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Geometric Corrections in Knudsen Layer Expansion

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Abstract. Due to singularity from the grazing set, the classical diffusive expansion (5) has been shown invalid in L^∞ for non-flat domains [1]. We justify the diffusive limit of neutron transport equation in bounded domains by developing a new boundary layer theory which captures the geometric effects of the boundary.

INTRODUCTION

The theory of hydrodynamic limits concerns the rigorous derivation of fluid equations (Navier-Stokes equations, Euler equations , etc.) and thermodynamic equations (diffusion equation , etc.) from kinetic theory (Boltzmann equation , Landau equation, neutron transport equation, etc.). This is a key ingredient to tackle Hilbert's 6th Problem, i.e. to derive fundamental equations in physics in an axiomatic manner.

For a large number ($\sim 10^{23}$) of particles, kinetic equations help track the position and velocity of each particle using probabilistic tools. Intuitively, when the collisions occur more and more frequently, the overall behavior of particles is closer and closer to fluid motion. The goal of a hydrodynamic limit is to justify such a limiting process. Specifically, we intend to study the asymptotic behavior of kinetic equations when the Knudsen number, which measures the relative distance a particle can travel between two collisions, shrinks to zero. In this work, we focus on the bounded domain case, where the kinetic corrections – boundary layers, play a key role.

Problem Formulation

We consider the steady neutron transport equation for the neutron density $u^\varepsilon(x,w)$ in a three-dimensional bounded domain $\Omega \ni x = (x_1,x_2,x_3)$ with one-speed velocity $w = (w_1,w_2,w_3) \in \mathbb{S}^2$, which reads

$$w\cdot\nabla_x u^\varepsilon + \frac{1}{\varepsilon}\Big(u^\varepsilon - \overline{u^\varepsilon}\Big) = 0, \tag{1}$$

where

$$\overline{u^\varepsilon}(x) = \fint_{\mathbb{S}^2} u^\varepsilon(x,w)\mathrm{d}w, \tag{2}$$

with the Knudsen number $0<\varepsilon\ll 1$. Equation (1) is accompanied by either in-flow boundary condition (when the incoming density is fixed)

$$u^\varepsilon(x_0,w) = g(x_0,w) \text{ for } w\cdot n<0 \text{ and } x_0\in\partial\Omega, \tag{3}$$

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or diffuse-reflection boundary condition (when the particles are reflected from the boundary in random directions)

$$u^{\varepsilon}(x_0,w)=\mathscr{P}[u^{\varepsilon}](x_0)+\varepsilon g(x_0,w):=\fint_{w\cdot n>0}u^{\varepsilon}(x_0,w)(w\cdot n)\mathrm{d}w+\varepsilon g(x_0,w)\ \text{ for }\ w\cdot n<0\ \text{ and }\ x_0\in\partial\Omega, \tag{4}$$

for outward unit normal vector n on $\partial\Omega$ and some given function $g(x_0,w)$. We intend to study the asymptotic behavior of u^{ε} as $\varepsilon\to 0^{+}$.

Background

The study of neutron transport equation (NTE) in bounded domains, has attracted a lot of attention since the dawn of the atomic age. Besides its significance in nuclear sciences and medical imaging, NTE is usually regarded as a linear prototype of the more important yet more complicated nonlinear Boltzmann equation, and thus, is an ideal starting point to develop new theories and techniques. The early investigation of NTE focuses on its formal expansion with respect to ε, explicit solution and numerical methods. We refer to [2], [3], [4], [5], [6], [7], [8], [9], [10] for more details.

Generally speaking, the solution u^{ε} to (1) varies smoothly and slowly in the interior of Ω, and behaves like $u^{\varepsilon}-\overline{u^{\varepsilon}}=0$ which ignores $w\cdot\nabla_x u^{\varepsilon}$. However, u^{ε} changes dramatically when approaching the boundary $\partial\Omega$ and in this regime, $w\cdot\nabla_x u^{\varepsilon}$ plays a crucial role. The smaller ε is, the more violently u^{ε} changes.

This phenomenon indicates that u^{ε} can actually be described in two distinct regimes with different scalings, namely, the interior solution U and the boundary layer $\mathscr{U}$. The interior solution satisfies certain fluid equations or thermodynamic equations, and the boundary layer satisfies a half-space kinetic equation, which decays rapidly when it is away from the boundary.

