

ANNALS OF K-THEORY

vol. 8 no. 4 2023

**Delocalized eta invariants, cyclic cohomology
and higher rho invariants**

Xiaoman Chen, Jinmin Wang, Zhizhang Xie and Guoliang Yu



A JOURNAL OF THE K-THEORY FOUNDATION

Delocalized eta invariants, cyclic cohomology and higher rho invariants

Xiaoman Chen, Jinmin Wang, Zhizhang Xie and Guoliang Yu

We prove that the convergence of Lott’s delocalized eta invariant holds for all self-adjoint first order elliptic differential operators with a sufficiently large spectral gap at zero. Furthermore, to each delocalized cyclic cocycle, we define a higher analogue of Lott’s delocalized eta invariant and prove its convergence when the delocalized cyclic cocycle has at most exponential growth. As an application, for each cyclic cocycle of at most exponential growth, we prove a formal higher Atiyah–Patodi–Singer index theorem on manifolds with boundary, under the condition that the operator on the boundary has a sufficiently large spectral gap at zero.

Our second main result is to obtain an explicit formula of the delocalized Connes–Chern character of all C^* -algebraic secondary invariants for word hyperbolic groups. Equivalently, we give an explicit formula for the pairing between C^* -algebraic secondary invariants and delocalized cyclic cocycles of the group algebra. When the C^* -algebraic secondary invariant is a K -theoretic higher rho invariant of an invertible differential operator, we show this pairing is precisely the higher analogue of Lott’s delocalized eta invariant alluded to above. Our work uses Puschnigg’s smooth dense subalgebra for word hyperbolic groups in an essential way. We emphasize that our construction of the delocalized Connes–Chern character is at the C^* -algebra K -theory level. This is of essential importance for applications to geometry and topology. As a consequence, we compute the pairing between delocalized cyclic cocycles and C^* -algebraic Atiyah–Patodi–Singer index classes for manifolds with boundary, when the fundamental group of the given manifold is hyperbolic. In particular, this improves the formal delocalized higher Atiyah–Patodi–Singer theorem from above and removes the condition that the spectral gap of the operator on the boundary is sufficiently large.

Chen is partially supported by NSFC 11420101001. Wang is partially supported by NSFC 11420101001. Xie is partially supported by NSF 1500823, NSF 1800737. Yu is partially supported by NSF 1700021, NSF 1564398.

MSC2020: 19D55, 19K56.

Keywords: higher eta invariants, higher rho invariants, cyclic cohomology, higher index, higher secondary invariants, Connes–Chern characters, higher index theorem for manifolds with boundary.

1. Introduction

Higher index theory is a far-reaching generalization of the classic Fredholm index theory by taking into consideration of the symmetries of the underlying space. Let X be a complete Riemannian manifold of dimension n with a discrete group G acting on it properly and cocompactly by isometries. Each G -equivariant elliptic differential operator D on X gives rise to a higher index class $\text{Ind}_G(D)$ in the K -group $K_n(C_r^*(G))$ of the reduced group C^* -algebra $C_r^*(G)$. This higher index is an obstruction to the invertibility of D . The higher index theory plays a fundamental role in the studies of many problems in geometry and topology such as the Novikov conjecture, the Baum–Connes conjecture and the Gromov–Lawson–Rosenberg conjecture. Higher index classes are invariant under homotopy and often referred to as primary invariants.

When the higher index class of an operator is trivial and given a specific trivialization, a secondary index theoretic invariant naturally arises. One such example is the associated Dirac operator on the universal covering \tilde{M} of a closed spin manifold M equipped with a positive scalar curvature metric. It follows from the Lichnerowicz formula that the Dirac operator on \tilde{M} is invertible. In this case, there is a natural C^* -algebraic secondary invariant introduced by Roe [1996] and Higson and Roe [2005a; 2005b; 2005c], called the higher rho invariant, which lies in $K_n(C_{L,0}^*(\tilde{M})^G)$, where G is the fundamental group $\pi_1(M)$ of M and $C_{L,0}^*(\tilde{M})^G$ is a certain geometric C^* -algebra. The precise definition of $C_{L,0}^*(\tilde{M})^G$ and that of the higher rho invariant are given in Section 2. This higher rho invariant is an obstruction to the inverse of the Dirac operator being local and has important applications to geometry and topology.

Parallel to the C^* -algebraic approach above, Lott [1999] developed a theory of secondary invariants in the framework of noncommutative differential forms. Lott's theory was in turn very much inspired by the work of Bismut and Cheeger [1989] on eta forms, which naturally arise in the index theory for families of manifolds with boundary. Despite the fact that Lott's higher eta invariant is defined by an explicit integral formula of noncommutative differential forms, it is difficult to compute in general. It is only after one pairs Lott's higher eta invariant with cyclic cocycles of $\pi_1(M)$ that it becomes more computable and more applicable to problems in geometry and topology. However, due to certain convergence issues, the question when such a pairing can actually be rigorously defined is often very subtle. We shall devote the first half of the current paper to these convergence issues. We show that the pairing of Lott's higher eta invariant and a (delocalized) cyclic cocycle is well-defined, under the condition that the operator on \tilde{M} has a sufficiently large spectral gap at zero and the cyclic cocycle has at most exponential growth. In particular, as a special case, if both $\pi_1(M)$ and the cyclic cocycle have subexponential growth, then the pairing is always well-defined for all invertible operators on \tilde{M} .

The second goal of our paper is to obtain an explicit formula for the delocalized Connes–Chern character of all C^* -algebraic secondary invariants for hyperbolic groups. Equivalently, this amounts to computing the pairing between C^* -algebraic secondary invariants and delocalized cyclic cocycles of the group algebra. In the case where the C^* -algebraic secondary invariant is a K -theoretic higher rho invariant, the pairing is given explicitly in terms of Lott’s higher eta invariant,¹ or rather its periodic version.² As mentioned above, one of the main technical difficulties is to resolve various convergence issues. In the case of hyperbolic groups, we overcome these convergence issues with the help of Puschnigg’s smooth dense subalgebra. As a consequence, we compute the pairing between delocalized cyclic cocycles and Atiyah–Patodi–Singer higher index classes for manifolds with boundary in terms of delocalized higher eta invariants, when the fundamental group of the given manifold is hyperbolic. The details of these results will occupy the second half of the paper.

In the following, we shall give a more precise overview of some of the main results of this paper. Let us recall the definition of Lott’s delocalized eta invariant, which shall be thought of (at least formally) as a pairing between Lott’s higher eta invariant and traces (i.e., degree zero cyclic cocycles). Suppose $\langle h \rangle$ is a nontrivial conjugacy class in $\pi_1(M)$ in the sense that the group element h is not equal to the identity in $\pi_1(M)$. If D is a self-adjoint elliptic differential operator on M and \tilde{D} is the lifting of D to \tilde{M} , then the delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ is defined by the formula

$$\eta_{\langle h \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{tr}_{\langle h \rangle}(\tilde{D}e^{-t^2\tilde{D}^2}) dt. \quad (1.1)$$

Here $\mathrm{tr}_{\langle h \rangle}$ is the following trace map

$$\mathrm{tr}_{\langle h \rangle}(A) = \sum_{g \in \langle h \rangle} \int_{x \in \mathcal{F}} A(x, gx) dx$$

on G -equivariant kernels $A \in C^\infty(\tilde{M} \times \tilde{M})$, where \mathcal{F} is a fundamental domain of \tilde{M} under the action of $G = \pi_1(M)$. As it stands, the above definition of delocalized eta invariant does not require a choice of a smooth dense subalgebra of $C_r^*(G)$. Of course, in the special event that $\tilde{D}e^{-t^2\tilde{D}^2}$ lies in an appropriate smooth dense subalgebra to which the trace map $\mathrm{tr}_{\langle h \rangle}$ continuously extends, this delocalized eta invariant indeed coincides with the pairing of $\mathrm{tr}_{\langle h \rangle}$ with Lott’s higher eta invariant.

¹In the literature, the delocalized part of Lott’s noncommutative-differential higher eta invariant sometimes is also referred to as higher rho invariant. To avoid confusion, we shall refer this noncommutative-differential higher rho invariant as *Lott’s higher eta invariant*.

²See [Lott 1992, Section 4.6] for a discussion of the periodic version of Lott’s higher eta invariant. As K -theory is 2-periodic, the periodic higher eta invariant is the correct version to be used here.

The convergence of the above formula is proved by Lott [1999] under the assumption that $\langle h \rangle$ has polynomial growth or is hyperbolic,³ and that \tilde{D} is invertible or more generally \tilde{D} has a spectral gap at zero. Recall that \tilde{D} is said to have a spectral gap at zero if there exists an open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ such that $\text{spectrum}(\tilde{D}) \cap (-\varepsilon, \varepsilon)$ is either $\{0\}$ or empty. In general, the convergence of (1.1) fails. For example, Piazza and Schick gave an explicit example where the convergence of (1.1) fails when \tilde{D} does not have a spectral gap at zero.

As the first main result of this paper, we show that if \tilde{D} has a sufficiently large spectral gap at zero, then Lott's delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ in line (1.1) converges absolutely. We refer to Definition 3.7 in Section 3 for a more precise quantitative explanation of what it means for a spectral gap to be “sufficiently large”.

Theorem 1.2. *Let M be a closed manifold and \tilde{M} the universal covering over M . Suppose D is a self-adjoint first-order elliptic differential operator over M and \tilde{D} the lift of D to \tilde{M} . If $\langle h \rangle$ is a nontrivial conjugacy class of $\pi_1(M)$ and \tilde{D} has a sufficiently large spectral gap at zero, then the delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ defined in line (1.1) converges absolutely.*

We would like to emphasize that the theorem above works for all fundamental groups. In the special case where the conjugacy class $\langle h \rangle$ has subexponential growth, then any nonzero spectral gap is in fact sufficiently large, hence in this case $\eta_{\langle h \rangle}(\tilde{D})$ converges absolutely as long as \tilde{D} is invertible.

Now a special feature of traces is that they always have uniformly bounded representatives, when viewed as degree zero cyclic cocycles. In fact, the techniques used to prove Theorem 1.2 above can be generalized to all delocalized cyclic cocycles of higher degrees, as long as they have at most exponential growth. Recall that the cyclic cohomology of a group algebra decomposes into a direct product with respect to the conjugacy classes of the group. A cyclic cocycle in a component of this direct product decomposition that corresponds to a nontrivial conjugacy class $\langle h \rangle$ will be called a delocalized cyclic cocycle at $\langle h \rangle$; see Definition 3.27. Moreover, see Definition 3.38 in Section 3 for the precise definition of exponential growth for cyclic cocycles.

Theorem 1.3. *Assume the same notation as in Theorem 1.2. Let φ be a delocalized cyclic cocycle at a nontrivial conjugacy class $\langle h \rangle$. If φ has exponential growth and \tilde{D} has a sufficiently large spectral gap at zero, then a higher analogue $\eta_\varphi(\tilde{D})$ (see Definition 3.30) of the delocalized eta invariant converges absolutely.*

³Lott [1999, Proposition 8] stated that the convergence of the formula (3.3) holds for both groups with polynomial growth and hyperbolic groups. However, his proof for hyperbolic groups contained a technical problem which was later fixed by Puschnigg [2010] when he constructed a different smooth dense subalgebra of the reduced group C^* -algebra for hyperbolic groups.

The explicit formula for $\eta_\varphi(\tilde{D})$ is described in terms of the transgression formula for Connes–Chern characters [Connes 1985; 1994; Jaffe et al. 1988]. It is closely related to the periodic version of Lott’s noncommutative-differential higher eta invariant. In the case where the fundamental group G has polynomial growth, we shall show that our formula for $\eta_\varphi(\tilde{D})$ is equivalent to the periodic version of Lott’s noncommutative-differential higher eta invariant; see Section 8. As $\eta_\varphi(\tilde{D})$ is an analogue for higher degree cyclic cocycles of Lott’s delocalized eta invariant, we shall call $\eta_\varphi(\tilde{D})$ a *delocalized higher eta invariant* from now on. Again, we refer to Definition 3.42 for a more precise quantitative explanation of what it means for a spectral gap to be “sufficiently large” in this context. For now, let us just point out that if both G and φ have subexponential growth, then any nonzero spectral gap is in fact sufficiently large, hence in this case $\eta_\varphi(\tilde{D})$ converges absolutely as long as \tilde{D} is invertible.

Formally speaking, just as Lott’s delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ can be interpreted as the pairing between the degree zero cyclic cocycle $\text{tr}_{\langle h \rangle}$ and the K -theoretic higher rho invariant $\rho(\tilde{D})$ (or the noncommutative differential higher eta invariant), so can the delocalized higher eta invariant $\eta_\varphi(\tilde{D})$ be interpreted as the pairing between the cyclic cocycle φ and the K -theoretic higher rho invariant $\rho(\tilde{D})$ (or the noncommutative differential higher eta invariant). As pointed out in the discussion above, a key analytic difficulty here is to verify when such a pairing is well-defined, or more ambitiously, to verify when one can extend this pairing to a pairing between the cyclic cohomology of $\mathbb{C}G$ and the K -theory group $K_*(C_{L,0}^*(\tilde{M})^G)$. The group $K_*(C_{L,0}^*(\tilde{M})^G)$ consists of C^* -algebraic secondary invariants; in particular, it contains all higher rho invariants from the discussion above. As pointed out above, such an extension of the pairing is important, often necessary, for many interesting applications to geometry and topology; see [Piazza and Schick 2014; Xie and Yu 2017; Weinberger et al. 2021].

Xie and Yu [2021] established a pairing between delocalized cyclic cocycles of degree zero (i.e., delocalized traces) and the K -theory group $K_*(C_{L,0}^*(\tilde{M})^G)$, under the assumption that the relevant conjugacy class has polynomial growth. In this paper, we shall construct a pairing between delocalized cyclic cocycles of *all degrees* and the K -theory group $K_*(C_{L,0}^*(\tilde{M})^G)$ for hyperbolic groups. Before we state the theorem, let us recall some notations that will be used in the statement of the next theorem. The cyclic cohomology of a group algebra $\mathbb{C}G$ has a decomposition with respect to the conjugacy classes of G [Burghelea 1985; Nistor 1990]:

$$HC^*(\mathbb{C}G) \cong \prod_{\langle h \rangle} HC^*(\mathbb{C}G, \langle h \rangle),$$

where $HC^*(\mathbb{C}G, \langle h \rangle)$ denotes the component that corresponds to the conjugacy class $\langle h \rangle$; see Definition 3.27 below.

Theorem 1.4. *Let M be a closed manifold whose fundamental group G is hyperbolic. Suppose $\langle h \rangle$ is a nontrivial conjugacy class of G . Then every element $[\alpha] \in HC^{2k+1-i}(\mathbb{C}G, \langle h \rangle)$ induces a natural map*

$$\tau_{[\alpha]}: K_i(C_{L,0}^*(\tilde{M})^G) \rightarrow \mathbb{C}$$

such that the following are satisfied:

- (i) $\tau_{[S\alpha]} = \tau_{[\alpha]}$, where S is Connes' periodicity map

$$S: HC^*(\mathbb{C}G, \langle h \rangle) \rightarrow HC^{*+2}(\mathbb{C}G, \langle h \rangle).$$

- (ii) *Suppose D is a first-order elliptic differential operator on M such that the lift \tilde{D} of D to the universal cover \tilde{M} of M is invertible. Then we have*

$$\tau_{[\alpha]}(\rho(\tilde{D})) = -\eta_\alpha(\tilde{D}),$$

where $\rho(\tilde{D})$ is the C^ -algebraic higher rho invariant of \tilde{D} and $\eta_\alpha(\tilde{D})$ is the delocalized higher eta invariant defined in Definition 3.30. In particular, in this case, the delocalized higher eta invariant $\eta_\alpha(\tilde{D})$ converges absolutely.*

The construction of the map $\tau_{[\alpha]}$ in the above theorem uses Puschnigg's [2010] smooth dense subalgebra for hyperbolic groups in an essential way. In more conceptual terms, the above theorem provides an explicit formula to compute the delocalized Connes–Chern character of C^* -algebraic secondary invariants. More precisely, the same techniques developed in this paper actually imply⁴ that there is a well-defined delocalized Connes–Chern character

$$\text{ch}_{\text{deloc}}: K_i(C_{L,0}^*(\tilde{M})^\Gamma) \rightarrow \overline{HC}_*^{\text{deloc}}(B(\mathbb{C}G)),$$

where $B(\mathbb{C}G)$ is Puschnigg's smooth dense subalgebra of the reduced group C^* -algebra of G and $\overline{HC}_*^{\text{deloc}}(B(\mathbb{C}G))$ is the delocalized part of the cyclic homology of $B(\mathbb{C}G)$.⁵ Now for Gromov's hyperbolic groups, every cyclic cohomology class of $\mathbb{C}\Gamma$ continuously extends to cyclic cohomology class of $B(\mathbb{C}G)$; see [Puschnigg 2010] for the case of degree zero cyclic cocycles and Section 4 of this paper for the case of higher degree cyclic cocycles. Thus the map $\tau_{[\alpha]}$ can be viewed as a pairing between cyclic cohomology and delocalized Connes–Chern characters of C^* -algebraic secondary invariants. We point out that, although the spectral gap of

⁴In fact, even more is true. It follows from the proof of Lemma 6.9 in Section 6 that if \mathcal{A} is smooth dense subalgebra of $C_r^*(\Gamma)$ for any group Γ (not necessarily hyperbolic) and in addition \mathcal{A} is a Fréchet locally m -convex algebra, then there is a well-defined delocalized Connes–Chern character $\text{ch}_{\text{deloc}}: K_i(C_{L,0}^*(\tilde{M})^\Gamma) \rightarrow \overline{HC}_*^{\text{deloc}}(\mathcal{A})$. Of course, in order to pair such a delocalized Connes–Chern character with a cyclic cocycle of $\mathbb{C}\Gamma$, a key remaining challenge is to continuously extend this cyclic cocycle of $\mathbb{C}\Gamma$ to a cyclic cocycle of \mathcal{A} .

⁵Here the definition of cyclic homology of $B(\mathbb{C}G)$ takes the topology of $B(\mathbb{C}G)$ into account; see [Connes 1985, Section II.5].

\tilde{D} is required to be sufficiently large in Theorems 1.2 and 1.3 in order for $\eta_\alpha(\tilde{D})$ to converge, such a requirement is *not* needed in the case of hyperbolic groups. This is again a consequence of some essential properties of Puschnigg's smooth dense subalgebra.

As an application, we use this delocalized Connes–Chern character map to obtain a delocalized higher Atiyah–Patodi–Singer index theorem for manifolds with boundary. More precisely, let W be a compact n -dimensional spin manifold with boundary ∂W . Suppose W is equipped with a Riemannian metric g_W which has a product structure near ∂W and in addition has positive scalar curvature on ∂W . Let \tilde{W} be the universal covering of W and $g_{\tilde{W}}$ the Riemannian metric on \tilde{W} lifted from g_W . With respect to the metric $g_{\tilde{W}}$, the associated Dirac operator \tilde{D}_W on \tilde{W} naturally defines a higher index $\text{Ind}_G(\tilde{D}_W)$ in $K_n(C^*(\tilde{W})^G) = K_n(C_r^*(G))$, where $G = \pi_1(W)$; see [Xie and Yu 2014, Section 3]. Since the metric $g_{\tilde{W}}$ has positive scalar curvature on $\partial \tilde{W}$, it follows from the Lichnerowicz formula that the associated Dirac operator \tilde{D}_∂ on $\partial \tilde{W}$ is invertible, hence naturally defines a higher rho invariant $\rho(\tilde{D}_\partial)$ in $K_{n-1}(C_{L,0}^*(\tilde{W})^\Gamma)$. We have the following delocalized higher Atiyah–Patodi–Singer index theorem.

Theorem 1.5. *With the same notations as above, if $G = \pi_1(W)$ is hyperbolic and $\langle h \rangle$ is a nontrivial conjugacy class of G , then for any $[\varphi] \in HC^*(\mathbb{C}G, \langle h \rangle)$, we have*

$$\text{ch}_{[\varphi]}(\text{Ind}_G(\tilde{D}_W)) = \frac{1}{2} \eta_{[\varphi]}(\tilde{D}_\partial), \quad (1.6)$$

where $\text{ch}_{[\varphi]}(\text{Ind}_G(\tilde{D}_W))$ is the Connes–Chern pairing between the cyclic cohomology class $[\varphi]$ and the C^* -algebraic index class $\text{Ind}_G(\tilde{D}_W)$.

There have been various versions of higher Atiyah–Patodi–Singer theorem in the literature [Leichtnam and Piazza 1999; Wahl 2013; Gorokhovskiy et al. 2016]. See the discussion after Theorem 7.6 for more details on the relations and differences of the above theorem with those existing results.

The methods developed in this paper can also be applied to prove analogues of Theorems 1.4 and 1.5 above for virtually nilpotent groups. The complete details can be found in the thesis of Sheagan John [2022].

We would like to point out that our proof of Theorem 1.4 does *not* rely on the Baum–Connes isomorphism for hyperbolic groups [Lafforgue 2012; Mineyev and Yu 2002], although the theorem is closely connected to the Baum–Connes conjecture and the Novikov conjecture. On the other hand, if one is willing to use the full power of the Baum–Connes isomorphism for hyperbolic groups, there is in fact a different, but more indirect, approach to the delocalized Connes–Chern character map. First, observe that the map $\tau_{[\alpha]}$ factors through a map

$$\tau_{[\alpha]} : (K_i(C_{L,0}^*(\underline{E}G)^G) \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

where $\underline{E}G$ is the universal space for proper G -actions. Now the Baum–Connes isomorphism

$$\mu: K_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^*(G))$$

for hyperbolic groups implies that one can identify $K_i(C_{L,0}^*(\underline{E}G)^G) \otimes \mathbb{C}$ with $\bigoplus_{\langle h \rangle \neq 1} HC_*(\mathbb{C}G, \langle h \rangle)$, where $HC_*(\mathbb{C}G, \langle h \rangle)$ is the delocalized cyclic homology at $\langle h \rangle$ (a cyclic homology analogue of Definition 3.27) and the direct sum is taken over all nontrivial conjugacy classes. In particular, after this identification, it follows that the map $\tau_{[\alpha]}$ becomes the usual componentwise pairing between cyclic cohomology and cyclic homology. However, for a specific element, e.g., the higher rho invariant $\rho(\tilde{D})$, in $K_i(C_{L,0}^*(\underline{E}G)^G)$, its identification with an element in $\bigoplus_{\langle h \rangle \neq 1} HC_*(\mathbb{C}G, \langle h \rangle)$ is rather abstract and implicit. More precisely, the computation of the number $\tau_{[\alpha]}(\rho(\tilde{D}))$ essentially amounts to the following process. Observe that if a closed spin manifold M is equipped with a positive scalar curvature metric, then stably it bounds (more precisely, the universal cover \tilde{M} of M becomes the boundary of another G -manifold, after finitely many steps of cobordisms and vector bundle modifications). In principle, the number $\tau_{[\alpha]}(\rho(\tilde{D}))$ can be derived from a higher Atiyah–Patodi–Singer index theorem for this bounding manifold. The drawback of such an approach is that there is no explicit formula for $\tau_{[\alpha]}(\rho(\tilde{D}))$, since there is no explicit procedure for producing such a bounding manifold. Deeley and Goffeng [2016] also constructed a delocalized Connes–Chern character for C^* -algebra secondary invariants. Their approach is in spirit similar to the indirect method just described above (making use of the Baum–Connes isomorphism for hyperbolic groups), although their actual technical implementation is different. The key feature of our approach in this paper is that we obtain an explicit and intrinsic formula for the delocalized Connes–Chern character of C^* -algebraic secondary invariants.

The paper is organized as follows. In Section 2, we review some standard geometric C^* -algebras and a construction of Higson–Roe’s K -theoretic higher rho invariants. In Section 3, we prove the convergence of Lott’s delocalized eta invariant holds for all operators with a sufficiently large spectral gap at zero. More generally, for each higher degree delocalized cyclic cocycle, we define a higher analogue of Lott’s delocalized eta invariant, and prove its convergence for all operators with a sufficiently large spectral gap at zero, provided that the given cyclic cocycle has at most exponential growth. In Section 4, we review Puschnigg’s construction of smooth dense subalgebras of reduced group C^* -algebras for hyperbolic groups. Puschnigg showed that any trace on the group algebra of a hyperbolic group extends continuously onto this smooth dense subalgebra. We shall generalize this result to cyclic cocycles of all degrees, provided that the cyclic cocycles have polynomial growth; see Proposition 4.20 below. In Section 5, we show that every cyclic

cohomology class of a hyperbolic group has a representative of polynomial growth. Furthermore, if the cyclic cohomology class has degree ≥ 2 , then it admits a uniformly bounded representative. In Section 6, we give an explicit formula for the pairing between C^* -algebraic secondary invariants and delocalized cyclic cocycles of the group algebra for word hyperbolic groups. When the C^* -algebraic secondary invariant is a K -theoretic higher rho invariant of an invertible differential operator, we show this pairing is precisely the higher delocalized eta invariant of the given operator. In Section 7, we compute the pairing between delocalized cyclic cocycles and C^* -algebraic Atiyah–Patodi–Singer index classes for manifolds with boundary, when the fundamental group of the given manifold is hyperbolic. In Section 8, we identify our definition of delocalized higher eta invariant with Lott’s higher eta invariant.

We would like to thank Denis Osin for providing us a proof of a useful result on hyperbolic groups (Lemma 5.1).

We would like to mention that about five months after the first version of this paper appeared on arXiv [Chen et al. 2019], Piazza, Schick and Zenobi posted a preprint [Piazza et al. 2019] dealing with the same questions in Section 6 of our paper.

2. Preliminaries

In this section, we review the construction of some geometric C^* -algebras and Higson–Roe’s higher rho invariants. We refer the reader to [Roe 1993; Yu 1997; Higson and Roe 2005a; 2005b; 2005c] for more details.

Let X be a proper metric space, that is, every closed metric ball in X is compact. An X -module is a separable Hilbert space equipped with a $*$ -representation of $C_0(X)$, the algebra of all continuous functions on X which vanish at infinity. An X -module is called nondegenerate if the $*$ -representation of $C_0(X)$ is nondegenerate. An X -module is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

Definition 2.1. Let H_X be an X -module and T a bounded linear operator acting on H_X :

- (1) The propagation of T is defined to be $\sup\{d(x, y) \mid (x, y) \in \text{supp}(T)\}$, where $\text{supp}(T)$ is the complement (in $X \times X$) of the set of points $(x, y) \in X \times X$ for which there exist $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x)g(y) \neq 0$.
- (2) T is said to be locally compact if fT and Tf are compact for all $f \in C_0(X)$.
- (3) T is said to be pseudolocal if $[T, f]$ is compact for all $f \in C_0(X)$.

Definition 2.2. Let H_X be a standard nondegenerate X -module and $B(H_X)$ the set of all bounded linear operators on H_X :

- (1) The Roe algebra of X , denoted by $C^*(X)$, is the C^* -algebra generated by all locally compact operators with finite propagations in $B(H_X)$.
- (2) $D^*(X)$ is the C^* -algebra generated by all pseudolocal operators and with finite propagations in $B(H_X)$. In particular, $D^*(X)$ is a subalgebra of the multiplier algebra of $C^*(X)$.
- (3) $C_L^*(X)$ (resp. $D_L^*(X)$) is the C^* -algebra generated by all bounded and uniformly norm-continuous functions $f : [0, \infty) \rightarrow C^*(X)$ (resp. $f : [0, \infty) \rightarrow D^*(X)$) such that

$$\text{propagation of } f(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

$D_L^*(X)$ is a subalgebra of the multiplier algebra of $C_L^*(X)$.

- (4) The kernel of the following evaluation map

$$ev : C_L^*(X) \rightarrow C^*(X), \quad f \mapsto f(0)$$

is defined to be $C_{L,0}^*(X)$. In particular, $C_{L,0}^*(X)$ is an ideal of $C_L^*(X)$. Similarly, we define $D_{L,0}^*(X)$ as the kernel of the evaluation map from $D_L^*(X)$ to $D^*(X)$.

Now in addition we assume that there is a countable discrete group G , and acts properly on X by isometries. In particular, if the action of Γ is free, then X is simply a G -covering of the compact space X/G . Let H_X be an X -module equipped with a covariant unitary representation of G . Let the representation of $C_0(X)$ be ϕ and let the action of G be π . We call (H_X, G, ϕ, π) a covariant system is to say

$$\pi(g)\phi(f) = \phi(g^*f)\pi(g),$$

where $g^*f(x) = f(g^{-1}x)$ for any $f \in C_0(X)$, $g \in G$.

Definition 2.3. A covariant system (H_X, G, ϕ) is called *admissible* if:

- (1) The action of G is proper and cocompact.
- (2) H_X is a nondegenerate standard X -module.
- (3) For each $x \in X$, the stabilizer group G_x acts on H_X regularly in the sense that the action is isomorphic to the obvious action of G_x on $l^2(G_x) \otimes H$ for some infinite dimensional Hilbert space H . Here G_x acts on $l^2(G_x)$ by (left) translations and acts on H trivially.

We remark that for each locally compact metric space X with a proper and cocompact isometric action of G , there exists an admissible covariant system (H_X, G, ϕ) . Also, we point out that the condition 3 above is automatically satisfied if G acts freely on X . If no confusion arises, we will denote an admissible covariant system (H_X, G, ϕ) by H_X and call it an admissible (X, G) -module.

Definition 2.4. Let X be a locally compact metric space X with a proper and cocompact isometric action of G . If H_X is an admissible (X, G) -module, we denote by $\mathbb{C}[X]^G$ to be $*$ -algebra of all G -invariant locally compact operators with finite propagations in $B(H_X)$. We define $C^*(X)^G$ to be the completion of $\mathbb{C}[X]^G$ in $B(H_X)$.

Since the action of G on X is cocompact, it is known that $C^*(X)^G$ is $*$ -isomorphic to $\mathcal{K} \otimes C_r^*(G)$, where \mathcal{K} is the algebra of all compact operators and $C_r^*(G)$ is the reduced group C^* -algebra.

Similarly, we can define $D^*(X)^G$, $C_L^*(X)^G$, $D_L^*(X)^G$, $C_{L,0}^*(X)^G$ and $D_{L,0}^*(X)^G$.

Remark 2.5. Up to isomorphism, $C^*(X)$ does not depend on the choice of the standard nondegenerate X -module H_X . The same holds for $D^*(X)$, $C_L^*(X)$, $D_L^*(X)$, $C_{L,0}^*(X)$, $D_{L,0}^*(X)$ and their G -equivariant versions.

