

## Large Inflationary Logarithms in a Nontrivial Nonlinear Sigma Model

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### ABSTRACT

Loops of inflationary gravitons are known to induce large temporal and spatial logarithms which can cause perturbation theory to break down. Nonlinear sigma models possess the same kind of derivative interactions and induce the same sorts of large logarithms, without the complicated index structure and potential gauge problem. Previous studies have examined models with zero field space curvature which can be reduced to free field theories by local, invertible field redefinitions. Here we study a model which cannot be so reduced and still shows the same sorts of large logarithms. We compute the evolution of the background at 1-loop and 2-loop orders, and we find the 1-loop  $\beta$  and  $\gamma$  functions.

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# 1 Introduction

The background geometry of cosmology is characterized by scale factor  $a(t)$ , Hubble parameter  $H(t)$  and first slow roll parameter  $\epsilon(t)$ ,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x} \quad \implies \quad H(t) \equiv \frac{\dot{a}}{a} , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} . \quad (1)$$

The accelerated expansion of inflation ( $H(t) > 0$  with  $0 \leq \epsilon(t) < 1$ ) rips virtual scalars and gravitons out of the vacuum. This is the basis for the primordial spectra of scalars [1] and tensors [2]. At some level these quanta must interact among themselves and with other particles. Gravitons are especially interesting because their couplings are universal and because their tensor structure allows them to mediate effects which scalars cannot. For example, on de Sitter background ( $\epsilon(t) = 0$ ) one loop gravitons modify the plane wave mode function  $u(t, k)$  of gravitational radiation [3] and the gravitational response  $\Psi(t, r)$  to a point mass [4] to,

$$u(t, k) = u_0(t, k) \left\{ 1 + \frac{16GH^2}{3\pi} \ln^2(a) + O(G^2) \right\} , \quad (2)$$

$$\Psi(t, r) = -\frac{GM}{ar} \left\{ 1 + \frac{103G}{15\pi a^2 r^2} - \frac{8GH^2}{\pi} \ln^3(a) + O(G^2) \right\} . \quad (3)$$

Similar results have been obtained for the corrections of inflationary gravitons to fermions [5], to electrodynamics [6, 7], and to massless, minimally coupled scalars [8].

A fascinating aspect of these corrections is that the steady production of inflationary gravitons endows them with a secular growth which must eventually overwhelm the loop-counting parameter  $GH^2$  provided inflation persists long enough. One must develop a non-perturbative resummation technique in order to evolve past that point. Nonlinear sigma models provide a useful testing ground for developing such a technique because they possess the same sorts of derivative interactions as gravitons, and induce the same sorts of large logarithms, without the complicating tensor structures or the potential for gauge dependence [9–12]. A successful technique has been devised through combining a variant of Starobinsky’s stochastic formalism [13, 14] with a variant of the renormalization group [15, 16]. Even better, the technique can be generalized from de Sitter to an arbitrary cosmological background (1) which has undergone primordial inflation [17]. Applying this technique shows that the large scales of primordial inflation can be transmitted to late time [18].

The obvious next step is generalizing the resummation technique from nonlinear sigma models to quantum gravity. This seems to be entirely possible, and works for the one graviton loop correction on which it has been checked [8]. However, our purpose here is to clear up a worry concerning the sorts of nonlinear sigma models on which the resummation technique has so far been applied. Specifically, both of those models can be reduced to free theories by means of local invertible field redefinitions, which means that their flat space S-matrices are unity by Borchers Theorem [19]. That in no way precludes interesting evolutions for the scalar backgrounds, and for the 1-particle states, and it was these evolutions which suggested and confirmed the resummation technique. One might still worry that the technique only

works for models which can be reduced to free theories. The purpose of this paper is to demonstrate that the resummation technique applies for a model whose flat space  $S$ -matrix is nontrivial.

This paper contains five sections, of which this Introduction is the first. In Section 2 we describe the old and new models, and we give the Feynman rules. Section 3 computes the time-evolving background at 1-loop and 2-loop orders. It also computes and renormalizes the 1PI (one-particle-irreducible) 2-point and 3-point functions at 1-loop order. The resummation technique is applied in Section 4. Our conclusions comprise Section 5.

## 2 The Model

The purpose of this section is to define the model and give those of its Feynman rules which are required for our work. We begin by presenting the old models and explaining why their  $S$ -matrices are trivial. Then a new model is proposed by making a small variation which preserves the lowest order interactions.

The resummation technique was developed based on work with two nonlinear sigma models [15]. One of these was based on a single field. All models of this sort can be made free by a local, invertible field redefinition,

$$\mathcal{L} = -\frac{1}{2}f^2(\Phi)\partial_\mu\Phi\partial_\nu\Phi g^{\mu\nu}\sqrt{-g} \quad , \quad d\Psi \equiv f(\Phi)d\Phi \quad \Rightarrow \quad \mathcal{L} = -\frac{1}{2}\partial_\mu\Psi\partial_\nu\Psi g^{\mu\nu}\sqrt{-g} . \quad (4)$$

A second model was based on two fields,

$$\mathcal{L}_{\text{old}} = -\frac{1}{2}\partial_\mu A\partial_\nu A g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\left(1 + \frac{\lambda}{2}A\right)^2\partial_\mu B\partial_\nu B g^{\mu\nu}\sqrt{-g} . \quad (5)$$

Although it was not initially realized, this model can be also reduced to a theory of two free scalars by making a local, invertible field redefinition,<sup>1</sup>

$$X \equiv \frac{2}{\lambda}\left(1 + \frac{\lambda}{2}A\right)\cos\left(\frac{\lambda}{2}B\right) , \quad (6)$$

$$Y \equiv \frac{2}{\lambda}\left(1 + \frac{\lambda}{2}A\right)\sin\left(\frac{\lambda}{2}B\right) . \quad (7)$$

Hence the flat space  $S$ -matrices of (4) and (5) are both unity by Borchers Theorem [19]. That in no way precludes these models from manifesting interesting evolutions of their backgrounds and of the single-particle kinematics.

To be certain that the resummation technique does not rely on having a trivial  $S$ -matrix we devised a slight modification of the 2-field model (5) whose field space curvature implies that it cannot be reduced to a free theory,

$$\mathcal{L}_{\text{new}} = -\frac{1}{2}\partial_\mu A\partial_\nu A g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\left(1 + \frac{\lambda}{4}A\right)^4\partial_\mu B\partial_\nu B g^{\mu\nu}\sqrt{-g} . \quad (8)$$

The 3-point coupling in this model is identical to that of the old model (5), and the 4-point coupling has the same field content with the old numerical coefficient of  $\frac{1}{8}$  replaced by  $\frac{3}{16}$ .

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<sup>1</sup>We thank Arkady Tseytlin for this observation.

There are additional 5-point and 6-point interactions which make only simple contributions to the diagrams we evaluate in section 3,

$$\frac{1}{2} \left(1 + \frac{\lambda}{4} A\right)^4 (\partial B)^2 - \frac{1}{2} (\partial B)^2 = \frac{\lambda}{2} A (\partial B)^2 + \frac{3\lambda^2}{16} A^2 (\partial B)^2 + \frac{\lambda^3}{32} A^3 (\partial B)^2 + \frac{\lambda^4}{512} A^4 (\partial B)^2. \quad (9)$$

A diagrammatic representation of the Feynman rules is shown in Figure 1.

