\mathfrak{e}}{n} \Delta c^\varepsilon, \bar{n} a^\varepsilon \right\rangle \right| = \left| \left\langle \frac{\mathfrak{e}}{n} \nabla_x c^\varepsilon, \nabla_x (\bar{n} a^\varepsilon) \right\rangle \right| \leq o(1) \|\nabla_x a^\varepsilon\|^2 + C \|\nabla_x c^\varepsilon\|^2.$$

We multiply (6.23) by $\bar{n} a^\varepsilon$ and integrate the resulting equalities to have

$$\begin{aligned} & \frac{3c}{4} \left\| \sqrt{\bar{n}} \nabla_x a^\varepsilon \right\|^2 + \bar{n}^2 \left\| \sqrt{\frac{4\pi}{T}} a^\varepsilon \right\|^2 \\ & \leq \sum_{i=1}^3 \langle \ell_{ai}^\varepsilon, \bar{n} \partial_i a^\varepsilon \rangle + \sum_{i=1}^3 \langle h_{ai}^\varepsilon, \bar{n} \partial_i a^\varepsilon \rangle + \left| \langle \Xi_5[a^\varepsilon, b^\varepsilon, c^\varepsilon], \bar{n} a^\varepsilon \rangle \right| + C \|\nabla_x c^\varepsilon\|^2. \end{aligned} \quad (6.24)$$

For the first term in (6.24), we have

$$\begin{aligned}
& \langle \ell_{ai}^\varepsilon, \bar{n} \partial_i a^\varepsilon \rangle \\
&= \left\langle \left(-(\partial_t + \hat{p} \cdot \nabla_x) [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ai} \right), \bar{n} \partial_i a^\varepsilon \right\rangle \\
&= \left\langle \left(-\partial_t [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_i a^\varepsilon \right\rangle \\
&\quad + \left\langle \left(-\hat{p} \cdot \nabla_x [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ai} \right), \bar{n} \partial_i a^\varepsilon \right\rangle \\
&\leq -\frac{d}{dt} \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_i a^\varepsilon \right\rangle + \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_i \partial_t a^\varepsilon \right\rangle \\
&\quad + o(1) \|\nabla_x a^\varepsilon\|^2 + C \left(\|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right).
\end{aligned} \tag{6.25}$$

By (6.19) and (6.20), we have

$$\begin{aligned}
& \left| \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_i \partial_t a^\varepsilon \right\rangle \right| \\
&\leq \left| \left\langle \partial_i \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_t a^\varepsilon \right\rangle \right| \lesssim \|\partial_t a^\varepsilon\|^2 + \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 \\
&\lesssim \varepsilon^{-\frac{1}{3}} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2 + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 \right) \\
&\quad + \varepsilon^k (1+t)^{2k+1}.
\end{aligned} \tag{6.26}$$

Therefore,

$$\begin{aligned}
& \sum_{i=1}^3 \langle \ell_{ai}^\varepsilon, \bar{n} \partial_i a^\varepsilon \rangle \\
&\leq -\sum \frac{d}{dt} \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ai} \right), \bar{n} \partial_i a^\varepsilon \right\rangle + o(1) \|\nabla_x a^\varepsilon\|^2 + C \varepsilon^{-\frac{1}{3}} \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
&\quad + C \left[\varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2 \right. \\
&\quad \left. + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 \right) + \varepsilon^k (1+t)^{2k+1} \right].
\end{aligned} \tag{6.27}$$

For the second and third terms in (6.24), we use (6.20) to have

$$\begin{aligned}
& \sum_{i=1}^3 \langle h_{ai}^\varepsilon, \bar{n} \partial_i a^\varepsilon \rangle \lesssim o(1) \|\nabla_x a^\varepsilon\|^2 + \varepsilon^{-\frac{1}{3}} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|f_R^\varepsilon\|^2 \\
&\quad + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) + \varepsilon^k (1+t)^{2k+1},
\end{aligned} \tag{6.28}$$

and

$$\sum \left| \langle \Xi_5[a^\varepsilon, b^\varepsilon, c^\varepsilon], n a^\varepsilon \rangle \right| \lesssim o(1) \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + C(1+t)^{-\beta_0} \left(\|f_R^\varepsilon\|^2 + \|E_R^\varepsilon\|^2 \right). \tag{6.29}$$

Collecting the estimates (6.27), (6.28) and (6.29) in (6.24), we obtain

$$\begin{aligned}
 & \frac{1}{2} \left\| \sqrt{\bar{n}} \nabla_x a^\varepsilon \right\|^2 + \bar{n}^2 \left\| \sqrt{\frac{4\pi}{T}} a^\varepsilon \right\|^2 \\
 & \leq -\frac{d}{dt} \left\langle \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), \partial_i a^\varepsilon \right\rangle + C\varepsilon^{-\frac{1}{3}} \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
 & \quad + C \left(\varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|_{H^1}^2 \right. \\
 & \quad \left. + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|_{H^1}^2 + \|B_R^\varepsilon\|_{H^1}^2 \right) + \varepsilon^k (1+t)^{2k+1} \right).
 \end{aligned} \tag{6.30}$$

□

6.2. Electromagnetic dissipation. In this subsection, we derive the dissipation of the electromagnetic field $(E_R^\varepsilon, B_R^\varepsilon)$.

PROPOSITION 6.2. It holds that, for $i = 0, 1$

$$\begin{aligned}
 & \frac{1}{2} \varepsilon^{i+1} \left\| \sqrt{\frac{u^0}{T}} \nabla_x^i E_R^\varepsilon \right\|^2 \\
 & \leq -\sum_{j=1}^3 \varepsilon^{i+1} \frac{d}{dt} \left\langle \left(\nabla_x^i (\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{aj} \right), \nabla_x^i E_{R,j}^\varepsilon \right\rangle + C\varepsilon^{i-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \\
 & \quad + C \left[\varepsilon^{i+1} \|\nabla_x^{i+1} \mathbf{P}[f_R^\varepsilon]\|^2 + \varepsilon^{\frac{2}{3}} (\mathcal{E} + \mathcal{D}) + \varepsilon^{k+i+1} (1+t)^{2k+1} \right].
 \end{aligned} \tag{6.31}$$

Proof. For brevity, we only prove (6.31) for $i = 0$ since the case $i = 1$ can be proved in the same way. From (6.18), we have

$$\left\| \sqrt{\frac{u^0}{T}} E_R^\varepsilon \right\|^2 \leq \sum_{i=1}^3 \langle \ell_{ai}^\varepsilon, E_{R,i}^\varepsilon \rangle + \sum_{i=1}^3 \left| \left\langle \partial_i a^\varepsilon - \frac{\rho_2^\varepsilon}{\rho_1^\varepsilon} \partial_i c^\varepsilon, E_{R,i}^\varepsilon \right\rangle \right| + \sum_{i=1}^3 |\langle h_{ai}^\varepsilon, E_{R,i}^\varepsilon \rangle|. \tag{6.32}$$