Let M be a closed Riemannian manifold. Let G be a discrete finitely generated countable group. Suppose \tilde{M} is a regular G -cover of M . For example, \tilde{M} is the universal covering of M and G is the fundamental group of M . Let p be the associated covering map from \tilde{M} to M . Suppose E is an Hermitian vector bundle over M and \tilde{E} the lifting of E to \tilde{M} . Write \mathcal{H} the collection of all L^2 -sections of \tilde{E} . The equivariant Roe algebra $C^*(\tilde{M})^G$ is defined to be the operator-norm completion of all G -equivariant locally compact operators of finite propagation acting on \mathcal{H} . The localization algebra $C_L^*(\tilde{M})^G$ and $C_{L,0}^*(\tilde{M})^G$ can be defined similarly.

Suppose that M is spin and E is the corresponding spinor bundle over M . Let D be the Dirac operator acting on E and \tilde{D} its lifting to \tilde{E} . If the scalar curvature of the metric over M is strictly positive, the Dirac operator naturally defines a K -theory class called the higher rho invariant in $K_*(C_{L,0}^*(\tilde{M})^G)$. For simplicity, we will only discuss the case where M is odd dimensional; the even dimensional case is completely similar; see [Xie and Yu 2014].

Define the following functions

$$F_t(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/t} e^{-s^2} ds \quad \text{and} \quad U_t(x) = \exp(2\pi i F_t(x)). \quad (2.6)$$

Since the scalar curvature over \tilde{M} is uniformly bounded below by a positive number, it follows from the Lichnerowicz formula that the Dirac operator \tilde{D} is invertible. This implies that $F_t(\tilde{D})$ converges to $\frac{1}{2}(1 + \tilde{D}|\tilde{D}|^{-1})$ in operator norm, as $t \rightarrow 0$. Thus the path $\{U_t(\tilde{D})\}_{0 \leq t < \infty}$ lies in $C_{L,0}^*(\tilde{M})^G$.

Definition 2.7 [Higson and Roe 2005a; 2005b; 2005c]. The Higson–Roe higher rho invariant $\rho(\tilde{D})$ of \tilde{D} is defined to be the K -theory class $[\{U_t(\tilde{D})\}_{0 \leq t < \infty}] \in K_1(C_{L,0}^*(\tilde{M})^G)$.

3. Higher eta invariants

In this section, we show that Lott's delocalized eta invariant converges absolutely for operators with a sufficiently large spectral gap at zero. More generally, we prove the convergence of a higher analogue of Lott's delocalized eta invariant for operators with a sufficiently large spectral gap at zero and higher degree cyclic cocycles that have at most exponential growth.

3.1. Convergence of delocalized eta invariants with a large enough spectral gap.

Let M be a closed Riemannian manifold. Let G be a finitely generated discrete group. Suppose \tilde{M} is a regular G -covering space of M . Let p be the associated covering map from \tilde{M} to M . Choose any fundamental domain \mathcal{F} of G -action on \tilde{M} . Suppose D is a first-order self-adjoint elliptic differential operator acting on some Hermitian vector bundle E over M and \tilde{E} (resp. \tilde{D}) is the lifting of E (resp. D) to \tilde{M} . Assume G acts on \tilde{E} as well. Let \mathcal{H} be the space of L^2 -sections of \tilde{E} . Then the operator $\tilde{D}e^{-t^2\tilde{D}}$ lies in $B(\mathcal{H})$, the algebra of bounded operators on \mathcal{H} . Moreover, the associated Schwartz kernel $k_t(x, y)$ of $\tilde{D}e^{-t^2\tilde{D}}$ is smooth.

Any conjugacy class $\langle h \rangle$ of G naturally induces a trace map

$$tr_{\langle h \rangle}: \mathbb{C}G \rightarrow \mathbb{C}, \quad \text{by } \sum_{g \in G} a_g g \mapsto \sum_{g \in \langle h \rangle} a_g.$$

In particular, when $\langle h \rangle$ is the trivial conjugacy class, $tr_{\langle e \rangle}$ is the canonical trace on $\mathbb{C}G$. The trace map $tr_{\langle h \rangle}$ generalizes (formally) to a trace map on G -equivariant integral operators T with a smooth Schwartz kernel $T(x, y)$ as follows:

$$tr_{\langle h \rangle}(T) = \sum_{g \in \langle h \rangle} \int_{x \in \mathcal{F}} tr(T(x, gx)) dx, \quad (3.1)$$

provided that the right-hand side converges.

Definition 3.2 [Lott 1999]. For any nontrivial conjugacy class $\langle h \rangle$ of G , Lott's *delocalized eta invariant* $\eta_{\langle h \rangle}(\tilde{D})$ of \tilde{D} is defined to be

$$\eta_{\langle h \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty tr_{\langle h \rangle}(\tilde{D}e^{-t^2\tilde{D}^2}) dt. \quad (3.3)$$

The terminology “delocalized” refers to the fact that $\langle h \rangle$ is a nontrivial conjugacy class. If we were to take the trivial conjugacy class in Definition 3.2, we would recover the L^2 -eta invariant of Cheeger and Gromov [1985].

Lott [1999] proved the convergence of the integral in line (3.3) under the assumption that G has polynomial growth or is hyperbolic, and \tilde{D} is invertible (or more generally has a spectral gap at zero) Piazza and Schick [2007, Section 3] gave an example where the formula (3.3) diverges for noninvertible \tilde{D} . They then raised the

question of whether a divergent example still exists if one assumes the invertibility of \tilde{D} [loc. cit., Remark 3.2].

Our first main result states that the answer to this question of Piazza and Schick is negative, as long as the spectral gap of \tilde{D} is sufficiently large. Before we give the precise statement of the theorem, we first fix some notation.

Fix a finite generating set S of G . Let ℓ be the corresponding word length function on G determined by S . Since S is finite, there exist $C > 0$ and $K_{\langle h \rangle} > 0$ such that

$$\#\{g \in \langle h \rangle : \ell(g) = n\} \leq C e^{K_{\langle h \rangle} \cdot n}. \quad (3.4)$$

We define $\tau_{\langle h \rangle}$ to be

$$\tau_{\langle h \rangle} = \liminf_{\substack{g \in \langle h \rangle \\ \ell(g) \rightarrow \infty}} \left(\inf_{x \in \mathcal{F}} \frac{\text{dist}(x, gx)}{\ell(g)} \right). \quad (3.5)$$

Since the action of G on \tilde{M} is free and cocompact, we have $\tau_{\langle h \rangle} > 0$.

Given a first-order differential operator D on M , we denote the principal symbol of D by $\sigma_D(x, v)$, for $x \in M$ and cotangent vector $v \in T_x^*M$. We define the propagation speed of D to be the positive number

$$c_D = \sup\{\|\sigma_D(x, v)\| : x \in M, v \in T_x^*M, \|v\| = 1\}. \quad (3.6)$$

When D is the Dirac operator on a spin manifold, we have $c_D = 1$.

Definition 3.7. With the above notation, let us define

$$\sigma_{\langle h \rangle} := \frac{2K_{\langle h \rangle} \cdot c_D}{\tau_{\langle h \rangle}}. \quad (3.8)$$

We say the spectral gap of \tilde{D} is sufficiently large if the spectral gap of \tilde{D} at zero is larger than $\sigma_{\langle h \rangle}$, i.e., $\text{spectrum}(\tilde{D}) \cap [-\sigma_{\langle h \rangle}, \sigma_{\langle h \rangle}]$ is either $\{0\}$ or empty.

In the following, we shall show that the convergence of the formula (3.3) holds if \tilde{D} has a spectral gap at zero larger than $\sigma_{\langle h \rangle}$. In fact, the convergence of the formula (3.3) for the case where \tilde{D} has a spectral gap at zero can be deduced from the invertible case by replacing \tilde{D} with its restriction to the orthogonal complement of the kernel of \tilde{D} . Without loss of generality, we will only give the details of the proof for the case where \tilde{D} is invertible and its spectral gap at zero is larger than $\sigma_{\langle h \rangle}$.

Theorem 3.9. *With the same notation as above, for any nontrivial conjugacy class $\langle h \rangle$, suppose that \tilde{D} is invertible and its spectral gap at zero is larger than $\sigma_{\langle h \rangle}$. Then the delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ given in line (3.3) converges absolutely.*

Proof. The proof is divided into three steps. In the first step, we show that $\text{tr}_{\langle h \rangle}(\tilde{D}e^{-t^2\tilde{D}^2})$ is finite for any fixed $t > 0$ (Proposition 3.20). In the second step,

we prove the convergence of the integral for small t (Proposition 3.21). In the last step, we show the convergence of the integral for large t (Proposition 3.23). In fact, only the last step requires the assumption that the spectral gap of \tilde{D} is larger than $\sigma_{\langle h \rangle}$. \square

We say that $\langle h \rangle$ has subexponential growth if we can choose $K_{\langle h \rangle}$ in line (3.4) to be arbitrarily small. In this case, we have the following corollary.

Corollary 3.10. *If $\langle h \rangle$ has subexponential growth, then $\eta_{\langle h \rangle}(\tilde{D})$ given in line (3.3) converges absolutely, as long as \tilde{D} has a spectral gap at zero.*

In Theorem 3.9, the condition that the spectral gap of \tilde{D} is larger than $\sigma_{\langle h \rangle}$ might first appear to be rather ad hoc. In the following, we shall show that such a condition in fact holds for an abundance of natural examples whose higher rho invariant is nontrivial.

Suppose that N is a closed spin manifold equipped with a positive scalar curvature metric g_N , whose fundamental group $F = \pi_1(N)$ is finite and its higher rho invariant $\rho(\tilde{D}_N)$ is nontrivial. Here \tilde{D}_N is the Dirac operator on the universal covering \tilde{N} of N . For instance, let N to be a lens space, that is, the quotient of the 3-dimensional sphere by a free action of a finite cyclic group. In this case, the classical equivariant Atiyah–Patodi–Singer index theorem implies that the higher rho invariant of N is nontrivial; see [Donnelly 1978].

Now let V be an even dimensional closed spin manifold, whose Dirac operator D_V has nontrivial higher index in $K_0(C_r^*(\Gamma))$, where $\Gamma = \pi_1(V)$. In particular, it follows that D_V defines a nonzero element in the equivariant K -homology $K_0(C_L^*(E\Gamma)^\Gamma)$ of the universal space $E\Gamma$ for free Γ actions. Consider the product space $M = V \times N$ equipped with a metric $g_M = g_V + \varepsilon \cdot g_N$, where g_V is an arbitrary Riemannian metric on V and the metric g_N on N is scaled by a positive number ε . Denote the Dirac operator on the universal covering \tilde{M} of M by \tilde{D}_M . The spectral gap of \tilde{D}_M at zero can always be made sufficiently large, as long as we choose ε to be sufficiently small. To see that $\rho(\tilde{D}_M)$ is nonzero in $K_1(C_{L,0}^*(E(\Gamma \times F))^{\Gamma \times F})$, we apply the product formula for secondary invariants (see [Xie and Yu 2014, Claim 2.19; Zeidler 2016, Corollary 4.15]), which states that the higher rho invariant $\rho(\tilde{D}_M)$ is the product of the K -homology class of D_V and the higher rho invariant $\rho(\tilde{D}_N)$. It follows from the above construction that $\rho(\tilde{D}_M)$ is nonzero in $K_1(C_{L,0}^*(E(\Gamma \times F))^{\Gamma \times F})$.

Now let us return to the proof of Theorem 3.9. First, we need a few technical lemmas. Let M be a closed Riemannian manifold. Suppose T is an integral operator on the space $L^2(M)$ of L^2 functions on M , and assume the Schwartz kernel of T is a continuous function on $M \times M$. In particular, we have

$$T(f)(x) = \int_M T(x, y) f(y) dy,$$

for all $f \in L^2(M)$. We denote the operator norm of an operator A on $L^2(M)$ by $\|A\|_{\text{op}}$.

Lemma 3.11. *Let M be a closed Riemannian manifold and D a first-order self-adjoint elliptic differential operator on M . Suppose T is a bounded linear operator on $L^2(M)$ such that*

$$\sup_{k+j \leq \frac{3}{2} \dim M + 3} \|D^k T D^j\|_{\text{op}} < \infty.$$

Then T is an integral operator with a continuous Schwartz kernel $K_T(x, y)$, and there exists a positive number C (independent of T) such that

$$\sup_{x, y \in M} |K_T(x, y)| \leq C \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|D^k T D^j\|_{\text{op}}. \quad (3.12)$$

Proof. Let p be the smallest even integer that is greater than $\frac{1}{2} \dim M$. Then $(1 + D^p)^{-1}$ is a Hilbert–Schmidt operator on $L^2(M)$. Denote the Hilbert–Schmidt norm of $(1 + D^p)^{-1}$ by $\|(1 + D^p)^{-1}\|_{\text{HS}}$,

By assumption, $(1 + D^p)T$ is bounded. It follows that $T = (1 + D^p)^{-1} \circ (1 + D^p)T$ is also a Hilbert–Schmidt operator, and furthermore

$$\|T\|_{\text{HS}} \leq \|(1 + D^p)^{-1}\|_{\text{HS}} \cdot \|(1 + D^p)T\|_{\text{op}}.$$

It follows that T is an integral operator whose Schwartz kernel $K_T(x, y)$ is an L^2 -function on $M \times M$.

To see that K_T is continuous on $M \times M$, we consider the elliptic differential operator

$$\mathcal{D} = D \otimes 1 + 1 \otimes D$$

on $M \times M$. Observe that

$$\begin{aligned} \|\mathcal{D}^n K_T\|_{L^2(M \times M)} &= \left\| \sum_{r=0}^n \binom{n}{r} D^r T D^{n-r} \right\|_{\text{HS}} \\ &\leq \sum_{r=0}^n \binom{n}{r} \|(1 + D^p)^{-1}\|_{\text{HS}} \cdot \|(1 + D^p) D^r T D^{n-r}\|_{\text{op}}. \end{aligned}$$

Therefore, our assumption

$$\sup_{k+j \leq \frac{3}{2} \dim M + 3} \|D^k T D^j\|_{\text{op}} < \infty$$

implies that $\|\mathcal{D}^n K_T\|_{L^2(M \times M)}$ is finite for all $n \leq \dim M + 1$. It follows from the Sobolev embedding theorem that there exists $C_1 > 0$ such that

$$\sup_{x, y \in M} |K_T(x, y)| \leq C_1 \cdot \sup_{n \leq \dim M + 1} \|\mathcal{D}^n K_T\|_{L^2(M \times M)}$$

where the right hand side is dominated by

$$C \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|D^k T D^j\|_{\text{op}}$$

for some constant $C > 0$. This finishes the proof. \square

Let E be a Hermitian vector bundle over M and D a first order self-adjoint elliptic differential operator acting on the space $L^2(M, E)$ of L^2 -sections of E . Let \tilde{M} be a regular G -covering space of M . Denote the lift of E to \tilde{M} by \tilde{E} and the lift of D to \tilde{M} by \tilde{D} . The same argument from above also proves the following analogue for bounded operators acting on $L^2(\tilde{M}, \tilde{E})$.

Lemma 3.13. *Suppose T is a bounded linear operator on $L^2(\tilde{M}, \tilde{E})$ such that*

$$\sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k T \tilde{D}^j\|_{\text{op}} < \infty.$$

Then T is an integral operator with a continuous Schwartz kernel $K_T(x, y)$, and there exists a positive number C (independent of T) such that

$$\sup_{x, y \in \tilde{M}} \|K_T(x, y)\| \leq C \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k T \tilde{D}^j\|_{\text{op}},$$

where $\|K_T(x, y)\|$ denotes the operator norm of elements in $\text{End}(\tilde{E}_y, \tilde{E}_x)$.

Proof. For each pair of points x_0 and y_0 in \tilde{M} , let φ_{x_0} (resp. φ_{y_0}) be a compactly supported smooth function that is constant 1 in a neighborhood of x_0 (resp. y_0). Consider the operator $\varphi_{x_0} T \varphi_{y_0}$ acting on $L^2(\tilde{M}, \tilde{E})$, which maps functions supported near x_0 to functions supported near y_0 . Since $[\tilde{D}, \varphi_{x_0}]$, $[\tilde{D}, \varphi_{y_0}]$, $\tilde{D}^{2k}[\tilde{D}, \varphi_{x_0}](1 + \tilde{D}^2)^{-k}$ and $\tilde{D}^{2k}[\tilde{D}, \varphi_{y_0}](1 + \tilde{D}^2)^{-k}$ are all bounded operators, it follows that

$$\sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k \varphi_{x_0} T \varphi_{y_0} \tilde{D}^j\|_{\text{op}} \leq C_{x_0, y_0} \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k T \tilde{D}^j\|_{\text{op}}$$

for some positive constant C_{x_0, y_0} . By restricting the proof of Lemma 3.11 near small neighborhoods of x_0 and y_0 , we see that $\varphi_{x_0} T \varphi_{y_0}$ is an integral operator with a continuous Schwartz kernel $K_{T, x_0, y_0}(x, y)$ and there is a positive number C'_{x_0, y_0} such that

$$\sup_{(x, y) \in \text{supp}(\varphi_{x_0}) \times \text{supp}(\varphi_{y_0})} \|K_{T, x_0, y_0}(x, y)\| \leq C'_{x_0, y_0} \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k T \tilde{D}^j\|_{\text{op}}.$$

Note that the metric on \tilde{M} is the lift of a metric on M and \tilde{D} is the lift of D . Therefore, by choosing appropriate φ_{x_0} and φ_{y_0} for all $x_0, y_0 \in \tilde{M}$, we see that the constant C'_{x_0, y_0} may be chosen independent of x_0 and y_0 in \tilde{M} . Moreover, for each pair of points x_0 and y_0 , by testing on functions supported in sufficiently small neighborhoods of x_0 and y_0 , we see that the value of K_{T, x_0, y_0} at (x_0, y_0) does not

depend on the choice of the above cutoff functions φ_{x_0} and φ_{y_0} . In particular, it follows that there is a continuous function K_T on $\tilde{M} \times \tilde{M}$ such that

$$K_{T,x_0,y_0} = \varphi_{x_0} K_T \varphi_{y_0}.$$

Now by testing on all compactly supported smooth functions on \tilde{M} , we see that K_T is precisely the Schwartz kernel of T . By the above discussion, we have

$$\sup_{x,y \in \tilde{M}} \|K_T(x, y)\| \leq C \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k T \tilde{D}^j\|_{\text{op}}$$

for some constant C that is independent of T . This finishes the proof. \square

Remark 3.14. As an immediate consequence of the above lemma, we see that for a closed Riemannian manifold M and any bounded linear operator T on $L^2(M)$, if

$$\sup_{k+j \leq \frac{3}{2} \dim M + 3} \|D^k T D^j\|_{\text{op}} < \infty,$$

then T is of trace class. In fact, in this case, we have

$$\text{tr}(T) = \int_M T(x, x) dx;$$

see [Bernat et al. 1972, Chapter V, Proposition 3.1.1].

Suppose D is a first-order self-adjoint elliptic differential operator acting on a vector bundle E over M , and \tilde{E} (resp. \tilde{D}) is the lift of E (resp. D) to \tilde{M} . If f is a function on \mathbb{R} such that

$$\|x^m f(x)\|_{L^\infty} < \infty \quad (3.15)$$

for all $m \leq \frac{3}{2} \dim M + 3$, then the corresponding Schwartz kernel of the operator $f(\tilde{D})$ is continuous. Denote the Schwartz kernel of $f(\tilde{D})$ by K_f .

Lemma 3.16. *With the notations above, for any $\mu > 1$ and $r > 0$, there exists a constant $C > 0$ such that*

$$\|K_f(x, y)\| \leq C \cdot F_f\left(\frac{\text{dist}(x, y)}{\mu \cdot c_D}\right),$$

for $\forall x, y \in \tilde{M}$ with $\text{dist}(x, y) > r$ and any f satisfying line (3.15). Here $\text{dist}(x, y)$ stands for the distance between x and y , the notation $\|\cdot\|$ denotes the operator norm of elements in $\text{End}(\tilde{E}_y, \tilde{E}_x)$, and the function F_f is defined by

$$F_f(s) := \sup_{n \leq \frac{3}{2} \dim M + 3} \int_{|\xi| > s} \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| d\xi,$$

where \hat{f} is the Fourier transform of f .

Proof. The condition on f implies that $F_f(s) < \infty$ for any $s \in \mathbb{R}$ and $f(\tilde{D})$ is an integral operator with continuous Schwartz kernel; see Lemma 3.13.

By the Fourier inverse transform formula, we have

$$f(\tilde{D}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi \tilde{D}} d\xi.$$

Fix $r > 0$. Let $x_0, y_0 \in \tilde{M}$ such that $\lambda := \text{dist}(x_0, y_0) \geq r$. Choose a smooth function φ over \mathbb{R} such that $\varphi(\xi) = 1$ for $|\xi| \geq 1$ and $\varphi(\xi) = 0$ for $|\xi| \leq 1/\mu$. Let $\varphi_\lambda(\xi) = \varphi(\xi \cdot c_D/\lambda)$. Let g be the function with Fourier transform

$$\hat{g}(\xi) = \varphi_\lambda(\xi) \hat{f}(\xi),$$

and $L \in C(\tilde{M} \times \tilde{M})$ be the Schwartz kernel corresponding to the operator $g(\tilde{D})$. It follows from standard finite propagation estimates of wave operators that $L(x, y) = K_f(x, y)$ for all $x, y \in \tilde{M}$ with $\text{dist}(x, y) \geq \lambda$. In particular, we have $L(x_0, y_0) = K_f(x_0, y_0)$.

By Lemma 3.11, it suffice to estimate the operator norm of

$$\tilde{D}^k g(\tilde{D}) \tilde{D}^j = \tilde{D}^{k+j} g(\tilde{D})$$

for all $k+j \leq \frac{3}{2} \dim M + 3$. Now for a given $n \leq \frac{3}{2} \dim M + 3$, define $\psi_n(x) = x^n g(x)$. We have

$$\widehat{\psi}_n(\xi) = \left(\frac{1}{i} \frac{d}{d\xi} \right)^n (\varphi_\lambda \hat{f})(\xi).$$

Since φ is supported on $|\xi| \geq \lambda/(\mu c_D)$, there exist positive numbers C_1 and C such that

$$\begin{aligned} \|\psi_n(\tilde{D})\|_{\text{op}} &\leq \frac{1}{2\pi} \int_{|\xi| \geq \lambda/(\mu c_D)} |\widehat{\psi}_n(\xi)| d\xi \\ &\leq C_1 \sum_{j=0}^n \left(\frac{c_D}{r} \right)^j \int_{|\xi| \geq \lambda/(\mu c_D)} |\hat{f}^{(n-j)}(\xi)| d\xi \\ &\leq C \cdot F_f \left(\frac{\lambda}{\mu \cdot c_D} \right) \end{aligned}$$

By Lemma 3.13, we see that

$$\|K_f(x_0, y_0)\| = \|L(x_0, y_0)\| \leq C \cdot F_f \left(\frac{\text{dist}(x_0, y_0)}{\mu \cdot c_D} \right).$$

This finishes the proof. □

To streamline our estimates later, we consider the following class of functions in $C_0(\mathbb{R})$ whose Fourier transform has exponential decay.

Definition 3.17. Let $\mathcal{A}_{\Lambda, N}$ be the subspace of $C_0(\mathbb{R})$ consisting of functions f satisfying the following conditions:

- (1) f admits an analytic continuation \tilde{f} on the strip $\{|\operatorname{Im}(z)| < \Lambda\}$.
- (2) For any $n \leq N$, $|z^n \tilde{f}(z)|$ is uniformly bounded on the strip.

Equip $\mathcal{A}_{\Lambda, N}$ with the norm $\|\cdot\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}} = \sup_{0 \leq n \leq N} \sup_{|\operatorname{Im}(z)| < \Lambda} |z^n \tilde{f}(z)|$$

For any fixed Λ and N , it is easy to verify that $\mathcal{A}_{\Lambda, N}$ is closed under multiplication and conjugation. In fact, $\mathcal{A}_{\Lambda, N}$ is a Banach $*$ -subalgebra of $C_0(\mathbb{R})$ under the norm $\|\cdot\|_{\mathcal{A}}$. Clearly, e^{-x^2} and xe^{-x^2} lie in $\mathcal{A}_{\Lambda, N}$.

Lemma 3.18. Suppose $f \in \mathcal{A}_{\Lambda, N}$ for some $N \geq 2$. Let \hat{f} be its Fourier transform. Then for any $0 < \lambda < \Lambda$ and $0 \leq n \leq N - 2$ there exists some constant $C = C_{\lambda, n}$ such that

$$\int_{|\xi| > s} \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| d\xi \leq C \cdot \|f\|_{\mathcal{A}} \cdot e^{-\lambda s}. \quad (3.19)$$

Proof. For notational simplicity, let us denote the analytic continuation of f on the strip $\{|\operatorname{Im}(z)| < \Lambda\}$ still by f . For any $|y| < \Lambda$, $f(x - iy)$ is a smooth L^1 -integrable function since $|z^2 f(z)|$ is uniformly bounded. Denote the Fourier transform of $f(x - iy)$ with respect to x by \hat{f}_y , namely

$$\hat{f}_y(\xi) = \int_{-\infty}^{+\infty} f(x - iy) e^{-i\xi x} dx.$$

Since the right-hand side is differentiable in y , and uniformly differentiable in x and y by Cauchy inequality, the left-hand side is also differentiable in y with

$$\frac{\partial}{\partial y} \hat{f}_y(\xi) = \int_{-\infty}^{+\infty} \frac{\partial}{\partial y} [f(x - iy)] e^{-i\xi x} dx.$$

Since f is holomorphic, we have that $\frac{\partial}{\partial y} f = i \frac{\partial}{\partial x} f$ by the Cauchy–Riemann equation. Thus

$$\frac{\partial}{\partial y} \hat{f}_y(\xi) = \int_{-\infty}^{+\infty} i \frac{\partial}{\partial x} [f(x - iy)] e^{-i\xi x} dx = \xi \hat{f}_y(\xi).$$

It follows that

$$\hat{f}_y(\xi) = \hat{f}(\xi) e^{y\xi}.$$

Therefore by our assumption, for any $n \leq N - 2$ and $0 < \lambda < \Lambda$ there exists some constant $C_1 > 0$ such that

$$\begin{aligned} \left| \frac{d^n}{d\xi^n} (\hat{f}(\xi) e^{\lambda\xi}) \right| &= \left| \frac{d^n}{d\xi^n} \hat{f}_y(\xi) \right| \\ &= \int_{-\infty}^{+\infty} e^{ix\xi} (ix)^n f(x - i\lambda) dx \\ &\leq \int_{-\infty}^{\infty} |x^n f(x - i\lambda)| dx \\ &\leq C_1 \|f\|_A. \end{aligned}$$

Thus by induction on n , for any $n \leq N - 2$ and $0 < \lambda < \Lambda$, there exists a constant $C_2 > 0$ such that

$$\left| \frac{d^n}{d\xi^n} (\hat{f}(\xi)) \right| \leq C_2 \|f\|_A e^{-\lambda\xi}.$$

Hence for $\xi > s$, we have

$$\int_{\xi > s} \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| d\xi \leq C e^{-\lambda s}.$$

The estimates for the part $\int_{\xi < -s} \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| d\xi$ are completely similar. This finishes the proof. \square

If we fix a fundamental domain $\mathcal{F} \subset \tilde{M}$ for the action of G , then one naturally identifies $L^2(\tilde{M})$ with $L^2(\mathcal{F}) \otimes \ell^2(G)$ through the mapping $\tilde{h} \mapsto h$ by $h(x, \gamma) = \tilde{h}(\gamma x)$ for $x \in \mathcal{F}$ and $\gamma \in G$. In particular, every G -equivariant Schwartz kernel A on $\tilde{M} \times \tilde{M}$ becomes a formal sum

$$A = \sum_{g \in G} A_g R_g$$

where $A_g(x, y) = A(x, gy)$ for $x, y \in \mathcal{F}$ and R_g denotes the right translation of g on $\ell^2(G)$ corresponding to the right regular representation of G on $\ell^2(G)$. Now suppose $A = f(\tilde{D})$ as in Lemma 3.16 above. In this case, each A_g is a trace class operator and

$$\text{tr}(A_g) = \int_{\mathcal{F}} A_g(x, x) dx = \int_{\mathcal{F}} A(x, gx) dx.$$

Now we are ready to proceed with the first step of the proof for Theorem 3.9.

Proposition 3.20. *Suppose $\langle h \rangle$ is a nontrivial conjugacy class of G . If $f \in \mathcal{A}_{\Lambda, N}$ with $N \geq \frac{3}{2} \dim M + 5$ and Λ sufficiently large, then $\text{tr}_{\langle h \rangle}(f(\tilde{D}))$ is finite.*

Proof. Fix a symmetric generating set S of G . Let ℓ be the length function on G determined by S . Since G acts freely and cocompactly on \tilde{M} , there exists $\varepsilon > 0$

such that $\text{dist}(x, gx) > \varepsilon$ for any $x \in \tilde{M}$ and $g \in \langle h \rangle$. Let $k(x, y)$ be the Schwartz kernel of $f(\tilde{D})$. Since the action of G on \tilde{M} is free and cocompact, there exist $C_1, C_2 > 0$ such that

$$\text{dist}(x, gx) > C_1 \cdot \ell(g) - C_2.$$

It follows from Lemma 3.16 that there exists $C_3 > 0$ such that

$$\begin{aligned} |\text{tr}_{\langle h \rangle}(f(\tilde{D}))| &\leq \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} |\text{tr}(k(x, gx))| dx \\ &\leq C_3 \sum_{g \in \langle h \rangle} F_f(\max\{\varepsilon, C_1 \cdot \ell(g) - C_2\}) \\ &\leq C_3 \sum_{n=1}^{\infty} |S|^n F_f(\max\{\varepsilon, C_1 \cdot n - C_2\}), \end{aligned}$$

where $|S|$ is the cardinality of the generating set S . By Lemma 3.18, when $N \geq \frac{3}{2} \dim M + 5$, for any $\lambda < \Lambda$, there exists $C > 0$ such that

$$F_f(x) \leq C e^{-\lambda x}.$$

Observe that the summation

$$\sum_{n=1}^{\infty} |S|^n e^{-\lambda \cdot \max\{\varepsilon, C_1 \cdot n - C_2\}}$$

converges absolutely, as long as λ is sufficiently large. This finishes the proof. \square

Now let us prove the convergence of the integral (3.3) for small t .

Proposition 3.21. *Suppose $\langle h \rangle$ is a nontrivial conjugacy class of G . If $f \in \mathcal{A}_{\Lambda, N}$ with $N \geq \frac{3}{2} \dim M + 5$ and Λ sufficiently large, then the following integral*

$$\int_0^1 t^k \cdot \text{tr}_{\langle h \rangle}(f(t\tilde{D})) dt$$

is absolutely convergent for any $k \in \mathbb{R}$.