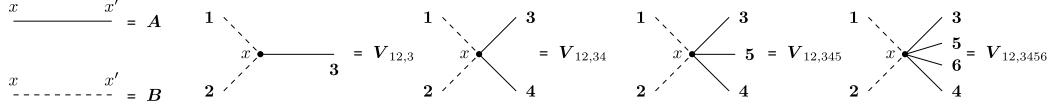


Figure 1: Primitive interactions of the bare Lagrangian (8). A lines are solid and  $B$  lines are dashed.

The new model (8) is no more renormalizable than the old one (5). Hence divergences must be subtracted, order-by-order, using BPHZ counterterms (Bogoliubov-Parasiuk-Hepp-Zimmermann [20–23]). The 1-loop counterterms we require are shown in Figure 2. The first

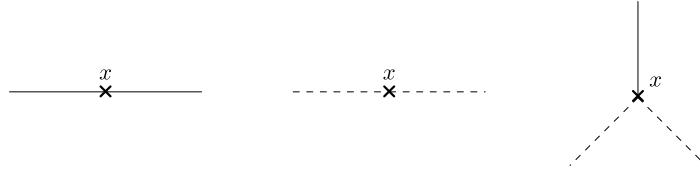


Figure 2: Diagrammatic representation of the 1-loop counterterms we require. The first two diagrams renormalize the  $A$  and  $B$  self-masses while the rightmost diagram renormalizes the vertex function. Recall that  $A$  lines are solid and  $B$  lines are dashed.

and second diagrams renormalize the  $A$  and  $B$  self-masses and correspond to the counterterm,

$$\begin{aligned} \Delta\mathcal{L}^{(2)} = & -\frac{1}{2} C_{1A^2} \square A \square A \sqrt{-g} - \frac{1}{2} C_{2A^2} R \partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} \\ & -\frac{1}{2} C_{1B^2} \square B \square B \sqrt{-g} - \frac{1}{2} C_{2B^2} R \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g}. \end{aligned} \quad (10)$$

The 3rd diagram is required to renormalize the 3-point vertex and corresponds to,

$$\begin{aligned} \delta\mathcal{L}^{(3)} = & -\frac{1}{2} C_{1AB^2} \square A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} - C_{2AB^2} \partial_\mu A \partial_\nu B \square B g^{\mu\nu} \sqrt{-g} \\ & -\frac{1}{2} C_{3AB^2} A \square B \square B \sqrt{-g} - \frac{1}{2} C_{4AB^2} R A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g}. \end{aligned} \quad (11)$$

Here  $R = D(D-1)H^2$  is the Ricci scalar. Section 3 will determine the various coefficients in expressions (10-11) as functions of  $\lambda$ ,  $D$  and the dimensional regularization scale  $\mu$ .

In  $D$  spacetime dimensions, the propagators of both fields obey the equation,

$$\partial^\mu [a^{D-2} \partial_\mu i\Delta(x; x')] \equiv \mathcal{D} i\Delta(x; x') = i\delta^D(x - x'). \quad (12)$$

The solution consists of a de Sitter invariant part plus a de Sitter breaking logarithm [24, 25],

$$i\Delta(x; x') = F(\mathcal{Y}(x; x')) + k \ln(aa') \quad , \quad k \equiv \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}. \quad (13)$$

Here  $\mathcal{Y}(x; x') \equiv aa'H^2\Delta x^2(x; x') \equiv aa'H^2(x - x')^2$  is the de-Sitter length function, and the first derivative of  $F(\mathcal{Y}(x; x'))$  obeys,

$$F'(\mathcal{Y}) = -\frac{H^{D-2}}{4(4\pi)^{D/2}} \left\{ \Gamma\left(\frac{D}{2}\right) \left(\frac{4}{\mathcal{Y}}\right)^{\frac{D}{2}} + \Gamma\left(\frac{D}{2}+1\right) \left(\frac{4}{\mathcal{Y}}\right)^{\frac{D}{2}-1} \right. \\ \left. + \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n+\frac{D}{2}+2)}{\Gamma(n+3)} \left(\frac{\mathcal{Y}}{4}\right)^{n-\frac{D}{2}+2} - \frac{\Gamma(n+D)}{\Gamma(n+\frac{D}{2}+1)} \left(\frac{\mathcal{Y}}{4}\right)^n \right] \right\}. \quad (14)$$

In dimensional regularization the coincidence limits of the propagator and its derivatives are,

$$i\Delta(x; x) = k \left[ -\pi \cot\left(\frac{D\pi}{2}\right) + 2\ln(a) \right] \quad , \quad \partial_\mu i\Delta(x; x') \Big|_{x=x'} = kHa\delta_\mu^0, \quad (15)$$

$$\partial_\mu \partial'_\nu i\Delta(x; x') \Big|_{x'=x} = -\left(\frac{D-1}{D}\right) kH^2 g_{\mu\nu} \quad , \quad \partial_\mu i\Delta(x; x) = 2kHa\delta_\mu^0. \quad (16)$$

We close by commenting on notation. Because the de Sitter metric  $g_{\mu\nu} = a^2\eta_{\mu\nu}$  is conformal to the Minkowski metric  $\eta_{\mu\nu}$ , we adopt a notation where  $\partial_\mu$  stands for  $\frac{\partial}{\partial x^\mu}$ , no matter what sort of tensor it acts upon. Further, we raise and lower its indices using the Minkowski metric,  $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$ . And we define  $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ . To save space we sometimes write coordinate arguments of the metric and its scale factor using a subscript or a superscript, as in  $\sqrt{-g(x)} \equiv \sqrt{-g_x} \equiv a_x^D$  and  $g^{\mu\nu}(y) \equiv g_y^{\mu\nu} \equiv a_y^{-2}\eta^{\mu\nu}$ . The same notation applies to derivatives, as in  $\frac{\partial}{\partial z^\mu} \equiv \partial_\mu^z$ .

## 3 Explicit Results

The purpose of this section is to carry out the same explicit calculations for (8) that were done for the old model (5) [15, 16]. We begin with the 1-loop and 2-loop expectation values of  $A(x)$ . Next, the self-masses of  $A$  and  $B$  are computed and renormalized at 1-loop order. Finally, we evaluate the 1-loop vertex function.

### 3.1 The 1-Loop and 2-Loop Background

We start with the 1-loop expectation value of  $A(x)$  whose diagrammatic representation is shown in Figure 3. Because the 3-point couplings of the old and new models agree, this

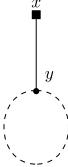


Figure 3: 1-loop contribution to the expectation value of  $A(x)$ .

diagram is unchanged from the old result [15],

$$\langle \Omega | A(x) | \Omega \rangle = \frac{\lambda H^2}{16\pi^2} \left[ \ln(a) - \frac{1}{3} + \frac{1}{3a^3} \right] + \mathcal{O}(\lambda^3).$$

(17)

The 2-loop contributions are shown in Figure 4. The first 7 diagrams (which we label

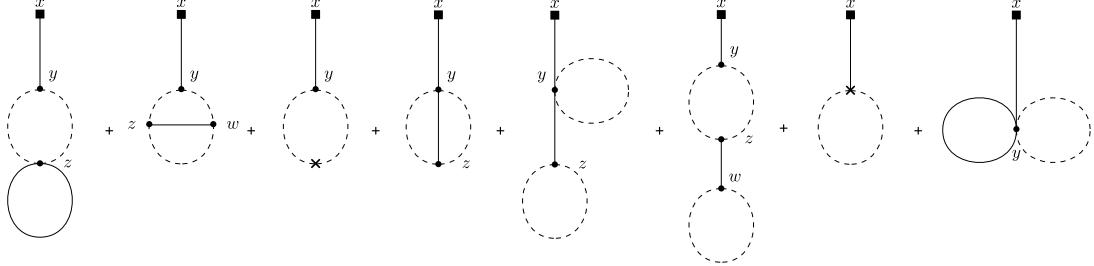


Figure 4: Contributions to the expectation value of  $A(x)$  at 2-loop order.