For the first term in (6.32), we use (2.13) to have

$$\begin{aligned}
 & \langle \ell_{ai}^\varepsilon, E_{R,i}^\varepsilon \rangle \\
 & = \left\langle \left(-\partial_t (\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), E_{R,i}^\varepsilon \right\rangle + \left\langle \left(-\hat{p} \cdot \nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ai} \right), E_{R,i}^\varepsilon \right\rangle \\
 & \leq -\frac{d}{dt} \left\langle \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), E_{R,i}^\varepsilon \right\rangle + \left\langle \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), \partial_t E_{R,i}^\varepsilon \right\rangle \\
 & \quad + o(1) \|E_{R,i}^\varepsilon\|^2 + C \left(\|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right) \\
 & \leq -\frac{d}{dt} \left\langle \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), E_{R,i}^\varepsilon \right\rangle + \varepsilon^2 (\|\nabla_x B_R^\varepsilon\|^2 + \|f_R^\varepsilon\|^2) \\
 & \quad + o(1) \|E_{R,i}^\varepsilon\|^2 + C \left(\|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right).
 \end{aligned}$$

For the second and third terms in (6.32), we bound them as follows:

$$\begin{aligned} \sum_{i=1}^3 \left| \left\langle \partial_i a^\varepsilon - \frac{\rho_2}{\rho_1} \partial_i c^\varepsilon, E_{R,i}^\varepsilon \right\rangle \right| &\lesssim o(1) \|E_R^\varepsilon\|^2 + C \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2, \\ \sum_{i=1}^3 |\langle h_{ai}^\varepsilon, E_{R,i}^\varepsilon \rangle| &\lesssim o(1) \|E_R^\varepsilon\|^2 + \varepsilon^{-\frac{1}{3}} (\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|f_R^\varepsilon\|^2) \\ &\quad + \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) + \varepsilon^k (1+t)^{2k+1}. \end{aligned}$$

We collect the above estimates in (6.32) to obtain

$$\begin{aligned} \frac{1}{2} \left\| \sqrt{\frac{u^0}{T}} E_R^\varepsilon \right\|^2 &\leq - \sum_{i=1}^3 \frac{d}{dt} \left\langle \left((\mathbf{I} - \mathbf{P})[f_R^\varepsilon], \zeta_{ai} \right), E_{R,i}^\varepsilon \right\rangle \\ &\quad + C \left(\|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right) \\ &\quad + C \left(\varepsilon^2 \|\nabla_x B_R^\varepsilon\|^2 + \varepsilon^{-\frac{1}{3}} \|f_R^\varepsilon\|^2 + \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 \right) \\ &\quad + C \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{2}{3}} \right] \left(\|E_R^\varepsilon\|^2 + \|B_R^\varepsilon\|^2 \right) + \varepsilon^k (1+t)^{2k+1}. \end{aligned} \quad (6.33)$$

This verifies (6.31) for $i = 0$. □

PROPOSITION 6.3. It holds that

$$\frac{1}{2} \varepsilon^2 \|\nabla_x B_R^\varepsilon\|^2 \leq \varepsilon^2 \frac{d}{dt} \langle E_R^\varepsilon, \nabla_x \times B_R^\varepsilon \rangle + \varepsilon^2 \|\nabla_x E_R^\varepsilon\|^2 + C \varepsilon^2 \mathcal{E}. \quad (6.34)$$

Proof. From (2.13), we have

$$\begin{aligned} \|\nabla_x \times B_R^\varepsilon\|^2 &= \langle \partial_t E_R^\varepsilon, \nabla_x \times B_R^\varepsilon \rangle + \left\langle -4\pi \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}} f_R^\varepsilon dp, \nabla_x \times B_R^\varepsilon \right\rangle \\ &\leq \frac{d}{dt} \langle E_R^\varepsilon, \nabla_x \times B_R^\varepsilon \rangle - \langle E_R^\varepsilon, \nabla_x \times \partial_t B_R^\varepsilon \rangle + C \|f_R^\varepsilon\| \|\nabla_x \times B_R^\varepsilon\| \\ &\leq \frac{d}{dt} \langle E_R^\varepsilon, \nabla_x \times B_R^\varepsilon \rangle + \langle E_R^\varepsilon, \nabla_x \times (\nabla_x \times E_R^\varepsilon) \rangle + o(1) \|\nabla_x \times B_R^\varepsilon\|^2 + C \|f_R^\varepsilon\|^2. \end{aligned} \quad (6.35)$$

Noting

$$\langle E_R^\varepsilon, \nabla_x \times (\nabla_x \times E_R^\varepsilon) \rangle = \|\nabla_x \times E_R^\varepsilon\|^2, \quad (6.36)$$

we have

$$\frac{1}{2} \|\nabla_x \times B_R^\varepsilon\|^2 \leq \frac{d}{dt} \langle E_R^\varepsilon, \nabla_x \times B_R^\varepsilon \rangle + \|\nabla_x \times E_R^\varepsilon\|^2 + C \|f_R^\varepsilon\|^2. \quad (6.37)$$

Noting $\nabla \cdot B_R^\varepsilon = 0$, this further implies (6.34). □

REMARK 6.1. By proper linear combination of (6.3), (6.31) and (6.34), we can obtain the macroscopic dissipation and the electromagnetic field dissipation together. However, the dissipation of the electromagnetic field is too weak to be necessarily included in $\mathcal{D}$.

7. Proof of Proposition 2.1.

Proof of energy estimates. We multiply (6.3) by a sufficiently small constant κ_1 and collect the resulting inequality, (4.1), (4.7) and (4.12) to obtain that for some small positive constant δ_1 ,

$$\begin{aligned} & \frac{d}{dt} \left\{ \sum_{i=0}^2 \varepsilon^i \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \nabla_x^i f_R^\varepsilon \right\|^2 + \|\nabla_x^i E_R^\varepsilon\|^2 + \|\nabla_x^i B_R^\varepsilon\|^2 \right) - \kappa_1 \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) \right\} \\ & + \delta_1 \varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \delta \varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 \\ & + \delta_1 \left(\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right) \\ & \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right] \mathcal{E} + \varepsilon^{\frac{2}{3}} \mathcal{D} + \varepsilon^{2k+1} (1+t)^{4k+2} + \varepsilon^k (1+t)^{2k} \sqrt{\mathcal{E}}. \end{aligned} \quad (7.1)$$

Multiplying (7.1) by a large constant C_1 and adding it to the sum of (5.2), (5.8), and (5.12), we have

$$\frac{d}{dt} \mathcal{E} + \frac{3}{2} \mathcal{D} \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} \right] \mathcal{E} + \varepsilon^{\frac{2}{3}} \mathcal{D} + \varepsilon^{2k+1} (1+t)^{4k+2} + \varepsilon^k (1+t)^{2k} \sqrt{\mathcal{E}}, \quad (7.2)$$

where $\mathcal{D}$ is given in (2.36), and

$$\begin{aligned} \mathcal{E} = & C_1 \left[\sum_{i=0}^2 \varepsilon^i \left(\left\| \sqrt{\frac{4\pi T}{u^0}} \nabla_x^i f_R^\varepsilon \right\|^2 + \|\nabla_x^i E_R^\varepsilon\|^2 + \|\nabla_x^i B_R^\varepsilon\|^2 \right) - \kappa_1 \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) \right] \\ & + Y \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 + \varepsilon \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2}^2 \right). \end{aligned}$$

Note that

$$\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \lesssim \sum_{i=1}^2 \varepsilon^i \|\nabla_x^{i-1} f_R^\varepsilon\| \|\nabla_x^i f_R^\varepsilon\| \lesssim \varepsilon^{\frac{1}{2}} \left(\|f_R^\varepsilon\|^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 \right)$$

by (6.2). This verifies (2.35). Then for sufficiently small constant $\varepsilon > 0$, we use (7.2) to have

$$\frac{d}{dt} \mathcal{E} + \mathcal{D} \lesssim \left[(1+t)^{-\beta_0} + \varepsilon^{\frac{1}{3}} + \varepsilon^{2k+1} (1+t)^{4k+2} \right] \mathcal{E} + \varepsilon^{2k+1} (1+t)^{4k+2} + \varepsilon^k (1+t)^{2k}.$$

Then for $k \geq 3$, we apply Gronwall's inequality to the above inequality to have

$$\sup_{s \in [0, \bar{t}]} \left[\mathcal{E}(s) + \int_0^s \mathcal{D}(s) ds \right] \lesssim \mathcal{E}(0) + 1$$

for $\bar{t} = \varepsilon^{-1/3}$. This verifies (2.34) and (2.37).