Proof. Let us denote $f_t(x) = f(tx)$. Clearly, $f_t \in \mathcal{A}_{\Lambda, N}$ since $f \in \mathcal{A}_{\Lambda, N}$. Similar to the proof of Proposition 3.20, we have

$$|t^k \text{tr}_{\langle h \rangle}(f_t(\tilde{D}))| \leq C_3 \sum_{n=1}^{\infty} |S|^n t^k F_{f_t}(\max\{\varepsilon, C_1 \cdot n - C_2\}).$$

Recall that $\hat{f}_t(\xi) = t^{-1} \cdot \hat{f}(\xi/t)$. In particular, we have

$$\frac{d^n}{d\xi^n} \hat{f}_t(\xi) = \frac{1}{t^{n+1}} \hat{f}^{(n)}(\xi/t),$$

where $\hat{f}^{(n)}$ is the n -th derivative of \hat{f} . It follows from Lemma 3.18 that there exists $C_\Lambda > 0$ such that

$$\begin{aligned} F_{f_t}(s) &= \sup_{n \leq \frac{3}{2} \dim M + 3} \frac{1}{t^{n+1}} \int_{|\xi| > s} |\hat{f}^{(n)}(\xi/t)| d\xi \\ &\leq \frac{F_f(s \cdot t^{-1})}{t^{(3/2) \dim M + 3}} \\ &\leq \frac{C_\Lambda}{t^{(3/2) \dim M + 3}} \cdot \|f\|_{\mathcal{A}} \cdot \exp\left(\frac{-\Lambda s}{2t}\right) \end{aligned} \quad (3.22)$$

for all $t \in (0, 1]$. The following summation

$$\sum_{n=1}^{\infty} |S|^n t^{k - ((3/2) \dim M + 3)} \exp\left(\frac{-\Lambda \cdot \max\{\varepsilon, C_1 \cdot n - C_2\}}{2t}\right)$$

is integrable on $(0, 1]$, as long as Λ is sufficiently large. This finishes the proof. \square

Proposition 3.23. *Let σ^2 be the infimum of the spectrum of \tilde{D}^2 . If $\sigma > \sigma_{\langle h \rangle}$ defined in line (3.8), then the following integral*

$$\int_1^{+\infty} \text{tr}_{\langle h \rangle}(\tilde{D}e^{-t^2 \tilde{D}^2}) dt$$

is absolutely convergent.

Proof. View $\tilde{D}e^{-t^2 \tilde{D}^2}$ as an element in $\mathcal{K} \otimes C_r^*(G)$ and write

$$\tilde{D}e^{-t^2 \tilde{D}^2} = \sum_{g \in G} A_{g,t} g.$$

Note that $A_{g,t}$ are compact operators for all $g \in G$ and $t \geq 1$. By Lemma 3.11, $A_{g,t}$ is of trace class and there exist $C_1, N_1 > 0$ such that

$$|A_{g,t}|_1 \leq C_1 t^{N_1} \cdot e^{-t^2 \sigma^2},$$

where $|\cdot|_1$ stands for the trace norm.

By the definition of $\tau_{\langle h \rangle}$ in line (3.5), for any $\varepsilon > 0$ there exists $L > 0$ such that

$$\text{dist}(x, gx) \geq (\tau_{\langle h \rangle} - \varepsilon)\ell(g),$$

for all $x \in \mathcal{F}$ and $g \in \langle h \rangle$ satisfying $\ell(g) > L$. By Lemma 3.16 and Proposition 3.20, if $\ell(g) > L$, then there exist $C_2, N_2 > 0$ such that

$$|A_{g,t}|_1 \leq C_2 (t \cdot \ell(g))^{N_2} \cdot \exp\left(-\frac{(\tau_{\langle h \rangle} - \varepsilon)^2 \ell^2(g)}{4t^2(\mu \cdot c_D)^2}\right). \quad (3.24)$$

Write $N = \max\{N_1, N_2\}$ and $C = \max\{C_1, C_2\}$. Then for $\delta > 0$ sufficiently small, we have

$$\begin{aligned} |A_{g,t}|_1 &\leq C(t \cdot \ell(g))^N \cdot \exp\left(-\frac{1}{2} \frac{(\tau_{\langle h \rangle} - \varepsilon)^2 \ell^2(g)}{4t^2(\mu \cdot c_D)^2} - \frac{t^2 \sigma^2}{2}\right) \\ &\leq C(t \cdot \ell(g))^N \cdot \exp\left(-\frac{(\tau_{\langle h \rangle} - \varepsilon)(\sqrt{\sigma^2 - \delta^2}) \ell(g)}{2\mu \cdot c_D}\right) \cdot e^{-t^2 \delta^2}, \end{aligned}$$

as long as $\ell(g) > L$. By the assumption that $\sigma > \sigma_{\langle h \rangle}$, we may find suitable μ, ε, δ such that

$$(\tau_{\langle h \rangle} - \varepsilon)(\sqrt{\sigma^2 - \delta^2}) > 2\mu \cdot c_D \cdot K_{\langle h \rangle}.$$

Therefore, by line (3.4), there exist C_3, C_4 such that

$$\begin{aligned} |\mathrm{tr}_{\langle h \rangle}(\tilde{D}e^{-t^2 \tilde{D}^2})| &\leq \sum_{g \in \langle h \rangle, \ell(g) \leq L} |A_{g,t}|_1 + \sum_{g \in \langle h \rangle, \ell(g) > L} |A_{g,t}|_1 \\ &\leq C_3 e^{K_{\langle h \rangle} L} \cdot C(tL)^N e^{-t^2 \sigma^2} \\ &\quad + \sum_{n > L} C_3 e^{K_{\langle h \rangle} n} \cdot C(nt)^N \exp\left(-\frac{(\tau_{\langle h \rangle} - \varepsilon)(\sqrt{\sigma^2 - \delta^2})n}{2\mu c_D}\right) e^{-t^2 \delta^2} \\ &\leq C_4 t^N e^{-\delta^2 t^2}. \end{aligned}$$

Hence the integral

$$\int_0^\infty \mathrm{tr}_{\langle h \rangle}(\tilde{D}e^{-t^2 \tilde{D}^2}) dt$$

converges absolutely. This finishes the proof. \square

3.2. Delocalized higher eta invariants. In this subsection, we shall generalize the results of the previous subsection to higher degree cyclic cocycles. For simplicity, we only give details for the odd case; the even case is similar.

Let us first recall the definition of cyclic cocycles.

Definition 3.25. Let $C^n(\mathbb{C}G)$ be the space spanned by all $(n+1)$ -linear functionals φ on $\mathbb{C}G$ such that

$$\varphi(g_1, g_2, \dots, g_n, g_0) = (-1)^n \varphi(g_0, g_1, \dots, g_n).$$

The coboundary map $b : C^n(\mathbb{C}G) \rightarrow C^{n+1}(\mathbb{C}G)$ is defined to be

$$\begin{aligned} b\varphi(g_0, g_1, \dots, g_{n+1}) \\ = \sum_{j=0}^n (-1)^j \varphi(g_0, g_1, \dots, g_j g_{j+1}, \dots, g_{n+1}) + (-1)^{n+1} \varphi(g_{n+1} g_0, g_1, \dots, g_n). \end{aligned}$$

The cohomology of this cochain complex $(C^n(\mathbb{C}G), b)$ is the cyclic cohomology of $\mathbb{C}G$, denoted by $HC^*(\mathbb{C}G)$.

Definition 3.26. Fix any conjugacy class $\langle h \rangle$ of G . Let $C^n(\mathbb{C}G, \langle h \rangle)$ be the space spanned by all elements $\varphi \in C^n(\mathbb{C}G)$ satisfying the condition

$$g_0 g_1 \cdots g_n \notin \langle h \rangle \implies \varphi(g_0, g_1, \dots, g_n) = 0.$$

It is easy to verify that $(C^n(\mathbb{C}G, \langle h \rangle), b)$ is a subcomplex of $(C^n(\mathbb{C}G), b)$. We denote the cohomology of $(C^n(\mathbb{C}G, \langle h \rangle), b)$ by $HC^*(\mathbb{C}G, \langle h \rangle)$.

Definition 3.27. If the conjugacy class $\langle h \rangle$ is nontrivial, then a cyclic cocycle in $C^n(\mathbb{C}G, \langle h \rangle)$ is called a *delocalized cyclic cocycle* at $\langle h \rangle$, and $HC^*(\mathbb{C}G, \langle h \rangle)$ is called the *delocalized cyclic cohomology* of $\mathbb{C}G$ at $\langle h \rangle$.

Recall that (see [Nistor 1990])

$$HC^*(\mathbb{C}G) \cong \prod_{\langle h \rangle} HC^*(\mathbb{C}G, \langle h \rangle).$$

Moreover, it is easy to verify that $HC^0(\mathbb{C}G, \langle h \rangle)$ is a one dimensional vector space generated by $tr_{\langle h \rangle}$.

Let us write $w = \sum_g w^g g$ for an element $w \in C^*(\tilde{M})^G$. Given $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ and $w = w_0 \otimes w_1 \otimes \cdots \otimes w_n$ in $(C^*(\tilde{M})^G)^{\otimes n+1}$, we define the following map

$$(\varphi \# tr)(w) = \sum_{g_0, \dots, g_n \in G} tr(w_0^{g_0} \cdots w_n^{g_n}) \varphi(g_0, \dots, g_n) \quad (3.28)$$

whenever the above formula converges. Here $tr(w_0^{g_0} \cdots w_n^{g_n})$ stands for the trace of the operator $w_0^{g_0} \cdots w_n^{g_n}$.

Following the definition of higher rho invariant in line (2.6), we define

$$u_t(x) = U_{1/t}(x) = \exp\left(2\pi i \frac{1}{\sqrt{\pi}} \int_{-\infty}^{xt} e^{-s^2} ds\right). \quad (3.29)$$

Note that the functions $u_t(x) - 1$, $u_t^{-1}(x) - 1$ and $\dot{u}_t(x)u_t^{-1}(x) = 2\sqrt{\pi}ixe^{-t^2x^2}$ are Schwartz functions. We define the delocalized higher eta invariant as follows.

Definition 3.30. For any $\varphi \in C^{2m}(\mathbb{C}G, \langle h \rangle)$ with $\langle h \rangle$ nontrivial, we define the delocalized higher eta invariant of \tilde{D} with respect to φ to be

$$\eta_\varphi(\tilde{D}) := \frac{m!}{\pi i} \int_0^\infty \eta_\varphi(\tilde{D}, t) dt, \quad (3.31)$$

where

$$\eta_\varphi(\tilde{D}, t) = \varphi \# tr(\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D}) \otimes ((u_t(\tilde{D}) - 1) \otimes (u_t^{-1}(\tilde{D}) - 1))^{\otimes m}). \quad (3.32)$$

More precisely, we have

$$\begin{aligned} \eta_\varphi(\tilde{D}, t) &:= \sum_{g_t \in G} \left[\varphi(g_0, g_1, \dots, g_{2m}) \int_{\mathcal{F}^{2m+1}} \text{tr}(k_{0,t}(x_0, g_0 x_1) k_{1,t}(x_1, g_1 x_2) \right. \\ &\quad \left. \times \dots \times k_{2m,t}(x_{2m}, g_{2m} x_0)) dx_0 \dots dx_{2m} \right]. \end{aligned} \quad (3.33)$$

where $k_{i,t}(x, y)$ is the corresponding Schwartz kernel of $\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D})$, $u_t(\tilde{D}) - 1$ and $u_t^{-1}(\tilde{D}) - 1$ respectively.

Remark 3.34. Clearly, if $m = 0$, then $\eta_{\text{tr}_{\langle h \rangle}}(\tilde{D}) = \eta_{\langle h \rangle}(\tilde{D})$. Hence the delocalized higher eta invariant is indeed a natural generalization of Lott's delocalized eta invariant.

Remark 3.35. In fact, the delocalized higher eta invariant can be defined for a more general class of representatives besides $u_t(\tilde{D})$ above. We shall deal with the general case in Section 6.

Remark 3.36. The map $\varphi \# \text{tr}$ naturally extends to the unitization $(C^*(\tilde{M})^G)^+$ of $C^*(\tilde{M})^G$ by defining $\varphi \# \text{tr}$ to be the map that vanishes on the identity element in $(C^*(\tilde{M})^G)^+$. With this notation, the formula of $\eta_\varphi(\tilde{D})$ becomes

$$\eta_\varphi(\tilde{D}) = \frac{m!}{\pi i} \int_0^\infty \varphi \# \text{tr}(\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D}) \otimes (u_t(\tilde{D}) \otimes u_t^{-1}(\tilde{D})^{\otimes m}) dt.$$

Remark 3.37. We have only discussed the odd case so far. The even case is completely analogous. In this case, in the construction of the higher rho invariant $\rho(\tilde{D})$, the path of invertibles $\{u_t(\tilde{D})\}_{0 \leq t < \infty}$ is replaced by a path of projections $\{p_t(\tilde{D})\}_{0 \leq t < \infty}$; see for example [Xie and Yu 2014]. Given $\varphi \in C^{2m+1}(\mathbb{C}G, \langle h \rangle)$, the delocalized higher eta invariant of \tilde{D} with respect to φ is defined to be

$$\eta_\varphi(\tilde{D}) = \frac{1}{\pi i} \frac{(2m)!}{m!} \int_0^\infty \eta_\varphi(\tilde{D}, t) dt,$$

where

$$\eta_\varphi(\tilde{D}, t) = \varphi \# \text{tr}([\dot{p}_t(\tilde{D}), p_t(\tilde{D})] \otimes p_t(\tilde{D})^{\otimes 2m+1}).$$

The integral formula in line (3.31) does not converge in general. In the following, we shall show that the convergence holds whenever \tilde{D} has a sufficiently large spectral gap at zero.

Definition 3.38. Let $\langle h \rangle$ be a nontrivial conjugacy class of G . An element $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ is said to have exponential growth if there exist C and $K_\varphi > 0$ such that

$$|\varphi(g_0, g_1, \dots, g_n)| \leq C e^{K_\varphi \cdot (\ell(g_0) + \ell(g_1) + \dots + \ell(g_n))} \quad (3.39)$$

for $\forall (g_0, g_1, \dots, g_n) \in G^{n+1}$.

Similar to the definition of $\tau_{\langle h \rangle}$ in line (3.5), we define τ to be the following positive number

$$\tau = \liminf_{\ell(g) \rightarrow \infty} \left(\inf_{x \in \mathcal{F}} \frac{\text{dist}(x, gx)}{\ell(g)} \right). \quad (3.40)$$

Since G is finitely generated, there exist C and $K_G > 0$ such that

$$\#\{g \in G : \ell(g) = n\} \leq C e^{K_G \cdot n}. \quad (3.41)$$

Definition 3.42. We define

$$\sigma_\varphi =: \frac{2(K_G + K_\varphi) \cdot c_D}{\tau}, \quad (3.43)$$

where c_D is the propagation speed of D as defined in line (3.6).

Theorem 3.44. Suppose that $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ has exponential growth. With the same notation from above, if the spectral gap of \tilde{D} at zero is larger than σ_φ given in line (3.43) above, then the delocalized higher eta invariant $\eta_\varphi(\tilde{D})$ given in line (3.31) converges absolutely.

Proof. The proof is divided into three steps. First we show that $\eta_\varphi(\tilde{D}, t)$ is well-defined for any fixed $t > 0$ (Proposition 3.46). Next we show that its integral for small t converges absolutely (Proposition 3.48). The last step is to show the convergence of the integral for large t (Proposition 3.51). In fact, only the last step requires the spectral gap of \tilde{D} to be larger than σ_φ . \square

We say G (resp. φ) has subexponential growth if we may choose K_G in line (3.41) (resp. K_φ in line (3.39)) to be arbitrarily small. The following corollary is an immediate consequence of Theorem 3.44 above.

Corollary 3.45. With the notation above, if both G and φ have subexponential growth, then the delocalized higher eta invariant $\eta_\varphi(\tilde{D})$ given in line (3.31) converges absolutely, as long as \tilde{D} has a spectral gap at zero.

Proposition 3.46. Suppose that $\langle h \rangle$ be a nontrivial conjugacy class of G and $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ has exponential growth. If $N \geq \frac{3}{2} \dim M + 5$ and Λ is sufficiently large, then there exists $C > 0$ such that for any $f_i \in \mathcal{A}_{\Lambda, N}$ ($i = 0, 1, \dots, n$),

$$\varphi \# \text{tr}(f_0(\tilde{D}) \otimes \cdots \otimes f_n(\tilde{D})) \leq C \|f_0\|_{\mathcal{A}} \cdots \|f_n\|_{\mathcal{A}}.$$

Proof. Fix a symmetric generating set S of G . Let ℓ be the length function on G determined by S . Denote the cardinality of S by $|S|$.

For each $0 \leq i \leq n$, let $w_i = f_i(\tilde{D})$ for some function $f_i \in \mathcal{A}_{\Lambda, N}$. Let us write w_i as a formal sum $w_i = \sum_g w_i^g g$. If we denote by $k_i(x, y)$ the Schwartz kernel of w_i , then

$$\text{tr}(w_0^{g_0} \cdots w_n^{g_n}) = \int_{\mathcal{F}^{n+1}} \text{tr}(k_0(x_0, g_0 x_1) \cdots k_n(x_n, g_n x_0)) dx_0 \cdots dx_n. \quad (3.47)$$

It follows from Lemma 3.11 and the definition of $\|\cdot\|_{\mathcal{A}}$ (see Definition 3.17) that there exists a constant C_1 such that

$$\|k_i(x, y)\| \leq C_1 \|f_i\|_{\mathcal{A}} \quad \text{for } 0 \leq i \leq n.$$

For any $(g_0, g_1, \dots, g_n) \in G^{n+1}$, we divide \mathcal{F}^{n+1} into $(n+1)$ disjoint (possibly empty) Borel sets $\mathcal{F}_{(j), (g_0, g_1, \dots, g_n)}^{n+1}$ such that $\text{dist}(x_j, g_j x_{j+1})$ is the maximum of the set $\{\text{dist}(x_i, g_i x_{i+1})\}_{0 \leq i \leq n}$, where $x_{n+1} = x_0$. In other words,

$$\text{dist}(x_j, g_j x_{j+1}) \geq \text{dist}(x_i, g_i x_{i+1}) \quad \text{on } \mathcal{F}_{(j), (g_0, g_1, \dots, g_n)}^{n+1} \quad \text{for all } 0 \leq i \leq n.$$

If no confusion is likely to arise, we shall write $\mathcal{F}_{(j)}^{n+1}$ in place of $\mathcal{F}_{(j), (g_0, g_1, \dots, g_n)}^{n+1}$.

Since the action of G on \tilde{M} is free and cocompact, there exist C_1, C_2 such that

$$\text{dist}(x, gy) \geq C_1 \ell(g) - C_2, \quad \text{for } \forall x, y \in \mathcal{F}.$$

It follows that

$$\text{dist}(x_j, g_j x_{j+1}) \geq C_1 \frac{\sum_{i=0}^n \ell(g_i)}{n+1} - C_2$$

on $\mathcal{F}_{(j)}^{n+1}$.

Again, as the action of G on \tilde{M} is free and cocompact, there exists $\varepsilon > 0$ such that $\text{dist}(x, gx) > \varepsilon$ for all $x \in \tilde{M}$ and all $g \neq e$. Note that, since the metric on \tilde{M} is G -equivariant, we have

$$\sum_{i=0}^n \text{dist}(x_i, g_i x_{i+1}) \geq \text{dist}(x_0, g_0 g_1 \cdots g_n x_0).$$

To summarize, we have

$$\text{dist}(x_j, g_j x_{j+1}) \geq \max \left\{ \frac{\varepsilon}{n+1}, C_1 \frac{\sum_{i=0}^n \ell(g_i)}{n+1} - C_2 \right\},$$

on $\mathcal{F}_{(j)}^{n+1}$. It follows from Lemma 3.16 that, for any $g_0 g_1 \cdots g_n \in \langle h \rangle$, there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned} & \left| \int_{\mathcal{F}_{(j)}^{n+1}} \text{tr}(k_0(x_0, g_0 x_1) \cdots k_n(x_n, g_n x_0)) dx_0 \cdots dx_n \right| \\ & \leq C_3 \cdot F_{f_j} \left(C_4 \cdot \max \left\{ \frac{\varepsilon}{n+1}, C_1 \frac{\sum_{i=0}^n \ell(g_i)}{n+1} - C_2 \right\} \right) \cdot \prod_{i \neq j} \|f_i\|_{\mathcal{A}}. \end{aligned}$$

Since φ has exponential growth, there exist $C, C', K_\varphi > 0$ such that

$$\begin{aligned}
& |\varphi \# \operatorname{tr}(w_0 \otimes w_1 \otimes \cdots \otimes w_n)| \\
& \leq C \sum_{g_0 g_1 \cdots g_n \in \langle h \rangle} e^{K_\varphi \sum_{i=0}^n \ell(g_i)} \cdot |\operatorname{tr}(w_0^{g_0} \cdots w_n^{g_n})| \\
& \leq C \sum_{g_0 g_1 \cdots g_n \in \langle h \rangle} e^{K_\varphi \sum_{i=0}^n \ell(g_i)} \\
& \quad \times \left(\sum_{j=0}^n \int_{\mathcal{F}_{(j)}^{n+1}} |\operatorname{tr}(k_0(x_0, g_0 x_1) \cdots k_n(x_n, g_n x_0))| dx_0 \cdots dx_n \right) \\
& \leq C C_3 \sum_{m=1}^{\infty} e^{K_\varphi \cdot m} \cdot |S|^{(n+1)m} \\
& \quad \times \left(\sum_{j=0}^n \left[F_{f_j} \left(C_4 \cdot \max \left\{ \frac{\varepsilon}{n+1}, \frac{m C_1}{n+1} - C_2 \right\} \right) \cdot \prod_{i \neq j} \|f_i\|_{\mathcal{A}} \right] \right) \\
& \leq C' C_3 \sum_{m=1}^{\infty} e^{K_\varphi \cdot m} \cdot |S|^{(n+1)m} \\
& \quad \times \exp \left(-\frac{\Lambda}{2} C_4 \cdot \max \left\{ \frac{\varepsilon}{n+1}, \frac{m C_1}{n+1} - C_2 \right\} \right) \prod_{i=0}^n \|f_i\|_{\mathcal{A}},
\end{aligned}$$

where the last summation converges for sufficiently large Λ by Lemma 3.18. This finishes the proof. \square

Let us now prove the convergence of the integral in line (3.31) for small t .

Proposition 3.48. *Suppose that $\langle h \rangle$ be a nontrivial conjugacy class of G and $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ has exponential growth. If $N \geq \frac{3}{2} \dim M + 5$ and Λ is sufficiently large, then the following integral*

$$\int_0^1 t^k \varphi \# \operatorname{tr}(f_0(t\tilde{D}) \otimes \cdots \otimes f_n(t\tilde{D})) dt$$

is absolutely convergent for all $f_i \in \mathcal{A}_{\Lambda, N}$ and any $k \in \mathbb{R}$.

Proof. Define $f_{j,t}(x) = f_j(tx)$ for $j = 0, 1, \dots, n$. From the definition of $\|\cdot\|_{\mathcal{A}}$ (see Definition 3.17), we have

$$\|f_{j,t}\|_{\mathcal{A}} \leq t^{-N} \|f_j\|_{\mathcal{A}} \quad \text{for all } t \in (0, 1] \text{ and } f_j \in \mathcal{A}_{\Lambda, N}.$$

Recall the inequality in line (3.22):

$$F_{f_{j,t}}(s) \leq \frac{F_{f_j}(s \cdot t^{-1})}{t^{(3/2) \dim M + 3}} \leq \frac{C_\Lambda}{t^{(3/2) \dim M + 3}} \cdot \|f_j\|_{\mathcal{A}} \cdot \exp\left(\frac{-\Lambda s}{2t}\right).$$

Similar to the proof of Proposition 3.46, there exist positive constants K_φ , C_1 , C_2 , C_3 , C_4 , C_5 , C_6 and C_7 such that

$$\begin{aligned}
 & |t^k \varphi \# \text{tr}(f_0(t\tilde{D}) \otimes \cdots \otimes f_n(t\tilde{D}))| \\
 & \leq t^k C_3 \sum_{m=1}^{\infty} e^{K_\varphi \cdot m} |S|^{(n+1)m} \left(\sum_{j=0}^n \left[F_{f_j, t} \left(C_4 \cdot \max \left\{ \frac{\varepsilon}{n+1}, \frac{mC_1}{n+1} - C_2 \right\} \right) \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \times \prod_{i \neq j} \|f_{i, t}\|_{\mathcal{A}} \right] \right) \\
 & \leq t^{k - ((3/2) \dim M + 3)n - nN} C_5 \prod_{i=0}^n \|f_i\|_{\mathcal{A}} \sum_{m=1}^{\infty} e^{mC_6} \exp \left(-mC_7 \cdot \frac{\Lambda}{2t} \right),
 \end{aligned}$$

which proves the proposition as long as Λ is sufficiently large. \square

To prove the convergence of the integral in line (3.31) for large t , we need to fix some notation.

Definition 3.49. For any $K > 0$, let \mathcal{L}_K be the subspace of $\mathcal{K} \otimes C_r^*(G)$ consisting of operators $A = \sum A_g g$ such that

- (1) for any $g \in G$, A_g is of trace class;
- (2) and we have

$$\sum_{g \in G} e^{K \cdot \ell(g)} |A_g|_1 < \infty,$$

where $|\cdot|_1$ is the trace norm.

Equip \mathcal{L}_K with the following norm

$$\|A\|_{\mathcal{L}} = \sum_{g \in G} e^{K \cdot \ell(g)} |A_g|_1.$$

Lemma 3.50. *The space \mathcal{L}_K is a Banach algebra with the norm $\|\cdot\|_{\mathcal{L}}$.*

Proof. It is not difficult to see that \mathcal{L}_K is a Banach space under the norm $\|\cdot\|_{\mathcal{L}}$. It remains to show that $\|\cdot\|_{\mathcal{L}}$ is submultiplicative. Indeed, given $A_1 = \sum_{g \in G} A_{1,g} g$ and $A_2 = \sum_{g \in G} A_{2,g} g$ in \mathcal{L}_K , we have

$$\begin{aligned}
 \|A_1 A_2\|_{\mathcal{L}} & \leq \sum_{g \in G} e^{K \cdot \ell(g)} \sum_{g_1 \in G} |A_{1, g g_1}|_1 |A_{2, g_1^{-1}}|_1 \\
 & = \sum_{g_1 \in G} \sum_{g_2 \in G} e^{K \cdot (\ell(g_1 g_2) - \ell(g_1) - \ell(g_2))} (e^{K \cdot \ell(g_1)} |A_{1, g_1}|_1) (e^{K \cdot \ell(g_2)} |A_{2, g_2}|_1) \\
 & \leq \|A_1\|_{\mathcal{L}} \|A_2\|_{\mathcal{L}}.
 \end{aligned}$$

\square

Now we prove the convergence of the integral in (3.31) for large t .

Proposition 3.51. *Suppose that $\varphi \in C^{2m}(\mathbb{C}G)$ has exponential growth.⁶ If the spectral gap of \tilde{D} at zero is larger than σ_φ given in line (3.43), then the integral*

$$\int_1^\infty \eta_\varphi(\tilde{D}, t) dt$$

is absolutely convergent.

Proof. The case where \tilde{D} has a spectral gap at zero can be deduced from the invertible case by replacing \tilde{D} with its restriction to the orthogonal complement of the kernel of \tilde{D} . Without loss of generality, let us assume \tilde{D} is invertible.

Since φ has exponential growth, there exist C and K_φ such that

$$|\varphi(g_0, g_1, \dots, g_n)| \leq C e^{K_\varphi(\ell(g_0) + \ell(g_1) + \dots + \ell(g_n))}$$

for $\forall(g_0, g_1, \dots, g_n) \in G^{n+1}$. We first show that $\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D})$, $u_t(\tilde{D}) - 1$ and $u_t^{-1}(\tilde{D}) - 1$ all lie in \mathcal{L}_{K_φ} , where \mathcal{L}_{K_φ} is given in Definition 3.49 above.

Let σ be the spectral gap of \tilde{D} at zero. Note that

$$\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D}) = 2\sqrt{\pi}i\tilde{D}e^{-t^2\tilde{D}^2}.$$

Since $\sigma > \sigma_\varphi = 2(K_G + K_\varphi) \cdot c_D/\tau$, the same proof of Proposition 3.23 shows that $\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D})$ lies in \mathcal{L}_{K_φ} and furthermore there exists sufficiently small $\omega > 0$ such that

$$\|\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D})\|_{\mathcal{L}} \leq 2\sqrt{\pi}e^{-\omega t^2}. \quad (3.52)$$

Observe that

$$u_t(\tilde{D}) - 1 = \exp\left(-\int_t^{+\infty} \dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D}) ds\right) - 1.$$

By the inequality in line (3.52) above, we have

$$\int_t^{+\infty} \|\dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D})\|_{\mathcal{L}} ds \leq \frac{2\pi e^{-\omega t^2}}{\sqrt{\omega}}.$$

Thus for any $t > 1$, we have

$$\int_t^{+\infty} \dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D}) ds \in \mathcal{L}_{K_\varphi}$$

and

$$\left\| \int_t^{+\infty} \dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D}) ds \right\|_{\mathcal{L}} \leq \frac{2\pi e^{-\omega t^2}}{\sqrt{\omega}}.$$

⁶Here φ is not necessarily delocalized.

By Lemma 3.50, \mathcal{L}_{K_φ} is a Banach algebra under the norm $\|\cdot\|_{\mathcal{L}}$. It follows that

$$u_t(\tilde{D}) - 1 = \exp\left(-\int_t^{+\infty} \dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D}) ds\right) - 1 \in \mathcal{L}_{K_\varphi}.$$

Moreover, we have

$$\|u_t(\tilde{D}) - 1\|_{\mathcal{L}} \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(\left\| \int_t^{+\infty} \dot{u}_s(\tilde{D})u_s^{-1}(\tilde{D}) ds \right\|_{\mathcal{L}} \right)^n \leq e^{C_1 \cdot e^{-\omega t^2}} - 1,$$

where $C_1 = 2\pi/\sqrt{\omega}$. A similar estimate holds for $u_t^{-1}(\tilde{D}) - 1$ as well. A straightforward calculation shows that

$$|\eta_\varphi(\tilde{D}, t)| \leq C \cdot \|\dot{u}_t(\tilde{D})u_t^{-1}(\tilde{D})\|_{\mathcal{L}} \cdot \|u_t(\tilde{D}) - 1\|_{\mathcal{L}}^m \cdot \|u_t^{-1}(\tilde{D}) - 1\|_{\mathcal{L}}^m.$$

It follows that

$$|\eta_\varphi(\tilde{D}, t)| \leq 2\sqrt{\pi} e^{-\omega t^2} (e^{C_1 \cdot e^{-\omega t^2}} - 1)^{2m},$$

where the right hand side is clearly integrable over $[1, \infty)$. This finishes the proof. \square

The following proposition shows that the delocalized higher eta invariant in Definition 3.30 is independent of the choice of representative within the same cyclic cohomology class.