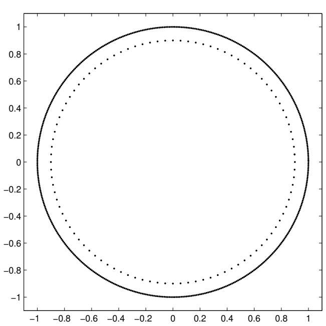

FIGURE 1. Boundary Layer in a Disk

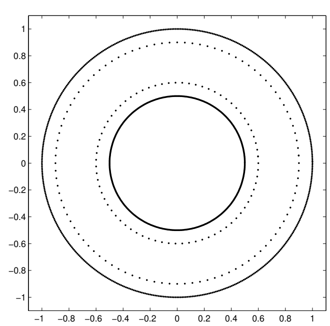

FIGURE 2. Boundary Layers in an Annulus

In Figure 1 and Figure 2, the solid circles represent the physical boundary $\partial\Omega$. The regions between the solid and dotted circles are the regime of boundary layers. Here we exaggerate the thickness of boundary layer regions for clarity; it is actually very thin and depends on ε.

The justification of this approximation, i.e. the so-called diffusive limit usually involves two steps:

1. Hilbert expansion: expanding $U=\sum_{k=0}^{\infty}\varepsilon^k U_k$ and $\mathscr{U}=\sum_{k=0}^{\infty}\varepsilon^k\mathscr{U}_k$ as power series of ε and proving the coefficients U_k and $\mathscr{U}_k$ are well-defined. On the one hand, the estimates of the interior solutions U_k are relatively straightforward. On the other hand, boundary layers $\mathscr{U}_k$ satisfy one-dimensional half-space problems which presents more challenges than the original problem. The well-posedness of boundary layer equations can sometimes be extremely difficult and it is possible that they are actually ill-posed (e.g. certain type of Prandtl layers [11]).

2. Remainder estimates: proving that

$$R=u^{\varepsilon}-U_0-\mathscr{U}_0=o(1) \tag{5}$$

as $\varepsilon \to 0$. Ideally, this should be done just by expanding to the leading-order level U_0 and $\mathscr{U}_0$. However, in singular perturbation problems like ours, the estimates of the remainder R usually involve negative powers of ε, which requires an expansion to higher-order terms U_N and $\mathscr{U}_N$ for $N \geq 1$ such that we have a sufficiently high power of ε. In other words, we define

$$R = u^{\varepsilon} - \sum_{k=0}^{N} \varepsilon^k U_k - \sum_{k=0}^{N} \varepsilon^k \mathscr{U}_k \tag{6}$$

for $N \geq 1$ instead of $R = u^{\varepsilon} - U_0 - \mathscr{U}_0$ to get better estimates of R.

CLASSICAL APPROACH

Interior Solution

Inserting ansatz for interior solution $U = \sum_{k=0}^{\infty} \varepsilon^k U_k$ into (1) and comparing order of ε, we obtain

$$U_0 - \overline{U}_0 = 0, \quad U_k - \overline{U}_k = -w \cdot \nabla_x U_{k-1} \text{ for } k \geq 1. \tag{7}$$

We may rearrange the terms to arrive at the cleaner form

$$\begin{cases} U_0(x,w) = \overline{U}_0(x) \text{ for } (x,w) \in \Omega \times \mathbb{S}^2, \\ \Delta_x \overline{U}_0 = 0 \text{ for } x \in \Omega, \end{cases} \tag{8}$$

and

$$\begin{cases} U_k(x,w) = \overline{U}_k(x) - w \cdot \nabla_x U_{k-1} \text{ for } (x,w) \in \Omega \times \mathbb{S}^2, \\ \Delta_x \overline{U}_k = 0 \text{ for } x \in \Omega. \end{cases} \tag{9}$$

However, $\overline{U}_k(x)$ cannot match the boundary conditions at each order of ε due to the presence of non-trivial w dependence in $g(x_0, w)$. Hence, we have to introduce the boundary layer.

Classical Approach for Boundary Layer in the Disk

The kinetic boundary layer theory has long been thought to be complete, thanks to the remarkable paper [12] in 1979. Unfortunately, in [1], we demonstrated that both the proof and result of this formulation are invalid due to lack of regularity for the Milne problem.