Proof of positivity. First we show that there exists $F_R^\varepsilon(0, x, p)$ such that $F^\varepsilon(0, x, p) \geq 0$. The procedure is motivated by the analysis in [43, Lemma A.2]. We first estimate the microscopic part of the coefficients $(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_i \right]$, $1 \leq i \leq 2k-1$. By (2.3) and the definition of $\mathcal{L}$ in (2.14), we have

$$\mathcal{L} \left[(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right] = -\mathbf{M}^{-\frac{1}{2}} \left[\partial_t \mathbf{M} + \hat{p} \cdot \nabla_x \mathbf{M} - (E + \hat{p} \times B) \cdot \nabla_p \mathbf{M} \right].$$

Then, we use Lemma 3.3 to have

$$\left| (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma} \lesssim |\nabla_x(n, u, T)| + |E| + |B|. \quad (7.3)$$

By similar arguments in the proof of Lemma 3.5, we can obtain that for any $\kappa < 1$,

$$\begin{aligned} & \left(\mathbf{M}^{-\kappa} \mathcal{L} \left[(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right], (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right) \\ & \gtrsim \left| \mathbf{M}^{-\frac{\kappa}{2}} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma}^2 - C \left(\left| (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma}^2 + |E|^2 + |B|^2 \right). \end{aligned}$$

This together with similar estimation in (7.3) implies

$$\begin{aligned} & \left| \mathbf{M}^{-\frac{\kappa}{2}} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma}^2 - C \left| (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma}^2 \\ & \lesssim o(1) \left| \mathbf{M}^{-\frac{\kappa}{2}} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma}^2 + C \left(|\nabla_x(n, u, T)|^2 + |E|^2 + |B|^2 \right). \end{aligned} \quad (7.4)$$

Now we combine (7.3) and (7.4) to have

$$\left| \mathbf{M}^{-\frac{\kappa}{2}} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right|_{\sigma} \lesssim |\nabla_x(n, u, T)| + |E| + |B|.$$

Similarly, we can obtain that

$$\sum_{0 \leq j \leq 2} \left| \nabla_p^j \left(\mathbf{M}^{-\frac{\kappa}{2}} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \right) \right|_{\sigma} \lesssim |\nabla_x(n, u, T)| + |E| + |B|.$$

By the Sobolev imbedding, this implies

$$(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_1 \right] \lesssim \mathbf{M}^{\frac{\kappa}{2}} \left[|\nabla_x(n, u, T)| + |E| + |B| \right]. \quad (7.5)$$

By induction, we can use equations (A.3), (A.4) and (A.5) in Appendix A to obtain

$$\begin{aligned} (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_i \right] & \lesssim \mathbf{M}^{\frac{\kappa}{2}} \left[(|\nabla_x^i(n, u, T)| + |\nabla_x^{i-1} E| + |\nabla_x^{i-1} B|) \right. \\ & \quad \left. + \sum_{1 \leq j \leq i-1} \left(|\nabla_x^{i-j}(a_j, b_j, c_j)| + |\nabla_x^{i-j-1} E_j| + |\nabla_x^{i-j-1} B_j| \right) \right], \end{aligned} \quad (7.6)$$

for all $\kappa < 1$ and $2 \leq i \leq 2k-1$. Note that here a_j, b_j, c_j , the coefficients of the macroscopic part of $\mathbf{P}(\mathbf{M}^{-\frac{1}{2}} F_j)$, are defined in (A.1). Then we use (7.6) to have

$$\begin{aligned} F_i(0, x, p) & \lesssim \mathbf{M}^{\kappa} \left[(|\nabla_x^i(n, u, T)| + |\nabla_x^{i-1} E| + |\nabla_x^{i-1} B|) + |(a_i, b_i, c_i)| \right. \\ & \quad \left. + \sum_{1 \leq j < i} \left(|\nabla_x^{i-j}(a_j, b_j, c_j)| + |\nabla_x^{i-j-1} E_j| + |\nabla_x^{i-j-1} B_j| \right) \right]. \end{aligned} \quad (7.7)$$

Now we choose $F_R^{\varepsilon}(0, x, p)$ in the following form

$$\begin{aligned} F_R^{\varepsilon}(0, x, p) & = \mathbf{M}^{\tau}(0, x, p) \left[\sum_{j=1}^{2k-1} \left(|\nabla_x^j(n, u, T)| + |\nabla_x^{j-1} E_j| + |\nabla_x^{j-1} B_j| \right) \right. \\ & \quad \left. + \sum_{i=1}^{2k-1} |(a_i, b_i, c_i)| + \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-1-i} \left(|\nabla_x^j(a_i, b_i, c_i)| + |\nabla_x^{j-1} E_i| + |\nabla_x^{j-1} B_i| \right) \right] \end{aligned} \quad (7.8)$$

with $0 < \tau < 1$. We choose $\kappa < 1$ such that

$$k(1 - \kappa) + \tau < \kappa. \quad (7.9)$$

From (7.7), we have

$$\begin{aligned} \sum_{i=1}^{2k-1} \varepsilon^i F_i(0, x, p) &\leq C\varepsilon \mathbf{M}^\kappa(0, x, p) \left[\sum_{j=1}^{2k-1} (|\nabla_x^j(n, u, T)(0, x)| + |\nabla_x^{j-1}(E_j, B_j)(0, x)|) \right. \\ &\quad \left. + \sum_{i=1}^{2k-1} |(a_i, b_i, c_i)| + \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-1-i} (|\nabla_x^j(a_i, b_i, c_i)| + |\nabla_x^{j-1}(E_i, B_i)|) \right] \\ &\leq C_0 \varepsilon \mathbf{M}^\kappa(0, x, p) \end{aligned} \quad (7.10)$$

for some uniform constant $C_0 \geq 1$. We discuss the positivity of $F_R^\varepsilon(0, x, p)$ in two domains in $\mathbb{R}_x^3 \times \mathbb{R}_p^3$:

$$\begin{aligned} A &:= \left\{ (x, p) : \mathbf{M}(0, x, p) \geq C_0 \varepsilon \mathbf{M}^\kappa(0, x, p) \right\}, \\ B &:= \left\{ (x, p) : \mathbf{M}(0, x, p) < C_0 \varepsilon \mathbf{M}^\kappa(0, x, p) \right\}. \end{aligned}$$

In the domain A , by the expression of the Hilbert expansion (2.1), we have $F_R^\varepsilon(0, x, p) \geq 0$. In the domain B , for the chosen κ , we use (7.9) to have

$$\begin{aligned} \varepsilon^k \mathbf{M}^\tau(0, x, p) &> C_0^{k+1} \varepsilon^{k+1} \mathbf{M}^\tau(0, x, p) \\ &\geq C_0 \varepsilon \mathbf{M}^{k(1-\kappa)}(0, x, p) \mathbf{M}^\tau(0, x, p) \\ &\geq C_0 \varepsilon \mathbf{M}^\kappa(0, x, p). \end{aligned}$$

This implies that the remainder term is the dominant term in (2.1) and $F^\varepsilon(0, x, p) \geq 0$ for ε small enough. Therefore we have $F^\varepsilon(0, x, p) \geq 0$ for all (x, p) .