Proposition 3.53. *Let $\langle h \rangle$ be a nontrivial conjugacy class of G . Suppose that φ_1 and φ_2 are two cyclic cocycles in $C^{2m}(\mathbb{C}G, \langle h \rangle)$ with exponential growth and φ_1 and φ_2 are cohomologous via a cochain with exponential growth. If \tilde{D} has a sufficiently large spectral gap at zero, then $\eta_{\varphi_1}(\tilde{D}) = \eta_{\varphi_2}(\tilde{D})$.*

Proof. It suffices to show that if $\varphi \in C^{2m-1}(\mathbb{C}G, \langle h \rangle)$ has exponential growth, then $\eta_{b\varphi}(\tilde{D}) = 0$. By the definition of $\widetilde{\varphi \# \text{tr}}$, if $w_j = 1$ for some j , then

$$\widetilde{\varphi \# \text{tr}}(w_0 \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_n) = 0.$$

For notational simplicity, let us write u_t in place of $u_t(\tilde{D})$ in the following.

Observe that

$$\begin{aligned} & \frac{d}{dt} \widetilde{\varphi \# \text{tr}}((u_t \otimes u_t^{-1})^{\otimes m}) \\ &= m \widetilde{\varphi \# \text{tr}}(\dot{u}_t \otimes u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m-1}) - m \widetilde{\varphi \# \text{tr}}(u_t \otimes u_t^{-1} \dot{u}_t u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m-1}), \end{aligned}$$

and

$$\begin{aligned} & (\widetilde{b\varphi \# \text{tr}})(\dot{u}_t u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m}) \\ &= \widetilde{\varphi \# \text{tr}}(\dot{u}_t \otimes u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m-1}) - \widetilde{\varphi \# \text{tr}}(u_t \otimes u_t^{-1} \dot{u}_t u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m-1}). \end{aligned}$$

It follows that

$$\frac{d}{dt} \widetilde{\varphi \# \text{tr}}((u_t \otimes u_t^{-1})^{\otimes m}) = m \cdot \eta_{b\varphi}(\widetilde{D}, t). \quad (3.54)$$

In particular, we have

$$\begin{aligned} & \int_{t_0}^{t_1} \eta_{b\varphi}(\widetilde{D}, t) dt \\ &= \widetilde{\varphi \# \text{tr}}((u_t \otimes u_t^{-1})^{\otimes m}) \Big|_{t=t_1} - \widetilde{\varphi \# \text{tr}}((u_t \otimes u_t^{-1})^{\otimes m}) \Big|_{t=t_0} \\ &= \varphi \# \text{tr}(((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) \Big|_{t=t_1} - \varphi \# \text{tr}(((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) \Big|_{t=t_0}. \end{aligned}$$

By the proof of Proposition 3.48, we have

$$\lim_{t \rightarrow 0} \varphi \# \text{tr}(((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) = 0.$$

Furthermore, it follows from the proof of Proposition 3.51 that

$$\lim_{t \rightarrow \infty} \varphi \# \text{tr}(((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) = 0,$$

as long as the spectral gap of \widetilde{D} is sufficiently large. This finishes the proof. \square

In the remaining part of this section, we show that the delocalized higher eta invariant is stable under Connes' periodicity map. First, let us recall the definition of Connes' periodicity map; see [Connes 1985, Page 121]:

$$S: HC^{2m}(\mathbb{C}G) \rightarrow HC^{2m+2}(\mathbb{C}G)$$

First, let us fix some notation; see Definition 3.25. For $0 \leq i \leq n$, we define $b_i: C^n(\mathbb{C}G) \rightarrow C^{n+1}(\mathbb{C}G)$ by

$$b_i f(g_0, g_1, \dots, g_{n+1}) = (-1)^i f(g_0, g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}),$$

and for $i = n + 1$, we define $b_{n+1}: C^n(\mathbb{C}G) \rightarrow C^{n+1}(\mathbb{C}G)$ by

$$b_{n+1} f(g_0, g_1, \dots, g_{n+1}) = (-1)^{n+1} f(g_{n+1} g_0, g_1, \dots, g_n).$$

Definition 3.55. Connes' periodicity map $S: C^n(\mathbb{C}G) \rightarrow C^{n+2}(\mathbb{C}G)$ is defined to be

$$S := \frac{-1}{(n+2)(n+1)} \sum_{0 \leq i < j \leq n+2} b_j b_i.$$

We point out that the above explicit formula of S differs by a constant from Connes' formula [1985]; compare with the formula for S in the proof of [loc. cit., Part II, Corollary]. This is because the constants appearing in the Connes–Chern character of the current paper are different from those in Connes' original paper [loc. cit.].

The following proposition shows that the delocalized higher eta invariant is stable under Connes' periodicity map.

Proposition 3.56. *Let $\langle h \rangle$ be a nontrivial conjugacy class of G . If a cyclic cocycle $\varphi \in C^{2m}(\mathbb{C}G, \langle h \rangle)$ has exponential growth and the spectral gap of \tilde{D} at zero is sufficiently large, then $\eta_{S\varphi}(\tilde{D}) = \eta_\varphi(\tilde{D})$.*

Proof. We will prove the proposition by a direct computation. It is much easier to follow Connes [1985, Part II] and carry out the computation in the context of universal graded differential algebras. Let us recall the construction of the universal graded differential algebra $\Omega(\mathcal{A})$ associated to an algebra \mathcal{A} . Denote by $\tilde{\mathcal{A}}$ the algebra obtained from \mathcal{A} by adjoining a unit: $\tilde{\mathcal{A}} = \{a + \lambda I \mid a \in \mathcal{A}, \lambda \in \mathbb{C}\}$. Let $\Omega^0(\mathcal{A}) = \mathcal{A}$ and

$$\Omega^n(\mathcal{A}) = \tilde{\mathcal{A}} \otimes (\mathcal{A})^{\otimes n}$$

for $n \geq 1$. The differential $\tilde{d}: \Omega^n(\mathcal{A}) \rightarrow \Omega^{n+1}(\mathcal{A})$ is given by

$$\tilde{d}((a_0 + \lambda I) \otimes a_1 \otimes \cdots \otimes a_n) = I \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n.$$

Clearly, one has $\tilde{d}^2 = 0$. The product structure; (see [Connes 1985, Part II, Proposition 1]) on $\Omega^*(\mathcal{A})$ is defined so that the following are satisfied:

- (1) $\tilde{d}(\omega_1 \omega_2) = (\tilde{d}\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 \tilde{d}\omega_2$ for $\omega_1 \in \Omega^i$ and $\omega_2 \in \Omega^j$, where $|\omega_1| = i$ is the degree of ω_1 .
- (2) $\tilde{a}_0 \tilde{d}a_1 \tilde{d}a_2 \cdots \tilde{d}a_n = \tilde{a}_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $\Omega^n(\mathcal{A})$.

An $(n+1)$ -linear functional φ on \mathcal{A} induces a linear functional $\widehat{\varphi}$ on $\Omega^n(\mathcal{A})$ by setting

$$\widehat{\varphi}((a_0 + \lambda I) da_1 \cdots da_n) = \varphi(a_0, a_1, \dots, a_n).$$

By [loc. cit., Part II, Proposition 1], since φ is a cyclic cocycle, $\widehat{\varphi}$ is a closed graded trace on $\Omega^n(\mathcal{A})$. In particular, we have

$$\widehat{\varphi}(\omega_1 \omega_2) = (-1)^{|\omega_1| \cdot |\omega_2|} \widehat{\varphi}(\omega_2 \omega_1).$$

By using the equality $\tilde{d}(\omega_1 \omega_2) = (\tilde{d}\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 \tilde{d}\omega_2$ above, a straightforward calculation shows that the formula for the periodicity operator S becomes the following; compare with [loc. cit., Part II, Corollary 10]:

$$\widehat{S\varphi}(a_0 \tilde{d}a_1 \cdots \tilde{d}a_{n+2}) = c_n \sum_{j=1}^{n+1} \widehat{\varphi}(a_0 \tilde{d}a_1 \cdots \tilde{d}a_{j-1} (a_j a_{j+1}) \tilde{d}a_{j+2} \cdots \tilde{d}a_{n+2}), \quad (3.57)$$

where $c_n = 1/((n+2)(n+1))$.

Now we shall prove the proposition. Let $\varphi \in C^{2m}(\mathbb{C}G, \langle h \rangle)$ be a cyclic cocycle with exponential growth, then $S\varphi$ also has exponential growth. Recall that the delocalized higher eta invariant of \tilde{D} with respect to φ to be⁷

$$\eta_\varphi(\tilde{D}) = \frac{m!}{\pi i} \int_0^\infty \varphi \# \text{tr}(\dot{u}_t u_t^{-1} \otimes ((u_t - I) \otimes (u_t^{-1} - I))^{\otimes m}) dt,$$

where $u_t = u_t(\tilde{D})$ as in line (3.29). For notational simplicity, let us write

$$\varphi(\dot{u}_t u_t^{-1} \otimes ((u_t - I) \otimes (u_t^{-1} - I))^{\otimes m})$$

in place of $\varphi \# \text{tr}(\dot{u}_t u_t^{-1} \otimes ((u_t - I) \otimes (u_t^{-1} - I))^{\otimes m})$, and furthermore write

$$a_0 = \dot{u}_t u_t^{-1} \quad \text{and for } j \geq 1, a_j = \begin{cases} u_t - I & \text{if } j \text{ is odd,} \\ u_t^{-1} - I & \text{if } j \text{ is even.} \end{cases}$$

By the above discussion, we have

$$\widehat{\varphi}(a_0 da_1 \cdots da_{2m}) = \varphi(\dot{u}_t u_t^{-1} \otimes ((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}).$$

The following observations will be useful in the computation below:

- (1) $(u_t - I)(u_t^{-1} - I) = 2I - u_t - u_t^{-1}$.
- (2) $\dot{d}(u_t - I) = \dot{d}u_t$ and $\dot{d}(u_t^{-1} - I) = \dot{d}u_t^{-1}$.
- (3) $(\dot{d}u_t)u_t^{-1} = -u_t(\dot{d}u_t^{-1})$ and $(\dot{d}u_t^{-1})u_t = -u_t^{-1}(\dot{d}u_t)$.

Observation (1) immediately implies that

$$\begin{aligned} \widehat{\varphi}(a_0 \dot{d}a_1 \cdots \dot{d}a_{j-1} (a_j a_{j+1}) \dot{d}a_{j+2} \cdots \dot{d}a_{2m+2}) \\ = 2\widehat{\varphi}(a_0 \dot{d}a_1 \cdots \dot{d}a_{j-1} \dot{d}a_{j+2} \cdots \dot{d}a_{2m+2}) \\ - \widehat{\varphi}(a_0 \dot{d}a_1 \cdots \dot{d}a_{j-1} (u_t) \dot{d}a_{j+2} \cdots \dot{d}a_{2m+2}) \\ - \widehat{\varphi}(a_0 \dot{d}a_1 \cdots \dot{d}a_{j-1} (u_t^{-1}) \dot{d}a_{j+2} \cdots \dot{d}a_{2m+2}). \end{aligned}$$

Now observations (2) and (3) imply the following:

- (i) If j is odd, then

$$\begin{aligned} \widehat{\varphi}(a_0 \dot{d}a_1 \cdots \dot{d}a_{j-1} (u_t) \dot{d}a_{j+2} \cdots \dot{d}a_{2m+2}) \\ = \widehat{\varphi}(\underbrace{\dot{u}_t \dot{d}u_t^{-1} \dot{d}u_t \cdots \dot{d}u_t^{-1} \dot{d}u_t}_{(j-1) \text{ terms}} \underbrace{\dot{d}u_t \dot{d}u_t^{-1} \cdots \dot{d}u_t \dot{d}u_t^{-1}}_{(2m-j+1) \text{ terms}}), \end{aligned}$$

⁷Here we use the notation $u_t^{-1} - I$ for the corresponding term $u_t^{-1} - 1$ in Definition 3.30. This is to emphasize that I is the identity operator, which is the unit adjoined.

and

$$\begin{aligned} \widehat{\varphi}(a_0 \dot{a}_1 \cdots \dot{a}_{j-1}(u_t^{-1}) \dot{a}_{j+2} \cdots \dot{a}_{2m+2}) \\ = -\widehat{\varphi}(\underbrace{\dot{u}_t^{-1} \dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(j-1) \text{ terms}} \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(2m-j+1) \text{ terms}}). \end{aligned}$$

(ii) If j is even, then

$$\begin{aligned} \widehat{\varphi}(a_0 \dot{a}_1 \cdots \dot{a}_{j-1}(u_t) \dot{a}_{j+2} \cdots \dot{a}_{2m+2}) \\ = \widehat{\varphi}(\underbrace{\dot{u}_t^{-1} \dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(j-1) \text{ terms}} \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(2m-j+1) \text{ terms}}), \end{aligned}$$

and

$$\begin{aligned} \widehat{\varphi}(a_0 \dot{a}_1 \cdots \dot{a}_{j-1}(u_t^{-1}) \dot{a}_{j+2} \cdots \dot{a}_{2m+2}) \\ = -\widehat{\varphi}(\underbrace{\dot{u}_t \dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(j-1) \text{ terms}} \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{(2m-j+1) \text{ terms}}). \end{aligned}$$

Since $\widehat{\varphi}$ is a closed graded trace, it follows that

$$\sum_{j=1}^{2m+1} \widehat{\varphi}(a_0 \dot{a}_1 \cdots \dot{a}_{j-1}(u_t) \dot{a}_{j+2} \cdots \dot{a}_{2m+2}) = \frac{d}{dt} \widehat{\varphi}((u_t - 1) \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{2m \text{ terms}})$$

and

$$\begin{aligned} \sum_{j=1}^{2m+1} \widehat{\varphi}(a_0 \dot{a}_1 \cdots \dot{a}_{j-1}(u_t^{-1}) \dot{a}_{j+2} \cdots \dot{a}_{2m+2}) \\ = \frac{d}{dt} \widehat{\varphi}((u_t^{-1} - 1) \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{2m \text{ terms}}). \end{aligned}$$

On the other hand, by the proof of Propositions 3.48 and 3.51, we have

$$\begin{aligned} \int_0^\infty \frac{d}{dt} \widehat{\varphi}((u_t - 1) \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{2m \text{ terms}}) dt \\ = \lim_{t \rightarrow \infty} \varphi((u_t - 1) \otimes ((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) \\ \quad - \lim_{t \rightarrow 0} \varphi((u_t - 1) \otimes ((u_t - 1) \otimes (u_t^{-1} - 1))^{\otimes m}) \\ = 0 \end{aligned}$$

Similarly, we have

$$\int_0^\infty \frac{d}{dt} \widehat{\varphi}((u_t^{-1} - 1) \underbrace{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{j-1} \dot{a}_{j+2} \cdots \dot{a}_{2m+2}}_{2m \text{ terms}}) dt = 0.$$

To summarize, we have

$$\begin{aligned} \int_0^\infty \widehat{S\varphi}(\dot{u}_t u_t^{-1} \underbrace{\dot{u}_t \dot{u}_t^{-1} \cdots \dot{u}_t \dot{u}_t^{-1}}_{(2m+2) \text{ terms}}) dt \\ = \frac{1}{m+1} \int_0^\infty \widehat{\varphi}(\dot{u}_t u_t^{-1} \underbrace{\dot{u}_t \dot{u}_t^{-1} \cdots \dot{u}_t \dot{u}_t^{-1}}_{2m \text{ terms}}) dt, \end{aligned}$$

which implies that

$$\eta_{S\varphi}(\widetilde{D}) = \eta_\varphi(\widetilde{D}).$$

This finishes the proof. \square

3.3. A higher Atiyah–Patodi–Singer index formula. In this subsection, for each cyclic cocycle of at most exponential growth, we prove a formal higher Atiyah–Patodi–Singer index theorem (abbr. higher APS index theorem) on manifolds with boundary, under the condition that the operator on the boundary has a sufficiently large spectral gap at zero. We point out that there is no condition on the fundamental group in this formal higher APS index theorem.

Leichtnam and Piazza [1999, Theorem 4.1] proved a higher APS index theorem in terms of noncommutative differential forms on a certain smooth dense subalgebra of the reduced C^* -algebra of the fundamental group. Heuristically speaking, our version of higher APS theorem is the pairing between their version of higher APS theorem and cyclic cocycles of the fundamental group. However, this is not the approach we take in this section. In fact, in general it is rather difficult to make this heuristic argument rigorous. A main difficulty here is whether cyclic cocycles of a group algebra extends continuously to cyclic cocycles on a given smooth dense subalgebra of the reduced group C^* -algebra. In this section, our approach is based on the convergence results from the previous sections, and avoids the subtle issue of continuous extension of cyclic cocycles. On the other hand, in order to apply the higher APS index theorem in this subsection to problems in geometry and topology (see [Piazza and Schick 2014; Xie and Yu 2017; Weinberger et al. 2021]), one actually needs to extend the pairing to be defined at the level of (periodic) cyclic cohomology and K -theory of C^* -algebras. In later sections, we shall use Puschnigg’s smooth dense subalgebra to define such a pairing at the level of (periodic) cyclic cohomology and K -theory of C^* -algebras for all hyperbolic groups. As a consequence, in the case of hyperbolic groups, we shall prove a higher APS index theorem without the assumption that the operator on the boundary has a sufficiently large spectral gap at zero; see Section 7.

For notational simplicity, we shall only discuss the case of even dimensional spin manifolds. The exact same strategy clearly works for the more general case of

Dirac-type operators acting on Clifford modules over Riemannian manifolds of all dimensions.

Let W be an even dimensional compact spin manifold with boundary M , and D the Dirac operator on W . Suppose the metric of W is a product metric when restricted to the boundary M . Let G be a finitely presented discrete group and \tilde{W} a regular G -covering space of W . Let \tilde{D} be the lift of D to \tilde{W} and \tilde{D}_∂ the restriction \tilde{D} to the boundary of \tilde{W} .

Let us briefly review Lott's noncommutative differential higher eta invariant. We shall follow closely the notation in Lott's paper [1992]. For each $q > 0$, we define \mathfrak{B}_q^ω to be the following dense subalgebra of $C_r^*(G)$:

$$\mathfrak{B}_q^\omega = \left\{ f : G \rightarrow \mathbb{C} \mid \sum_{g \in G} e^{q \cdot \ell(g)} |f(g)| < \infty \right\},$$

where ℓ is a word-length function on G . Note that \mathfrak{B}_q^ω is generally *not* closed under holomorphic functional calculus in $C_r^*(G)$. The universal graded differential algebra of \mathfrak{B}_q^ω is

$$\Omega_*(\mathfrak{B}_q^\omega) = \bigoplus_{k=0}^{\infty} \Omega_k(\mathfrak{B}_q^\omega)$$

where as a vector space, $\Omega_k(\mathfrak{B}_q^\omega) = \mathfrak{B}_q^\omega \otimes (\mathfrak{B}_q^\omega / \mathbb{C})^{\otimes k}$. As \mathfrak{B}_q^ω is a Banach algebra (see Lemma 3.50 above), we consider the Banach completion of $\Omega_*(\mathfrak{B}_q^\omega)$, which will still be denoted by $\Omega_*(\mathfrak{B}_q^\omega)$.

Let S be the restriction of spinor bundle of W on M . We denote the corresponding \mathfrak{B}_q^ω -vector bundle by $\mathfrak{S} = (\tilde{M} \times_G \mathfrak{B}_q^\omega) \otimes S$ and the space of smooth sections by $C^\infty(M; \mathfrak{S})$. Now suppose ψ is a smooth function on \tilde{M} with compact support such that

$$\sum_{g \in G} g^* \psi = 1.$$

Then we have a superconnection $\nabla : C^\infty(M; \mathfrak{S}) \rightarrow C^\infty(M; \mathfrak{S} \otimes_{\mathfrak{B}_q^\omega} \Omega_1(\mathfrak{B}_q^\omega))$ given by

$$\nabla(f) = \sum_{g \in G} (\psi \cdot g^* f) \otimes_{\mathfrak{B}_q^\omega} dg.$$

See [Quillen 1985] for more details of the superconnection formalism.

Definition 3.58 [Lott 1992, Sections 4.4 and 4.6]. Lott's higher eta invariant $\tilde{\eta}(\tilde{D}_\partial)$ is defined by the formula

$$\tilde{\eta}(\tilde{D}_\partial) = \int_0^\infty \text{STR}(\tilde{D} e^{-(t\tilde{D}_\partial + \nabla)^2}) dt,$$

where STR is the corresponding supertrace; see [Lott 1992, Proposition 22].

Remark 3.59. The above integral formula for $\tilde{\eta}(\tilde{D}_\partial)$ generally does not converge to define an element in $\Omega_*(\mathfrak{B}_q^\omega)$. On the other hand, the estimates in the previous subsections show that the above integral converges absolutely and defines an element in $\Omega_*(\mathfrak{B}_q^\omega)$, provided the spectral gap of \tilde{D}_∂ at zero is sufficiently large.

Now let W_∞ be the complete Riemannian manifold obtained by attaching an infinite cylinder $M \times [0, \infty)$ to W . We still denote the associated Dirac operator on W_∞ by D and its lift to \tilde{W}_∞ by \tilde{D} . Combining Quillen's superconnection formalism with Melrose's b -calculus formalism [1993], for each $t > 0$, one defines the b -Connes–Chern character of \tilde{D} on \tilde{W}_∞ to be⁸

$$b\text{-Ch}_t(\tilde{D}) = b\text{-STR}(e^{-(t\tilde{D}+\nabla)^2}) \in \Omega_*(\mathfrak{B}_q^\omega),$$

where $b\text{-STR}$ is the corresponding b -supertrace in this b -calculus setting. See for example [Leichtnam and Piazza 1997] for more details.

Theorem 3.60. Assume that $\varphi \in C^{2m}(\mathbb{C}G)$ is a cyclic cocycle with exponential growth. Let σ_φ be the positive number from Definition 3.42. If the spectral gap of \tilde{D}_∂ at zero is larger than σ_φ , then the pairing $\langle \varphi, \tilde{\eta}(\tilde{D}_\partial) \rangle$ converges, the limit $\lim_{t \rightarrow \infty} \langle \varphi, b\text{-Ch}_t(\tilde{D}) \rangle$ exists, and furthermore

$$\lim_{t \rightarrow \infty} \langle \varphi, b\text{-Ch}_t(\tilde{D}) \rangle = \left\langle \varphi, \int_W \hat{A} \wedge \omega \right\rangle - \frac{1}{2} \langle \varphi, \tilde{\eta}(\tilde{D}_\partial) \rangle, \quad (3.61)$$

where \hat{A} is the associated \hat{A} -form on W and ω is an element in $\Omega^*(W) \otimes \Omega_*(\mathfrak{B}_q^\omega)$ for some $q > 0$; see [Leichtnam and Piazza 1997, Theorem 13.6]. In particular, if both G and φ have subexponential growth, then the equality in line (3.61) holds as long as \tilde{D}_∂ is invertible.

Proof. By Propositions 3.46 and 3.48, we have that, for any $\varepsilon > 0$, $\mu > 1$ and $k \in \mathbb{N}$, there exist C, N_1, N_2 such that

$$\|(\tilde{D}^k e^{-t^2 \tilde{D}^2})(x, y)\| \leq C \frac{d(x, y)^{N_1}}{t^{N_2}} \exp\left(-\frac{d(x, y)^2}{4\mu^2 c_D^2 t^2}\right), \quad (3.62)$$

for all $x, y \in \tilde{W}_\infty$ with $d(x, y) > \varepsilon$, and

$$\|(\tilde{D}_\partial^k e^{-t^2 \tilde{D}_\partial^2})(x, y)\| \leq C \frac{d(x, y)^{N_1}}{t^{N_2}} \exp\left(-\frac{d(x, y)^2}{4\mu^2 c_{D_\partial}^2 t^2}\right). \quad (3.63)$$

for all $x, y \in \tilde{M}$ with $d(x, y) > \varepsilon$, where c_D (resp. c_{D_∂}) is the propagation speed of D (resp. D_∂); see line (3.6).

⁸It is not difficult to adapt the estimates from the previous subsections to the b -calculus setting and show that $b\text{-tr}_s(e^{-(t\tilde{D}+\nabla)^2})$ indeed defines an element in $\Omega_*(\mathfrak{B}_q^\omega)$.

Since the spectral gap of \tilde{D}_∂ is larger than $\sigma_\varphi = 2(K_G + K_\varphi) \cdot c_{D_\partial}/\tau$, the proof of Proposition 3.23 shows that for each $k \in \mathbb{N}$, the operator $\tilde{D}_\partial^k e^{-t^2 \tilde{D}_\partial^2}$ lies in \mathcal{L}_{K_φ} (see Definition 3.49) and furthermore there exists sufficiently small $\omega > 0$ such that

$$\|\tilde{D}_\partial^k e^{-t^2 \tilde{D}_\partial^2}\|_{\mathcal{L}} \leq 2\sqrt{\pi} e^{-\omega t^2}.$$

Now we apply the commutator formula for b -trace (see [Melrose 1993, (In.22) on Page 8]), and a straightforward calculation shows that for any $0 < t_0 < t_1$, the equality

$$\begin{aligned} b\text{-Ch}_{t_1}(\tilde{D}) - b\text{-Ch}_{t_0}(\tilde{D}) \\ = -\frac{1}{2} \int_{t_0}^{t_1} \text{STR}(\tilde{D}_\partial e^{-(s\tilde{D}_\partial + \nabla)^2}) ds + d \int_{t_0}^{t_1} b\text{-STR}(\tilde{D} e^{-(\nabla + s\tilde{D})^2}) ds \end{aligned} \quad (3.64)$$

holds in $\Omega_*(\mathfrak{B}_q^\omega)$ with $q = K_\varphi$, where $d: \Omega_*(\mathfrak{B}_q^\omega) \rightarrow \Omega_{*+1}(\mathfrak{B}_q^\omega)$ is the differential on $\Omega_*(\mathfrak{B}_q^\omega)$; see [Getzler 1993, Section 6]. In particular, by pairing both sides of (3.64) with φ , we have

$$\langle \varphi, b\text{-Ch}_{t_1}(\tilde{D}) \rangle - \langle \varphi, b\text{-Ch}_{t_0}(\tilde{D}) \rangle = -\frac{1}{2} \int_{t_0}^{t_1} \langle \varphi, \text{STR}(\tilde{D}_\partial e^{-(s\tilde{D}_\partial + \nabla)^2}) \rangle ds.$$

Let us write $\varphi = \varphi_e + \varphi_d$, where φ_d is the delocalized part of φ , i.e.,

$$\varphi_d(g_0, g_1, \dots, g_{2m}) = \begin{cases} \varphi(g_0, g_1, \dots, g_{2m}) & \text{if } g_0 g_1 \cdots g_{2m} \neq e, \\ 0 & \text{otherwise.} \end{cases}$$

A similar argument as in the proof of Proposition 3.46, combined with Getzler's symbol calculus [1986], shows that

$$\lim_{t \rightarrow 0} \langle \varphi_e, b\text{-Ch}_t(\tilde{D}) \rangle = \int_W \hat{A} \wedge \omega,$$

where ω is an element in $\Omega^*(W) \otimes \Omega_*(\mathfrak{B}_q^\omega)$; see [Leichtnam and Piazza 1997, Theorem 13.6]. Moreover, it follows from the inequality in line (3.62) that

$$\lim_{t \rightarrow 0} \langle \varphi_d, b\text{-Ch}_t(\tilde{D}) \rangle = 0.$$

Therefore, as $t_0 \rightarrow 0$, we obtain the following formula:

$$\langle \varphi, b\text{-Ch}_{t_1}(\tilde{D}) \rangle - \left\langle \varphi, \int_W \hat{A} \wedge \omega \right\rangle = -\frac{1}{2} \int_0^{t_1} \langle \varphi, \text{STR}(\tilde{D}_\partial e^{-(s\tilde{D}_\partial + \nabla)^2}) \rangle ds.$$

Now it follows from the discussion above that the integral on the right hand side converges absolutely as $t_1 \rightarrow \infty$, under the condition that the spectral gap of \tilde{D}_∂ is larger than σ_φ . This finishes the proof. \square

Remark 3.65. Formally speaking, the term $\lim_{t \rightarrow \infty} \langle \varphi, b\text{-Ch}_t(\tilde{D}) \rangle$ represents the pairing between the higher index class of \tilde{D} and the cyclic cocycle φ .⁹ However, to make this formal assertion rigorous, one needs to extend the pairing in (3.61) from \mathfrak{B}_q^ω to a smooth dense subalgebra of $C_r^*(G)$, which is a rather subtle issue in general. In the remaining sections below, we will show the existence of such an extension of the pairing, in the special case where G is hyperbolic.

4. Puschnigg smooth dense subalgebra for hyperbolic groups

In this section, we review the construction of Puschnigg's [2010] smooth dense algebra of $C_r^*(G)$ for hyperbolic groups. One particular feature is that every trace on $\mathbb{C}G$ admits a continuous extension to this Puschnigg smooth dense subalgebra; see [Puschnigg 2010, Theorem 5.2]. We shall generalize this extension result to cyclic cocycles of all degrees.

4.1. Unconditional seminorms and tensor products. In this subsection, we review the construction of Puschnigg's [2010] smooth dense subalgebra of $C_r^*(G)$ for hyperbolic groups G .

Let X be a set and R a normed algebra equipped with a submultiplicative norm $|\cdot|$. We denote by RX the algebra consisting of all finitely supported functions on X with values in R . For each element $A = \sum A_x x \in RX$, we define its absolute value to be

$$|A| = \sum |A_x| x \in \mathbb{C}X.$$

Define a partial order on elements in RX by

$$A \leq A' \iff |A_x| \leq |A'_x| \quad \text{for } \forall x \in X.$$

Recall the following notion of unconditional seminorm due to Bost and Lafforgue; see [Lafforgue 2002].

Definition 4.1. A seminorm $\|\cdot\|$ on RX is called unconditional if

$$|A| \leq |A'| \implies \|A\| \leq \|A'\|, \quad \forall A, A' \in RX.$$

Any seminorm $\|\cdot\|$ on RX naturally determines an unconditional seminorm $\|\cdot\|^+$ by

$$\|A\|^+ := \inf_{|A'| \geq |A|} \|A'\|. \quad (4.2)$$

⁹In general, there is no natural way to define the higher index class of a Dirac operator on a manifold with boundary. However, in our setup above, due to the invertibility of the operator \tilde{D}_∂ on the boundary, there is a natural higher index class associated to \tilde{D} .