We start from the unit disk domain $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ with $w \in \mathbb{S}^1$. Let $\eta = \varepsilon^{-1}(1-r)$ denote the rescaled normal variable for r the radial distance, θ the tangential variables. (1) is equivalent to

$$-(w \cdot n)\frac{\partial \mathscr{U}}{\partial \eta} + (w \cdot \tau)\frac{\varepsilon}{1-\varepsilon\eta}\frac{\partial \mathscr{U}}{\partial \theta} + \mathscr{U} - \overline{\mathscr{U}} = 0, \tag{10}$$

where n is unit outer normal vector and τ is unit counterclockwise tangential vector.

Inserting the ansatz for boundary layer $\mathscr{U} = \sum_{k=0}^{\infty} \varepsilon^k \mathscr{U}_k$ into (10) and comparing the order of ε, we may obtain the flat Milne problem,

$$-(w \cdot n)\frac{\partial \mathscr{U}_0}{\partial \eta} + \mathscr{U}_0 - \overline{\mathscr{U}}_0 = 0, \tag{11}$$

$$-(w \cdot n)\frac{\partial \mathscr{U}_k}{\partial \eta} + \mathscr{U}_k - \overline{\mathscr{U}}_k = -(w \cdot \tau)\frac{1}{1-\varepsilon\eta}\frac{\partial \mathscr{U}_{k-1}}{\partial \theta} \text{ for } k \geq 1. \tag{12}$$

This classical formulation was widely used and intuitively reasonable.

The remainder estimates require $\mathscr{U}_1 \in L^\infty$ which needs $\dfrac{\partial \mathscr{U}_0}{\partial \theta} \in L^\infty$. However, though [12] shows that $\mathscr{U}_0 \in L^\infty$, it is well-known that the normal derivative $\dfrac{\partial \mathscr{U}_0}{\partial \eta}$ is singular due to the grazing set $w \cdot n = 0$ at the boundary. Furthermore, this singularity $\dfrac{\partial \mathscr{U}_0}{\partial \eta} \notin L^\infty$ will be transferred to $\dfrac{\partial \mathscr{U}_0}{\partial \theta} \notin L^\infty$. A careful construction of boundary data (see [1]) justifies this invalidity, i.e. the chain of estimates

$$R = o(1) \;\Leftarrow\; \mathscr{U}_1 \in L^\infty \;\Leftarrow\; \frac{\partial \mathscr{U}_0}{\partial \theta} \in L^\infty \;\Leftarrow\; \frac{\partial \mathscr{U}_0}{\partial \eta} \in L^\infty, \tag{13}$$

is broken since the rightmost estimate is problematic.

Also, this invalidity was further captured by numerical tests in [13], which pulled the whole research back to the starting point. Any later results based on this type of boundary layers should be reexamined.

Note that the difficulty of the above classical approach is purely due to the geometry of the curved boundary $\partial\Omega$. When $\partial\Omega$ is flat, i.e. when Ω is the half space $\mathbb{T} \times \mathbb{R}^+$, the flat Milne problem (11) provides the correct description of the kinetic boundary layer.

GEOMETRIC CORRECTION

It is notable that the proof of the diffusive/hydrodynamic limit relies on two contradictory requirements:

- The remainder estimates require sufficiently high order in the expansion. The higher order we expand to, the better estimates we may obtain.
- While the interior solutions are usually smooth, the boundary layers lack regularity and thus cannot be expanded to arbitrarily higher order.

Hence, in order to resolve the difficulty, we have to attack the problem from both sides – improving the remainder estimates and modifying the boundary layer construction.

Improved Remainder Estimates for General Domains

From the remainder equation with R defined in (6)

$$\varepsilon w \cdot \nabla_x R + R - \overline{R} = S, \tag{14}$$

with either in-flow boundary condition $R = h$ or the diffuse-reflection boundary condition $R = \mathscr{P}[R] + h$ on $w \cdot n < 0$, we may derive the estimate in the form of

$$\|R\|_{L^\infty} \lesssim \frac{1}{\varepsilon^p} \|S\|_{L^q} + \text{other good terms}, \tag{15}$$

for some constant $p > 0, q \geq 1$. Clearly, the smaller $p > 0$ is, the better estimate we may obtain. In addition, since $S \approx \varepsilon^{N+1} \dfrac{\partial \mathscr{U}_N}{\partial \theta}\left(\dfrac{x}{\varepsilon}\right)$, we have $\left\| \dfrac{\partial \mathscr{U}_N}{\partial \theta}\left(\dfrac{x}{\varepsilon}\right) \right\|_{L^q} \sim \left(\displaystyle\int_0^\infty \left(\dfrac{\partial \mathscr{U}_N}{\partial \theta}\right)^q \left(\dfrac{x}{\varepsilon}\right) \mathrm{d}x \right)^{\frac{1}{q}} \sim \varepsilon^{\frac{1}{q}}$, and thus the smaller $q \geq 1$ is, the better estimate we have.