Based on the proof of [90, Lemma 9, Page 307–308], we may rearrange equation (1.17) as

$$\begin{aligned} &\partial_t F^\varepsilon + \hat{p} \cdot \nabla_x F^\varepsilon - \left(E^\varepsilon + \hat{p} \times b^\varepsilon \right) \nabla_p F^\varepsilon \\ &= \frac{1}{\varepsilon} \left(\int_{\mathbb{R}^3} \Phi^{ij}(p, q) F^\varepsilon(q) dq \right) \partial_{p_i} \partial_{p_j} F^\varepsilon \\ &\quad + \frac{1}{\varepsilon} \left(\int_{\mathbb{R}^3} \partial_{p_i} \Phi^{ij}(p, q) F^\varepsilon(q) dq \right) \partial_{p_j} F^\varepsilon - \frac{1}{\varepsilon} \left(\int_{\mathbb{R}^3} \Phi^{ij}(p, q) \partial_{q_j} F^\varepsilon(q) dq \right) \partial_{p_i} F^\varepsilon \\ &\quad - \frac{1}{\varepsilon} \partial_{p_i} \left(\int_{\mathbb{R}^3} \Phi^{ij}(p, q) \partial_{q_j} F^\varepsilon(q) dq \right) F^\varepsilon. \end{aligned} \quad (7.11)$$

Then clearly, there is an elliptic structure on the R.H.S. of (7.11). Therefore, using the maximum principle (see the proof of [90, Lemma 9, Page 308] and [69, Theorem 1.1, Page 201]), we have

$$\min_{t, x, p} \{ F^\varepsilon \} = \min_{x, p} \{ F_0^\varepsilon \} \geq 0.$$

Then for sufficiently smooth F^ε , as long as the initial data $F^\varepsilon \geq 0$, we naturally have $F^\varepsilon \geq 0$. For general F^ε , a standard mollification and approximation argument leads to the desired result.

8. Relativistic Landau equation.

8.1. *No-weight energy estimates.* In this section, we derive the no-weight energy estimates. Since the estimates can be derived via arguments similar to the r-VML case, we omit most of the details of the proof and only point out main differences.

Corresponding to Proposition 4.1, we have

PROPOSITION 8.1. For the remainder f_R^ε , it holds that

$$\frac{d}{dt} \|f_R^\varepsilon\|^2 + \varepsilon^{-1} \delta \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \lesssim (\varepsilon + Z)\mathcal{E} + \varepsilon\mathcal{D} + \varepsilon^{2k+3}. \quad (8.1)$$

Proof. The whole proof can be done in a similar and much simpler way as in Proposition 4.1. We only proceed to the following two estimates:

$$\begin{aligned} & \left| \left\langle \mathbf{M}^{-\frac{1}{2}} \left(\partial_t \mathbf{M}^{\frac{1}{2}} + \hat{p} \cdot \nabla_x \mathbf{M}^{\frac{1}{2}} \right) f_R^\varepsilon, f_R^\varepsilon \right\rangle \right| \lesssim Z \left\| \sqrt{p^0} f_R^\varepsilon \right\|^2 \\ & \lesssim Z \left(\left\| \sqrt{p^0} \mathbf{P}[f_R^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{p^0 \leq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 + \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right) \\ & \lesssim Z \left(\left\| \mathbf{P}[f_R^\varepsilon] \right\|^2 + \varepsilon^{-1} o(1) \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \right) \\ & \lesssim Z \|f_R^\varepsilon\|^2 + o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + Z \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1}\kappa} p^0 |(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]|^2 \\ & \lesssim Z \|f_R^\varepsilon\|^2 + o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + Z \varepsilon \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w_0}^2, \end{aligned} \quad (8.2)$$

by Lemma 3.1, and

$$|\langle \overline{S}, f_R^\varepsilon \rangle| = |\langle \overline{S}, (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \rangle| \lesssim o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{2k+3},$$

by (B.3) in Appendix B. □

For derivatives of f_R^ε , we can obtain the following estimates.

PROPOSITION 8.2. For the remainder f_R^ε , it holds that

$$\begin{aligned} & \varepsilon \left(\frac{d}{dt} \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^{-1} \delta \|(\mathbf{I} - \mathbf{P})[\nabla_x f_R^\varepsilon]\|_\sigma^2 \right) \\ & \lesssim (\varepsilon + Z)\mathcal{E} + o(1) \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon\mathcal{D} + \varepsilon^{2k+3}, \end{aligned} \quad (8.3)$$

and

$$\begin{aligned} & \varepsilon^2 \left(\frac{d}{dt} \|\nabla_x^2 f_R^\varepsilon\|^2 + \varepsilon^{-1} \delta \|(\mathbf{I} - \mathbf{P})[\nabla_x^2 f_R^\varepsilon]\|_\sigma^2 \right) \\ & \lesssim (\varepsilon + Z)\mathcal{E} + o(1) \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^1}^2 + \varepsilon\mathcal{D} + \varepsilon^{2k+4}. \end{aligned} \quad (8.4)$$

8.2. *Weighted energy estimates.* In this subsection, we will derive the weighted energy estimates of f^ε .

8.2.1. *Weighted basic energy estimate.* We take microscopic projection onto (i.e. apply operator $\mathbf{I} - \mathbf{P}$ on both sides of) (2.44) to have

$$\begin{aligned} & \partial_t((\mathbf{I} - \mathbf{P})[f_R^\varepsilon]) + \hat{p} \cdot \nabla_x((\mathbf{I} - \mathbf{P})[f_R^\varepsilon]) + \frac{1}{\varepsilon} \mathcal{L}[(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] \\ &= \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon] + \Gamma[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i] \right\} \\ & \quad - \mathbf{M}^{-\frac{1}{2}} \left(\partial_t \mathbf{M}^{\frac{1}{2}} + \hat{p} \cdot \nabla_x \mathbf{M}^{\frac{1}{2}} \right) ((\mathbf{I} - \mathbf{P})[f_R^\varepsilon]) + (\mathbf{I} - \mathbf{P})[\bar{S}] + [\mathbf{P}, \tau][f_R^\varepsilon], \end{aligned} \quad (8.5)$$

where

$$[\mathbf{P}, \tau] = \mathbf{P}\tau - \tau\mathbf{P} = (\mathbf{I} - \mathbf{P})\tau - \tau(\mathbf{I} - \mathbf{P}) \quad (8.6)$$

denotes a commutator of operators $\mathbf{P}$ and τ which is given by

$$\tau = \partial_t + \hat{p} \cdot \nabla_x + \mathbf{M}^{-\frac{1}{2}} \left(\partial_t \mathbf{M}^{\frac{1}{2}} + \hat{p} \cdot \nabla_x \mathbf{M}^{\frac{1}{2}} \right). \quad (8.7)$$