Lemma 4.3 [Puschnigg 2010, Lemma 2.3]. *Let X, Y be two sets and $\|\cdot\|_X, \|\cdot\|_Y$ be seminorms on RX and RY respectively. Let $\varphi : (RX, \|\cdot\|_X) \rightarrow (RY, \|\cdot\|_Y)$ be a bounded linear map. Assume that φ is expressed by a positive integral kernel, that is*

$$\varphi\left(\sum A_y y\right)(x) = \sum_{y \in Y} \varphi_{x,y} A_y y, \quad (4.4)$$

where $\varphi_{x,y} \in \mathbb{R}_{\geq 0}$, for $\forall x \in X, \forall y \in Y$. Then φ is also bounded with respect to the corresponding unconditional seminorms $\|\cdot\|_X^+, \|\cdot\|_Y^+$, and

$$\|\varphi\|^+ \leq \|\varphi\|. \quad (4.5)$$

Now we recall the notion of unconditional tensor product seminorm.

Definition 4.6. Let X, Y be sets and $\|\cdot\|_X, \|\cdot\|_Y$ be unconditional seminorms on RX and RY respectively. Let $R \otimes R'$ be the algebraic tensor product of R and R' equipped with the projective seminorm. The unconditional tensor product seminorm $\|\cdot\|_{uc}$ on $RX \otimes R'Y \cong (R \otimes R')(X \times Y)$ is defined to be the unconditional norm determined by this projective seminorm. More precisely, $\|\cdot\|_{uc}$ is given by

$$\|A\|_{uc} := \inf_{|A| \leq \sum |A'_i| \otimes |A''_i|} \sum_i \|A'_i\|_X \|A''_i\|_Y, \quad \forall A \in RX \otimes R'Y.$$

This norm is less than or equal to projective seminorm over $RX \otimes R'Y$. An example where these two are not equal is given in [Puschnigg 2010, Example 2.4].

Given a finitely generated group G , we fix a symmetric generating set S of G . Let ℓ be the corresponding word metric on G . In the following, let \mathcal{S} be the collection of all trace class operators equipped with the trace norm $|\cdot|_1$. Let \mathcal{SG} be the subalgebra of $\mathcal{K} \otimes C_r^*(G)$ consisting of all finite sums $\sum A_g g$ with $A_g \in \mathcal{S}$.

Definition 4.7. For any fixed $p \geq 1$, we define an unconditional norm $\|\cdot\|_{RD,p}$ on \mathcal{SG} by

$$\|A\|_{RD,p}^2 = \sum_g |A_g|_1^2 (1 + \ell(g))^{2p}, \quad (4.8)$$

for $A = \sum_g A_g g \in \mathcal{SG}$.

We denote the completion of \mathcal{SG} with respect to $\|\cdot\|_{RD,p}$ by $RD_p(\mathcal{SG})$. Similarly, the same formula also defines an unconditional norm $\|\cdot\|_{RD,p}$ on $\mathbb{C}G$. We denote the completion of $\mathbb{C}G$ under this norm by $RD_p(G)$. In the following, if no confusion is likely to arise, we shall omit p from the notation.

Let us assume G is hyperbolic for the rest of this section. In this case, it is known that $RD(G)$ is a smooth dense subalgebra of $C_r^*(G)$; see [Jolissaint 1990; de la Harpe 1988; Lafforgue 2000]. Similarly, $RD(\mathcal{SG})$ is a smooth dense algebra of $\mathcal{K} \otimes C_r^*(G)$.

Recall the following quasiderivation map defined by Puschnigg:

$$\Delta : SG \rightarrow SG \otimes \mathbb{C}G \cong \mathcal{S}(G \times G), \quad A_g g \mapsto \sum_{\substack{g_1 g_2 = g \\ \ell(g_1) + \ell(g_2) = \ell(g)}} A_g g_1 \otimes g_2.$$

Definition 4.9. Let $\|\cdot\|_{B,p}$ be the norm on SG given by

$$\|A\|_{B,p} := \|A\|_{RD,p} + \|\Delta A\|_{uc}, \quad \forall A \in SG. \quad (4.10)$$

Here $\|\cdot\|_{uc}$ is the unconditional tensor product norm on $SG \otimes \mathbb{C}G \cong \mathcal{S}(G \times G)$ determined by the unconditional norm $\|\cdot\|_{RD,p}$ on both SG and $\mathbb{C}G$.

Let $B_p(SG)$ be the completion of SG with respect to $\|\cdot\|_{B,p}$. Apply the same construction to $\mathbb{C}G$ and we obtain $B_p(\mathbb{C}G)$. If no confusion is likely to arise, we shall omit p from the notation.

We define a more flexible quasiderivation as follows.

Definition 4.11. For any $g \in G$ and $q \geq 0$, let $C(q, g)$ be the collection of all pairs $(g_1, g_2) \in G \times G$ satisfying the following conditions:

- (1) $g_1 g_2 = g$.
- (2) There exists a geodesic $[e, g]$ connecting the identity e and g in the Cayley graph of G such that g_1 lies in the q -neighborhood of $[e, g]$.

We define

$$\Delta_q : \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G \cong \mathbb{C}(G \times G), \quad g \mapsto \sum_{(g_1, g_2) \in C(q, g)} g_1 \otimes g_2.$$

If $q = 0$, then Δ_0 agrees with Δ . By definition, for any $(g_1, g_2) \in C(q, g)$, there exists a group element $v \in G$ with $\ell(v) \leq q$ such that $\ell(g_1 v^{-1}) + \ell(v g_2) = \ell(g)$. For each $v \in G$, define a map

$$s_v : \mathbb{C}G \otimes \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G, \quad g_1 \otimes g_2 \mapsto g_1 v \otimes v^{-1} g_2.$$

Then we have for $A \in \mathbb{C}G$

$$\Delta_q |A| \leq \sum_{\ell(v) \leq q} s_v \Delta |A|.$$

By Lemma 4.3, the operator norm of s_v (with respect to the unconditional norm $\|\cdot\|_{uc}$) does not exceed $(1 + \ell(v))^2$. Since the number of elements in G of length $\leq q$ is finite, we see that there exists a constant K_q such that $\|(\Delta_q |A|)\|_{uc} \leq K_q \cdot \|(\Delta |A|)\|_{uc}$ for all $A \in \mathbb{C}G$.

Proposition 4.12 [Puschnigg 2010, Proposition 3.5]. *If G is a hyperbolic group whose Cayley graph is δ -hyperbolic, then there exists $C > 0$ such that*

$$\|\Delta(AA')\|_{uc} \leq C(\|\Delta(A)\|_{uc} \|A'\|_{RD} + \|A\|_{RD} \|\Delta(A')\|_{uc}),$$

for $A, A' \in SG$.

Proof. By the discussion above, it suffices to prove the following pointwise inequality

$$\Delta|AA'| \leq \Delta_\delta(|A|)(1 \otimes |A'|) + (|A| \otimes 1)\Delta_\delta(|A'|), \quad (4.13)$$

for $A, A' \in SG$. Without loss of generality, it suffices to consider the case where $A = g$ and $A' = g'$, for $g, g' \in G$.

Let $k = gg'$. If a term $k_1 \otimes k_2$ appears in the summation expression of $\Delta|AA'|$, then k_1 is a point on the geodesic $[e, k]$. Since the Cayley graph of G is δ -hyperbolic, k_1 lies in the δ -neighborhood of the union of $[e, g]$ and $[g, k]$. Either there is a group element $g_1 \in [e, g]$ such that $\text{dist}(k_1, g_1) < \delta$, or there is a group element $g_2 \in [g, k]$ such that $\text{dist}(k_1, g_2) < \delta$. We prove the former case; the latter case is similar. In the former case, we see that the term $k_1 \otimes k_1^{-1}g$ appears in the summation expression of $\Delta_\delta(g)$. This implies that the term $k_1 \otimes k_2 = (k_1 \otimes k_1^{-1}g)(1 \otimes g')$ appears in $\Delta_\delta(|A|)(1 \otimes |A'|)$. This finishes the proof. \square

Remark 4.14. The above proof in fact shows that the pointwise inequality in line (4.13) is equivalent to the hyperbolicity of the group.

Proposition 4.15. $B(SG)$ and $B(\mathbb{C}G)$ are smooth dense algebras of $\mathcal{K} \otimes C_r^*(G)$ and $C_r^*(G)$ respectively.

Proof. We prove the case of $B(\mathbb{C}G)$; the other case is similar.

Since $RD(G)$ is a smooth dense subalgebra of $C_r^*(G)$, it suffices to show that if an element $T \in B(\mathbb{C}G)^+$ is invertible in $RD(G)^+$, then T is invertible in $B(\mathbb{C}G)^+$. In fact, it suffices to show that there exists a constant $\varepsilon > 0$ such that if an element $T \in B(\mathbb{C}G)^+$ satisfies $\|T - 1\|_{RD} < \varepsilon$, then T is invertible in $B(\mathbb{C}G)^+$. Indeed, let S be an element in $B(\mathbb{C}G)^+$ such that S is invertible in $RD(G)^+$ with inverse R . Since $B(\mathbb{C}G)^+$ is dense in $RD(G)^+$, there exists an invertible element U of $RD(G)^+$ such that $\|U - R\|_{RD} < \varepsilon \cdot \|S\|_{RD}$. It follows that SU is invertible in $RD(G)^+$ and $\|SU - 1\|_{RD} < \varepsilon$. Then by our assumption, SU is invertible in $B(\mathbb{C}G)^+$, which implies S is invertible in $B(\mathbb{C}G)^+$.

Now suppose $A \in B(\mathbb{C}G)^+$ such that $\|A - 1\|_{RD} < \min\{1/C, 1\}$, where C is the same constant as in Proposition 4.12. It follows from Proposition 4.12 that

$$\|\Delta((A - 1)^n)\|_{uc} \leq n(C\|A - 1\|_{RD})^{n-1}\|\Delta(A - 1)\|_{uc}.$$

This immediately implies that $A^{-1} = (1 - (1 - A))^{-1} = \sum_{n=0}^{\infty} (1 - A)^n$ lies in $B(\mathbb{C}G)^+$. Therefore $B(\mathbb{C}G)$ is a smooth dense subalgebra of $RD(G)$. \square

4.2. Continuous extension of traces. In this subsection, we review Puschnigg's result [2010, Theorem 5.2] on continuous extension of traces from $\mathbb{C}G$ to $B(\mathbb{C}G)$ for hyperbolic groups. In fact, for the purposes of this paper, we only need a weaker version of Puschnigg's theorem, to which we give a slightly different proof.

Lemma 4.16 [Puschnigg 2010, Lemma 4.1]. *Let G be a group whose Cayley graph is δ -hyperbolic. Given $h \in G$, if g lies in the conjugacy class $\langle h \rangle$, then there exist $g_1, g_2 \in G$ such that $g_1 g_2 = g$, $\ell(g_1) + \ell(g_2) = \ell(g)$ and $\ell(g_2 g_1) \leq 6\delta + 6 + 3\ell(h)$.*

Proof. Suppose $h = ugu^{-1}$ for some $u \in G$. In the following, we denote by $[a, b]$ a geodesics connecting $a, b \in G$. By hyperbolicity, there exist vertices $w \in [e, ug]$, $u_1 \in [e, u]$ and $ug_1 \in [u, ug]$ such that $\text{dist}(w, u_1) < \delta + 1$ and $d(w, ug_1) < \delta + 1$. Moreover, there exists $v_1 \in G$ such that $ugv_1 \in [ug, h]$ and $\text{dist}(w, ugv_1) < \delta + 1 + \ell(h)$, since the $[e, ug]$ lies entirely in the $\delta + \ell(h)$ neighborhood of $[ug, h]$. Let u_2, g_2, v_2 be elements in G such that $u = u_1 u_2$, $g = g_1 g_2$ and $u^{-1} = v_1 v_2$. Let us write

$$g_2 g_1 = (g_2 v_1)(v_1^{-1} u_2^{-1})(u_2 g_1).$$

Clearly, we have that

$$\begin{aligned} \ell(g_2 v_1) &= \text{dist}(ug_1, ug_1 g_2 v_1) = \text{dist}(ug_1, ugv_1) < 2\delta + 2 + \ell(h) \quad \text{and} \\ \ell(u_2 g_1) &= \text{dist}(u_1, u_1 u_2 g_1) = \text{dist}(u_1, ug_1) < 2\delta + 2. \end{aligned}$$

Furthermore, observe that v_2^{-1} is a vertex on the geodesic $[e, u]$. It follows that

$$\ell(v_1^{-1} u_2^{-1}) = \ell(v_2 u_1) = \text{dist}(v_2^{-1}, u_1) = |\ell(v_2^{-1}) - \ell(u_1)|.$$

Therefore, we have

$$\begin{aligned} \ell(v_1^{-1} u_2^{-1}) &= |\ell(v_2^{-1}) - \ell(u_1)| \\ &= |\text{dist}(h, hv_2^{-1}) - \text{dist}(e, u_1)| \\ &\leq |\text{dist}(e, h) + \text{dist}(e, hv_2^{-1}) - \text{dist}(e, u_1)| \\ &\leq |\text{dist}(e, h)| + |\text{dist}(u_1, hv_2^{-1})| \\ &< 2\delta + 2 + 2\ell(h). \end{aligned}$$

This finishes the proof. □

Theorem 4.17 [Puschnigg 2010, Theorem 5.2]. *Let G be a hyperbolic group and $B(\mathbb{C}G)$ the Puschnigg smooth dense subalgebra of $C_r^*(G)$. For any conjugacy class $\langle h \rangle$ of G , the map*

$$\text{tr}_{\langle h \rangle} : \mathbb{C}G \rightarrow \mathbb{C}, \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in \langle h \rangle} a_g,$$

admits a continuous extension to $B(\mathbb{C}G)$.

Proof. Define a map $\mu : \mathbb{C}G \rightarrow \mathbb{C}(G \times G)$ as follows: if $g \in \langle h \rangle$, then

$$\mu(g) := g_1 \otimes g_2$$

where $(g_1, g_2) \in G \times G$ is a pair of elements as given in Lemma 4.16; if $g \notin \langle h \rangle$, define $\mu(g) = 0$. Clearly, $\mu(|A|) \leq \Delta|A|$. Thus μ admits a continuous extension from $B(\mathbb{C}G)$ to $RD(G) \otimes_{uc} RD(G)$, which we will still denote by μ .

By Lemma 4.3, the maps

$$\begin{aligned} T: RD(G) \otimes_{uc} RD(G) &\rightarrow RD(G) \otimes_{uc} RD(G), & g_1 \otimes g_2 &\mapsto g_2 \otimes g_1, \\ M: RD(G) \otimes_{uc} RD(G) &\rightarrow RD(G), & g_1 \otimes g_2 &\mapsto g_1 g_2, \end{aligned}$$

are continuous.

We define an evaluation map $E: RD(G) \rightarrow \mathbb{C}$ as follows:

$$E(g) = \begin{cases} 1 & \text{if } \ell(g) \leq 6\delta + 6 + 3\ell(h), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, E is also well-defined and continuous. It follows that the composition

$$B(\mathbb{C}G) \xrightarrow{\mu} RD(G) \otimes_{uc} RD(G) \xrightarrow{T} RD(G) \otimes_{uc} RD(G) \xrightarrow{M} RD(G) \xrightarrow{E} \mathbb{C}$$

is a continuous extension of $\text{tr}_{\langle h \rangle}$. This finishes the proof. \square

The previous theorem has the following obvious analogue where the coefficient \mathbb{C} is replaced by the algebra of trace class operators \mathcal{S} .

Proposition 4.18. *Let $B(SG)$ be the smooth dense subalgebra of $\mathcal{K} \otimes C_r^*(G)$ defined above. For any conjugacy class $\langle h \rangle$ in G , let $\text{tr}_{\langle h \rangle}: SG \rightarrow \mathbb{C}$ be the trace map defined by*

$$\text{tr}_{\langle h \rangle}(A) = \sum_{g \in \langle h \rangle} \text{tr}(A_g), \quad \text{for } A = \sum A_g g \in SG.$$

Then $\text{tr}_{\langle h \rangle}$ extends to a continuous trace map on $B(SG)$.

4.3. Continuous extension of higher degree cyclic cochains. In this subsection, we generalize the continuous extension result for traces to higher degree cyclic cochains.

Definition 4.19. Fix a length function ℓ on G . For any $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$, we say φ has polynomial growth if there exist constants C and k such that

$$|\varphi(g_0, g_1, \dots, g_n)| \leq C \prod_{i=0}^n (1 + \ell(g_i))^k.$$

In Section 5 below, we will show that, when G is hyperbolic, every element in $HC^n(\mathbb{C}G, \langle h \rangle)$ have a representative with polynomial growth.

Denote by $(SG)^{\otimes n+1}$ the algebraic tensor product of $(n+1)$ copies of SG . Recall the unconditional tensor product defined in Definition 4.6. We construct the unconditional tensor product norms $\|\cdot\|_{RD}$ and $\|\cdot\|_B$ over $(SG)^{n+1}$, and denote their completions by $RD(SG)^{\otimes n+1}$ and $B(SG)^{\otimes n+1}$ respectively.

Proposition 4.20. *Let G be a hyperbolic group whose Cayley graph is δ -hyperbolic. If $\varphi \in C^n(\mathbb{C}G, \langle h \rangle)$ has polynomial growth, then the map $\varphi \# \text{tr}: (SG)^{\otimes n+1} \rightarrow \mathbb{C}$ given by*

$$A_{g_0}g_0 \otimes A_{g_1}g_1 \otimes \cdots \otimes A_{g_n}g_n \mapsto \text{tr}(A_{g_0}A_{g_1} \cdots A_{g_n})\varphi(g_0, g_1, \dots, g_n) \quad (4.21)$$

extends continuously to $B(SG)^{\otimes n+1}$.

Remark 4.22. Before we prove the proposition, let us point out that the construction of $B_p(SG)$ involves a choice of some sufficiently large p . In order to extend $\varphi \# \text{tr}$ continuously to $B_p(SG)^{\otimes n+1}$, we assume that p is sufficiently large so that it “dominates” the growth rate of φ . Hence, strictly speaking, the algebra $B(SG)^{\otimes n+1}$ may vary for different cyclic cochains.

Proof. Suppose that

$$|\varphi(g_0, g_1, \dots, g_n)| \leq C \prod_{i=0}^n (1 + \ell(g_i))^k$$

for $(g_0, g_1, \dots, g_n) \in G^{n+1}$.

Define the following maps:

(1) $\pi_\varphi: (SG)^{\otimes n+1} \rightarrow (SG)^{\otimes n+1}$ by

$$A_{g_0}g_0 \otimes A_{g_1}g_1 \otimes \cdots \otimes A_{g_n}g_n \mapsto \varphi(g_0, g_1, \dots, g_n)A_{g_0}g_0 \otimes A_{g_1}g_1 \otimes \cdots \otimes A_{g_n}g_n.$$

(2) $M: (SG)^{\otimes n+1} \rightarrow SG$ by

$$A_{g_0}g_0 \otimes A_{g_1}g_1 \otimes \cdots \otimes A_{g_n}g_n \mapsto A_{g_0}A_{g_1} \cdots A_{g_n}g_0g_1 \cdots g_n.$$

Clearly, the composition

$$(SG)^{\otimes n+1} \xrightarrow{\pi_\varphi} (SG)^{\otimes n+1} \xrightarrow{M} SG \xrightarrow{\text{tr}_{(h)}} \mathbb{C}$$

is exactly the map $\varphi \# \text{tr}$. Therefore, it suffices to show that $M \circ \pi_\varphi$ extends to a continuous map $B_p(SG)^{\otimes n+1}$ to $B_{p-k}(SG)$. It follows from Lemma 4.3 that the map $M \circ \pi_\varphi$ extends to a continuous map from $RD_p(SG)^{\otimes n+1}$ to $RD_{p-k}(SG)$.

In the following, let us prove the case where $k = 0$, that is, φ is uniformly bounded over G^{n+1} . The general case is similar. If φ is uniformly bounded, then for any $A \in (SG)^{\otimes n+1}$, we observe that

$$|\Delta(M \circ \pi_\varphi A)| = \Delta(|M \circ \pi_\varphi A|) \leq \Delta(M|\pi_\varphi A|) \leq C \cdot \Delta(M|A|).$$

Define another multiplication map $M': \mathbb{C}G^{2(n+1)} \rightarrow \mathbb{C}G^2$ by

$$g_0 \otimes g'_0 \otimes g_1 \otimes g'_1 \otimes \cdots \otimes g_n \otimes g'_n \mapsto g_0g_1 \cdots g_n \otimes g'_0g'_1 \cdots g'_n.$$

This is also bounded with respect to the unconditional norm. We claim that

$$\Delta(M|A|) \leq M' \Delta_{n\delta}^{\otimes(n+1)}(|A|), \quad \forall A \in \mathbb{C}G^{n+1}. \quad (4.23)$$

Here $\Delta_{n\delta}$ is the quasiderivation from Definition 4.11, and $\Delta_{n\delta}^{\otimes(n+1)}$ stands for the tensor product of $(n+1)$ -copies of $\Delta_{n\delta}$ from $\mathbb{C}G^{n+1}$ to $\mathbb{C}(G^2)^{(n+1)} \cong \mathbb{C}G^{2(n+1)}$.

Assume the claim holds for the moment. Clearly, the map M' is bounded with respect to the unconditional norm. Moreover, by the discussion before Proposition 4.12, there exists a constant K such that

$$\|\Delta_{n\delta}^{\otimes(n+1)}(|A|)\|_{uc} \leq K \|\Delta^{\otimes(n+1)}(|A|)\|_{uc}$$

for all $A \in \mathbb{C}G^{n+1}$. This proves the proposition.

Now let us prove the claim. It suffices to prove the inequality (4.23) when $|A| = g_0 \otimes \cdots \otimes g_n$. Denote $g_0 g_1 \cdots g_n$ by g . Suppose $g' \otimes g''$ appears on the left-hand side of the inequality (4.23), where by definition g' is a point on the geodesic $[e, g]$. We will show that $g' \otimes g''$ also appears on the right-hand side of the inequality (4.23). Indeed, by hyperbolicity, there exists a point x on the path $[e, g_0], [g_0, g_0 g_1], \dots, [g_0 g_1 \cdots g_{n-1}, g]$ such that the distance from x to g' is less than $n\delta$. More precisely, there exist $j \geq 0$ and $v, v' \in G$ such that $vv' = g_j$, $\ell(v) + \ell(v') = \ell(g_j)$ and $d(g', g_0 g_1 \cdots g_{j-1} v) = d(v, (g_0 g_1 \cdots g_{j-1})^{-1} g') < n\delta$. Thus the following element

$$(g_0 \otimes 1) \otimes \cdots \otimes (g_{j-1} \otimes 1) \otimes ((g_0 g_1 \cdots g_{j-1})^{-1} g' \otimes g'' (g_{j+1} g_{j+2} \cdots g_n)^{-1}) \otimes (1 \otimes g_{j+1}) \otimes \cdots \otimes (1 \otimes g_n)$$

appears in the summation expression of $\Delta_{n\delta}^{\otimes(n+1)}(g_0 \otimes g_1 \otimes \cdots \otimes g_n)$. After applying the map M' , we see that $g' \otimes g''$ indeed appears on the right-hand side of the inequality (4.23). This proves the claim, hence finishes the proof of the proposition. \square

5. Cyclic cohomology of hyperbolic groups

In this section, we show that every cyclic cohomology class of a hyperbolic group has a uniformly bounded representative if its degree is ≥ 2 . Since for any group, the equivalence class of a cyclic cocycle of degree ≤ 1 always has a representative of polynomial growth, it follows that all cyclic cohomology classes of a hyperbolic group can be represented by cyclic cocycles of polynomial growth.

We will need the following results on the geometry of hyperbolic groups [Gromov 1987]. Suppose G is a word hyperbolic group. For each $h \in G$, let G_h be the centralizer of h in G , and N_h the quotient of G_h by the cyclic group generated by h :

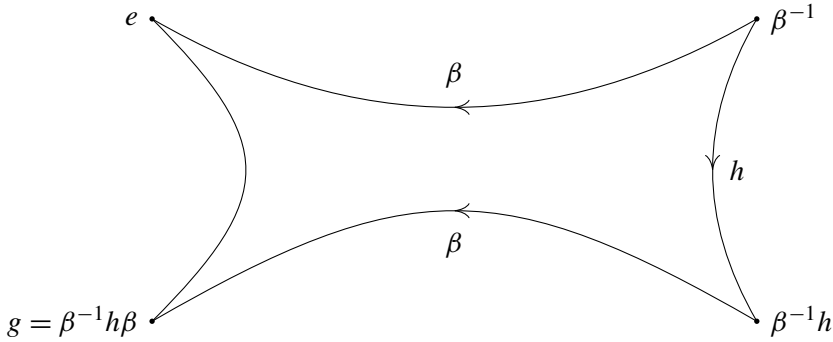


Figure 1. Geodesic quadrilateral.

- (G1) If $h \in G$ has infinite order, then N_h is finite.
- (G2) For any $h \in G$, the centralizer G_h is a quasiconvex subspace of G , that is, there exists some $K > 0$ such that any geodesic in G connecting a pair of points in G_h lies in a K -neighborhood of G_h .
- (G3) For any $h \in G$, its centralizer G_h is also word hyperbolic, and the inclusion $G_h \hookrightarrow G$ is a quasiisometry.

Moreover, we will use the following result of Mineyev [2001, Theorem 11] in an essential way:

- (M1) Suppose G is a word hyperbolic group with a given length function ℓ . If $n \geq 2$, then every element in $H^n(G; \mathbb{C})$ — the group cohomology of G — admits a uniformly bounded representative. Here a cocycle element φ is said to be uniformly bounded if there exists $C > 0$ such that $|\varphi(g_0, g_1, \dots, g_n)| \leq C$ for $g_i \in G$.

We will also need the following lemma, the proof of which is communicated to us by Denis Osin.

Lemma 5.1. *Let G be a δ hyperbolic group with a word length function ℓ . For each element $h \in G$, there exists a constant $K_h > 0$ such that*

$$\min\{\ell(\gamma) \mid \gamma^{-1}h\gamma = g\} \leq \ell(g) + K_h.$$

Proof. Let β be a group element of minimal length such that $\beta^{-1}h\beta = g$. Consider the geodesic quadrilateral $[e, \beta^{-1}]$, $[\beta^{-1}, \beta^{-1}h]$, $[\beta^{-1}h, g]$ and $[g, e]$ in the Cayley graph of G ; see Figure 1 below.

By continuity, there exists a point x on $[e, g]$ such that x is equidistant from the two sides $[e, \beta^{-1}]$ and $[\beta^{-1}h, g]$. Let y (resp. z) be a closest point on $[e, \beta^{-1}]$ (resp. $[\beta^{-1}h, g]$) to x , that is, $d(x, y) = d(x, z)$ equals the distance between x and

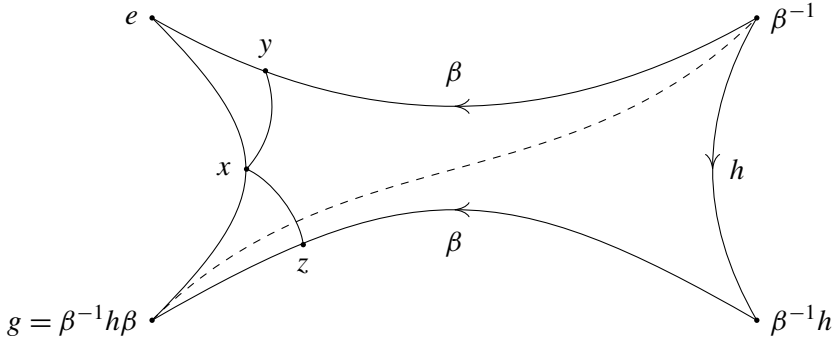


Figure 2. $d(x, y) = d(x, z) \leq 2\delta + \ell(h)$.

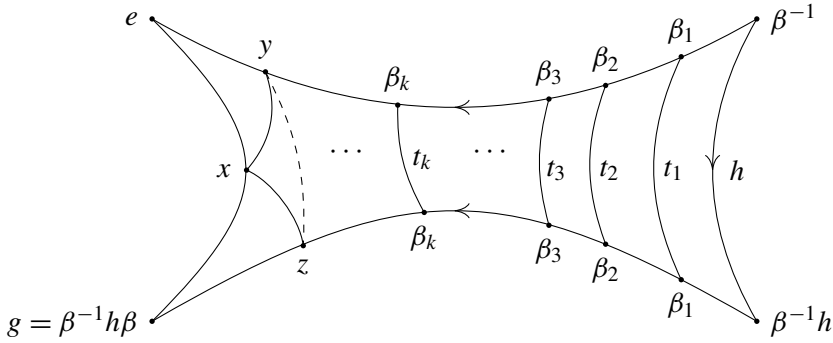


Figure 3. Estimates for the length of $[y, \beta^{-1}]$.

the geodesic $[e, \beta^{-1}]$. By hyperbolicity, it is not difficult to see that

$$d(x, y) = d(x, z) \leq 2\delta + \ell(h);$$

see Figure 2. It follows immediately that both $d(e, y)$ and $d(z, g)$ are less than $2\delta + \ell(h) + \ell(g)$.

It remains to estimate the length of either $[y, \beta^{-1}]$ or $[z, \beta^{-1}h]$. Suppose $\beta_1\beta_2 \cdots \beta_m$ is a word of minimal length that represents β . Let t_i be a geodesic connecting the points labeled by β_i on $[e, \beta^{-1}]$ and $[\beta^{-1}h, g]$; see Figure 3 below.

If $t_i = t_j$ for some $i < j$, we can cut the shaded region in Figure 4 and it follows that the element $\alpha = \beta_1 \cdots \beta_i \beta_{j+1} \cdots \beta_m$ satisfies that $\alpha^{-1}h\alpha = g$ and $\ell(\alpha) < \ell(\beta)$. But this contradicts the assumption that β is a group element of minimal length such that $\beta^{-1}h\beta = g$. Thus all the t_i are pairwise distinct.