In a series of papers [1, 14, 15, 16, 17], we gradually improve the remainder estimate by a modified $L^2 - L^p - L^\infty$ framework. So far, the best result we obtain is as follows:

Theorem 1 *In d dimension, there exists a unique solution $R(x,w) \in L^\infty$ to* (14) *satisfying*

$$\|R\|_{L^\infty} \lesssim_m \frac{1}{\varepsilon^{2+\frac{d}{2m}}} \|S\|_{L^{\frac{2m}{2m-1}}} + \frac{1}{\varepsilon^{1+\frac{d}{2m}}} \|S\|_{L^2} + \|S\|_{L^\infty} + \frac{1}{\varepsilon^{k+\frac{d}{2m}}} \|h\|_{L^2} + \frac{1}{\varepsilon^{\frac{d}{2m}}} \|h\|_{L^m} + \|h\|_{L^\infty}. \tag{16}$$

Here the notation $A \lesssim_m B$ indicates $A \leq C(m)B$ for a constant $C(m)$ depending on m. In 2D, the above estimate holds for any integer $m \geq 1$, and in 3D, it only holds for $1 \leq m < 3$. Also, $k = \dfrac{1}{2}$ for the in-flow boundary condition and $k = 1$ for the diffuse-reflection boundary condition.

Based on Theorem 1, in order to show $\|R\|_{L^\infty} = o(1)$, we need $N+1+\frac{2m-1}{2m}-2-\frac{d}{2m} > 0 \Rightarrow N \geq 1$, and henceforth we at least need to expand the boundary layer to $\mathscr{U}_0 + \varepsilon\mathscr{U}_1$, and estimate $\frac{\partial \mathscr{U}_1}{\partial\theta}$ in L^∞.

Boundary Layer with Geometric Correction in the Disk

While the classical approach does not provide the desired bound $\frac{\partial \mathscr{U}_1}{\partial\theta} \in L^\infty$, in [1], we introduced a novel framework on the boundary layer construction. In particular, we take the geometric effects into consideration and design an ε-Milne problem with geometric correction.

To be more specific, the failure of the classical approach is mainly due to that n is not a constant vector on $\partial\Omega$ and thus we are not free to take tangential θ derivative on $w\cdot n$. Therefore, we define an additional substitution

$$w \to (w_n, w_\tau) := (w\cdot n, w\cdot\tau). \tag{17}$$

Then (1) is equivalent to

$$-w_n\frac{\partial \mathscr{U}}{\partial\eta} + \frac{\varepsilon}{1-\varepsilon\eta}\left(w_\tau\frac{\partial \mathscr{U}}{\partial\theta} - w_\tau^2\frac{\partial \mathscr{U}}{\partial w_n} + w_n w_\tau\frac{\partial \mathscr{U}}{\partial w_\tau}\right) + \mathscr{U} - \overline{\mathscr{U}} = 0. \tag{18}$$

Inserting the ansatz for boundary layer $\mathscr{U} = \mathscr{U}_0 + \varepsilon\mathscr{U}_1$ into (18) and comparing the order of ε, we may obtain the ε-Milne problem with geometric correction,

$$-w_n\frac{\partial \mathscr{U}_0}{\partial\eta} - \frac{\varepsilon}{1-\varepsilon\eta}\left(w_\tau^2\frac{\partial \mathscr{U}_0}{\partial w_n} - w_n w_\tau\frac{\partial \mathscr{U}_0}{\partial w_\tau}\right) + \mathscr{U}_0 - \overline{\mathscr{U}}_0 = 0, \tag{19}$$

$$-w_n\frac{\partial \mathscr{U}_1}{\partial\eta} - \frac{\varepsilon}{1-\varepsilon\eta}\left(w_\tau^2\frac{\partial \mathscr{U}_1}{\partial w_n} - w_n w_\tau\frac{\partial \mathscr{U}_1}{\partial w_\tau}\right) + \mathscr{U}_1 - \overline{\mathscr{U}}_1 = -\frac{1}{1-\varepsilon\eta}w_\tau\frac{\partial \mathscr{U}_0}{\partial\theta}. \tag{20}$$