PROPOSITION 8.3. For the remainder f_R^ε , it holds that

$$\begin{aligned} & \frac{d}{dt} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 + \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0, \sigma}^2 + Y \left\| \sqrt{p^0} (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^0}^2 \\ & \lesssim \varepsilon \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+3}, \end{aligned} \quad (8.8)$$

$$\begin{aligned} & \varepsilon \left(\frac{d}{dt} \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \varepsilon^{-1} \delta \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1, \sigma}^2 + Y \left\| \sqrt{p^0} \nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon] \right\|_{w^1}^2 \right) \\ & \lesssim \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+4}, \end{aligned} \quad (8.9)$$

and

$$\begin{aligned} & \varepsilon^3 \left(\frac{d}{dt} \|\nabla_x^2 f_R^\varepsilon\|_{w^2}^2 + \varepsilon^{-1} \delta \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2, \sigma}^2 + Y \left\| \sqrt{p^0} \nabla_x^2 f_R^\varepsilon \right\|_{w^2}^2 \right) \\ & \lesssim \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+6}. \end{aligned} \quad (8.10)$$

8.3. *Macroscopic dissipation.* In this subsection, we study the macroscopic structure of (2.44).

As in (2.17), denote

$$\mathbf{P}[f_R^\varepsilon] := \mathbf{M}^{\frac{1}{2}} \left\{ \left[a^\varepsilon(t, x) - \frac{\rho_2(t, x)}{\rho_1(t, x)} c^\varepsilon(t, x) \right] + p \cdot b^\varepsilon(t, x) + p^0 c^\varepsilon(t, x) \right\} \in \mathcal{N}.$$

PROPOSITION 8.4. There are two functionals $\mathcal{E}_i^{mac}$ for $i = 1, 2$ satisfying

$$\mathcal{E}_i^{mac} \lesssim \|\nabla_x^{i-1} f_R^\varepsilon\| \|\nabla_x f_R^\varepsilon\|, \quad (8.11)$$

such that

$$\begin{aligned} & -\frac{d}{dt} (\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac}) + \left(\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 \right) \\ & \lesssim \varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon(\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+3}. \end{aligned} \quad (8.12)$$

Proof. As in Proposition 6.1, it can be proved via local conservation laws and the macroscopic equations of f_R^ε . Write (2.44) as

$$\partial_t f_R^\varepsilon + \hat{p} \cdot \nabla_x f_R^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon] = \bar{h}^\varepsilon, \quad (8.13)$$

where

$$\begin{aligned} \bar{h}^\varepsilon &= \varepsilon^{k-1} \Gamma[f_R^\varepsilon, f_R^\varepsilon] + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma[\mathbf{M}^{-\frac{1}{2}} F_i, f_R^\varepsilon] + \Gamma[f_R^\varepsilon, \mathbf{M}^{-\frac{1}{2}} F_i] \right\} \\ &\quad - \mathbf{M}^{-\frac{1}{2}} (\partial_t \mathbf{M}^{\frac{1}{2}} + \hat{p} \cdot \nabla_x \mathbf{M}^{\frac{1}{2}}) f_R^\varepsilon + \bar{S}. \end{aligned}$$

Local conservation laws: Similar to the derivation of (6.11), (6.12) and (6.13), we can obtain

$$nu^0 \partial_t a^\varepsilon + \frac{n}{\gamma} \nabla_x \cdot b^\varepsilon = \Xi_1[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}} (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] dp + \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp, \quad (8.14)$$

$$n \left(\frac{K_1(\gamma)}{\gamma K_2(\gamma)} + \frac{4}{\gamma^2} \right) \partial_t b^\varepsilon + \frac{n}{\gamma} \nabla_x a^\varepsilon \quad (8.15)$$

$$= \Xi_2[a^\varepsilon, b^\varepsilon, c^\varepsilon] - \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} p \mathbf{M}^{\frac{1}{2}} (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] dp + \int_{\mathbb{R}^3} p \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp,$$

$$n \left(-\frac{K_1^2(\gamma)}{K_2^2(\gamma)} - \frac{3}{\gamma} \frac{K_1(\gamma)}{K_2(\gamma)} + 1 + \frac{3}{\gamma^2} \right) \partial_t c^\varepsilon + \frac{n}{\gamma^2} \nabla_x \cdot b^\varepsilon \quad (8.16)$$

$$\begin{aligned} &= \Xi_3[a^\varepsilon, b^\varepsilon, c^\varepsilon] + \int_{\mathbb{R}^3} p^0 \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp \\ &\quad - u^0 \left(\frac{K_1(\gamma)}{K_2(\gamma)} + \frac{3}{\gamma} \right) \left(-\nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}} (\mathbf{I} - \mathbf{P}) [f_R^\varepsilon] dp + \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}} \bar{h}^\varepsilon dp \right). \end{aligned}$$

Macroscopic equations: Secondly, we turn to the macroscopic equations of f_R^ε . Splitting f_R^ε as the macroscopic part $\mathbf{P}[f_R^\varepsilon]$ and the microscopic $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$ part in (8.13), we have

$$\begin{aligned} &\left(\partial_t \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + \hat{p} \cdot \partial_t b^\varepsilon + p^0 \partial_t c^\varepsilon \right) \mathbf{M}^{\frac{1}{2}} \\ &\quad + \hat{p} \cdot \left\{ \nabla_x \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + \nabla_x b^\varepsilon \cdot \hat{p} + p^0 \nabla_x c^\varepsilon \right\} \mathbf{M}^{\frac{1}{2}} = \ell^\varepsilon + h^\varepsilon, \end{aligned} \quad (8.17)$$

where

$$\ell^\varepsilon := -(\partial_t + \hat{p} \cdot \nabla_x) [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \quad (8.18)$$

$$h^\varepsilon := - \left\{ \left(a^\varepsilon - \frac{\rho_2}{\rho_1} c^\varepsilon \right) + p \cdot b^\varepsilon + p^0 c^\varepsilon \right\} (\partial_t + \hat{p} \cdot \nabla_x) \mathbf{M}^{\frac{1}{2}} + \bar{h}^\varepsilon. \quad (8.19)$$

For fixed t, x , we compare the coefficients in front of

$$\left\{ \mathbf{M}^{\frac{1}{2}}, p_i \mathbf{M}^{\frac{1}{2}}, p^0 \mathbf{M}^{\frac{1}{2}}, \frac{p_i}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i^2}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i p_j}{p^0} \mathbf{M}^{\frac{1}{2}} \right\}, \quad 1 \leq i, j \leq 3 \quad (8.20)$$

on both sides of (8.17) and get the following macroscopic equations:

$$\begin{aligned}
 \partial_t a^\varepsilon - \frac{\rho_2}{\rho_1} \partial_t c^\varepsilon &= \ell_a^\varepsilon + h_a^\varepsilon + \partial_t \left(\frac{\rho_2}{\rho_1} \right) c^\varepsilon, \\
 \partial_t b_i^\varepsilon + \partial_i c^\varepsilon &= \ell_{bi}^\varepsilon + h_{bi}^\varepsilon, \\
 \partial_t c^\varepsilon &= \ell_c^\varepsilon + h_c^\varepsilon, \\
 \partial_i a^\varepsilon - \frac{\rho_2}{\rho_1} \partial_i c^\varepsilon &= \ell_{ai}^\varepsilon + h_{ai}^\varepsilon + c \partial_i \left(\frac{\rho_2}{\rho_1} \right) c^\varepsilon, \\
 c \partial_i b_i^\varepsilon &= \ell_{ii}^\varepsilon + h_{ii}^\varepsilon, \\
 \partial_i b_j^\varepsilon + \partial_j b_i^\varepsilon &= \ell_{ij}^\varepsilon + h_{ij}^\varepsilon, \quad i \neq j.
 \end{aligned} \tag{8.21}$$