Using hyperbolicity on the quadrilateral with vertices $\{y, z, \beta^{-1}h, \beta^{-1}\}$ (see [Bridson and Haefliger 1999, Chapter III.H, Lemma 1.15]), it is not difficult to see that there exists a constant $C_1 > 0$ such that the following is satisfied: if β_i is to the right of both y and z as shown in Figure 3, then $\ell(t_i) \leq C_1$. Here C_1 only

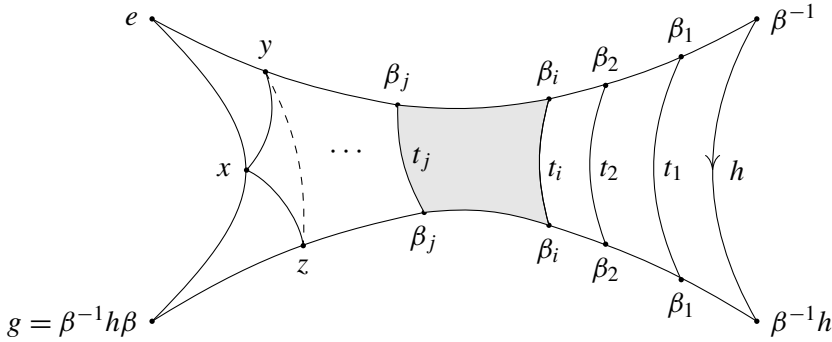


Figure 4. If $t_i = t_j$ for some $i < j$, we can shorten β .

depends on $\ell(h)$ and δ , and in particular is independent of g . It follows immediately the length of either $[y, \beta^{-1}]$ or $[z, \beta^{-1}h]$ is $\leq C_2 + 1$, where C_2 is the number of elements of G of length at most C_1 .

Combining the above estimates together, we see that there exists a constant K_h (only dependent on h and δ) such that

$$\ell(\beta) \leq \ell(g) + K_h.$$

This finishes the proof. \square

Theorem 5.2. Suppose G is a word hyperbolic group. Fix a conjugacy class $\langle h \rangle$ of G . Then every element in $HC^n(\mathbb{C}G, \langle h \rangle)$ has a representative $\varphi : G^{n+1} \rightarrow \mathbb{C}$ such that φ is of polynomial growth. Furthermore, when $n \geq 2$, such representative can be chosen to be uniformly bounded over G^{n+1} .

Proof. Elements of $HC^n(\mathbb{C}G, \langle h \rangle)$ have the following description; see [Lott 1992, Section 4.1]. Let $C^n(G, G_h, h)$ be the space spanned by all $(n+1)$ -linear maps on $\mathbb{C}G$ satisfying the following conditions:

$$\phi(g_{\sigma(0)}, g_{\sigma(1)}, \dots, g_{\sigma(n)}) = (-1)^\sigma \phi(g_0, g_1, \dots, g_n), \quad \forall \sigma \in S_n; \quad (5.3)$$

$$\phi(zg_0, zg_1, \dots, zg_n) = \phi(g_0, g_1, \dots, g_n), \quad \forall z \in G_h; \quad (5.4)$$

$$\phi(hg_0, g_1, \dots, g_n) = \phi(g_0, g_1, \dots, g_n). \quad (5.5)$$

Define a coboundary map $\partial : C^n(G, G_h, h) \rightarrow C^{n+1}(G, G_h, h)$ by

$$\partial\phi(g_0, g_1, \dots, g_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \phi(g_0, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_{n+1}).$$

Denote the resulting cohomology groups by $H^*(G, G_h, h)$. For each cocycle ϕ in $C^n(G, G_h, h)$, there is a cyclic cocycle $\mathcal{T}_\phi \in C^n(\mathbb{C}G, \langle h \rangle)$ given by

$$\mathcal{T}_\phi(g_0, g_1, \dots, g_n) = \begin{cases} 0 & \text{if } g_0 g_1 \cdots g_n \notin \langle h \rangle, \\ \phi(\gamma, \gamma g_0, \dots, \gamma g_0 \cdots g_{n-1}) & \text{if } g_0 g_1 \cdots g_n = \gamma^{-1} h \gamma. \end{cases} \quad (5.6)$$

Let us first consider the case of cyclic cocycles with degree ≥ 2 . Observe that if ϕ is uniformly bounded, then \mathcal{T}_ϕ is also uniformly bounded. Therefore, it suffices to show that ϕ is uniformly bounded. It is not difficult to see that $H^*(G, G_h, h)$ is isomorphic to $H^*(N_h; \mathbb{C})$. If the order of h is infinite, then $H^n(N_h; \mathbb{C})$ vanishes for $n > 0$, since N_h is a finite group by item (G1) above. Thus in the case, $H^n(G, G_h, h)$ with $n > 0$ does not contribute to the cyclic cohomology of $\mathbb{C}G$.

Let us assume h has finite order for the rest of the proof. In fact, it is more convenient for us to work with the group cohomology $H^n(G_h; \mathbb{C})$ of G_h . By applying the transfer map, we immediately see that $H^n(G_h; \mathbb{C})$ surjects onto $H^n(N_h; \mathbb{C})$. More precisely, consider the chain complex $(E^n(G, G_h), b)$ by removing the condition in line (5.5). There are two natural chain morphisms: the inclusion map $\iota: (C^n(G, G_h, h), b) \rightarrow (E^n(G, G_h), b)$ and the transfer map $\tau: (E^n(G, G_h), b) \rightarrow (C^n(G, G_h, h), b)$ defined by

$$\tau(\psi)(g_0, g_1, \dots, g_n) = \sum_{j=1}^{\text{ord}(h)} \psi(h^j g_0, g_1, \dots, g_n).$$

Since $\tau \circ \iota = \text{ord}(h) \cdot \text{Id}$, it follows that τ induces a surjection on cohomology groups. Clearly, if ψ is uniformly bounded, then $\tau(\psi)$ is also uniformly bounded. Therefore it suffices to show that for $n \geq 2$, every element $H^n(E^*(G, G_h))$ admits a uniformly bounded representative.

Let Y (resp. Y_h) be the Δ -complex consisting of all simplices of ordered $(n+1)$ -tuple $\{g_0, g_1, \dots, g_n\}$, where $g_i \in G$ (resp. $g_i \in G_h$). Observe that G_h acts freely on both Y and Y_h . Moreover, we see that the cochain complex of G_h -equivariant simplicial cochain on Y is essentially $E^*(G, G_h)$,¹⁰ and the cochain complex of G_h -equivariant simplicial cochain on Y_h gives the standard resolution cochain complex for the group cohomology of G_h .

Let $\pi: Y \rightarrow Y_h$ be any G_h -equivariant projection, that is, $\pi \circ i = \text{Id}$ on Y_h , where $i: Y_h \rightarrow Y$ is the inclusion map; such map always exists (see the discussion below for a specific construction of such a map). Then π induces a chain map from the standard resolution cochain complex for the group cohomology of G_h to $E^*(G, G_h)$, which is an isomorphism $\pi^*: H^n(G_h; \mathbb{C}) \xrightarrow{\cong} H^n(E^*(G, G_h))$ at the level of cohomology. In particular, any uniformly bounded group cocycle of

¹⁰To be precise, elements of $E^*(G, G_h)$ are assumed to be skew-symmetric (i.e., the condition in line (5.3)). But this can be easily fixed by applying a standard antisymmetrization map.

G_h pulls back to a uniformly bounded cocycle of the complex $E^*(G, G_h)$. On the other hand, by item (G3) above, G_h is hyperbolic. Therefore, by the item (M1) above, every element of $H^n(G_h; \mathbb{C})$ has a uniformly bounded representative, when $n \geq 2$. This finishes the proof for cyclic cocycles of degree $n \geq 2$.

If $n = 0$, $HC^0(\mathbb{C}G, \langle h \rangle)$ is a one dimensional linear space spanned by $\text{tr}_{\langle h \rangle}$; and $\text{tr}_{\langle h \rangle}$ is clearly uniformly bounded on G .

The only remaining case is when $n = 1$. We divide the proof of this case as follows. First, we shall show that every element of $H^1(E^*(G, G_h))$ has a representative of polynomial growth. Second, we shall construct a G_h -equivariant projection $\pi : Y \rightarrow Y_h$ such that π is simplicial and furthermore Lipschitz with respect to the word length metric on G and the corresponding subspace metric on G_h . More precisely, we say π is Lipschitz if there exists a constant $C > 0$ such that

$$d(\pi(g_1), \pi(g_2)) \leq Cd(g_1, g_2),$$

for all $g_1, g_2 \in G$, where d is the given word length metric on G . Now by item (G3), the metric on G_h is quasiisometric to the subspace metric inherited from G . It follows that if a degree 1 group cocycle φ of G_h has polynomial growth, then the pullback $\pi^*(\varphi)$ of φ to the complex $E^*(G, G_h)$ also has polynomial growth. In this case, by applying Lemma 5.1 above, it is not difficult to see that the corresponding $\mathcal{T}_{\pi^*(\varphi)} \in C^n(\mathbb{C}G, \langle h \rangle)$ has polynomial growth.

Let us now show that every element of $H^1(E^*(G, G_h))$ has a representative of polynomial growth. By definition, a degree 1 group cocycle φ of G_h is a G_h -equivariant function $\varphi : G_h \times G_h \rightarrow \mathbb{C}$, such that $\varphi(g_2, g_3) - \varphi(g_1, g_3) + \varphi(g_1, g_2) = 0$. In particular, it follows that φ is determined by the function $\psi : G_h \rightarrow \mathbb{C}$, where $\psi(g) = \varphi(1, g)$. The cocycle condition implies that $\psi(g_1 g_2) = \psi(g_1) + \psi(g_2)$, i.e., ψ is a group homomorphism from G_h to \mathbb{C} . As any homomorphism from G_h to \mathbb{C} factors through the abelianization of G_h , it follows that ψ has polynomial growth, so does φ .

Now let us construct a G_h -equivariant Lipschitz projection $\pi : Y \rightarrow Y_h$, which will finish the proof of the theorem by the above discussion. Fix an element, say $\alpha_i \in G$, for each coset of G_h in G . Let α'_i be an element of G_h such that $d(\alpha_i, \alpha'_i) = d(\alpha_i, G_h)$. For simplicity, if $\alpha_i = e \in G$, then we map e to itself. Define a G_h -equivariant map $\pi : G \rightarrow G_h$ by mapping $\beta \cdot \alpha_i \mapsto \beta \cdot \alpha'_i$, where $\beta \in G_h$. Clearly, π extends by linear combination to a G_h -equivariant map from Y to Y_h , which will still be denoted by π .

Let us show that $\pi : Y \rightarrow Y_h$ is Lipschitz. For two distinct points $g_1, g_2 \in G$, choose a geodesic $[\pi(g_1), \pi(g_2)]$ in the Cayley graph of G . By the quasiconvexity of G_h from item (G2) above, $[\pi(g_1), \pi(g_2)]$ lies in a K -neighborhood of G_h . By assumption, G is hyperbolic. More specifically, let us assume the Cayley graph of G is δ -hyperbolic. Then the geodesic $[\pi(g_1), \pi(g_2)]$ lies in the 2δ -neighborhood

of the union of geodesics $[\pi(g_1), g_1]$, $[g_1, g_2]$ and $[g_2, \pi(g_2)]$. Choose $\gamma \in G$ to be a “midpoint” of $[\pi(g_1), \pi(g_2)]$, that is,

$$|d(\pi(g_1), \gamma) - d(\pi(g_2), \gamma)| \leq 1.$$

Then there exists a point β on one of the geodesics $[\pi(g_1), g_1]$, $[g_1, g_2]$ or $[g_2, \pi(g_2)]$ such that $d(\gamma, \beta) \leq 2\delta$:

- (1) We claim that, if there exists a point β on $[\pi(g_1), g_1]$ such that $d(\gamma, \beta) \leq 2\delta$, then we have $d(\beta, \pi(g_1)) \leq 2\delta + K$. Indeed, otherwise, we could find an element $h_1 \in G_h$ such that $d(g_1, h_1) < d(g_1, \pi(g_1))$, which contradicts the fact $d(g_1, \pi(g_1)) = d(g_1, G_h)$. Therefore, in this case, we see that

$$d(\pi(g_1), \pi(g_2)) \leq 2(4\delta + K) + 1.$$

Things are similar for p' lies on $[g_2, \pi(g_2)]$.

- (2) Similarly, if there exists β on $[\pi(g_1), g_1]$ such that $d(\gamma, \beta) \leq 2\delta$, then we also have

$$d(\pi(g_1), \pi(g_2)) \leq 2(4\delta + K) + 1.$$

- (3) If there exists β on $[g_1, g_2]$ such that $d(\gamma, \beta) \leq 2\delta$, then both $d(g_1, G_h)$ and $d(g_2, G_h)$ are $\leq d(g_1, g_2) + 2\delta + K$. It follows immediately that

$$d(\pi(g_1), \pi(g_2)) \leq 3d(g_1, g_2) + 4\delta + 2K.$$

To summarize, we see that there exists a constant $C > 0$ such that

$$d(\pi(g_1), \pi(g_2)) \leq Cd(g_1, g_2),$$

for all $g_1, g_2 \in G$. This finishes the proof. \square

Remark 5.7. By a theorem of Meyer [2006, Theorem 5.2 and Corollary 5.3], the same strategy in the above proof can be used to show the following: given a cyclic cohomology class $[\alpha]$ of a hyperbolic group G , if φ_1 and φ_2 are two representatives with polynomial growth of $[\alpha]$, then there exists a cyclic cocycle ψ such that $\varphi_1 - \varphi_2 = b\psi$ and ψ has polynomial growth. Here $b : C^n(\mathbb{C}G) \rightarrow C^{n+1}(\mathbb{C}G)$ is the coboundary map of the cyclic cochain complex.

6. Delocalized Connes–Chern character of secondary invariants

In this section, we construct a delocalized Connes–Chern character map for C^* -algebraic secondary invariants and prove the second main theorem (Theorem 6.1) of the paper. We will only give the details for the odd dimensional case; the even dimensional case is completely similar.

Theorem 6.1. *Let M be a closed manifold whose fundamental group G is hyperbolic. Suppose $\langle h \rangle$ is nontrivial conjugacy class of G . Then every element $[\alpha] \in HC^{2k+1-i}(\mathbb{C}G, \langle h \rangle)$ induces a natural map*

$$\tau_{[\alpha]}: K_i(C_{L,0}^*(\tilde{M})^G) \rightarrow \mathbb{C}$$

such that the following are satisfied:

- (a) $\tau_{[S\alpha]} = \tau_{[\alpha]}$, where S is Connes' periodicity map

$$S: HC^*(\mathbb{C}G, \langle h \rangle) \rightarrow HC^{*+2}(\mathbb{C}G, \langle h \rangle).$$

- (b) *Suppose \tilde{D} is an elliptic operator on M such that the lift \tilde{D} of D to the universal cover \tilde{M} of M is invertible. Let φ be a representative of $[\alpha]$ with polynomial growth. Then the delocalized higher eta invariant $\eta_\varphi(\tilde{D})$ (see Definition 3.30) converges absolutely. Moreover, we have*

$$\tau_{[\alpha]}(\rho(\tilde{D})) = -\eta_\varphi(\tilde{D}),$$

where $\rho(\tilde{D})$ is the higher rho invariant of \tilde{D} .

In more conceptual terms, the above theorem provides a formula to compute the Connes–Chern character of elements of $K_i(C_{L,0}^*(\tilde{M})^G)$. Moreover, the theorem establishes a precise connection between Higson–Roe's K -theoretic higher rho invariants and Lott's higher eta invariants.

Remark 6.2. Note that, in part (b) of the theorem, $\eta_\varphi(\tilde{D})$ converges absolutely for all invertible \tilde{D} . In particular, it is not necessary for the spectral gap of \tilde{D} to be sufficiently large.

This section is organized as follows. First, we show that each element in $K_i(C_{L,0}^*(\tilde{M})^G)$ has a particular type of nice representatives. Second, we construct an explicit formula for the map $\tau_{[\alpha]}$ by using such nice representatives, and prove that the formula is well-defined. We shall only give the details for the case of $K_1(C_{L,0}^*(\tilde{M})^G)$; the even case is completely similar.

Definition 6.3. Let $B_L(\tilde{M})^G$ to be the subalgebra of $C_L^*(\tilde{M})^G$ consisting of elements $f(t) \in C_L^*(\tilde{M})^G$ such that $f(t) \in B(SG)$ for all $t \in [0, \infty)$ and $f(t)$ is piecewise smooth with respect to $\|\cdot\|_B$.

$B_L(\tilde{M})^G$ is a smooth dense subalgebra of $C_L^*(\tilde{M})^G$. Similarly, we define $B_{L,0}(\tilde{M})^G$ to be the kernel of the evaluation map

$$ev: B_L(\tilde{M})^G \rightarrow B(SG), \quad f \mapsto f(0).$$

Note that $B_{L,0}(\tilde{M})^G$ is a smooth dense subalgebra of $C_{L,0}^*(\tilde{M})^G$. In particular, it follows that

$$K_*(B_{L,0}(\tilde{M})^G) \cong K_*(C_{L,0}^*(\tilde{M})^G).$$

Definition 6.4. Let $SC^*(\tilde{M})^G$ be the suspension of $C^*(\tilde{M})^G$, and $\varphi \in$ an invertible element in $SC^*(\tilde{M})^G$, that is, a loop $\varphi: S^1 = [0, 1]/\{0, 1\} \rightarrow (C^*(\tilde{M})^G)^+$ of invertible elements such that $\varphi(1) = 1$, where $(C^*(\tilde{M})^G)^+$ is the unitization $C^*(\tilde{M})^G$. We say φ is *local* if it is the image of an invertible element $\psi \in SC_L^*(\tilde{M})^G$ under the evaluation map $SC_L^*(\tilde{M})^G \rightarrow SC^*(\tilde{M})^G$. Similarly, an invertible element $\varphi \in SB(SG)$ is called local if it is the image of an invertible element of $\psi \in SB_L(\tilde{M})^G$ under the evaluation map.

Definition 6.5 [Keswani 1999, Definition 3.3]. A path $\zeta \in B_L(\tilde{M})^G$ is said to have *polynomial B-norm control* if:

- (1) The propagation of $\zeta(t)$ is finite and goes to zero as $t \rightarrow \infty$.
- (2) There exists some polynomial q such that $\|\zeta(t)\|_B \leq q(1/\text{prop } \zeta(t))$ for sufficiently large $t \gg 0$. Here $\text{prop } \zeta(t)$ stands for the propagation of $\zeta(t)$.

In the following, we shall prove a sharpened version of [Xie and Yu 2021, Proposition 3.6]. We show that every element of $K_1(B_{L,0}(\tilde{M})^G)$ has nice representatives that satisfy certain regularity properties, in particular, the polynomial control property above.

Let us first prove the following technical lemma.

Lemma 6.6. *Suppose D is a self-adjoint first order elliptic differential operator over M and \tilde{D} is the lifting of D to the universal cover \tilde{M} of M . If $G = \pi_1(M)$ is hyperbolic and $f \in \mathcal{A}_{\Lambda,N}$ (see Definition 3.17) with $N \geq \frac{3}{2} \dim M + 5$ and Λ sufficiently large, then $f(\tilde{D}) \in B(SG)$.*

Proof. Fix a symmetric generating set S of G . Let ℓ be the corresponding word length function of G determined by S and $|S|$ the cardinality of S . Suppose $f(\tilde{D}) = \sum_{g \in G} A_g g$. By Lemmas 3.16 and 3.18, there exist $C_1, C_2 > 0$ such that

$$|A_g|_1 \leq C_1 \cdot e^{-C_2 \Lambda \ell(g)},$$

for all but finitely many $g \in G$, where $|A_g|_1$ stands for the trace norm of A_g . Let us denote

$$A^{(n)} = \sum_{\ell(g) \leq n} A_g g.$$

It suffices to show that $\{A^{(n)}\}$ is a Cauchy sequence under the norm $\|\cdot\|_{B,p}$; see Definition 4.9. Now for any $m < n$, we have

$$\|A^{(n)} - A^{(m)}\|_{RD,p}^2 = \sum_{m < \ell(g) \leq n} |A_g|_1^2 (1 + \ell(g))^{2p} \leq C_1 \cdot \sum_{j=m+1}^n e^{-C_2 \Lambda j} (1 + j)^{2p} |S|^j,$$

and

$$\begin{aligned}
& \|\Delta(A^{(n)} - A^{(m)})\|_{\text{uc}} \\
& \leq \sum_{m < \ell(g) \leq n} |A_g|_1 \sum_{\substack{g_1 g_2 = g \\ \ell(g_1) + \ell(g_2) = \ell(g)}} (1 + \ell(g_1))^p (1 + \ell(g_2))^p \\
& \leq \sum_{m < \ell(g) \leq n} |A_g|_1 (1 + \ell(g))^{2p} \cdot \#\{(g_1, g_2) : g_1 g_2 = g, \ell(g_1) + \ell(g_2) = \ell(g)\} \\
& \leq C_1 \cdot C \cdot \sum_{m < \ell(g) \leq n} e^{-C_2 \Lambda \ell(g)} (1 + \ell(g))^{2p+1} \\
& \leq C_1 \cdot C \cdot \sum_{j=m+1}^n e^{-C_2 \Lambda j} (1 + j)^{2p+1} |S|^j,
\end{aligned}$$

where we have used the fact that there exists $C > 0$ such that

$$\#\{(g_1, g_2) : g_1 g_2 = g, \ell(g_1) + \ell(g_2) = \ell(g)\} \leq C \cdot \ell(g),$$

since G is hyperbolic. It follows that, as long as Λ is sufficiently large, both $\|A^{(n)} - A^{(m)}\|_{RD,p}^2$ and $\|\Delta(A^{(n)} - A^{(m)})\|_{\text{uc}}$ go to zero, as $m, n \rightarrow \infty$. \square

Now let us show that every element of $K_1(B_{L,0}(\tilde{M})^G)$ has nice representatives that satisfy certain regularity properties, in particular, the polynomial control property above. The main motivation for choosing such nice representatives is to justify the explicit construction of $\tau_{[\alpha]} : K_1(B_{L,0}(\tilde{M})^G) \rightarrow \mathbb{C}$ below (Definition 6.10). Moreover, we show that for a given element of $K_1(B_{L,0}(\tilde{M})^\Gamma)$, two different such regularized representatives can be connected by a family of representatives of the same kind. This allows us to show that the integral in line (6.11) in Definition 6.10 is independent of the choice of such representatives.

Proposition 6.7. *Every element $[u] \in K_1(B_{L,0}(\tilde{M})^G)$ has a representative $w \in (B_{L,0}(\tilde{M})^G)^+$ such that*

$$w(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1, \\ h(t) & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i(F(t-1)+1)/2} & \text{if } t \geq 2, \end{cases}$$

where h is a path of invertible elements connecting $u(1)$ to $\exp(2\pi i(F(t-1)+1)/2)$, and F is a piecewise smooth map $F : [1, \infty) \rightarrow D^*(\tilde{M})^G$ satisfying:

- (1) $F(t)^2 - 1 \in B(SG)$ and $F(t)^* = F(t)$.
- (2) Its derivative $F'(t) \in B(SG)$.
- (3) $\|F(t)\|_{\text{op}} \leq 1$, where $\|\cdot\|_{\text{op}}$ stands for operator norm.
- (4) propagation of $F(t)$ is finite, and goes to zero as $t \rightarrow \infty$.

(5) Both $F(t)^2 - 1$ and $F'(t)$ have polynomial B -norm control in the sense of Definition 6.5 above.

Moreover, if v is another such representative, then there exists a piecewise smooth path of invertibles $u_s \in (B_{L,0}(\tilde{M})^G)^+$ and piecewise smooth maps $F_s : [1, \infty) \rightarrow D^*(\tilde{M})^G$ satisfying conditions above, with $s \in [0, 1]$, such that:

- (I) $u_0 = w$, $u_1(t) = v(t)$ for $t \notin (1, 2)$.
- (II) $u_s(t) = \exp(2\pi i(F_s(t-1)+1)/2)$ for all $t \geq 2$.
- (III) $u_1 v^{-1} : [1, 2] \rightarrow (B(SG))^+$ is a local loop of invertible elements,
- (IV) $\partial_s(F_s)$ has polynomial B -norm control.
- (V) The operator norm of $F_s(t)$ is uniformly bounded, and the degrees of polynomials used for the polynomial B -norm control of F_s and $\partial_s F_s$ are uniformly bounded, and the propagation of $F_s(t)$ goes to zero uniformly in s , as $t \rightarrow \infty$.

Remark 6.8. We shall call a representative appearing in the proposition above a *regularized* representative from now on.

Proof. View the invertible element $u \in (B_{L,0}(\tilde{M})^G)^+$ as an invertible element in $(B_L(\tilde{M})^G)^+$. Consider the element $\hat{u} = u : [1, \infty) \rightarrow (B(\tilde{M})^G)^+$ in $K_1(B_L(\tilde{M})^G)$. Since the K -theory of $B_L(\tilde{M})^G$ is the K -homology of M , it follows from the geometric description of K -homology by Baum and Douglas [1982] that \hat{u} can be represented by a twisted Dirac operator over a $spin^c$ manifold. More precisely, let X be a $spin^c$ manifold together with a vector bundle E over X and a continuous map $\psi : X \rightarrow M$. Suppose D_E is the associated twisted Dirac operator on X . Let \tilde{X} be the G -covering space of X induced by ψ , and \tilde{D} be the lift of D_E to \tilde{X} . Choose an odd continuous function $\chi : \mathbb{R} \rightarrow [-1, 1]$ such that $\chi(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$ and its distributional Fourier transform $\hat{\chi}$ has compact support. We define $F(t) = \psi_*(\chi(\tilde{D}/t))$, where $\psi_* : D^*(\tilde{X})^G \rightarrow D^*(\tilde{M})^G$ is the natural map induced by ψ . It is not difficult to see that F satisfies the properties (1)-(5) listed above.¹¹ Moreover, we have $[\hat{u}] = [e^{2\pi i(F(t)+1)/2}] \in K_1(B_L(\tilde{M})^G)$. In particular, u is homotopic to the invertible element w defined by

$$w(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1, \\ h(t) & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i(F(t-1)+1)/2} & \text{if } t \geq 2, \end{cases}$$

where h is a path of invertible elements connecting $u(1)$ and $e^{2\pi i(F(1)+1)/2}$.

¹¹Since by construction the propagation of $F(t)$ is uniformly bounded (in particular finite) for all $t \in [1, \infty)$, the polynomial B -norm control in property (5) follows from the work of Keswani [1999, Section 4]. Roughly speaking, the polynomial B -norm control is a consequence of the existence of partition of unity $\{\psi_{n,j}\}$ on the manifold X for each $n \in \mathbb{N}$ such that the diameter of $\psi_{n,j}$ is $\leq 1/n$ and the norm of $d\psi_{n,j}$ is bounded by $q(1/n)$ for some polynomial q .

Now suppose v is another representative of $[u]$ such that

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1, \\ g(t) & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i(G(t-1)+1)/2} & \text{if } t \geq 2, \end{cases}$$

where g is a path of invertible elements connecting $u(1)$ and $e^{2\pi i(G(1)+1)/2}$, and G_t is a piecewise smooth map from $[1, \infty)$ to $D^*(\tilde{M})^G$ satisfying properties (1)–(5) above.

By Theorem 3.8 in [Keswani 1999], there exists a piecewise smooth family $F_s : [1, \infty) \rightarrow D^*(\tilde{M})^G$ with $s \in [0, 1]$ such that $F_0 = F$ and $F_1 = G$; $F_s(t)^* = F_s(t)$; propagation of $F_s(t)$ goes to zero, as $t \rightarrow \infty$; and all $F_s(t)^2 - 1$, $\partial_t F_s(t)$, and $\partial_s F_s(t)$ lie in $B(SG)$. Furthermore, since the propagation of $\partial_s F_s(t)$ (resp. $F_s(t)$) is finite and the propagation is bounded uniformly in t , it is not difficult to see that $\partial_s F_s$ has polynomial B -norm control and the degrees of polynomials used for the polynomial B -norm control of F_s and $\partial_s F_s$ are uniformly bounded.

Let $\varpi : [0, \infty) \rightarrow (B(SG))^+$ be the path of invertibles defined as

$$\varpi(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1, \\ h(t) & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i(F_s(1)+1)/2} & \text{if } 2 \leq t = s+2 \leq 3, \\ e^{2\pi i(G(t-2)+1)/2} & t \geq 3. \end{cases}$$

Clearly, w is homotopic to ϖ . On the other hand, after a reparametrization, it is not difficult to see that ϖ differs from v by the loop $f : [0, 1] \rightarrow (B(SG))^+$ defined by

$$f(t) = \begin{cases} g(t)^{-1}h(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g(t)^{-1}e^{2\pi i(F_{2t-1}(1)+1)/2} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Moreover, f is a local loop in the sense of Definition 6.4. This finishes the proof. \square

The following lemma will be useful in the proof of Theorem 6.1.

Lemma 6.9. *Let $u = u_s(t)$ be the family of invertible elements from Proposition 6.7 above. Then for any delocalized cyclic cocycle φ with polynomial growth,*

$$\lim_{t \rightarrow +\infty} \varphi \# \widetilde{\text{tr}}(u_s(t)^{-1} \partial_s(u_s(t)) \otimes (u_s(t) \otimes u_s(t)^{-1})^{\otimes m}) = 0$$

uniformly in $s \in [0, 1]$.

Proof. By definition,

$$u_s(t) = e^{2\pi i(F_s(t-1)+1)/2},$$

where F_s and $\partial_s F_s$ have polynomial B -norm control. Let $P = P_s(t) = \frac{1}{2}(F_s(t) + 1)$. Then $P_s^2 - P_s = (F_s^2 - 1)/4$. Let

$$f_n(x) = \sum_{k=0}^n \frac{(2\pi i x)^k}{k!}.$$

Note that we have

$$f_n(P) = \sum_{k=0}^n \frac{(2\pi i)^k}{k!} P^k = 1 + \left(\sum_{k=1}^n \frac{(2\pi i)^k}{k!} \right) P + \left(\sum_{k=1}^n \frac{(2\pi i)^k}{k!} \sum_{j=0}^{k-2} P^j \right) (P^2 - P).$$

Define

$$A_n = f_n(P) - \left(\sum_{k=1}^n \frac{(2\pi i)^k}{k!} \right) P^2.$$

Clearly, $A_n \in B(SG)^+$ for all $n \geq 1$, and

$$u - A_n = \left(\sum_{k=n+1}^{\infty} \frac{(2\pi i)^k}{k!} \right) (P - P^2) + \left(\sum_{k=n+1}^{\infty} \frac{(2\pi i)^k}{k!} \sum_{j=0}^{k-2} P^j \right) (P^2 - P).$$

Recall that, by construction, there exists a polynomial q such that

$$\|P^2(t) - P(t)\|_B \leq q(1/\text{prop } F(t)).$$

Since the operator norm of $P(t)$ is uniformly bounded and the propagation of $P(t)$ goes to zero as $t \rightarrow \infty$, a routine calculation shows that there exists $C > 0$ such that

$$\|P^j(t)(P^2(t) - P(t))\|_B \leq C^j \cdot q(1/\text{prop } F(t))$$

for all $j \in \mathbb{N}$ and sufficiently large $t \gg 0$. It follows that there exists $K > 0$ such that

$$\|u(t) - A_n(t)\|_B < K \frac{C^n}{(n+1)!} \cdot q(1/\text{prop } F(t)),$$

and

$$\|u(t) - 1\|_B < K e^C \cdot q(1/\text{prop } F(t))$$

for sufficiently large $t \gg 0$. The same type of estimates also apply to u^{-1} and $\partial_s u$.