The extra $-\frac{\varepsilon}{1-\varepsilon\eta}\left(w_\tau^2\frac{\partial \mathscr{U}_k}{\partial w_n} - w_n w_\tau\frac{\partial \mathscr{U}_k}{\partial w_\tau}\right)$ is called the geometric correction term. They seem of extra order ε compared with other terms. However, the singularity near the grazing set indicates that they are actually of the same order in the L^∞ sense.

Now we are free to take θ derivatives on $\mathscr{U}_0$ equation and presumably we may obtain $\frac{\partial \mathscr{U}_1}{\partial\theta} \in L^\infty$. However, the difficulty is transferred to the proof of well-posedness for such new type of Milne problems.

Through a delicate analysis using energy method and characteristics, we justify the L^∞ well-posedness and exponential decay of the solution to the ε-Milne problem with geometric correction (see [1])

$$\begin{cases} -w_n\dfrac{\partial f}{\partial\eta} - \dfrac{\varepsilon}{1-\varepsilon\eta}\left(w_\tau^2\dfrac{\partial f}{\partial w_n} - w_n w_\tau\dfrac{\partial f}{\partial w_\tau}\right) + f - \overline{f} = S, \\ f(0,w_n,w_\tau) = h(w_n,w_\tau) \ \text{ for } \ w_n > 0, \\ f(L,w_n,w_\tau) = f(L,-w_n,w_\tau), \end{cases} \tag{21}$$

where $L = \varepsilon^{-\frac{1}{2}}$.

Theorem 2 *Assume that* $\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty} \lesssim 1$ *for some constant* $K_0 > 0$ *and* $\|h\|_{L^\infty} \lesssim 1$. *Then there exists a unique solution* $f \in L^\infty$ *to* (21). *In addition, there exists a constant* f_L *depending on* S *and* h *such that*

$$\left\|\mathrm{e}^{K\eta}(f - f_L)\right\|_{L^\infty} \lesssim 1, \tag{22}$$

for some constant $0 < K < K_0$. *The estimate is uniform in* ε.

With both Theorem 1 and Theorem 2 in hand, we may deduce the diffusive limit in unit disk (see [1]):

Theorem 3 *Let $\Omega = \mathbb{D}$. Assume $g(x_0, w) \in C^3$. Then for the steady neutron transport equation with either in-flow or diffuse-reflection boundary conditions, there exists a unique solution $u^\varepsilon(x,w) \in L^\infty$ satisfying*

$$\left\| u^\varepsilon - U_0 - \mathscr{U}_0 \right\|_{L^\infty} = O(\varepsilon). \tag{23}$$

Remark 4 *As [13] reveals, the discrepancy between the expansion using our new boundary layer and the classical one in [12] only occurs in the regime near the boundary. The two types of interior solutions agree as $\varepsilon \to 0$.*

GENERAL CONVEX DOMAINS

In general convex domains, a similar substitution based on the normal and tangential variables reveals that the boundary layer should satisfy

$$-w_n \frac{\partial \mathscr{U}_0}{\partial \eta} - \frac{\varepsilon}{R_\kappa - \varepsilon\eta}\left(w_\tau^2 \frac{\partial \mathscr{U}_0}{\partial w_n} - w_n w_\tau \frac{\partial \mathscr{U}_0}{\partial w_\tau}\right) + \mathscr{U}_0 - \overline{\mathscr{U}}_0 = 0, \tag{24}$$

$$-w_n \frac{\partial \mathscr{U}_1}{\partial \eta} - \frac{\varepsilon}{R_\kappa - \varepsilon\eta}\left(w_\tau^2 \frac{\partial \mathscr{U}_1}{\partial w_n} - w_n w_\tau \frac{\partial \mathscr{U}_1}{\partial w_\tau}\right) + \mathscr{U}_1 - \overline{\mathscr{U}}_1 = -\frac{1}{R_\kappa - \varepsilon\eta} w_\tau \frac{\partial \mathscr{U}_0}{\partial \theta}, \tag{25}$$

where $R_\kappa > 0$ is the radius of curvature on $\partial\Omega$, η is the rescaled normal variable and θ is the tangential variable (see [14] for detailed definitions).