Here $\ell_a^\varepsilon, h_a^\varepsilon, \ell_{bi}^\varepsilon, h_{bi}^\varepsilon, \ell_c^\varepsilon, h_c^\varepsilon, \ell_{ai}^\varepsilon, h_{ai}^\varepsilon, \ell_{ii}^\varepsilon, h_{ii}^\varepsilon$, and $\ell_{ij}^\varepsilon, h_{ij}^\varepsilon$ take the form of

$$(\ell^\varepsilon, \zeta) \quad \text{and} \quad (h^\varepsilon, \zeta),$$

where ζ is linear combinations of

$$\left\{ \mathbf{M}^{\frac{1}{2}}, p_i \mathbf{M}^{\frac{1}{2}}, p^0 \mathbf{M}^{\frac{1}{2}}, \frac{p_i}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i^2}{p^0} \mathbf{M}^{\frac{1}{2}}, \frac{p_i p_j}{p^0} \mathbf{M}^{\frac{1}{2}} \right\}. \tag{8.22}$$

For $m = 0, 1$, we have the following estimates:

$$\begin{aligned}
 &\|\nabla_x^m h_a^\varepsilon\| + \|\nabla_x^m h_{bi}^\varepsilon\| + \|\nabla_x^m h_c^\varepsilon\| + \|\nabla_x^m h_{ai}^\varepsilon\| + \|\nabla_x^m h_{ii}^\varepsilon\| + \|\nabla_x^m h_{ij}^\varepsilon\| \\
 &\lesssim \varepsilon^{\frac{1}{2}} \|f_R^\varepsilon\|_{H^m} \\
 &\quad + \sum_{l=1}^{2k-1} \left(\|\mathbf{M}^{-\frac{1}{2}} F_l\|_{W_x^{m,\infty} L_p^2} \|f_R^\varepsilon\|_{H_\sigma^m} + \|\mathbf{M}^{-\frac{1}{2}} |F_l|_\sigma\|_{W_x^{m,\infty}} \|\nabla_x^j f_R^\varepsilon\| \right) + \varepsilon^{k+1} \\
 &\lesssim \|\nabla_x^j (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H_\sigma^m} + \|f_R^\varepsilon\|_{H^m} + Z \|f_R^\varepsilon\|_{H^m} + \varepsilon^{k+1}.
 \end{aligned} \tag{8.23}$$

For brevity, we only give the estimate of $\|\nabla_x b^\varepsilon\|$ in (8.12) since other estimates can be derived similarly. From the last two equalities in (6.16), we have

$$-\Delta b_j^\varepsilon - \partial_j \nabla_x \cdot b^\varepsilon = - \sum_{i=1}^3 \partial_i (\ell_{ij}^\varepsilon + h_{ij}^\varepsilon) (1 + \delta_{ij}). \tag{8.24}$$

We multiply b_j^ε and integrate over $\mathbb{R}^3$ to get

$$\begin{aligned}
 \|\nabla_x b^\varepsilon\|^2 + \|\nabla_x \cdot b^\varepsilon\|^2 &= \sum_{i=1}^3 \langle (\ell_{ij}^\varepsilon + h_{ij}^\varepsilon) (1 + \delta_{ij}), \partial_i b_j^\varepsilon \rangle \\
 &= \sum_{i=1}^3 (1 + \delta_{ij}) \left\langle \left(-(\partial_t + \hat{p} \cdot \nabla_x) [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle \\
 &\quad + \sum_{i=1}^3 (1 + \delta_{ij}) \langle (h_{ij}^\varepsilon, \zeta_{ij}), \partial_i b_j^\varepsilon \rangle.
 \end{aligned} \tag{8.25}$$

For the first term related to $(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]$, we have

$$\begin{aligned}
 & \left\langle \left(-(\partial_t + \hat{p} \cdot \nabla_x) [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle \\
 &= \left\langle \left(-\partial_t [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle + \left\langle \left(-\hat{p} \cdot \nabla_x [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]] - \frac{1}{\varepsilon} \mathcal{L}[f_R^\varepsilon], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle \\
 &\leq -\frac{d}{dt} \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle + \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_i \partial_t b_j^\varepsilon \right\rangle \\
 &\quad + o(1) \|\nabla_x b^\varepsilon\|^2 + C \left(\|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right).
 \end{aligned} \tag{8.26}$$

By (6.12) and (6.20) with terms related to the electromagnetic field be zero, we have

$$\begin{aligned}
 & \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_i \partial_t b_j^\varepsilon \right\rangle = - \left\langle \left(\partial_i [(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_t b_j^\varepsilon \right\rangle \\
 & \lesssim o(1) (\|\nabla_x a^\varepsilon\|^2 + \|\nabla_x c^\varepsilon\|^2) + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{H^1}^2 + \|f_R^\varepsilon\|^2 + \varepsilon^{k+1}.
 \end{aligned} \tag{8.27}$$

Then we use (8.23) again to obtain

$$\begin{aligned}
 & \frac{1}{2} \|\nabla_x b^\varepsilon\|^2 + \|\nabla_x \cdot b^\varepsilon\|^2 \\
 & \leq -\frac{d}{dt} \left\langle \left([(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]], \zeta_{ij} \right), \partial_i b_j^\varepsilon \right\rangle + o(1) (\|\nabla_x a^\varepsilon\|^2 + \|\nabla_x c^\varepsilon\|^2) \\
 & \quad + \varepsilon^{-2} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + (1 + Z) \|f_R^\varepsilon\|^2 + \varepsilon^{k+1}.
 \end{aligned} \tag{8.28}$$

□

8.4. Proof of Proposition 2.2.

Proof of energy estimates. Multiplying (6.3) by a small constant κ_2 and adding it to the sum of (8.1), (8.3) and (8.4), we obtain that for some small constant $\delta_2 > 0$,

$$\begin{aligned}
 & \frac{d}{dt} \left(\left(\|f_R^\varepsilon\|^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 \right) - \kappa_2 \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) \right) \\
 & + \delta_2 \left(\varepsilon \|\nabla_x \mathbf{P}[f_R^\varepsilon]\|^2 + \varepsilon^2 \|\nabla_x^2 \mathbf{P}[f_R^\varepsilon]\|^2 \right) \\
 & + \delta_2 \left(\varepsilon^{-1} \|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \|\nabla_x (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 + \varepsilon \|\nabla_x^2 (\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_\sigma^2 \right) \\
 & \lesssim (\varepsilon + Z) \mathcal{E} + \varepsilon \mathcal{D} + \varepsilon^{2k+3}.
 \end{aligned} \tag{8.29}$$