Now fix $\varepsilon > 0$ so that, if $a_i \in B(SG)^+$ has propagation $\leq \varepsilon$ for all $0 \leq i \leq 2m$, then

$$\widetilde{\varphi\#} \text{tr}(a_0 \otimes a_1 \otimes \cdots \otimes a_{2m}) = 0.$$

By the proof of Proposition 3.46, such an ε exists. Note that there exists an increasing sequence of positive number $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\frac{\varepsilon}{n+1} \leq \text{prop } F_s(t) \leq \frac{\varepsilon}{n}$$

for $t_n \leq t \leq t_{n+1}$. In particular, we have that

- (1) $\text{prop } A_n(t) \leq \varepsilon$,
- (2) $\|u(t) - A_n(t)\|_B < K \frac{C^n}{(n+1)!} \cdot q\left(\frac{n+1}{\varepsilon}\right)$,
- (3) $\|u(t) - 1\|_B < K e^C \cdot q\left(\frac{n+1}{\varepsilon}\right)$

for all $t_n \leq t \leq t_{n+1}$. Similarly, the same type of estimates hold for u^{-1} and $\partial_s u$. The lemma easily follows from these estimates. This finishes the proof. \square

Now for each class $[\alpha] \in HC^{2m}(\mathbb{C}G, \langle h \rangle)$, we define a map

$$\tau_{[\alpha]}: K_1(B_{L,0}(\tilde{M})^G) \rightarrow \mathbb{C}$$

as follows.

Definition 6.10. Let φ be a representative of $[\alpha]$ with polynomial growth. For each $[u] \in K_1(B_{L,0}(\tilde{M})^G)$, let w be a regularized representative of $[u]$. We define

$$\tau_{[\alpha]}([u]) := \tau_\varphi(w) = \frac{m!}{\pi i} \int_0^\infty \widetilde{\varphi \# \text{tr}(\dot{w}(t)w(t)^{-1} \otimes (w(t) \otimes w(t)^{-1})^{\otimes m})} dt. \quad (6.11)$$

The convergence of the integral in line (6.11) follows from the following two observations:

- (1) By Proposition 4.20, the integrand

$$\widetilde{\varphi \# \text{tr}(\dot{w}(t)w(t)^{-1} \otimes (w(t) \otimes w(t)^{-1})^{\otimes m})}$$

is a piecewise smooth function with respect to t on $[0, \infty)$. In particular, this implies that the integral in line (6.11) converges absolutely for small t .

- (2) By the proof of Proposition 6.7, when $t \geq 2$, we have that

$$w(t) = e^{2\pi i(F(t)+1)/2} = e^{2\pi i(\chi(\tilde{D}/t)+1)/2}.$$

Set $s = 1/t$, and then we have

$$\begin{aligned} & \int_2^\infty \widetilde{\varphi \# \text{tr}(\dot{w}(t)w(t)^{-1} \otimes (w(t) \otimes w(t)^{-1})^{\otimes m})} dt \\ &= \pi i \int_{1/2}^0 \widetilde{\varphi \# \text{tr}(\dot{\chi}(s\tilde{D})\tilde{D} \otimes (e^{2\pi i(\chi(s\tilde{D})+1)/2} \otimes e^{-2\pi i(\chi(s\tilde{D})+1)/2})^{\otimes m})} ds. \end{aligned}$$

Since the Fourier transform of χ has compact support, it follows that $x\chi'(x)$ and $\exp(\pm 2\pi i(\chi(x) + 1)/2) - 1$ lies in $\mathcal{A}_{\Lambda, N}$ for any Λ, N . Therefore by Proposition 3.46 and 3.48, the integral with respect to s converges absolutely for small s . Consequently the integral from line (6.11) converges for large t .

Proof of Theorem 6.1. Let φ be a representative of $[\alpha]$ with polynomial growth. Let us first show that $\tau_\varphi([u])$ is independent of the choice of regularized representative of $[u]$. Suppose w and v are two regularized representatives of $[u]$. By Proposition 6.7, there exists a piecewise smooth family of invertibles $u_s \in B_{L,0}(\tilde{M})^G$ with the stated properties (I)–(V).

Now a straightforward calculation shows that

$$\partial_s(\widetilde{\varphi\#} \operatorname{tr}(u^{-1} \partial_t u \otimes (u \otimes u^{-1})^{\otimes m})) = \partial_t(\widetilde{\varphi\#} \operatorname{tr}(u^{-1} \partial_s u \otimes (u \otimes u^{-1})^{\otimes m})). \quad (6.12)$$

It follows that

$$\begin{aligned} \int_0^T \widetilde{\varphi\#} \operatorname{tr}(\dot{u}_1 u_1^{-1} \otimes (u_1 \otimes u_1^{-1})^{\otimes m}) dt - \int_0^T \widetilde{\varphi\#} \operatorname{tr}(\dot{u}_0 u_0^{-1} \otimes (u_0 \otimes u_0^{-1})^{\otimes m}) dt \\ = \int_0^1 \widetilde{\varphi\#} \operatorname{tr}(u^{-1} \partial_s u \otimes (u \otimes u^{-1})^{\otimes m}) \Big|_{t=0}^{t=T} ds. \end{aligned}$$

By Lemma 6.9 below, we have

$$\widetilde{\varphi\#} \operatorname{tr}(u^{-1} \partial_s u \otimes (u \otimes u^{-1})^{\otimes m}) \rightarrow 0$$

as $t \rightarrow \infty$. Also, note that $u_s(0) \equiv 1$ for all s . It follows that

$$\tau_\varphi(u_1) = \tau_\varphi(u_0).$$

On the other hand, u_1 differs from v by a local loop $f: S^1 \rightarrow B(SG)^+$. By [Xie and Yu 2021, Lemma 3.4], for $\forall \varepsilon > 0$, there exists an idempotent $p \in B(SG)^+$ such that the propagation of p is $\leq \varepsilon$ and f is homotopic, in the algebra $SB(SG)$, to the element

$$\beta(t) = e^{2\pi i t} p + (1 - p), \quad \text{where } 0 \leq t \leq 1.$$

It follows that

$$\begin{aligned} \int_0^1 \widetilde{\varphi\#} \operatorname{tr}(\dot{f}(t) f(t)^{-1} \otimes (f(t) \otimes f(t)^{-1})^{\otimes m}) dt \\ = \int_0^1 \widetilde{\varphi\#} \operatorname{tr}(\dot{\beta}(t) \beta(t)^{-1} \otimes (\beta(t) \otimes \beta(t)^{-1})^{\otimes m}) dt \\ = \int_0^1 \varphi\# \operatorname{tr}(2\pi i p \otimes ((e^{2\pi i t} - 1)p \otimes (e^{-2\pi i t} - 1)p)^{\otimes m}) dt \end{aligned}$$

where the last integral is clearly zero, as long as ε is sufficiently small. Therefore, $\tau_{[\alpha]}([u])$ is independent of the choice of regularized representative of $[u]$.

For a given regularized representative w of $[u]$, the same proof as Proposition 3.53 shows that $\tau_{[\alpha]}(w)$ is independent of the choice of polynomial growth representative

φ of $[\alpha]$; see Remark 5.7. This proves that the map

$$\tau_{[\alpha]}: K_1(B_{L,0}(\tilde{M})^G) \rightarrow \mathbb{C}$$

is well-defined. Furthermore, the same proof from Proposition 3.56 shows that

$$\tau_\varphi(w) = \tau_{S\varphi}(w).$$

This proves part (a) of the theorem.

We shall prove part (b) of the theorem in three steps:

- (i) Recall that the definition of $\eta_\varphi(\tilde{D})$ (see Definition 3.30) uses the representative $u_t = U_{1/t}(\tilde{D})$ of the higher rho invariant $\rho(\tilde{D})$, where

$$U_t = e^{2\pi i F_t(\tilde{D})} \quad \text{with } F_t(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/t} e^{-s^2} ds \text{ for } t > 0 \text{ and } U_0 = 1.$$

We first prove that the path U_t is an element of $(B_{L,0}(\tilde{M})^G)^+$.

- (ii) Second, we prove that the integral

$$\eta_\varphi(\tilde{D}) = (-1) \frac{m!}{\pi i} \int_0^\infty \widetilde{\varphi\#} \text{tr}(\dot{U}_t U_t^{-1} \otimes (U_t \otimes U_t^{-1})^{\otimes m}) dt$$

absolutely converges, where the minus sign is due to the change of variables $1/t \rightarrow t$. Note that here we do not require the spectral gap of \tilde{D} to be sufficiently large. In other words, the convergence of the integral holds as long as \tilde{D} is invertible.

- (iii) Recall that we defined $\tau_{[\alpha]}(\rho(\tilde{D}))$ by using a regularized representative of $\rho(\tilde{D})$. In the third step, we use a transgression formula as in line (6.12) to prove that $\tau_{[\alpha]}(\rho(\tilde{D})) = -\eta_\varphi(\tilde{D})$.

The first and second steps are proved in Proposition 6.13 below. Let us now turn to the third step. Let χ be a normalizing function from the proof of Proposition 6.7, that is, an odd continuous function $\chi: \mathbb{R} \rightarrow [-1, 1]$ such that $\chi(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$ and its distributional Fourier transform $\widehat{\chi}$ has compact support. Furthermore, without loss of generality, we can assume in addition $x \cdot \widehat{\chi}(x)$ is a smooth function. Denote $E_t(\tilde{D}) = (\chi(\tilde{D}/t) + 1)/2$. It follows from Lemmas 3.18 and 6.6 that $e^{2\pi i E_t(\tilde{D})}$ is a smooth path in $(B(SG))^+$. Let us define

$$V_t = \begin{cases} U_t & \text{if } 0 \leq t \leq 1, \\ e^{2\pi i ((2-t)F_1(\tilde{D}) + (t-1)E_1(\tilde{D}))} & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i E_{t-1}(\tilde{D})} & \text{if } t \geq 2. \end{cases}$$

Then the path V_t is a regularized representative of $\rho(\tilde{D})$ in $B_{L,0}(SG)^+$. Furthermore, V_t and U_t are homotopic in $B_{L,0}(SG)^+$ by the following family of elements H_s ,

with $0 \leq s \leq 1$:

$$H_s(t) = \begin{cases} U_t & \text{if } 0 \leq t \leq 1, \\ e^{2\pi i((2-t)F_1 + (t-1)(sE_1 + (1-s)F_1))} & \text{if } 1 \leq t \leq 1+s, \\ e^{2\pi i((1-s)F_{t-1} + sE_{t-1})} & \text{if } t \geq 1+s. \end{cases}$$

Now the same transgression formula in line (6.12) can be applied to show that

$$\tau_\varphi(V) = \tau_\varphi(U) = -\eta_\varphi(\tilde{D}).$$

This finishes the proof. \square

Proposition 6.13. *Under the same assumptions as Theorem 6.1, let U_t be the representative of $\rho(\tilde{D})$ given by*

$$U_t = e^{2\pi i F_t(\tilde{D})} \quad \text{with } F_t(x) = \frac{1}{\sqrt{\pi}} \int_0^{x/t} e^{-s^2} ds \text{ for } t > 0 \text{ and } U_0 = 1.$$

Then the path U_t defines an invertible element of $B_{L,0}(SG)^+$. Furthermore, if φ in $C^{2m}(\mathbb{C}G, \langle h \rangle)$ has polynomial growth, then the following integral

$$\int_0^\infty \varphi \# \text{tr}(\dot{U}_t U_t^{-1} \otimes (U_t \otimes U_t^{-1})^{\otimes m}) dt$$

converges absolutely.

Proof. Since $U_t(x) - 1$ admits an analytic continuation to an entire function, it follows from Lemma 6.6 that $U_t = U_t(\tilde{D}) \in B(SG)^+$ for each $t > 0$ and the path U_t is smooth with respect to the norm $\|\cdot\|_B$ on $(0, \infty)$. It remains to show that U_t is continuous at $t = 0$ with respect to the norm $\|\cdot\|_B$.

Since \tilde{D} is invertible, let $\sigma > 0$ be the spectral gap of \tilde{D} at zero. Then the spectral radius of $e^{-\tilde{D}^2}$ as an element in $B(SG)$ is $e^{-\sigma^2}$, since $B(SG)$ is a smooth dense subalgebra of $C^*(\tilde{M})^G$. Recall that $B(SG)$ is a Banach algebra with respect to the norm¹² $\|\cdot\|_B$ (see Proposition 4.12), that is,

$$\|A_1 A_2\|_B \leq \|A_1\|_B \|A_2\|_B, \quad \text{for any } A_1, A_2 \in B(SG).$$

By the spectral radius formula

$$\lim_{n \rightarrow \infty} (\|(e^{-\tilde{D}^2})^n\|_B)^{1/n} = e^{-\sigma^2},$$

there exists $C_1 > 0$ such that

$$\|e^{-(1/t)\tilde{D}^2}\|_B \leq C_1 e^{-(1/(2t))\sigma^2}$$

¹²Rescale the norm $\|\cdot\|_B$ if necessary.

for all sufficiently small $t > 0$. It follows that there exists $C > 0$ such that

$$\begin{aligned}\|\dot{U}_t U_t^{-1}\|_B &= \left\| -2\sqrt{\pi}i \frac{\tilde{D}}{t^2} e^{-\tilde{D}^2/t^2} \right\|_B \\ &\leq \frac{1}{t^2} \|2\sqrt{\pi} \tilde{D} e^{-\tilde{D}^2}\|_B \cdot \|e^{-(1/t^2-1)\tilde{D}^2}\|_B \\ &\leq C \frac{1}{t^2} e^{-\frac{1}{2}\sigma^2/t^2}.\end{aligned}\tag{6.14}$$

By the definition of U_t , we have

$$U_t - 1 = \exp\left(\int_0^t \dot{U}_s U_s^{-1} ds\right) - 1.$$

In fact, the integral on the right hand side converges in $B(SG)$, thanks to the inequality in line (6.14). In particular, it follows that

$$\begin{aligned}\|U_t - 1\|_B &= \left\| \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_0^t \dot{U}_s U_s^{-1} ds \right)^n \right\|_B \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{C}{t^2} e^{-\sigma^2/(2t^2)} \right)^n \\ &= \exp\left(\frac{C}{t^2} e^{-\sigma^2/(2t^2)}\right) - 1,\end{aligned}\tag{6.15}$$

which goes to zero as t goes to zero. It follows that the path U_t , with $t \in [0, \infty)$, gives an invertible element in $B_{L,0}(SG)^+$. Furthermore, by Proposition 3.48 and Proposition 4.20, the following integral

$$\int_0^\infty \widetilde{\varphi\#} \text{tr}(\dot{U}_t U_t^{-1} \otimes (U_t \otimes U_t^{-1})^{\otimes m}) dt$$

converges absolutely. This finishes the proof. \square

7. Delocalized higher Atiyah–Patodi–Singer index theorem

In this section, we apply the results from previous sections to prove a delocalized higher Atiyah–Patodi–Singer index theorem.

Let us first review the Connes–Chern character map in our context. We shall only discuss the even dimensional case; the odd case is similar. Let G be a discrete group, and $[\alpha] \in HC^{2m}(\mathbb{C}G)$. The $[\alpha]$ -component of the Connes–Chern character of an idempotent $p \in SG$ is given by

$$\text{ch}_{[\alpha]}(p) = \frac{(2m)!}{m!} \varphi\# \text{tr}(p^{\otimes 2m+1}),\tag{7.1}$$

where φ is a cyclic cocycle representative of $[\alpha]$. It has been implied that $\text{ch}_{[\varphi]}(p)$ is independent of the choice of representative of $[\alpha]$. Indeed, for a cyclic coboundary $b\psi$, we have

$$b\psi \# \text{tr}(p^{\otimes 2m+1}) = \psi(p^{\otimes 2m}) = 0.$$

The last equality follows from the fact ψ is cyclic, which in particular implies that $\psi(p^{\otimes 2m}) = -\psi(p^{\otimes 2m})$.

If G is hyperbolic, then by Proposition 4.20 the formula in line (7.1) continues to make sense for idempotents in $B(SG)$, as long as φ has polynomial growth. In fact, in this case, $\text{ch}_{[\alpha]}$ defines a Connes–Chern character map at the level of K -theory:

$$\text{ch}_{[\alpha]}: K_0(B(SG)) \rightarrow \mathbb{C}.$$

Indeed, suppose $[p_0] = [p_1] \in K_0(B(SG))$. Let p_t be a piecewise smooth path of idempotents in $B(SG)^+$ connecting p_0 and p_1 . Suppose φ has polynomial growth. Then a routine calculation shows that

$$\frac{d}{dt} \varphi \# \text{tr}(p_t^{\otimes 2m+1}) = (2m+1)(b\varphi \# \text{tr})((\dot{p}_t p_t - p_t \dot{p}_t) \otimes p_t^{\otimes 2m+1}) = 0,$$

since φ is a cyclic cocycle. It follows immediately that

$$\varphi \# \text{tr}(p_0^{\otimes 2m+1}) = \varphi \# \text{tr}(p_1^{\otimes 2m+1}).$$

Therefore, the map $\text{ch}_{[\alpha]}: K_0(B(SG)) \rightarrow \mathbb{C}$ is well-defined.

By Theorem 5.2, every cyclic cohomology class of a hyperbolic group has a representative with polynomial growth. Hence, to summarize the above, we have the following proposition.

Proposition 7.2. *Suppose that G is a hyperbolic group and $\langle h \rangle$ is a conjugacy class of G . For each class $[\alpha] \in HC^{2m}(\mathbb{C}, \langle h \rangle)$, the Connes–Chern character map*

$$\text{ch}_{[\alpha]}: K_0(C_r^*(G)) \rightarrow \mathbb{C} \tag{7.3}$$

given by the formula (7.1) is well-defined. Moreover, we have $\text{ch}_{S[\alpha]} = \text{ch}_{[\alpha]}$, where $S: HC^{2m}(\mathbb{C}G, \langle h \rangle) \rightarrow HC^{2m+2}(\mathbb{C}G, \langle h \rangle)$ is Connes’ periodicity map.

Proof. The formula for Connes’s periodicity map is given in Definition 3.55. A straightforward computation shows that

$$(S\varphi \# \text{tr})(p^{\otimes 2m+3}) = \frac{1}{2(2m+1)} \varphi \# \text{tr}(p^{\otimes 2m+1}),$$

from which the second statement of the proposition immediately follows. \square

The Connes–Chern character $\text{ch}_{[\alpha]}: K_0(C_r^*(G)) \rightarrow \mathbb{C}$ above and the delocalized Connes–Chern character map $\tau_{[\alpha]}: K_i(C_{L,0}^*(\tilde{M})^G) \rightarrow \mathbb{C}$ from Theorem 6.1 are related as follows.

Proposition 7.4. *Suppose G is hyperbolic and $\langle h \rangle$ is a nontrivial conjugacy class of G . Given $[\alpha] \in HC^{2m}(\mathbb{C}G, \langle h \rangle)$, we have the following commutative diagram:*

$$\begin{array}{ccc} K_0(C_r^*(G)) & \xrightarrow{\text{ch}_{[\alpha]}} & \mathbb{C} \\ \partial \downarrow & & \downarrow \times (-2) \\ K_1(C_{L,0}^*(\tilde{M})^G) & \xrightarrow{\tau_{[\alpha]}} & \mathbb{C} \end{array}$$

where $\partial: K_0(C_r^*(G)) \rightarrow K_1(C_{L,0}^*(\tilde{M})^G)$ is the connecting map in the six-term K -theoretical exact sequence for the short exact sequence

$$0 \rightarrow C_{L,0}^*(\tilde{M})^G \rightarrow C_L^*(\tilde{M})^G \rightarrow C^*(\tilde{M})^G \rightarrow 0.$$

Proof. Each element of $K_0(C_r^*(G))$ is represented by the formal difference of two idempotents in $B(\mathcal{S}G)^+$. For notational simplicity, let us carry out the computation for an idempotent p in $B(\mathcal{S}G)$.

Recall that $\partial[p]$ is defined as follows: let $\{a_t\}_{t \in [0, \infty)}$ be the following lift of p in $B_L(\tilde{M})^G$:

$$a_t = \begin{cases} (1-t)p & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

Then we have

$$\partial p := u \quad \text{with } u(t) = e^{2\pi i a_t} \text{ for } t \in [0, \infty).$$

Note that for $0 \leq t \leq 1$, we have

$$u_t = e^{2\pi i(1-t)p} = 1 + (e^{2\pi i(1-t)} - 1)p.$$

It follows that

$$\begin{aligned} & \int_0^\infty \widetilde{\varphi\#} \text{tr}(\dot{u}_t u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m}) dt \\ &= \int_0^1 \widetilde{\varphi\#} \text{tr}(\dot{u}_t u_t^{-1} \otimes (u_t \otimes u_t^{-1})^{\otimes m}) dt \\ &= -\varphi\# \text{tr}(p^{\otimes 2m+1}) \int_0^1 (2\pi i)(e^{2\pi i(1-t)} - 1)^m (e^{-2\pi i(1-t)} - 1)^m dt \\ &= -2\pi i \frac{(2m)!}{(m!)^2} \varphi\# \text{tr}(p^{\otimes 2m+1}) \\ &= -\frac{2\pi i}{m!} \text{ch}_{[\alpha]}([p]). \end{aligned}$$

It follows that

$$\text{ch}_{[\alpha]}([p]) = -\frac{1}{2} \tau_{[\alpha]}(\partial[p]).$$

This finishes the proof. □

Let W be a compact n -dimensional spin manifold with boundary $M = \partial W$. Suppose W is equipped with a Riemannian metric which has product structure near M and in addition has positive scalar curvature on M . Let \tilde{W} be the universal covering of W equipped with the metric lift from W . Denote $\pi_1(W)$ by G . The associated Dirac operator \tilde{D}_W naturally defines a higher index in $K_n(C^*(\tilde{W})^G)$, denoted by $\text{Ind}_G(\tilde{D}_W)$ as in [Xie and Yu 2014, Section 3]. Denote the lift of M with respect to the covering map by $\tilde{M} = \partial \tilde{W}$. The associated Dirac operator \tilde{D}_M naturally defines a higher rho invariant $\rho(\tilde{D}_M)$ in $K_{n-1}(C_{L,0}^*(\tilde{M})^G)$. The image of $\rho(\tilde{D}_M)$ under the natural homomorphism $K_{n-1}(C_{L,0}^*(\tilde{M})^G) \rightarrow K_{n-1}(C_{L,0}^*(\tilde{W})^G)$ will still be denoted by $\rho(\tilde{D}_M)$.

We denote by $\partial : K_n(C^*(\tilde{W})^G) \rightarrow K_{n-1}(C_{L,0}^*(\tilde{W})^G)$ the connecting map in the six-term K-theoretical exact sequence for the short exact sequence

$$0 \rightarrow C_{L,0}^*(\tilde{W})^G \rightarrow C_L^*(\tilde{W})^G \rightarrow C^*(\tilde{W})^G \rightarrow 0.$$

By [Piazza and Schick 2014, Theorem 1.14] and [Xie and Yu 2014, Theorem A], we have

$$\partial(\text{Ind}_G(\tilde{D}_W)) = \rho(\tilde{D}_M) \quad \text{in } K_{n-1}(C_{L,0}^*(\tilde{W})^G). \quad (7.5)$$

This together with Proposition 7.4 implies the following delocalized Atiyah–Patodi–Singer index theorem.

Theorem 7.6. *Let W be a compact even-dimensional spin manifold with boundary M . Suppose W is equipped with a Riemannian metric which has product structure near M and in addition has positive scalar curvature on M . Suppose $G = \pi_1(W)$ is hyperbolic and $\langle h \rangle$ is a nontrivial conjugacy class of G . Then for any $[\alpha] \in HC^{2m}(\mathbb{C}G, \langle h \rangle)$, we have*

$$\text{ch}_{[\alpha]}(\text{Ind}_G(\tilde{D}_W)) = \frac{1}{2} \eta_{[\alpha]}(\tilde{D}_M). \quad (7.7)$$

Proof. Observe that Proposition 7.4 still holds if we replace \tilde{M} by \tilde{W} . In particular, we have the following commutative diagram:

$$\begin{array}{ccc} K_0(C_r^*(G)) & \xrightarrow{\text{ch}_{[\alpha]}} & \mathbb{C} \\ \partial \downarrow & & \downarrow \times(-2) \\ K_1(C_{L,0}^*(\tilde{W})^G) & \xrightarrow{\tau_{[\alpha]}} & \mathbb{C} \end{array}$$

Now the theorem follows immediately from Theorem 6.1 and the equality

$$\partial(\text{Ind}_G(\tilde{D}_W)) = \rho(\tilde{D}_M) \quad \text{in } K_{n-1}(C_{L,0}^*(\tilde{W})^G). \quad \square$$

By using Theorem 6.1, we have derived Theorem 7.6 as a consequence of a K -theoretic counterpart. This is possible only because we have realized $\eta_{[\alpha]}(\tilde{D}_M)$ as the pairing between the cyclic cocycle $[\alpha]$ and the C^* -algebraic secondary invariant $\rho(\tilde{D}_M)$ in $K_1(C_{L,0}^*(\tilde{W})^G)$.

Alternatively, one can also derive Theorem 7.6 from a version of higher Atiyah–Patodi–Singer index theorem due to Leichtnam and Piazza [1999, Theorem 4.1] and Wahl [2013, Theorems 9.4 and 11.1]. This version of higher Atiyah–Patodi–Singer index theorem is stated in terms of noncommutative differential forms on a smooth dense subalgebra of $C_r^*(G)$; or noncommutative differential forms on a certain class of smooth dense subalgebras (if exist) of general C^* -algebras (not just group C^* -algebras) in Wahl’s version. In the case of Gromov’s hyperbolic groups, one can choose such a smooth dense subalgebra to be Puschnigg’s smooth dense subalgebra $B(\mathbb{C}G)$. For hyperbolic groups, every cyclic cohomology class of $\mathbb{C}G$ continuously extends to a cyclic cohomology class of $B(\mathbb{C}G)$; see Sections 4 and 5. Now Theorem 7.6 follows by pairing the higher Atiyah–Patodi–Singer index formula of Leichtnam and Piazza and Wahl with the delocalized cyclic cocycles of $\mathbb{C}\Gamma$.

One can also try to pair the higher Atiyah–Patodi–Singer index formula of Leichtnam and Piazza and Wahl with group cocycles of Γ , or equivalently cyclic cocycles in $HC^*(\mathbb{C}\Gamma, \langle 1 \rangle)$, where $\langle 1 \rangle$ stands for the conjugacy class of the identity element of Γ . In this case, for fundamental groups with property RD, Gorokhovsky, Moriyooshi and Piazza proved a higher Atiyah–Patodi–Singer index theorem for group cocycles with polynomial growth [Gorokhovsky et al. 2016, Theorem 7.2].

8. Delocalized higher eta invariant and its relation to Lott’s higher eta invariant

In this section, we shall establish the relation between our definition of the delocalized higher eta invariant (see Definition 3.30) and Lott’s higher eta invariant [1992, Sections 4.4 and 4.6]. In particular, we prove that our definition of the delocalized higher eta invariant is equal to Lott’s higher eta invariant up to a constant $\frac{1}{\sqrt{\pi}}$. The main techniques used in this section are due to Connes [1988; 1991].

Let M be a closed manifold and \tilde{M} the universal covering over M . Suppose D is a first order self-adjoint elliptic differential operator acting on a vector bundle E over M and \tilde{D} the lift of D to \tilde{M} . Suppose that $\langle h \rangle$ is a nontrivial conjugacy class of $G = \pi_1(M)$ and \tilde{D} is invertible. Throughout this section, we assume that G has polynomial growth.

Let \mathcal{B} be the following dense subalgebra of $C_r^*(G)$:

$$\mathcal{B} = \left\{ f : G \rightarrow \mathbb{C} \mid \sum_{g \in G} (1 + \ell(g))^{2k} |f(g)|^2 < \infty \quad \text{for all } k \in \mathbb{N} \right\},$$

where ℓ is a word-length function on G . \mathcal{B} is Fréchet locally m -convex algebra. Moreover, since G has polynomial growth, \mathcal{B} is a smooth dense subalgebra of $C_r^*(G)$. The universal graded differential algebra of \mathcal{B} is

$$\Omega_*(\mathcal{B}) = \bigoplus_{k=0}^{\infty} \Omega_k(\mathcal{B})$$

where as a vector space, $\Omega_k(\mathcal{B}) = \mathcal{B} \otimes (\mathcal{B})^{\otimes k}$. As \mathcal{B} is a Fréchet algebra, we consider the completion of $\Omega_*(\mathcal{B})$, which will still be denoted by $\Omega_*(\mathcal{B})$.

Let $\mathfrak{E} = (\tilde{M} \times_G \mathcal{B}) \otimes E$ be the associated \mathcal{B} -vector bundle and $C^\infty(M; \mathfrak{E})$ its space of smooth sections. Now suppose ψ is a smooth function on \tilde{M} with compact support such that

$$\sum_{g \in G} g^* \psi = 1.$$

Note that every compactly supported smooth function on \tilde{M} may be equivalently viewed an element in $C^\infty(M; \mathfrak{E})$. We have a superconnection $\nabla: C^\infty(M; \mathfrak{E}) \rightarrow C^\infty(M; \mathfrak{E} \otimes_{\mathcal{B}} \Omega_1(\mathcal{B}))$ given by

$$\nabla(f) = \sum_{g \in G} (\psi \cdot g^* f) \otimes_{\mathcal{B}} dg. \quad (8.1)$$

Definition 8.2 [Lott 1992, Sections 4.4 and 4.6]. For each $\beta > 0$, Lott's higher eta invariant $\tilde{\eta}(\tilde{D})$ is defined by the formula

$$\tilde{\eta}(\tilde{D}, \beta) = \beta^{1/2} \int_0^\infty \text{STR}(\tilde{D} e^{-\beta(t\tilde{D} + \nabla)^2}) dt.$$

Here we follow the superconnection formalism, and STR is the corresponding supertrace; see [Lott 1992, Proposition 22].

We recall the following periodic version of Lott's higher eta invariant.

Definition 8.3 [Lott 1992, Section 4.6]. Define $\tilde{\eta}(\tilde{D}) \in \Omega_*(\mathcal{B})$ to be

$$\tilde{\eta}(\tilde{D}) = \int_0^\infty e^{-\beta} \tilde{\eta}(\tilde{D}, \beta^2) d\beta.$$

Similar estimates as those in Section 3 show that, under the assumption G has polynomial growth, the above integral converges in $\Omega_*(\mathcal{B})$, hence $\tilde{\eta}(\tilde{D})$ is well-defined.