Note that R_κ may vary for different boundary points. Therefore, taking θ derivative in $\mathscr{U}_0$ equation can be very risky since the derivative might hit θ and an extra term involving velocity (w_n, w_τ) derivatives will appear. In contrast to the flat case, our new contribution is to establish the regularity for the solution to (24) for general convex domains.

Diffuse-Reflection Boundary Condition

For the diffuse-reflection boundary condition (4), the boundary layer should satisfy

$$U_0 + \mathscr{U}_0 = \mathscr{P}\left[U_0 + \mathscr{U}_0\right], \quad U_1 + \mathscr{U}_1 = \mathscr{P}\left[U_1 + \mathscr{U}_1\right] + g. \tag{26}$$

The analysis is greatly simplified based on the crucial observation that the leading-order boundary layer $\mathscr{U}_0 = 0$. Therefore, it suffices to study $\mathscr{U}_1$ equation

$$-w_n \frac{\partial \mathscr{U}_1}{\partial \eta} - \frac{\varepsilon}{R_\kappa - \varepsilon\eta}\left(w_\tau^2 \frac{\partial \mathscr{U}_1}{\partial w_n} - w_n w_\tau \frac{\partial \mathscr{U}_1}{\partial w_\tau}\right) + \mathscr{U}_1 - \overline{\mathscr{U}}_1 = 0, \tag{27}$$

and bound $\dfrac{\partial \mathscr{U}_1}{\partial \theta}$ in L^∞. However, still we encounter the velocity derivative estimates of $\mathscr{U}_1$.

Note that $W = \dfrac{\partial \mathscr{U}_1}{\partial \theta}$ satisfies

$$-w_n \frac{\partial W}{\partial \eta} - \frac{\varepsilon}{R_\kappa - \varepsilon\eta}\left(w_\tau^2 \frac{\partial W}{\partial w_n} - w_n w_\tau \frac{\partial W}{\partial w_\tau}\right) + W - \overline{W} = -\frac{\varepsilon \partial_\theta R_\kappa}{(R_\kappa - \varepsilon\eta)^2}\left(w_\tau^2 \frac{\partial \mathscr{U}_1}{\partial w_n} - w_n w_\tau \frac{\partial \mathscr{U}_1}{\partial w_\tau}\right). \tag{28}$$

We observe that estimating the non-trivial source term in (28) requires far less than direct bounds of $\dfrac{\partial \mathscr{U}_1}{\partial w_n}$ and $\dfrac{\partial \mathscr{U}_1}{\partial w_\tau}$. In fact, from (27), it suffices to estimate $w_n \dfrac{\partial \mathscr{U}_1}{\partial \eta}$. Here w_n is the key to suppress the singularity in $\dfrac{\partial \mathscr{U}_1}{\partial \eta}$.

For weight function $\zeta(\eta, w_n, w_\tau)$, denote $A = \zeta \dfrac{\partial \mathscr{U}_1}{\partial \eta}$. Then A satisfies

$$-w_n \frac{\partial A}{\partial \eta} - \frac{\varepsilon}{R_\kappa - \varepsilon\eta}\left(w_\tau^2 \frac{\partial A}{\partial v_n} - w_n w_\tau \frac{\partial A}{\partial w_\tau}\right) + A - \zeta \frac{\partial \overline{\mathscr{U}}_1}{\partial \eta} = 0. \tag{29}$$

Unfortunately, the non-local term $\dfrac{\partial\overline{\mathscr{U}}_1}{\partial\eta}$ is incompatible with ζ, i.e.

$$\zeta(\eta,v)\frac{\partial\overline{\mathscr{U}}_1}{\partial\eta}(\eta)=\frac{\zeta(\eta,v)}{2\pi}\int_{\mathbb{S}^1}\frac{\partial\mathscr{U}_1}{\partial\eta}(\eta,v')\mathrm{d}v'=\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\zeta(\eta,v)}{\zeta(\eta,v')}A(\eta,v')\mathrm{d}v'\neq\overline{\zeta\frac{\partial\mathscr{U}_1}{\partial\eta}}(\eta)=\overline{A}(\eta). \tag{30}$$

Hence, we cannot cite the previous estimates for ε-Milne problem with geometric correction, and have to attack A-equation directly. In particular, $\zeta(\eta,v')$ may introduce a strong singularity in the integral.