$A_{2a}$  through  $A_{2g}$ ) were calculated in [15]. We need only include a relative factor of  $\frac{3}{2}$  for each 4-point coupling to express their final contributions, at leading logarithm order, as multiples of the factor  $S \equiv \lambda^3 H^4 \ln^2(a)/2^{10}\pi^4$ ,

$$A_{2a} \longrightarrow -3 \cdot S, \quad A_{2b} \longrightarrow +8 \cdot S, \quad A_{2c} \longrightarrow 0 \cdot S, \quad A_{2d} \longrightarrow -6 \cdot S, \quad (18)$$

$$A_{2e} \longrightarrow +\frac{3}{2} \cdot S, \quad A_{2f} \longrightarrow -2 \cdot S, \quad A_{2g} \longrightarrow 0 \cdot S. \quad (19)$$

Only the last diagram,  $A_{2h}$  is really new. It has a symmetry factor of  $\frac{1}{4}$  and its initial expression is,

$$A_{2h} = \frac{-i}{4} \cdot \frac{3\lambda^3}{8} \cdot \int d^D y \sqrt{-g(y)} g^{\mu\nu}(y) \cdot i\Delta(x; y) \cdot i\Delta(y; y) \cdot \partial_\mu^z \partial_\nu^y i\Delta(y; z) \Big|_{y=z}. \quad (20)$$

The two coincidence limits can be read from (15-16) to give,

$$A_{2h} = \frac{3i\lambda^3}{2^5} (D-1) k H^2 I_2, \quad (21)$$

where  $I_2$ , and its leading logarithm result, was given in [18],

$$I_2 \equiv \int d^D y \sqrt{-g(y)} i\Delta(x; y) i\Delta(y; y) \longrightarrow -\frac{i}{24\pi^2} \times \ln^2(a). \quad (22)$$

This is finite and does not require renormalization, so we can set  $D = 4$ ,

$$A_{2h} \longrightarrow \frac{3}{2} \cdot S. \quad (23)$$

Adding this to the leading logarithm results for the first seven diagrams gives,

$$A_{2a} \longrightarrow -3 \cdot S, \quad A_{2b} \longrightarrow +8 \cdot S, \quad A_{2c} \longrightarrow 0 \cdot S, \quad A_{2d} \longrightarrow -6 \cdot S, \quad (24)$$

$$A_{2e} \longrightarrow +\frac{3}{2} \cdot S, \quad A_{2f} \longrightarrow -2 \cdot S, \quad A_{2g} \longrightarrow 0 \cdot S, \quad A_{2h} \longrightarrow +\frac{3}{2} \cdot S. \quad (25)$$

At leading logarithm order the eight diagrams of Figure 4 sum to zero, so our result for the expectation value of  $A$  is,

$$\langle \Omega | A(x) | \Omega \rangle = \frac{\lambda H^2 \ln(a)}{16\pi^2} [1 + 0] + \mathcal{O}(\lambda^4).$$

### 3.2 The 1-Loop Self-Masses

We now move on to the 1-loop self-masses. Absorbing the divergences of these reveals curvature-dependent field strength renormalizations  $Z_A$  and  $Z_B$  from the terms proportional to  $C_{2A^2}$  and  $C_{2B^2}$  in expression (10). These give the  $\gamma$  functions that will be used in the Renormalization Group (RG) analysis of Section 4.

The 1-loop contributions to  $-iM_A^2(x; x')$  are shown in Figure 5. The first diagram is



Figure 5: 1-loop contributions to the  $A$  self mass  $-iM_A^2(x; x')$ .

unchanged from the old model, and the second diagram is just  $\frac{3}{2}$  times the previous result [5],

$$\begin{aligned} -iM_{A_3}^2(x; x') &= \frac{(-i\lambda)^2}{2} \left\{ \frac{1}{4} \mathcal{D} \mathcal{D}' \left[ i\Delta(x; x') \right]^2 \right. \\ &\quad \left. - \frac{1}{2} \mathcal{D} \left[ i\Delta(x; x') i\delta^D(x-x') \right] - kH a^{D-1} \partial_0 i\delta^D(x-x') \right\}, \quad (27) \end{aligned}$$

$$-iM_{A_4}^2(x; x') = -\frac{i3\lambda^2}{8} \delta^D(x-x') \cdot -(D-1)kH^2 a^D. \quad (28)$$

The counterterm is  $i$  times the second variation of (10),

$$-iM_{Ac}^2(x; x') = -C_{1A^2} \mathcal{D} \mathcal{D}' \left[ \frac{i\delta^D(x-x')}{(aa')^{\frac{D}{2}}} \right] + C_{2A^2} \partial^\mu \left[ Ra^{D-2} \partial_\mu i\delta^D(x-x') \right]. \quad (29)$$

Because only (27) diverges in dimensional regularization, and this diagram is identical to the old model, so too are the coefficients  $C_{A_1}$  and  $C_{A_2}$ ,

$$C_{1A^2} = -\frac{\lambda^2 \mu^{D-4}}{32\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}, \quad C_{2A^2} = \frac{\lambda^2 \mu^{D-4}}{4(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)}. \quad (30)$$

We move to the  $B$  self-mass, whose 1-loop contributions are shown in Figure 6. As before,



Figure 6: 1-loop contributions to the  $B$  self-mass  $-iM_B^2(x; x')$ .

the 3-point diagram is unchanged from the old model and the 4-point diagram is just the result from the old model times  $\frac{3}{2}$ ,

$$-iM_{B_3}^2 = \frac{(i\lambda)^2}{2} \mathcal{D} \mathcal{D}' \left[ i\Delta(x; x') \right]^2 - (i\lambda)^2 \partial^\mu \partial'^\rho \left[ (aa')^{D-2} \partial_\mu i\Delta(x; x') \partial'_\rho i\Delta(x; x') \right] \quad (31)$$

$$-iM_{B_4}^2 = -\frac{3\lambda^2 k\pi \cot(\frac{D\pi}{2})\mathcal{D}}{8} [i\delta^D(x-x')] + \frac{3\lambda^2 H^2 \partial^\mu}{32\pi^2} [\ln(a)a^2 \partial_\mu i\delta^4(x-x')] + \mathcal{O}(D-4). \quad (32)$$

The counterterm is  $i$  times the second variation of (10) with respect to  $B$ ,

$$-iM_{B_c}^2(x; x') = -C_{1B^2}\mathcal{D}\mathcal{D}' \left[ \frac{i\delta^D(x-x')}{(aa')^{\frac{D}{2}}} \right] + C_{2B^2}\partial^\mu \left[ Ra^{D-2}\partial_\mu i\delta^D(x-x') \right]. \quad (33)$$

Multiplying the divergences from (32) by  $\frac{3}{2}$  and combining with those from (31) gives,

$$C_{1B^2} = -\frac{\lambda^2 \mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}, \quad (34)$$

$$C_{2B^2} = \frac{3\lambda^2 \mu^{D-4}}{8(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2 \mu^{D-4}}{32\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \frac{D-2}{D-1}. \quad (35)$$