Let us write

$$\tilde{\eta}(\tilde{D}) = \sum_{m \geq 0} \tilde{\eta}_m(\tilde{D}) = \sum_{m \geq 0} \int_0^\infty e^{-\beta} \tilde{\eta}_{2m}(\tilde{D}, \beta^2) d\beta,$$

where $\tilde{\eta}_{2m}(\tilde{D})$ and $\tilde{\eta}_{2m}(\tilde{D}, \beta^2)$ are the $2m$ -th components of $\tilde{\eta}(\tilde{D})$ and $\tilde{\eta}(\tilde{D}, \beta^2)$ in $\Omega_{2m}(\mathcal{B})$ respectively.

For each $m \geq 0$, only a finite number of terms in the Duhamel expansion of $\tilde{D}e^{-\beta^2(t\tilde{D}+\nabla)^2}$ will contribute to $\tilde{\eta}_{2m}(\tilde{D}, \beta^2)$. Suppose $\varphi \in C^{2m}(G, \langle h \rangle)$ is a cyclic cocycle with polynomial growth. Without loss of generality, we assume that φ is normalized, that is,

$$\varphi(g_0, g_1, \dots, g_{2k}) = 0 \quad \text{if } g_i = 1 \text{ for some } i \geq 1.$$

Let us consider the pairing $\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle$. Observe that, since $\tilde{\eta}_{2m}(\tilde{D}, \beta^2)$ is paired with φ , we can relax the smoothness condition on ψ in the definition of the connection ∇ above and choose ψ to be the characteristic function of a fundamental domain of \tilde{M} under the action G . More precisely, for such a choice of ψ , we should treat the summands $t\nabla\tilde{D}$ and $t\tilde{D}\nabla$ in the supercommutator $[\nabla, t\tilde{D}] = t(\nabla\tilde{D} + \tilde{D}\nabla)$ separately so that we avoid taking the differential of ψ . As we have seen in Section 3, under the assumption that \tilde{D} has sufficiently large spectral gap in the sense of Definitions 3.7 and 3.42, the pairing

$$\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle$$

converges absolutely for any choice of the smooth cutoff function ψ that appears in line (8.1). It is also not difficult to see that the value $\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle$ is independent of the choice of such a smooth cutoff function ψ . Now by choosing a sequence of such smooth cutoff functions that approximate the characteristic function of a fundamental domain of \tilde{M} , it follows that, under the same sufficiently large spectral gap assumption, the pairing $\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle$ still makes sense and takes the same value even if we choose ψ to be the characteristic function of a fundamental domain. Moreover, observe that $\nabla^2: C^\infty(M; \mathfrak{E}) \rightarrow C^\infty(M; \mathfrak{E} \otimes_{\mathcal{B}} \Omega_2(\mathcal{B}))$ given by

$$\nabla^2(f) = \sum_{g_1 \in G} \sum_{g_2 \in G} \psi \cdot g_1^*(\psi \cdot g_2^*f) \otimes_{\mathcal{B}} dg_1 \otimes_{\mathcal{B}} dg_2$$

which becomes

$$\sum_{g_2 \in G} (\psi \cdot g_2^*f) \otimes_{\mathcal{B}} de \otimes_{\mathcal{B}} dg_2$$

if ψ is the characteristic function of a fundamental domain of \tilde{M} , where $e = 1$ stands for the identity element of G . Since φ is a normalized cyclic cocycle, it follows that the term ∇^2 does not contribute to the pairing $\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle$ when ψ is the characteristic function of a fundamental domain of \tilde{M} . To summarize, we

have

$$\begin{aligned}
 & \langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle \\
 &= \int_0^\infty \beta^{2m+1} \int_{(\sum_{j=0}^{2m} s_j) = \beta} \langle \varphi, \text{STR}(\tilde{D} e^{-s_0 \beta t^2 \tilde{D}^2} [\nabla, t \tilde{D}] e^{-s_1 \beta t^2 \tilde{D}^2} [\nabla, t \tilde{D}] \cdots \\
 & \quad \times [\nabla, t \tilde{D}] e^{-s_{2m} \beta t^2 \tilde{D}^2}) \rangle ds_0 ds_1 \cdots ds_{2m} dt \\
 &= \int_0^\infty \beta^{m+1/2} \int_{(\sum_{j=0}^{2m} s_j) = \beta} \langle \varphi, \text{STR}(\tilde{D} e^{-s_0 t^2 \tilde{D}^2} [\nabla, t \tilde{D}] e^{-s_1 t^2 \tilde{D}^2} [\nabla, t \tilde{D}] \cdots \\
 & \quad \times [\nabla, t \tilde{D}] e^{-s_{2m} t^2 \tilde{D}^2}) \rangle ds_0 ds_1 \cdots ds_{2m} dt,
 \end{aligned}$$

where the second equality follows from the change of variables $t \mapsto \sqrt{\beta}t$. Here $[\nabla, t \tilde{D}]$ is the supercommutator (i.e., graded-commutator) of ∇ and $t \tilde{D}$.

Let $f_0(x) = \tilde{D} e^{-x t^2 \tilde{D}^2}$ and $f_j(x) = [\nabla, t \tilde{D}] e^{-x t^2 \tilde{D}^2}$ for $j > 0$. From the above calculation, we see that

$$\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle = \beta^{m+1/2} \int_0^\infty \langle \varphi, \text{STR}(f_0 * f_1 * \cdots * f_{2m}(\beta)) \rangle dt,$$

where $*$ stands for the convolution

$$(f * h)(\beta) = \int_0^\beta f(x) h(\beta - x) dx.$$

Recall that the Laplace transform

$$f \mapsto \mathcal{L}(f)(s) = \int_0^\infty e^{-s\beta} f(\beta) d\beta$$

converts convolutions of functions into pointwise products of functions. Let us define

$$H(s) := \left\langle \varphi, \text{STR} \left(\prod_{j=0}^{2m} \mathcal{L}(f_j)(s) \right) \right\rangle,$$

which is the Laplace transform of

$$\int_0^\infty \langle \varphi, \text{STR}(f_0 * f_1 * \cdots * f_{2m}(\beta)) \rangle dt.$$

Recall that

$$\mathcal{L}(e^{-a\beta})(s) = \frac{1}{a + s}.$$

Therefore, we have

$$\begin{aligned} H(s) &= \int_0^\infty \left\langle \varphi, \text{STR} \left(\frac{\tilde{D}}{t^2 \tilde{D}^2 + s} [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + s} \cdots [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + s} \right) \right\rangle dt \\ &= s^{-m-1/2} \int_0^\infty \left\langle \varphi, \text{STR} \left(\frac{\tilde{D}}{t^2 \tilde{D}^2 + 1} [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \cdots [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \right) \right\rangle dt \\ &= s^{-m-1/2} H(1) \end{aligned}$$

where the second equality follows from the change of variables $t \rightarrow t/\sqrt{s}$. Apply the inverse Laplace transform to $H(s)$ and we obtain

$$\int_0^\infty \langle \varphi, \text{STR}(f_0 * f_1 * \cdots * f_{2m}(\beta)) \rangle dt = \frac{\beta^{m-1/2}}{\Gamma(m+1/2)} H(1)$$

where $\Gamma(m + \frac{1}{2}) = \frac{1}{4^m} \frac{(2m)!}{m!} \sqrt{\pi}$. It follows that

$$\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle = \frac{\beta^{2m}}{\Gamma(m+1/2)} H(1).$$

Hence we have

$$\begin{aligned} &\langle \varphi, \tilde{\eta}_{2m}(\tilde{D}) \rangle \\ &= \int_0^\infty \langle \varphi, e^{-\beta} \tilde{\eta}_{2m}(\tilde{D}, \beta^2) \rangle d\beta \\ &= \frac{4^m m!}{\sqrt{\pi}} \int_0^\infty \left\langle \varphi, \text{STR} \left(\frac{\tilde{D}}{t^2 \tilde{D}^2 + 1} [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \cdots [\nabla, t \tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \right) \right\rangle dt. \end{aligned}$$

We shall identify this formula with our formula for delocalized higher eta invariant in Definition 3.30. To this end, let us first define

$$w_t(x) := \frac{tx - i}{tx + i}.$$

Note that the path $w_{1/t}(\tilde{D})$ is a representative of the higher rho invariant $\rho(\tilde{D})$. A direct computation shows that

$$\dot{w}_t(\tilde{D}) w_t(\tilde{D})^{-1} = \frac{2i \tilde{D}}{t^2 \tilde{D}^2 + 1}$$

and

$$[w_t(\tilde{D}), \nabla] \cdot [w_t(\tilde{D})^{-1}, \nabla] = 4(t \tilde{D} + i)^{-1} [\nabla, t \tilde{D}] (t^2 \tilde{D}^2 + 1)^{-1} [\nabla, t \tilde{D}] (t \tilde{D} - i)^{-1},$$

where $[w_t(\tilde{D}), \nabla]$ and $[w_t(\tilde{D})^{-1}, \nabla]$ are the usual ungraded commutator.¹³ For notational simplicity, let us write w_t in place of $w_t(\tilde{D})$. The above computation

¹³In the superconnection formalism of Definition 8.2 above, the relevant bundle has been “doubled”. In other words, the superconnection is formulated on a superbundle, which is the direct sum of two

implies that

$$\begin{aligned} & \operatorname{tr}(\dot{w}_t w_t^{-1} ([w_t, \nabla] [w_t^{-1}, \nabla])^m) \\ &= 4^m \operatorname{tr} \left(\frac{2i\tilde{D}}{t^2 \tilde{D}^2 + 1} \left(\frac{1}{t\tilde{D} + i} [\nabla, t\tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} [\nabla, t\tilde{D}] \frac{1}{t\tilde{D} - i} \right)^m \right) \\ &= 4^m \operatorname{tr} \left(\frac{2i\tilde{D}}{t^2 \tilde{D}^2 + 1} [\nabla, t\tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \cdots [\nabla, t\tilde{D}] \frac{1}{t^2 \tilde{D}^2 + 1} \right). \end{aligned}$$

Since φ is normalized, again ∇^2 does not contribute to the pairing

$$\langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} ([w_t, \nabla] [w_t^{-1}, \nabla])^m) \rangle.$$

It follows that

$$\begin{aligned} & \langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} ([w_t, \nabla] [w_t^{-1}, \nabla])^m) \rangle \\ &= \langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} (\nabla w_t \nabla w_t^{-1})^m) \rangle + \langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} (w_t \nabla w_t^{-1} \nabla)^m) \rangle, \end{aligned}$$

when $m \geq 1$. By the definition of the connection ∇ and the definition of the trace, we have

$$\langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} (\nabla w_t \nabla w_t^{-1})^m) \rangle = \widetilde{\varphi\#} \operatorname{tr}(\dot{w}_t w_t^{-1} \otimes (w_t \otimes w_t^{-1})^{\otimes m}).$$

On the other hand, we have

$$\langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} (w_t \nabla w_t^{-1} \nabla)^m) \rangle = \langle \varphi, \operatorname{tr}(\dot{w}_t (\nabla w_t^{-1} \nabla w_t)^{m-1} \nabla w_t^{-1} \nabla) \rangle = 0.$$

Therefore, for all $m \geq 0$, we have

$$\begin{aligned} \int_0^\infty \langle \varphi, \operatorname{tr}(\dot{w}_t w_t^{-1} ([w_t, \nabla] [w_t^{-1}, \nabla])^m) \rangle dt &= \int_0^\infty \widetilde{\varphi\#} \operatorname{tr}(\dot{w}_t w_t^{-1} \otimes (w_t \otimes w_t^{-1})^{\otimes m}) dt \\ &= \frac{\pi i}{m!} \tau_\varphi(w), \end{aligned}$$

where τ_φ is the map from Definition 6.10. Now Lemma 8.5 below proves the convergence of $\tau_\varphi(w)$. To summarize, we have established the following precise relation between our definition of the delocalized higher eta invariant (see Definition 3.30) and Lott's higher eta invariant [1992, Section 4.4 and 4.6].

Proposition 8.4. *Suppose D is a first order self-adjoint elliptic differential operator acting on a vector bundle E over M and \tilde{D} the lift of D to \tilde{M} . Assume that $\langle h \rangle$ is a nontrivial conjugacy class of $G = \pi_1(M)$, \tilde{D} is invertible and $G = \pi_1(M)$ has polynomial growth. Then for any $[\varphi] \in HC^*(\mathbb{C}G, \langle h \rangle)$, we have*

$$\tau_\varphi(\rho(\tilde{D})) = \frac{1}{\sqrt{\pi}} \langle \varphi, \tilde{\eta}_{2m}(\tilde{D}) \rangle.$$

copies of the original bundle together with the obvious $\mathbb{Z}/2$ -grading. On the other hand, the operator $w_t(\tilde{D})$ is defined on the original bundle, instead of the superbundle.

In the remaining part of this section, we prove Lemma 8.5 below and hence complete the proof of the above proposition. The proof of Lemma 8.5 uses the technical assumption that $G = \pi_1(M)$ has polynomial growth. It remains an open question how to identify our formulation of higher eta invariants with Lott's higher eta invariant in general.

Lemma 8.5. *With the same notation as above, the following integral*

$$\int_0^\infty \widetilde{\varphi\#} \operatorname{tr}(\dot{w}_t w_t^{-1} \otimes (w_t \otimes w_t^{-1})^{\otimes m}) dt \quad (8.6)$$

converges absolutely.

Proof. let $a_t(x) = w_t(x) - 1 = -2i(tx + i)^{-1}$. For each $t > 0$, the Fourier transform of a_t is

$$\widehat{a}_t(\xi) = -\frac{2}{t} e^{\xi/t} \theta(\xi),$$

where θ is the characteristic function of the interval $(-\infty, 0)$. The function \widehat{a}_t and all its derivatives are smooth away from $\xi = 0$ and decay exponentially as $|\xi| \rightarrow \infty$. It follows from the proof of Lemma 3.16 that the Schwartz kernel of $w_t(\widetilde{D}) - 1$ is smooth away from the diagonal of $\widetilde{M} \times \widetilde{M}$. The same holds for $w_t^{-1} - 1$ and $\dot{w}_t w_t^{-1}$. Similar arguments to the proofs of Propositions 3.46 and 3.48 show that $\widetilde{\varphi\#} \operatorname{tr}(\dot{w}_t w_t^{-1} \otimes (w_t \otimes w_t^{-1})^{\otimes m})$ is finite for each $t > 0$ and furthermore the integral in line (8.6) converges absolutely for small t .

Now we prove the integral in line (8.6) converges absolutely for large t . Since G acts freely and cocompactly on \widetilde{M} , there exists a constant $\varepsilon > 0$ such that $\operatorname{dist}(x, gx) > \varepsilon$ for all $x \in \widetilde{M}$ and all $g \neq e \in G$. Fix a point $x_0 \in \widetilde{M}$. For $x \in \widetilde{M}$, let $v(x)$ be a smooth approximation of the distance from x to x_0 . More precisely, let v be a smooth function on \widetilde{M} satisfying the following:

- (1) $\operatorname{dist}(x, x_0) \leq v(x) \leq 2 \operatorname{dist}(x, x_0)$ if $\operatorname{dist}(x, x_0) \geq \varepsilon$.
- (2) v has uniformly bounded derivatives up to order N with N sufficiently large.

Let δ be the unbound derivation on $C^*(\widetilde{M})^G$ defined by

$$\delta(T) := [T, v] = T \circ v - v \circ T$$

for $T \in C^*(\widetilde{M})^G$. If T admits a distributional Schwartz kernel which is smooth away from the diagonal of $\widetilde{M} \times \widetilde{M}$, then we have

$$\delta^k(T)(x, y) = T(x, y)(v(x) - v(y))^k,$$

for all $(x, y) \in \widetilde{M} \times \widetilde{M}$.

Denote $A_t = w_t(\widetilde{D}) - 1$. We have

$$\delta(A_t) = 2it \frac{1}{t\widetilde{D} + i} [\widetilde{D}, v] \frac{1}{t\widetilde{D} + i}.$$

Since ν has uniformly bounded derivatives and \tilde{D} is invertible, there exists $C_1 > 0$ such that

$$\|\tilde{D}\delta(A_t)\tilde{D}\|_{\text{op}} \leq \frac{C_1}{t},$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm. By induction, we see that there exists $C_2 > 0$ such that

$$\sup_{k+j \leq N+1} \|\tilde{D}^k \delta^N(A_t) \tilde{D}^j\|_{\text{op}} \leq \frac{C_2}{t}.$$

Let $K_t(x, y)$ be the distributional Schwartz kernel of A_t . Then the Schwartz kernel of $\delta^N(A_t)$ is

$$K_t(x, y)(\nu(y) - \nu(x))^N$$

for all $x, y \in \tilde{M}$. It follows from Lemma 3.11 that there exists $C_2 > 0$ such that

$$|K_t(x, y)| \cdot |\nu(y) - \nu(x)|^N \leq \frac{C_2}{t}$$

for all $x, y \in \tilde{M}$, where $|K_t(x, y)|$ is the norm of the matrix $K_t(x, y)$. In particular, if $(x, y) = (x_0, gx_0)$ with $g \neq e$, then we have

$$|K_t(x_0, gx_0)| \leq \frac{C_2}{t \cdot (\text{dist}(x_0, gx_0))^N}.$$

Now for each $x \in \tilde{M}$, use a smooth approximation of the distance function centered at x and apply the same estimates above. Since the action of G on \tilde{M} is cocompact, we may choose C_2 so that

$$|K_t(x, gx)| \leq \frac{C_2}{t \cdot (\text{dist}(x, gx))^N}$$

for all $x \in \mathcal{F}$ and all $g \neq e \in G$, where \mathcal{F} is a fundamental domain of \tilde{M} under the action of G . Similar estimates hold for the Schwartz kernels of $w_t^{-1} - 1$ and $\dot{w}_t w_t^{-1}$.

By assumption, G has polynomial growth and φ is a delocalized cyclic cocycle with polynomial growth. A straightforward computation shows that there exists a constant $C > 0$ such that

$$|\varphi \# \text{tr}(\dot{w}_t w_t^{-1} \otimes (w_t \otimes w_t^{-1})^{\otimes m})| \leq \frac{C}{t^{2m+2}},$$

which implies that the integral in line (8.6) converges absolutely for large t . This finishes the proof. \square

References

- [Baum and Douglas 1982] P. Baum and R. G. Douglas, “ K homology and index theory”, pp. 117–173 in *Operator algebras and applications, Part 1* (Kingston, Ont., 1980), Proc. Sympos. Pure Math. **38**, Amer. Math. Soc., Providence, RI, 1982. MR Zbl
- [Bernat et al. 1972] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Raïs, P. Renouard, and M. Vergne, *Représentations des groupes de Lie résolubles*, Monographies de la Société Mathématique de France **4**, Dunod, Paris, 1972. MR Zbl
- [Bismut and Cheeger 1989] J.-M. Bismut and J. Cheeger, “ η -invariants and their adiabatic limits”, *J. Amer. Math. Soc.* **2**:1 (1989), 33–70. MR Zbl
- [Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundle Math. Wissen. **319**, Springer, 1999. MR Zbl
- [Burghelaea 1985] D. Burghelaea, “The cyclic homology of the group rings”, *Comment. Math. Helv.* **60**:3 (1985), 354–365. MR Zbl
- [Cheeger and Gromov 1985] J. Cheeger and M. Gromov, “Bounds on the von Neumann dimension of L^2 -cohomology and the Gauss–Bonnet theorem for open manifolds”, *J. Differential Geom.* **21**:1 (1985), 1–34. MR Zbl
- [Chen et al. 2019] X. Chen, J. Wang, Z. Xie, and G. Yu, “Delocalized eta invariants, cyclic cohomology and higher rho invariants”, preprint, 2019. arXiv 1901.02378
- [Connes 1985] A. Connes, “Noncommutative differential geometry”, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360. MR Zbl
- [Connes 1988] A. Connes, “Entire cyclic cohomology of Banach algebras and characters of θ -summable Fredholm modules”, *K-Theory* **1**:6 (1988), 519–548. MR Zbl
- [Connes 1991] A. Connes, “On the Chern character of θ summable Fredholm modules”, *Comm. Math. Phys.* **139**:1 (1991), 171–181. MR Zbl
- [Connes 1994] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994. MR Zbl
- [Deeley and Goffeng 2016] R. J. Deeley and M. Goffeng, “Realizing the analytic surgery group of Higson and Roe geometrically part III: higher invariants”, *Math. Ann.* **366**:3-4 (2016), 1513–1559. MR Zbl
- [Donnelly 1978] H. Donnelly, “Eta invariants for G -spaces”, *Indiana Univ. Math. J.* **27**:6 (1978), 889–918. MR Zbl
- [Getzler 1986] E. Getzler, “A short proof of the local Atiyah–Singer index theorem”, *Topology* **25**:1 (1986), 111–117. MR Zbl
- [Getzler 1993] E. Getzler, “Cyclic homology and the Atiyah–Patodi–Singer index theorem”, pp. 19–45 in *Index theory and operator algebras* (Boulder, CO, 1991), edited by J. Fox and P. Haskell, Contemp. Math. **148**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl
- [Gorokhovskiy et al. 2016] A. Gorokhovskiy, H. Moriyoshi, and P. Piazza, “A note on the higher Atiyah–Patodi–Singer index theorem on Galois coverings”, *J. Noncommut. Geom.* **10**:1 (2016), 265–306. MR Zbl
- [Gromov 1987] M. Gromov, “Hyperbolic groups”, pp. 75–263 in *Essays in group theory*, edited by S. M. Gersten, Math. Sci. Res. Inst. Publ. **8**, Springer, 1987. MR Zbl
- [de la Harpe 1988] P. de la Harpe, “Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint”, *C. R. Acad. Sci. Paris Sér. I Math.* **307**:14 (1988), 771–774. MR Zbl
- [Higson and Roe 2005a] N. Higson and J. Roe, “Mapping surgery to analysis, I: Analytic signatures”, *K-Theory* **33**:4 (2005), 277–299. MR Zbl

- [Higson and Roe 2005b] N. Higson and J. Roe, “Mapping surgery to analysis, II: Geometric signatures”, *K-Theory* **33**:4 (2005), 301–324. MR Zbl
- [Higson and Roe 2005c] N. Higson and J. Roe, “Mapping surgery to analysis, III: Exact sequences”, *K-Theory* **33**:4 (2005), 325–346. MR Zbl
- [Jaffe et al. 1988] A. Jaffe, A. Lesniewski, and K. Osterwalder, “Quantum K -theory, I: The Chern character”, *Comm. Math. Phys.* **118**:1 (1988), 1–14. MR Zbl
- [John 2022] S. A. K. A. John, “Secondary higher invariants and cyclic cohomology for groups of polynomial growth”, *J. Noncommut. Geom.* **16**:4 (2022), 1283–1335. MR Zbl
- [Jolissaint 1990] P. Jolissaint, “Rapidly decreasing functions in reduced C^* -algebras of groups”, *Trans. Amer. Math. Soc.* **317**:1 (1990), 167–196. MR Zbl
- [Keswani 1999] N. Keswani, “Geometric K -homology and controlled paths”, *New York J. Math.* **5** (1999), 53–81. MR Zbl
- [Lafforgue 2000] V. Lafforgue, “A proof of property (RD) for cocompact lattices of $SL(3, \mathbb{R})$ and $SL(3, \mathbb{C})$ ”, *J. Lie Theory* **10**:2 (2000), 255–267. MR Zbl
- [Lafforgue 2002] V. Lafforgue, “ K -théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes”, *Invent. Math.* **149**:1 (2002), 1–95. MR Zbl
- [Lafforgue 2012] V. Lafforgue, “La conjecture de Baum–Connes à coefficients pour les groupes hyperboliques”, *J. Noncommut. Geom.* **6**:1 (2012), 1–197. MR Zbl
- [Leichtnam and Piazza 1997] E. Leichtnam and P. Piazza, “The b -pseudodifferential calculus on Galois coverings and a higher Atiyah–Patodi–Singer index theorem”, *Mém. Soc. Math. Fr. (N.S.)* **68** (1997), iv+121. MR Zbl
- [Leichtnam and Piazza 1999] E. Leichtnam and P. Piazza, “Homotopy invariance of twisted higher signatures on manifolds with boundary”, *Bull. Soc. Math. France* **127**:2 (1999), 307–331. MR Zbl
- [Lott 1992] J. Lott, “Higher eta-invariants”, *K-Theory* **6**:3 (1992), 191–233. MR Zbl
- [Lott 1999] J. Lott, “Delocalized L^2 -invariants”, *J. Funct. Anal.* **169**:1 (1999), 1–31. MR Zbl
- [Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics **4**, A K Peters, Ltd., Wellesley, MA, 1993. MR Zbl
- [Meyer 2006] R. Meyer, “Combable groups have group cohomology of polynomial growth”, *Q. J. Math.* **57**:2 (2006), 241–261. MR Zbl
- [Mineyev 2001] I. Mineyev, “Straightening and bounded cohomology of hyperbolic groups”, *Geom. Funct. Anal.* **11**:4 (2001), 807–839. MR Zbl
- [Mineyev and Yu 2002] I. Mineyev and G. Yu, “The Baum–Connes conjecture for hyperbolic groups”, *Invent. Math.* **149**:1 (2002), 97–122. MR Zbl
- [Nistor 1990] V. Nistor, “Group cohomology and the cyclic cohomology of crossed products”, *Invent. Math.* **99**:2 (1990), 411–424. MR Zbl
- [Piazza and Schick 2007] P. Piazza and T. Schick, “Groups with torsion, bordism and rho invariants”, *Pacific J. Math.* **232**:2 (2007), 355–378. MR Zbl
- [Piazza and Schick 2014] P. Piazza and T. Schick, “Rho-classes, index theory and Stolz’ positive scalar curvature sequence”, *J. Topol.* **7**:4 (2014), 965–1004. MR Zbl
- [Piazza et al. 2019] P. Piazza, T. Schick, and V. F. Zenobi, “Higher rho numbers and the mapping of analytic surgery to homology”, preprint, 2019. arXiv 1905.11861v1
- [Puschnigg 2010] M. Puschnigg, “New holomorphically closed subalgebras of C^* -algebras of hyperbolic groups”, *Geom. Funct. Anal.* **20**:1 (2010), 243–259. MR Zbl

- [Quillen 1985] D. Quillen, “Superconnections and the Chern character”, *Topology* **24**:1 (1985), 89–95. MR Zbl
- [Roe 1993] J. Roe, “Coarse cohomology and index theory on complete Riemannian manifolds”, pp. x+90 in *Mem. Amer. Math. Soc.* **497**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl
- [Roe 1996] J. Roe, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics **90**, American Mathematical Society, Providence, RI, 1996. MR Zbl
- [Wahl 2013] C. Wahl, “The Atiyah–Patodi–Singer index theorem for Dirac operators over C^* -algebras”, *Asian J. Math.* **17**:2 (2013), 265–319. MR Zbl
- [Weinberger et al. 2021] S. Weinberger, Z. Xie, and G. Yu, “Additivity of higher rho invariants and nonrigidity of topological manifolds”, *Comm. Pure Appl. Math.* **74**:1 (2021), 3–113. MR Zbl
- [Xie and Yu 2014] Z. Xie and G. Yu, “Positive scalar curvature, higher rho invariants and localization algebras”, *Adv. Math.* **262** (2014), 823–866. MR Zbl
- [Xie and Yu 2017] Z. Xie and G. Yu, “Higher rho invariants and the moduli space of positive scalar curvature metrics”, *Adv. Math.* **307** (2017), 1046–1069. MR Zbl
- [Xie and Yu 2021] Z. Xie and G. Yu, “Delocalized eta invariants, algebraicity, and K -theory of group C^* -algebras”, *Int. Math. Res. Not.* **2021**:15 (2021), 11731–11766. MR Zbl
- [Yu 1997] G. Yu, “Localization algebras and the coarse Baum–Connes conjecture”, *K-Theory* **11**:4 (1997), 307–318. MR Zbl
- [Zeidler 2016] R. Zeidler, “Positive scalar curvature and product formulas for secondary index invariants”, *J. Topol.* **9**:3 (2016), 687–724. MR Zbl

Received 3 Apr 2023. Accepted 10 Sep 2023.

XIAOMAN CHEN: xchen@fudan.edu.cn

School of Mathematical Sciences, Fudan University, Shanghai, China

JINMIN WANG: wangjm15@fudan.edu.cn

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China

ZHIZHANG XIE: xie@tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX, United States

GUOLIANG YU: guoliangyu@tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX, United States

ANNALS OF K-THEORY

msp.org/akt

EDITORIAL BOARD

Joseph Ayoub	(Managing Editor) Universität Zürich Zürich, Switzerland joseph.ayoub@math.uzh.ch
Paul Balmer	University of California, Los Angeles, USA balmer@math.ucla.edu
Guillermo Cortiñas	(Managing Editor) Universidad de Buenos Aires and CONICET, Argentina gcorti@dm.uba.ar
Hélène Esnault	Freie Universität Berlin, Germany liveesnault@math.fu-berlin.de
Eric Friedlander	University of Southern California, USA ericmf@usc.edu
Max Karoubi	Institut de Mathématiques de Jussieu – Paris Rive Gauche, France max.karoubi@imj-prg.fr
Moritz Kerz	Universität Regensburg, Germany moritz.kerz@mathematik.uni-regensburg.de
Huaxin Lin	University of Oregon, USA livehlin@uoregon.edu
Alexander Merkurjev	University of California, Los Angeles, USA merkurev@math.ucla.edu
Birgit Richter	Universität Hamburg, Germany birgit.richter@uni-hamburg.de
Jonathan Rosenberg	University of Maryland, USA jmr@math.umd.edu
Marco Schlichting	University of Warwick, UK m.schlichting@warwick.ac.uk
Charles Weibel	Rutgers University, USA weibel@math.rutgers.edu
Guoliang Yu	Texas A&M University, USA guoliangyu@math.tamu.edu

PRODUCTION

Silvio Levy	(Scientific Editor) production@msp.org
-------------	---

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2023 is US \$570/year for the electronic version, and \$640/year (+\$30, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

AKT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY
 **mathematical sciences publishers**
nonprofit scientific publishing
<http://msp.org/>
© 2023 Mathematical Sciences Publishers

ANNALS OF K-THEORY

2023

vol. 8

no. 4

Delocalized eta invariants, cyclic cohomology and higher rho invariants 531

XIAOMAN CHEN, JINMIN WANG, ZHIZHANG XIE and
GUOLIANG YU

The fundamental theorem of localizing invariants 609

VICTOR SAUNIER

Riemannian embeddings in codimension one as unbounded KK -cycles 645

WALTER D. VAN SUIJLEKOM and LUUK S. VERHOEVEN

Braid groups and symplectic Steinberg groups 669

FRANÇOIS DIGNE and CHRISTIAN KASSEL