Our strategy consists of two key ingredients:

1. The weight $\zeta=w_n$ is way too singular to close the proof, and we need to find an alternative based on kinetic distance

$$\zeta(\eta,v):=\left(\left(w_n^2+w_\tau^2\right)-\left(\frac{R_\kappa-\varepsilon\eta}{R_\kappa}\right)^2w_\tau^2\right)^{\frac{1}{2}}, \tag{31}$$

   which is stronger than w_n and commutes with the derivative operators.

2. Even if with the modified weight function above, there is still no way to directly bound the integral $\zeta(\eta,v)\dfrac{\partial\overline{\mathscr{U}}_1}{\partial\eta}$ in L^∞. Instead, we rewrite (29) with mild formulation and track along the characteristics. This offers an additional integral on $\zeta(\eta,v)\dfrac{\partial\overline{\mathscr{U}}_1}{\partial\eta}$ and help suppress the singularity, based on the general belief that the integral is less singular than the integrand.

In summary, we can show the weighted $W^{1,\infty}$ estimates of $\mathscr{U}_1$ (see [14, 15]):

$$\zeta\frac{\partial\mathscr{U}_1}{\partial\eta}\in L^\infty\Rightarrow w_n\frac{\partial\mathscr{U}_1}{\partial\eta}\in L^\infty\Rightarrow\frac{\varepsilon}{R_\kappa-\varepsilon\eta}\left(w_\tau^2\frac{\partial\mathscr{U}_1}{\partial w_n}-w_nw_\tau\frac{\partial\mathscr{U}_1}{\partial w_\tau}\right)\in L^\infty. \tag{32}$$

Theorem 5 *Let Ω be a smooth convex domain. Assume $g(x_0,w)\in C^3$. Then for the steady neutron transport equation with diffuse-reflection boundary condition, there exists a unique solution $u^\varepsilon(x,w)\in L^\infty$ satisfying for any $\delta>0$*

$$\left\|u^\varepsilon-U_0\right\|_{L^\infty}=O\left(\varepsilon^{\frac{1}{3}-\delta}\right). \tag{33}$$

In-Flow Boundary Condition

For the in-flow boundary condition (3), the boundary layer should satisfy

$$U_0+\mathscr{U}_0=g. \tag{34}$$

Hence, $\mathscr{U}_0$ does not necessarily vanish and we have to study the $W^{2,\infty}$ estimates of $\mathscr{U}_0$ in order to control $\dfrac{\partial\mathscr{U}_1}{\partial\theta}$. However, this is essentially not attainable (see [18]).

Observing that the singularity of $\dfrac{\partial\mathscr{U}_1}{\partial\eta}$ is restricted to the near-grazing-set region, we may further decompose $\mathscr{U}_0$ into a regular part which achieves $W^{2,\infty}$ estimates, and a singular part which only achieves $W^{1,\infty}$ but has small support.

We first decompose the boundary data $g=g_R+g_S$, and correspondingly $\mathscr{U}_0=\mathscr{U}_R+\mathscr{U}_S$. The decomposition guarantees that g_R is constant for $-\varepsilon^\alpha<w_n<0$ and $\dfrac{\partial\mathscr{U}_R}{\partial\eta}\in L^\infty$, and $g_S=0$ for $w_n<-\varepsilon^{-\alpha}$ for some $0<\alpha\leq 1$ which means a small support.

The regular boundary layer $\mathscr{U}_R$ has better regularity up to $\dfrac{\partial^2\mathscr{U}_R}{\partial\theta^2}$:

$$\frac{\partial^2\mathscr{U}_R}{\partial\theta^2}\Leftarrow\zeta\frac{\partial}{\partial\eta}\left(\frac{\partial\mathscr{U}_R}{\partial\theta}\right)\Leftarrow\frac{\varepsilon}{R_\kappa-\varepsilon\eta}\left(w_\tau^2\frac{\partial}{\partial w_n}\left(\frac{\partial\mathscr{U}_R}{\partial\eta}\right)-w_nw_\tau\frac{\partial}{\partial w_\tau}\left(\frac{\partial\mathscr{U}_R}{\partial\eta}\right)\right)\Leftarrow\frac{\varepsilon}{R_\kappa-\varepsilon\eta}\left(w_\tau^2\frac{\partial\mathscr{U}_R}{\partial w_n}-w_nw_\tau\frac{\partial\mathscr{U}_R}{\partial w_\tau}\right). \tag{35}$$

Here $\dfrac{\partial \mathscr{U}_R}{\partial \eta} \in L^\infty$ plays a key role.