### 3.3 The 1-Loop Vertex Function

In this subsection we first isolate the primitive divergences of the 3-point vertex  $-iV(x; y; z)$  at 1-loop order. These divergences are removed by the counterterm (11), which determines the coefficients  $C_{1AB^2}$ ,  $C_{2AB^2}$ ,  $C_{3AB^2}$  and  $C_{4AB^2}$ . We regard the  $C_{4AB^2}$  counterterm as a curvature-dependent renormalization of the bare 3-point coupling and infer the corresponding beta function as,

$$\delta\lambda = C_{4AB^2} \times R + O(\lambda^5) \quad \Rightarrow \quad \beta \equiv \frac{\partial\delta\lambda}{\partial\ln(\mu)}. \quad (36)$$

The tree order vertex can be inferred from the Feynman rules,

$$-iV_0(x; y; z) = -i\lambda\sqrt{-g(x)} g_x^{\mu\nu} \partial_\mu \delta^D(x-y) \partial_\nu \delta^D(x-z). \quad (37)$$

The various 1-loop contributions are shown in Figure 7. Because the leftmost diagram

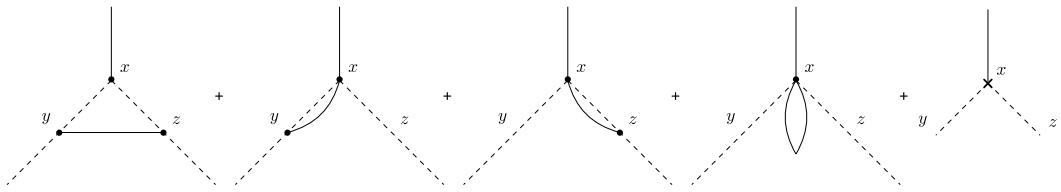


Figure 7: 1-loop contributions to the  $ABB$  vertex  $-iV(x; y; z)$ .

involves only 3-point couplings, it is unchanged from the old model [16],

$$\begin{aligned} -iV_{1a}(x; y; z) &= \frac{i\lambda^3}{2} \mathcal{D}_x \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(x; y) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(x; z) i\Delta(y; z) \right\} \\ &\quad + \frac{\lambda^3}{4} \mathcal{D}_y \mathcal{D}_z \left\{ [i\Delta(y; z)]^2 [\delta^D(x-y) + \delta^D(x-z)] \right\} \\ &\quad - \frac{\lambda^3}{2} \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(y; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(y; z) [\delta^D(x-y) + \delta^D(x-z)] \right\}. \quad (38) \end{aligned}$$

The 2nd and 3rd diagrams involve a single 4-point coupling and are therefore  $\frac{3}{2}$  times the results of the old model [16],

$$\begin{aligned} -iV_{1b}(x; y; z) &= \frac{3\lambda^3}{8}\mathcal{D}_y \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x \left[ i\Delta(x; y) \right]^2 \partial_\nu \delta^D(x-z) \right\} \\ &\quad + \frac{3\lambda^3}{4} \partial_\nu^z \partial_\beta^y \left\{ \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(z; y) \sqrt{-g_y} g_y^{\alpha\beta} \partial_\alpha^y i\Delta(z; y) \delta^D(x-z) \right\}, \end{aligned} \quad (39)$$

$$\begin{aligned} -iV_{1c}(x; y; z) &= \frac{3\lambda^3}{8}\mathcal{D}_z \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x [i\Delta(x; z)]^2 \partial_\nu \delta^D(x-y) \right\} \\ &\quad + \frac{3\lambda^3}{4} \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(y; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(y; z) \delta^D(x-y) \right\}. \end{aligned} \quad (40)$$

Before considering the 4th diagram, we combine and reduce expressions (38-40). Because the last two differ from the old model, the reduction is more complicated. Adding all three terms and performing some judicious partial integrations gives,

$$\begin{aligned} -iV_{1abc} &= \frac{i\lambda^3}{2} \mathcal{D}_x \partial_y^\rho \partial_z^\alpha \left\{ (a_y a_z)^{D-2} i\Delta(y; z) \partial_\rho^y i\Delta(x; y) \partial_\alpha^z i\Delta(x; z) \right\} \\ &\quad + \frac{3\lambda^3}{8} \mathcal{D}_y \partial_z^\alpha \left\{ a_z^{D-2} \left[ i\Delta(y; z) \right]^2 \partial_\alpha^z \delta^D(x-z) \right\} + \frac{3\lambda^3}{8} \mathcal{D}_z \partial_y^\alpha \left\{ a_y^{D-2} \left[ i\Delta(y; z) \right]^2 \partial_\alpha^y \delta^D(x-y) \right\} \\ &\quad - \frac{\lambda^3}{8} \mathcal{D}_y \mathcal{D}_z \left\{ \left[ i\Delta(y; z) \right]^2 \left[ \delta^D(x-y) + \delta^D(x-z) \right] \right\} \\ &\quad + \frac{\lambda^3}{4} \partial_y^\rho \partial_z^\alpha \left\{ (a_y a_z)^{D-2} \partial_\rho^y i\Delta(y; z) \partial_\alpha^z i\Delta(y; z) \left[ \delta^D(x-y) + \delta^D(x-z) \right] \right\}. \end{aligned} \quad (41)$$

After considerable manipulations explained in the Appendix expression the divergent part of (41) can be brought to the form,

$$-iV_{1abc}(x; y; z) \longrightarrow -\frac{i\lambda^3 \Gamma(\frac{D}{2}+1) \mu^{D-4} H^2 a_x^2}{32\pi^{\frac{D}{2}} (D-3)(D-4)} \partial^\mu \delta^D(x-y) \partial_\mu \delta^D(x-z) - i\tilde{V}(x; y; z), \quad (42)$$

where  $-i\tilde{V}(x; y; z)$  consists of higher derivative divergences,

$$\begin{aligned} -i\tilde{V}(x; y; z) &= \frac{i\lambda^3 \mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{D(D-3)(D-4)} \mathcal{D}_x \left[ \frac{\partial_\mu \delta^D(x-y) \partial^\mu \delta^D(x-z)}{a_x^{D-2}} \right] \\ &\quad + \frac{5i\lambda^3 \mu^{D-4} \Gamma(\frac{D}{2}-1)}{128\pi^{\frac{D}{2}} (D-3)(D-4)} \left\{ \mathcal{D}_y \partial_x^\mu \left[ \frac{\delta^D(x-y) \partial_\mu \delta^D(x-z)}{a_x^{D-2}} \right] + \mathcal{D}_z \partial_x^\mu \left[ \frac{\partial_\mu \delta^D(x-y) \cdot \delta^D(x-z)}{a_x^{D-2}} \right] \right\} \\ &\quad - \frac{\lambda^3 \Gamma(\frac{D}{2}-1) \mu^{D-4}}{64\pi^{\frac{D}{2}} (D-3)(D-4)} \mathcal{D}_y \mathcal{D}_z \left[ \frac{i\delta^D(x-z) \delta^D(x-y)}{a_x^{2D-4}} \right]. \end{aligned} \quad (43)$$

We call the new diagram  $-iV_{1d}(x; y; z)$  and its contribution is,

$$-iV_{1d} = -i \frac{3\lambda^3}{16} \sqrt{-g(x)} g^{\mu\nu}(x) \partial_\mu \delta^D(x-y) \partial_\nu \delta^D(x-z) \cdot i\Delta(x; x). \quad (44)$$