Multiplying (8.29) by a large constant C_2 and adding it to the sum of (8.8), (8.9), and (8.10), we have

$$\frac{d}{dt} \left(\mathcal{E} - \kappa_2 \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) \right) + \frac{3}{2} \mathcal{D} \lesssim (\varepsilon + Z) \mathcal{E} + \varepsilon \mathcal{D} + \varepsilon^{2k+3}, \tag{8.30}$$

where $\mathcal{D}$ is given in (2.53), and

$$\begin{aligned} \mathcal{E} = & C_1 \left[\sum_{i=0}^2 \varepsilon^i \left\| \sqrt{\frac{4\pi T}{u^0}} \nabla_x^i f_R^\varepsilon \right\|^2 - \kappa_2 \left(\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \right) \right] \\ & + Y \left(\|(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^0}^2 + \varepsilon \|\nabla_x(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^1}^2 + \|\nabla_x^2(\mathbf{I} - \mathbf{P})[f_R^\varepsilon]\|_{w^2}^2 \right). \end{aligned}$$

Note that

$$\varepsilon \mathcal{E}_1^{mac} + \varepsilon^2 \mathcal{E}_2^{mac} \lesssim \sum_{i=1}^2 \varepsilon^i \|\nabla_x^{i-1} f_R^\varepsilon\| \|\nabla_x^i f_R^\varepsilon\| \lesssim \varepsilon^{\frac{1}{2}} \left(\|f_R^\varepsilon\|^2 + \varepsilon \|\nabla_x f_R^\varepsilon\|^2 + \varepsilon^2 \|\nabla_x^2 f_R^\varepsilon\|^2 \right)$$

by (8.11). This verifies (2.52). When ε is sufficiently small, we know $\varepsilon \lesssim Z$. Thus, we have

$$\frac{d}{dt} \mathcal{E} + \mathcal{D} \lesssim Z \mathcal{E} + \varepsilon^{2k+3}. \quad (8.31)$$

By Gronwall's inequality, for $t \leq t_0$, we have

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s) ds \lesssim e^{Zt} \mathcal{E}(0) + \varepsilon^{2k+3} \int_0^t e^{Z(t-s)} ds \lesssim e^{Zt} \mathcal{E}(0) + Z^{-1} \varepsilon^{2k+3}. \quad (8.32)$$

Due to (2.49), we know $Zt_0 \lesssim 1$. Hence, we have

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s) ds \lesssim \mathcal{E}(0) + \varepsilon^{2k+3}, \quad t \leq t_0. \quad (8.33)$$

This verifies the validity of (2.51) and (2.55).

Proof of positivity. It is analogous to the corresponding part in the proof of Proposition 2.1. We omit the proof for brevity.

Appendix A. Expansion of the relativistic Vlasov-Maxwell-Landau system.

In this part, we list our result about the construction and regularity estimates of the coefficients in the Hilbert expansion (2.1). For any integer $n \in [1, 2k-1]$, we decompose $\mathbf{M}^{-\frac{1}{2}} F_n$ as the sum of macroscopic and microscopic parts:

$$\begin{aligned} \mathbf{M}^{-\frac{1}{2}} F_n &= \mathbf{P} \left[\mathbf{M}^{-\frac{1}{2}} F_n \right] + (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_n \right] \\ &= \left(a_n(t, x) + b_n(t, x) \cdot p + c_n(t, x) p^0 \right) \mathbf{M}^{\frac{1}{2}} + (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_n \right]. \end{aligned} \quad (A.1)$$

PROPOSITION A.1. For any integer $n \in [0, 2k-2]$, assume that (F_i, E_i, B_i) have been constructed for all $0 \leq i \leq n$. Then the microscopic part $(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right]$ can be

written as:

$$\begin{aligned}
 (\mathbf{I} - \mathbf{P})\left(\mathbf{M}^{-\frac{1}{2}}F_{n+1}\right) &= \mathcal{L}^{-1}\left[-\mathbf{M}^{-\frac{1}{2}}\left(\partial_t F_n + \hat{p} \cdot \nabla_x F_n - \frac{1}{\varepsilon} \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} [\mathcal{C}(F_i, F_j) \right. \right. \\
 &\quad \left. \left. + \mathcal{C}(F_i, F_j)\right] + \sum_{\substack{i+j=n \\ i,j \geq 0}} \left(E_i + \hat{p} \times B_i\right) \cdot \nabla_p F_j\right]. \quad (\text{A.2})
 \end{aligned}$$

And $a_{n+1}(t, x), b_{n+1}(t, x), c_{n+1}(t, x), E_{n+1}(t, x), B_{n+1}(t, x)$ satisfy the following system:

$$\begin{aligned}
 &\partial_t (nu^0 a_{n+1} + (\mathfrak{e} + P)u^0(u \cdot b_{n+1}) + (\mathfrak{e}(u^0)^2 + P|u|^2)c_{n+1}) \\
 &+ \nabla_x \cdot (nua_{n+1} + (\mathfrak{e} + P)u(u \cdot b_{n+1}) + Pb_{n+1} + (\mathfrak{e} + P)u^0 uc_{n+1}) \\
 &+ \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}}F_{n+1}\right] dp = 0, \quad (\text{A.3})
 \end{aligned}$$

$$\begin{aligned}
 &\partial_t \left\{ (\mathfrak{e} + P)u^0 u_j a_{n+1} + \frac{n}{\gamma K_2(\gamma)} \left((6K_3(\gamma) + \gamma K_2(\gamma))u^0 u_j (u \cdot b_{n+1}) + K_3(\gamma)u^0 b_{n+1,j} \right) \right. \\
 &+ \left. \frac{n}{\gamma K_2(\gamma)} \left((5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2 \right) u_j c_{n+1} \right\} \\
 &+ \nabla_x \cdot \left((\mathfrak{e} + P)u_j u a_{n+1} + \frac{n}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma))u_j u \left((u \cdot b_{n+1}) + u^0 c_{n+1} \right) \right) \\
 &+ \partial_{x_j} (Pa_{n+1}) + \nabla_x \cdot \left(\frac{nK_3(\gamma)}{\gamma K_2(\gamma)} (ub_{n+1,j} + u_j b_{n+1}) \right) \\
 &+ \partial_{x_j} \left(\frac{nK_3(\gamma)}{\gamma K_2(\gamma)} (u \cdot b_{n+1} + u^0 c_{n+1}) \right) \\
 &+ E_{0,j} (nu^0 a_{n+1} + (\mathfrak{e} + P)u^0(u \cdot b_{n+1}) + (\mathfrak{e}(u^0)^2 + P|u|^2)c_{n+1}) \\
 &+ \left((nua_{n+1} + (\mathfrak{e} + P)u(u \cdot b_{n+1}) + Pb_{n+1} + (\mathfrak{e} + P)u^0 uc_{n+1}) \times B \right)_j \\
 &+ \left(nu^0 E_{n+1,j} + (nu \times B_{n+1})_j \right) \\
 &+ \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} E_{k,j} (nu^0 a_l + (\mathfrak{e} + P)u^0(u \cdot b_l) + (\mathfrak{e}(u^0)^2 + P|u|^2)c_l) \\
 &+ \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \left((nua_l + (\mathfrak{e} + P)u(u \cdot b_l) + Pb_l + (\mathfrak{e} + P)u^0 uc_l) \times B_k \right)_j \\
 &+ \nabla_x \cdot \int_{\mathbb{R}^3} \frac{p_j p}{p^0} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}}F_{n+1}\right] dp + \left(\int_{\mathbb{R}^3} \hat{p} \times B \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}}F_{n+1}\right] dp \right)_j \\
 &+ \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \left(\int_{\mathbb{R}^3} \hat{p} \times B_k \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}}F_l\right] dp \right)_j = 0, \quad (\text{A.4})
 \end{aligned}$$