The singular boundary layer $\mathscr{U}_S$ can only be estimated up to $\dfrac{\partial \mathscr{U}_S}{\partial \theta}$. Note that the non-local operator propagates the smallness from boundary to interior. Hence, in the region $0 < \zeta < \varepsilon^\alpha$, $\dfrac{\partial \mathscr{U}_S}{\partial \theta}$ is $O(1)$, but the region itself is very small, and in the region $\varepsilon^\alpha < \zeta < 1$, $\dfrac{\partial \mathscr{U}_S}{\partial \theta}$ is small, but the region is $O(1)$. Then in the L^p norm, the overall integral is small.

In summary, in the boundary layer expansion, we expand the regular boundary layer up to second order to gain power of ε, and expand the singular boundary layer only to the first order to gain power of ε by the small support. Then it is available to close the proof by optimizing α (see [16, 17]).

Theorem 6 *Let Ω be a smooth convex domain. Assume $g(x_0, w) \in C^3$. Then for the steady neutron transport equation with in-flow boundary condition, there exists a unique solution $u^\varepsilon(x, w) \in L^\infty$ satisfying for any $\delta > 0$*

$$\|u^\varepsilon - U_0 - \mathscr{U}_0\|_{L^\infty} = O\left(\varepsilon^{\frac{1}{3}-\delta}\right). \tag{36}$$

NON-CONVEX DOMAINS

The key differences between convex and non-convex domains lie in the regularity estimates. From the boundary layer viewpoint, non-convex domains imply that $R_\kappa < 0$. However, this sign flip dramatically changes the characteristics.

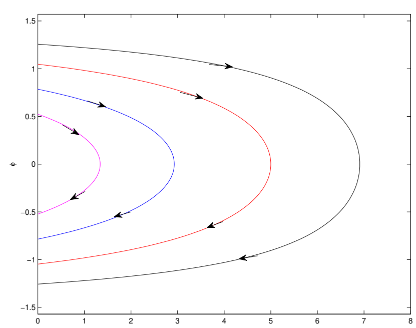

FIGURE 3. Characteristics in Convex Domains

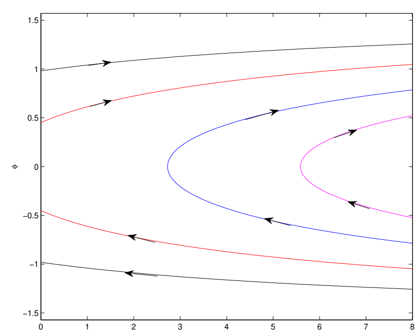

FIGURE 4. Characteristics in Non-Convex Domains

In Figure 3 and Figure 4, the horizontal direction represents the rescaled normal variable η and the vertical direction represents velocity w_n. We can clearly see that there is a special region in Figure 4 that the characteristics may never track back to the left boundary $\eta = 0$. This hollow region makes even the $W^{1,\infty}$ estimates impossible.

In our forthcoming work [19], we develop a new method to justify the diffusive limits (5) of (1) in non-convex domains in L^2 norm.

Theorem 7 *Let Ω be a smooth non-convex domain. Assume $g(x_0, w) \in C^3$. Then for the steady neutron transport equation with in-flow or diffuse-reflection boundary condition, there exists a unique solution $u^\varepsilon(x, w) \in L^\infty$ satisfying*

$$\|u^\varepsilon - U_0\|_{L^2} = O(\varepsilon^{\frac{1}{2}}). \tag{37}$$

CONCLUSION

In this work, through a detailed analysis of boundary layer, we justify the diffusive limit of steady neutron transport equation in convex and non-convex domains. Our techniques developed, including the geometric correction, boundary layer regularity estimates, and boundary layer decomposition, may potentially be implemented to other fields.

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