This is just the bare vertex (37) times the simple factor  $\frac{3\lambda^2}{16}i\Delta(x; x)$ . Recall that the coincident propagator is given in equation (15),

$$-iV_{1d} = -i\frac{3\lambda^3}{16}a_x^{D-2}\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z)k\left[-\pi\cot\left(\frac{D\pi}{2}\right) + 2\ln(a)\right], \quad (45)$$

$$= -i\frac{3\lambda^3k}{16}\times -\pi\cot\left(\frac{D\pi}{2}\right)\times a_x^2\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z) + O(D-4). \quad (46)$$

Adding the primitive divergences from (42) to this gives,

$$\begin{aligned} -iV_{1\text{pdv}}(x; y; z) &= -i\frac{\lambda^3H^2a_x^2}{8}\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z) \\ &\quad \times \left[\frac{3k}{2H^2}\times -\pi\cot\left(\frac{D\pi}{2}\right) + \frac{\mu^{D-4}\Gamma(\frac{D}{2}+1)}{4\pi^{\frac{D}{2}}(D-3)(D-4)}\right] - i\tilde{V}(x; y; z). \end{aligned} \quad (47)$$

The primitive divergences (47) are absorbed by the third variation of (11),

$$\begin{aligned} \frac{i\delta^3S_{AB^2}[A, B]}{\delta A(x)\delta B(y)\delta B(z)} &= -iC_{1AB^2}\mathcal{D}_x\left[\frac{\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z)}{a_x^2}\right] \\ &\quad + iC_{2AB^2}\mathcal{D}_y\partial_x^\mu\left[\frac{\delta^D(x-y)\partial_\mu\delta^D(x-z)}{a_x^2}\right] + iC_{2AB^2}\mathcal{D}_z\partial_x^\mu\left[\frac{\partial_\mu\delta^D(x-y)\times\delta^D(x-z)}{a_x^2}\right] \\ &\quad - iC_{3AB^2}\mathcal{D}_y\mathcal{D}_z\left[\frac{\delta^D(x-y)\delta^D(x-z)}{a_x^D}\right] - iC_{4AB^2}Ra_x^2\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z). \end{aligned} \quad (48)$$

Adding (48) to (47) and demanding regularity as  $D \rightarrow 4$  implies,

$$C_{1AB^2} = \frac{\lambda^3\mu^{D-4}\Gamma(\frac{D}{2}-1)}{16\pi^{\frac{D}{2}}D(D-3)(D-4)}, \quad (49)$$

$$C_{2AB^2} = -\frac{5\lambda^3\mu^{D-4}\Gamma(\frac{D}{2}-1)}{128\pi^{\frac{D}{2}}(D-3)(D-4)}, \quad (50)$$

$$C_{3AB^2} = -\frac{\lambda^3\mu^{D-4}\Gamma(\frac{D}{2}-1)}{64\pi^{\frac{D}{2}}(D-3)(D-4)}, \quad (51)$$

$$C_{4AB^2} = \frac{\lambda^3\mu^{D-4}}{32\pi^{\frac{D}{2}}D(D-1)}\left[\frac{3\pi\cot(\frac{D\pi}{2})\Gamma(D-1)}{8\Gamma(\frac{D}{2})} - \frac{\Gamma(\frac{D}{2}+1)}{(D-3)(D-4)}\right]. \quad (52)$$

Because we have suppressed the finite contributions the renormalized result will not be given but let us take note of the fact that logarithms of  $\mu$  come in the form  $\ln(\frac{\mu a}{H})$ . We can regard  $C_{4AB^2} \times R = \delta\lambda$  as a curvature-dependent renormalization. The associated 1-loop beta function (36) is,

$$\boxed{\beta = \frac{\partial\delta\lambda}{\partial\ln(\mu)} = -\frac{\lambda^3H^2}{64\pi^2} + O(\lambda^5)}. \quad (53)$$

## 4 Summing the Logarithms

Here we demonstrate that the results of the previous section for the new model (8) can be explained using the same combination of Starobinsky's stochastic formalism and the renormalization group that worked for the old model (5). We begin by deriving the curvature-dependent effective potential and showing that it explains the evolution of the background. We next apply our explicit results for the 1-loop counterterms to work out the curvature-dependent renormalization group corrections.

### 4.1 Curvature-Dependent Effective Potential

The shift symmetry of the field  $B(x)$  evident in (8) prevents it from developing an effective potential. However,  $A(x)$  does acquire one from integrating out the differentiated  $B$  fields from the  $A$  field equation in the presence of a constant  $A(x) = A_0$  background,

$$\frac{\delta S[A, B]}{\delta A(x)} = \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu A \right] - \frac{\lambda}{2} \left( 1 + \frac{\lambda}{4} A \right)^3 \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} = 0. \quad (54)$$

One can see from the Lagrangian (8) that a constant  $A(x) = A_0$  background just renormalizes the  $B$  field strength,

$$\left\langle \Omega \left| T \left[ B(x) B(x') \right] \right| \Omega \right\rangle_{A=A_0} = \frac{i \Delta(x; x')}{(1 + \frac{\lambda}{4} A_0)^4}. \quad (55)$$

Hence the  $A$  equation can be recognized as that of a scalar potential model,

$$\frac{\delta S[A, B]}{\delta A(x)} \longrightarrow \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu A \right] - \frac{\lambda}{2} \left( 1 + \frac{\lambda}{4} A \right)^3 \times \frac{\sqrt{-g} g^{\mu\nu} \partial_\mu \partial'_\nu i \Delta(x; x')|_{x'=x}}{(1 + \frac{\lambda}{4} A)^4}, \quad (56)$$

$$= \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu A \right] + \frac{(\frac{D-1}{2}) \lambda k H^2 \sqrt{-g}}{1 + \frac{\lambda}{4} A} \equiv \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu A \right] - V'_{\text{eff}}(A) \sqrt{-g}. \quad (57)$$

Note that we have employed expression (16) to evaluate the coincidence limit of the doubly differentiated propagator. This is free of divergences, so we can set  $D = 4$  to find,

$$V'_{\text{eff}}(A) = -\frac{3\lambda H^4}{16\pi^2} \left( 1 + \frac{\lambda}{4} A \right)^{-1} \implies V_{\text{eff}}(A) = -\frac{3H^4}{4\pi^2} \ln \left( 1 + \frac{\lambda}{4} A \right). \quad (58)$$

We see that the new model's curvature-dependent effective potential (58) is almost the same as the old model's result of  $-\frac{3H^4}{8\pi^2} \ln(1 + \frac{\lambda}{2} A)$  [15].