for $j = 1, 2, 3$ with $b_{n+1} = (b_{n+1,1}, b_{n+1,2}, b_{n+1,3})$, $E_{n+1} = (E_{n+1,1}, E_{n+1,2}, E_{n+1,3})$,

$$\begin{aligned}
 & \partial_t \left\{ (\epsilon(u^0)^2 + P|u|^2)a_{n+1} + \frac{n(u \cdot b_{n+1})}{\gamma K_2(\gamma)} \left((5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2 \right) \right. \\
 & + \frac{n}{\gamma K_2(\gamma)} \left((3K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + 3K_3(\gamma)|u|^2 \right) u^0 c_{n+1} \Big\} \\
 & + \nabla_x \cdot ((\epsilon + P)u^0 a_{n+1}) \\
 & + \nabla_x \cdot \left\{ \frac{n}{\gamma K_2(\gamma)} (6K_3(\gamma) + \gamma K_2(\gamma)) u^0 u (u \cdot b_{n+1}) + \frac{nu^0 K_3(\gamma)}{\gamma K_2(\gamma)} u^0 b_{n+1} \right. \\
 & + \frac{n}{\gamma K_2(\gamma)} \left((5K_3(\gamma) + \gamma K_2(\gamma))(u^0)^2 + K_3(\gamma)|u|^2 \right) u c_{n+1} \Big\} \\
 & + \left(nu \cdot E_{n+1} + nu \cdot E a_{n+1} + (\epsilon + P)(u \cdot b_{n+1})(u \cdot E) \right. \\
 & + PE \cdot b_{n+1} + (\epsilon + P)u^0(u \cdot E)c_{n+1} \Big) \\
 & + \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right] dp \cdot E \\
 & + \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \left(nu \cdot E_k a_l + (\epsilon + P)(u \cdot b_l)(u \cdot E_k) + PE_k \cdot b_l + (\epsilon + P)u^0(u \cdot E_k)c_l \right. \\
 & + \left. \int_{\mathbb{R}^3} \hat{p} \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_l \right] dp \cdot E_k \right) + \nabla_x \cdot \int_{\mathbb{R}^3} p \mathbf{M}^{\frac{1}{2}}(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right] dp = 0,
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 & \partial_t E_{n+1} - \nabla_x \times B_{n+1} \\
 & = 4\pi (nu a_{n+1} + P b_{n+1} + (\epsilon + P)u(u \cdot b_{n+1}) + (\epsilon + P)u^0 u c_{n+1}) \\
 & + 4\pi \int_{\mathbb{R}^3} \left(\hat{p} \mathbf{M}^{\frac{1}{2}} \{ \mathbf{I} - \mathbf{P} \} \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right] \right) dp, \\
 & \partial_t B_{n+1} + \nabla_x \times E_{n+1} = 0, \\
 & \nabla_x \cdot E_{n+1} = -4\pi (nu^0 a_{n+1} + (\epsilon + P)u^0(u \cdot b_{n+1}) + (\epsilon(u^0)^2 + P|u|^2)c_{n+1}), \\
 & \nabla_x \cdot B_{n+1} = 0.
 \end{aligned} \tag{A.6}$$

Furthermore, assume $a_{n+1}(0, x), b_{n+1}(0, x), c_{n+1}(0, x), E_{n+1}(0, x), B_{n+1}(0, x) \in H^N$, $N \geq 1$, be given initial data to the system consisted of equations (A.3), (A.4), (A.5) and (A.6). Then the linear system is well-posed in $C^0([0, \infty); H^N)$. Moreover, it holds that

$$\begin{aligned}
 & |F_{n+1}| \lesssim (1+t)^{n+1} \mathbf{M}^{1-}, \quad |\nabla_p F_{n+1}| \lesssim (1+t)^{n+1} \mathbf{M}^{1-}, \\
 & |\nabla_x F_{n+1}| \lesssim (1+t)^{n+1} \mathbf{M}^{1-}, \quad |\nabla_x \nabla_p F_{n+1}| \lesssim (1+t)^{n+1} \mathbf{M}^{1-}, \\
 & |\nabla_x^2 F_{n+1}| \lesssim (1+t)^n \mathbf{M}^{1-}, \quad |\nabla_x^2 \nabla_p F_{n+1}| \lesssim (1+t)^{n+1} \mathbf{M}^{1-}, \\
 & |E_{n+1}| + |B_{n+1}| + |\nabla_x E_{n+1}| + |\nabla_x B_{n+1}| + |\nabla_x^2 E_{n+1}| + |\nabla_x^2 B_{n+1}| \lesssim (1+t)^{n+1}.
 \end{aligned} \tag{A.7}$$

Proof. The whole proof follows from analogous arguments as in [54, Appendix 3]. We omit the details for brevity. $\square$

Appendix B. Expansion of the relativistic Landau equation. In this part, we list our result about the construction and regularity estimates of the coefficients in the Hilbert expansion (2.1). The proof can be done in a similar way as that in [54, Appendix 3], so we only record the results. For any integer $n \in [1, 2k - 1]$, we decompose $\mathbf{M}^{-\frac{1}{2}} F_n$ as the sum of macroscopic and microscopic parts:

$$\begin{aligned} \frac{F_n}{\mathbf{M}^{\frac{1}{2}}} &= \mathbf{P} \left[\mathbf{M}^{-\frac{1}{2}} F_n \right] + (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_n \right] \\ &= \left(a_n(t, x) + b_n(t, x) \cdot p + c_n(t, x) p^0 \right) \mathbf{M}^{\frac{1}{2}} + (\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_n \right]. \end{aligned} \quad (\text{B.1})$$

PROPOSITION B.1. For any integer $n \in [0, 2k - 2]$, assume that F_i have been constructed for all $0 \leq i \leq n$. Then the microscopic part $(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right]$ can be written as:

$$(\mathbf{I} - \mathbf{P}) \left[\mathbf{M}^{-\frac{1}{2}} F_{n+1} \right] = \mathcal{L}^{-1} \left[-\mathbf{M}^{-\frac{1}{2}} \left(\partial_t F_n + \hat{p} \cdot \nabla_x F_n - \sum_{\substack{i+j=n+1 \\ i, j \geq 1}} \mathcal{C}[F_i, F_j] \right) \right]. \quad (\text{B.2})$$

And $a_{n+1}(t, x), b_{n+1}(t, x), c_{n+1}(t, x)$ satisfy (A.4), (A.4) and (A.5) by deleting all terms related to the electromagnetic field. Furthermore, assume $a_{n+1}(0, x), b_{n+1}(0, x), c_{n+1}(0, x) \in H^N$ with $N \geq 1$ are given initial data to the corresponding linear system. Then this linear system is well-posed in $C^0([0, \infty); H^N)$. Moreover, it holds that for sufficiently large N

$$\begin{aligned} |F_{n+1}| &\lesssim \mathbf{M}^{1-}, & |\nabla_p F_{n+1}| &\lesssim \mathbf{M}^{1-}, \\ |\nabla_x F_{n+1}| &\lesssim \mathbf{M}^{1-}, & |\nabla_x \nabla_p F_{n+1}| &\lesssim \mathbf{M}^{1-}, \\ |\nabla_x^2 F_{n+1}| &\lesssim \mathbf{M}^{1-}, & |\nabla_x^2 \nabla_p F_{n+1}| &\lesssim \mathbf{M}^{1-}. \end{aligned} \quad (\text{B.3})$$

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