Starobinsky long ago showed how to sum the leading logarithms of a scalar potential model like (57) [13, 14]. The leading logarithms contained in correlators of the quantum field  $A(t, \vec{x})$  turn out to agree, to all orders [9], with those of stochastic random field  $\mathcal{A}(t, \vec{x})$  which obeys the Langevin equation,

$$3H \left[ \dot{\mathcal{A}} - \dot{\mathcal{A}}_0 \right] = V'_{\text{eff}}(\mathcal{A}). \quad (59)$$

The “stochastic jitter” in this equation is supplied by the time derivative of the infrared-truncated, free field expansion of  $A(t, \vec{x})$ ,

$$\mathcal{A}_0(t, \vec{x}) \equiv \int_H^{aH} \frac{d^3 k}{(2\pi)^3} \frac{H}{\sqrt{2k^3}} \left\{ \alpha(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + \alpha^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\} , \quad [\alpha(\vec{k}), \alpha^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') . \quad (60)$$

If we turn off the stochastic jitter then equation (59) is simple to solve, adopting the initial condition  $\mathcal{A}(0, \vec{x}) = 0$  and noting that  $\ln(a) = Ht$ ,

$$\mathcal{A}(t, \vec{x}) \Big|_{\mathcal{A}_0=0} = \frac{4}{\lambda} \left[ \sqrt{1 + \frac{\lambda^2 H^2 \ln(a)}{32\pi^2}} - 1 \right] = \frac{\lambda H^2 \ln(a)}{16\pi^2} - \frac{\lambda^3 H^4 \ln^2(a)}{2^{11}\pi^4} + O(\lambda^5) . \quad (61)$$

Because it is easier to fluctuate down the potential than up, we expect that the effect of adding stochastic jitter is to accelerate the evolution of  $\mathcal{A}$  down its potential. Solving (59) perturbatively gives,

$$\mathcal{A} = \mathcal{A}_0 + \frac{\lambda H^2 \ln(a)}{16\pi^2} - \frac{\lambda^2 H^3}{2^6 \pi^2} \int_0^t dt' \mathcal{A}_0(t', \vec{x}) - \frac{\lambda^3 H^4 \ln^2(a)}{2^{11} \pi^4} + \frac{\lambda^3 H^3}{2^8 \pi^2} \int_0^t dt' \mathcal{A}_0^2(t', \vec{x}) + O(\lambda^4) . \quad (62)$$

By taking the expectation value of the previous equation, and using the mode sum (60) to conclude,

$$\langle \Omega | \mathcal{A}_0(t, \vec{x}) | \Omega \rangle = 0 \quad , \quad \langle \Omega | \mathcal{A}_0^2(t, \vec{x}) | \Omega \rangle = \frac{H^2 \ln(a)}{4\pi^2} , \quad (63)$$

we find,

$$\langle \Omega | \mathcal{A}(t, \vec{x}) | \Omega \rangle = \frac{\lambda H^2 \ln(a)}{16\pi^2} + O(\lambda^5) . \quad (64)$$

This is exactly the result (26) we got by explicit computation, which is impressive confirmation of the stochastic prediction.

## 4.2 Curvature-Dependent Renormalization Group

In a theory with derivative interactions not all of the large logarithms derive from stochastic effects. Some of them arise instead from the incomplete cancellation between primitive divergences — which have no  $D$ -dependent powers of the scale factor — and counterterms — which inherit a  $a^D$  from the measure factor  $\sqrt{-g}$ ,

$$\left( \frac{H^{D-4}}{D-4} \right) - \left( \frac{\mu^{D-4} a^{D-4}}{D-4} \right) = -\ln\left(\frac{\mu a}{H}\right) + O(D-4) . \quad (65)$$

These logarithms can be recovered using the renormalization group, which follows factors of  $\ln(\mu)$ . Because we are only interested in cases where the factors of  $\ln(a)$  are not suppressed by inverse powers of the scale factor, the counterterms of concern are those which can be viewed as curvature-dependent renormalizations of bare parameters [15]. Of the four counterterms (11) required to renormalize the vertex function at 1-loop order we have already seen that the contribution proportional to  $C_{4AB^2}$  can be viewed as a curvature-dependent renormalization of the bare vertex, and we used this to compute the associated beta function (36). Of the

four counterterms (10) required to renormalize the 1-loop self-masses it is the contributions proportional to  $C_{2A^2}$  and  $C_{2B^2}$  which can be regarded as curvature-dependent field strength renormalizations,

$$Z_A \equiv 1 + C_{2A^2} \times R + O(\lambda^4) \quad , \quad Z_B \equiv 1 + C_{2B^2} \times R + O(\lambda^4) . \quad (66)$$

Our explicit results (30) and (35) give the associated gamma functions,

$$\boxed{\gamma_A \equiv \frac{\partial \ln(Z_A)}{\partial \ln(\mu^2)} = \frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4) \quad , \quad \gamma_B \equiv \frac{\partial \ln(Z_B)}{\partial \ln(\mu^2)} = -\frac{\lambda^2 H^2}{64\pi^2} + O(\lambda^4) .} \quad (67)$$

We are ready to investigate the Callan-Symanzik equations for the n-point Green's functions  $G_n(x_1; x_2; \dots; x_n; \lambda; \mu)$  of the field  $A$ ,<sup>2</sup>

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + n\gamma_A \right] G_n(x_1; x_2; \dots; x_n; \lambda; \mu) = 0 . \quad (68)$$

This equation can be solved using the method of characteristics. We first find a running coupling constant  $\bar{\lambda}(\mu)$  which obeys the differential equation and initial condition,

$$\mu \frac{d\bar{\lambda}}{d\mu} = -\beta(\bar{\lambda}(\mu)) \quad , \quad \bar{\lambda}(\mu_0) = \lambda \quad \Rightarrow \quad \beta(\lambda) \frac{\partial \bar{\lambda}}{\partial \lambda} = \beta(\bar{\lambda}) . \quad (69)$$

We can then write the solution as,

$$G_n(x_1; x_2; \dots; x_n; \lambda; \mu) = G_n(x_1; x_2; \dots; x_n; \bar{\lambda}(\mu); \mu_0) \times \exp \left[ -n \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\bar{\lambda}(\mu')) \right] . \quad (70)$$

Inserting the  $\beta$  function (53) into (69), and ignoring higher loop corrections yields,

$$\bar{\lambda}(\mu) = \frac{\lambda}{\sqrt{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(\frac{\mu}{\mu_0})}} . \quad (71)$$

Substituting (71) and (67) into (70), and again ignoring higher loop corrections, gives,

$$G_n(x_1; x_2; \dots; x_n; \lambda; \mu) = G_n(x_1; x_2; \dots; x_n; \bar{\lambda}(\mu); \mu_0) \times \left[ 1 - \frac{\lambda^2 H^2}{32\pi^2} \ln\left(\frac{\mu}{\mu_0}\right) \right]^n . \quad (72)$$

Similar results could be derived for Green's functions which also, or even exclusively, involve the field  $B(x)$ .

Having a negative beta function traditionally means that the theory runs towards weak coupling in the ultraviolet because logarithms of the scale  $\mu$  are associated with inverse factors of some characteristic momentum in the process. In cosmology we are interested in how things behave at *late times*, and we note from expression (65) that the scale  $\mu$  is associated with the scale factor  $a(t)$  in the form  $\ln[\mu a/H]$ . It should therefore be that having a negative beta function means the theory evolves towards strong coupling at late times.

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<sup>2</sup>1PI  $n$ -point functions obey a similar equation with the last term replaced by  $-n\gamma_A$ .

## 5 Conclusion

In this work, we examined a nontrivial nonlinear sigma model (defined in section 2) whose loop corrections on de Sitter background show the same large logarithms as have been reported from inflationary gravitons [3–8]. Previous work [15, 16] on nonlinear sigma models which can be reduced to free theories has shown that these large logarithms can be resummed by combining a variant of Starobinsky’s stochastic formalism [13, 14] — based on curvature-dependent effective potentials — with a variant of the renormalization group — based on the subset of counterterms which can be viewed as curvature-dependent renormalizations of bare parameters. Our analysis confirms that these techniques continue to apply for models whose S-matrix is nontrivial. In section 3.1, we explicitly computed the evolution of the background (26) at 1-loop and 2-loop orders. In section 4.1 we showed that the resummation techniques correctly predict these results.

One significant difference associated with a nontrivial S-matrix is that the beta function does not vanish. The 1-loop beta function (53) was derived in section 3.3. The 1-loop gamma functions (67) were derived in section 3.2, and combined with the beta function in section 4.2, to solve the Callan-Symanzik equation for  $n$ -point Green’s functions (72). Because the beta function (53) is negative, the running coupling constant  $\bar{\lambda}(\mu)$  grows with the scale  $\mu$  as shown in equation (71). From (65) we see that logarithms of  $\mu$  are associated with the scale factor  $a(t)$  in the form  $\ln[\mu a(t)/H]$ . This implies that the model becomes strongly coupled at late times.

An interesting higher loop phenomenon is that there can be mixing of large stochastic logarithms with large logarithms from the renormalization group. We speculate that the correct way to include these is to use the renormalization group to improve the curvature-dependent effective potential and then use that, without any extra RG corrections. From (67) we observe that, because  $\gamma_A \sim \lambda^2$ , an RG-improvement to the effective potential (58) will include lowest order corrections of the form  $\lambda^3 A$ . Those would not engender any changes in the background evolution at leading logarithm order.

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## A Reduction of the 1-loop vertex function

The aim of this appendix is to reduce the last line of equation (41) for  $-iV_{1abc}(x; y; z)$ ,

$$I(x; y; z) \equiv \frac{\lambda^3}{4} \partial_y^\rho \partial_z^\alpha \left\{ (a_y a_z)^{D-2} \partial_\rho^y i\Delta(y; z) \partial_\alpha^z i\Delta(y; z) \left[ \delta^D(x-y) + \delta^D(x-z) \right] \right\}. \quad (73)$$

Note that  $\partial_\rho^y i\Delta(y; z) \partial_\alpha^z i\Delta(y; z)$  is quadratically divergent, so extracting just the divergent part of  $I(x; y; z)$  requires only the two terms given on the first line of expansion (14) for each

propagator,

$$\partial_\rho^y i\Delta(y; z) = -\frac{\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}(a_y a_z)^{\frac{D}{2}-1}} \left[ \frac{(y-z)_\rho}{(y-z)^D} + \frac{a_y H \delta_\rho^0}{2(y-z)^{D-2}} + \frac{D a_y a_z H^2 (y-z)_\rho}{8(y-z)^{D-2}} + \dots \right], \quad (74)$$

$$\partial_\alpha^z i\Delta(y; z) = -\frac{\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}(a_y a_z)^{\frac{D}{2}-1}} \left[ -\frac{(y-z)_\alpha}{(y-z)^D} + \frac{a_z H \delta_\alpha^0}{2(y-z)^{D-2}} - \frac{D a_y a_z H^2 (y-z)_\alpha}{8(y-z)^{D-2}} + \dots \right]. \quad (75)$$

Multiplying (74) and (75), and retaining only potentially divergent terms yields,

$$\begin{aligned} \partial_\rho^y i\Delta(y; z) \times \partial_\alpha^z i\Delta(y; z) &= \frac{\Gamma^2(\frac{D}{2})}{4\pi^D (a_y a_z)^{D-2}} \left\{ -\frac{(y-z)_\rho (y-z)_\alpha}{(y-z)^{2D}} + \frac{a_z H \delta_\alpha^0 (y-z)_\rho}{2(y-z)^{2D-2}} \right. \\ &\quad \left. - \frac{a_y H \delta_\rho^0 (y-z)_\alpha}{2(y-z)^{2D-2}} + \frac{a_y a_z H^2 \delta_\rho^0 \delta_\alpha^0}{4(y-z)^{2D-4}} - \frac{D a_y a_z H^2 (y-z)_\rho (y-z)_\alpha}{4(y-z)^{2D-2}} + \dots \right\}. \end{aligned} \quad (76)$$

The terms highlighted in red and blue require separate reductions which we give below.

We first extract two derivatives from the red terms of expression (76),

$$\begin{aligned} &\frac{\Gamma^2(\frac{D}{2})}{16\pi^D (a_y a_z)^{D-2}} \left\{ -\frac{(y-z)_\rho (y-z)_\alpha}{(y-z)^{2D}} + \frac{a_z H \delta_\alpha^0 (y-z)_\rho}{2(y-z)^{2D-2}} - \frac{a_y H \delta_\rho^0 (y-z)_\alpha}{2(y-z)^{2D-2}} + \frac{a_y a_z H^2 \delta_\rho^0 \delta_\alpha^0}{4(y-z)^{2D-4}} \right\} \\ &= \frac{\Gamma^2(\frac{D}{2}-1)}{64\pi^D} \left\{ \partial_\rho^y \partial_\alpha^z \left[ \frac{1}{(a_y a_z)^{D-2} (y-z)^{2D-4}} \right] + \frac{[\partial_\rho^y \partial_\alpha^y - \eta_{\rho\alpha} \partial_y^2]}{(D-1)(a_y a_z)^{D-2}} \left[ \frac{1}{(y-z)^{2D-4}} \right] \right\}. \end{aligned} \quad (77)$$

The denominator  $(y-z)^{2D-4}$  is logarithmically divergent so we can extract a local divergence from it by adding the flat space propagator equation [24, 25],

$$\frac{1}{(y-z)^{2D-4}} = \frac{\mu^{D-4}}{2(D-3)(D-4)} \frac{4\pi^{\frac{D}{2}} i \delta^D (y-z)}{\Gamma(\frac{D}{2}-1)} - \frac{\partial_y^2}{4} \left[ \frac{\ln[\mu^2 (y-z)^2]}{(y-z)^2} \right] + O(D-4). \quad (78)$$

Substituting (78) in (77), and retaining only the divergences, reduces the red terms of expression (76) to,

$$\begin{aligned} &\frac{\Gamma^2(\frac{D}{2})}{16\pi^D (a_y a_z)^{D-2}} \left\{ -\frac{(y-z)_\rho (y-z)_\alpha}{(y-z)^{2D}} + \frac{a_z H \delta_\alpha^0 (y-z)_\rho}{2(y-z)^{2D-2}} - \frac{a_y H \delta_\rho^0 (y-z)_\alpha}{2(y-z)^{2D-2}} + \frac{a_y a_z H^2 \delta_\rho^0 \delta_\alpha^0}{4(y-z)^{2D-4}} \right\} \\ &= \frac{\Gamma(\frac{D}{2}-1) \mu^{D-4}}{32\pi^{\frac{D}{2}} (D-3)(D-4)} \left\{ \partial_\rho^y \partial_\alpha^z \left[ \frac{i \delta^D (y-z)}{(a_y a_z)^{D-2}} \right] + \frac{[\partial_\rho^y \partial_\alpha^y - \eta_{\rho\alpha} \partial_y^2] i \delta^D (y-z)}{(D-1)(a_y a_z)^{D-2}} \right\} + (\text{Finite}). \end{aligned} \quad (79)$$

We now extract derivatives from the blue term in (76),

$$\begin{aligned} &-\frac{\Gamma(\frac{D}{2}) \Gamma(\frac{D}{2}+1) H^2}{8\pi^D (a_y a_z)^{D-3}} \times \frac{(y-z)_\rho (y-z)_\alpha}{(y-z)^{2D-2}} \\ &= -\frac{\Gamma(\frac{D}{2}-1) \Gamma(\frac{D}{2}+1) H^2}{32\pi^D (a_y a_z)^{D-3}} \times \left[ \frac{\partial_\rho^y \partial_\alpha^y}{2(D-3)} + \frac{\eta_{\rho\alpha} \partial_y^2}{2(D-3)(D-4)} \right] \frac{1}{(y-z)^{2D-6}}. \end{aligned} \quad (80)$$

Of the two terms inside the square brackets at the end of (80), only the one proportional to  $\eta_{\rho\alpha}\partial_y^2$  is divergent. Using the same reduction as (78) we therefore reduce the blue term of expression (76) to,

$$-\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)H^2}{8\pi^D(a_ya_z)^{D-3}} \times \frac{(y-z)_\rho(y-z)_\alpha}{(y-z)^{2D-2}} = -\frac{\Gamma(\frac{D}{2}+1)}{16\pi^{\frac{D}{2}}} \frac{\mu^{D-4}\eta_{\rho\alpha}H^2i\delta^D(y-z)}{(D-3)(D-4)(a_ya_z)^{D-3}} + (\text{Finite}). \quad (81)$$

We are now ready to employ expressions (79) and (81) in (73). Note that the transverse projection operator in the second term of (79) vanishes upon canceling the scale factors and exploiting the  $i\delta^D(y-z)$  to reflect derivatives where necessary,

$$\begin{aligned} & \frac{\lambda^3}{4} \partial_y^\rho \partial_z^\alpha \left\{ (a_ya_z)^{D-2} \times \frac{\Gamma(\frac{D}{2}-1)\mu^{D-4}}{32\pi^{\frac{D}{2}}(D-3)(D-4)} \frac{[\partial_\rho^y \partial_\alpha^y - \eta_{\rho\alpha} \partial_y^2] i\delta^D(y-z)}{(D-1)(a_ya_z)^{D-2}} [\delta^D(x-y) + \delta^D(x-z)] \right\} \\ &= \frac{\lambda^3 \mu^{D-4} \Gamma(\frac{D}{2}-1) \partial_y^\rho \partial_z^\alpha}{128\pi^{\frac{D}{2}}(D-1)(D-3)(D-4)} \left\{ \left[ \partial_\rho^z \partial_\alpha^z - \eta_{\rho\alpha} \partial_z^2 \right] i\delta^D(y-z) \times \delta^D(x-y) \right. \\ & \quad \left. + \left[ \partial_\rho^y \partial_\alpha^y - \eta_{\rho\alpha} \partial_y^2 \right] i\delta^D(y-z) \times \delta^D(x-z) \right\} = 0. \end{aligned} \quad (82)$$

The divergent part of  $I(x; y; z)$  comes from the first term of (79) and from (80). After some judicious partial integrations it can be written as,

$$\begin{aligned} I_{\text{div}} &= \frac{i\lambda^3 \mu^{D-4} \Gamma(\frac{D}{2}-1)}{128\pi^{\frac{D}{2}}(D-3)(D-4)} \left\{ 2\mathcal{D}_y \mathcal{D}_z \left[ \frac{\delta^D(x-y) \delta^D(x-z)}{(a_ya_z)^{D-2}} \right] - \mathcal{D}_y \partial_z^\alpha \left[ \frac{\delta^D(y-z) \partial_\alpha^z \delta^D(x-z)}{a_y^{D-2}} \right] \right. \\ & \quad \left. - \mathcal{D}_z \partial_y^\rho \left[ \frac{\delta^D(y-z) \partial_\rho^y \delta^D(x-y)}{a_z^{D-2}} \right] \right\} - \frac{i\lambda^3 H^2 a_x^2 \mu^{D-4} \Gamma(\frac{D}{2}+1)}{32\pi^{\frac{D}{2}}(D-3)(D-4)} \partial_x^\mu \delta^D(x-y) \partial_\mu^x \delta^D(x-z). \end{aligned} \quad (83)$$

## References

- [1] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532-535 (1981)
- [2] A. A. Starobinsky, JETP Lett. **30**, 682-685 (1979)
- [3] L. Tan, N. C. Tsamis and R. P. Woodard, Phil. Trans. Roy. Soc. Lond. A **380**, 0187 (2021) doi:10.1098/rsta.2021.0187 [arXiv:2107.13905 [gr-qc]].
- [4] L. Tan, N. C. Tsamis and R. P. Woodard, Universe **8**, no.7, 376 (2022) doi:10.3390/universe8070376 [arXiv:2206.11467 [gr-qc]].
- [5] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 024021 (2006) doi:10.1103/PhysRevD.74.024021 [arXiv:gr-qc/0603135 [gr-qc]].
- [6] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. **31**, 175002 (2014) doi:10.1088/0264-9381/31/17/175002 [arXiv:1308.3453 [gr-qc]].

[7] C. L. Wang and R. P. Woodard, Phys. Rev. D **91**, no.12, 124054 (2015) doi:10.1103/PhysRevD.91.124054 [arXiv:1408.1448 [gr-qc]].

[8] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, JHEP **03**, 088 (2022) doi:10.1007/JHEP03(2022)088 [arXiv:2112.00959 [gr-qc]].

[9] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **724**, 295-328 (2005) doi:10.1016/j.nuclphysb.2005.06.031 [arXiv:gr-qc/0505115 [gr-qc]].

[10] H. Kitamoto and Y. Kitazawa, Phys. Rev. D **83**, 104043 (2011) doi:10.1103/PhysRevD.83.104043 [arXiv:1012.5930 [hep-th]].

[11] H. Kitamoto and Y. Kitazawa, Phys. Rev. D **85**, 044062 (2012) doi:10.1103/PhysRevD.85.044062 [arXiv:1109.4892 [hep-th]].

[12] H. Kitamoto, Phys. Rev. D **100**, no.2, 025020 (2019) doi:10.1103/PhysRevD.100.025020 [arXiv:1811.01830 [hep-th]].

[13] A. A. Starobinsky, Lect. Notes Phys. **246**, 107-126 (1986) doi:10.1007/3-540-16452-9\_6

[14] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357-6368 (1994) doi:10.1103/PhysRevD.50.6357 [arXiv:astro-ph/9407016 [astro-ph]].

[15] S. P. Miao, N. C. Tsamis and R. P. Woodard, JHEP **03**, 069 (2022) doi:10.1007/JHEP03(2022)069 [arXiv:2110.08715 [gr-qc]].

[16] R. P. Woodard and B. Yesilyurt, [arXiv:2302.11528 [gr-qc]].

[17] E. Kasdagli, M. Ulloa and R. P. Woodard, Phys. Rev. D **107**, no.10, 105023 (2023) doi:10.1103/PhysRevD.107.105023 [arXiv:2302.04808 [gr-qc]].

[18] R. P. Woodard and B. Yesilyurt, [arXiv:2305.17641 [gr-qc]].

[19] H. J. Borchers, Il Nuovo Cimento **15**, 784-794 (1960) doi.org/10.1007/BF02732693.

[20] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97**, 227-266 (1957) doi:10.1007/BF02392399

[21] K. Hepp, Commun. Math. Phys. **2**, 301-326 (1966) doi:10.1007/BF01773358

[22] W. Zimmermann, Commun. Math. Phys. **11**, 1-8 (1968) doi:10.1007/BF01654298

[23] W. Zimmermann, Commun. Math. Phys. **15**, 208-234 (1969) doi:10.1007/BF01645676

[24] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. **19**, 4607 (2002) doi:10.1088/0264-9381/19/17/311 [arXiv:gr-qc/0204065 [gr-qc]].

[25] V. K. Onemli and R. P. Woodard, Phys. Rev. D **70**, 107301 (2004) doi:10.1103/PhysRevD.70.107301 [arXiv:gr-qc/0406098 [gr-qc]].